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## Measurements of Thermal Constants of Soil

### — Thermal Diffusivity —

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#### Abstract

To determine thermal diffusivity of soil, two methods called amplitude and time lag methods have been used. These methods, however, have some assumptions which do not agree with actual state of soil. Then the new method is shown in this paper. The writer considered boundary conditions at two finite depths and initial condition shown in the soil between the two boundaries in order to avoid the unsuitable assumptions in the amplitude and time lag methods, and solved the differential equation of heat conduction in soil under these conditions.

Application of the method was illustrated by using data measured at Hokkaido Prefecture Agricultural Experimental Station in Takikawa. The result derived from the method was compared with the values obtained from the amplitude and time lag methods, and it was found that the values from the amplitude and time lag methods was very different from the value by the new method.

#### 1. Introduction

In the amplitude and time lag methods<sup>1)</sup>, the differential equation of heat conduction in soil was solved under the following conditions:

1) Variation of soil temperature was regarded sine curve and the period was taken one day or one year.

2) Two boundary conditions were given by the temperatures at soil surface and infinite depth, and these conditions were as follows:

$$v = \bar{v} + A \sin \frac{2\pi}{T} t, \quad \text{at } x = 0, \quad (1-1)$$

$$v = \bar{v} + \alpha x, \quad \text{at } x = \infty, \quad (1-2)$$

where  $t$  and  $x$  represent time and depth from the soil surface, and  $A$  and  $T$  are amplitude and period of the variation of the temperature at the soil surface respectively,  $\bar{v}$  is mean value of the variation of the soil temperature at  $x=0$ , and  $\alpha$  is increasing rate of the temperature for the depth.

From (1-1), (1-2) and the differential equation of heat conduction, soil temperature  $v$  is given by

$$v = \bar{v} + \alpha x + A e^{-x \sqrt{\frac{\pi}{\kappa T}}} \sin \left( \frac{2\pi}{T} t - x \sqrt{\frac{\pi}{\kappa T}} \right), \quad (1-3)$$

where  $\kappa$  is thermal diffusivity of soil. Now, let  $B_1$ ,  $C_1$  and  $\varepsilon_1$ ,  $\varepsilon_1'$  be respectively amplitudes and time lags of the variation of the soil temperatures at  $x_1$  and  $x_2$  depths. From (1-3) the ratio of amplitude  $B_1/C_1$  becomes to

$$\frac{B_1}{C_1} = e^{-\sqrt{\frac{\pi}{\kappa T}} (x_1 - x_2)}, \quad (1-4)$$

accordingly

$$\log \left( \frac{B_1}{C_1} \right) = -\sqrt{\frac{\pi}{\kappa T}} (x_1 - x_2). \quad (1-5)$$

Then it follows from (1-5) that

$$\kappa = \frac{\pi}{T} \frac{(x_1 - x_2)^2}{\left\{ \log \left( \frac{B_1}{C_1} \right) \right\}^2}. \quad (1-6)$$

The method of determining the thermal diffusivity of soil from (1-6) is called the amplitude method. Secondary, from (1-3) the difference of time lag  $\varepsilon_1 - \varepsilon_1'$  is expressed as

$$\varepsilon_1 - \varepsilon_1' = \sqrt{\frac{\pi}{\kappa T}} (x_1 - x_2). \quad (1-7)$$

Therefore

$$\kappa = \frac{\pi}{T} \frac{(x_1 - x_2)^2}{(\varepsilon_1 - \varepsilon_1')^2}. \quad (1-8)$$

The method using (1-8) is called the time lag method.

Fig. 1 shows the variations of atmospheric and soil temperatures at Sapporo District Meteorological Observatory, and the weather during the observational period was very fine. From the figure it is clear that the variation of the atmospheric temperature is not sine curve, and the soil temperature influenced by the atmospheric temperature does not show either sine curve having the period of one day. Consequently this fact did not satisfy condition (1), and it became necessary that the initial condition of the soil temperature should be taken into consideration. Condition (2) which boundaries are taken at  $x=0$  and  $x=\infty$ , means that the soil is homogeneous from surface to infinite depth. Nevertheless Fig. 2 indicating a section of soil at the observational place, is apparently not consisted of the uniform soil. Then this

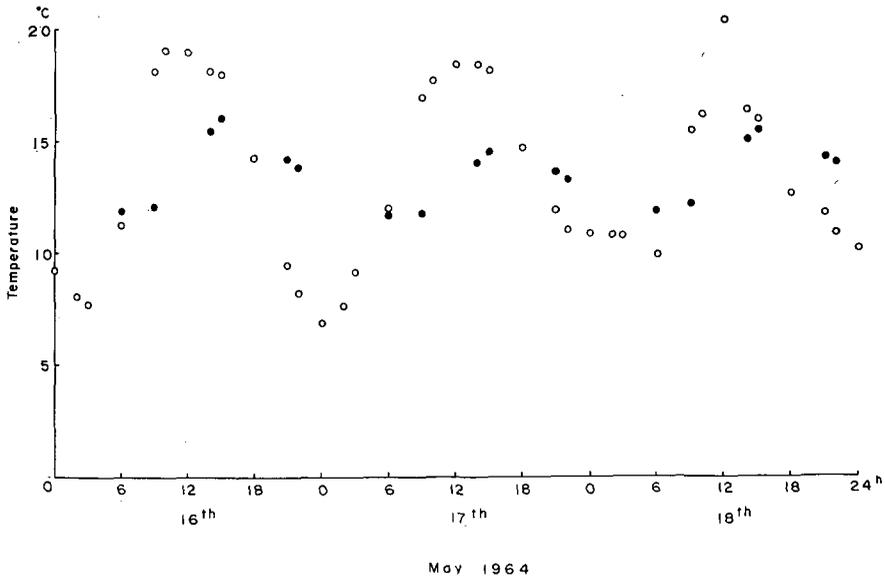


Fig. 1. Variations of atmospheric and soil temperatures.  
 ○: atmospheric temperature, ●: soil temperature at 10 cm depth.

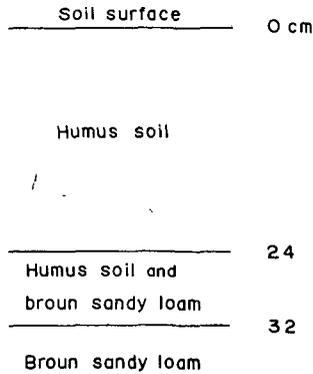


Fig. 2. Section of the soil at the observational place.

means that the condition (2) can not be identical with the actual soil section. Therefore one needs to select two boundaries between which the state of soil must be homogeneous, and the obtained value of  $\kappa$  shows the thermal diffusivity of the soil between the two boundaries. Fig. 3 represents the variation of soil temperature at 5 cm depth in Hokkaido Prefecture Agricultural Experimental Station in Takikawa. The variation does not coincide with

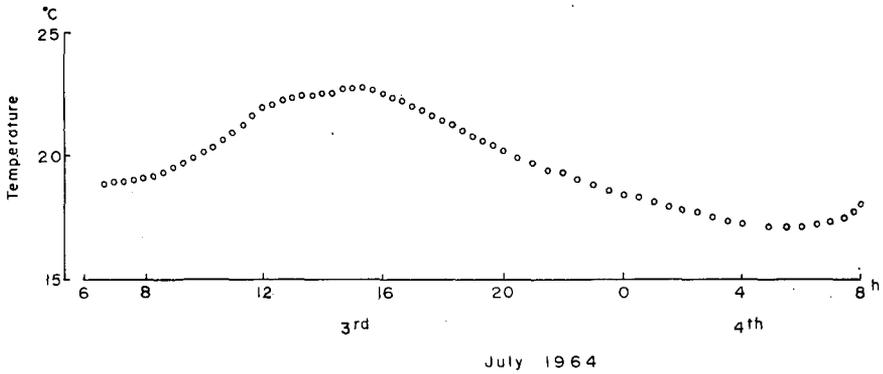


Fig. 3. Variation of soil temperature at Takikawa, Hokkaido.

sine curve, and does not express (1-3). This discrepancy is due to the boundary condition at  $x=0$  represented as (1-1), the boundary condition (1-1) needs to be given by other equation showing the actual soil surface temperature.

To avoid the above mentioned discrepancy the writer selected arbitrary time range, took the boundary conditions at two finite depths between which the soil is considered uniform, and gave the initial condition given in the soil between the two boundaries.

## 2. Variation of soil temperature influenced by the change of atmospheric temperature

When soil surface is flat and special heat source does not exist in the soil, variation of the soil temperature is influenced only by the change of atmospheric temperature, and may be considered as the functions of time and depth from the soil surface, but may not vary with the perpendicular direction to the depth. Then the variation of the soil temperature can be treated as the problem of one dimension, and differential equation of heat conduction in soil is expressed by

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} \quad (0 < x < l \text{ and } 0 < t < T_0), \quad (2-1)$$

where  $l$  is the distance between the two boundaries, and  $T_0$  is the time range of the variation of the soil temperature.

When the boundary and initial conditions are given by

$$v = g_1(t), \quad \text{at} \quad x = 0, \quad (2-2)$$

$$v = g_2(t), \quad \text{at} \quad x = l, \quad (2-3)$$

$$v = f(x), \quad \text{when } t = 0, \quad (2-4)$$

where  $x=0$  shows the one plane giving one boundary condition, and does not necessarily express the soil surface, the solution of (2-1) satisfying (2-2), (2-3) and (2-4) was obtained by CARSLAW and JAEGER<sup>2)</sup> as the following equation:

$$v = \frac{2}{l} \sum_{n=1}^{\infty} e^{-\kappa n^2 \pi t / l^2} \sin \frac{n \pi x}{l} \left[ \int_0^l f(x') \sin \frac{n \pi x'}{l} dx' + \frac{n \kappa \pi}{l} \int_0^l e^{\kappa n^2 \pi^2 \lambda / l^2} \{g_1(\lambda) - (-1)^n g_2(\lambda)\} d\lambda \right]. \quad (2-5)$$

If  $g_1(t)$  and  $g_2(t)$  are represented by the Fourier series and  $f(x)$  is equal to zero, namely

$$g_1(t) = C_0 + \sum_{m=1}^{\infty} C_m \sin(m \omega t + \varepsilon'_m), \quad (2-6)$$

$$g_2(t) = B_0 + \sum_{m=1}^{\infty} B_m \sin(m \omega t + \varepsilon_m), \quad (2-7)$$

$$f(x) = 0, \quad (2-8)$$

where

$$\omega = \frac{2 \pi}{T_0}, \quad (2-9)$$

and  $m$  is positive integer,  $\varepsilon_m$  and  $\varepsilon'_m$  show the phase lags of each sine curve, by substituting (2-6), (2-7) and (2-8) into (2-5),  $v$  becomes to

$$v = \sum_{m=1}^{\infty} B_m A_m \sin(m \omega t + \varepsilon_m + \phi_m) + F(x, t), \quad (2-10)$$

$$F(x, t) = \frac{2 \kappa \pi}{l^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n \sin \frac{n \pi x}{l} \left[ \frac{C_m D_n}{D_n^2 + m^2 \omega^2} \left\{ \sin(m \omega t + \varepsilon'_m) - \frac{m \omega}{D_n} \cos(m \omega t + \varepsilon'_m) \right\} - e^{-D_n t} \left( \sin \varepsilon'_m - \frac{m \omega}{D_n} \cos \varepsilon'_m \right) \right] + \left\{ \frac{B_m D_n}{D_n^2 + m^2 \omega^2} (-1)^n \left( \sin \varepsilon_m - \frac{m \omega}{D_n} \cos \varepsilon_m \right) - \frac{(C_0 - (-1)^n B_0)}{D_n} \right\} e^{-D_n t} \right] + G(x), \quad (2-11)$$

where

$$D_n = \frac{\kappa n^2 \pi^2}{l^2},$$

$$A_m = \left| \frac{\sinh k_m x (1+i)}{\sinh k_m l (1+i)} \right| = \left\{ \frac{\cosh 2 k_m x - \cos 2 k_m x}{\cosh 2 k_m l - \cos 2 k_m l} \right\}^{1/2}, \quad (2-12)$$

$$\phi_m = \arg. \left\{ \frac{\sinh k_m x (1+i)}{\sinh k_m l (1+i)} \right\}, \quad (2-13)$$

$$k_m = \left( \frac{m \omega}{2 \kappa} \right)^{1/2}, \quad (2-14)$$

$$G(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \{C_0 - (-1)^n B_0\} \frac{\sin \frac{n \pi x}{l}}{n}, \quad (2-15)$$

where  $i$  is unit of imaginary number.

From (2-12) and (2-13),  $A_m$  and  $\phi_m$  are found to be the functions of  $k_m$ ,  $x$  and  $l$ . When  $T_0$  is selected and substituted into (2-9)  $\omega$  becomes constant, therefore  $k_m$  is merely the function of  $m$  from (2-14). Then, in this case  $A_m$  and  $\phi_m$  become the functions of  $m$ ,  $x$  and  $l$ .

Now, let  $A_1$ ,  $\phi_1$  and  $k_1$  be the values of  $A_m$ ,  $\phi_m$  and  $k_m$  at  $m=1$ . It follows from (2-12), (2-13) and (2-14) that

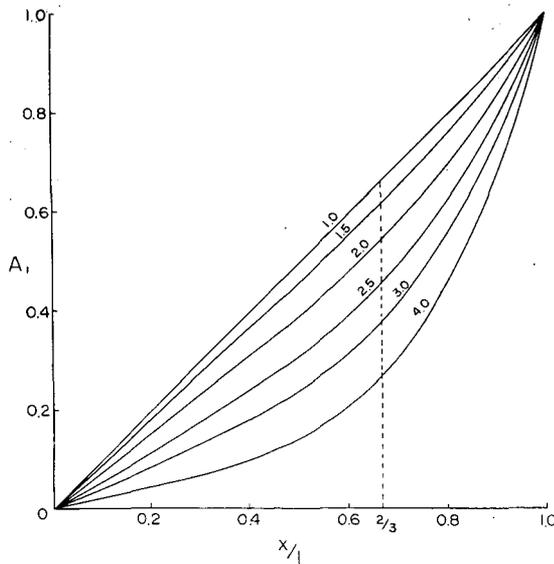


Fig. 4. Variation of  $A_1$  for  $x/l$ . Numbers on the curves are the values of  $k_1 l$ .

$$A_1 = \left\{ \frac{\cosh 2(k_1 l) \left(\frac{x}{l}\right) - \cos 2(k_1 l) \left(\frac{x}{l}\right)}{\cosh 2(k_1 l) - \cos 2(k_1 l)} \right\}^{1/2}, \tag{2-16}$$

$$\phi_1 = \arg. \left\{ \frac{\sinh(k_1 l) \left(\frac{x}{l}\right) (1+i)}{\sinh(k_1 l) (1+i)} \right\}, \tag{2-17}$$

$$k_1 = \left(\frac{\omega}{2\kappa}\right)^{1/2}. \tag{2-18}$$

Accordingly,  $A_1$  and  $\phi_1$  may be considered as the functions of  $x/l$  and  $k_1 l$  instead of  $x$  and  $l$ .  $A_1$  and  $\phi_1$ , for  $x/l$  and  $k_1 l$  are calculated from (2-16) and (2-17), Figs. 4 and 5 show respectively the variations of  $A_1$  and  $\phi_1$  for  $x/l$  in the case that the values of  $k_1 l$  are 1.0, 1.5, 2.0, 2.5, 3.0 and 4.0.

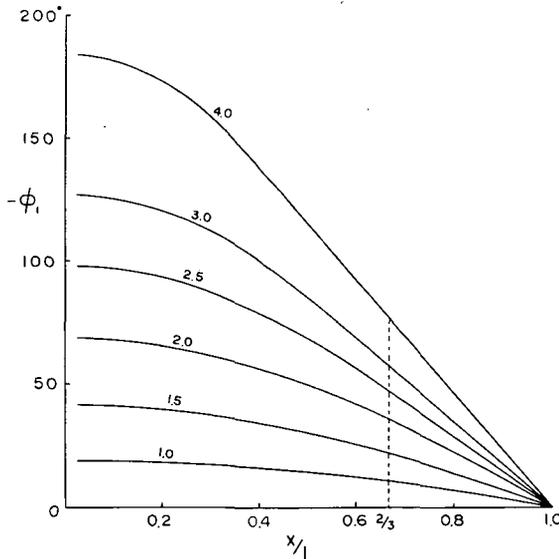


Fig. 5. Variation of  $-\phi_1$  for  $x/l$ . Numbers on the curves are the values of  $k_1 l$ .

### 3. Determination of $\kappa$

When  $F(x, t)$  in (2-10) is transposed to the left side, the right side of (2-10) become only sine series as

$$v - F(x, t) = \sum_{m=1}^{\infty} B_m A_m \sin(m\omega t + \epsilon_m + \phi_m), \tag{3-1}$$

and the writer uses (3-1) to determine the value of  $\kappa$ . In this case, the larger the variation of  $v-F(x, t)$  of (3-1) becomes, the easier the analysis becomes. In order to let the variation of  $v-F(x, t)$  be large, the value of  $B_m$  in (3-1) which is involved in the condition of  $v=g_2(t)$  at  $x=l$  should be taken large. And the variation of soil temperature at the boundary near the soil surface shows actually larger amplitude of the variation than the other boundary. Accordingly the boundary of  $x=l$  must be taken nearer to the soil surface than that of  $x=0$ .

When the soil temperatures are measured at  $z_1, z$  and  $z_2$  depths ( $z_1 < z < z_2$ ), the values of  $x$  corresponding to these depths are taken as  $l, x$  and 0 respectively as shown in Fig. 6. If  $l$  and  $x$  are fixed, the value of  $x/l$  is decided, and from (2-16) and (2-17)  $A_1$  and  $\phi_1$  are represented by the function of  $k_1 l$ . In the case the value of  $x/l$  in Figs. 4 and 5 is fixed, the values of  $A_1$  and  $\phi_1$  for various  $k_1 l$  are given by the intersecting points of  $A_1$  and  $\phi_1$  curves and the dashed lines expressing  $x/l = \text{const.}$  as shown in the same figures. Fig. 7 is the variations of  $A_1$  and  $\phi_1$  for  $k_1 l$  when the value of  $x/l$  is equal to  $2/3$ , and the value of  $\kappa$  shown in the same figure is transformed from (2-18) when  $l=15\text{ cm}$  and  $\omega=2\pi/(9.12 \times 10^4)\text{ sec}^{-1}$ .

Now, let the origin of time be the time when all the soil temperatures at  $x=0, x$  and  $l$  take equal temperature and suppose this temperature is zero, and if the soil temperatures at  $x=0$  and  $l$  are represented by the Fourier series, then the boundary and initial conditions can satisfy (2-6), (2-7) and (2-8), and from (2-9)  $\omega$  is determined by selecting suitable time range of the variation of the soil temperature in order to make  $B_m$  and  $C_m$  as small as possible except  $m=1$ .

When the value of  $\kappa$  is assumed,  $k_1 l$  is decided from (2-18),  $F(x, t)$  for  $t$  is

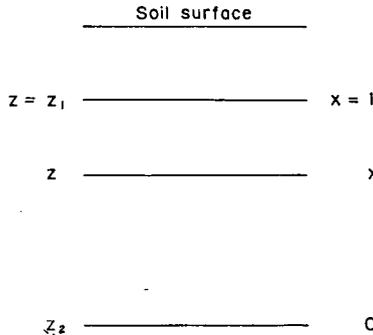


Fig. 6. Values of  $x$  corresponding to depths  $z$ .

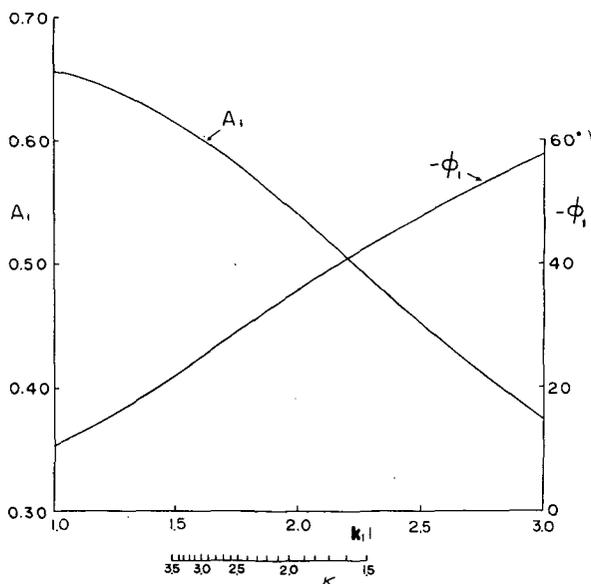


Fig. 7. Variations of  $A_1$  and  $-\phi_1$  for  $k_1 l$  or  $\kappa$  at  $x/l=2/3$  when  $l=15$  cm and  $\omega=2\pi/(9.12 \times 10^4)$  sec $^{-1}$ .

calculated by substituting constants  $B_m$ ,  $C_m$ ,  $\varepsilon_m$  and  $\varepsilon_m'$  in (2-6) and (2-7) into (2-11), and the values of  $v-F(x, t)$  for  $t$  are derived. The curve of  $v-F(x, t)$  is expanded again in sine series and the amplitude and phase lag of the fundamental sine curve are put as  $D_1$  and  $E_1$ . But,  $G(x)$  in (2-11) is only the function of  $x$ , and is constant for  $t$ . Then, suppose  $x$  is fixed,  $D_1$  and  $E_1$  can be obtained by expanding  $v-\{F(x, t)-G(x)\}$  instead of  $v-F(x, t)$ . One puts  $D_1$  and  $E_1$  as

$$D_1 = B_1 A_1', \quad (3-2)$$

$$E_1 = \phi_1' + \varepsilon_1, \quad (3-3)$$

therefore

$$A_1' = \frac{D_1}{B_1}, \quad (3-4)$$

$$\phi_1' = E_1 - \varepsilon_1. \quad (3-5)$$

Equation (3-1) is given only in the case that  $F(x, t)$  for the proper value of  $\kappa$  is substituted into the left side. Then, if the value of  $\kappa$  is assumed properly, since (3-1) be satisfied,  $D_1$  and  $E_1$  become equal to  $B_1 A_1$  and  $\phi_1 + \varepsilon_1$  in (3-1),  $A_1'$  and  $\phi_1'$  in (3-2) and (3-3) become equal to  $A_1$  and  $\phi_1$

respectively, and the values of  $A_1'$  and  $\phi_1'$  are to be plotted on the  $A_1$  and  $\phi_1$  curves in Fig. 7.

Accordingly the values of  $D_1$  and  $E_1$  are calculated for various  $\kappa$ , and the values of  $A_1'$  and  $\phi_1'$  gained from (3-4) and (3-5) are plotted, the plotted points are connected, and an intersecting point with the  $A_1$  and  $\phi$  curves is found as shown in Fig. 10. Consequently the value of  $k_1 l$  corresponding to the intersecting point can be derived and  $k_1$  is obtained by means of division by  $l$ , and the value of  $\kappa$  is determined by substituting the value of  $k_1$  into (2-18).

#### 4. Data and analysis

Fig. 8 shows the variations of soil temperature at 5, 10 and 20 cm depths at Hokkaido Prefecture Experimental Station in Takikawa, and the temperatures were measured by means of a thermistor thermometer with the precision of 0.1°C. The values of  $x$  corresponding to 5, 10 and 20 cm depths are taken respectively 15, 10 and 0 cm. The values of  $x$  and  $l$  are taken respectively 10 and 15 cm, and  $x/l$  becomes 2/3. Fig. 7 is the curves of  $A_1$  and  $\phi_1$  for  $k_1 l$  in the case that  $x/l$  is equal to 2/3. The soil from the surface to 24 cm depth is humus soil as shown in Fig. 2, and the soil between 5 and 20 cm depths is considered to be homogeneous, and the above method can be applied.

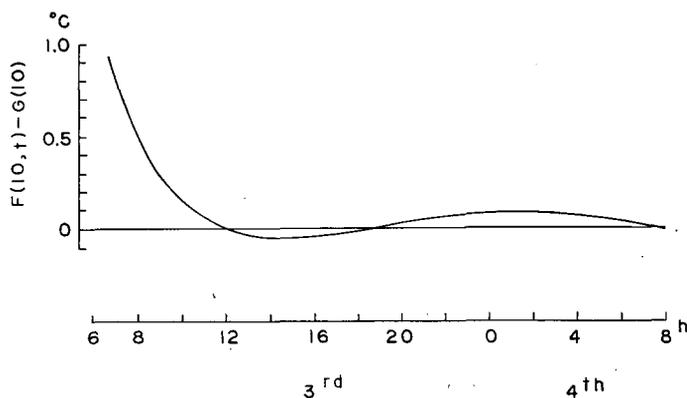


Fig. 8. Variations of soil temperature at 5, 10 and 20 cm depths.  
 —: at 5 cm, — —: at 10 cm, — · —: at 20 cm.

Now, let the soil temperatures at 5, 10 and 20 cm depths be represented respectively  $g_2(t)$ ,  $v$  and  $g_1(t)$ , let the origin of time be 6<sup>h</sup>40<sup>m</sup> a.m. of July 3 when the soil temperatures at these depths took equal values, and let the end of the time range be 8<sup>h</sup>00<sup>m</sup> a.m. of July 4. Then, the values of  $T_0$  and  $\omega$  become

Table 1. Values of  $B_m$ ,  $C_m$ ,  $\epsilon_m$  and  $\epsilon'_m$ .

constants	$m=0$	1	2	3
$B_m$	0.90	2.60	0.38	0.13
$C_m$	-0.16	0.34	0	0
$\epsilon_m$	—	- 31°30'	-100°30'	-57°32'
$\epsilon'_m$	—	-125°30'	—	—



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Fig. 9. A example of the variation of  $F(10, t) - G(10)$  in the case of  $\kappa = 2.5 \times 10^{-3}$  C.G.S..

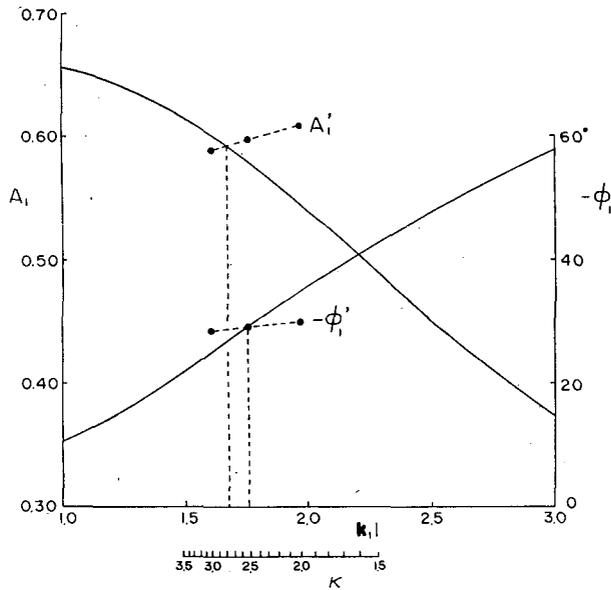
$9.12 \times 10^4$  sec and  $2\pi / (9.12 \times 10^4)$  sec<sup>-1</sup>, and initial condition (2-8) is satisfied.  $g_1(t)$  and  $g_2(t)$  are expanded by the Fourier series, when the time range  $T_0$  are divided into 152 equal parts, the values of  $B_0$ ,  $B_m$ ,  $C_0$ ,  $C_m$ ,  $\epsilon_m$  and  $\epsilon'_m$  in (2-6) and (2-7) are obtained as shown in Table 1.

When  $B_1 = 2.60$  and  $C_1 = 0.34$  are substituted into (1-6) of amplitude method the value of  $\kappa$  becomes  $1.9 \times 10^{-3}$  C.G.S.. And when  $\epsilon_1 = -31^\circ 30'$  and  $\epsilon'_1 = -125^\circ 30'$  are substituted into (1-8) of time lag method, the value of  $\kappa$  becomes  $2.9 \times 10^{-3}$  C.G.S..

First the constants in Table 1 are substituted into (2-11), and  $F(10, t) - G(10)$  are calculated when the value of  $\kappa$  are equal to  $2.0 \times 10^{-3}$ ,  $2.5 \times 10^{-3}$  and  $3.0 \times 10^{-3}$  C.G.S. which are near the values of  $1.9 \times 10^{-3}$  and  $2.9 \times 10^{-3}$  C.G.S.. Fig. 9 shows a example of the variation of  $F(10, t) - G(10)$  in the case of  $\kappa = 2.5 \times 10^{-3}$  C.G.S., and the digital computer used in the calculation is HIPAC 103. Second the values of  $v - \{F(10, t) - G(10)\}$  for various values of  $\kappa$  are calculated and the each curve of  $v - \{F(10, t) - G(10)\}$  are expanded in sine series, and

Table 2. Values of  $D_1$ ,  $E_1$ ,  $A'_1$ ,  $\phi'_1$  and  $k_1 l$  for various values of  $\kappa$ .

constants	$\kappa=2.0 \times 10^{-3}$ C.G.S.	$2.5 \times 10^{-3}$	$3.0 \times 10^{-3}$
$D_1$	1.57	1.56	1.55
$E_1$	$-61^\circ 45'$	$-60^\circ 50'$	$-60^\circ 10'$
$A'_1$	0.60	0.60	0.59
$\phi'_1$	$-30^\circ 15'$	$-29^\circ 20'$	$-28^\circ 40'$
$k_1 l$	1.97	1.75	1.61

Fig. 10. Determination of the values of  $(k_1 l)_A$ ,  $(k_1 l)_\phi$ ,  $\kappa_A$  and  $\kappa_\phi$ .

the time range are divided equally by 152 as in the cases of  $g_1(t)$  and  $g_2(t)$ . Table 2 shows the values of  $D_1$ ,  $E_1$ ,  $A'_1$ ,  $\phi'_1$  and  $k_1 l$  for various values of  $\kappa$ , and the results are plotted in Fig. 10. Now, let  $k_1 l$  and  $\kappa$  for the intersecting point of the curves of  $A_1$  and  $A'_1$  be respectively  $(k_1 l)_A$  and  $\kappa_A$ , and similarly let  $k_1 l$  and  $\kappa$  for the intersecting point of the curves of  $\phi_1$  and  $\phi'_1$  be respectively  $(k_1 l)_\phi$  and  $\kappa_\phi$ . Then, from Fig. 10, the following values are gained.

$$(k_1 l)_A = 1.67,$$

$$(k_1 l)_\phi = 1.75,$$

and

$$\kappa_A = 2.8 \times 10^{-3} \text{ C.G.S. ,}$$

$$\kappa_\phi = 2.5 \times 10^{-3} \text{ C.G.S. .}$$

Generally, values of  $\kappa_A$  and  $\kappa_\phi$  are slightly different. This reason may be considered that the soil was disturbed by inserting thermistor thermometers into the soil, then the soil near the thermometers will not be homogeneous, or in the case of expanding a curve by the Fourier series it will be somewhat rough dividing that the time range is divided into 152 equal parts.

### 5. Conclusion

To get thermal diffusivity of soil, the writer derived the new method in which initial and boundary conditions were taken into consideration to the actual state of soil temperature, and the differential equation of heat conduction in soil was solved under these conditions. The method and application were described in sections 3 and 4.

From the amplitude and time lag methods for humus soil, the value of thermal diffusivity was obtained respectively  $1.9 \times 10^{-3}$  and  $2.9 \times 10^{-3}$  C.G.S.. But the values computed from the new method were  $2.8 \times 10^{-3}$  and  $2.5 \times 10^{-3}$  C.G.S., and this discrepancy could not be avoided.

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