Jour. Fac. Sci., Hokkaido Univ., Ser. VII (Geophysics), Vol. 10, No. 1, 21-30, 1996.

Simple Quasi-periodic Functions and an Inverse Power Law

Itaru Maeda

Division of Easth and Planetary Sciences, Graduate School of Science, Hokkaido University, Sapporo 060, Japan

(Received November 13, 1995)

Abstract

Several quasi-periodic functions are examined in relation to a so-called inverse power law. The functions are constructed by addition and multiplication of sin functions with irrational frequencies. Characteristics of the functions are examined by Lorenz maps and number-amplitude plots. The amplitude is the local maximum value of a function and the number is the number of the local maxima reached to an assigned level. It is found that only triply multiplied sin functions obey the inverse power law on the amplitude-number relation. The slope in log-log scale or the value of the power is independent of frequencies.

1. Introduction

There are many phenomena repeatedly occur and governed by nonlinear equations of motion. Such occurrence is not periodic in the exact sense. For example, earthquakes seem to occur repeatedly in a region but no clear period of its occurrence in the usual sense is discovered. Inter-plates earthquakes are modeled as a stick-slip motion (Brace and Byerlee, 1966) or a relaxation oscillation. Such oscillatory motions can be described by nonlinear equations of motion, that is, nonlinear differential equations.

Nonlinear differential equations are nearly impossible to solve in an analytic closed form. When the nonlinearity is weak, the solutions for the nonlinear oscillations of some type have a form of quasi-periodic or almost-periodic functions.

On the other hands, the occurrence of earthquakes obeys several statistical laws the most famous of which is Gutenberg-Richter relation (Utsu, 1977). The Gutenberg-Richter relation is one of the so-called inverse-power laws. Although no theory is known which elicits the relation from sound physical basis, many numerical models are proposed since the work by Burridge and

Knopoff (1967). These models exhibit the repeated occurrence of earthquakes and the Gutenberg-Richter relation.

In this report, I examine several simple quasi-periodic functions and if they have a characteristic that can be expressed as an inverse-power law.

2. Quasi-periodic functions

The quasi-periodic function, f, is defined as follow:

1) f is defined on $\mathbb{R}$ and continuous,
2) there exists $\tau(\varepsilon)$ for which $|f(t+\tau(\varepsilon))-f(t)|<\varepsilon$ for $^\forall\varepsilon>0$ and $t\in\mathbb{R}$,
3) the set $\tau(\varepsilon)$ is dense in $\mathbb{R}$,

where $\mathbb{R}$ is one dimensional Euclid space (Minorsky, 1962). The word *dense* is used not as a daily word but as a technical one in mathematics.

An example of the quasi-periodic function is given by

$$f(t)=\exp(i\xi t)+\exp(i\zeta t),$$

where the ratio ξ/ζ is irrational. We can construct many quasi-periodic functions from the $f(t)$ using the property that a sum or product of quasi-periodic functions is a quasi-periodic function (Minorsky, 1962).

The quasi-periodic functions examined in this study are constructed from sin-function: sum of three sin functions with different frequencies, which will be denoted S+S+S, product two sins plus one sin, SS+S, and product three to seven sin functions. In order to obtain different frequencies with irrational ratio, the square roots of arbitrary floating point numbers with only one or two figures are used and for simplicity one of the number is fixed to be 1.

In the followings, we examine these functions numerically. We count the number of peaks reached to the assigned range with finite width of ε for the numerical functions calculated over finite time length. From the condition 2) and boundedness of the functions, the set of the counted numbers can approximate statistical characters of the functions, though the condition 3) is not fulfilled for finite time length. We will examine if this set has an inverse-power law character, that is, if the counted number is proportional to inverse of the amplitude.

We also produce so-called Lorenz map from the set. Lorenz map is an iterated map of the peak values and may exhibit some characteristics of the functions (Lorenz, 1963; Jackson, 1989 and 1990). In the present context, the peak means local maximum of a function examined. In addition, we examine Lorenz map for time difference between times of consecutive peaks.

3. Results

Examples of the wave forms are given in Fig. 1. We call these traces orbits. The uppermost function is the type SSS, the middle is SS+S and the lowest is S+S+S. All are calculated with frequencies of 1, $\sqrt{0.55}$, and $\sqrt{0.77}$. Because of the irrational frequencies, "beats" in the case S+S+S is clearly seen

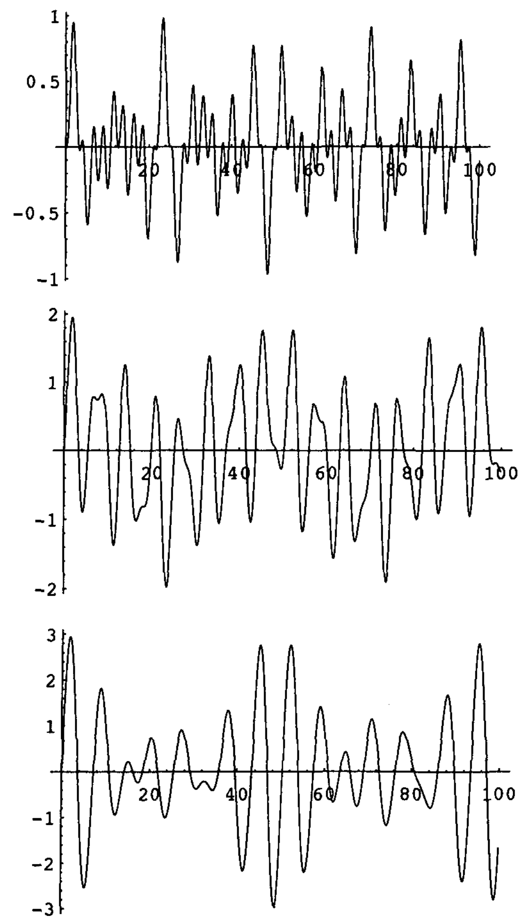

Fig. 1. Examples of traces of quasi-periodic functions. The upper trace is triply multiplied sin function (SSS). The middle is doubly multiplied sin plus a sin function (SS+S). The lower is triply summed sin functions (S+S+S). Frequencies used are 1, $\sqrt{0.77}$, and $\sqrt{0.55}$.

but the orbit of SSS is complex and the "beats" or any regularity is difficult to recognize for this time span.

Next, we compare Lorenz maps and number of peaks-amplitude relation (MF-relation for short). Fig. 2 shows these maps for S+S+S and SS+S cases with the same parameters as Fig. 1. Upper three figures (a, b, c) are for S+S+S and the lowers (d, e, f) are for SS+S. The left two (a, d) are Lorenz maps of orbits, the middle (b, e) are Lorenz maps of peak time difference, and the right (c, f) the MF-relation.

Lorenz maps in Fig. 2 have quite an interesting pattern but mapped points may be dense in some region. This means that the points will fill the region if the calculation is made for sufficiently long time span. From this pattern, we expect that the MF-relation is not in an inverse-power form. Calculated results are shown in c and f which are plotted in log-log scale. The vertical axis in logarithm of the number of peaks having amplitudes within assigned range and the horizontal axis is logarithm of amplitude of the peakes. Each

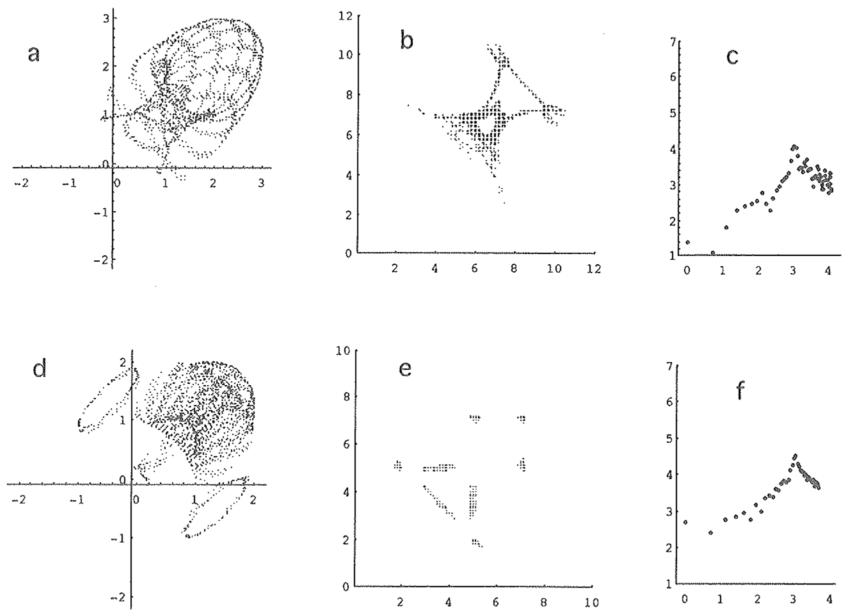

Fig. 2. Lorenz maps and number-amplitude plots for the case of S+S+S and SS+S. The upper three are for S+S+S and the lower are SS+S. Figures a and d are Lorenz maps for orbit, b and e are for peak-time difference, and c and f are number-amplitude plots in log-log scale. *Number* is the number of visits by peaks (local maxima) to an assigned amplitude range. The frequencies are the same as in Fig. 1.

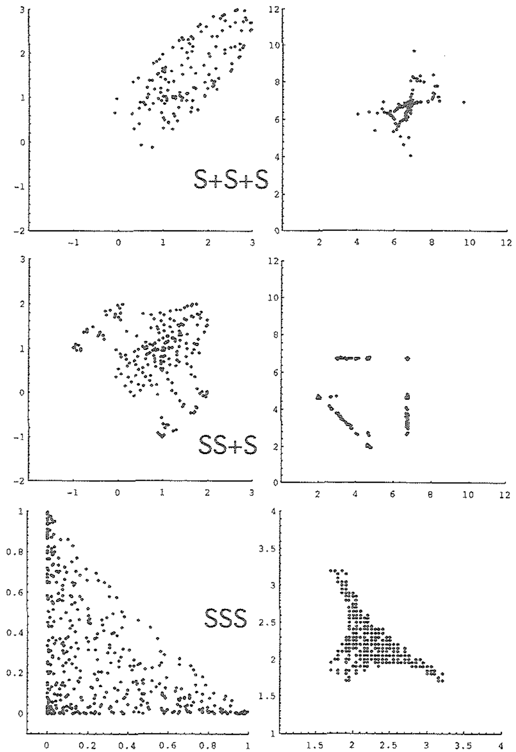

Fig. 3. Lorenz maps for orbits and iterated maps for peak time difference. Cases from triply summed (S+S+S) to triply multiplied (SSS) sin functions are given. The frequencies used are 1, $\sqrt{0.9}$, and $\sqrt{0.7}$.

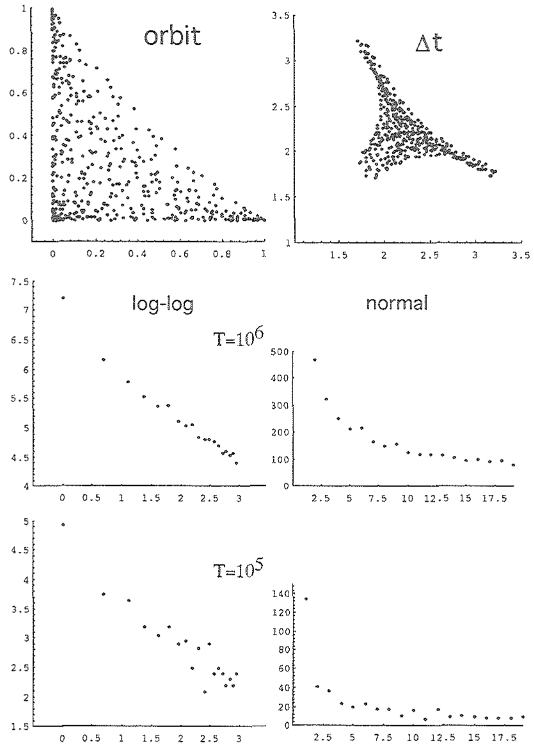

Fig. 4. Lorenz maps and number-amplitude plots in normal and in log-log scales for the case of SSS calculated over two time lengths, 10^5 and 10^6 in arbitrary time unit. Figure assigned Δt is an iterated map for peak to peak times. The frequencies are 1, $\sqrt{0.89}$, and $\sqrt{0.73}$.

figure c or f shows a maximum in the middle and is not linear as expected.

Lorenz maps for the peak-time-differece have also interesting patterns the iterates of which are distributed one dimensionally in the sense of the fractal dimension. The distribution is seemingly sparse but by the definition 3) it must be dense if the calculation is sufficiently long.

Fig. 3 shows Lorenz maps with frequencies 1, $\sqrt{0.9}$, $\sqrt{0.7}$. Left three figures are for orbit and the rights are for peak-time-difference. The type of function is given in the figure. Differences between type SSS and the other two are clear. The distribution of iterates of the peak-time-difference of SSS is two dimensional. Judging from the pattern for orbit, we can expect an inverse power law character for the case of SSS. We will examine this case a bit closely.

Fig. 4 shows the MF-relation together with Lorenz maps of the case SSS. MF relations are plotted in log-log and normal scales calculated over two time lengths as assigned in the figure. The longer the time length, the smaller the

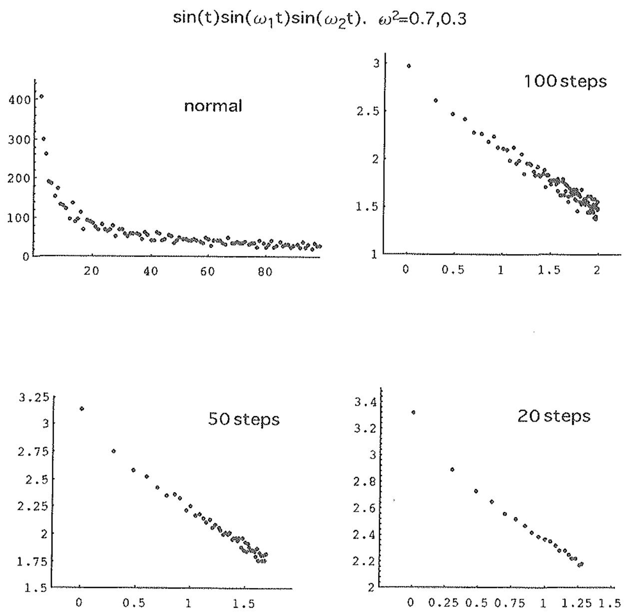

Fig. 5. Number-amplitude plots for the case of SSS with different widths of amplitude range. Because the maximum amplitude of the function is 1, to change the width is to change the number of steps for counting. The value of steps is attached to each figure. The original data is the same for all figures.

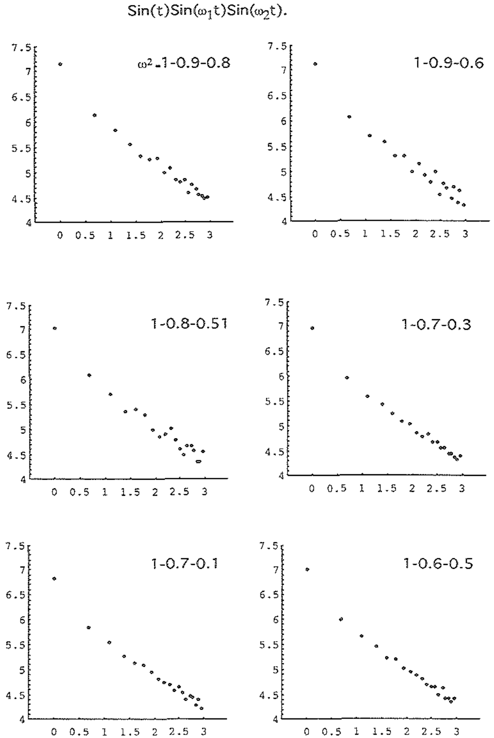

Fig. 6. Number-amplitude plots in log-log scale for the case of SSS with different frequencies which are attached to the plots. All figures show clear linearity.

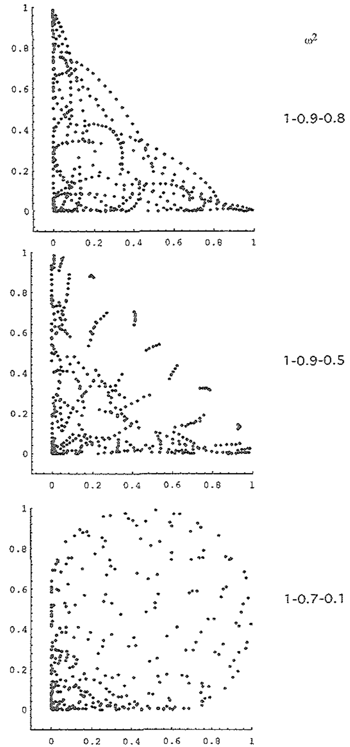

Fig. 7. Lorenz maps of SSS for three cases of frequencies.

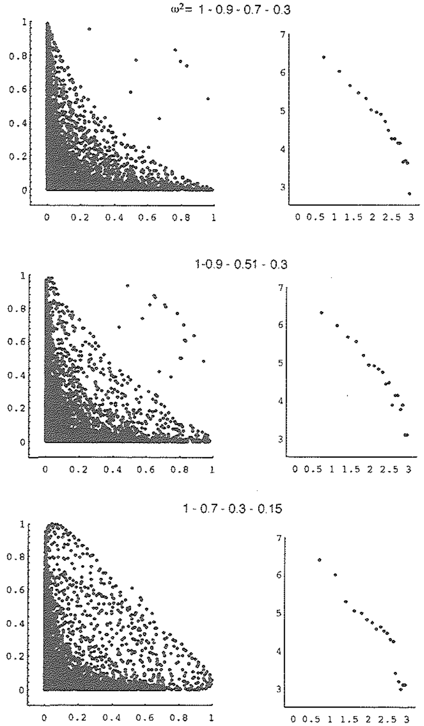

Fig. 8. Lorenz maps and number-amplitude plots in log-log scale for quadruply multiplied sin functions. Quadraple number to each plot is square of frequencies. The linearity is not satisfactory.

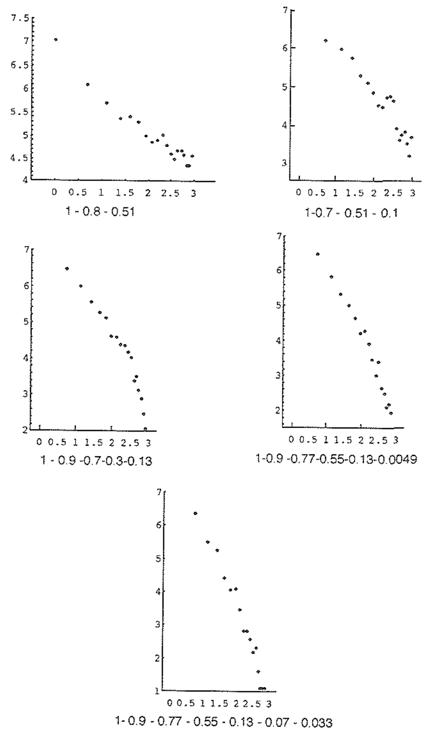

Fig. 9. Lorenz maps for functions constructed by multiplying from three to seven sin functions.

variance. The relation is linear in log-log scale except the point at zero amplitude range. This is the expected inverse power relation.

In order to examine if the zero point is truly exceptional, the amplitude width for the counting is changed (Fig. 5). For the data set given in upper-left figure plotted in normal scale, three plots with the widths of 1/100, 1/50, and 1/20 are produced. By reducing the width, the aperture between zero and the next points becomes narrow, though the variance increases. Judging from the figures, we conclude that the zero point is exceptional. In the followings, we will ignore the zero points when we examine the inverse power relation.

The next step is to examine if the relation holds for the type SSS with specific values of frequencies or with general irrational frequencies. Fig. 6 shows MF-relations of the type SSS with various frequencies which are assigned in each figures. Although these frequencies are arbitrarily selected, all cases show the linear relation in log-log scale with different variances. The slopes are also the same for different maps in Fig. 6. We can conclude that the inverse power law holds for the function SSS in general.

The different variances in Fig. 6 are the results of different orbits calculated with different frequencies. Fig. 7 shows Lorenz maps for three frequency sets as examples. As already pointed out, the sparse distribution of iterates is caused by the finite time calculation. It is interesting to observe that in spite of the difference in Fig. 7, all the functions obey the inverse power law with the same slope.

We examine next the case of multiplied four sin functions. Examples for three frequency sets are shown in Fig. 8 which exhibits nearly a linear relation but the variance is the different type from the case of SSS. That is to say, the variance is not caused by the finite time calculation. Fig. 9 shows MF relations of multiplied three to seven sin functions. It can be seen that the deviations from the linearity for the cases of four to seven sin functions are different from the case SSS.

4. Concluding remarks

The triply multiplied sin function $\sin(\xi t)\sin(\zeta t)\sin(\eta t)$ is decomposed into

$$[\sin(\xi-\zeta+\eta)t-\sin(\xi+\zeta+\eta)t+\sin(\xi-\zeta-\eta)t-\sin(\xi+\zeta-\eta)t]/4.$$

This means that the function has the four lines Fourier spectrum. In this sense, the function cannot satisfy the so-called $1/f$ spectrum, an inverse power relation. The analysis in section 3 showed the function had an inverse power law character though its Fourier spectrum did not have. It is important to recognize the following; Whether or not a solution of a given differential equation has an inverse power law character depends on what kind of analysis we would make.

We analyzed several quasi-periodic functions and found that only a function of triply multiplied sin had an inverse power relation. The relation does not depend on frequencies; linearity in log-log scale holds for all irrational frequencies and the slop is independent from the frequencies.

References

Brace, W.F. and J.D. Byerlee, 1966. Stick-slip as a mechanism for earthquakes. Science, **153**(3739), 990-992.

Burridge, R. and L. Knopoff, 1967. Model and theoretical seismicity. Bull. Seism. Soc. Amer., **57**, 341-371.

Byerlee, J.D., 1970. The mechanics of stick-slip. Tectonophys., **9**, 475-486.

Byerlee, J.D. and W.F. Brace, 1968. Stick slip, stable sliding, and earthquakes-effect of rock type, pressure, strain rate, and stiffness. J. Geophys. Res., **73**, 6031-6037.

Jackson, E.T., 1989, 1990. Perspectives of nonlinear Dynamics, vol. 1 and 2. Cambridge UP, 495 pp, 632 pp.

Lorenz, E.N., 1963. Deterministic nonperiodic flow. J. Atmos. Sci., **20**, 130-141.

Minorsky, N., 1962. Nonlinear oscillations. Krieger, U.S.A., 714 pp.

Utsu, T., 1977. Seismology. Kyouritu Shuppann, 286 pp.