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Pressure Variation due to Crack Extension with Fixed Fluid Mass Inside

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Abstract

In this paper, we give a simple analytic expression for pressure variation in a thin elliptic crack when it extends suddenly. It is assumed that the crack is filled with a compressible fluid and is in an equilibrium initially. The extension is assumed to be uniform and small. This means that the fluid inside the crack does not need to flow. The expression is given as

$$p=p_0+3K\ln\left[\frac{c_0}{c}\right]$$

where p is pressure, c the length of the crack, and K the incompressibility of the fluid. The suffix 0 means the initial value of the corresponding quantity.

1. Introduction

Transport of magma through cracks or fissures is one of the most important mechanisms of volcanic activities such as volcanic earthquakes, tremors, and fissure eruptions. Dikes are the frozen form of the transport. The magma transport through a crack or with a moving crack is first investigated by Weertman (1971a, 1971b). The model is regarded as almost a solid theory by Japanese- so- called- leading- researchers who copied the model without critical considerations [for example, Hujii, 1979 ; The figure 2.6 in this article is a complete copy of Fig. 11 given in Weertman (1971a)]. Actually the model can only be applied to cases in which cracks contain fluid having infinitely low viscosity. One of the examples is the case like water transport in cracked glaciers, to which Weertman originally intended to apply his model. Weertman assumes a steady-state motion of a crack and equates the velocity to the fluid velocity averaged over the crack length (in 1971b, p 8546, R-sector, 2-nd paragraph and eq.(11)). This averaging ignores the fact that the velocity of fluid flowing a narrower section of the crack is slower than that of wider section.

Then he concludes that the cross section of the moving crack become tadpole shaped. Here after we will call this cross sectional shape Weertman's tadpole.

There are two problems arise when we apply the model to the magma transport. The first one is viscosity stated above. Viscosities of magma are usually so large that we cannot ignore them as can in the case of water. It is clear that the results by Weertman can be applied only in the limit of infinite time future if the viscosity of the fluid is not sufficiently low. This means that the image given in the article by Hujii cannot be real. The reason why a crack needs infinite time to achieve the Weertman's tadpole is due to the viscous fluid flow which obeys a parabolic partial differential equation when the crack is thin (the aspect ratio is very small). In reality, cracks cannot be broad and is considered to be very thin indeed. Its solution may have a factor of the form $\exp(-\alpha t)$, where t is time.

The second problem relates to compressibility of fluid in the crack considered. If the crack motion is in a steady state and its velocity is controlled by the inside fluid flow, then the compressibility will not take part in the motion. Several researches after the Weertman's assumes the steady state in some sense because of the difficulty of handling coupled problems of elasticity and viscous fluid dynamics. For example, Lister and Kerr (1991) considered a quasi-steady state motion of a crack front which is far from the boundary from which magma is supplied at a constant rate. This model treats a single crack but the crack is not isolated in an infinite medium. Mériaux and Jaupart (1998) solved a quasi-dynamic problem of a situation similar to the case treated by Lister and Kerr. All studies treat incompressible fluid.

In realistic situations of volcanos, the crack motion or the crack extension will not be stationary because we nearly always observe volcanic earthquakes when the volcanos are in active. The earthquakes are the results of the crack extension, though we don't know whether all crack extensions causing volcanic earthquakes involve with the magma transport. We know through fracture experiments that there are many types of cracking; slow continuous fatigue crack growth, irregular intermittent growth etc. There must be cases in which dikes or fissures grow intermittently. The growth must be very quick because it causes earthquakes. In this case, we must take into account of compressibility of the fluid contained in the crack. In this paper, we consider pressure variations of an isolated crack filled with compressible fluid when the crack extends uniformly.

2. Formulation

We consider a problem how much of the pressure is changed when a crack is extended by a given amount. In other words, we want to obtain a relation between pressure and crack length under the fixed fluid mass inside the crack and a given initial pressure. Consider a penny shaped crack in 2-dimension with the z-coordinate axis taken to be the major axis and x the minor axis (in the y-direction, we take unit length as a crack width). We call the half length of the major axis the crack length. When the length$=c_0$, and the pressure inside the crack$=p_0$, the displacement in the direction x at $y=0$ is given by

$$u(0, z)=Ap_0c_0\sqrt{1-\left(\frac{z}{c_0}\right)^2} \tag{1}$$

(Sneddon and Lowengrub, 1969). Then, the volume of the crack with unit width, i.e., in y- direction is

$$V_0=\pi Ap_0c_0^2$$

where A is a constant defined by

$$A=\frac{2(1-\nu^2)}{E}$$

and ν is Poisson's ratio, E the Yang's modulus. For the case of a three-dimensional penny shaped crack, the coefficient corresponding to A is

$$A'=\frac{2}{\pi}A$$

This results in a rather different crack volume:

$$V_0'=2\int_0^{c_0}2\,\pi z\left(\frac{2}{\pi}Ap_0c_0\sqrt{1-\left(\frac{z}{c_0}\right)^2}\right)dz=8Ap_0c_0^3/3$$

This is different from V_0, the reason of which comes from the fact that for V_0 only unit thickness is taken into account, in the third direction c_0 is compressed by $1/c_0$. This consideration shows that we can use a conversion factor which adjusts the geometrical difference.

The strain energy E_1 of the medium containing the crack is expressed as

$$E_1=\frac{\pi}{2}Ap_0^2c_0^2 \tag{2}$$

Consider next the fluid inside the crack. The incompressibility is K and the volume at zero pressure is assumed to be Q. We consider the pressure inside

the crack is supported by compressing the fluid. By definition,

$$dp = -K\frac{dv}{V}$$

Integration of this expression gives

$$p = -K\ln[V] + \text{Const.}$$

By definition at $p=0$, the volume is Q;

$$0 = -K\ln[Q] + \text{Const.}$$

From this the const. $= K\ln[Q]$. This gives

$$p = K\ln[\frac{Q}{V}]$$

For the initial situation this is read as

$$p_0 = K\ln\left[\frac{Q_0}{V_0}\right]$$

We think $Q_0 \approx V_0$, so that x defined by $1+x = Q_0/V_0$ will be sufficiently small positive quantity. Expand $\ln(1+x)$ in x, i.e., $\ln(1+x) \approx x$, we obtain

$$p_0 = K(Q_0 - V_0)/V_0$$

The above linearized approximation will be used in the following. A simple evaluation seems to show that non approximate expression $K\ln[Q/V]$ would not improve or simplify following equations as a whole.

The strain energy of this fluid E_2 will be

$$E_2 = \frac{1}{2}P_0\Delta V_0 = \frac{K}{2}(Q_0 - V_0)^2/V_0 \tag{3}$$

Then, the total energy E will be expressed as

$$E_0 = \frac{\pi}{2}Ap_0^2c_0^2 + \frac{K}{2}(Q_0 - \pi Ap_0c_0^2)^2/V_0$$

Now we consider a situation in which a constraint suppressing the extension of a crack is removed and the crack extends. The work to extend the crack is done by the fluid which exerts pressure on the crack surface from inside to outside. The strain energy of the fluid is converted to the strain energy of the medium. The total energy will be conserved. In the present case, the fluid pressure p is a function of the crack length c, i.e., $p = p(c)$ under the condition of constant energy. The total energy of the system

$$E(p(c)) = \frac{\pi}{2}Ap^2c^2 + \frac{K}{2}\frac{(Q - \pi Apc^2)^2}{\pi Apc^2} \tag{4}$$

is constant, i.e., $E_0=E$. We think that the above equation (4) expresses a relation between p and c. We differentiate equation (4) with respect c and equate the result to zero. Since

$$dE(p(c))/dc=(\partial E/\partial p)\ dp/dc+(\partial E/\partial c)=0$$

we obtain a first order strongly nonlinear differential equation

$$\frac{dp}{dc}=-\frac{2p(-KQ^2+A^2Kp^2\pi^2c^4-A^2p^3\pi^2c^4)}{c(-KQ^2+A^2Kp^2\pi^2c^4+2\ A^2p^3\pi^2c^4)} \tag{5}$$

with the initial condition

$$E_0=\frac{\pi}{2}Ap_0{}^2c_0{}^2+\frac{K}{2}\frac{(Q-\pi Ap_0c_0{}^2)^2}{\pi Ap_0c_0{}^2}$$

This condition is a relation to determine p_0 as a function of c_0 and E_0. The explicit formula for p_0 can be obtained as follow but it is messy.

$$\begin{aligned}
p_0=&-\frac{K}{3}-(2^{1/3}(-6A^3\pi^3(E_0+KQ)\ c_0{}^6-A^4K^2\pi^4c_0{}^8))/(3A^2\pi^2c_0{}^4\\
&(-27A^4K\pi^4Q^2c_0{}^8-18A^5K\pi^5(E_0+KQ)c_0{}^{10}-2A^6K^3\pi^6c_0{}^{12}+\\
&\sqrt{(4(-6A^3\pi^3(E_0+KQ)\ c_0{}^6-A^4K^2\pi^4c_0{}^8)\text{^}3+}\\
&(27A^4K\pi^4Q^2c_0{}^8-\\
&18A^5K\pi^5(E_0+KQ)c_0{}^{10}-2A^6K^3\pi^6c_0{}^{12})\text{^}2))\text{^}(1/3))+\\
&\frac{1}{32^{1/3}A^2\pi^2c_0{}^4}((-27A^4K\pi^4Q^2c_0{}^8-18A^5K\pi^5(E_0+KQ)c_0{}^{10}-2A^6K^3\pi^6c_0{}^{12}+\\
&\sqrt{(4(-6A^3\pi^3(E_0+KQ)\ c_0{}^6-A^4K^2\pi^4c_0{}^8)\text{^}3+}\\
&(-27A^4K\pi^4Q^2c_0{}^8-\\
&18A^5K\pi^5(E_0+KQ)c_0{}^{10}-2A^6K^3\pi^6c_0{}^{12})\text{^}2))\wedge(1/3))
\end{aligned}$$

where $(\ )\text{^}n=(\ )^n$. In actual calculations, we only need numerical values for c_0, E_0, and p_0. We can calculate E_0, for given p_0 and c_0 values which may be rather arbitrarily assigned. Therefore, we can avoid using the above expression directly. These expressions are too complex. We try to evaluate the size of the factors in order to get an approximate more concise expression. Since $V=\pi Apc^2$, the equation (5) can be rewritten as

$$\begin{aligned}
\frac{dp}{dc}&=-\frac{2p(-KQ^2+KV^2-pV^2)}{c(-KQ^2+KV^2+2pV^2)}\\
&=\frac{2p\left(-\left(\frac{Q}{V}\right)^2+\frac{K-p}{K}\right)}{c\left(-\left(\frac{Q}{V}\right)^2+\frac{K+2p}{K}\right)}
\end{aligned} \tag{6}$$

Let us define

$$\frac{Q}{V}=1+x$$

where $0<x<<1$. Using the relation $p=K\ln\left[\frac{Q}{V}\right]$, we obtain

$$\left(\frac{Q}{V}\right)^2=(1+x)^2=1+2x+x^2$$

$$\frac{p}{K}=\ln\left[\frac{Q}{V}\right]\approx x-\frac{x^2}{2}$$

Substituting these expressions into the DE. (6), we obtain

$$\frac{dp}{dc}\simeq-\frac{2p}{c}\frac{-2x-x^2-x+\frac{x^2}{2}}{-2x-x^2+2x-x^2}=-\frac{2p}{c}\frac{-3-\frac{x}{2}}{-2x}$$

$$\simeq-\frac{2p}{c}\frac{3}{2x}\simeq-\frac{2p}{c}\frac{3}{2\frac{p}{K}}=-\frac{3K}{c} \tag{7}$$

Replacement of x with p/K corresponds to an approximation $p\ll K$, which is a good approximation in normal situations. This approximate DE. (7) gives

$$p=-3K\ln[c]+\text{const}$$

where the integration constant is determined by

$$p_0=-3K\ln[c_0]+\text{const}$$

The solution of the equation (7) is

$$p=p_0+3K\ln\left[\frac{c_0}{c}\right] \tag{8}$$

Crack extension means $c>c_0$, $\ln[c_0/c]<0$, and therefore $p<p_0$. That is, the pressure is reduced by the crack extension as expected.

3. Discussion

Note first that, if we simply neglect the third terms in the numerator and the denominator of the original DE. (6), we obtain the coefficient to be 2 instead of 3. Note second that the following relation is directly obtained from the definition

$$p=K\ln\left[\frac{Q}{V}\right]$$

$$p_0=K\ln\left[\frac{Q}{V_0}\right]$$

$$p-p_0=K\ln\left[\frac{Q}{V}\frac{V_0}{Q}\right]=K\ln\left[\frac{V_0}{V}\right]$$

Because $V\sim O(c^3)$, the above result (8) seems to be not sufficiently superb. In spite of this observation, it is not so simple matter as it seems because the exact expression $V=\pi Apc^2$ contains p. If we define $\mathrm{Pln}[z]$ as a solution w of the equation $z=w\exp(w)$, we can solve the above equation formally;

$$p=K\,\mathrm{Pln}\left[\frac{p_0c_0^2\exp\left[\frac{p_0}{K}\right]}{c^2K}\right]\Bigg]$$

For practical purpose, it is of no use because function Pln is simply a complete formal inverse function. Our result is more practical.

The above formula (8) can be applied only to the cases of uniform extension of an elliptical crack. In real situations, cracks will extend unidirectionally; for a vertical crack, the upper edge moves upward with the other edge, for example, fixed. In this case, sudden extension causes non-uniform pressure distribution and results in a non-elliptical cross section. To recover an elliptical cross section, the inside fluid must flow. The flow state is determined by pressure distribution exerted by the surrounding elastic medium. The pressure distribution is a function of the difference between a final equilibrium cross section and temporal one when the crack extension just occurred. The final cross section must be expressed by the Weertman's tadpole. The formula for the tadpole contains uniform pressure which results in an elliptical section if there is no gravitational force. In this process, the only constraint must be the conservation of fluid mass. Pressure change considered in the present work will be relaxed after a time elapse which the internal fluid needs to flow.

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