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# Mandelstam-Chaikin-Kaidanovsky Theory and Unsymmetric Motion of a Rayleigh Oscillator

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## Abstract

A stick-slip mechanism as a building block of an earthquake simulator is analyzed. The mechanism consists of a mass block connected to a fixed wall through a linear spring and placed on a conveyer belt. We call the velocity of the conveyer the driving velocity which is constant in the present model and determines the position of a nullcline on the phase plane. The friction law between the block and the belt has the form given by Rayleigh. The law is expressed on the phase plane a third order algebraic curve and defines the other nullcline. We call the mechanical system Rayleigh oscillator. It is already known that the oscillator bifurcates from a limit cycle oscillation to a stable sliding as the driving velocity increases.

We integrate the equation of motion numerically with the theory of Mandelstam-Chaikin-Kaidanovsky (MCK) in mind for the selection of the parameter values. When the mass is large, Rayleigh oscillator has a normal symmetric limit cycle and therefore its motion is different from a stick-slip motion expected as an earthquake simulator. As the mass is reduced with large dissipative friction the orbit becomes angulate and unsymmetrical and becomes a bit expected one.

When the driving velocity approaches to the bifurcation point, a new cyclic but small orbit appears which we call a trapped orbit. We gave a plausible argument that the trapped orbit is not a numerical fallacy. We examined whether the MCK-theory approximately holds when the mass is sufficiently small and found that the recovery jump to the stick state was not correct for the present friction law.

## 1. Introduction

Though the idea of stick-slip motion was introduced for earthquake events by Brace and Byerlee (1966), analyses for the stick-slip motion have a long history. At 1883, Rayleigh proposed a theory for sustaining vibration caused by a non-oscillatory external force. We call the Rayleigh's model the Rayleigh oscillator. The settings he analyzed is equivalent to that for a model earth-

quake. For the latter, a block having a mass  $m$  and being connected to a fixed point through a spring is placed on a moving belt which moves in one direction and exerts frictional force to the block. The driving belt is an idealization of an oceanic plate which drags the continental plate colliding to it. The block and spring system corresponds to the continental plate or a part of it tangent to the oceanic plate.

Maeda and Yokomori (1999) constructed a model consisting of multiply connected Rayleigh oscillators. The model well simulates not only statistical nature of seismic activities, such as the magnitude-frequency relation, but also its time series such as energy accumulation curves. In the model, the motion of each Rayleigh oscillator is smooth, so discrete events corresponding earthquakes must be defined by somewhat artificial criteria. The motion is approximately symmetric. With symmetric, we means that the displacement and velocity are symmetric with respect their average values.

Can the Rayleigh oscillator oscillate non smoothly and unsymmetrically? If it can, the identification of an event can be made more naturally. The stick-slip motion is considered to be a kind of relaxation oscillation. There are theories on the relaxation oscillation. The van der Pol oscillator is famous, which can oscillate non smoothly if the nonlinearity is sufficiently high but which seems to be symmetric. This oscillator is equivalent to Rayleigh oscillator mathematically (Drazin 1994).

For unsymmetric and nonsmooth oscillations, Mandelstam-Chaikin-Kaidanovsky theory exists, which is cited by Minorsky (1962). The basic idea is to use degenerate equation of motion for analyzing smooth part of an orbit and allow jump at the nonsmooth part of the orbit under a condition which is not controlled by the degenerate equation.

In this report, we first outlined the Mandelstam-Chaikin-Kaidanovsky theory, then apply it to the block-spring-belt model given above with some friction law. We then analyze the Rayleigh oscillator with these ideas to get conditions for quasi-discontinuous unsymmetric motion. Numerical example is also given.

## 2. Outline of Mandelstam-Chaikin-Kaidanovsky theory

This section is a summary of the Mandelstam-Chaikin-Kaidanovsky theory. Here after we call the theory MCK theory for short. Although the theory can be applied to various kind of phenomena having piecewise analytic orbit, we only describe it in relation to mechanical phenomena, especially a mass-spring-

slider model analyzed by Chaikin and Kaidanovsky cited by Minorsky (1962). Then, the equation of motions are second order ordinary differential equations.

$$a\ddot{x} + f(\dot{x}) + kx = 0 \quad (1)$$

Consider that the system stays in a state of slow variation, i.e., the orbit is analytic, for the most part of time and experiences a fast variation only for very short time. For the case of stick-slip, the slow variation is in the stick state and the fast variation corresponds to slip. During slow variation, the inertial term will be negligible and the orbit may be controlled by a first order differential equation (D.E. for short), called degenerate D.E.

$$f(\dot{x}) + kx = 0 \quad (2)$$

To determine the state, i.e., the orbit, it is needed two initial constants for equation (1) but only one is relevant to the degenerate D.E. (2). The theory prescribes how to deal with the second constant. The orbit determined through equation (2) goes into jump to reconcile the second constant required in the original equation (1). The jump condition is called Mandelstam condition and is determined by invariants of the system under consideration. The condition of the present case is a function of  $x$  but not velocity because the potential energy cannot vary during the jump and it can only be expressed by position variable  $x$ . This will be the elastic potential energy  $kx^2/2$ . This means the variable which can jump is the velocity.

The timing to jump must be determined by the degenerate D.E. (2). Differentiate Eq.(2) and rewrite, we obtain

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{kv}{f'(v)} \end{aligned} \quad (3)$$

where  $f'$  means differentiation with respect to its variable  $v$ . If there exist a  $v = v_c$ , a critical point, such that  $f'(v_c) = 0$ ,  $\dot{v}$  becomes infinite and the velocity  $v = \dot{x}$  can jump. If  $f'(v) < 0$ , the velocity  $v$  and the acceleration have the same sign, so the system is unstable. Because this system is driven not by a periodic external force but by a constant unidirectional moving belt, the relaxation oscillation is sustained by the internal instability. In other word, the system is autonomous.

In order to occur the instability, the friction law  $f(v)$  must have a minimum at certain  $v$ . This corresponds to the well known fact that the dynamic friction for small  $v$  is smaller than the static friction and for sufficiently large  $v$  it increases. In order to make discussion a bit quantitative, we need to introduce

a functional form of the friction law. To this end, now we consider the Rayleigh oscillator.

### 3. Rayleigh oscillator

Rayleigh introduced the following friction law in order to sustain vibration under the action of constant force (given in the text, Rayleigh, 1894),

$$f(v) = -\eta_1 v + \eta_2 v^3 \quad (4)$$

with  $v = \dot{x}$ .  $\eta_1$  and  $\eta_2$  are frictional constants and positive. The first term of RHS has the opposite sign to the usual friction. This function satisfies the condition for jump to occur.

For the case of our block-spring-friction model,  $v = \dot{x} - V$ , where  $V$  is the velocity of the driving belt. The equation of motion is

$$\begin{aligned} \dot{x} &= v + V \\ m\dot{v} &= -kx + \eta_1 v - \eta_2 v^3 \end{aligned} \quad (5)$$

This is one of the Lienard system but does not satisfy the conditions of Lienard's theorem for a limit cycle to exist (Perko, 1996). This means that Eq. (5) may have a limit cycle and the number may not be limited to one. Numerical calculations show at least one main limit cycle exist in a certain range of the parameters.

On  $x$ - $v$  phase plane, the system has two nullclines defined either  $\dot{x} = 0$  or  $\dot{v} = 0$ , on which the phase orbit cross either vertically or horizontally. The cross point of these nullclines is the fixed point. Fig.1 shows the nullclines  $r$ ,  $s$  (bold straight line and S-curve), the fixed point  $F$ , the local minimum of friction  $B$ , (and  $D$  for return motion) and the limit cycle  $A$ - $B$ - $C$ - $D$ .

The unstable condition for oscillation  $f'(v) < 0$  given by MCK-theory is the same as that the fixed point is unstable. That is to say if it is in the interval

$$-\sqrt{\eta_1/3\eta_2} < v < +\sqrt{\eta_1/3\eta_2} \quad (6)$$

the point is unstable. The jump points are the end points of this interval. These points are simply the local minimum or maximum of the nullcline  $s$ .

The motion of a phase point in standard situation is well known and as follow: The phase point having begun in the right side region of the nullcline  $s$  goes down to cross  $s$ , then goes left along the underside of  $s$ . If the fixed point is on the branch lower than  $B$ , i.e., outside of the interval (6), the phase point stops but if the fixed point is above  $B$  as depicted in Fig. 1, the phase point will

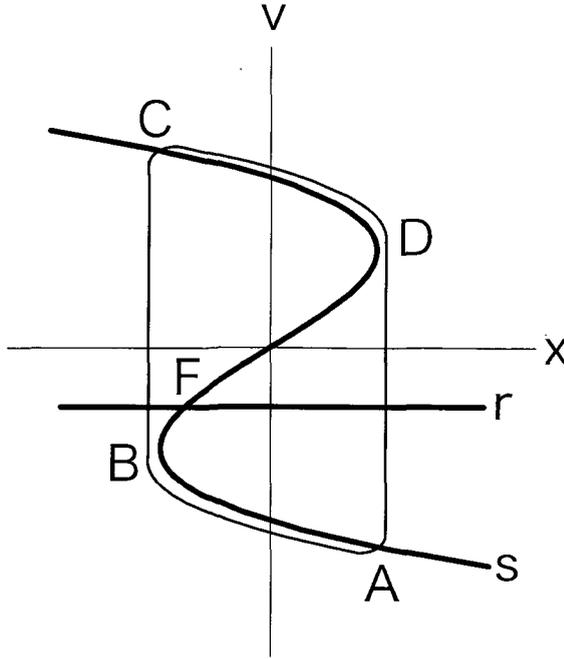


Fig. 1. A schematic phase diagram of a limit cycle for the relaxation oscillation. The horizontal axis is displacement and the vertical is velocity. The bold horizontal straight line  $r$  and S-like curve  $s$  are the nullclines. The closed curve  $A-B-C-D$  is the limit cycle. The point  $B$  (and  $D$ ) is the local minimum of the friction law.  $F$  is the fixed point.

jump to  $C$ . It seems that the system have only two kinds of states, a simple oscillation corresponding to a single limit cycle and a stable sliding.

Now we seek the condition for the unsymmetrical and quasi-discontinuous motion. In order to hold the degenerate equation for Eq. (5), the inertial term must be negligible, according to MCK-theory. In other word, cases of small mass and large dissipative friction will satisfy the conditions.

#### 4. Numerical Integration

We examine whether the condition of small mass and large dissipation actually result in an unsymmetrical and quasi-discontinuous motion by numerical integration of Eq. (5). Because "large" or "small" have only relative meaning for the present model, to establish actual numerics we use physics argument. If the mass of the block  $m$  is large, and the spring is strong, the

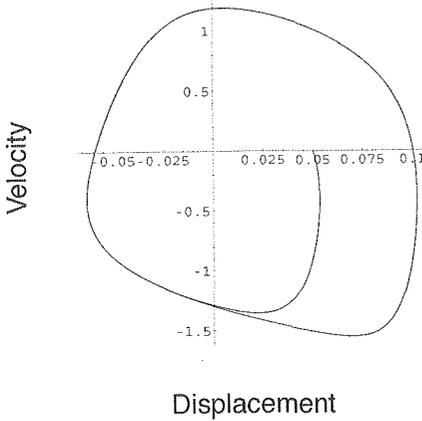


Fig. 2. Phase orbit for a case of large inertia (mass  $m$ ). In this model,  $m=0.01$  and the spring constant  $k=4$ , the energy accumulating frictional constant  $\eta_1=0.3$ , the dissipative frictional constant  $\eta_2=0.2$ . The driving velocity  $V=0.4$ .

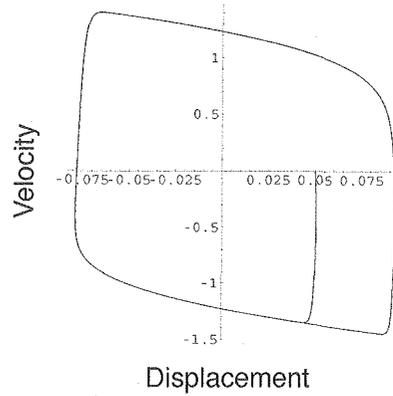


Fig. 3. Phase orbit for smaller mass,  $m=0.001$  with other parameters being the same as Fig. 2.

oscillation will be sinusoidal and the phase orbit will look like a circle. Fig. 2 shows such an orbit with  $m=0.01$ ,  $k=4$ ,  $\eta_1=0.3$ ,  $\eta_2=0.2$ , and the driving velocity  $V=0.4$ .

Small mass  $m=0.001$  makes the orbit angulate but rather symmetric (Fig. 3) with other parameters being the same as in Fig. 2. The angulate phase orbit means nonsmooth motion of the block. When the dissipative friction  $\eta_2$  is larger than the energy accumulative friction  $\eta_1$ , the phase orbit is deformed and the corresponding displacement becomes nonsmooth and unsymmetric as shown in Fig. 4-(a) and -(b). The parameter values are  $m=0.001$ ,  $k=3$ ,  $\eta_1=0.3$ ,  $\eta_2=0.6$ , and the driving velocity  $V=0.4$ . This is the expected orbit by MCK-theory.

We proceed further to see what happens if the fixed point approaches to the boundary of stable and unstable region given by Eq. (6) and the point  $B$  in Fig. 1, i.e., a kind of bifurcation point, by adjusting the driving velocity  $V$  with other parameters not changed. In the present values of the model system, the critical velocity at the bifurcation point is  $\sqrt{1/6} \approx 0.408248$

When the fixed point approaches to the bifurcation point from the unstable region, the phase point would not jump immediately at the boundary and moves along the nullcline  $s$  (see Fig. 1) for a while then jumps (Fig. 5 at  $V=0.406501$ ).

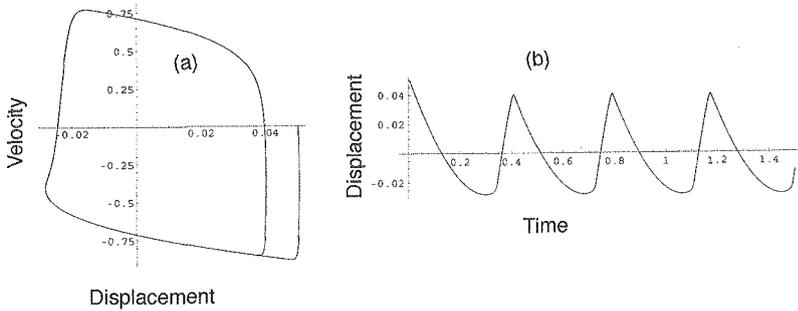


Fig. 4. Unsymmetrical and nonsmooth phase orbit (a) for the case of larger dissipation,  $\eta_1=0.3$  and  $\eta_2=0.6$ . Other parameters are the same as Fig. 3. (b) is the displacement variation.

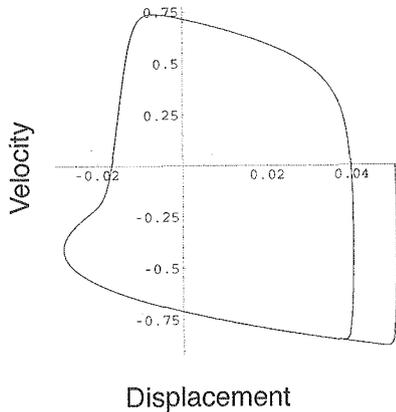


Fig. 5. Deformed limit cycle at the neighbor of a new bifurcation point.  $V=0.406501$ .

If the fixed point approach a bit nearer ( $V=0.406502$ ) to the bifurcation point, a new cyclic orbit (we call it the trapped orbit) appears. Fig. 6-(a) shows the phase orbit, (b) shows the displacement as a function of time and (c) the corresponding velocity variation. The advent of this trapped orbit indicates the existence of another bifurcation point. So we call the original stable-unstable bifurcation point SUB in order to distinguish it from the trapped orbit.

As the fixed point is nearing to SUB at  $\sqrt{1/6}$ , the trapped orbit seems to shrink continuously but not to reach the fixed point (Fig. 7-(b)). Although numerical integration cannot prove this along with the advent of the trapped orbit, we can make a feasibility argument. Different from exactly degenerate cases, in the present model, the phase point is not on the nullcline  $s$  but underside

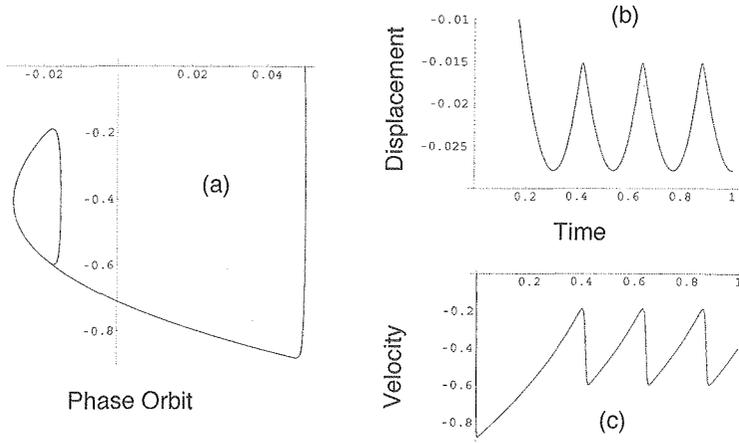


Fig. 6. A new cyclic orbit at  $V=0.406502$ , which we call the trapped orbit. (a) is the phase orbit, (b) the displacement variation and (c) the velocity variation.

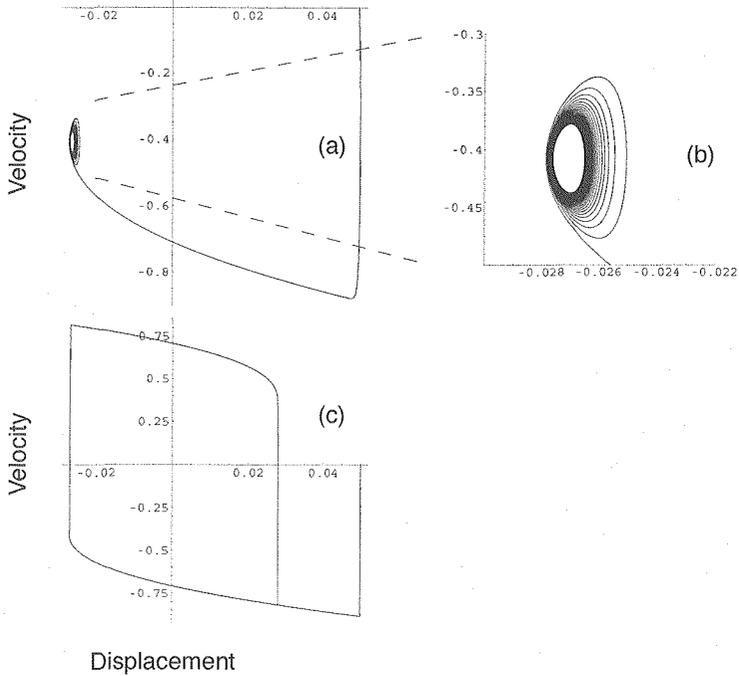


Fig. 7. Shrinking trapped orbit (a) and (b) at  $V=0.408$  and its release (c). The release is attained by reducing the mass, from  $m=0.01$  for (a) to  $m=0.001$  for (c).

of the branch  $A-B$  in Fig. 1. This means that the phase point simply passes through neighborhood of the SUB, and after that it moves away from the SUB. But soon after the passage, it passes the other nullcline  $r$  at the neighbor of the nullcline  $s$ . In the upper side region of the nullcline  $r$ , the phase point can cross the nullcline  $s$  from left to right and actually it occurs because the phase point is still sufficiently near the nullcline  $s$ . After the crossing, the point quickly moves to the branch  $A-B$  and crosses it.

Deformation of the orbit (see Fig. 5) when the fixed point approaches to SUB is caused by the inertia which prevents rapid change of the velocity and therefore it takes some time to change during which the block can rebound and the orbit may go along the nullcline  $s$ . It can happen that the orbit crosses the nullcline  $s$  when the mass is large. If this theory is true, reduction of inertia (mass) may prevent the deformation and induce to jump at point  $B$  (SUB). This actually is observed by means of numerical integration. Fig. 7-(a) is a trapped orbit with  $m=0.001$ , Fig. 7-(b) is the enlarged orbit and Fig. 7-(c) is the released orbit, i.e. a normal limit cycle, by reducing mass to 0.00001.

It is not clear that the trapped orbit is really cyclic. The case shown Fig. 7-(a) and (b) ostensibly suggests that the orbit will eventually sink into some point. But there seems to be no candidate for the point, the fixed point is unstable and the minimum of friction  $B$  on the nullcline  $s$  cannot be reached because, to be able to do that, the orbit must cross the line  $s$  before to cross the other nullcline  $r$  but it is impossible. This suggests that the trapped orbit is a kind of a limit cycle.

Because it seems to depend not on the initial conditions but only on the values of system parameters, the trapped orbit may be an isolated orbit. There is a supporting result for the existence of this new cycle. Kloeden et al. (1997) give an example showing a similar orbit for a case of Bonhoeffer-van der Pol oscillator with random agitation. We reproduce the result in Fig. 8. In order to clearly distinguish from the other figures, we depicted it in somewhat different manner. Because the way of calculation and setting are completely different but show similar orbit, it is quite probable that the new orbit is real.

## V. Conclusions

We analyzed a Rayleigh oscillator by using the theory of Mandelstam-Chaikin-Kaidanovsky and by numerical integration. The behavior of the oscillator is expressed on the phase plane. When the mass of the block is large, the phase orbit is a limit cycle and approximately a circle. As the mass is

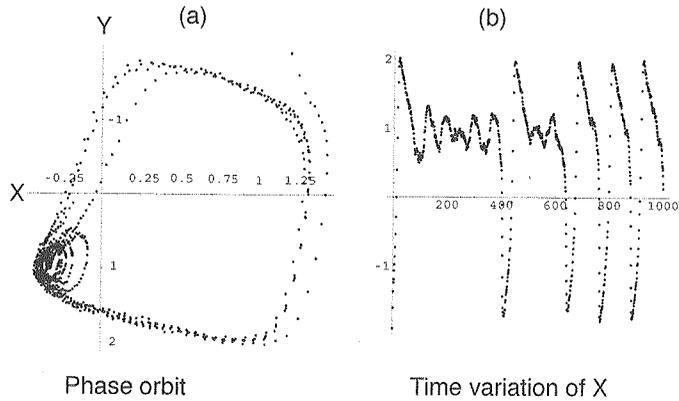


Fig. 8. Phase orbit for a stochastic nonlinear system showing trapped orbit. The system is nearly the same as the present model mathematically except the stochastic term.

reduced, the orbit becomes angulate at crossing points on the nullcline defined by  $dy/dt=0$  but is still a usual limit cycle. When the dissipative friction term is larger than the accumulative term with mass kept to be small, Rayleigh oscillator can oscillate unsymmetrically and nonsmoothly.

As the driving velocity is increased with other parameters fixed, the phase orbit deformed and the jump in the velocity components is retarded, and eventually a new kind of cyclic orbit (trapped orbit) appears without jump. We made a plausible argument for the existence of the trapped orbit. To increase the driving velocity corresponds to the process on the phase plane that the unstable fixed point approaches to the branch point at which the stability of the fixed point changes and the friction takes its minimum. The trapped phase point is released by the further reduction of the mass with other parameters fixed, but if the fixed point approaches more closely to the branch point a trapped orbit appears again.

For sufficiently small mass, the phase point makes a sharp turn at the cross point of the nullcline  $s$ , then goes along the underside of the nullcline but, after having reached the minimum in friction, the phase point jumps to the opposite branch of the nullcline. This contradicts to the prediction by MCK-theory. It will be needed to introduce other type of friction law to resolve the situation.

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