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Measurements of Thermal Constants of Soil

—— Thermal Diffusivity ——

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Abstract

To determine thermal diffusivity of soil, two methods called amplitude and time lag methods have been used. These methods, however, have some assumptions which do not agree with actual state of soil. Then the new method is shown in this paper. The writer considered boundary conditions at two finite depths and initial condition shown in the soil between the two boundaries in order to avoide the unsuitable assumptions in the amplitude and time lag methods, and solved the differential equation of heat conduction in soil under these conditions.

Application of the method was illustrated by using data measured at Hokkaido Prefecture Agricaltural Experimental Station in Takikawa. The result derived from the method was compared with the values obtained from the amplitude and time lag methods, and it was found that the values from the amplitude and time lag methods was very different from the value by the new method.

1. Introduction

In the amplitude and time lag methods[1], the differential equation of heat conduction in soil was solved under the following conditions:

1) Variation of soil temperature was regarded sine curve and the period was taken one day or one year.

2) Two boundary conditions were given by the temperatures at soil surface and infinite depth, and these conditions were as follows:

$$v = \bar{v} + A \sin \frac{2\pi}{T} t, \qquad \text{at} \quad x = 0, \tag{1–1}$$

$$v = \bar{v} + \alpha x, \qquad \text{at} \quad x = \infty, \tag{1–2}$$

where t and x represent time and depth from the soil surface, and A and T are amplitude and period of the variation of the temperature at the soil surface respectively, $\bar{v}$ is mean value of the variation of the soil temperature at $x=0$, and α is increasing rate of the temperature for the depth.

From (1–1), (1–2) and the differential equation of heat conduction, soil temperature v is given by

$$v = \bar{v} + a x + A e^{-x\sqrt{\frac{\pi}{\kappa T}}} \sin\left(\frac{2\pi}{T} t - x\sqrt{\frac{\pi}{\kappa T}}\right), \tag{1-3}$$

where κ is thermal diffusivity of soil. Now, let B_1, C_1 and ε_1, ε_1' be respectively amplitudes and time lags of the variation of the soil temperatures at x_1 and x_2 depths. From (1–3) the ratio of amplitude B_1/C_1 becomes to

$$\frac{B_1}{C_1} = e^{-\sqrt{\frac{\pi}{\kappa T}}(x_1 - x_2)}, \tag{1-4}$$

accordingly

$$\log\left(\frac{B_1}{C_1}\right) = -\sqrt{\frac{\pi}{\kappa T}}(x_1 - x_2). \tag{1-5}$$

Then it follows from (1–5) that

$$\kappa = \frac{\pi}{T} \frac{(x_1 - x_2)^2}{\left\{\log\left(\frac{B_1}{C_1}\right)\right\}^2}. \tag{1-6}$$

The method of determining the thermal diffusivity of soil from (1–6) is called the amplitude method. Secondary, from (1–3) the difference of time lag ε_1–ε_1' is expressed as

$$\varepsilon_1 - \varepsilon_1' = \sqrt{\frac{\pi}{\kappa T}}(x_1 - x_2). \tag{1-7}$$

Therefore

$$\kappa = \frac{\pi}{T} \frac{(x_1 - x_2)^2}{(\varepsilon_1 - \varepsilon_1')^2}. \tag{1-8}$$

The method using (1–8) is called the time lag method.

Fig. 1 shows the variations of atmospheric and soil temperatures at Sapporo District Meteorological Observatory, and the weather during the observational period was very fine. From the figure it is clear that the variation of the atmospheric temperature is not sine curve, and the soil temperature influenced by the atmospheric temperature does not show either sine curve having the period of one day. Consequently this fact did not satisfy condition (1), and it became necessary that the initial condition of the soil temperature should be taken into consideration. Condition (2) which boundaries are taken at $x=0$ and $x=\infty$, means that the soil is homogeneous from surface to infinite depth. Neverthless Fig. 2 indicating a section of soil at the observational place, is apparentely not consisted of the uniformal soil. Then this

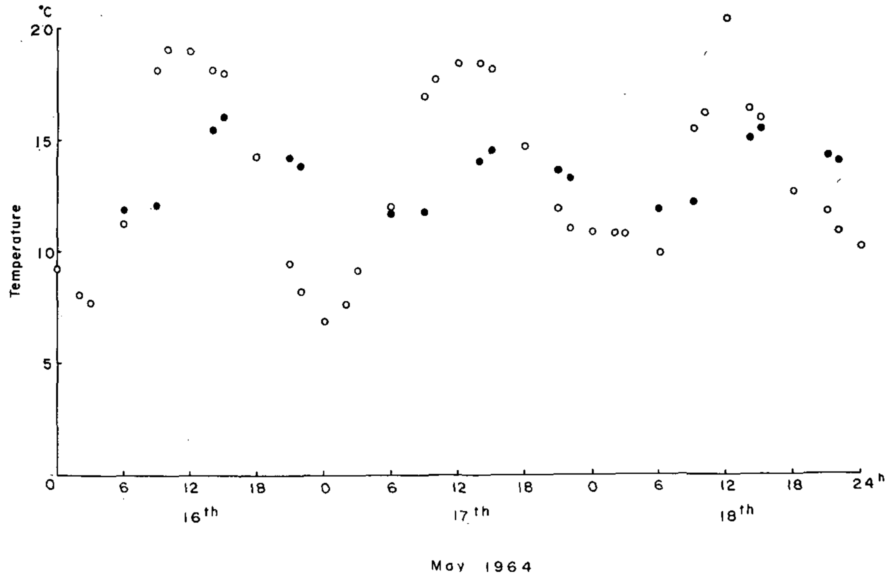

Fig. 1. Variations of atmospheric and soil temperatures.
○: atmospheric temperature, ●: soil temperature at 10 cm depth.

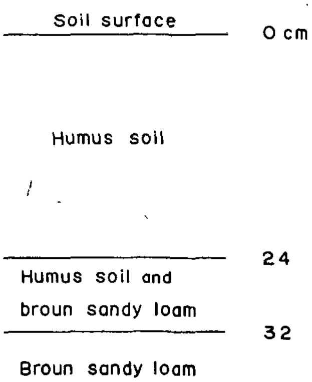

Fig. 2. Section of the soil at the observational place.

means that the condition (2) can not be identical with the actual soil section. Therefore one needs to select two boundaries between which the state of soil must be homogeneous, and the obtained value of κ shows the thermal diffusivity of the soil between the two boundaries. Fig. 3 represents the variation of soil temperature at 5 cm dpeth in Hokkaido Prefecture Agricaltural Experimental Station in Takikawa. The variation does not coincide with

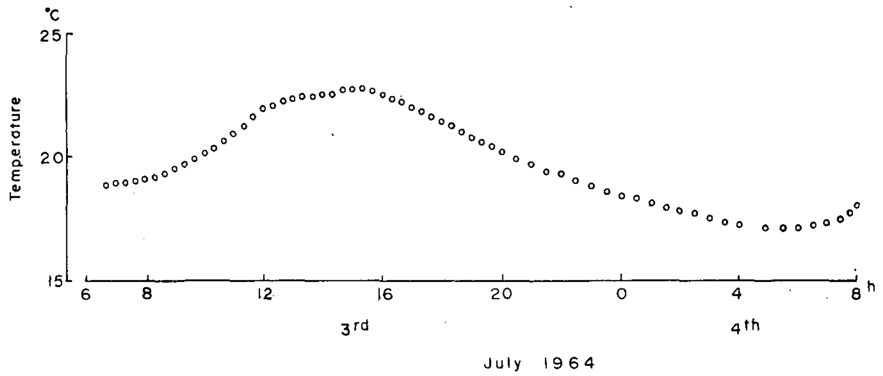

Fig. 3. Variation of soil temperature at Takikawa, Hokkaido.

sine curve, and does not express (1–3). This discrepancy is due to the boundary condition at $x=0$ represented as (1–1), the boundary condition (1–1) needs to be given by other equation showing the actual soil surface temperature.

To avoide the above mentioned discrepancy the writer selected arbitrary time range, took the boundary conditions at two finite depths between which the soil is considered uniform, and gave the initial condition given in the soil between the two boundaries.

2. Variation of soil temperature influenced by the change of atmospheric temperature

When soil surface is flat and special heat source does not exist in the soil, variation of the soil temperature is influenced only by the change of atmospheric temperature, and may be considered as the functions of time and depth from the soil surface, but may not vary with the perpendicular direction to the depth. Then the variation of the soil temperature can be treated as the problem of one dimension, and differential equation of heat conduction in soil is expressed by

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} \quad (0 < x < l \text{ and } 0 < t < T_0), \tag{2–1}$$

where l is the distance between the two boundaries, and T_0 is the time range of the variation of the soil temperature.

When the boundary and initial conditions are given by

$$v = g_1(t), \quad \text{at} \quad x = 0, \tag{2–2}$$

$$v = g_2(t), \quad \text{at} \quad x = l, \tag{2–3}$$

$$v = f(x), \quad \text{when} \quad t = 0, \tag{2-4}$$

where $x=0$ shows the one plane giving one boundary condition, and does not necessarily express the soil surface, the solution of (2–1) satisfying (2–2), (2–3) and (2–4) was obtained by CARSLAW and JAEGER[2] as the following equation:

$$v = \frac{2}{l} \sum_{n=1}^{\infty} e^{-\kappa n^2 \pi t / l^2} \sin \frac{n \pi x}{l} \left[\int_0^l f(x') \sin \frac{n \pi x'}{l} \, d x' + \frac{n \kappa \pi}{l} \int_0^t e^{\kappa n^2 \pi^2 \lambda / l^2} \{ g_1(\lambda) - (-1)^n g_2(\lambda) \} \, d\lambda \right]. \tag{2-5}$$

If $g_1(t)$ and $g_2(t)$ are represented by the Fourier series and $f(x)$ is equal to zero, namely

$$g_1(t) = C_0 + \sum_{m=1}^{\infty} C_m \sin(m \omega t + \varepsilon'_m), \tag{2-6}$$

$$g_2(t) = B_0 + \sum_{m=1}^{\infty} B_m \sin(m \omega t + \varepsilon_m), \tag{2-7}$$

$$f(x) = 0, \tag{2-8}$$

where

$$\omega = \frac{2\pi}{T_0}, \tag{2-9}$$

and m is positive integer, ε_m and ε_m' show the phase lags of each sine curve, by substituting (2–6), (2–7) and (2–8) into (2–5), v becomes to

$$v = \sum_{m=1}^{\infty} B_m A_m \sin(m \omega t + \varepsilon_m + \phi_m) + F(x, t), \tag{2-10}$$

$$\begin{aligned} F(x, t) = \frac{2 \kappa \pi}{l^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n \sin \frac{n \pi x}{l} & \left[\frac{C_m D_n}{D_n^2 + m^2 \omega^2} \left[\left\{ \sin(m \omega t + \varepsilon_m') \right. \right. \right. \\ & - \left. \frac{m \omega}{D_n} \cos(m \omega t + \varepsilon_m') \right\} - e^{-D_n t} \left(\sin \varepsilon_m' - \frac{m \omega}{D_n} \cos \varepsilon_m' \right) \Big] \\ & + \left\{ \frac{B_m D_n}{D_n^2 + m^2 \omega^2} (-1)^n \left(\sin \varepsilon_m - \frac{m \omega}{D_n} \cos \varepsilon_m \right) \right. \\ & - \left. \left. \frac{(C_0 - (-1)^n B_0)}{D_n} \right\} e^{-D_n t} \right] + G(x), \end{aligned} \tag{2-11}$$

where

$$D_n = \frac{\kappa n^2 \pi^2}{l^2},$$

$$A_m = \left| \frac{\sinh k_m x (1+i)}{\sinh k_m l (1+i)} \right| = \left\{ \frac{\cosh 2 k_m x - \cos 2 k_m x}{\cosh 2 k_m l - \cos 2 k_m l} \right\}^{1/2}, \tag{2-12}$$

$$\phi_m = \arg. \left\{ \frac{\sinh k_m x (1+i)}{\sinh k_m l (1+i)} \right\}, \tag{2-13}$$

$$k_m = \left(\frac{m \omega}{2 \kappa} \right)^{1/2}, \tag{2-14}$$

$$G(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \{C_0 - (-1)^n B_0\} \frac{\sin \frac{n \pi x}{l}}{n}, \tag{2-15}$$

where i is unit of imaginary number.

From (2-12) and (2-13), A_m and ϕ_m are found to be the functions of k_m, x and l. When T_0 is selected and substituted into (2-9) ω becomes constant, therefore k_m is merely the function of m from (2-14). Then, in this case A_m and ϕ_m become the functions of m, x and l.

Now, let A_1, ϕ_1 and k_1 be the values of A_m, ϕ_m and k_m at $m=1$. It follows from (2-12), (2-13) and (2-14) that

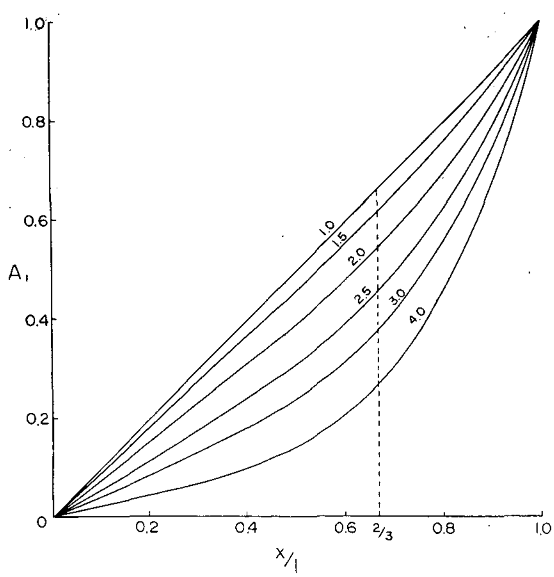

Fig. 4. Variation of A_1 for x/l. Numbers on the curves are the values of $k_1 l$.

$$A_1 = \left\{ \frac{\cosh 2\,(k_1 l)\left(\frac{x}{l}\right) - \cos 2\,(k_1 l)\left(\frac{x}{l}\right)}{\cosh 2\,(k_1 l) - \cos 2\,(k_1 l)} \right\}^{1/2}, \tag{2–16}$$

$$\phi_1 = \arg. \left\{ \frac{\sinh\,(k_1 l)\left(\frac{x}{l}\right)(1+i)}{\sinh\,(k_1 l)\,(1+i)} \right\}, \tag{2–17}$$

$$k_1 = \left(\frac{\omega}{2\,\kappa}\right)^{1/2}. \tag{2–18}$$

Accordingly, A_1 and ϕ_1 may be considered as the functions of x/l and $k_1 l$ instead of x and l. A_1 and ϕ_1, for x/l and $k_1 l$ are calculated from (2–16) and (2–17), Figs. 4 and 5 show respectively the variations of A_1 and ϕ_1 for x/l in the case that the values of $k_1 l$ are 1.0, 1.5, 2.0, 2.5, 3.0 and 4.0.

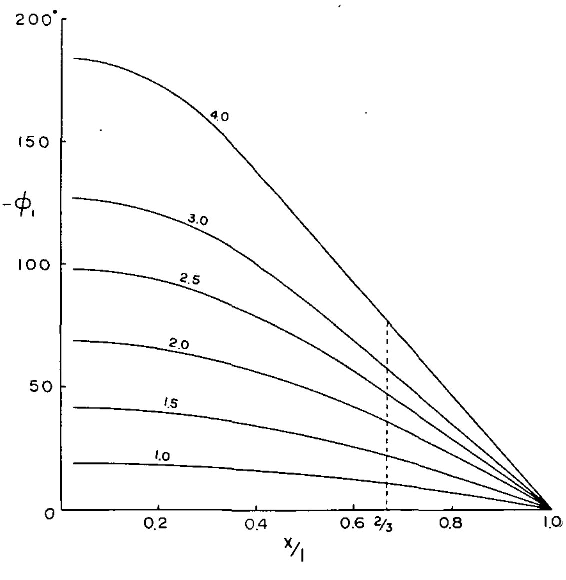

Fig. 5. Variation of $-\phi_1$ for x/l. Numbers on the curves are the values of $k_1 l$.

3. Determination of κ

When $F(x, t)$ in (2–10) is transposed to the left side, the right side of (2–10) become only sine series as

$$v - F\,(x, t) = \sum_{m=1}^{\infty} B_m\,A_m \sin\,(m\,\omega\,t + \varepsilon_m + \phi_m)\,, \tag{3–1}$$

and the writer uses (3–1) to determine the value of κ. In this case, the larger the variation of $v-F(x, t)$ of (3–1) becomes, the easier the analysis becomes. In order to let the variation of $v-F(x, t)$ be large, the value of B_m in (3–1) which is involved in the condition of $v=g_2(t)$ at $x=l$ should be taken large. And the variation of soil temperature at the boundary near the soil surface shows actually larger amplitude of the variation than the other boundary. Accordingly the boundary of $x=l$ must be taken nearer to the soil surface than that of $x=0$.

When the soil temperatures are measured at z_1, z and z_2 depths ($z_1<z<z_2$), the values of x corresponding to these depths are taken as l, x and 0 respectively as shown in Fig. 6. If l and x are fixed, the value of x/l is decided, and from (2–16) and (2–17) A_1 and ϕ_1 are represented by the function of k_1l. In the case the value of x/l in Figs. 4 and 5 is fixed, the values of A_1 and ϕ_1 for various k_1l are given by the intersecting points of A_1 and ϕ_1 curves and the dashed lines expressing x/l=const. as shown in the same figures. Fig. 7 is the variations of A_1 and ϕ_1 for k_1l when the value of x/l is equal to 2/3, and the value of κ shown in the same figure is transformed from (2–18) when $l=15$ cm and $\omega=2\pi/(9.12\times10^4)$ sec^{-1}.

Now, let the origin of time be the time when all the soil temperatures at $x=0$, x and l take equal temperature and suppose this temperature is zero, and if the soil temperatures at $x=0$ and l are represented by the Fourier series, then the boundary and initial conditions can satisfy (2–6), (2–7) and (2–8), and from (2–9) ω is determined by selecting suitable time range of the variation of the soil temperature in order to make B_m and C_m as small as possible except $m=1$.

When the value of κ is assumed, k_1l is decided from (2–18), $F(x, t)$ for t is

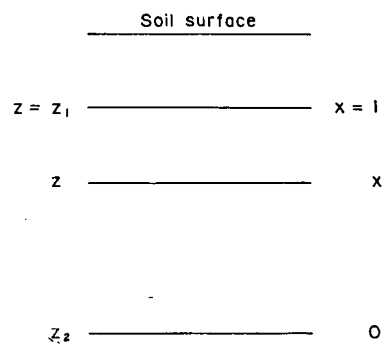

Fig. 6. Values of x corresponding to depths z.

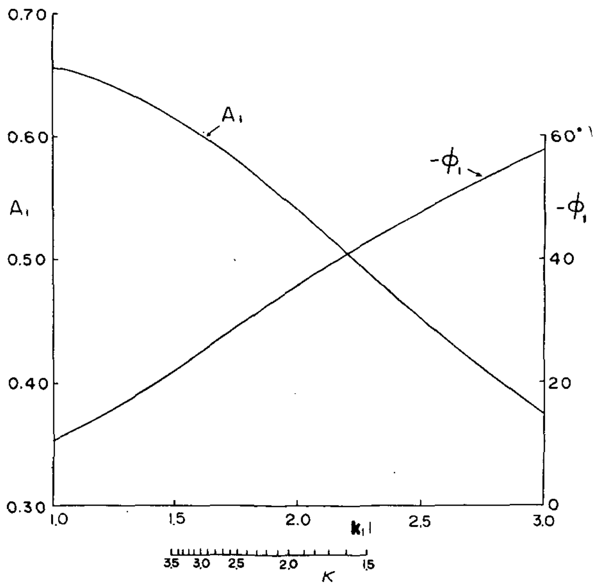

Fig. 7. Variations of A_1 and $-\phi_1$ for k_1l or κ at $x/l=2/3$ when $l=15$ cm and $\omega=2\pi/(9.12\times10^4)$ sec^{-1}.

calculated by substituting constants B_m, C_m, ε_m and ε_m' in (2–6) and (2–7) into (2–11), and the values of v–$F(x, t)$ for t are derived. The curve of v–$F(x, t)$ is expanded again in sine series and the amplitude and phase lag of the fundamental sine curve are put as D_1 and E_1. But, $G(x)$ in (2–11) is only the function of x, and is constant for t. Then, suppose x is fixed, D_1 and E_1 can be obtained by expanding $v-\{F(x, t)-G(x)\}$ instead of v–$F(x, t)$. One puts D_1 and E_1 as

$$D_1 = B_1 A_1' , \tag{3–2}$$

$$E_1 = \phi_1' + \varepsilon_1 , \tag{3–3}$$

therfore

$$A_1' = \frac{D_1}{B_1} , \tag{3–4}$$

$$\phi_1' = E_1 - \varepsilon_1 . \tag{3–5}$$

Equation (3–1) is given only in the case that $F(x, t)$ for the proper value of κ is substituted into the left side. Then, if the value of κ is assumed properly, since (3–1) be satisfied, D_1 and E_1 become equal to B_1A_1 and $\phi_1+\varepsilon_1$ in (3–1), A_1' and ϕ_1' in (3–2) and (3–3) become equal to A_1 and ϕ_1

respectively, and the values of A_1' and ϕ_1' are to be plotted on the A_1 and ϕ_1 cruves in Fig. 7.

Accordingly the values of D_1 and E_1 are calculated for various κ, and the values of A_1' and ϕ_1' gained from (3–4) and (3–5) are plotted, the plotted points are connected, and a intersecting point with the A_1 and ϕ curves is found as shown in Fig. 10. Consequently the value of k_1l corresponding to the intersecting point can be derived and k_1 is obtained by means of division by l, and the value of κ is determined by substituting the value of k_1 into (2–18).

4. Data and analysis

Fig. 8 shows the variations of soil temperature at 5, 10 and 20 cm depths at Hokkaido Prefecture Experimental Station in Takikawa, and the temperatures were measured by means of a thermistor thermometer with the precision of 0.1°C. The values of x corresponding to 5, 10 and 20 cm depths are taken respectively 15, 10 and 0 cm. The values of x and l are taken respectively 10 and 15 cm, and x/l becomes 2/3. Fig. 7 is the curves of A_1 and ϕ_1 for k_1l in the case that x/l is equal to 2/3. The soil from the surface to 24 cm depth is humus soil as shown in Fig. 2, and the soil between 5 and 20 cm depths is considered to be homogeneous, and the above method can be applied.

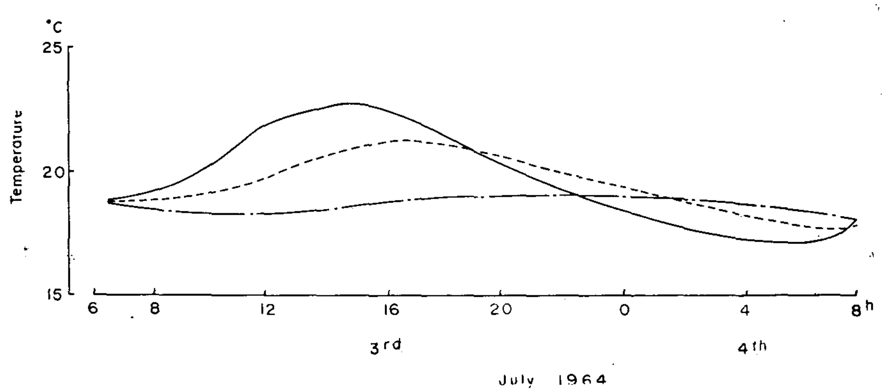

Fig. 8. Variations of soil temperature at 5, 10 and 20 cm depths.
———: at 5 cm, — — —: at 10 cm, — · —: at 20 cm.

Now, let the soil temperatures at 5, 10 and 20 cm depths be represented respectively $g_2(t)$, v and $g_1(t)$, let the origin of time be 6^h40^m a.m. of July 3 when the soil temperatures at these depths took equal values, and let the end of the time range be 8^h00^m a.m. of July 4. Then, the values of T_0 and ω become

Table 1. Values of B_m, C_m, ε_m and ε_m'.

constants	m=0	1	2	3
B_m	0.90	2.60	0.38	0.13
C_m	−0.16	0.34	0	0
ε_m	—	− 31°30′	−100°30′	−57°32′
ε'_m	—	−125°30′	—	—

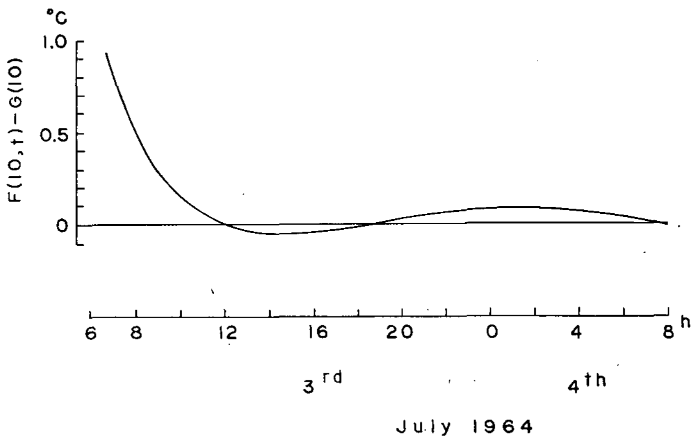

Fig. 9. A example of the variation of F (10, t)−G(10) in the case of $\kappa=2.5\times10^{-3}$ C.G.S..

9.12×10^4 sec and $2\pi/(9.12\times10^4)$ sec^{-1}, and initial condition (2–8) is satisfied. $g_1(t)$ and $g_2(t)$ are expanded by the Fourier series, when the time range T_0 are divided into 152 equal parts, the values of B_0, B_m, C_0, C_m, ε_m and ε_m' in (2–6) and (2–7) are obtained as shown in Table 1.

When B_1=2.60 and C_1=0.34 are substituted into (1–6) of amplitude method the value of κ becomes 1.9×10^{-3} C.G.S.. And when ε_1=−31°30′ and ε_1'=−125°30′ are substituted into (1–8) of time lag method, the value of κ becomes 2.9×10^{-3} C.G.S..

First the constants in Table 1 are substituted into (2–11), and $F(10, t)$−$G(10)$ are calculated when the value of κ are equal to 2.0×10^{-3}, 2.5×10^{-3} and 3.0×10^{-3} C.G.S. which are near the values of 1.9×10^{-3} and 2.9×10^{-3} C.G.S.. Fig. 9 shows a example of the variation of $F(10, t)$−$G(10)$ in the case of $\kappa=2.5\times10^{-3}$ C.G.S., and the digital computer used in the calculation is HIPAC 103. Second the values of v−{$F(10, t)$−$G(10)$} for various values of κ are calculated and the each curve of v−{$F(10, t)$−$G(10)$} are expanded in sine series, and

Table 2. Values of D_1, E_1, A'_1, ϕ'_1 and k_1l for various values of κ.

constants	$\kappa=2.0\times10^{-3}$C.G.S.	2.5×10^{-3}	3.0×10^{-3}
D_1	1.57	1.56	1.55
E_1	$-61°45'$	$-60°50'$	$-60°10'$
A'_1	0.60	0.60	0.59
ϕ'_1	$-30°15'$	$-29°20'$	$-28°40'$
k_1l	1.97	1.75	1.61

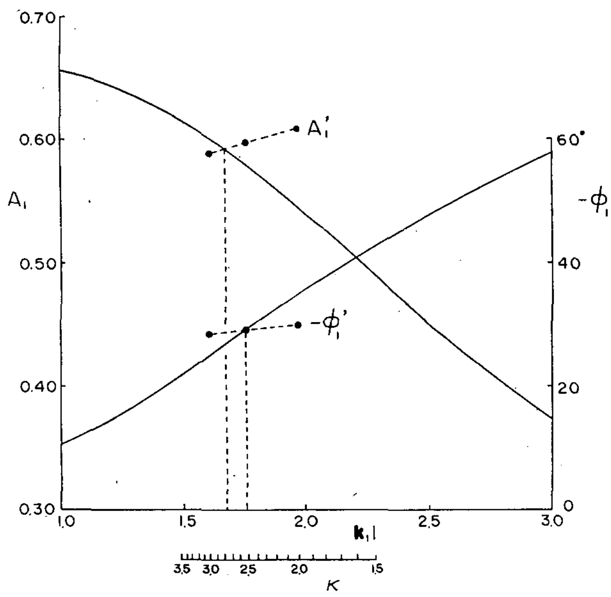

Fig. 10. Determination of the values of $(k_1l)_A$, $(k_1l)_\phi$, κ_A and κ_ϕ.

the time range are divided equally by 152 as in the cases of $g_1(t)$ and $g_2(t)$. Table 2 shows the values of D_1, E_1, A_1', ϕ_1' and k_1l for various values of κ, and the results are plotted in Fig. 10. Now, let k_1l and κ for the intersecting point of the curves of A_1 and A_1' be respectively $(k_1l)_A$ and κ_A, and similarly let k_1l and κ for the intersecting point of the curves of ϕ_1 and ϕ_1' be respectively $(k_1l)_\phi$ and κ_ϕ. Then, from Fig. 10, the following values are gained.

$$(k_1 l)_A = 1.67\,,$$

$$(k_1 l)_\phi = 1.75\,,$$

and

$$\kappa_A = 2.8 \times 10^{-3} \text{ C.G.S.},$$

$$\kappa_\phi = 2.5 \times 10^{-3} \text{ C.G.S.}.$$

Generally, values of κ_A and κ_ϕ are slightly different. This reason may be considered that the soil was disturbed by inserting thermistor thermometers into the soil, then the soil near the thermometers will not be homogeneous, or in the case of expanding a curve by the Fourier series it will be somewhat rough dividing that the time range is divided into 152 equal parts.

5. Conclusion

To get thermal diffusivity of soil, the writer derived the new method in which initial and boundary conditions were taken into consideration to the actual state of soil temperature, and the differential equation of heat conduction in soil was solved under these conditions. The method and application were described in sections 3 and 4.

From the amplitude and time lag methods for humus soil, the value of thermal diffusivity was obtained respectively 1.9×10^{-3} and 2.9×10^{-3} C.G.S.. But the values computed from the new method were 2.8×10^{-3} and 2.5×10^{-3} C.G.S., and this discrepancy could not be avoided.

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References

1) SINGER I.A. and BROWN R.M.: The annual variation of sub-soil temperatures about a 600-foot circle. Trans. Amer. Geophys. Union, **37** (1956), 743–748.
2) CARSLAW H.S. and JAEGER J.C.: Conduction of heat in solids (2nd edition). Oxford University Press, (1959), 102–107.