博士学位論文

On a mathematical treatment of a particle-reaction-diffusion model

(ある粒子反応拡散系モデルの数学的取り扱いについて)

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Abstract

In the nature, we can state that the collective motion is one of the universal phenomena. Here, collective motion is the complex phenomena as the aggregation of several individuals which moves following ones rule, and is not described as simply sum of each individuals. We can see some example of such a collective motion in flog of birds, school of fishes, and bacteria, etc. For the purpose of understanding the basic mechanism of such collective motion, many experiments of self-driven materials, which are inanimate systems that do not depend on characteristics peculiar to living things, have been reported. At the same time, many theoretical analyzes using mathematical models are performed. In this paper, we state the result for a model equation derived from the motion of camphor disk. The first result shows the existence of unique solution to initial value problem for the model equation. In preceding studies, some mathematical results are obtained for the model equation, and the existence of the solutions to initial value problem are referred. But the proof does not exist. The second result states the existence of the solutions corresponds to non-trivial motion of two camphor disks. Our result reveals the conditions for existence and non-existence for the solution.

Contents

1 Introduction

When a large number of individuals move cooperatively, functionalities quite often arise. Examples around in many biological systems such as flocks of birds [1], schools of fish[53], insect swarms, bacterial colonies [48], and collective migrations of cells [38]. It is vicsek's pioneering work[52] that has shown the appearance of collective motions by using a model of motile elements. Subsequently, theoretical researches have been conducted in the context of nonlinear physics for understanding the mechanism of collective motions performed by living organisms [39, 52, 53].

On the other hand, many researchers have focused on non-biological systems of self-propelled materials to study collective motions from an experimental point of view. For example, extensive studies have been done on the surfactant particles or droplets driven by the difference of the surrounding surface tension [2, 4, 6, 11, 20, 25, 29, 30, 31, 34, 35, 44, 47, 51]. It has also been reported that the movements of self-propelled materials can be controlled by chemical reactions [32, 33, 49].

For the theoretical understanding of these experimental self-propelled materials, a framework of mathematical models was introduced and analyzed. For instance, consider a disk made of the surfactant material. Let $\boldsymbol{x}_c(t)$ and $u(\boldsymbol{x}, t)$ be the center of the disk and the surface concentration of the surfactant layer, respectively. Due to the constraints of the experimental system, we are interested in a two-dimensional space at most, but the formulation itself can be considered in an arbitrary dimensional space. Then, a general structure of mathematical models for self-propelled motions is described as follows:

$$\begin{aligned} &\rho \frac{d^2 \boldsymbol{x}_c}{dt^2}(t) = \boldsymbol{G}[u](\boldsymbol{x}_c(t), t) - \mu \frac{d\boldsymbol{x}_c}{dt}(t), \\ &\frac{\partial u}{\partial t}(\boldsymbol{x}, t) = d_u \Delta u(\boldsymbol{x}, t) + f(u(\boldsymbol{x}, t)) - F[\boldsymbol{x}_c](\boldsymbol{x}, t), \end{aligned} \tag{1.1}$$

where ρ, μ, and d_u denote the area density of the disk, the viscosity coefficient, and the diffusion coefficient, respectively. The operator F represents the effect of the disk on the surface concentration of the surfactant layer, such as, the supply of the surfactant molecules. A simple example of it is described by

$$F[\boldsymbol{x}_c](\boldsymbol{x}, t) = \begin{cases} k_u u_0, & |\boldsymbol{x} - \boldsymbol{x}_c(t)| \leq r, \\ 0, & |\boldsymbol{x} - \boldsymbol{x}_c(t)| > r, \end{cases}$$

or

$$F[\boldsymbol{x}_c](\boldsymbol{x}, t) = k_u u_0 \delta(\boldsymbol{x} - \boldsymbol{x}_c(t)),$$

where k_u and u_0 are the supply rate and the effective density of the solid surfactant, respectively, and $\delta(\boldsymbol{x})$ is the Dirac delta function. The operator $\boldsymbol{G}$ is the driving force exerted on the disk. Examples in a one-dimensional space are given by

$$G[u](x, t) = \frac{1}{2r} \left(\gamma(u(t, x + r)) - \gamma(u(t, x - r)) \right)$$

or

$$G[u](x,t) = \left.\frac{\partial \gamma(u(y,t))}{\partial y}\right|_{y=x}.$$

In the case of a two-dimensional space, examples are as follows:

$$\boldsymbol{G}[u](\boldsymbol{x},t) = \int_{\partial(\boldsymbol{x}+\omega_r)} \gamma(u(t,\boldsymbol{y}))\nu(\boldsymbol{y})d\sigma(\boldsymbol{y}),$$

or

$$\boldsymbol{G}[u](\boldsymbol{x},t) = \lim_{r\to+0}\int_{\partial(\boldsymbol{x}+\omega_r)} \gamma(u(t,\boldsymbol{y}))\nu(\boldsymbol{y})d\sigma(\boldsymbol{y}).$$

Here, $\boldsymbol{x}+\omega_r = \{\boldsymbol{y} \in \mathbb{R}^2 \mid \exists \boldsymbol{y}_0 \in \omega_r,\ s.t.\ \boldsymbol{y} = \boldsymbol{y}_0 + \boldsymbol{x}\}$ with $\omega_r = \{\boldsymbol{x} \in \mathbb{R}^2 \mid |\boldsymbol{x}| \leq r\}$, which denotes the closed disk centered at the origin with radius $r > 0$. $\nu(\boldsymbol{y})$ and $\sigma(\boldsymbol{y})$ are the unit normal vector at point $\boldsymbol{y}$ and the line segment, respectively[28].

In the above examples, the function $\gamma(u)$ represents the surface tension of the water surface which changes with the concentration of surfactant. A possible functional form is

$$\gamma(u) = \frac{a^m}{a^m+u^m}(\gamma_0-\gamma_1)+\gamma_1, \tag{1.2}$$

where $a > 0$, $m \in \mathbb{N}$, and γ_0, $\gamma_1 > 0$ represent the surface tension of pure water and that of the critical micelle concentration of the surfactant layer, respectively. Although it is difficult to determine the surface tension function $\gamma(u)$ from experimental measurements, the surface tension decreases monotonically as the concentration of the surfactant increases as a general trend. Thus, we assume that $\gamma(u)$ is strictly deceasing in mathematical models. Indeed, this assumption has been employed in previous researches, where the following functions have been proposed as candidates for $\gamma(u)$ of (1.2) [21, 45]:

$$\gamma(u) = \frac{1}{2}(\gamma_0-\gamma_1)(\tanh(-(u-u_0))+1)+\gamma_1, \tag{1.3}$$

in which u_0 is a positive constant, and

$$\gamma(u) = \gamma_0 - au. \tag{1.4}$$

To understand the mechanism of self-propelled motions theoretically, the mathematical model (1.1) has been studied by use of the computer-aided analysis [3, 11, 20, 21, 24, 26, 41]. Moreover, a mathematical model for two-dimensional problems, such as motions on the water surface, has been constructed [5, 17, 19, 35] and studied theoretically [15, 18, 22, 23]. Also, the experiments to control motions of self-propelled materials by a chemical reaction have been reported [32, 33, 49], and their theoretical studies have been conducted by using the mathematical model (1.1) coupled with a chemical reaction model [14, 27, 40]. These studies suggest that the particle reaction-diffusion system (1.1) is considered as a physically relevant model for describing self-propelled motions widely observed in nonlinear physics and physical chemistry.

One of the remarkable phenomena observed in self-propelled motions is the appearance of collective motions. Indeed, several patterns of collective motions have been reported in the previous researches [12, 42, 43, 45, 46, 50]. For example, Suematsu et al. have observed that the camphor boats exhibit traffic jam-like phenomena [45] and that oscillatory motions of camphor disks appear depending on the number of disks and their surface area [46]. Nakata et al. have also reported collective motions of camphor disks such as billiard motions and traffic jam phenomena in an annular water channel [16].

Our concern is whether these collective motions appear in the mathematical model (1.1) as well. Since camphor disks used in the experiments are very light, Nishi et al. have analyzed motions of two camphor disks by using the following dimensionless mathematical model without the inertia term [36]:

$$\begin{aligned} \mu\frac{dx_c^i}{dt} &= \frac{\gamma(u(t,\pi_L(x_c^i+r))) - \gamma(u(t,\pi_L(x_c^i-r)))}{2r}, \\ \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - u + F(x-x_c^1) + F(x-x_c^2), \end{aligned} \tag{1.5}$$

for $i=1,2$, and $x \in [0,L) \setminus \{\pi_L(x_c^1+r), \pi_L(x_c^1-r), \pi_L(x_c^2+r), \pi_L(x_c^2-r)\}$, where $\mu > 0$ and $L > 4r > 0$. The function $\gamma(u) > 0$ is strictly decreasing with respect to $u > 0$ and the function $F(x)$ is given by

$$\begin{aligned} F(x) &= \begin{cases} 1, & |x|_L \le r, \\ 0, & |x|_L > r, \end{cases} \\ |x|_L &= \min_{n\in\mathbb{Z}} |x+nL|. \end{aligned} \tag{1.6}$$

Meanwhile, π_L denotes the map from $\mathbb{R}$ to $[0,L)$ defined by following recursive form:

$$\pi_L(x) = \begin{cases} \pi_L(x+L), & x < 0, \\ x, & 0 \le x < L, \\ \pi_L(x-L), & L \le x. \end{cases} \tag{1.7}$$

The periodic boundary condition is imposed by $u(t,L) = u(t,0)$ and $\partial u/\partial x(t,L) = \partial u/\partial x(t,0)$. Note that a solution of (1.5) satisfies $u \in C([0,T]\times[0,L])$ and $u(t,\cdot) \in C[0,L]$ for any $t > 0$.

The previous research [36] has clarified the mechanism of the emergence of billiard motions and traffic jam motions in the model (1.5) by the computer-aided analysis. Besides, they have also reported notable motions: symmetrically and asymmetrically oscillating motions, symmetrically and asymmetrically rotating motions (see Figure 1), rotating motion with oscillations, and so on. Furthermore, both the experimental measurement and the numerical computation have shown that the asymmetrically rotating motion of two camphor disks is stable as shown in Figure 1(b).

The existence of the asymmetrically rotating motion is counter-intuitive: The density of camphor molecules in the rear of the moving camphor disk is higher than that in the other regions in general, because camphor molecules are left behind for a while

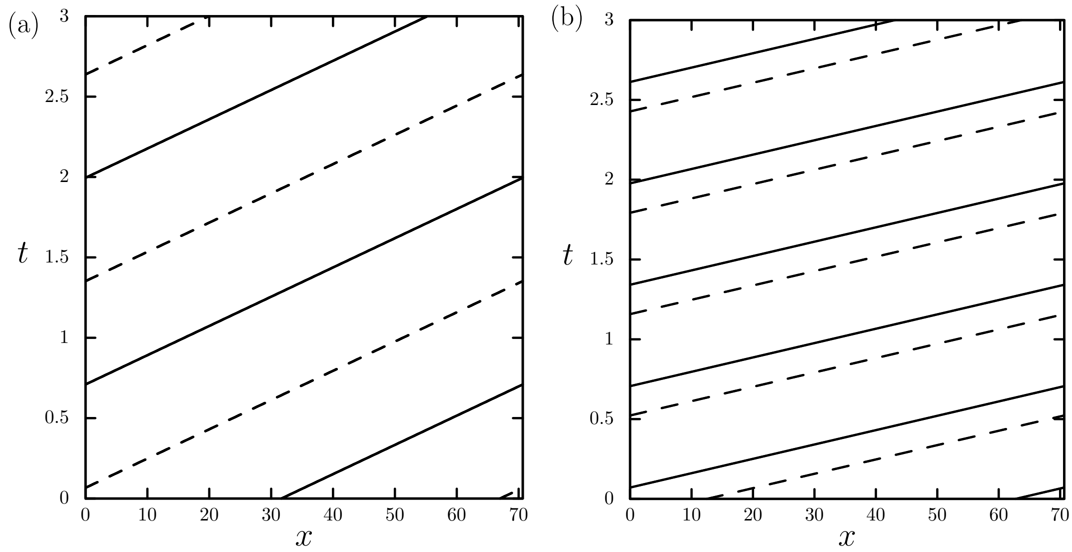

Figure 1: The trajectories of (a) a symmetrically rotating motion of two camphor disks for $\mu = 0.004$ and (b) an asymmetrically rotating motion of two camphor disks for $\mu = 0.001$. Both solutions are obtained by the numerical computation for (1.5) with $r = 0.5$, $L = 70.665$ and $\gamma(u) = a^2/(a^2 + u^2)$, where $a = 0.05$. The solid line and dashed line show the trajectories of two camphor disks.

after the passing of the disk. Thus, the driving force for the rear disk generated by the difference in the surface tensions, seems to be weaker for the rear disk than for the front disk, and consequently the rear disk moves slower than the front disk. Hence, it seems reasonable that the distance between the two camphor disks increases gradually and their positions become symmetric in the end. The analysis of the reduced equations for camphor motions, derived based on the weak interaction theory, suggests that repulsive forces act on the camphor particles through the reaction-diffusion field [7, 8, 9]. From the above considerations, the appearance of asymmetrically rotating motions suggested numerically in [36] seems to be curious, because this usually requires attractive forces between camphor disks. Moreover, the computer-aided analysis in [36] has shown that the asymmetrically rotating solution appears via a pitch-fork bifurcation of the symmetrically rotating solution. Our motivation in this study is, through the analysis of the model (1.5), to make it clear with mathematical rigor what is essential for the existence of asymmetrically rotating motions for a strictly decreasing function γ with mathematical rigor.

There is another topic in this paper. We can see by numerical simulation that an asymmetrically rotating motion of two camphor disks still exists in an extremely long channel (see Figure 2.). In that case, the tail of the camphor layer seems to vanish before reaching the former camphor disk. It suggests that the asymmetrically rotating motion of two camphor disks is the intrinsic motion of two camphor disks. In other words, the asymmetrically rotating motion of two camphor disks is independent of the finiteness of the circular channel. For this reason, we discuss (1.5) in $\mathbb{R}$ and show the existence of the solution corresponding to asymmetrically rotating motions in a finite periodic interval.

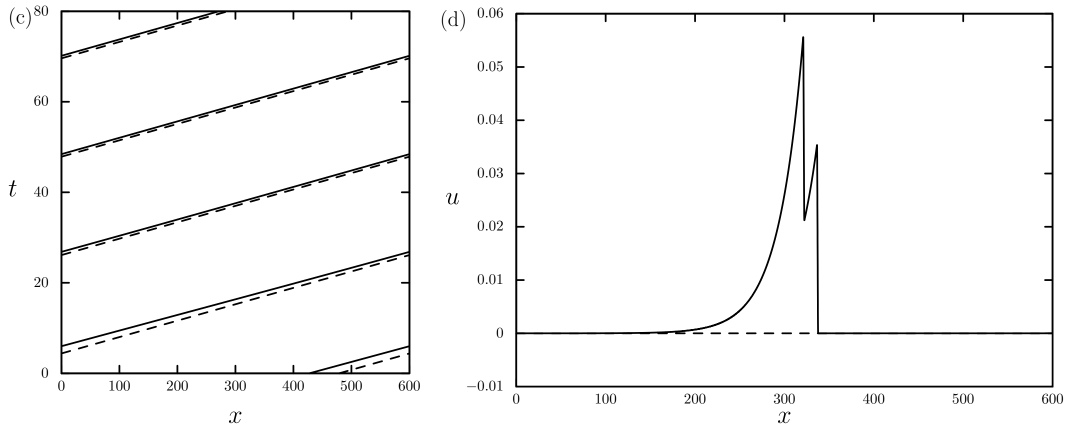

Figure 2: (c) The trajectory of the asymptotic solution x_c^1 and x_c^2 to an asymmetrically rotating motion of two camphor disks. The solid line and dashed line show the trajectory of two camphor disks. (d) The profile of u corresponds to it. Both solutions are obtained by the numerical computation for (1.5) with $\mu = 0.015$, $r = 0.5$, $L = 600$ and $\gamma(u) = a^2/(a^2 + u^2)$, where $a = 0.05$.

Before these topics, we state the unique existence of time global solution to (1.1). The existence of the solution is mentioned in [7, 8, 9], but no proof has been given so far. Fan et al. have dealt with the initial value problems for similar equations [10, 28]. One of the fixed point theorems is utilized in the proof in our result in the same way as the previous results in our result. The difference from these previous results is the point that the existence of the unique time global solution is shown directly by the direct evaluation using the heat kernel and the weighted norm. In comparison, the previous results have only shown the existence of either time local or non-unique solution. Although the proof is shown for the case where the region is taken along the entire real axis, the same method is believed to be valid when considering the periodic boundary condition on the bounded interval.

This thesis is organized as follows. In section 2, we state the main results of this thesis and the target equation because some variations of (1.1) are discussed in this thesis. In section 3, we state the general theorem for the solvability of the initial value problem for (1.5). The result will show the basic estimation for general particle-reaction-diffusion equations. In section 4, we state the unique existence of a time global solution to the initial value problem for (1.5). In section 5, we state the existence of the solution corresponding to asymmetrically rotating motions. The proof is performed for the case where the periodic boundary condition is imposed on the bounded interval, and the sufficient conditions for existence and non-existence are shown. In section 6, we state the existence of a bimodal solution, which is one of the special traveling wave solutions.

From the clarified conditions, it is suggested that asymmetrically rotating motion is a phenomenon that does not originate from the boundedness of the region.

2 Main results

In this section, we formulate the target equation discussed in this paper and state the main results.

In section 3, we are concerned the solvability of a general particle-reaction-diffusion equation. We state the sufficient conditions for an existence of unique solution to initial boundary value problem.

In section 4, we are concerned with the following equation:

$$\begin{aligned}
\rho\frac{d^2x_c^i}{dt^2} &= \frac{\gamma(u(x_c^i(t)+r,t))-\gamma(u(x_c^i(t)-r,t))}{2r} - \mu\frac{dx_c^i}{dt}, \quad i=1,\cdots,N,\\
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - u + \sum_{i=1}^{N} F(x-x_c^i(t)), \qquad x\in\mathbb{R},\\
F(x) &= \begin{cases} 1, & |x|<r,\\ 0, & |x|\geq r,\end{cases}
\end{aligned} \tag{2.1}$$

with the initial condition:

$$x_c^i(0)=X_0^i, \quad \frac{dx_c^i}{dt}(0)=V_0^i, \quad i=1,\cdots,N, \tag{2.2}$$

$$u(x,0)=u_0(x), \quad x\in\mathbb{R}, u_0\in C^b(\mathbb{R})\cap L^1(\mathbb{R}). \tag{2.3}$$

where $C^b(\mathbb{R})$ denotes the space of continuous and bounded functions. It is obtained as a special case of (1.1) where

$$G(u) = \frac{\gamma(u(x_c^i(t)+r,t))-\gamma(u(x_c^i(t)-r,t))}{2r}, \tag{2.4}$$

$$F(x) = \begin{cases} 1, & |x|<r,\\ 0, & |x|\geq r,\end{cases} \tag{2.5}$$

with dimensionless variables:

$$\widetilde{t}=kt, \qquad \widetilde{x}=\sqrt{\frac{k}{d}}x, \qquad \widetilde{u}=\frac{k}{k_uS_0}u.$$

For simplicity of notation, we use the original variables and parameters in the dimensionless model (2.1).

Because (2.1) has a discontinuous term $F(x-x_c^i(t))$, there are no classical solutions. Thus, we need a definition of a weak solution to (2.1) with N disks on the whole space. In this paper, we define the weak solution to (2.1) as follows:

Definition 2.1. *Let $r, \mu > 0$ and $\gamma \in C[0,\infty)$ be Lipschitz continuous. For a given $u_0 \in C^b(\mathbb{R}) \cap L^1(\mathbb{R})$, functions $u \in L^1_{\text{loc}}((0,\infty); L^1(\mathbb{R})) \cap C((0,\infty); C_0(\mathbb{R}))$ and $\boldsymbol{x}_c = (x_c^1, \cdots, x_c^N) \in (C^2(0,\infty))^N$ are called a weak solution to* (2.1)–(2.3) *provided that*

(i) $x_c^i \in C^2(0,\infty)$ *satisfies*

$$\frac{d^2 x_c^i}{dt^2} = -\mu \frac{dx_c^i}{dt} + \frac{\gamma(u(x_c^i + r, t)) - \gamma(u(x_c^i - r, t))}{2r} \tag{2.6}$$

with the initial condition (2.2) *for $i = 1, \cdots, N$,*

(ii) $u \in L^1_{\text{loc}}((0,\infty); L^1(\mathbb{R})) \cap C((0,\infty); C_0(\mathbb{R}))$ *satisfies*

$$0 = \int_{\mathbb{R}} \varphi(x,0) u_0(x) dx + \int_0^\infty \int_{\mathbb{R}} \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} - \varphi \right) u \tag{2.7}$$

$$+ \varphi \sum_{i=1}^{N} F(x - x_c^i) dx dt \tag{2.8}$$

for any $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$.

Then we obtain the following theorem based on the above definition as the main result in section 4.

Theorem 2.2. *Let μ, $r > 0$ and $\gamma \in C[0,\infty)$ be Lipschitz continuous. For a given $u_0 \in C^b(\mathbb{R}) \cap L^1(\mathbb{R})$, there exist functions $u \in L^1_{\text{loc}}((0,\infty); L^1(\mathbb{R})) \cap C((0,\infty); C_0(\mathbb{R}))$ and $\boldsymbol{x}_c = (x_c^1, \cdots, x_c^N) \in \left(C^2(0,\infty)\right)^N$ which solve* (2.6)–(2.8).

In section 5, we discuss some kind of traveling wave solution to (1.5). The expression "some kind of" means that a little trick is needed before defining such a solution because the equation (1.5) is defined on a bounded interval.

We can expand the equation (1.5) periodically to the whole space as follows:

$$\begin{aligned}
\mu \frac{dx_c^{1,n}}{dt} &= \frac{\gamma(u(t, x_c^{1,n} + r)) - \gamma(u(t, x_c^{1,n} - r))}{2r}, \\
\mu \frac{dx_c^{2,n}}{dt} &= \frac{\gamma(u(t, x_c^{2,n} + r)) - \gamma(u(t, x_c^{2,n} - r))}{2r}, \\
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - u + \sum_{n=-\infty}^{\infty} F(x - x_c^{1,n}) + \sum_{n=-\infty}^{\infty} F(x - x_c^{2,n}), \\
& t > 0, \qquad x \in \mathbb{R},
\end{aligned} \tag{2.9}$$

where $x_c^{1,n}(0) = x_c^{1,n-1}(0) + L$, $x_c^{2,n}(0) = x_c^{2,n-1}(0) + L$ and $u(0,x) = u(0, x - L)$. For this periodically expanded equation, we can define the traveling wave solution in the usual way, i.e., the traveling wave solution with speed c is defined by introducing the moving coordinate $z = x - ct$:

$$
\begin{aligned}
0 &= \frac{\gamma(u(t, z_c^{1,n} + r)) - \gamma(u(t, z_c^{1,n} - r))}{2r} - \mu c, \\
0 &= \frac{\gamma(u(t, z_c^{2,n} + r)) - \gamma(u(t, z_c^{2,n} - r))}{2r} - \mu c, \\
0 &= \frac{\partial^2 U}{\partial z^2} + c\frac{\partial U}{\partial z} - U + \sum_{n=-\infty}^{\infty} F(z - z_c^{1,n}) + \sum_{n=-\infty}^{\infty} F(z - z_c^{2,n}), \\
& t > 0, \qquad z \in \mathbb{R}.
\end{aligned} \tag{2.10}
$$

If there exists a periodic solution $(U, c, z_c^{1,n}, z_c^{2,n})$ such that $z_c^{1,n} = z_c^{1,n-1} + L$, $z_c^{2,n} = z_c^{2,n-1} + L$ and $U(z) = U(z + L)$ to (2.10), we can recover the solution $(u, x_c^{1,n}, x_c^{2,n})$ to (2.9) from such a solution $(U, c, z_c^{1,n}, z_c^{2,n})$. From this discussion, we state the definition of a *rotating solution* as follows, which is the special case of a traveling wave solution with periodic boundary condition.

Definition 2.3. *The quadruple $(U(z), c, z_c^1, z_c^2)$ is called a rotating solution to* (1.5) *if it satisfies the following equations:*

$$
\begin{aligned}
0 &= \gamma(U(z_c^i + r)) - \gamma(U(z_c^i - r)) - 2r\mu c, \\
0 &= \frac{\partial^2 U}{\partial z^2} + c\frac{\partial U}{\partial z} - U + F(z - z_c^1) + F(z - z_c^2),
\end{aligned} \tag{2.11}
$$

for $i = 1, 2$, and $z \in [0, L) \setminus \{\pi_L(z_c^1 + r), \pi_L(z_c^1 - r), \pi_L(z^2 + r), \pi_L(z_c^2 - r)\}$, where U is a C^1-periodic function on $[0, L]$. In particular, the rotating solution $(U(z), c, z_c^1, z_c^2)$ satisfying $|z_c^1 - z_c^2| = L/2$ is called a **symmetrically rotating solution**, *and otherwise the solution is called an* **asymmetrically rotating solution**.

On the basis of the above definition, we state the following theorem about the existence of symmetrically and asymmetrically rotating solutions as the main results of section 5.

Theorem 2.4. *Assume $|z_c^1 - z_c^2|_L > 2r$ and that $\gamma \in C^1[0, \infty)$ satisfies $\gamma(u) > 0$ and $\gamma'(u) < 0$ for $u > 0$. Then the following statements hold.*

(a) *For any $c > 0$, there exists a unique $\mu > 0$ such that* (2.11) *has a symmetrically rotating solution. In the case of $c = 0$, there always exists a symmetrically rotating solution for any value of $\mu > 0$.*

(b) *For any $\mu > 0$,* (2.11) *has no asymmetrically rotating solution with $c = 0$.*

(c) *Suppose $\gamma \in C^2(0, \infty)$ and $\gamma'' \geq 0$. Then, for any $c > 0$ and $\mu > 0$,* (2.11) *has no asymmetrically rotating solution.*

(d) *Suppose that $\gamma \in C^2(0, 1)$ satisfies*

$$
\frac{1}{2}\left(1 + \frac{1}{4r^2(1 - \rho)}\right)\gamma'(\rho) < (1 - \rho)\gamma''(\rho) < \gamma'(\rho), \tag{2.12}
$$

where $\rho = 4r/L$. Then, (2.11) *has an asymmetrically rotating solution for sufficiently large c.*

In section 6, we discuss the traveling wave solutions of (1.5). Unlike the case of rotating solutions, we state the definition of traveling wave solutions as follows:

Definition 2.5. *For a given constant $c > 0$, $\{Z_c^i\}_{i=1}^N \in \mathbb{R}^N$ and $U \in C_0^1(\mathbb{R})$ are called a traveling wave solution to* (2.1)–(2.3) *with a uniform velocity $c > 0$ provided that they satisfy*

$$\begin{cases} 0 = \dfrac{\gamma(U(Z_c^i + r)) - \gamma(U(Z_c^i - r))}{2r} - \mu c, \quad i = 1, \cdots, N, \\ 0 = U'' + cU' - U + \displaystyle\sum_{i=1}^{N} F(z - Z_c^i), \quad z \in \mathbb{R}. \end{cases} \tag{2.13}$$

In particular, a traveling wave solution to (2.1) with $N = 2$ is called a bimodal traveling wave solution. The main result of section 6 is the following theorem about the existence of a bimodal traveling wave solution.

Theorem 2.6. *Suppose $|Z_c^1 - Z_c^2| > 2r$ and that $\gamma \in C[0, \infty)$ is a Lipschitz continuous and strictly decreasing function satisfying $\gamma > 0$. Then, the following statements hold for* (2.13) *with $N = 2$:*

1. *For any $\mu \in \mathbb{R}$, there is no bimodal traveling wave solution with $c = 0$.*
2. *Suppose $\gamma \in C^1(0, \infty)$ and that γ' is strictly increasing. Then, for any $\mu > 0$, there is no bimodal traveling wave solution with $c > 0$.*
3. *Suppose that $\gamma \in C^2[0, \infty)$ satisfies*

$$\frac{1 + 4r^2}{8r^2} \gamma'(0) < \gamma''(0) < \gamma'(0) < 0. \tag{2.14}$$

Then, there exists a bimodal traveling wave solution for sufficiently large $c > 0$.

Theorem 2.6 reveals that the existence of bimodal traveling wave solutions is closely related to the shape of γ. Indeed, for a smooth function γ, a traveling wave solution can exist under the condition that γ has a concave part. In the preceding study [37], the result similar to Theorem 2.6 has been shown for the case of a bounded interval whose length is $L > 0$ with the periodic boundary condition. Although we find that (2.14) coincides with the limit of the corresponding result in [37] as $L \to \infty$, this extension is not trivial in terms of the mathematical analysis. One of the main purposes of the present study is to clarify a condition for the existence of bimodal traveling solutions of (2.11) and to show its consistency with the result in [37].

3 The general theorem for initial value problem of particle-reaction-diffusion model

Let us consider the following general model equation on an open set $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \dfrac{d\boldsymbol{x}_c}{dt} = \boldsymbol{G}[u](\boldsymbol{x}_c, t), \\ \quad \boldsymbol{x}_c(0) = \boldsymbol{x}_0, \\ \dfrac{\partial u}{\partial t} = Du + F[\boldsymbol{x}_c], \\ \quad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \end{cases} \quad t > 0, \quad \boldsymbol{x} \in \Omega, \quad \boldsymbol{x}_c \in \Omega^N, \tag{3.1}$$

where $\boldsymbol{G}$ is the external force term caused by u and F describes the change in u caused by $\boldsymbol{x}_c$. D is the operator which describes the diffusion and reaction process independent on $\boldsymbol{x}_c$. In addition, the smoothness of u is specified by the smoothness of F and the property of D, but $\boldsymbol{x}_c$ always requires continuity because $\boldsymbol{x}_c$ denotes the trajectory of each particle. Note that we can treat the equation (3.1) with the second order ODE by a well known method. Typically, D has a form:

$$Du = d_u \Delta u + f(u).$$

In particular, from the next section on, this thesis will deal with:

$$Du = d_u \frac{\partial^2 u}{\partial x^2} - ku, \tag{3.2}$$

where d_u and k are positive real number.

In this section, our purpose is to show the existence of a unique solution to (3.1) under assumptions as weak as possible. The first idea is to divide (3.1) into two parts:

1. ODE part. For given U,

$$\begin{cases} \dfrac{d\boldsymbol{X}_c}{dt} = \boldsymbol{G}[U](\boldsymbol{X}_c, t), \\ \quad \boldsymbol{X}_c(0) = \boldsymbol{x}_0, \end{cases} \tag{3.3}$$
$$t > 0, \quad \boldsymbol{X}_c \in \Omega^N.$$

2. PDE part. For given $\boldsymbol{X}_c$,

$$\begin{cases} \dfrac{\partial U}{\partial t} = DU + F[\boldsymbol{X}_c], \\ \quad U(x, 0) = u_0(x), \end{cases} \tag{3.4}$$
$$t > 0, \quad x \in \Omega.$$

In the PDE part, it is difficult to show the existence of unique solutions for general D. Hence, we decide to discuss D for which a solution exists, and leave the discussion as to whether such a solution actually exists for a specific D. Furthermore, we also leave the discussion of the space in which the solution exists because F in the general particle-reaction-diffusion model takes various forms from a highly singular distribution such as the Dirac delta function to relatively modest but discontinuous functions such as piecewise constant functions. Thus, it should be discussed separately whit is appropriate space considering the solution of a PDE part. We will require the solution to PDE part (3.4) to belong to $L^p(\Omega)$ at each time t. In contrast, a solution to the ODE part needs to be at least continuous because it shows the trajectory of the particle. Because of this, we consider the following as space to which the solution of an ODE part should belong:

$$C_X = \{\boldsymbol{X}_c \in C([0,\infty);\Omega^N) \mid \boldsymbol{X}_c(0) = \boldsymbol{x}_0\}. \tag{3.5}$$

Let us consider a simple example. In the case of (3.2) with $\Omega = \mathbb{R}$, it is easily seen that the fundamental solution of such a D has following form:

$$\tilde{H}(x,t) = \exp(-kt)H(x,t),$$

where $H(x,t)$ denotes the heat kernel in $\mathbb{R}$. Then, the solution to (3.4) can be written formally as follows:

$$U[\boldsymbol{X}_c](\cdot,t) = \left(\tilde{H}(\cdot,t) * u_0(\cdot)\right) + \int_0^t \left(\tilde{H}(\cdot,t-\tau) * F[\boldsymbol{X}_c](\cdot,\tau)\right) d\tau.$$

In this case, the important fact is that the following evaluation holds by Young's inequality for convolutions:

$$\begin{aligned}
&\left\|U[\boldsymbol{X}_c](\cdot,t) - U[\boldsymbol{X}'_c](\cdot,t)\right\|_{L^r(\Omega)} \\
&\leq \int_0^t \left\|\tilde{H}(\cdot,t-\tau)\right\|_{L^p(\Omega)} \left\|F[\boldsymbol{X}_c](\cdot,\tau) - F[\boldsymbol{X}'_c](\cdot,\tau)\right\|_{L^q(\Omega)} d\tau,
\end{aligned}$$

for any $p,q,r \geq 1$ such that $1/p + 1/q = 1 + 1/r$ and $F[\boldsymbol{X}_c](\cdot,t) \in L^q(\Omega)$. If $\|F[\boldsymbol{X}_c](\cdot,\tau) - F[\boldsymbol{X}'_c](\cdot,\tau)\|_{L^q(\Omega)}$ can be bounded as

$$\left\|F[\boldsymbol{X}_c](\cdot,\tau) - F[\boldsymbol{X}'_c](\cdot,\tau)\right\|_{L^q(\Omega)} \leq C \left\|\boldsymbol{X}_c(\tau) - \boldsymbol{X}'_c(\tau)\right\|,$$

where $\|\boldsymbol{X}\| = \max_{i=1,\cdots,N} |X^i|$, then, we obtain the following estimate:

$$\begin{aligned}
&\left\|U[\boldsymbol{X}_c](\cdot,t) - U[\boldsymbol{X}'_c](\cdot,t)\right\|_{L^r(\Omega)} \\
\leq& C\int_0^t \left\|\tilde{H}(\cdot,t-\tau)\right\|_{L^p(\Omega)} d\tau \cdot \left\|\boldsymbol{X}_c - \boldsymbol{X}'_c\right\|_{[0,t]}, \\
=& C'(t) \left\|\boldsymbol{X}_c - \boldsymbol{X}'_c\right\|_{[0,t]},
\end{aligned}$$

where $\|\boldsymbol{X}_c\|_{[0,t]} = \max_{\tau\in[0,t]} \|\boldsymbol{X}_c(\tau) - \boldsymbol{X}'_c(\tau)\|$. Thus, it holds that

$$\left\|U[\boldsymbol{X}_c] - U[\boldsymbol{X}'_c]\right\|_{L^s([0,t];L^r(\Omega))} = C''(t) \left\|\boldsymbol{X}_c - \boldsymbol{X}'_c\right\|_{[0,t]}. \tag{3.6}$$

It means that the solution U is continuously depend on $\boldsymbol{X}_c$. In this special case of (3.2), the result seems essential and important for unique solution to (3.1). Thus, we will require that a solution to PDE part $U[\boldsymbol{X}_c]$ is continuously depend on $\boldsymbol{X}_c$.

Next, let us consider the ODE part with the above result. We introduce the integral form of (3.3):

$$\boldsymbol{X}_c(t) = P_t \boldsymbol{X}_c := \boldsymbol{x}_0 + \int_0^t \boldsymbol{G}[U](\boldsymbol{X}_c(\tau), \tau) d\tau. \tag{3.7}$$

We try to apply the Picard's iterative method and Banach's fixed point theorem for uniquness of the solution. Thus, we consider the following estimate:

$$\begin{aligned} \|\boldsymbol{X}_c(t) - \boldsymbol{X}'_c(t)\| &\le \int_0^t \|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}_c(\tau), \tau) - \boldsymbol{G}[U[\boldsymbol{X}'_c]](\boldsymbol{X}'_c(\tau), \tau)\| d\tau \\ &\le \int_0^t \|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}_c(\tau), \tau) - \boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}'_c(\tau), \tau)\| d\tau \\ &\quad + \int_0^t \|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}'_c(\tau), \tau) - \boldsymbol{G}[U[\boldsymbol{X}'_c]](\boldsymbol{X}'_c(\tau), \tau)\| d\tau. \end{aligned} \tag{3.8}$$

Now, we assume the following two properties for $\boldsymbol{G}$:

1. Uniform Lipschitz continuity with respects to $\boldsymbol{X}$: There exists a $L_1 > 0$ for all U which solves to (3.4) and $\boldsymbol{X}, \boldsymbol{X}' \in \Omega$ such that,

$$\|\boldsymbol{G}[U](\boldsymbol{X}, \tau) - \boldsymbol{G}[U](\boldsymbol{X}', \tau)\| \le L_1 \|\boldsymbol{X} - \boldsymbol{X}'\|. \tag{3.9}$$

2. Uniform Lipschitz continuity with respect to $\boldsymbol{U}$: There exists $1 \le s, r \le \infty$ and $L_2 > 0$ for all $\boldsymbol{X}_c \in C([0,t]; \mathbb{R}^N)$ which satisfy $\boldsymbol{X}_c(0) = \boldsymbol{x}_0$ and U, U' which solves to (3.4) such that,

$$\|\boldsymbol{G}[U](\cdot, \boldsymbol{X}_c(\cdot)) - \boldsymbol{G}[U'](\cdot, \boldsymbol{X}_c(\cdot))\|_{L^1[0,t]} \le L_2 \left\| U - U' \right\|_{L^s([0,t];L^r(\Omega))}. \tag{3.10}$$

These properties can be used to further evaluate (3.8):

$$\|\boldsymbol{X}_c(t) - \boldsymbol{X}'_c(t)\| \le \left(tL_1 + L_2 C''(t)\right) \left\| \boldsymbol{X}_c - \boldsymbol{X}'_c \right\|_{[0,t]}. \tag{3.11}$$

Note that $C''(t) = \int_0^t C \int_0^s \left\| \tilde{H}(\cdot, t-\tau) \right\|_{L^p(\Omega)} d\tau ds$; thus the right hand side of (3.11) is monotonically increasing. The following esitmate holds:

$$\|\boldsymbol{X}_c - \boldsymbol{X}'_c\|_{[0,t]} \le \left(tL_1 + L_2 C''(t)\right) \left\| \boldsymbol{X}_c - \boldsymbol{X}'_c \right\|_{[0,t]}.$$

It means that, P_t is a contraction map for sufficiently small $t > 0$. As a trivial fact, $C([0,t]; \mathbb{R}^N)$ with $\| \cdot \|_{[0,t]}$ is a complete functional space. In conclusion, we can obtain a unique time local solution to (3.1).

We summarize the above discussion in the next theorem:

Theorem 3.1. *Consider the equation* (3.1). *Let* $T > 0$, $N \in \mathbb{N}$, $\boldsymbol{X}_0 \in \mathbb{R}^N$, $1 \leq p, q \leq \infty$, $u_0 \in L^p(\Omega)$ *and* $C_X = \{\boldsymbol{X}_c \in C([0,T];\mathbb{R}^N) \mid \boldsymbol{X}_c(0) = \boldsymbol{X}_0\}$. *Suppose:*

1. *For any* $\boldsymbol{X}_c \in C_X$, *there exists a unique solution* $U[\boldsymbol{X}_c]$ *to PDE part such that* $U[\boldsymbol{X}_c] \in L^q([0,T]; L^p(\Omega))$.
2. *There exists* $k \in C[0,T]$ *which satisfy* $k(0) = 0$ *such that for any* $\boldsymbol{X}_c, \boldsymbol{X}'_c \in C_X$,
$$\|U[\boldsymbol{X}_c](\cdot,t) - U[\boldsymbol{X}'_c](\cdot,t)\|_{L^p(\Omega)} \leq k(t)\|\boldsymbol{X}_c - \boldsymbol{X}'_c\|_{[0,t]}. \tag{3.12}$$
3. *There exists* $L_1 > 0$ *such that for any* $\boldsymbol{X}, \boldsymbol{X}' \in \Omega$, $\boldsymbol{X}_c \in C_X$ *and* $\tau \in [0,T]$,
$$\|G[U[\boldsymbol{X}_c]](\boldsymbol{X},\tau) - G[U[\boldsymbol{X}_c]](\boldsymbol{X}',\tau)\| \leq L_1\|\boldsymbol{X} - \boldsymbol{X}'\|. \tag{3.13}$$
4. *There exists* $L_2 > 0$ *and such that for any* $\boldsymbol{X}_c, \boldsymbol{X}'_c \in C_X$,
$$\begin{aligned}&\|G[U[\boldsymbol{X}_c]](\boldsymbol{X}_c,\cdot) - G[U[\boldsymbol{X}'_c]](\boldsymbol{X}_c,\cdot)\|_{L^1[0,t]} \\ &\quad \leq L_2\|U[\boldsymbol{X}_c] - U[\boldsymbol{X}'_c]\|_{L^q([0,t];L^p(\Omega))},\end{aligned}$$
for all $t \in [0,T]$.

Then, there exists unique $\tilde{\boldsymbol{X}}_c \in C_X$ *for sufficiently small* $t > 0$ *such that the pair* $(\tilde{\boldsymbol{X}}_c|_{[0,t]}, U[\tilde{\boldsymbol{X}}_c]|_{\Omega\times[0,t]})$ *solves to* (3.1) *on interval* $[0,t]$.

Proof. What is needed is to apply Picard's iterative method and Banach's fixed point theorem. Let $P_t : C_X \to C_X$ as:
$$P_t\boldsymbol{X}_c = \boldsymbol{x}_0 + \int_0^t \boldsymbol{G}[U](\boldsymbol{X}_c(\tau),\tau)d\tau, \tag{3.14}$$
for $t \in [0,T]$ and $\boldsymbol{X}_c \in C_X$.

For any $\boldsymbol{X}_c, \boldsymbol{X}'_c \in C_X$,
$$\begin{aligned}\|P_t\boldsymbol{X}_c - P_t\boldsymbol{X}'_c\| &= \left\|\int_0^t \boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}_c(\tau),\tau) - \boldsymbol{G}[U[\boldsymbol{X}'_c]](\boldsymbol{X}'_c(\tau),\tau)d\tau\right\| \\ &\leq \int_0^t \left\|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}_c(\tau),\tau) - \boldsymbol{G}[U[\boldsymbol{X}'_c]](\boldsymbol{X}'_c(\tau),\tau)\right\| d\tau \\ &\leq \int_0^t \left\|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}_c(\tau),\tau) - \boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}'_c(\tau),\tau)\right\| d\tau \\ &\qquad + \int_0^t \left\|\boldsymbol{G}[U[\boldsymbol{X}_c]](\boldsymbol{X}'_c(\tau),\tau) - \boldsymbol{G}[U[\boldsymbol{X}'_c]](\boldsymbol{X}'_c(\tau),\tau)\right\| d\tau \\ &\leq L_1 t\|\boldsymbol{X}_c - \boldsymbol{X}'_c\|_{[0,t]} + L_2\|U[\boldsymbol{X}_c] - U[\boldsymbol{X}'_c]\|_{L^q([0,t];L^p(\Omega))} \\ &\leq (L_1 t + k'(t))\|\boldsymbol{X}_c - \boldsymbol{X}'_c\|_{[0,t]},\end{aligned}$$
where $k'(t) = L_2\left(\max_{\tau\in[0,t]}(k(t))^q\right)^{1/q}$. Note that $k \in C[0,T]$ and $k(0) = 0$ imply $k' \in C[0,T]$ and $k'(0) = 0$. Thus, we can choose $0 < t'$ such that $L_1 t' + k'(t') < 1$. It means that $P_{t'}$ is a contraction map with the sup norm. Therefore, we can apply the Banach's fixed point theorem to $C_X|_{[0,t']}$ and $P_{t'}$, then we can obtain a unique fixed point $\tilde{\boldsymbol{X}}_c$ on $[0,t']$. It is clearly seen that the pair $(\tilde{\boldsymbol{X}}_c, U[\tilde{\boldsymbol{X}}_c]|)$ on $[0,t']$. □

Let us give you an example. Let $\Omega = \mathbb{R}$, $N = 1$, and

$$
\begin{aligned}
G[u](x,t) &= \frac{\gamma(u(x+r,t)) - \gamma(u(x-r,t))}{2r}, \\
Du &= \frac{\partial^2 u}{\partial x^2} - u, \\
F[x_c](x,t) &= \begin{cases} 1, & |x - x_c(t)| \le r, \\ 0, & |x - x_c(t)| > r, \end{cases}
\end{aligned}
$$

with initial value condition:

$$
u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \quad u_0 \in C^b(\mathbb{R}) \cap L^1(\mathbb{R}),
$$

where γ is a Lipschitz continuous function with boundary condition $\lim_{|x|\to\infty} u(x,t) = 0$. We will deal with this example in more detail in later sections, but we can quickly see that it falls into the category of Theorem 3.1.

4 Proof of the existence of a solution to the initial value problem

We state the outline of the proof of Theorem 2.2. For a given $\boldsymbol{X}_c(t) = \{X_c^i(t)\}_{i=1}^N$, (2.8) has a unique solution represented by

$$
\begin{aligned}
U[\boldsymbol{X}_c](x,t) = e^{-t} &\int_{\mathbb{R}} H(x-y,t) u_0(y) dy \\
&+ e^{-t} \int_0^t \int_{\mathbb{R}} H(x-y,t-\tau) e^{\tau} \sum_{i=1}^N F(y - X_c^i(\tau)) dy d\tau,
\end{aligned}
$$

where $H(x,t) \equiv (4\pi t)^{-\frac{1}{2}} e^{-x^2/(4t)}$ denotes the Gaussian kernel. Then $U[\boldsymbol{X}_c]$ is rewritten as

$$
U[\boldsymbol{X}_c](\cdot,t) = \left(\widetilde{H}(\cdot,t) * u_0(\cdot)\right) + \int_0^t \left(\widetilde{H}(\cdot,t-\tau) * F[\boldsymbol{X}_c](\cdot,\tau)\right) d\tau,
$$

where

$$
\widetilde{H}(x,t) \equiv e^{-t} H(x,t), \qquad F[\boldsymbol{X}_c](x,t) \equiv \sum_{i=1}^N F(x - X_c^i(t)).
$$

Substituting this formula into (2.6), we find

$$
\frac{d^2 X_c^i}{dt^2}(t) = -\mu \frac{dX_c^i}{dt}(t) + \frac{\gamma\left(U[\boldsymbol{X}_c](X_c^i(t)+r,t)\right) - \gamma\left(U[\boldsymbol{X}_c](X_c^i(t)-r,t)\right)}{2r}, \tag{4.1}
$$

for $i = 1, \cdots, N$. We prove the existence of a unique solution $\boldsymbol{X}_c$ of (4.1) by applying the Picard's iterative method. Consider the following system, which is equivalent to

(4.1):

$$X_c^i(t) = X_0^i + \int_0^t V_c^i(\tau)d\tau, \tag{4.2}$$

$$V_c^i(t) = V_0^i \tag{4.3}$$

$$+ \int_0^t -\mu V_c^i(\tau) + \frac{\gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau)+r,\tau)\right) - \gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau)-r,\tau)\right)}{2r} d\tau, \tag{4.4}$$

where $V_c^i(t)$ is an auxiliary function corresponding to the derivative of $X_c^i(t)$, and define the map P by

$$P : \begin{pmatrix} X_c^i(t) \\ V_c^i(t) \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X_0^i + \int_0^t V_c^i(\tau)d\tau \\ V_0^i + \int_0^t -\mu V_c^i(\tau) + \frac{\gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau)+r,\tau)\right) - \gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau)-r,\tau)\right)}{2r} d\tau \end{pmatrix}.$$

Using the Banach fixed point theorem, we prove the existence of a fixed point $\boldsymbol{X}_c^*$ of the map P in a proper functional space so that $(\boldsymbol{X}_c^*, U[\boldsymbol{X}_c^*])$ is a unique solution of (4.1).

As the first step of the proof, we introduce the functional space in which we show P is a contraction mapping. Let $\phi \in C[0,\infty)$ be a positive function. Then, the functional space $C_\phi[0,\infty) \equiv \{f \in C[0,\infty) \mid \|f\|_\phi < \infty\}$ with $\|f\|_\phi \equiv \|\phi f\|_{L^\infty[0,\infty)}$ is a Banach space. Indeed, for any Cauchy sequence $\{f_n\} \subset C_\phi[0,\infty)$, there exists $F \in C^b[0,\infty)$ such that $\|\phi f_n - F\|_{L^\infty[0,\infty)} \to 0$ as $n \to \infty$. Thus, setting the function $f \in C_\phi[0,\infty)$ by $f = F/\phi$, we find $\|f_n - f\|_\phi \to 0$ as $n \to \infty$. Thus, $\{f_n\}$ is a convergent sequence in $C_\phi[0,\infty)$. In what follows, we consider the functional space $C_\phi[0,\infty)$ for $\phi(t) = e^{-\alpha t}$, $\alpha > 0$ and, for convenience, we introduce the following notation:

Definition 4.1. *For any $f \in C[0,\infty)$, we define*

$$\|f\|_\alpha \equiv \sup_{t\in[0,\infty)} \left|e^{-\alpha t} f(t)\right|,$$

and $C_\alpha \equiv \{f \in C[0,\infty) \mid \|f\|_\alpha < \infty\}$.

In addition to the above definition, we use the notations,

$$\|\boldsymbol{X}(t)\| = \max_{1\le i\le N} |X^i(t)|, \qquad \|\boldsymbol{X}\|_\alpha = \max_{1\le i\le N} \|X^i\|_\alpha,$$

for any $\boldsymbol{X} = (X^1, \cdots, X^N) \in (C_\alpha)^N$. Note that we omit the domain in the L^p norm when it is given by $\mathbb{R}$.

Next, to show that P is a contraction mapping, we introduce maps P_1 and P_2 on C_α:

$$P_1 : V \mapsto \int_0^t V(\tau)d\tau, \qquad P_2 : V \mapsto \int_0^t -\mu V(\tau)d\tau, \quad \mu > 0.$$

Then, the maps P_1 and P_2 are continuous on C_α. More precisely, we have

$$\|P_1V - P_1V'\|_\alpha \le \frac{1}{\alpha}\|V - V'\|_\alpha, \qquad \|P_2V - P_2V'\|_\alpha \le \frac{\mu}{\alpha}\|V - V'\|_\alpha, \tag{4.5}$$

for any V, $V' \in C_\alpha$, since it follows that

$$\left|e^{-\alpha t}\left(\int_0^t V(\tau)d\tau - \int_0^t V'(\tau)d\tau\right)\right| \le \|V - V'\|_\alpha \int_0^t e^{-\alpha(t-\tau)}d\tau \le \frac{1}{\alpha}\|V - V'\|_\alpha.$$

We also define the map P_3 on $(C_\alpha)^N$ by

$$P_3 : \boldsymbol{X}_c \mapsto \left(\frac{1}{2r}\int_0^t \left(\gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau) + r, \tau)\right) - \gamma\left(U[\boldsymbol{X}_c](X_c^i(\tau) - r, \tau)\right)\right) d\tau\right)_{1\le i\le N}.$$

Before deriving an estimate for P_3, we show the following lemma.

Lemma 4.2. For any $\boldsymbol{X}_c$, $\boldsymbol{X}_c' \in (C_\alpha)^N$, we have

$$\left\|\frac{\partial U[\boldsymbol{X}_c]}{\partial x}(\cdot, t)\right\|_{L^\infty} \le \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}}\|u_0\|_{L^\infty} + N,$$
$$\|U[\boldsymbol{X}_c](\cdot, t) - U[\boldsymbol{X}_c'](\cdot, t)\|_{L^\infty} \le \frac{Ne^{\alpha t}}{(1+\alpha)^{\frac{1}{2}}}\|\boldsymbol{X}_c - \boldsymbol{X}_c'\|_\alpha.$$

Proof. Note that the derivative of $U[\boldsymbol{X}_c](x, t)$ with respect to x is expressed by

$$\frac{\partial U[\boldsymbol{X}_c]}{\partial x}(\cdot, t) = \left(\frac{\partial \widetilde{H}}{\partial x}(\cdot, t) * u_0(\cdot)\right) + \int_0^t \left(\frac{\partial \widetilde{H}}{\partial x}(\cdot, t-\tau) * F[\boldsymbol{X}_c](\cdot, \tau)\right) d\tau.$$

The first term in the right-hand side is estimated by the Young's inequality:

$$\left\|\frac{\partial \widetilde{H}}{\partial x}(\cdot, t) * u_0\right\|_{L^\infty} \le e^{-t}\left\|\frac{\partial H}{\partial x}(\cdot, t)\right\|_{L^1}\|u_0\|_{L^\infty} \le \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}}\|u_0\|_{L^\infty}.$$

The second term is estimated by

$$\begin{aligned}\left\|\int_0^t \frac{\partial \widetilde{H}}{\partial x}(\cdot, t-\tau) * F[\boldsymbol{X}_c](\cdot, \tau)d\tau\right\|_{L^\infty} &\le \int_0^t e^{-(t-\tau)}\left\|\frac{\partial H}{\partial x}(\cdot, t-\tau)\right\|_{L^1}\|F[\boldsymbol{X}_c](\cdot, \tau)\|_{L^\infty} \\ &\le \frac{N}{\sqrt{\pi}}\int_0^t \frac{e^{-(t-\tau)}}{(t-\tau)^{1/2}}d\tau \le \frac{N}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = N,\end{aligned}$$

where $\Gamma(s)$ denotes the Gamma function. Thus, we obtain

$$\left\|\frac{\partial U[\boldsymbol{X}_c]}{\partial x}(\cdot, t)\right\|_{L^\infty} \le \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}}\|u_0\|_{L^\infty} + N.$$

We show the continuity of $U[\boldsymbol{X}_c]$ with respect to $\boldsymbol{X}_c$. Note that

$$\big|U[\boldsymbol{X}_c](x,t) - U[\boldsymbol{X}'_c](x,t)\big| = \int_0^t \Big(\widetilde{H}(\cdot, t-\tau) * \big(F[\boldsymbol{X}_c](\cdot,\tau) - F[\boldsymbol{X}'_c](\cdot,\tau)\big)\Big)\, d\tau$$

$$\leq \int_0^t \int_{\mathbb{R}} \widetilde{H}(x-y, t-\tau) \sum_{i=1}^N \Big|\chi_{(X_c^i(\tau)-r, X_c^i(\tau)+r)}(y) - \chi_{({X'}_c^i(\tau)-r, {X'}_c^i(\tau)+r)}(y)\Big|\, dy d\tau,$$

where $\chi_A(x)$ is the indicator function, that is, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. Using the notations: $A \bigtriangleup B \equiv (A \cup B) \setminus (A \cap B)$ and

$$I_c^i(t) \equiv (X_c^i(t) - r, X_c^i(t) + r), \qquad {I'}_c^i(t) \equiv ({X'}_c^i(t) - r, {X'}_c^i(t) + r),$$

we find

$$|\chi_{(X_c^i(t)-r, X_c^i(t)+r)}(x) - \chi_{({X'}_c^i(t)-r, {X'}_c^i(t)+r)}(x)| = \chi_{I_c^i(t) \bigtriangleup {I'}_c^i(t)}(x),$$

and

$$|I_c^i(t) \bigtriangleup {I'}_c^i(t)| = \begin{cases} 4r, & I_c^i(t) \cap {I'}_c^i(t) = \emptyset, \\ 2|X_c^i(t) - {X'}_c^i(t)|, & I_c^i(t) \cap {I'}_c^i(t) \neq \emptyset. \end{cases}$$

Since $I_c^i(t) \cap {I'}_c^i(t) = \emptyset$ is equivalent to $|X_c^i(t) - {X'}_c^i(t)| > 2r$, we obtain

$$|I_c^i(t) \bigtriangleup {I'}_c^i(t)| \leq 2|X_c^i(t) - {X'}_c^i(t)|.$$

Thus, it follows from the Hölder inequality that

$$\big|U[\boldsymbol{X}_c](x,t) - U[\boldsymbol{X}'_c](x,t)\big| \leq \frac{N}{\sqrt{\pi}} \int_0^t \frac{e^{-(t-\tau)}}{(t-\tau)^{\frac{1}{2}}} \big\|\boldsymbol{X}_c(\tau) - \boldsymbol{X}'_c(\tau)\big\| d\tau,$$

and we obtain

$$\begin{aligned}\big\|U[\boldsymbol{X}_c](\cdot,t) - U[\boldsymbol{X}'_c](\cdot,t)\big\|_{L^\infty} &\leq \frac{Ne^{\alpha t}}{\sqrt{\pi}} \big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha \int_0^t \frac{e^{-(1+\alpha)(t-\tau)}}{(t-\tau)^{\frac{1}{2}}} d\tau \\ &\leq \frac{Ne^{\alpha t}}{\sqrt{\pi}(1+\alpha)^{\frac{1}{2}}} \big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha \int_0^{(1+\alpha)t} \tau^{-\frac{1}{2}} e^{-\tau} d\tau \\ &\leq \frac{Ne^{\alpha t}}{(1+\alpha)^{\frac{1}{2}}} \big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha,\end{aligned}$$

in which we have used $\Gamma(1/2) = \sqrt{\pi}$ in the last inequality. □

We now show the continuity of P_3 on $(C_\alpha)^N$.

Lemma 4.3. For any $\boldsymbol{X}_c$, $\boldsymbol{X}'_c \in (C_\alpha)^N$, there exist constants $q > 2$ and $C(q) > 0$ such that

$$\big\|P_3\boldsymbol{X}_c - P_3\boldsymbol{X}'_c\big\|_\alpha \leq \frac{M}{\alpha^{1/q}} \left(\frac{2N}{\alpha^{1-1/q}} + C(q)\|u_0\|_{L^\infty}\right) \big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha.$$

Proof. For $\boldsymbol{X}_c, \boldsymbol{X}'_c \in (C_\alpha)^N$, it follows from Lemma 4.2 that

$$\begin{aligned}
&\Big|\gamma(U[\boldsymbol{X}_c](X_c^i(t)+r,t)) - \gamma(U[\boldsymbol{X}'_c](X'^i_c(t)+r,t))\Big| \\
&\le M\big|U[\boldsymbol{X}_c](X_c^i(t)+r,t) - U[\boldsymbol{X}'_c](X_c^i(t)+r,t)\big| \\
&\quad + M\Big|U[\boldsymbol{X}'_c](X_c^i(t)+r,t) - U[\boldsymbol{X}'_c](X'^i_c(t)+r,t)\Big| \\
&\le M\big\|U[\boldsymbol{X}_c](\cdot,t) - U[\boldsymbol{X}'_c](\cdot,t)\big\|_{L^\infty} + M\left\|\frac{\partial U[\boldsymbol{X}'_c]}{\partial x}(\cdot,t)\right\|_{L^\infty}\Big|X_c^i(t) - X'^i_c(t)\Big| \\
&\le M\big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha \left[\frac{Ne^{\alpha t}}{(1+\alpha)^{\frac{1}{2}}} + \left(\frac{e^{-t}}{(\pi t)^{\frac{1}{2}}}\|u_0\|_{L^\infty} + N\right)e^{\alpha t}\right],
\end{aligned}$$

and thus

$$\begin{aligned}
&\left\|\int_0^t \gamma(U[\boldsymbol{X}_c](X_c^i(\tau)+r,\tau)) - \gamma(U[\boldsymbol{X}'_c](X'^i_c(\tau)+r,\tau))d\tau\right\|_\alpha \\
&\le M\big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha \sup_{t\ge 0}\left[N\left(\frac{1}{(1+\alpha)^{\frac{1}{2}}}+1\right)\int_0^t e^{-\alpha(t-\tau)}d\tau + \|u_0\|_{L^\infty}\int_0^t \frac{e^{-\tau}e^{-\alpha(t-\tau)}}{(\pi\tau)^{\frac{1}{2}}}d\tau\right] \\
&= M\big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha \left[\frac{N}{\alpha}\left(\frac{1}{(1+\alpha)^{\frac{1}{2}}}+1\right) + \frac{\|u_0\|_{L^\infty}}{\sqrt{\pi}}\sup_{t\ge 0}\int_0^t \tau^{-\frac{1}{2}}e^{-\tau}e^{-\alpha(t-\tau)}d\tau\right].
\end{aligned}$$

Owing to the Hölder inequality, we have

$$\begin{aligned}
\int_0^t \tau^{-\frac{1}{2}}e^{-\tau}e^{-\alpha(t-\tau)}d\tau &\le \left(\int_0^t \tau^{-\frac{p}{2}}e^{-p\tau}d\tau\right)^{\frac{1}{p}}\left(\int_0^t e^{-\alpha q(t-\tau)}d\tau\right)^{\frac{1}{q}} \\
&\le (\alpha q)^{-\frac{1}{q}}\left(\int_0^\infty \tau^{-\frac{p}{2}}e^{-p\tau}d\tau\right)^{\frac{1}{p}} = C(q)\alpha^{-\frac{1}{q}},
\end{aligned}$$

for $1 < p < 2$ and $q > 2$ satisfying $1/p + 1/q = 1$. Hence, we obtain

$$\big\|P_3\boldsymbol{X}_c - P_3\boldsymbol{X}'_c\big\|_\alpha \le \frac{M}{\alpha^{1/q}}\left(\frac{2N}{\alpha^{1-1/q}} + C(q)\|u_0\|_{L^\infty}\right)\big\|\boldsymbol{X}_c - \boldsymbol{X}'_c\big\|_\alpha.$$

□

On the basis of the continuity of P_1, P_2 and P_3, we show that the map P is a contraction mapping on $(C_\phi)^{2N}$. Let $\overline{\boldsymbol{X}}_c \equiv (\boldsymbol{X}_c, \boldsymbol{V}_c) \in (C_\alpha)^N \times (C_\alpha)^N = (C_\alpha)^{2N}$. For any $\overline{\boldsymbol{X}}_c$, $\overline{\boldsymbol{X}'}_c \in (C_\alpha)^{2N}$, we have

$$\big\|P\overline{\boldsymbol{X}}_c - P\overline{\boldsymbol{X}'}_c\big\|_\alpha = \left\|\begin{pmatrix} P_1V_c^1 - P_1V'^1_c \\ \vdots \\ P_1V_c^N - P_1V'^N_c \\ \begin{pmatrix} P_2V_c^1 - P_2V'^1_c \\ \vdots \\ P_2V_c^N - P_2V'^N_c \end{pmatrix} + P_3\boldsymbol{X}_c - P_3\boldsymbol{X}'_c \end{pmatrix}\right\|_\alpha.$$

It follows from (4.5) and Lemma 4.3 that there exist constants $q > 2$ and $C(q,N) > 0$ such that

$$\left\|P\overline{\boldsymbol{X}}_c - P\overline{\boldsymbol{X}'}_c\right\|_\alpha \leq \frac{C(q,N)}{\alpha^{1/q}}\left\|\overline{\boldsymbol{X}}_c - \overline{\boldsymbol{X}'}_c\right\|_\alpha. \tag{4.6}$$

Since we have $C(q,N)\alpha^{-1/q} < 1$ for sufficiently large $\alpha > 0$, P is a contraction mapping on $(C_\alpha)^{2N}$. Thus, there exists a unique fixed point $\overline{\boldsymbol{X}}_c^* = (\boldsymbol{X}_c^*, \boldsymbol{V}_c^*) \in (C_\alpha)^{2N}$ of the map P, that is, $\overline{\boldsymbol{X}}_c^*$ satisfies (4.2) and (4.4), which concludes that $(\boldsymbol{X}_c^*, U[\boldsymbol{X}_c^*])$ is a unique solution to (2.6)–(2.8).

5 Proof of the existence and non-existence of asymmetrically rotating solutions

5.1 Reformulation

We consider the following equations:

$$0 = \gamma(U(\pi_L(Z^i + r))) - \gamma(U(\pi_L(Z^i - r))) - 2r\mu c, \tag{5.1}$$

$$0 = U'' + cU' - U + F(z - Z^1) + F(z - Z^2), \tag{5.2}$$

for $i = 1, 2$, and

$$z \in [0, L) \setminus \{\pi_L(Z^1 + r), \pi_L(Z^1 - r), \pi_L(Z^2 + r), \pi_L(Z^2 - r)\},$$

where $U \in C^1[0, L]$ with $U(L) = U(0)$ and $U'(L) = U'(0)$. To see the translational symmetry of (5.1) and (5.2), we introduce the following periodically extended equations:

$$\begin{cases} 0 = \gamma(U^p(Z^{i,n} + r)) - \gamma(U^p(Z^{i,n} - r)) - 2r\mu c, \\ Z^{i,n+1} - Z^{i,n} = L, \\ Z^{2,n} - Z^{1,n} > 2r, \quad Z^{1,n+1} - Z^{2,n} > 2r, \end{cases} \tag{5.3}$$

for $i = 1, 2$, $n \in \mathbb{Z}$, and

$$\begin{cases} 0 = U^{p\prime\prime} + cU^{p\prime} - U + \displaystyle\sum_{i\in\{1,2\},\ n\in\mathbb{Z}} F(z - Z^{i,n}), \\ U^p(z + L) = U^p(z), \end{cases} \tag{5.4}$$

for $z \in \mathbb{R} \setminus \bigcup_{i\in\{1,2\},\ n\in\mathbb{Z}} \{Z^{i,n} + r, Z^{i,n} - r\}$, where $U^p \in C^1(\mathbb{R})$. It is easily confirmed that if (Z^{1*}, Z^{2*}, U^*) is a solution to (5.1) and (5.2), then $(Z^{1*} + nL, Z^{2*} + nL, U^{*p})$ is a solution to (5.3) and (5.4), where U^{*p} is a periodically extended function of U^*. Conversely, if $(Z^{1,n*}, Z^{2,n*}, U^{p*})$ is a solution to (5.3) and (5.4), then $(Z^{1,0}, Z^{2,0}, U^{p*}|_{[0,L)})$ satisfies (5.1) and (5.2). In this sense, the system (5.1) and (5.2) is equivalent to (5.3) and (5.4). Since the system (5.3) and (5.4) has the translational symmetry, it is sufficient to consider the system (5.1) and (5.2) with $Z^1 = r$. Then, (5.2) is rewritten by

$$
\begin{aligned}
0 &= U'' + cU' - U + 1, && z \in (0, l_1),\\
0 &= U'' + cU' - U, && z \in (l_1, l_2),\\
0 &= U'' + cU' - U + 1, && z \in (l_2, l_3),\\
0 &= U'' + cU' - U, && z \in (l_3, L),
\end{aligned}
\tag{5.5}
$$

where $l_1 = 2r$, $l_2 = 2r + d$, $l_3 = 4r + d$ and $d = Z^2 - r - (Z^1 + r)$. We assume that $U \in C^1[0, L]$ satisfies the periodic boundary condition given by $U(L) = U(0)$ and $U'(L) = U'(0)$. Note that, in the above formulation, symmetrically and asymmetrically rotating solutions correspond with $d = L/2 - 2r$ and $d \neq L/2 - 2r$, respectively.

For the sake of simplicity, we define the following functions:

$$
\begin{aligned}
\phi(c) = \frac{-c}{2}, \quad \theta(c) = \frac{\sqrt{c^2+4}}{2}, \quad \lambda_+(c) = \phi(c) + \theta(c), \quad \lambda_-(c) = \phi(c) - \theta(c),\\
E_+(c, z) = \exp(-\lambda_+(c)z), \quad E_-(c, z) = \exp(\lambda_-(c)z),\\
U_+(c) = \frac{1}{2\theta(c)\lambda_+(c)} \frac{1 - E_+(c, 2r)}{1 - E_+(c, L)}, \quad U_-(c) = \frac{-1}{2\theta(c)\lambda_-(c)} \frac{1 - E_-(c, 2r)}{1 - E_-(c, L)}.
\end{aligned}
\tag{5.6}
$$

Throughout this paper, we will omit the discussion on c when it is fixed. We first show that the system (5.5) has a unique solution.

Lemma 5.1. Let the constants $l_1 < l_2 < l_3 < L$ be $l_1 = 2r$, $l_2 = 2r + d$ and $l_3 = 4r + d$ with $d, r > 0$. For any given $c \in \mathbb{R}$, there exists a unique solution of (5.5). Moreover, $U(0)$, $U(l_1)$, $U(l_2)$ and $U(l_3)$ are expressed by

$$
\begin{aligned}
U(0) &= U_+(1 + E_+(l_2)) + U_-(E_-(L - l_3) + E_-(L - l_1)),\\
U(l_1) &= U_+(E_+(L - l_1) + E_+(d)) + U_-(E_-(L - l_2) + 1),\\
U(l_2) &= U_+(1 + E_+(L - l_2)) + U_-(E_-(L - l_1) + E_-(d)),\\
U(l_3) &= U_+(E_+(L - l_3) + E_+(L - l_1)) + U_-(1 + E_-(l_2)).
\end{aligned}
$$

Proof. Let us fix $c \in \mathbb{R}$. By a classical theory, we can represent a solution of (5.5) by

$$
\begin{aligned}
U(z) &= a^1_+ E_+(-z) + a^1_- E_-(z) + 1, && z \in (0, l_1),\\
U(z) &= b^1_+ E_+(-(z - l_1)) + b^1_- E_-(z - l_1), && z \in (l_1, l_2),\\
U(z) &= a^2_+ E_+(-(z - l_2)) + a^2_- E_-(z - l_2) + 1, && z \in (l_2, l_3),\\
U(z) &= b^2_+ E_+(-(z - l_3)) + b^2_- E_-(z - l_3), && z \in (l_3, L).
\end{aligned}
\tag{5.7}
$$

Thus, it is sufficient to show the unique existence of constants $a^1_\pm$, $a^2_\pm$, $b^1_\pm$ and $b^2_\pm$. The periodic boundary condition, $U(L) = U(0)$ and $U'(L) = U'(0)$, requires that

$$
\lim_{z \to +0} U(z) = \lim_{z \to L-0} U(z), \qquad \lim_{z \to +0} U'(z) = \lim_{z \to L-0} U'(z),
$$

which is equivalent to

$$
\begin{aligned}
a^1_+ + a^1_- + 1 &= b^2_+ E_+(-(L-l_3)) + b^2_- E_-(L-l_3),\\
a^1_+\lambda_+ + a^1_-\lambda_- &= b^2_+\lambda_+ E_+(-(L-l_3)) + b^2_-\lambda_- E_-(L-l_3).
\end{aligned}
$$

These relations are rewritten by

$$\Lambda \boldsymbol{a}^1 + \boldsymbol{e} = \Lambda E(L-l_3)\boldsymbol{b}^2, \tag{5.8}$$

where

$$
\boldsymbol{a}^i = \begin{pmatrix} a^i_+ \\ a^i_- \end{pmatrix}, \qquad \boldsymbol{b}^i = \begin{pmatrix} b^i_+ \\ b^i_- \end{pmatrix}, \qquad \boldsymbol{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
\Lambda = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}, \qquad E(z) = \begin{pmatrix} E_+(-z) & 0 \\ 0 & E_-(z) \end{pmatrix}.
$$

Similarly, since U is continuously differentiable at l_1, l_2 and l_3, we have

$$
\begin{aligned}
\Lambda E(l_1)\boldsymbol{a}^1 + \boldsymbol{e} &= \Lambda \boldsymbol{b}^1,\\
\Lambda \boldsymbol{a}^2 + \boldsymbol{e} &= \Lambda E(l_2-l_1)\boldsymbol{b}^1,\\
\Lambda E(l_3-l_2)\boldsymbol{a}^2 + \boldsymbol{e} &= \Lambda \boldsymbol{b}^2.
\end{aligned} \tag{5.9}
$$

Note that it follows from $\det \Lambda = \lambda_- - \lambda_+ = -2\theta < 0$ that Λ is a regular matrix. We find that (5.8) and (5.9) are equivalent to

$$
\begin{aligned}
\boldsymbol{a}^1 &= E(L-l_3)\boldsymbol{b}^2 - \Lambda^{-1}\boldsymbol{e},\\
\boldsymbol{b}^1 &= E(l_1)\boldsymbol{a}^1 + \Lambda^{-1}\boldsymbol{e},\\
\boldsymbol{a}^2 &= E(l_2-l_1)\boldsymbol{b}^1 - \Lambda^{-1}\boldsymbol{e},\\
\boldsymbol{b}^2 &= E(l_3-l_2)\boldsymbol{a}^2 + \Lambda^{-1}\boldsymbol{e}.
\end{aligned} \tag{5.10}
$$

Substituting these equalities in order, we obtain

$$
\begin{aligned}
\boldsymbol{a}^1 &= -\Lambda^{-1}\boldsymbol{e}\\
&\quad + E(L-l_3)(E(l_3-l_2)(E(l_2-l_1)(E(l_1)\boldsymbol{a}^1 + \Lambda^{-1}\boldsymbol{e}) - \Lambda^{-1}\boldsymbol{e}) + \Lambda^{-1}\boldsymbol{e}),\\
&= E(L)\boldsymbol{a}^1 + (E(L-l_1) - E(L-l_2) + E(L-l_3) - I)\Lambda^{-1}\boldsymbol{e}.
\end{aligned}
$$

Here, we used $E(z_1)E(z_2) = E(z_1+z_2)$. Note that $\det(I - E(z)) = (1 - E_+(-z))(1 - E_-(z)) \neq 0$ for any $z \neq 0$. Then, we have

$$\boldsymbol{a}^1 = (I - E(L))^{-1}(E(L-l_1) - E(L-l_2) + E(L-l_3) - I)\Lambda^{-1}\boldsymbol{e},$$

which indicates that constants $a^1_\pm$ are uniquely determined. Using (5.10), we obtain the other constants $a^2_\pm$, $b^1_\pm$ and $b^2_\pm$ that are uniquely expressed by

$$
\begin{aligned}
\boldsymbol{b}^1 &= (I - E(L))^{-1}(-E(L-l_2+l_1) + E(L-l_3+l_1) - E(l_1) + I)\Lambda^{-1}\boldsymbol{e},\\
\boldsymbol{a}^2 &= (I - E(L))^{-1}(E(L-l_3+l_2) - E(l_2) + E(l_2-l_1) - I)\Lambda^{-1}\boldsymbol{e},\\
\boldsymbol{b}^2 &= (I - E(L))^{-1}(-E(l_3) + E(l_3-l_1) - E(l_3-l_2) + I)\Lambda^{-1}\boldsymbol{e}.
\end{aligned}
$$

Thus, we conclude that (5.5) has a unique solution $U \in C^1[0, L)$ satisfying the periodic boundary condition.

On the other hand, considering

$$(I - E(L))^{-1} = \begin{pmatrix} \dfrac{1}{1 - E_+(-L)} & 0 \\ 0 & \dfrac{1}{1 - E_-(L)} \end{pmatrix}, \quad \Lambda^{-1} = -\frac{1}{2\theta}\begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix},$$

we find

$$\begin{pmatrix} a_+^1 \\ a_-^1 \end{pmatrix} = \begin{pmatrix} \dfrac{-\lambda_-}{2\theta} \dfrac{E_+(-(L - l_1)) - E_+(-(L - l_2)) + E_+(-(L - l_3)) - 1}{1 - E_+(-L)} \\ \dfrac{\lambda_+}{2\theta} \dfrac{E_-(L - l_1) - E_-(L - l_2) + E_-(L - l_3) - 1}{1 - E_-(L)} \end{pmatrix}.$$

Substituting $a_\pm^1$ into the first line in (5.7), we obtain

$$\begin{aligned} U(z) = 1 &- \frac{\lambda_- E_+(-z)}{2\theta} \frac{E_+(-(L - l_1)) - E_+(-(L - l_2)) + E_+(-(L - l_3)) - 1}{1 - E_+(-L)} \\ &+ \frac{\lambda_+ E_-(z)}{2\theta} \frac{E_-(L - l_1) - E_-(L - l_2) + E_-(L - l_3) - 1}{1 - E_-(L)}, \end{aligned}$$

for $z \in (0, l_1)$. Since the constants $\boldsymbol{a}^1$, $\boldsymbol{a}^2$, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ are determined so that $U \in C^1[0, L]$, we have

$$\begin{aligned} U(0) &= \lim_{z \to +0} U(z) \\ &= 1 - \frac{\lambda_-}{2\theta} \frac{E_+(-(L - l_1)) - E_+(-(L - l_2)) + E_+(-(L - l_3)) - 1}{1 - E_+(-L)} \\ &\quad + \frac{\lambda_+}{2\theta} \frac{E_-(L - l_1) - E_-(L - l_2) + E_-(L - l_3) - 1}{1 - E_-(L)}. \end{aligned}$$

It follows from $\lambda_+ - \lambda_- = 2\theta$ and $\lambda_+ \lambda_- = -1$ that

$$\begin{aligned} &1 - \frac{-\lambda_-}{2\theta} \frac{1}{1 - E_+(-L)} - \frac{\lambda_+}{2\theta} \frac{1}{1 - E_-(L)} \\ &= -\frac{-\lambda_-}{2\theta} \frac{E_+(-L)}{1 - E_+(-L)} - \frac{\lambda_+}{2\theta} \frac{E_-(L)}{1 - E_-(L)}, \\ &= -\frac{1}{2\theta\lambda_+} \frac{E_+(-L)}{1 - E_+(-L)} - \frac{1}{-2\theta\lambda_-} \frac{E_-(L)}{1 - E_-(L)}. \end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
U(0) &= \frac{1}{2\theta\lambda_+}\frac{E_+(-(L-l_1)) - E_+(-(L-l_2)) + E_+(-(L-l_3)) - E_+(L)}{1-E_+(-L)} \\
&\quad - \frac{1}{2\theta\lambda_-}\frac{E_-(L-l_1) - E_-(L-l_2) + E_-(L-l_3) - E_-(L)}{1-E_-(L)} \\
&= \frac{1}{2\theta\lambda_+}\frac{E_+(-L)}{1-E_+(-L)}(E_+(l_1) - E_+(l_2) + E_+(l_3) - 1) \\
&\quad - \frac{1}{2\theta\lambda_-}\frac{E_-(L-l_1) - E_-(L-l_2) + E_-(L-l_3) - E_-(L)}{1-E_-(L)} \\
&= \frac{1}{2\theta\lambda_+}\frac{1 - E_+(2r) + E_+(2r+d) - E_+(4r+d)}{1-E_+(L)} \\
&\quad - \frac{1}{2\theta\lambda_-}\frac{E_-(L-2r) - E_-(L-2r-d) + E_-(L-4r-d) - E_-(L)}{1-E_-(L)} \\
&= \frac{1}{2\theta\lambda_+}\frac{1-E_+(2r)}{1-E_+(L)}(1+E_+(2r+d)) \\
&\quad - \frac{1}{2\theta\lambda_-}\frac{1-E_-(2r)}{1-E_-(L)}E_-(L-4r-d)(1+E_-(2r+d)).
\end{aligned}
$$

The definitions of U_+ and U_- yield

$$U(0) = (1+E_+(2r+d))U_+ + (E_-(L-4r-d) + E_-(L-2r))U_-.$$

Similarly, it is confirmed that

$$
\begin{aligned}
U(2r) &= (E_+(L-2r) + E_+(d))U_+ + (E_-(L-2r-d)+1)U_-, \\
U(2r+d) &= (1+E_+(L-2r-d))U_+ + (E_-(L-2r) + E_-(d))U_-, \\
U(4r+d) &= (E_+(L-4r-d) + E_+(L-2r))U_+ + (1+E_-(2r+d))U_-. \quad \square
\end{aligned}
$$

□

For later use, we introduce the following notations:

$$
\begin{aligned}
U_{1f}(c,d) \equiv U(2r), &\quad U_{1r}(c,d) \equiv U(0), \\
U_{2f}(c,d) \equiv U(4r+d), &\quad U_{2r}(c,d) \equiv U(2r+d), \\
\Delta U_1(c,d) \equiv U_{1f}(c,d) - U_{1r}(c,d), &\quad \Delta U_2(c,d) \equiv U_{2f}(c,d) - U_{2r}(c,d).
\end{aligned}
$$

We will omit the discussion on c and d when they are fixed. Before stating the proof of Theorem 2.4, we show some properties of U_{1f}, U_{1r}, U_{2f} and U_{2r}.

Lemma 5.2. Let $c > 0$ be a fixed constant. Then, we have

(a) $\Delta U_1(c, L/2-2r) = \Delta U_2(c, L/2-2r) < 0$.

(b) $\Delta U_1(c,d) - \Delta U_2(c,d) = 0$ if and only if $L/2-2r-d = 0$. If $L/2-2r-d \neq 0$, then the sign of $\Delta U_1(c,d) - \Delta U_2(c,d)$ is equal to that of $L/2-2r-d$.

(c) $U_{1f}(c,d) - U_{2f}(c,d) = 0$ if and only if $L/2 - 2r - d = 0$. If $L/2 - 2r - d \neq 0$, then the sign of $U_{1f}(c,d) - U_{2f}(c,d)$ is equal to that of $L/2 - 2r - d$.

Proof. We consider ΔU_i, U_{if}, and U_{ir} as functions of d, that is, $\Delta U_i(d)$, $U_{if}(d)$, and $U_{ir}(d)$ for $i = 1, 2$.

(a) It is straightforward to check that $U_{1f}(c, L/2-2r) = U_{2f}(c, L/2-2r)$ and $U_{1r}(c, L/2-2r) = U_{2r}(c, L/2-2r)$ so that $\Delta U_1(c, L/2-2r) = \Delta U_2(c, L/2-2r)$. We have

$$\begin{aligned}\Delta U_1(L/2-2r) &= (E_+(L-2r) + E_+(L/2-2r) - 1 - E_+(L/2))U_+ \\ &\quad + (E_-(L/2) + 1 - E_-(L/2-2r) - E_-(L-2r))U_- \\ &= -(1+E_+(L/2))(1-E_+(L/2-2r))U_+ \\ &\quad + (1+E_-(L/2))(1-E_-(L/2-2r))U_-.\end{aligned}$$

Note that

$$\begin{aligned}&\theta(1+E_+(L/2))(1-E_+(L/2-2r))U_+ \\ &= \frac{(1+\exp(-(L/2)\lambda_+))(1-\exp(-(L/2-2r)\lambda_+))}{2\lambda_+}\frac{1-\exp(-2r\lambda_+)}{1-\exp(-L\lambda_+)} \\ &= \frac{1}{\lambda_+}\frac{\sinh(r\lambda_+)}{\sinh((L/4)\lambda_+)}\sinh((L/4-r)\lambda_+) \\ &= \xi_2(\lambda_+; r, L/4),\end{aligned}$$

and $\theta U_-(1+E_-(L/2))(1-E_+(L/2-2r)) = \xi_2(-\lambda_-; r, L/4)$. Since it follows from $0 < 4r < L$ that $\xi_2(x; r, L/4)$ is strictly decreasing for $x > 0$ (see Appendix A.2), we find that $\xi_2(\lambda_+; r, L/4) < \xi_2(\lambda_-; r, L/4)$, that is, $U_+(1+E_+(L/2))(1-E_+(L/2-2r)) > U_-(1+E_-(L/2))(1-E_-(L/2-2r))$. Hence, we obtain $\Delta U_1(L/2-2r) < 0$.

(b) It follows that

$$\begin{aligned}\Delta U_1(d) - \Delta U_2(d) &= U_+E_+(d)(1-E_+(2r))(1-E_+(L-4r-2d)) \\ &\quad + U_-E_-(d)(1-E_-(2r))(1-E_-(L-4r-2d)).\end{aligned}$$

Since we have $U_\pm E_\pm(d)(1-E_\pm(2r)) > 0$ and $1 - E_\pm(z)$ has the same sign as that of z, we obtain the desired result.

(c) Note that

$$\begin{aligned}U_{1f}(d) - U_{2f}(d) &= E_+(d)(1-E_+(L-4r-2d))U_+ \\ &\quad - E_-(2r+d)(1-E_-(L-4r-2d))U_-.\end{aligned} \tag{5.11}$$

Since we have

$$
\begin{aligned}
&\theta E_+(r+d)(1-E_+(L-4r-2d))U_+ \\
&= \frac{\exp(-(r+d)\lambda_+)}{2\lambda_+}\frac{1-\exp(-2r\lambda_+)}{1-\exp(-L\lambda_+)}(1-\exp(-(L-4r-2d)\lambda_+)) \\
&= \frac{\sinh((L/2-2r-d)\lambda_+)}{\lambda_+}\frac{\sinh(r\lambda_+)}{\sinh((L/2)\lambda_+)} \\
&= \xi_3(\lambda_+; r, L/2, d/2),
\end{aligned}
$$

and, similarly, $\theta E_-(r+d)(1-E_-(L-4r-2d))U_- = \xi_3(-\lambda_-; r, L/2, d/2)$, we find that (5.11) is rewritten by

$$
\begin{aligned}
&U_{1f}(d) - U_{2f}(d) \\
&= \frac{E_+(-r)}{\theta}\xi_3(\lambda_+; r, L/2, d/2) - \frac{E_-(r)}{\theta}\xi_3(-\lambda_-; r, L/2, d/2).
\end{aligned}
$$

For the case of $L-4r-2d > 0$, owing to $0 < 4r < L$ and $0 < d < L - 4r$, $\xi_3(x; r, L/2, d/2)$ is strictly decreasing for $x > 0$ (see Appendix A.3). Thus, considering $E_+(-r) > 1 > E_-(r)$, we find

$$
\begin{aligned}
&U_{1f}(d) - U_{2f}(d) \\
&= \frac{1}{\theta}(E_+(-r)\xi_3(\lambda_+; r, L/2, d/2) - E_-(r)\xi_3(-\lambda_-; r, L/2, d/2)) > 0.
\end{aligned}
$$

For the case of $L-4r-2d < 0$, we have

$$
\xi_3(\lambda_+; r, L/2, d/2) < \xi_3(-\lambda_-; r, L/2, d/2) < 0,
$$

and thus

$$
\begin{aligned}
&U_{1f}(d) - U_{2f}(d) \\
&= \frac{1}{\theta}(E_+(-r)\xi_3(\lambda_+; r, L/2, d/2) - E_-(r)\xi_3(-\lambda_-; r, L/2, d/2)) < 0.
\end{aligned}
$$

Finally, we easily confirm that $L-4r-2d = 0$ is equivalent to $U_{1f}(c) - U_{2f}(c) = 0$.
□

□

5.2 Proofs of Theorem 2.4(a)–(b)

We first prove Theorem 2.4(a) for the case of $c > 0$. Note that (5.1) are rewritten by

$$
2r\mu c = \gamma(U_{1f}(c,d)) - \gamma(U_{1r}(c,d)), \tag{5.12}
$$

$$
0 = \gamma(U_{1f}(c,d)) - \gamma(U_{1r}(c,d)) - (\gamma(U_{2f}(c,d)) - \gamma(U_{2r}(c,d))). \tag{5.13}
$$

Since, for symmetric solutions satisfying $d = L/2 - 2r$, Lemma 5.2(a) gives $U_{1f}(c, L/2 - 2r) < U_{1r}(c, L/2 - 2r)$ and $\gamma(u)$ is strictly decreasing for $u > 0$, there exists a unique $\mu > 0$ satisfying (5.12). Thus, it is sufficient to show that $U_{1f}(c, L/2 - 2r)$, $U_{1r}(c, L/2 - 2r)$, $U_{2f}(c, L/2 - 2r)$ and $U_{2r}(c, L/2 - 2r)$ satisfy (5.13). It follows from Lemma 5.2(b)–(c) that

$$U_{1f}(c, L/2 - 2r) = U_{2f}(c, L/2 - 2r), \tag{5.14}$$

and

$$U_{1f}(c, L/2 - 2r) - U_{1r}(c, L/2 - 2r) = U_{2f}(c, L/2 - 2r) - U_{2r}(c, L/2 - 2r),$$

respectively. Thus, we find

$$U_{1r}(c, L/2 - 2r) = U_{2r}(c, L/2 - 2r). \tag{5.15}$$

Combining (5.14) with (5.15), we obtain (5.13).

Next, we consider the case of $c = 0$. Note that (5.1) are equivalent to

$$\gamma(U_{1f}(0, d)) = \gamma(U_{1r}(0, d)), \qquad \gamma(U_{2r}(0, d)) = \gamma(U_{2r}(0, d)). \tag{5.16}$$

Since $\gamma(u)$ is strictly decreasing for $u > 0$, (5.16) is satisfied if and only if $\Delta U_1(0, d) = \Delta U_2(0, d) = 0$ holds. Considering $\lambda_-(0) = -\lambda_+(0) = -1$ and $U_+(0, d) = U_-(0, d) = U_0$, where

$$U_0 = \frac{1}{2}\frac{1 - E_0(2r)}{1 - E_0(L)}, \qquad E_0(z) = \exp(-z),$$

we obtain

$$\begin{aligned}
U_{1f}(0, d) &= U_0(1 + E_0(L - 2r) + E_0(d) + E_0(L - 2r - d)),\\
U_{1r}(0, d) &= U_0(1 + E_0(L - 2r) + E_0(2r + d) + E_0(L - 4r - d)),\\
U_{2f}(0, d) &= U_0(1 + E_0(L - 2r) + E_0(2r + d) + E_0(L - 4r - d)),\\
U_{2r}(0, d) &= U_0(1 + E_0(L - 2r) + E_0(d) + E_0(L - 2r - d)),
\end{aligned}$$

which implies

$$\Delta U_1(0, d) = -\Delta U_2(0, d) = U_0 E_0(d)(1 - E_0(2r))(1 - E_0(L - 4r - 2d)).$$

Hence, (5.16) holds if and only if $d = L/2 - 2r$, which concludes Theorem 2.4(a) with $c = 0$ and Theorem 2.4(b).

5.3 Proof of Theorem 2.4(c)

We prove Theorem 2.4(c) by contradiction. Let $c > 0$ be a fixed constant. If there exists a rotating solution, then (5.13) is satisfied with $\Delta U_1 < 0$ and $\Delta U_2 < 0$. Thus, it is sufficient to prove that (5.13) is not satisfied provided that $\Delta U_1 < 0$ and $\Delta U_2 < 0$. For asymmetric solutions, that is, $d \neq L/2 - 2r$, we should consider the following three cases.

1. $U_{1f} > U_{2r}$.

2. $U_{1f} \leq U_{2r}$ and $L/2 - 2r - d < 0$.

3. $U_{1f} \leq U_{2r}$ and $L/2 - 2r - d > 0$.

For the case 1, (5.13) is equivalent to

$$\int_{U_{1f}}^{U_{1r}} -\gamma'(U)dU = \int_{U_{2f}}^{U_{2r}} -\gamma'(U)dU. \tag{5.17}$$

It follows from $\Delta U_1 < 0$, $\Delta U_2 < 0$ and $U_{1f} > U_{2r}$ that

$$U_{1r} > U_{1f} > U_{2r} > U_{2f}. \tag{5.18}$$

Since we have $\gamma'(u) < 0$ and $\gamma''(u) \geq 0$ for $u > 0$, (5.18) yields

$$0 < -\gamma'(U_{1r}) \leq -\gamma'(U_{1f}) \leq -\gamma'(U_{2r}) \leq -\gamma'(U_{2f}). \tag{5.19}$$

Owing to Lemma 5.2(c), $U_{1f} > U_{2f}$ in (5.18) implies $L/2 - 2r - d > 0$. Thus, Lemma 5.2(b) yields $0 > U_{1f} - U_{1r} > U_{2f} - U_{2r}$. Hence, we obtain

$$0 < U_{1r} - U_{1f} < U_{2r} - U_{2f}. \tag{5.20}$$

It follows from (5.19) and (5.20) that

$$\begin{aligned}\int_{U_{1f}}^{U_{1r}} -\gamma'(U)dU &\leq -\gamma'(U_{1f})(U_{1r} - U_{1f}) \\ &< -\gamma'(U_{2r})(U_{2r} - U_{2f}) \leq \int_{U_{2f}}^{U_{2r}} -\gamma'(U)dU,\end{aligned}$$

which contradicts (5.17).

For the case 2, (5.13) is equivalent to

$$\int_{U_{1f}}^{U_{2f}} -\gamma'(U)dU = \int_{U_{1r}}^{U_{2r}} -\gamma'(U)dU. \tag{5.21}$$

Owing to $L/2 - 2r - d < 0$, it follows from Lemma 5.2(c) that $U_{1f} < U_{2f}$. Thus, (5.21) and $\gamma' < 0$ yield $U_{1r} < U_{2r}$. In addition, the assumption $\Delta U_2 < 0$ gives

$$U_{1f} < U_{2f} < U_{2r}.$$

If $U_{2f} \leq U_{1r}$, then we have $U_{1f} < U_{2f} \leq U_{1r} < U_{2r}$ and it follows from $L/2 - 2r - d < 0$ and Lemma 5.2.(b) that $\Delta U_1 < \Delta U_2$, that is, $U_{2r} - U_{1r} < U_{2f} - U_{1f}$. Thus,

$$\begin{aligned}\int_{U_{1r}}^{U_{2r}} -\gamma'(U)dU &\leq -\gamma'(U_{1r})(U_{2r} - U_{1r}) \\ &< -\gamma'(U_{2f})(U_{2f} - U_{1f}) \leq \int_{U_{1f}}^{U_{2f}} -\gamma'(U)dU,\end{aligned}$$

which contradicts (5.21). As for $U_{2f} > U_{1r}$, we obtain $U_{1f} < U_{1r} < U_{2f} < U_{2r}$ and following inequality:

$$\int_{U_{2f}}^{U_{2r}} -\gamma'(U)dU \le -\gamma'(U_{2f})(U_{2r} - U_{2f})$$
$$< -\gamma'(U_{1r})(U_{1r} - U_{1f}) \le \int_{U_{1f}}^{U_{1r}} -\gamma'(U)dU,$$

owing to $0 < -\Delta U_2 < -\Delta U_1$, which contradicts (5.21). Thus, (5.13) is not satisfied in the case 2.

For the case 3, it follows from $L/2 - 2r - d > 0$ and Lemma 5.2(c) that $U_{2f} < U_{1f}$. Hence, we have $U_{2f} < U_{1f} \le U_{2r}$. In the same manner as that for the case 2, we assume that $U_{2r} < U_{1r}$. By $L/2 - 2r - d > 0$, Lemma 5.2(b) gives $\Delta U_1 > \Delta U_2$, that is, $U_{1f} - U_{2f} > U_{1r} - U_{2r}$. Hence, we find

$$\int_{U_{2r}}^{U_{1r}} -\gamma'(U)dU \le -\gamma'(U_{2r})(U_{1r} - U_{2r})$$
$$< -\gamma'(U_{1f})(U_{1f} - U_{2f}) \le \int_{U_{2f}}^{U_{1f}} -\gamma'(U)dU,$$

which contradicts (5.13).

5.4 Proof of Theorem 2.4(d)

Let $\Gamma(c, d) \equiv \gamma(U_{1f}(c, d)) - \gamma(U_{1r}(c, d)) - [\gamma(U_{2f}(c, d)) - \gamma(U_{2r}(c, d))]$. We show that there exists a constant $d_0 \in (0, L/2 - 2r)$ such that $\Gamma(c, d_0) = 0$. Our strategy is to investigate the asymptotic behavior of $\Gamma(c, d)$ as $d \to +0$ and $d \to L/2 - 2r - 0$ for sufficiently large c. We first consider the limit of $\Gamma(c, d)$ as $d \to +0$. It follows that

$$\begin{aligned}\Gamma(c, 0) &= \lim_{d\to 0} \Gamma(c, d)\\ &= \gamma(U_{1f}(c, 0)) - \gamma(U_{1r}(c, 0)) - [\gamma(U_{2f}(c, 0)) - \gamma(U_{2r}(c, 0))],\end{aligned}$$

and

$$\begin{aligned}U_{1f}(c, 0) &= U_+(c)(1 + E_+(c, L - 2r)) + U_-(c)(E_-(c, L - 2r) + 1),\\ U_{1r}(c, 0) &= U_+(c)(1 + E_+(c, 2r)) + U_-(c)(E_-(c, L - 4r) + E_-(c, L - 2r)),\\ U_{2f}(c, 0) &= U_+(c)(E_+(c, L - 4r) + E_+(c, L - 2r)) + U_-(c)(1 + E_-(c, 2r)),\\ U_{2r}(c, 0) &= U_+(c)(1 + E_+(c, L - 2r)) + U_-(c)(E_-(c, L - 2r) + 1),\end{aligned}$$

in the same limit. Note that

$$\begin{aligned}&U_{1f}(c, 0) - U_{1r}(c, 0)\\ &= -E_+(c, 2r)U_+(c)(1 - E_+(c, L - 4r)) + U_-(c)(1 - E_-(c, L - 4r)),\\ &U_{2f}(c, 0) - U_{2r}(c, 0)\\ &= -U_+(c)(1 - E_+(c, L - 4r)) + E_-(c, 2r)U_-(c)(1 - E_-(c, L - 4r)).\end{aligned}$$

Then, we have $U_{1f}(c,0) - U_{1r}(c,0) < 0$ and $U_{2f}(c,0) - U_{2r}(c,0) < 0$ for sufficiently large c. Indeed, since it follows that

$$\begin{aligned} &U_{1f}(c,0) - U_{1r}(c,0) \\ &= (1 - E_+(c, L-4r)) \left[-E_+(c,2r)U_+(c) + U_-(c)\frac{1 - E_-(c, L-4r)}{1 - E_+(c, L-4r)} \right], \end{aligned}$$

and Lemma C.1 in Appendix C gives

$$\lim_{c\to\infty} \left[-E_+(2r)U_+(c) + U_-(c)\frac{1 - E_-(c, L-4r)}{1 - E_+(c, L-4r)} \right] = -\frac{2r}{L} < 0,$$

we obtain $U_{1f}(c,0) - U_{1r}(c,0) < 0$ for sufficiently large c. On the other hand, we have

$$\begin{aligned} \theta U_+(1 - E_+(L-4r)) &= \frac{1}{2\lambda_+}\frac{1 - E_+(2r)}{1 - E_+(L)}(1 - E_+(L-4r)) \\ &= \frac{1}{\lambda_+}\frac{\sinh(r\lambda_+)}{\sinh(L/2\lambda_+)}\sinh((L/2 - 2r)\lambda_+)E_+(-r) \\ &= \xi_3(\lambda_+; r, L/2, 0)E_+(-r), \end{aligned}$$

and, similarly,

$$\theta U_-(1 - E_-(L-4r)) = \xi_3(-\lambda_-; r, L/2, 0)E_-(r).$$

Since $\xi_3(x; r, L/2, 0)$ is strictly decreasing for $x > 0$ (see Appendix B), we find

$$\begin{aligned} U_+(1 - E_+(L-4r)) &> \frac{\xi_3(\lambda_+; r, L/2, 0)}{\theta} \\ &> \frac{\xi_3(\lambda_-; r, L/2, 0)}{\theta} > E_-(2r)U_-(1 - E_-(L-4r)), \end{aligned}$$

in which we used $E_+(-r) > 1$ and $E_-(r) < 1$. Thus, $U_{2f}(c,0) - U_{2r}(c,0) < 0$ holds for sufficiently large c.

From the above estimates, we have $U_{2f}(c,0) < U_{2r}(c,0) = U_{1f}(c,0) < U_{1r}(c,0)$. Hence, the mean value theorem implies that there exist U^* and U^{**} such that

$$\begin{aligned} \gamma(U_{1r}(c,0)) - \gamma(U_{1f}(c,0)) &= (U_{1r}(c,0) - U_{1f}(c,0))\gamma'(U^*), \\ \gamma(U_{2r}(c,0)) - \gamma(U_{2f}(c,0)) &= (U_{2r}(c,0) - U_{2f}(c,0))\gamma'(U^{**}), \end{aligned} \tag{5.22}$$

where U^* and U^{**} satisfy

$$U_{2f}(c,0) < U^{**} < U_{2r}(c,0) = U_{1f}(c,0) < U^* < U_{1r}(c,0). \tag{5.23}$$

Similarly, it follows from $U^{**} < U^*$ that there exists U^{***} satisfying $U^{**} < U^{***} < U^*$ such that

$$\gamma'(U^*) - \gamma'(U^{**}) = (U^* - U^{**})\gamma''(U^{***}). \tag{5.24}$$

Using (5.22) and (5.24), we find

$$
\begin{aligned}
\Gamma(c,0) &= (U_{1r}(c,0)-U_{1f}(c,0))\gamma'(U^*)-(U_{2r}(c,0)-U_{2f}(c,0))\gamma'(U^{**})\\
&= (U_{1r}(c,0)-U_{2f}(c,0))\gamma'(U^{**})\\
&\quad + (U_{1r}(c,0)-U_{1f}(c,0))(U^*-U^{**})\gamma''(U^{***})\\
&= [U_+(1-E_+(2r))(1-E_+(L-4r))\\
&\qquad +U_-(1-E_-(2r))(1-E_-(L-4r))]\,\gamma'(U^{**})\\
&\quad - [E_+(2r)U_+(1-E_+(L-4r))\\
&\qquad -U_-(1-E_-(L-4r))]\,(U^*-U^{**})\gamma''(U^{***}).
\end{aligned}
$$

Here, we introduce the function $\Gamma_0(c)$ defined by

$$
\begin{aligned}
\Gamma_0(c) &\equiv \frac{1}{(U^*-U^{**})(1-E_+(2r))}\Gamma(c,0)\\
&= \left[\frac{U_+(1-E_+(L-4r))}{U^*-U^{**}}+\frac{U_-(1-E_-(2r))(1-E_-(L-4r))}{(U^*-U^{**})(1-E_+(2r))}\right]\gamma'(U^{**})\\
&\quad - \left[E_+(2r)U_+\frac{1-E_+(L-4r)}{1-E_+(2r)}-U_-\frac{1-E_-(L-4r)}{1-E_+(2r)}\right]\gamma''(U^{***}).
\end{aligned}
$$

Then, owing to $(U^*-U^{**})(1-E_+(2r))>0$, the sign of $\Gamma(c,0)$ coincides with that of $\Gamma_0(c)$. Since it follows from (5.23) that

$$
\begin{aligned}
&0<U^*-U^{**}\\
&<U_+(1-E_+(L-4r))(1+E_+(2r))-U_-(1-E_-(L-4r))(1+E_-(2r)),
\end{aligned}
$$

defining the function $\widetilde{\Gamma}_0(c)$ by

$$
\begin{aligned}
\widetilde{\Gamma}_0 &= \frac{U_+(1-E_+(L-4r))\gamma'(U^{**})}{U_+(1-E_+(L-4r))(1+E_+(2r))-U_-(1-E_-(L-4r))(1+E_-(2r))}\\
&+\frac{U_-(1-E_+(2r))^{-1}(1-E_-(2r))(1-E_-(L-4r))\gamma'(U^{**})}{U_+(1-E_+(L-4r))(1+E_+(2r))-U_-(1-E_-(L-4r))(1+E_-(2r))}\\
&-\left[E_+(2r)U_+\frac{1-E_+(L-4r)}{1-E_+(2r)}-\frac{U_-(1-E_-(L-4r))}{1-E_+(2r)}\right]\gamma''(U^{***})\\
&\equiv \widetilde{\Gamma}_1+\widetilde{\Gamma}_2-\widetilde{\Gamma}_3,
\end{aligned}
$$

we find $\Gamma_0<\widetilde{\Gamma}_0$, owing to $\gamma'<0$. We remark that if the limit of $\widetilde{\Gamma}_0$ as $c\to\infty$ is negative, then $\widetilde{\Gamma}_0(c)<0$ holds for sufficiently large c by the continuity of $\widetilde{\Gamma}_0$, which implies that $\Gamma_0(c)$ and $\Gamma(c,0)$ are also negative. Thus, it is enough to investigate the limit of $\widetilde{\Gamma}_0(c)$ as $c\to\infty$ to determine the condition on which $\Gamma(c,0)$ is negative for sufficiently large c. For simplicity of notation, we set $\rho\equiv 4r/L$. Then, we have

$$
\lim_{c\to\infty}U^{**}(c)=\lim_{c\to\infty}U^{***}(c)=\lim_{c\to\infty}U^{*}(c)=\rho.
$$

Indeed, it follows from (5.23) and $U^{**} < U^{***} < U^*$ that

$$\begin{aligned}&U_+(E_+(L-4r)+E_+(L-2r))+U_-(1+E_-(2r))\\&<U^{**}<U^{***}<U^*\\&<U_+(1+E_+(2r))+U_-(E_-(L-4r)+E_-(L-2r)),\end{aligned}$$

and Lemma C.1 in Appendix C gives

$$\begin{aligned}&\lim_{c\to\infty} U_+(c)(E_+(c,L-4r)+E_+(c,L-2r))+U_-(c)(1+E_-(c,2r))=\rho,\\&\lim_{c\to\infty} U_+(c)(1+E_+(c,2r))+U_-(c)(E_-(c,L-4r)+E_-(c,L-2r))=\rho.\end{aligned}$$

As for the limit of $\widetilde{\Gamma}_1(c)$ as $c\to\infty$, since we have

$$\begin{aligned}&\frac{U_+(1-E_+(L-4r))}{U_+(1-E_+(L-4r))(1+E_+(2r))-U_-(1-E_-(L-4r))(1+E_-(2r))}\\&=\left[1+E_+(2r)-\frac{U_-}{1-E_+(L-4r)}\frac{(1-E_-(L-4r))(1+E_-(2r))}{U_+}\right]^{-1},\end{aligned}$$

Lemma C.1 in Appendix C yields $\widetilde{\Gamma}_1(c)\to\frac{1}{2}\gamma'(\rho)$ as $c\to\infty$. In the same manner as that for $\widetilde{\Gamma}_1$, we confirm that $\widetilde{\Gamma}_2(c)\to\frac{1}{8r^2(1-\rho)}\gamma'(\rho)$ as $c\to\infty$. Owing to the estimate

$$\lim_{c\to\infty}\left[E_+(c,2r)U_+(c)\frac{1-E_+(c,L-4r)}{1-E_+(c,2r)}-U_-(c)\frac{1-E_-(c,L-4r)}{1-E_+(c,2r)}\right]=1-\rho,$$

we obtain $\widetilde{\Gamma}_3(c)\to(1-\rho)\gamma''(\rho)$ as $c\to\infty$. Summarizing the above estimates for $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_3$, we find

$$\lim_{c\to\infty}\widetilde{\Gamma}_0(c)=\frac{1}{2}\left(1+\frac{1}{4r^2(1-\rho)}\right)\gamma'(\rho)-(1-\rho)\gamma''(\rho).$$

Thus, we conclude that $\Gamma(c,0)<0$ holds for sufficiently large c provided that

$$\frac{1}{2}\left(1+\frac{1}{4r^2(1-\rho)}\right)\gamma'(\rho)-(1-\rho)\gamma''(\rho)<0. \tag{5.25}$$

Next, we estimate the limit of $\partial\Gamma(c,d)/\partial d$ as $d\to L/2-2r-0$, where

$$\begin{aligned}\frac{\partial\Gamma}{\partial d}(c,d)=&\frac{\partial U_{1f}}{\partial d}(c,d)\gamma'(U_{1f}(c,d))-\frac{\partial U_{1r}}{\partial d}(c,d)\gamma'(U_{1r}(c,d))\\&-\left[\frac{\partial U_{2f}}{\partial d}(c,d)\gamma'(U_{2f}(c,d))-\frac{\partial U_{2r}}{\partial d}(c,d)\gamma'(U_{2r}(c,d))\right].\end{aligned}$$

From Lemma 5.2(a), we have $U_{1f}(c,L/2-2r)=U_{2f}(c,L/2-2r)\equiv U_f(c)$ and $U_{1r}(c,L/2-2r)=U_{2r}(c,L/2-2r)\equiv U_r(c)$ with

$$\begin{aligned}U_f(c)&=U_+(c)(E_+(c,L-2r)+E_+(c,L/2-2r))+U_-(c)(E_-(c,L/2)+1),\\U_r(c)&=U_+(c)(E_+(c,L/2)+1)+U_-(c)(E_-(c,L/2-2r)+E_-(c,L-2r)),\end{aligned}$$

which yields $\Gamma(c, L/2-2r) = 0$ and $U_f(c) < U_r(c)$. Since it follows that

$$\begin{aligned}
\frac{\partial U_{1f}}{\partial d}(c,d) &= -\lambda_+(c)U_+(c)E_+(c,d) - \lambda_-(c)U_-(c)E_-(c, L-2r-d), \\
\frac{\partial U_{1r}}{\partial d}(c,d) &= -\lambda_+(c)U_+(c)E_+(c,2r+d) - \lambda_-(c)U_-(c)E_-(c, L-4r-d), \\
\frac{\partial U_{2f}}{\partial d}(c,d) &= \lambda_+(c)U_+(c)E_+(c,L-4r-d) + \lambda_-(c)U_-(c)E_-(c, 2r+d), \\
\frac{\partial U_{2r}}{\partial d}(c,d) &= \lambda_+(c)U_+(c)E_+(c,L-2r-d) + \lambda_-(c)U_-(c)E_-(c, d),
\end{aligned} \tag{5.26}$$

substituting $d = L/2 - 2r$ into (5.26), we obtain

$$\begin{aligned}
\frac{\partial U_{1f}}{\partial d}(c, L/2-2r) &= -\frac{\partial U_{2f}}{\partial d}(c, L/2-2r) = u_f(c), \\
\frac{\partial U_{1r}}{\partial d}(c, L/2-2r) &= -\frac{\partial U_{2r}}{\partial d}(c, L/2-2r) = u_r(c),
\end{aligned}$$

where

$$\begin{aligned}
u_f(c) &= -\lambda_+(c)U_+(c)E_+(c, L/2-2r) - \lambda_-(c)U_-(c)E_-(c, L/2), \\
u_r(c) &= -\lambda_+(c)U_+(c)E_+(c, L/2) - \lambda_-(c)U_-(c)E_-(c, L/2-2r).
\end{aligned}$$

Thus, we find

$$\frac{\partial \Gamma}{\partial d}(c, L/2-2r) = 2\left[u_f(c)\gamma'(U_f(c)) - u_r(c)\gamma'(U_r(c))\right].$$

Owing to $U_f(c) < U_r(c)$, the mean value theorem gives U^* satisfying

$$\gamma'(U_r(c)) - \gamma'(U_f(c)) = (U_r(c) - U_f(c))\gamma''(U^*),$$

and $U_f(c) < U^* < U_r(c)$. Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial \Gamma}{\partial d}(c, L/2-2r) \\
&= (u_f(c) - u_r(c))\gamma'(U_r(c)) - u_f(c)(U_r(c) - U_f(c))\gamma''(U^*) \\
&= u_f(c)(U_r(c) - U_f(c))\left[\frac{u_f(c) - u_r(c)}{u_f(c)(U_r(c) - U_f(c))}\gamma'(U_r(c)) - \gamma''(U^*)\right] \\
&\equiv u_f(c)(U_r(c) - U_f(c))\widetilde{\Gamma}_d(c).
\end{aligned}$$

Note that $u_f(c) < 0$. Indeed, if follows that

$$\begin{aligned}
2\theta u_f &= -\frac{1-\exp(-2r\lambda_+)}{1-\exp(-L\lambda_+)}\exp(-(L/2-2r)\lambda_+) \\
&\quad + \frac{1-\exp(2r\lambda_-)}{1-\exp(L\lambda_-)}\exp(L/2\lambda_-) \\
&= -\frac{\sinh(r\lambda_+)}{\sinh(L/2\lambda_+)}\exp(r\lambda_+) + \frac{\sinh(-r\lambda_-)}{\sinh(-L/2\lambda_-)}\exp(r\lambda_-) \\
&= -\xi_1(\lambda_+; r, L/2)\exp(r\lambda_+) + \xi_1(-\lambda_-; r, L/2)\exp(r\lambda_-).
\end{aligned}$$

Since $\xi_1(x, r, L/2)$ is strictly decreasing for $x > 0$, it follows from $0 < \exp(r\lambda_-) < 1 < \exp(r\lambda_+)$ that

$$\xi_1(\lambda_+; r, L/2) \exp(r\lambda_+) > \xi_1(-\lambda_-; r, L/2) \exp(r\lambda_-),$$

which concludes $u_f(c) < 0$. We now investigate the limit of $\widetilde{\Gamma}_d(c)$ as $c \to \infty$. Since Lemma C.1 in Appendix C gives $U_f(c)$, $U_r(c) \to \rho$ so that $U^*(c) \to \rho$ as $c \to \infty$, we have $\gamma'(U_r(c)) \to \gamma'(\rho)$ and $\gamma''(U^*(c)) \to \gamma''(\rho)$. Then,

$$\begin{aligned}
&\frac{u_f - u_r}{u_f} \\
&= \frac{-\lambda_+ U_+ E_+(L/2 - 2r)(1 - E_+(2r)) + \lambda_- U_- E_-(L/2 - 2r)(1 - E_-(2r))}{-\lambda_+ U_+ E_+(L/2 - 2r) - \lambda_- U_- E_-(L/2)} \\
&= \frac{U_+ E_+(L/2 - 2r)(1 - E_+(2r)) + \lambda_-^2 U_- E_-(L/2 - 2r)(1 - E_-(2r))}{U_+ E_+(L/2 - 2r) - \lambda_-^2 U_- E_-(L/2)} \\
&= \left[1 - \frac{\lambda_-^2 U_- E_-(L/2)}{U_+ E_+(L/2 - 2r)}\right]^{-1} \\
&\quad \times \left[(1 - E_+(2r)) + \frac{\lambda_-^2 U_-(1 - E_-(2r))}{U_+ E_+(L/2 - 2r)} E_-(L/2 - 2r)\right].
\end{aligned}$$

It follows from Lemma C.1 in Appendix C that

$$\begin{aligned}
\lim_{c \to \infty} \frac{\lambda_-(c)^2 U_-(c)(1 - E_-(2r))}{U_+(c) E_+(c, L/2 - 2r)} &= \frac{2}{L - 4r}, \\
\lim_{c \to \infty} \left[1 - \frac{\lambda_-(c)^2 U_-(c) E_-(c, L/2)}{U_+(c) E_+(c, L/2 - 2r)}\right] &= 1.
\end{aligned}$$

To estimate the limit of $(u_f(c) - u_r(c)) [u_f(c)(U_r(c) - U_f(c))]^{-1}$, it remains to see the limits of $(1 - E_+(c, 2r))(U_r(c) - U_f(c))^{-1}$ and $E_-(c, L/2)(U_r(c) - U_f(c))^{-1}$ as $c \to \infty$. It follows that

$$\begin{aligned}
\frac{1 - E_+(c, 2r)}{U_r(c) - U_f(c)} &= \frac{1 - E_+(c, 2r)}{1 - E_+(c, L/2 - 2r)} h(c), \\
\frac{E_-(c, L/2)}{U_r(c) - U_f(c)} &= \frac{E_-(c, L/2)}{1 - E_+(c, L/2 - 2r)} h(c),
\end{aligned}$$

where

$$h(c) \equiv \left[U_+(E_+(L/2) + 1) + \frac{U_-(E_-(L/2) + 1)(1 - E_-(L/2 - 2r))}{1 - E_+(L/2 - 2r)}\right]^{-1}.$$

Thus, we find from Lemma C.1 in Appendix C that

$$\lim_{c \to \infty} \frac{1 - E_+(c, 2r)}{U_r(c) - U_f(c)} = \frac{1}{1 - \rho}, \qquad \lim_{c \to \infty} \frac{E_-(c, L/2)}{U_r(c) - U_f(c)} = 0,$$

which leads to

$$\lim_{c\to\infty} \frac{u_f(c)-u_r(c)}{u_f(c)(U_r(c)-U_f(c))} = \frac{1}{1-\rho}.$$

As a consequence, we obtain

$$\lim_{c\to\infty} \widetilde{\Gamma}_d(c) = \frac{1}{1-\rho}\gamma'(\rho) - \gamma''(\rho), \tag{5.27}$$

and conclude that $\partial\Gamma/\partial d(c, L/2-2r) < 0$ holds for sufficiently large c provided that $\frac{1}{1-\rho}\gamma'(\rho) - \gamma''(\rho) < 0$. Then, owing to $\Gamma(c, L/2-2r) = 0$, there exists a constant $0 < d^*(c) < L/2-2r$ such that $\Gamma(c,d)$ is positive for any $d \in (d^*, L/2-2r)$ for sufficiently large c.

Summarizing the estimates (5.25) and (5.27), we find that, under the condition $\frac{1}{2}\left(1+\frac{1}{4r^2(1-\rho)}\right)\gamma'(\rho) < (1-\rho)\gamma''(\rho) < \gamma'(\rho)$, it follows that $\Gamma(c,0) < 0$ and $\Gamma(c,d) > 0$ with $d \in (d^*, L/2-2r)$ for sufficiently large c. Therefore, it follows from the continuity of $\Gamma(c,d)$ that there exists d_0 such that $0 < d_0 < L/2-2r$ and $\Gamma(c,d_0) = 0$.

6 Proof of the bimodal traveling wave solution

We first reformulate the system (2.13) with $N = 2$:

$$0 = \frac{\gamma(U(Z_c^i + r)) - \gamma(U(Z_c^i - r))}{2r} - \mu c, \quad i = 1,\ 2, \tag{6.1}$$

$$0 = U'' + cU' - U + F(z - Z_c^1) + F(z - Z_c^2). \tag{6.2}$$

As for (6.2), considering the function F given by (1.6), we find that (6.2) is rewritten by

$$0 = U'' + cU' - U, \quad z \in (-\infty, Z_c^1 - r) \cup (Z_c^1 + r, Z_c^2 - r) \cup (Z_c^2 + r, \infty),$$
$$0 = U'' + cU' - U + 1, \quad z \in (Z_c^1 - r, Z_c^1 + r) \cup (Z_c^2 - r, Z_c^2 + r).$$

Note that we have $d \equiv Z_c^2 - Z_c^1 - 2r > 0$ and that the solution $U \in C^1(\mathbb{R})$ satisfies $U(z) \to 0$ as $z \to \pm\infty$. Then, it is sufficient to consider the following system:

$$\begin{aligned}
0 &= U_{0,1}'' + cU_{0,1}' - U_{0,1}, \quad z \in (-\infty, Z_c^1 - r),\\
0 &= U_i'' + cU_i' - U_i + 1, \quad z \in (Z_c^1 - r, Z_c^1 + r),\\
0 &= U_{1,2}'' + cU_{1,2}' - U_{1,2}, \quad z \in (Z_c^1 + r, Z_c^2 - r),\\
0 &= U_i'' + cU_i' - U_i + 1, \quad z \in (Z_c^2 - r, Z_c^2 + r),\\
0 &= U_{2,3}'' + cU_{2,3}' - U_{2,3}, \quad z \in (Z_c^2 + r, \infty),
\end{aligned}$$

with the boundary and decay conditions:

$$\begin{aligned}
&U_{0,1}(Z_c^1 - r) = U_1(Z_c^1 - r), \quad U_1(Z_c^1 + r) = U_{1,2}(Z_c^1 + r),\\
&U_{1,2}(Z_c^2 - r) = U_2(Z_c^2 - r), \quad U_2(Z_c^2 + r) = U_{2,3}(Z_c^2 + r),\\
&U_{0,1}'(Z_c^1 - r) = U_1'(Z_c^1 - r), \quad U_1'(Z_c^1 + r) = U_{1,2}'(Z_c^1 + r),\\
&U_{1,2}'(Z_c^2 - r) = U_2'(Z_c^2 - r), \quad U_2'(Z_c^2 + r) = U_{2,3}'(Z_c^2 + r),
\end{aligned}$$

$$\lim_{z\to-\infty} U_{0,1}(z) = \lim_{z\to\infty} U_{2,3}(z) = 0, \tag{6.3}$$

Owing to the transnational symmetry of the system, (6.1) and (6.2) is equivalent to the following system:

$$0 = \gamma(U_i(2r)) - \gamma(U_i(0)) - \mu c, \quad i = 1,\ 2, \tag{6.4}$$

$$\begin{aligned}
0 &= U_i'' + cU_i' - U_i + 1, \quad z \in (0, 2r), \quad i = 1,\ 2,\\
0 &= U_{0,1}'' + cU_{0,1}' - U_{0,1}, \quad z \in (-\infty, 0),\\
0 &= U_{1,2}'' + cU_{1,2}' - U_{1,2}, \quad z \in (0, d),\\
0 &= U_{2,3}'' + cU_{2,3}' - U_{2,3}, \quad z \in (0, \infty),
\end{aligned} \tag{6.5}$$

with (6.3) and the boundary conditions:

$$\begin{aligned}
&U_1(0) = U_{0,1}(0), \quad U_1(2r) = U_{1,2}(0), \quad U_2(0) = U_{1,2}(d), \quad U_2(2r) = U_{2,3}(0),\\
&U_1'(0) = U_{0,1}'(0), \quad U_1'(2r) = U_{1,2}'(0), \quad U_2'(0) = U_{1,2}'(d), \quad U_2'(2r) = U_{2,3}'(0),
\end{aligned} \tag{6.6}$$

Next, We construct a solution of (6.5). By a classical theory, a solution of (6.5) is expressed by

$$\begin{aligned}
U_1(x) &= a_+^1 \exp(x\lambda_+) + a_-^1 \exp(x\lambda_-) + 1,\\
U_2(x) &= a_+^2 \exp(x\lambda_+) + a_-^2 \exp(x\lambda_-) + 1,\\
U_{0,1}(x) &= b_+^0 \exp(x\lambda_+) + b_-^0 \exp(x\lambda_-),\\
U_{1,2}(x) &= b_+^1 \exp(x\lambda_+) + b_-^1 \exp(x\lambda_-),\\
U_{2,3}(x) &= b_+^2 \exp(x\lambda_+) + b_-^2 \exp(x\lambda_-),
\end{aligned}$$

where $\lambda_\pm$ are the functions of $c > 0$ defined by

$$\lambda_\pm(c) = \phi(c) \pm \theta(c), \qquad \phi(c) = \frac{-c}{2}, \qquad \theta(c) = \frac{\sqrt{c^2+4}}{2}.$$

For convenience, we use the notation $\exp(x) \equiv e^x$ throughout this section and Appendix D. It follows from (6.6) that

$$\begin{aligned}
&a_+^1 + a_-^1 + 1 = b_+^0 + b_-^0,\\
&a_+^1\lambda_+ + a_-^1\lambda_- = b_+^0\lambda_+ + b_-^0\lambda_-,\\
&a_+^1 \exp(2r\lambda_+) + a_-^1 \exp(2r\lambda_-) + 1 = b_+^1 + b_-^1,\\
&a_+^1\lambda_+ \exp(2r\lambda_+) + a_-^1\lambda_- \exp(2r\lambda_-) = b_+^1\lambda_+ + b_-^1\lambda_-,\\
&a_+^2 + a_-^2 + 1 = b_+^1 \exp(d\lambda_+) + b_-^1 \exp(d\lambda_-),\\
&a_+^2\lambda_+ + a_-^2\lambda_- = b_+^1\lambda_+ \exp(d\lambda_+) + b_-^1\lambda_- \exp(d\lambda_-),\\
&a_+^2 \exp(2r\lambda_+) + a_-^2 \exp(2r\lambda_-) + 1 = b_+^2 + b_-^2,\\
&a_+^2\lambda_+ \exp(2r\lambda_+) + a_-^2\lambda_- \exp(2r\lambda_-) = b_+^2\lambda_+ + b_-^2\lambda_-.
\end{aligned} \tag{6.7}$$

For convenience, we introduce the following notations:

$$\boldsymbol{a}^i \equiv \begin{pmatrix} a_+^i \\ a_-^i \end{pmatrix}, \quad \boldsymbol{b}^i \equiv \begin{pmatrix} b_+^i \\ b_-^i \end{pmatrix}, \quad \boldsymbol{e} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$P \equiv \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}, \quad Q(x) \equiv \begin{pmatrix} \exp(x\lambda_+) & 0 \\ 0 & \exp(x\lambda_-) \end{pmatrix}.$$

Then, (6.7) is rewritten by

$$P\boldsymbol{a}^1 + \boldsymbol{e} = P\boldsymbol{b}^0, \quad PQ(2r)\boldsymbol{a}^1 + \boldsymbol{e} = P\boldsymbol{b}^1,$$
$$P\boldsymbol{a}^2 + \boldsymbol{e} = PQ(d)\boldsymbol{b}^1, \quad PQ(2r)\boldsymbol{a}^2 + \boldsymbol{e} = P\boldsymbol{b}^2,$$

that is,

$$\begin{aligned} \boldsymbol{a}^1 &= \boldsymbol{b}^0 - P^{-1}\boldsymbol{e}, \qquad \boldsymbol{b}^1 = Q(2r)\boldsymbol{a}^1 + P^{-1}\boldsymbol{e}, \\ \boldsymbol{a}^2 &= Q(d)\boldsymbol{b}^1 - P^{-1}\boldsymbol{e}, \qquad \boldsymbol{b}^2 = Q(2r)\boldsymbol{a}^2 + P^{-1}\boldsymbol{e}. \end{aligned} \tag{6.8}$$

Thus, considering $Q(p)Q(q) = Q(p+q)$, we find

$$\begin{aligned} \boldsymbol{b}^2 &= Q(4r+d)\boldsymbol{b}^0 + (I - Q(2r) + Q(2r+d) - Q(4r+d))P^{-1}\boldsymbol{e} \\ &= Q(4r+d)\boldsymbol{b}^0 + (I - Q(2r))(I + Q(d)Q(2r))P^{-1}\boldsymbol{e}, \end{aligned}$$

which yields

$$\boldsymbol{b}^0 = Q(-(4r+d))\boldsymbol{b}^2 + (I - Q(-2r))(I + Q(-d)Q(-2r))P^{-1}\boldsymbol{e}.$$

Since the condition (6.3) yields $b_-^0 = b_+^2 = 0$, we obtain

$$b_+^0 = U_+(c)(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)), \qquad b_-^2 = U_-(c)(1 + \exp(d\lambda_-)\exp(2r\lambda_-)),$$

where

$$U_\pm(c) = \frac{1}{2\theta(c)} \frac{1 - \exp(\mp 2r\lambda_\pm(c))}{\pm\lambda_\pm(c)}. \tag{6.9}$$

Then, it follows from (6.8) that

$$\begin{pmatrix} a_+^1 \\ a_-^1 \end{pmatrix} = \begin{pmatrix} U_+(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)) + \dfrac{\lambda_-}{2\theta} \\ -\dfrac{\lambda_+}{2\theta} \end{pmatrix}, \quad \begin{pmatrix} b_+^1 \\ b_-^1 \end{pmatrix} = \begin{pmatrix} U_+\exp(-d\lambda_+) \\ U_- \end{pmatrix},$$
$$\begin{pmatrix} a_+^2 \\ a_-^2 \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda_-}{2\theta}\exp(-2r\lambda_+) \\ \exp(-2r\lambda_-)(1 + \exp(d\lambda_-)\exp(2r\lambda_-))U_- - \dfrac{\lambda_+}{2\theta}\exp(-2r\lambda_-) \end{pmatrix}.$$

Thus, the functions U_i and $U_{i,i+1}$ with $\boldsymbol{a}^i$ and $\boldsymbol{b}^i$, which are uniquely determined for any given constants (c, d), satisfy (6.5). In particular, U_1 and U_2 are explicitly given by

$$\begin{aligned} U_1(x) &= \exp(x\lambda_+)(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+))U_+ \\ &\quad + \frac{\lambda_+}{2\theta}(1 - \exp(x\lambda_-)) - \frac{\lambda_-}{2\theta}(1 - \exp(x\lambda_+)), \\ U_2(x) &= \exp((x-2r)\lambda_-)(1 + \exp(d\lambda_-)\exp(2r\lambda_-))U_- \\ &\quad + \frac{\lambda_+}{2\theta}(1 - \exp((x-2r)\lambda_-) - \frac{\lambda_-}{2\theta}(1 - \exp((x-2r)\lambda_+)). \end{aligned}$$

Thus, we obtain

$$U_1(0) = U_+(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)), \qquad U_1(2r) = U_+\exp(-d\lambda_+) + U_-,$$
$$U_2(0) = U_+ + U_-\exp(d\lambda_-), \qquad U_2(2r) = U_-(1 + \exp(d\lambda_-)\exp(2r\lambda_-)).$$

To make it clear that these quantities depend on the parameters (c, d), we use the following notations:

$$\begin{aligned}
U_{pf}(c,d) &\equiv U_2(2r) = U_-(c)(1 + \exp(d\lambda_-(c))\exp(2r\lambda_-(c))),\\
U_{pr}(c,d) &\equiv U_2(0) = U_+(c) + U_-(c)\exp(d\lambda_-(c)),\\
U_{sf}(c,d) &\equiv U_1(2r) = U_+(c)\exp(-d\lambda_+(c)) + U_-(c),\\
U_{sr}(c,d) &\equiv U_1(0) = U_+(c)(1 + \exp(-d\lambda_+(c))\exp(-2r\lambda_+(c))),
\end{aligned} \tag{6.10}$$

and, for later use, we introduce the following functions:

$$\Delta U_p(c,d) \equiv U_{pf}(c,d) - U_{pr}(c,d), \quad \Delta U_s(c,d) \equiv U_{sf}(c,d) - U_{sr}(c,d).$$

We are now ready to prove Theorem 2.6. To show the (non-)existence of a solution to (2.13), we see whether the constructed solution (U_1, U_2) to (6.5) satisfies (6.4) or not.

Proof of Theorem 2.6–1 Note that (6.4) requires

$$\gamma(U_{pr}(0,d)) - \gamma(U_{pf}(0,d)) = 0, \qquad \gamma(U_{sr}(0,d)) - \gamma(U_{sf}(0,d)) = 0. \tag{6.11}$$

Since γ is strictly decreasing, (6.11) is equivalent to $\Delta U_p(0,d) = \Delta U_f(0,d) = 0$. For the case of $c = 0$, it follows from $\lambda_- = -\lambda_+ = -1$ that

$$\begin{aligned}
U_{pf}(0,d) &= U_{sr}(0,d) = U_0(1 + \exp(-d)\exp(-2r)),\\
U_{pr}(0,d) &= U_{sf}(0,d) = U_0(1 + \exp(-d)),
\end{aligned}$$

where $U_0 \equiv U_\pm(0) = (1 - \exp(-2r))/2 > 0$. Thus, we obtain

$$\Delta U_p(0,d) = -\Delta U_f(0,d) = -U_0\exp(-d)(1 - \exp(-2r)) < 0,$$

and conclude that there exists no solution of (2.13) with $c = 0$.

Proof of Theorem 2.6–2 Note that it follows from Lemma D.1(ii) that $\Delta U_p < \Delta U_s$. For the case that γ is a linear function, however, (6.14) is equivalent to $\Delta U_p = \Delta U_s$, and thus there exits no solution. For the case that γ' is strictly increasing, since γ is strictly decreasing, (6.4) yields $\Delta U_p < 0$ and $\Delta U_s < 0$, that is, $U_{pf} < U_{pr}$ and $U_{sf} < U_{sr}$. We first consider the case of $U_{pr} - U_{sf} \le 0$. Then, (6.14) is rewritten by

$$\int_{U_{pf}}^{U_{pr}} -\gamma'(U)dU = \int_{U_{sf}}^{U_{sr}} -\gamma'(U)dU. \tag{6.12}$$

Since γ' is strictly increasing, we find $\gamma'(U_{pr}(r,c,d)) \le \gamma'(U_{sf}(r,c,d))$ and

$$\min_{U\in[U_{pf},U_{pr}]} \big(-\gamma'(U)\big) = -\gamma'(U_{pr}), \qquad \max_{U\in[U_{sf},U_{sr}]} \big(-\gamma'(U)\big) = -\gamma'(U_{sf}).$$

Thus, considering Lemma D.1(ii), we obtain

$$\begin{aligned}\int_{U_{pf}}^{U_{pr}} -\gamma'(U)dU &\ge -\gamma'(U_{pr})(U_{pr}-U_{pf}) > -\gamma'(U_{pr})(U_{sr}-U_{sf}) \\ &\ge -\gamma'(U_{sf})(U_{sr}-U_{sf}) \ge \int_{U_{sf}}^{U_{sr}} -\gamma'(U)dU,\end{aligned}$$

which contradicts (6.12). Next, we consider the case of $U_{pr} - U_{sf} > 0$. Then, (6.14) is rewritten by

$$\int_{U_{pf}}^{U_{sf}} -\gamma'(U)dU = \int_{U_{pr}}^{U_{sr}} -\gamma'(U)dU. \tag{6.13}$$

Owing to Lemma D.1(iii), the left-hand side of (6.13) is positive. If $U_{sr} \le U_{pr}$, then the right-hand side of (6.13) is not positive, Thus, it is sufficient to consider the case of $U_{sr} > U_{pr}$. Note that

$$\min_{U\in[U_{pf},U_{sf}]} \big(-\gamma'(U)\big) = -\gamma'(U_{sf}), \qquad \max_{U\in[U_{pr},U_{sr}]} \big(-\gamma'(U)\big) = -\gamma'(U_{pr}),$$

and it follows from $U_{pr} - U_{sf} > 0$ that $-\gamma'(U_{pr}) \le -\gamma'(U_{sf})$. Since Lemma D.1(ii) yields $U_{sr} - U_{pr} < U_{sf} - U_{pf}$, we obtain

$$\begin{aligned}\int_{U_{pf}}^{U_{sf}} -\gamma'(U)dU &\ge -\gamma'(U_{sf})(U_{sf}-U_{pf}) > -\gamma'(U_{sf})(U_{sr}-U_{pr}) \\ &\ge -\gamma'(U_{pr})(U_{sr}-U_{pr}) \ge \int_{U_{pr}}^{U_{sr}} -\gamma'(U)dU,\end{aligned}$$

which contradicts (6.13). Thus, there is no solution of (2.13) provided that γ' is strictly increasing.

Proof of Theorem 2.6–3 In order to show Theorem 2.6–3, we introduce the following function:

$$\Gamma(c,d) \equiv \gamma(U_{sf}(c,d)) - \gamma(U_{sr}(c,d)) - (\gamma(U_{pf}(c,d)) - \gamma(U_{pr}(c,d))). \tag{6.14}$$

We investigate the properties of $\Gamma(c,d)$ in the limits of $d \to 0$ and $d \to \infty$. Note that it follows from (6.10) that

$$\begin{aligned}&\lim_{d\to 0} U_{pf}(c,d) = U_-(c)(1+\exp(2r\lambda_-(c))), \\ &\lim_{d\to 0} U_{pr}(c,d) = U_+(c) + U_-(c), \\ &\lim_{d\to 0} U_{sf}(c,d) = U_+(c) + U_-(c), \\ &\lim_{d\to 0} U_{sr}(c,d) = U_+(c)(1+\exp(-2r\lambda_+(c))).\end{aligned} \tag{6.15}$$

Then, we easily confirm that

$$\Gamma_0(c) \equiv \lim_{d\to 0}\Gamma(c,d)$$
$$= 2\gamma(U_+(c)+U_-(c)) - \left[\gamma\left(U_+(c)(1+\exp(-2r\lambda_+(c))\right) + \gamma\left(U_-(c)(1+\exp(2r\lambda_-(c))\right))\right].$$

Regarding the limit $d\to\infty$ limit of $\Gamma(c,d)$, we have the following lemma.

Lemma 6.1. For sufficiently large $d>0$, the sign of $\Gamma(c,d)$ coincides with that of

$$\Gamma_\infty(c) \equiv \gamma'(U_-(c))\exp(2r\lambda_+(c)) - \gamma'(U_+(c)),$$

provided that $\Gamma_\infty(c)\neq 0$.

Proof. Note that Lemma D.1(iii) gives $U_{pf}<U_{sf}$ and Lemma D.2(ii) implies $U_{pr}<U_{sr}$ for sufficiently large d. Then, It follows from the mean value theorem that there exist constants $U^*\in(U_{pf},U_{sf})$ and $U^{**}\in(U_{pr},U_{sr})$ such that

$$\gamma(U_{sf})-\gamma(U_{pf}) = \gamma'(U^*)(U_{sf}-U_{pf}), \quad \gamma(U_{sr})-\gamma(U_{pr}) = \gamma'(U^{**})(U_{sr}-U_{pr}).$$

Thus, we obtain

$$\begin{aligned}\Gamma(c,d) &= \gamma'(U^*)(U_{sf}-U_{pf}) - \gamma'(U^{**})(U_{sr}-U_{pr})) \\ &= (U_{sr}-U_{pr})\left[\gamma'(U^*)\frac{U_{sf}-U_{pf}}{U_{sr}-U_{pr}} - \gamma'(U^{**})\right].\end{aligned}$$

Considering $U_{pr}<U_{sr}$, we find that the sign of $\Gamma(c,d)$ corresponds to that of

$$\gamma'(U^*)\frac{U_{sf}-U_{pf}}{U_{sr}-U_{pr}} - \gamma'(U^{**}).$$

Note that it follows from (6.10) that

$$\lim_{d\to\infty}U_{pf}(c,d) = \lim_{d\to\infty}U_{sf}(c,d) = U_-(c), \qquad \lim_{d\to\infty}U_{pr}(c,d) = \lim_{d\to\infty}U_{sr}(c,d) = U_+(c),$$

which implies $U^*(d)\to U_-$ and $U^{**}(d)\to U_+$ in the $d\to\infty$ limit. Since we have

$$\begin{aligned}U_{sf}-U_{pf} &= \frac{\exp(-d\lambda_+)}{2\theta}\left[-\lambda_-(1-\exp(-2r\lambda_+))\right. \\ &\quad \left.-\lambda_+\exp(2r\lambda_-)\exp(d(\lambda_-+\lambda_+))(1-\exp(2r\lambda_-))\right], \\ U_{sr}-U_{pr} &= \frac{\exp(-d\lambda_+)}{2\theta}\left[-\lambda_-(1-\exp(-2r\lambda_+))\exp(-2r\lambda_+)\right. \\ &\quad \left.-\lambda_+\exp(d(\lambda_-+\lambda_+))(1-\exp(2r\lambda_-))\right],\end{aligned}$$

it follows that

$$\begin{aligned}&\lim_{d\to\infty}\frac{U_{sf}(c,d)-U_{pf}(c,d)}{U_{sr}(c,d)-U_{pr}(c,d)} \\ &= \lim_{d\to\infty}\frac{\lambda_-(1-\exp(-2r\lambda_+)) + \lambda_+(1-\exp(2r\lambda_-))\exp(d(\lambda_-+\lambda_+))\exp(2r\lambda_-)}{\lambda_-(1-\exp(-2r\lambda_+))\exp(-2r\lambda_+) + \lambda_+\exp(d(\lambda_-+\lambda_+))(1-\exp(2r\lambda_-))} \\ &= \exp(2r\lambda_+).\end{aligned}$$

Summarizing the above estimates, we obtain

$$\lim_{d\to\infty}\left[\gamma'(U^*)\frac{U_{sf}(c,d)-U_{pf}(c,d)}{U_{sr}(c,d)-U_{pr}(c,d)}-\gamma'(U^{**})\right]=\gamma'(U_-)\exp(2r\lambda_+)-\gamma'(U_+)=\Gamma_\infty.$$

Since $\Gamma(c,d)$ is continuous for d, the sign of $\Gamma(c,d)$ coincides with that of $\Gamma_\infty(c)$ for sufficiently large d when $\Gamma_\infty(c)\neq 0$ holds. □

Suppose that $\Gamma_0(c_0)\Gamma_\infty(c_0)<0$ for a constant $c_0>0$. Then, the intermediate value theorem implies that there exists a constant $d_0>0$ satisfying (6.14), that is, $\Gamma(c_0,d_0)=0$. Since we have Lemma D.1(i) and γ is strictly decreasing, there exists a constant $\mu>0$ such that (6.4) holds. To see the existence of a constant $c_0>0$ satisfying $\Gamma_0(c_0)\Gamma_\infty(c_0)<0$, we show the following lemma.

Lemma 6.2. (i) Suppose $\gamma'(0)-\gamma''(0)\neq 0$. Then, for sufficiently large $c>0$, the sign of $\gamma'(0)-\gamma''(0)$ coincides with that of $\Gamma_\infty(c)$.
(ii) Suppose that

$$\frac{1+4r^2}{8r^2}\gamma'(0)-\gamma''(0)<0.$$

Then, we have $\Gamma_0(c)<0$ for sufficiently large $c>0$.

Proof. (i) Note that

$$\Gamma_\infty(c)=\exp(2r\lambda_+(c))\left[\gamma'(U_-(c))-\gamma'(U_+(c))\exp(-2r\lambda_+(c))\right].$$

Since there exists a constant $U^*\in(U_-,U_+)$ such that

$$\gamma'(U_+)-\gamma'(U_-)=(U_+-U_-)\gamma''(U^*),$$

we have

$$\begin{aligned}\gamma'(U_-)-\gamma'(U_+)\exp(-2r\lambda_+)&=\left[\gamma'(U_+)-(U_+-U_-)\gamma''(U^*)\right]-\gamma'(U_+)\exp(-2r\lambda_+)\\&=\gamma'(U_+)(1-\exp(-2r\lambda_+))-(U_+-U_-)\gamma''(U^*)\\&=(1-\exp(-2r\lambda_+))\left[\gamma'(U_+)-\gamma''(U^*)\frac{U_+-U_-}{1-\exp(-2r\lambda_+)}\right].\end{aligned}$$

Thus, the sign of $\Gamma_\infty(c)$ coincides with that of

$$\gamma'(U_+(c))-\gamma''(U^*(c))\frac{U_+(c)-U_-(c)}{1-\exp(-2r\lambda_+(c))}.$$

Note that, taking the $c\to\infty$ limit, we have

$$\theta(c)\to\infty,\quad \lambda_+(c)\to 0,\quad \lambda_-(c)\to-\infty,\quad \frac{-\lambda_-(c)}{2\theta(c)}\to 1,\quad U_\pm(r,c)\to 0, \tag{6.16}$$

and $U_- < U^* < U_+$ yields $U^*(c) \to 0$. For later use, we introduce the function $g(x) = (1 - \exp(-2rx))/x$ for $x > 0$, which satisfies

$$\lim_{c\to\infty} g(\lambda_+(c)) = \lim_{c\to\infty} \frac{1 - \exp(-2r\lambda_+(c))}{\lambda_+(c)} = \lim_{c\to\infty} \frac{2r\lambda_+(c) + o(\lambda_+(c))}{\lambda_+(c)} = 2r. \tag{6.17}$$

Then, it follows from $\lambda_+\lambda_= - 1$ and (6.9), that

$$\begin{aligned}\lim_{c\to\infty} \frac{U_+(c)}{1 - \exp(-2r\lambda_+(c))} &= \lim_{c\to\infty} \frac{-\lambda_-(c)}{2\theta(c)} = 1,\\ \lim_{c\to\infty} \frac{U_-(c)}{1 - \exp(-2r\lambda_+(c))} &= \lim_{c\to\infty} \frac{1 - \exp(2r\lambda_-(c))}{2\theta(c)g(\lambda_+(c))} = 0,\end{aligned} \tag{6.18}$$

and thus

$$\lim_{c\to\infty} \left[\gamma'(U_+(c)) - \gamma''(U^*)\frac{U_+(c) - U_-(c)}{1 - \exp(-2r\lambda_+(c))}\right] = \gamma'(0) - \gamma''(0),$$

which concludes that if $\gamma'(0) - \gamma''(0) \neq 0$, the sign of $\Gamma_\infty(c)$ coincides with that of $\gamma'(0) - \gamma''(0)$ for sufficiently large $c > 0$.

(ii) We introduce the function $h(c) = U_-(c) - U_+(c)\exp(-2r\lambda_+(c))$. Then, we have

$$h = (1 - \exp(-2r\lambda_+))\left[\frac{U_-}{1 - \exp(-2r\lambda_+)} - \frac{U_+\exp(-2r\lambda_+)}{1 - \exp(-2r\lambda_+)}\right],$$

and it follows from (6.16) and (6.18) that

$$\lim_{c\to\infty} \left[\frac{U_-(c)}{1 - \exp(-2r\lambda_+(c))} - \frac{U_+(c)\exp(-2r\lambda_+(c))}{1 - \exp(-2r\lambda_+(c))}\right] = -1,$$

which implies that $h(c) < 0$ holds for sufficiently large $c > 0$. Hence, we find from Lemma D.2(ii) and (6.15) that $U_{sf}(c, 0) < U_{sr}(c, 0)$ and

$$U_-(c)(1 + \exp(2r\lambda_-(c))) < U_+(c) + U_-(c) < U_+(c)(1 + \exp(-2r\lambda_+(c))),$$

for sufficiently large $c > 0$. It follows from the mean value theorem that there exist constants U^* and U^{**} such that

$$U_-(1 + \exp(2r\lambda_-)) < U^* < U_+ + U_- < U^{**} < U_+(1 + \exp(-2r\lambda_+)), \tag{6.19}$$

and

$$\begin{aligned}\gamma(U_+ + U_-) - \gamma(U_-(1 + \exp(2r\lambda_-))) &= (U_+ - U_-\exp(2r\lambda_-))\gamma'(U^*),\\ \gamma(U_+(1 + \exp(-2r\lambda_+))) - \gamma(U_+ + U_-) &= (U_+\exp(-2r\lambda_+) - U_-)\gamma'(U^{**}).\end{aligned}$$

Thus, we obtain

$$\Gamma_0(c) = (U_+(c) - U_-(c)\exp(2r\lambda_-(c)))\gamma'(U^*) - (U_+(c)\exp(-2r\lambda_+(c)) - U_-(c))\gamma'(U^{**}).$$

Note that

$$\frac{U_+ - U_- \exp(2r\lambda_-)}{1-\exp(-2r\lambda_+)} = \frac{1}{2\theta}\left(-\lambda_- - \frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}\exp(2r\lambda_-)\right),$$
$$\frac{U_+ \exp(-2r\lambda_+) - U_-}{1-\exp(-2r\lambda_+)} = \frac{1}{2\theta}\left(-\lambda_- \exp(-2r\lambda_+) - \frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}\right),$$

and there exists a constant $U^{***} \in (U^*, U^{**})$ such that

$$\gamma'(U^{**}) - \gamma'(U^*) = (U^{**} - U^*)\gamma''(U^{***}).$$

Then, we find

$$\begin{aligned}
\widetilde{\Gamma}(c) &\equiv \frac{\Gamma_0(c)}{(1-\exp(-2r\lambda_+))(U^{**}-U^*)} \\
&= \frac{1}{2\theta\,(U^{**}-U^*)}\left[\left(-\lambda_- - \frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}\exp(2r\lambda_-)\right)\gamma'(U^*)\right. \\
&\quad \left. - \left(-\lambda_- \exp(-2r\lambda_+) - \frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}\right)\gamma'(U^{**})\right] \\
&= \frac{1}{2\theta\,(U^{**}-U^*)}\left[-\lambda_-(\gamma'(U^*) - \exp(-2r\lambda_+)\gamma'(U^{**}))\right. \\
&\quad \left. -\frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}(\exp(2r\lambda_-)\gamma'(U^*) - \gamma'(U^{**}))\right] \\
&= \frac{1}{2\theta\,(U^{**}-U^*)}\left[-\lambda_-(\gamma'(U^*) - \exp(-2r\lambda_+)(\gamma'(U^*) + (U^{**}-U^*)\gamma''(U^{***})))\right. \\
&\quad \left. -\frac{1-\exp(2r\lambda_-)}{g(\lambda_+)}(\exp(2r\lambda_-)\gamma'(U^*) - (\gamma'(U^*) + (U^{**}-U^*)\gamma''(U^{***})))\right] \\
&= \left[g(\lambda_+) + \frac{(1-\exp(2r\lambda_-))^2}{g(\lambda_+)}\right]\frac{\gamma'(U^*)}{2\theta\,(U^{**}-U^*)} \\
&\quad + \left[\frac{1-\exp(2r\lambda_-)}{2\theta g(\lambda_+)} - \exp(-2r\lambda_+)\frac{-\lambda_-}{2\theta}\right]\gamma''(U^{***}).
\end{aligned}$$

Owing to $(1-\exp(-2r\lambda_+))(U^{**}-U^*) > 0$, the sign of $\Gamma_0(c)$ coincides with that of $\widetilde{\Gamma}(c)$. Since it follows from (6.19) that

$$0 < U^{**} - U^* < U_+(1+\exp(-2r\lambda_+)) - U_-(1+\exp(2r\lambda_-)) \equiv \Delta U^*,$$

and $\gamma'(u) < 0$ for $u > 0$, we have

$$\begin{aligned}
\widetilde{\Gamma}(c) &< \frac{g\,(\lambda_+(c))}{2\theta(c)\Delta U^*(c)}\gamma'(U^*) + \frac{(1-\exp(2r\lambda_-(c)))^2}{2\theta(c)\Delta U^*(c) g\,(\lambda_+(c))}\gamma'(U^*) \\
&\quad + \left[\frac{1-\exp(2r\lambda_-(c))}{2\theta(c) g\,(\lambda_+(c))} - \exp(-2r\lambda_+(c))\frac{-\lambda_-(c)}{2\theta(c)}\right]\gamma''(U^{***}) \\
&\equiv \widetilde{\Gamma}_1(c) + \widetilde{\Gamma}_2(c) + \widetilde{\Gamma}_3(c).
\end{aligned}$$

It follows from (6.16) and (6.18)

$$\lim_{c\to\infty} 2\theta(c)U_+(c) = \lim_{c\to\infty} g\left(\lambda_+(c)\right) = 2r, \qquad \lim_{c\to\infty} 2\theta(c)U_-(c) = \lim_{c\to\infty} g\left(-\lambda_-(c)\right) = 0,$$

which yields $\lim_{c\to\infty} 2\theta(c)\Delta U^*(c) = 4r$. In addition, considering (6.19) and $U^* < U^{***} < U^{**}$, we have $\lim_{c\to\infty} U^*(c) = \lim_{c\to\infty} U^{***}(c) = 0$. Hence, we obtain

$$\lim_{c\to\infty} \widetilde{\Gamma}_1(c) = \frac{1}{2}\gamma'(0), \qquad \lim_{c\to\infty} \widetilde{\Gamma}_2(c) = \frac{1}{8r^2}\gamma'(0), \qquad \lim_{c\to\infty} \widetilde{\Gamma}_3(c) = -\gamma''(0),$$

which yields

$$\lim_{c\to\infty} \widetilde{\Gamma}(c) \le \frac{1+4r^2}{8r^2}\gamma'(0) - \gamma''(0) < 0.$$

Since $\Gamma_0(c)$ has the same sign as $\widetilde{\Gamma}(c)$, we conclude that $\Gamma_0(c) < 0$ holds for sufficiently large $c > 0$. □

Since the assumption of Theorem 2.6–3 gives

$$\gamma'(0) - \gamma''(0) > 0, \qquad \frac{1+4r^2}{8r^2}\gamma'(0) - \gamma''(0) < 0,$$

we have $\Gamma_0(c) < 0 < \Gamma_\infty(c)$ for sufficiently large $c > 0$. Thus, Lemma 6.1 concludes that there exists a constant $\mu > 0$ satisfying (6.4).

7 Conclusion

We provided the global existence and uniqueness of the weak solutions to model equations. As mentioned earlier, there was no proof that there is a unique time global solution although it was mentioned by some proceeding works. Our works can provide the mathematical guarantee for such works. Then, we showed the existence of symmetrically rotating solutions and gave necessary conditions for the existence and non-existence of asymmetrically rotating solutions for the two camphor model (1.5). In particular, it has been shown that a concave part of the function γ, which describes the surface tension, is necessary for the existence of asymmetrically rotating solutions. Our result clarifies an essential condition for the existence of solutions and provides a clue for the dependence between the concentration of the surfactant layer and the surface tension in the mathematical model, which is difficult to be measured in experiments. Moreover, it is suggested that the characteristic motion varies depending on the form of the surface tension function γ and, thus, the change of the qualitative motion is caused by γ in other mathematical models.

Finally, we gave sufficient conditions for the existence of the bimodal traveling solution. From the numerical calculation results in the very long interval (as shown in Figure 7), it is suggested that this solution is stable, because if the perturbation is given to the distance between the two self-propelled materials, it will continue to move back to the original state. By use of computer aided analysis, we were able to perform comparatively rigorous stability analysis in the same way as that in [36]. However, since it is difficult to evaluate the essential spectrum and the solution to the nonlinear equations satisfying the eigenvalues, the rigorous stability analysis of the bimodal traveling solution has not been completed yet, which is expected to be one of our future topics.

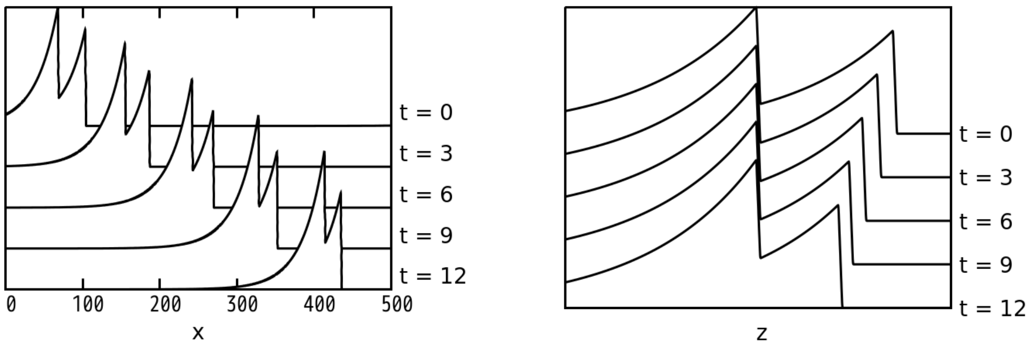

Figure 3: The numerical result for very long annular channel. The position of follower, z_c^1, is adjusted to the point shown by small arrow to see the profile of u. Each line shows the profile of u atdifferent time. The upper curve corresponds to the older profile of u.

8 Acknowledgement

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A part of the introduction and the proof are shared with author's paper "Existence and non-existence of asymmetrically rotating solutions to a mathematical model ofself-propelled motion"[1] published from Japan Journal of Industrial and Applied Mathematics.

A Existence for a single camphor disk

We consider the steady state problem of the single camphor model on the moving coordinate:

$$0 = \gamma(U(\pi_L(z_c + r))) - \gamma(U(\pi_L(z_c - r))) - 2r\mu c, \tag{A.1}$$

$$0 = \frac{\partial^2 U}{\partial z^2} + c\frac{\partial U}{\partial z} - U + F(z - z_c), \tag{A.2}$$

[1]URL: `https://link.springer.com/article/10.1007/s13160-020-00427-x`

for $z \in [0, L)\backslash\{\pi_L(z_c+r), \pi_L(z_c-r)\}$. We assume that $U \in C^1[0, L]$ satisfies $U(0) = U(L)$ and $U'(0) = U'(L)$.

Theorem A.1. For any $c > 0$, there exists a unique $\mu > 0$ such that (A.1) and (A.2) have a solution. In the case of $c = 0$, there always exists a solution for any value of $\mu > 0$.

Proof. Let $c \geq 0$ be a fixed constant. Since (A.1) and (A.2) have a translational symmetry, we may assume $z_c = r$ without loss of generality. We first derive a nonlinear equation that is equivalent to (A.1) and (A.2) in a similar manner as the proof of Theorem 2.4. A general solution to (A.2) is given by

$$\begin{aligned} &U(z) = a_+E_+(-z) + a_-E_-(z) + 1, && z \in (0, 2r), \\ &U(z) = b_+E_+(-(z-2r)) + b_-E_-(z-2r), && z \in (2r, L), \end{aligned} \tag{A.3}$$

where the constants $a_\pm$ and $b_\pm$ are determined by the C^1-matching conditions at $z = 0$, $2r$. Then, similarly to the proof of Lemma 5.1, we find that (A.1) and (A.2) are equivalent to

$$\Lambda \boldsymbol{a} + \boldsymbol{e} = \Lambda E(L-2r)\boldsymbol{b}, \qquad \Lambda \boldsymbol{b} = \Lambda E(2r)\boldsymbol{a} + \boldsymbol{e},$$

where $\boldsymbol{a}^i$, $\boldsymbol{b}^i$, $\boldsymbol{e}$, Λ and $E(x)$ are the same as those in Lemma 5.1. Hence, it follows that

$$\boldsymbol{a} = (I - E(L))^{-1}(E(L-2r) - I)\Lambda^{-1}\boldsymbol{e} = \begin{pmatrix} \dfrac{1}{2\theta\lambda_+}\dfrac{E_+(L) - E_+(2r)}{1 - E_+(L)} \\ -\dfrac{1}{2\theta\lambda_-}\dfrac{E_-(L-2r) - 1}{1 - E_-(L)} \end{pmatrix},$$

$$\boldsymbol{b} = (I - E(L))^{-1}(I - E(2r))\Lambda^{-1}\boldsymbol{e} = \begin{pmatrix} \dfrac{1}{2\theta\lambda_+}\dfrac{E_+(L-2r) - E_+(L)}{1 - E_+(L)} \\ -\dfrac{1}{2\theta\lambda_-}\dfrac{1 - E_-(2r)}{1 - E_-(L)} \end{pmatrix}.$$

Substituting these constants into (A.3), we obtain

$$\begin{aligned} U(z) &= \frac{E_+(-z)}{2\theta\lambda_+}\frac{E_+(L) - E_+(2r)}{1 - E_+(L)} - \frac{E_-(z)}{2\theta\lambda_-}\frac{E_-(L-2r) - 1}{1 - E_-(L)} + 1, && z \in (0, 2r), \\ U(z) &= \frac{E_+(-(z-2r))}{2\theta\lambda_+}\frac{E_+(L-2r) - E_+(L)}{1 - E_+(L)} - \frac{E_-(z-2r)}{2\theta\lambda_-}\frac{1 - E_-(2r)}{1 - E_-(L)}, && z \in (2r, L), \end{aligned}$$

which implies

$$U(0) = U_+ + E_-(L-2r)U_-, \qquad U(2r) = E_+(L-2r)U_+ + U_-. \tag{A.4}$$

To clarify the dependence of $c \geq 0$, we rewrite (A.4) by $U_r(c) = U_+(c) + E_-(c, L - 2r)U_-(c)$ and $U_f(c) = E_+(c, L-2r)U_+(c) + U_-(c)$. Then, (A.1) is rewritten by

$$0 = \gamma(U_f(c)) - \gamma(U_r(c)) - 2r\mu c. \tag{A.5}$$

Next, we show that, for any $c \geq 0$, there exists $\mu > 0$ satisfying (A.5). For the case of $c = 0$, we have (A.5) for any $\mu > 0$ since it follows from $\lambda_\pm(0) = \pm 1$ that $U_+(0) = U_-(0)$ and $E_+(0, z) = E_-(0, z)$. In the case of $c > 0$, (A.5) is rewritten by

$$\mu = \frac{\gamma(U_f(c)) - \gamma(U_r(c))}{2rc}. \tag{A.6}$$

Noting that

$$U_r(c) - U_f(c) = (1 - E_+(c, L - 2r))U_+(c) - (1 - E_-(c, L - 2r))U_-(c),$$

we have

$$\begin{aligned}
\theta(c)(1 - E_+(c, L - 2r))U_+(c) &= (1 - \exp(-(L - 2r)\lambda_+(c)))\frac{1}{2\lambda_+(c)}\frac{1 - \exp(-2r\lambda_+(c))}{1 - \exp(-L\lambda_+(c))} \\
&= \frac{(\exp((L/2 - r)\lambda_+(c)) - \exp(-(L/2 - r)\lambda_+(c)))}{2\lambda_+(c)} \\
&\quad \times \frac{\exp(r\lambda_+(c)) - \exp(-r\lambda_+(c))}{\exp((L/2)\lambda_+(c)) - \exp(-(L/2)\lambda_+(c))} \\
&= \frac{\sinh((L/2 - r)\lambda_+(c))}{\lambda_+(c)}\frac{\sinh(r\lambda_+)}{\sinh((L/2)\lambda_+)} \\
&= \xi_2(\lambda_+(c); r, L/2),
\end{aligned}$$

and, similarly,

$$\theta(c)(1 - E_-(c, L - 2r))U_-(c) = \xi_2(-\lambda_-(c); r, L/2).$$

Since $\xi_2(x; r, L/2)$ is strictly decreasing for $x > 0$ and $\lambda_+(c) < -\lambda_-(c)$ for $c > 0$, we obtain

$$\theta(c)\left(U_r(c) - U_f(c)\right) = \xi_2(\lambda_+(c); r, L/2) - \xi_2(\lambda_-(c); r, L/2) > 0,$$

so that $U_f(c) < U_r(c)$ for any $c > 0$. Thus, since $\gamma(u)$ is strictly decreasing for $u > 0$, we conclude that, for any $c > 0$, there exists a unique constant $\mu > 0$ such that (A.6) holds. □

B Properties of auxiliary functions

We first show that

$$\xi_0(x) = \xi_0(x; a) = \frac{x}{\tanh(ax)},$$

with a constant $a > 0$ is strictly increasing for $x > 0$. Indeed, we have

$$\xi_0'(x) = \frac{\tanh(ax) - ax(1 - \tanh^2(ax))}{\tanh^2(ax)},$$

and

$$\tanh(ax) - ax(1-\tanh^2(ax)) = \tanh(ax)\left[1 - ax\left(\frac{1}{\tanh(ax)} - \tanh(ax)\right)\right].$$

Since $\sinh(x)/x$ and $\cosh(x)$ attain the minimum value 1 at $x=0$, we obtain

$$1 - ax\left(\frac{1}{\tanh(ax)} - \tanh(ax)\right) = 1 - \frac{ax}{\sinh(ax)\cosh(ax)} > 0,$$

for $x > 0$. Hence, $\xi_0(x)$ is strictly increasing for $x > 0$.

1. The function $\xi_1(x)$. We show that

$$\xi_1(x) = \xi_1(x; a, b) = \frac{\sinh(ax)}{\sinh(bx)},$$

with $b > a > 0$ is strictly decreasing for $x > 0$. We have

$$\xi_1'(x) = \frac{a\cosh(ax)\sinh(bx) - b\sinh(ax)\cosh(bx)}{\sinh^2(bx)} = \xi_1(x; a, b)(\xi_0(a; x) - \xi_0(b; x)).$$

Since $\xi_0(x)$ is strictly increasing for $x > 0$, we have $\xi_0(a; x) < \xi_0(b; x)$ so that $\xi_1'(x) < 0$ for $x > 0$.

2. The function $\xi_2(x)$. We show that

$$\xi_2(x) = \xi_2(x; a, b) = \frac{\sinh(ax)}{\sinh(bx)}\frac{\sinh((b-a)x)}{x},$$

with $b > a > 0$ is strictly decreasing for $x > 0$. It follows that

$$\begin{aligned}\frac{1}{\xi_2(x)} &= \frac{x\sinh(xb)}{\sinh(xa)\sinh((b-a)x)}\\ &= x\frac{\sinh(ax)\cosh((b-a)x) + \sinh((b-a)x)\cosh(ax)}{\sinh(ax)\sinh((b-a)x)}\\ &= \frac{x}{\tanh((b-a)x)} + \frac{x}{\tanh(ax)} = \xi_0(x; b-a) + \xi_0(x; a).\end{aligned}$$

Since $\xi_0(x)$ is strictly increasing for $x > 0$, $\xi_2(x)$ is strictly decreasing for $x > 0$.

3. The function $\xi_3(x)$. Let $a, b, c > 0$ satisfy $c < b - 2a$. We show that

$$\xi_3(x) = \xi_3(x; a, b, c) = \frac{\sinh(ax)}{\sinh(bx)}\frac{\sinh((b-2a-2c)x)}{x}$$

satisfies the following properties for $x > 0$:

1. If $b-2a-2c>0$, then $\xi_3(x)$ is strictly decreasing for $x>0$.

2. If $b-2a-2c=0$, then $\xi_3(x)=0$.

3. If $b-2a-2c<0$, then $\xi_3(x)$ is strictly increasing for $x>0$.

We have

$$\xi_3(x;a,b,c)=\xi_2(x;a,b)\xi_1(x;b-2a-2c,b-a).$$

In the case of $b-2a-2c>0$, it follows from $b-a>b-2a-2c>0$ that $\xi_1(x;b-2a-2c,b-a)$ is strictly decreasing for $x>0$. Since $\xi_2(x;a,b)$ is also strictly decreasing for $x>0$, $\xi_3(x)$ is strictly decreasing for $x>0$. In the case of $b-2a-2c<0$, we have

$$\xi_3(x;a,b,c)=-\xi_2(x;a,b)\xi_1(x;-(b-2a-2c);b-a).$$

Owing to $b-a-(-(b-2a-2c))=2(b-2a-c)+a>0$, we find $b-a>-(b-2a-2c)>0$, which means that $\xi_1(x;-(b-2a-2c);b-a)$ is strictly decreasing for $x>0$. Hence, $-\xi_3(x)$ is positive and strictly decreasing for $x>0$, that is, $\xi_3(x)$ is strictly increasing for $x>0$. It is straightforward to check that $b/2-a-c=0$ yields $\xi_3=0$.

C Limiting estimates of auxiliary functions

We show useful formulae for the limits of $E_\pm(c,z)$, $U_\pm(c)$ and their related functions as $c\to\infty$. The following lemma is used for the proof of Theorem 2.4(d).

Lemma C.1. For any $z,z_1,z_2>0$, we have

$$\begin{gathered}
\lim_{c\to\infty}E_+(c,z)=1,\quad \lim_{c\to\infty}E_-(c,z)=0,\\
\lim_{c\to\infty}U_+(c)=\frac{2r}{L},\quad \lim_{c\to\infty}U_-(c)=0,\\
\lim_{c\to\infty}\frac{U_-(c)}{1-E_+(c,z)}=0,\quad \lim_{c\to\infty}\lambda_-(c)^2U_-(c)=1,\\
\lim_{c\to\infty}\frac{1-E_+(c,z_1)}{1-E_+(c,z_2)}=\frac{z_1}{z_2},\quad \lim_{c\to\infty}\frac{E_-(c,z_1)}{1-E_+(c,z_2)}=0,\\
\lim_{c\to\infty}\frac{U_-(c)}{(1-E_+(c,z_1))(1-E_+(c,z_2))}=\frac{1}{z_1z_2}.
\end{gathered}$$

Proof. It follows from (5.6) that

$$\begin{aligned}
\lim_{c\to\infty}\lambda_+(c)&=\lim_{c\to\infty}\frac{2}{c+\sqrt{4+c^2}}=0,\\
\lim_{c\to\infty}\lambda_-(c)&=\lim_{c\to\infty}\frac{-c-\sqrt{4+c^2}}{2}=-\infty,\\
\lim_{c\to\infty}\frac{\lambda_-(c)}{2\theta(c)}&=\lim_{c\to\infty}\frac{1}{2}\left(\frac{-c}{\sqrt{4+c^2}}-1\right)=-1,
\end{aligned}$$

and we have

$$\lim_{c\to\infty} E_+(c,z) = \lim_{c\to\infty} \exp(-z\lambda_+(c)) = 1,$$
$$\lim_{c\to\infty} E_-(c,z) = \lim_{c\to\infty} \exp(z\lambda_-(c)) = 0.$$

Thus, we obtain

$$\lim_{c\to\infty} U_-(c) = \lim_{c\to\infty} \frac{\lambda_+(c)}{2\theta(c)} \frac{1-E_-(c,2r)}{1-E_-(c,L)} = 0,$$
$$\lim_{c\to\infty} \lambda_-(c)^2 U_-(c) = \lim_{c\to\infty} \frac{-\lambda_-(c)}{2\theta(c)} \frac{1-E_-(c,2r)}{1-E_-(c,L)} = 1,$$

in which we used $\lambda_+\lambda_- = -1$. Note that $E_+(c,z) = 1 - z\lambda + (c) + o(\lambda_+(c))$. Then,

$$\lim_{c\to\infty} \frac{\lambda_+(c)}{1-E_+(c,z)} = \lim_{c\to\infty} \frac{\lambda_+(c)}{1-(1-z\lambda_+(c)+o(\lambda_+(c)))} = \frac{1}{z},$$
$$\lim_{c\to\infty} \frac{1-E_+(c,z_1)}{1-E_+(c,z_2)} = \lim_{c\to\infty} \frac{1-(1-z_1\lambda_+(c)+o(\lambda_+(c)))}{1-(1-z_2\lambda_+(c)+o(\lambda_+(c)))} = \frac{z_1}{z_2},$$

for $z > 0$, $z_1 > 0$, $z_2 > 0$, which yields

$$\lim_{c\to\infty} U_+(c) = \lim_{c\to\infty} \frac{-\lambda_-(c)}{2\theta(c)} \frac{1-E_+(c,2r)}{1-E_+(c,L)} = \frac{2r}{L},$$
$$\lim_{c\to\infty} \frac{U_-(c)}{1-E_+(c,z)} = \lim_{c\to\infty} \frac{1}{2\theta(c)} \frac{\lambda_+(c)}{1-E_+(c,z)} \frac{1-E_-(c,2r)}{1-E_-(c,L)} = 0,$$
$$\lim_{c\to\infty} \frac{E_-(c,z_1)}{1-E_+(c,z_2)} = \lim_{c\to\infty} \frac{-\lambda_-(c)\exp(\lambda_-(c)z_1)}{z_2 + \frac{o(\lambda_+(c))}{\lambda_+(c)}} = 0,$$
$$\lim_{c\to\infty} \frac{U_-(c)}{(1-E_+(c,z_1))(1-E_+(c,z_2))}$$
$$= \lim_{c\to\infty} \frac{-\lambda_-(c)}{2\theta(c)} \frac{\lambda_+(c)}{1-E_+(c,z_1)} \frac{\lambda_+(c)}{1-E_+(c,z_2)} \frac{1-E_-(c,2r)}{1-E_-(c,L)} = \frac{1}{z_1 z_2}.$$

□

D Auxiliary lemmas and their proofs

We show some useful lemmas for the proof of Theorem 2.6. Recall the following functions:

$$U_{pf}(c,d) = U_-(c)(1+\exp(d\lambda_-(c))\exp(2r\lambda_-(c))),$$
$$U_{pr}(c,d) = U_+(c) + U_-(c)\exp(d\lambda_-(c)),$$
$$U_{sf}(c,d) = U_+(c)\exp(-d\lambda_+(c)) + U_-(c),$$
$$U_{sr}(c,d) = U_+(c)(1+\exp(-d\lambda_+(c))\exp(-2r\lambda_+(c))),$$

and

$$\Delta U_p(c,d) = U_{pf}(c,d) - U_{pr}(c,d), \qquad \Delta U_s(c,d) = U_{sf}(c,d) - U_{sr}(c,d).$$

For any $c > 0$, $\lambda_\pm(c)$ satisfy $\lambda_-(c) < -\lambda_+(c) < 0$ and $\lambda_+(c)\lambda_-(c) = -1$, and $U_\pm(c)$ are given by

$$U_\pm(c) = \frac{1}{2\theta(c)} \frac{1 - \exp(\mp 2r\lambda_\pm(c))}{\pm\lambda_\pm(c)} > 0,$$

with $\theta(c) > 0$. For later use, we introduce the following function:

$$\begin{aligned}\Delta U(r) &= \Delta U(r;c) \\ &\equiv U_-(c) - U_+(c) = \frac{-1}{2\theta(c)} \left(\frac{1 - \exp(2r\lambda_-(c))}{\lambda_-(c)} + \frac{1 - \exp(-2r\lambda_+(c))}{\lambda_+(c)} \right).\end{aligned}$$

For any fixed $c > 0$, $\Delta U(r)$ is negative for $r > 0$. Indeed, we have $\theta(\Delta U)'(r) = \exp(2r\lambda_-) - \exp(-2r\lambda_+) < 0$ and $\Delta U(r) \to 0$ as $r \to 0$. Then, the following two lemmas hold.

Lemma D.1. For any c, $d > 0$, $(U_{pf}, U_{pr}, U_{sf}, U_{sr})$ satisfy

(i) $\Delta U_p(c,d) < 0$, (ii) $\Delta U_p(c,d) < \Delta U_s(c,d)$, (iii) $U_{pf}(c,d) < U_{sf}(c,d)$.

Proof. The claim directly follows from the following calculations:

$$\begin{aligned}\text{(i)}\quad \Delta U_p &= U_-(1 + \exp(d\lambda_-)\exp(2r\lambda_-)) - (U_+ + U_-\exp(d\lambda_-)) \\ &= -U_-\exp(d\lambda_-)(1 - \exp(2r\lambda_-)) + \Delta U < 0.\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad & 2\theta\,(\Delta U_p - \Delta U_s) \\ &= (\Delta U - \lambda_+ \exp(d\lambda_-)(1 - \exp(2r\lambda_-))^2) \\ &\quad - (\Delta U - \lambda_- \exp(-d\lambda_+)(1 - \exp(-2r\lambda_+))^2) \\ &= \lambda_- \exp(-d\lambda_+)(1 - \exp(-2r\lambda_+))^2 - \lambda_+ \exp(d\lambda_-)(1 - \exp(2r\lambda_-))^2 \\ &< 0.\end{aligned}$$

$$\begin{aligned}\text{(iii)}\quad U_{pf} - U_{sf} &= U_-(1 + \exp(d\lambda_-)\exp(2r\lambda_-)) - (U_+\exp(-d\lambda_+) + U_-) \\ &= U_-\exp(d\lambda_-)\exp(2r\lambda_-) - U_+(c)\exp(-d\lambda_+) \\ &< \exp(-d\lambda_+)\Delta U < 0.\end{aligned}$$

□

Lemma D.2. Let $h(c) = U_-(c) - U_+(c)\exp(-2r\lambda_+(c))$. Then, for any $c > 0$, there exist constants $d^*(c) > 0$ and $d^{**}(c) > 0$ such that

(i) $\Delta U_s(c,d) < 0$ holds if and only if

$$d > \begin{cases} 0 & h(c) \le 0, \\ d^*(c) & h(c) > 0. \end{cases}$$

(ii) $U_{pr}(c,d) < U_{sr}(c,d)$ holds if and only if

$$d > \begin{cases} 0 & h(c) \le 0, \\ d^{**}(c) & h(c) > 0. \end{cases}$$

Proof. (i) Since we have

$$\begin{aligned}\Delta U_s &= U_+ \exp(-d\lambda_+) + U_- - (U_+(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)))\\ &= U_+ \exp(-d\lambda_+)(1 - \exp(-2r\lambda_+)) + \Delta U,\end{aligned}$$

$\Delta U_s(c, d)$ is strictly decreasing for $d > 0$ and satisfies

$$\begin{aligned}&\lim_{d\to 0} \Delta U_s(c, d) = U_-(c) - U_+(c)\exp(-2r\lambda_+(c)) = h(c),\\ &\lim_{d\to\infty} \Delta U_s(c, d) = \Delta U < 0.\end{aligned}$$

If we have $h(c) \le 0$ for fixed $c > 0$, $\Delta U_s(c, d) < 0$ holds for any $d > 0$. In the case of $h(c) > 0$, there exists a constant $d^*(c) > 0$ such that $\Delta U_s(c, d^*(c)) = 0$ and $\Delta U_s(c, d) < 0$ holds for any $d > d^*$. Note that $d^*(c)$ is explicitly expressed by

$$d^*(c) = -\frac{1}{\lambda_+(c)} \log\left(\frac{U_+(c) - U_-(c)}{U_+(c)(1 - \exp(-2r\lambda_+(c)))}\right).$$

(ii) We consider the following function:

$$\begin{aligned}f_1(d) &\equiv 2\theta\,(U_{pr}(c, d) - U_{sr}(c, d))\\ &= \lambda_- \exp(-d\lambda_+)\exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) + \lambda_+ \exp(d\lambda_-)(1 - \exp(2r\lambda_-)).\end{aligned}$$

Then, $f_1'(d)$ is given by

$$\begin{aligned}f_1'(d) &= \exp(-d\lambda_+)\exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) - \exp(d\lambda_-)(1 - \exp(2r\lambda_-)))\\ &= \exp(-d\lambda_+)\left[\exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) - \exp(d(\lambda_- + \lambda_+))(1 - \exp(2r\lambda_-))\right]\\ &\equiv \exp(-d\lambda_+) f_2(d),\end{aligned}$$

and it follows from $\lambda_- < -\lambda_+ < 0$ that $f_2(d)$ is a strictly increasing function satisfying

$$\begin{aligned}&\lim_{d\to 0} f_2(d) = \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) - (1 - \exp(2r\lambda_-)) < 0,\\ &\lim_{d\to\infty} f_2(d) = \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) > 0.\end{aligned}$$

Hence, there exists a constant $d_0 > 0$ such that $f_1(d)$ is monotonically decreasing for $0 < d < d_0$ and monotonically increasing for $d > d_0$. Considering

$$\begin{aligned}&\lim_{d\to 0} f_1(d) = \lambda_- \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) + \lambda_+(1 - \exp(2r\lambda_-)) = 2\theta(c)h(c),\\ &\lim_{d\to\infty} f_1(d) = 0,\end{aligned}$$

we find that, for any $c > 0$ satisfying $h(c) \le 0$, $f_1(d)$ is negative for $d > 0$ and thus $f_1(d) = 2\theta(U_{pr}(c, d) - U_{sr}(c, d)) < 0$ holds for any $d > 0$. In the case of $h(c) > 0$, there exists a constant $d^{**}(c) > 0$ such that $f_1(d^{**}(c)) = 0$ and $f_1(d) = 2\theta(U_{pr}(c, d) - U_{sr}(c, d)) < 0$ for any $d > d^{**}$. Then, $d^{**}(c)$ is given by

$$d^{**}(c) = \frac{1}{\lambda_+(c) + \lambda_-(c)} \log\left(\exp(-2r\lambda_+(c))\frac{U_+(c)}{U_-(c)}\right).$$

□

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