博士学位論文

Compatibility of Carnot efficiency and finite power in an underdamped Brownian Carnot cycle

（Underdamped Brown 粒子 Carnot サイクルにおける
Carnot 効率と有限のパワーの両立について）

三浦　孝祐

北海道大学大学院理学院
物性物理学専攻

令和４年３月

DOCTORAL DISSERTATION

Compatibility of Carnot efficiency and finite power in an underdamped Brownian Carnot cycle

Kosuke Miura

Department of Condensed Matter Physics,
Graduate School of Science, Hokkaido University

March 2022

Acknowledgments

The author would like to express the appreciation to Professor Koji Okuda and Professor Koji Nemoto for continuous guidance and encouragement. The author is also grateful to Professor Migaku Oda and Professor Takafumi Kita for helpful suggestions and critical reading. The author thanks all members of the Statistical Physics Laboratory for their encouragement.

Finally, I would like to thank my parents for their mental and financial support.

Abstract

Heat engines are indispensable equipment in our modern society converting supplied heat into output work. Many efforts have been conducted to improve their performance in various scientific and engineering fields. The efficiency defined as the ratio of the work to the supplied heat and the power defined as the work per unit time are often used to characterize their performance. Carnot demonstrated that the efficiency of any heat engine is limited by the upper bound called the Carnot efficiency which is achievable in the quasistatical operation. Although there are many attempt to achieve the Carnot efficiency in the finite-power heat engines, it is proven that there is a trade-off relation between the efficiency and power in the heat engines described by the Markov process. The trade-off relation says that the power vanishes when the efficiency approaches the Carnot efficiency. On the other hand, the possibility of the compatibility of the Carnot efficiency and finite power without breaking the trade-off relation by focusing on relaxation times of a system is proposed. However, complicated dependency of the efficiency, power, and trade-off relation between them on the relaxation times of the system makes the feasibility of the scenario non-trivial.

In this thesis, we consider the possibility of the Carnot efficiency in the finite-power underdamped Brownian Carnot cycle based on the trade-off relation in the system. We first introduce the previous research of the linear irreversible heat engine as the motivation for our research. We consider the stochastic process to describe the fluctuating system and show that there exists the trade-off relation between the efficiency and power in the heat engines described by the Markov process, based on another previous research. We consider the Brownian dynamics and construct the Carnot cycle with the instantaneous adiabatic process. In the cycle, the Carnot efficiency is compatible with the finite power in the small temperature-difference regime in the vanishing limit of the relaxation times without breaking the trade-off relation. We also consider the Brownian Carnot cycle with the finite-time adiabatic process and show that the Carnot efficiency is compatible with the finite power in the arbitrary temperature difference in the vanishing limit of the relaxation times.

Contents

Chapter 1

Introduction

1.1 Finite-time heat engine

Heat engines constitute one of the indispensable technologies in our modern society, and many efforts have been conducted to improve their performance in various scientific or engineering fields [1]. Heat engines convert supplied heat into output work. Moreover, their ratio can be used as the efficiency to characterize the performance of heat engines. The Carnot cycle is one of the most important models of heat engines, which operates between hot and cold heat baths with constant temperatures T_h and T_c ($< T_h$). Moreover, the cycle is composed of two isothermal processes and two adiabatic processes. Carnot demonstrated that the efficiency of any heat engine is limited by the upper bound called the Carnot efficiency [2]:

$$\eta_{\mathrm{C}} \equiv 1 - \frac{T_c}{T_h}. \tag{1.1}$$

It is known that we can reach the Carnot efficiency by the reversible cycle, where the heat engine always remains at equilibrium and is typically operated quasistatically, which implies that the engine spends an infinitely long time per cycle. Moreover, power, defined as output work per unit time, is another important quantity for evaluating the performance of heat engines. When we operate the heat engines quasistatically, the power vanishes. Thus, several studies have been devoted to investigating the feasibility of finite-power heat engines with Carnot efficiency [3–16].

1.2 Stochastic thermodynamics

To consider the finite-time heat engine, we have to consider the system out of equilibrium and introduce the heat and work in that system. Brownian motion is one of the most important models used to study the nonequilibrium thermodynamics. The motion is modeled by the Wiener process, which is one of the stochastic processes, in mathematical terms. Then, we can introduce the equation of motion of the Brownian particle called the Langevin equation. By considering the kinetic and potential energy of the particle and their change based on its equation of motion, we can define the thermodynamic quantities such as internal energy, entropy, heat, and work. Thus, we can construct the heat engine called the Brownian heat engine by using the Brownian particle. There are many studies of the efficiency and power of the heat engine [5, 16–27]. The framework connecting the stochastic dynamics and thermodynamics is called the stochastic thermodynamics [28].

1.3 Trade-off relation between efficiency and power

Although there are many efforts to achieve the Carnot efficiency with the finite power in heat engines, their compatibility is denied by the trade-off relation between efficiency η and power P. It is recently proved in general heat engines described by the Markov process [29–31]. The trade-off relation is given by

$$P \leq A\eta(\eta_{\mathrm{C}} - \eta), \tag{1.2}$$

where A is a positive quantity depending on the heat engine details. Based on this relation, the power should vanish as the efficiency approaches the Carnot efficiency. Similar trade-off relations to Eq. (1.2) have been obtained in various heat engine models [25, 32–34].

On the other hand, the Carnot efficiency may be achievable in a general class of finite-power Carnot cycles in the vanishing limit of the relaxation times [23]. Although that seems to contradict the trade-off relation in Eq. (1.2), the power may remain finite if the quantity A in Eq. (1.2) diverges at the same time as η approaches η_{C} in that limit. Thus, the compatibility of the Carnot efficiency and finite power may be allowed without breaking the trade-off relation in Eq. (1.2). However, complicated dependence of the quantity A, efficiency η and power P on the relaxation times may make the feasibility of the scenario nontrivial.

1.4 Purpose of this thesis

In this thesis, we study the underdamped Brownian Carnot cycle and show the compatibility of the Carnot efficiency and finite power in the vanishing limit of the relaxation times. We first introduce an important study of the compatibility in the linear irreversible heat engine as a motivation of our study. We consider the fluctuating system described by the stochastic process and review the trade-off relation between the efficiency and power in the heat engines described by the Markov process. We also consider the thermodynamics of the Brownian motion and construct the Brownian Carnot cycle with the instantaneous adiabatic processes. In this cycle, we show that the compatibility of the Carnot efficiency and finite power is achievable in the vanishing limit of the relaxation time of the system only in the small temperature-difference regime. We also show that the compatibility does not contradict the trade-off relation. Moreover, we construct the Brownian Carnot cycle with the finite-time adiabatic processes instead of the instantaneous adiabatic processes. In this cycle, we show that the above compatibility is achievable in that limit in arbitrary temperature difference without breaking the trade-off relation.

Chapter 2

Review: Possibility of Carnot efficiency with finite power in linear irreversible heat engine

We review an important previous study of the possibility of the Carnot efficiency with finite power in the linear irreversible heat engine, which is close to equilibrium, as a motivation of our study [7]. There are many efforts to study the nonequilibrium system. Especially, when the system is close to equilibrium, we can consider the irreversible currents such as the heat current or electric current. The thermodynamics of a system close to equilibrium is often called the linear irreversible thermodynamics [35–39]. There are many study related to the heat engine within this regime [3, 7–14, 16, 18, 20, 29, 40–62]. By considering the system close to equilibrium and introducing the irreversible currents, we construct a heat engine in the system and show the possibility of the Carnot efficiency with the finite power.

2.1 Review : Thermodynamic Bounds on Efficiency for Systems with Broken Time-reversal Symmetry

We consider the steady-state irreversible heat engine close to equilibrium with the broken time-reversal symmetry. We show that the compatibility of the Carnot efficiency and finite power is achievable in the heat engine.

2.1.1 Model

We consider the system represented by the schematic illustration in Fig. 2.1. The system is in contact with the left bath with the temperature T_L and chemical potential of electron μ_L and the right one with $T_R(< T_L)$ and μ_R. In this system, there is the external magnetic field $\boldsymbol{B}$ to break the time-reversal symmetry. Note that we can use another force such as Coriolis force to break the time-reversal symmetry instead of the magnetic field. We assume that $T_L \simeq T_R \simeq T$ is satisfied. When the temperature and chemical potential of the left and right baths are different, there are affinities, which is often called thermodynamic

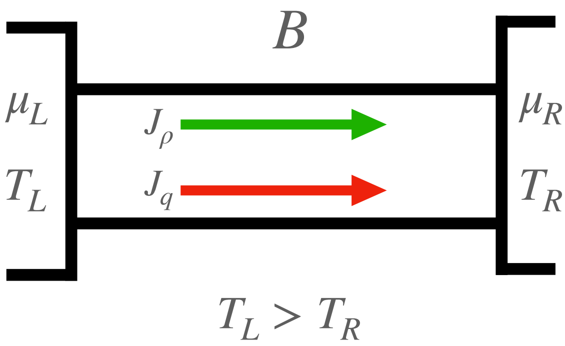

Figure 2.1: Schematic illustration of linear irreversible heat engine with the external magnetic field. Two currents, for example the electric and heat currents, flow. Because of the external magnetic field, the time reversal symmetry is broken.

forces, given by

$$X_1 \equiv -\frac{\mu_R}{T_R} - \left(-\frac{\mu_L}{T_L}\right) = -\frac{\Delta\mu}{T}, \tag{2.1}$$

$$X_2 \equiv \frac{1}{T_R} - \frac{1}{T_L} = -\frac{\Delta T}{T^2}, \tag{2.2}$$

where we defined $\Delta T \equiv T_R - T_L$ and $\Delta\mu \equiv \mu_R - \mu_L$. Because of $T_L > T_R$, X_2 in Eq. (2.2) is positive. In the steady-state system, the affinities are independent of time. Since the affinities in Eqs. (2.1) and (2.2) exist, there are two currents J_1 and J_2 denoting electric and heat currents corresponding to X_1 and X_2, respectively. We assume that J_1 and J_2 satisfy the Onsager relation given by

$$\begin{cases} J_1(\boldsymbol{B}) = L_{11}(\boldsymbol{B})X_1 + L_{12}(\boldsymbol{B})X_2, \\ J_2(\boldsymbol{B}) = L_{21}(\boldsymbol{B})X_1 + L_{22}(\boldsymbol{B})X_2, \end{cases} \tag{2.3}$$

where L_{ij} in Eq. (2.3) is called the Onsager coefficient. Since the system and affinities are independent of time, J_1 and J_2 are the stationary currents. By considering the time reversal of the system [36, 37, 63], we find that $L_{ij}(\boldsymbol{B})$ in this system satisfies

$$L_{ij}(\boldsymbol{B}) = L_{ji}(-\boldsymbol{B}). \tag{2.4}$$

If the external magnetic field $\boldsymbol{B}$ does not exist, the system has time reversal symmetry and satisfy the Onsager's reciprocal relation given by

$$L_{ij} = L_{ji}. \tag{2.5}$$

However, when the time-reversal symmetry is broken because of $\boldsymbol{B}$, the Onsager's reciprocal relation in Eq. (2.5) does not satisfied, and $L_{ij}(\boldsymbol{B}) \neq L_{ji}(\boldsymbol{B})$ $(i \neq j)$ is allowed.

The entropy production rate in this system [36, 37, 64–66] is defined by

$$\dot{\Sigma} \equiv J_1X_1 + J_2X_2 \geq 0, \tag{2.6}$$

where we used the second law of the thermodynamics at the last inequality. Then, by using Eq. (2.3), we obtain

$$\dot{\Sigma} = L_{11}\left(X_1 + \frac{L_{12}+L_{21}}{2L_{11}}X_2\right) + \frac{4L_{11}L_{22}-(L_{12}+L_{21})^2}{4L_{11}}X_2^2 \geq 0. \tag{2.7}$$

Since the entropy production rate should be positive in any value of X_1, X_2, and $\boldsymbol{B}$, we obtain the restriction for the Onsager coefficients as

$$L_{11} \geq 0, \quad L_{22} \geq 0, \quad L_{11}L_{22} - \frac{1}{4}(L_{12}+L_{21})^2 \geq 0. \tag{2.8}$$

2.1.2 Efficiency and power in the linear irreversible heat engine

We consider the heat engine in the linear irreversible thermodynamics. Because of $T_L > T_R$, the heat current J_2 should flow from left to right, and we can interpret it as the supplied heat per unit time. Thus, we consider that $J_2 > 0$ is satisfied. Moreover, in the above model, we assume that $\mu_L < \mu_R$ is satisfied. When the electronic current flows from left bath to right bath, the electron moves against the gradient of the chemical potential. Then, we can interpret that the electron is done the positive work. Thus, we can define power, which is the work per unit time, as

$$P \equiv J_1 \Delta\mu = -J_1 T X_1. \tag{2.9}$$

Since the supplied heat and work per unit time are defined and independent of the time, we can define the efficiency of this steady-state heat engine as

$$\eta \equiv \frac{P}{J_2} = -\frac{J_1 T X_1}{J_2}. \tag{2.10}$$

From Eq. (2.2), the Carnot efficiency, which is the upper bound of the efficiency in Eq. (2.10), is defined by

$$\eta_{\mathrm{C}} \equiv -\frac{\Delta T}{T} = T X_2. \tag{2.11}$$

Then, using Eqs. (2.3), (2.7), (2.10), and (2.11), we can obtain

$$\eta_{\mathrm{C}} - \eta = \frac{T}{J_2}\dot{\Sigma} \geq 0. \tag{2.12}$$

From Eq. (2.12), the efficiency approaches the Carnot efficiency when the entropy production rate $\dot{\Sigma}$ in Eq. (2.7) vanishes. Then, the Onsager coefficients should satisfy

$$X_1 = -\frac{L_{12}+L_{21}}{2L_{11}}X_2, \quad L_{11}L_{22} = \frac{1}{4}(L_{12}+L_{21})^2. \tag{2.13}$$

Thus, J_1 is given by

$$J_1 = L_{11}X_1 + L_{12}X_2 = \frac{L_{12}-L_{21}}{2}X_2. \tag{2.14}$$

Using Eqs. (2.9) and (2.14), we can obtain

$$P = \frac{T}{4}\frac{L_{12}^2 - L_{21}^2}{L_{11}}X_2^2. \tag{2.15}$$

When the system has time reversal symmetry, Eq. (2.5) is satisfied. Then, from $L_{12} = L_{21}$, the power in Eq. (2.15) vanishes. On the other hand, when the time reversal symmetry is broken because of the external magnetic field, the finite-power is allowed even when the entropy production vanishes. Thus, the compatibility of the Carnot efficiency and finite power is achievable in the system close to the equilibrium with the broken time-reversal symmetry.

We showed the compatibility of the Carnot efficiency and finite power in the steady-state irreversible heat engine. This research triggered a lot of study on whether the Carnot efficiency and finite power are compatible. Although many efforts have been conducted to show the compatibility in various models of the steady-state irreversible heat engines where the Onsager coefficients can be obtained, no model can realize the compatibility of the Carnot efficiency and finite power [8–13]. In recent years, since the trade-off relation between the efficiency and power is proved in general heat engines described by the Markov process, the compatibility may be impossible. In Chap. 6 and 7, however, we reconsider the compatibility of the Carnot efficiency and finite power based on the trade-off relation and show the compatibility in the Brownian Carnot cycle. To construct the Brownian Carnot cycle, we introduce the stochastic process and review the trade-off relation.

Chapter 3

Stochastic process

Stochastic thermodynamics is a framework connecting the stochastic process and thermodynamics and is used to describe the thermodynamics of the fluctuating system such as a Brownian particle [28]. The Brownian Carnot cycle is one of the models of heat engines within the stochastic thermodynamics, where the Brownian particle is used as a working substance. Since the position and velocity of the Brownian particle fluctuate, we have to use the stochastic process to describe the Brownian motion. In this chapter, we first introduce the stochastic process and Markov process. We show that all the Markov processes are described by the master equation. We introduce the Wiener process characterizing the mathematical aspect of the Brownian motion and show that the Langevin equation can be used to describe the motion. Moreover, we show that the Fokker-Planck equation can be used to describe the evolution of the probability distribution of the Brownian particle. Finally, we consider the equivalence between the Langevin and Fokker-Planck equations.

3.1 Stochastic process

We consider a fluctuating system and introduce the stochastic process to describe it [19,28, 39,67–74]. In the fluctuating system, we assume that the state of the system is described by N stochastic variables $\boldsymbol{X} \equiv (X_1, X_2, \cdots, X_N)$ depending on time. By measuring the system at $t_k \equiv k\Delta t$ $(k = 0, 1, 2, \cdots, n)$, where Δt is the duration of observation of the system, we can derive the results as

$$\boldsymbol{X}(t_0),\ \boldsymbol{X}(t_1),\ \cdots,\ \boldsymbol{X}(t_n). \tag{3.1}$$

Since these results fluctuate, we can use the stochastic process to describe them. We introduce a joint probability distribution of Eq. (3.1) as

$$p(\boldsymbol{X}(t_k), t_k; \cdots; \boldsymbol{X}(t_1), t_1; \boldsymbol{X}(t_0), t_0). \tag{3.2}$$

In general, $\boldsymbol{X}(t_{k+1})$ depends on $\boldsymbol{X}(t_0), \boldsymbol{X}(t_1), \cdots, \boldsymbol{X}(t_k)$. Thus, we can define the conditional probability distribution of $\boldsymbol{X}(t_{k+1})$ under measured $\boldsymbol{X}(t_0), \cdots, \boldsymbol{X}(t_k)$ as

$$W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k; \cdots; \boldsymbol{X}(t_0), t_0) \equiv \frac{p(\boldsymbol{X}(t_{k+1}), t_{k+1}; \boldsymbol{X}(t_k), t_k; \cdots; \boldsymbol{X}(t_0), t_0)}{p(\boldsymbol{X}(t_k), t_k; \cdots; \boldsymbol{X}(t_0), t_0)}. \tag{3.3}$$

We can often expect that $\boldsymbol{X}(t_{k+1})$ depends only on $\boldsymbol{X}(t_k)$. Then, the conditional probability distribution in Eq. (3.3) satisfies

$$W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k; \cdots ; \boldsymbol{X}(t_0), t_0) = W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k). \tag{3.4}$$

When the conditional probability distribution of the stochastic process satisfies Eq. (3.4), the process is called the Markov process. $W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k)$ is called the transition probability from $(\boldsymbol{X}(t_k), t_k)$ to $(\boldsymbol{X}(t_{k+1}), t_{k+1})$. From Eqs. (3.3) and (3.4), the probability distribution satisfies the Chapman-Kolmogorov equation:

$$p(\boldsymbol{X}(t_{k+1}), t_{k+1}) = \int d\boldsymbol{X}(t_k) W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k) p(\boldsymbol{X}(t_k), t_k), \tag{3.5}$$

where we defined $d\boldsymbol{X}(t_k) \equiv dX_1(t_k) \cdots dX_N(t_k)$. Note that unless otherwise specified, the integration interval is all the regions that variables of integration can take. From Eq. (3.5), when initial probability distribution $p(\boldsymbol{X}(t_0), t_0)$ is given, $p(\boldsymbol{X}(t_{k+1}), t_{k+1})$ is determined by only the transition probability $W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_0), t_0)$. Moreover, in the limit of $\Delta t \to 0$, we obtain

$$W(\boldsymbol{X}(t_{k+1}), t_k|\boldsymbol{X}(t_k), t_k) = \delta(\boldsymbol{X}(t_{k+1}) - \boldsymbol{X}(t_k)). \tag{3.6}$$

From Eqs. (3.3) and (3.4), we also find that $W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k)$ satisfies

$$1 = \int d\boldsymbol{X}(t_{k+1}) W(\boldsymbol{X}(t_{k+1}), t_{k+1}|\boldsymbol{X}(t_k), t_k). \tag{3.7}$$

3.2 Master equation

We obtain Master equation describing all the Markov processes [35,68,69]. Using $W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t)$, we can define the transition rate at t, which is the transition probability from $\boldsymbol{X}$ to $\boldsymbol{X}'$ $(\boldsymbol{X} \neq \boldsymbol{X}')$ per unit time, as

$$w_{\boldsymbol{X}\to\boldsymbol{X}'}(t) \equiv \lim_{\Delta t\to 0} \frac{W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t)}{\Delta t}. \tag{3.8}$$

Using $w_{\boldsymbol{X}\to\boldsymbol{X}'}(t)$, we can also define the escape rate as

$$e_{\boldsymbol{X}}(t) \equiv \int d\boldsymbol{X}' \; w_{\boldsymbol{X}\to\boldsymbol{X}'}(t). \tag{3.9}$$

The multiplication of $e_{\boldsymbol{X}}(t)$ by small Δt gives the probability where the system escapes from the state $\boldsymbol{X}$ in Δt. Then, we can define the probability where the system remains $\boldsymbol{X}$ between t to $t + \Delta t$ as

$$1 - e_{\boldsymbol{X}}(t)\Delta t. \tag{3.10}$$

Using Eqs. (3.8) and (3.9), we can expect that $W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t)$ is expanded as

$$W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t) = (1 - e_{\boldsymbol{X}}(t)\Delta t)\delta(\boldsymbol{X}' - \boldsymbol{X}) + w_{\boldsymbol{X}\to\boldsymbol{X}'}(t)\Delta t + O((\Delta t)^2). \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.5), we obtain

$$p(\boldsymbol{X}', t+\Delta t) = (1 - e_{\boldsymbol{X}'}(t)\Delta t)p(\boldsymbol{X}', t) + \int d\boldsymbol{X} w_{\boldsymbol{X}\to\boldsymbol{X}'}(t)\Delta t p(\boldsymbol{X}, t). \tag{3.12}$$

Considering $\Delta t \to 0$, we obtain the master equation given by

$$\frac{\partial p(\boldsymbol{X})}{\partial t} = \int d\boldsymbol{X}' \left[w_{\boldsymbol{X}' \to \boldsymbol{X}}(t) p(\boldsymbol{X}', t) - w_{\boldsymbol{X} \to \boldsymbol{X}'}(t) p(\boldsymbol{X}, t) \right], \tag{3.13}$$

using Eqs. (3.9) and (3.12). When we define

$$R_{\boldsymbol{X} \to \boldsymbol{X}'} \equiv \begin{cases} w_{\boldsymbol{X} \to \boldsymbol{X}'}(t) & (\boldsymbol{X} \neq \boldsymbol{X}') \\ -e_{\boldsymbol{X}'} & (\boldsymbol{X} = \boldsymbol{X}'), \end{cases} \tag{3.14}$$

we can rewrite Eq. (3.13) as

$$\frac{\partial p(\boldsymbol{X})}{\partial t} = \int d\boldsymbol{X}' R_{\boldsymbol{X}' \to \boldsymbol{X}}(t) p(\boldsymbol{X}', t). \tag{3.15}$$

Because of Eqs. (3.9) and (3.14), $R_{\boldsymbol{X}' \to \boldsymbol{X}}$ satisfies

$$\int d\boldsymbol{X} R_{\boldsymbol{X}' \to \boldsymbol{X}} = 0. \tag{3.16}$$

When $\boldsymbol{X}$ denotes the discretized state, the master equation is given by

$$\frac{d}{dt} p_{\boldsymbol{X}}(t) = \sum_{\boldsymbol{X}'} R_{\boldsymbol{X}' \to \boldsymbol{X}}(t) p_{\boldsymbol{X}'}(t), \tag{3.17}$$

$$\sum_{\boldsymbol{X}} R_{\boldsymbol{X}' \to \boldsymbol{X}}(t) = 0, \tag{3.18}$$

where $p_{\boldsymbol{X}}(t)$ is the probability where we find the system at discretized state $\boldsymbol{X}$. Since we do not assume the form of the transition probability for the derivation of the master equation, any Markov process can be described by the master equation.

3.3 Wiener process and Langevin equation

3.3.1 One-dimensional Wiener process

We consider the Wiener process, which is a mathematical model of the Brownian motion, and introduce the Langevin equation [28]. We first consider the one-dimensional Wiener process. We assume that $t \equiv n\Delta t = t_n$ in this section. The process has the following two properties:

(1). $B(s)$ is a stochastic process and continuous for any s.

(2). For $s = t_0, t_1, \cdots, t_k, \cdots, t_n = t, \cdots$, the increments $\{B(t_{k+1}) - B(t_k)\}_{k=0,1,\cdots}$ are independent of each other. Moreover, the probability distribution of the increments $B(t_{k+1}) - B(t_k)$ for all k is the Gaussian distribution:

$$W(B(t_{k+1}), t_{k+1} | B(t_k), t_k) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left\{ -\frac{|B(t_{k+1}) - B(t_k)|^2}{2\Delta t} \right\}. \tag{3.19}$$

By using the increments $B(t_{k+1}) - B(t_k)$, $B(t_n)$ is given by

$$B(t_n) = \sum_{k=0}^{n-1} (B(t_{k+1}) - B(t_k)) + B(0). \tag{3.20}$$

In the Wiener process, there is a famous lemma called Itô's lemma. To show that, we introduce the total variance as

$$\sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|^2. \tag{3.21}$$

Because of Eq. (3.19), the average of the quantity in Eq. (3.21) satisfies

$$\sum_{k=0}^{n-1} \left\langle |B(t_{k+1}) - B(t_k)|^2 \right\rangle = \sum_{k=0}^{n-1} \Delta t = t_n. \tag{3.22}$$

Moreover, when we use $t = n\Delta t = t_n$, we can obtain

$$\left\langle \left[\sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|^2 - t \right]^2 \right\rangle = \frac{2t_n^2}{n}. \tag{3.23}$$

In the limit of $\Delta t \to 0$ and $n \to \infty$ with fixed $t = n\Delta t = t_n$, since the right-hand side of Eq. (3.23) vanishes, the quantity in Eq. (3.21) satisfies

$$\lim_{\Delta t \to 0, n \to \infty} \sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|^2 = t. \tag{3.24}$$

with probability 1. In this chapter, we rewrite $\lim_{\Delta t \to 0, n \to \infty}$ as $\lim_{\Delta t \to 0}$ for simplicity. For infinitesimal increment of time dt, the infinitesimal increment of B can be defined as

$$dB(t) \equiv B(t + dt) - B(t) \equiv \lim_{\Delta t \to 0} (B(t + \Delta t) - B(t)). \tag{3.25}$$

Because of Eq. (3.24), $dB(t)$ has the order of $\sqrt{dt}$, and we obtain

$$(dB(t))^2 = dt, \quad dB(t)dt = 0, \quad (dt)^2 = 0. \tag{3.26}$$

This property of the infinitesimal quantities is called the Itô's lemma.

We introduce the function $f(B(t))$ and its integral with respect to B. We first consider the case of $f(B(t)) = B(t) - B(0)$ and calculate two quantities:

$$I_1 \equiv \sum_{l=0}^{n-1} \left[\sum_{k=0}^{l-1} (B(t_{k+1}) - B(t_k)) \right] (B(t_l) - B(t_{l-1})), \tag{3.27}$$

$$I_2 \equiv \sum_{l=0}^{n-1} \left[\sum_{k=0}^{l-1} (B(t_{k+1}) - B(t_k)) \right] (B(t_{l+1}) - B(t_l)). \tag{3.28}$$

If B is the bounded variation, $f(B)$ is the Riemann integrable, and I_1 and I_2 become same in the limit of $\Delta t \to 0$ and $n \to 0$ with keeping $t = n\Delta t$ constant. When B is the Wiener process, however, I_1 and I_2 satisfy

$$I_2 - I_1 = \sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|^2 + O(\Delta t) = t + O(\Delta t). \tag{3.29}$$

Even in the limit of $\Delta t \to 0$, I_1 and I_2 do not become same. Thus, we have to reconsider the integral of B. We introduce two important definition of integral [70, 71]. The first definition of integral is called Itô-type stochastic integral (or simply, Itô integral) given by

$$\int_{s=0}^{s=t} f(B) \cdot dB(s) \equiv \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f(B(t_k))(B(t_{k+1}) - B(t_k)). \tag{3.30}$$

Note that when we use the product " $\cdot$ " called the Itô-type product in the left-hand side of Eq. (3.30), the integral is calculated by Eq. (3.30). For example, we consider the Itô integral of $f(B(s)) = B(s) - B(0)$ as

$$\begin{aligned}
&\int_{s=0}^{s=t} (B(s) - B(0)) \cdot dB(s) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} [B(t_k) - B(0)] (B(t_{k+1}) - B(t_k)) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[\frac{B(t_k) + B(t_{k+1})}{2} + \frac{B(t_k) - B(t_{k+1})}{2} - B(0) \right] \times (B(t_{k+1}) - B(t_k)) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[\frac{B(t_{k+1})^2 - B(t_k)^2}{2} - \frac{|B(t_{k+1}) - B(t_k)|^2}{2} - B(0)(B(t_{k+1}) - B(t_k)) \right] \\
&= \frac{(B(t) - B(0))^2}{2} - \frac{t}{2},
\end{aligned} \tag{3.31}$$

using $t = n\Delta t$. The second definition of integral called Stratonovich-type stochastic integral (or simply, Stratonovich integral), which is very often used in the stochastic thermodynamics, is given by

$$\int_{s=0}^{s=t} f(B) \circ dB(s) \equiv \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f\left(B\left(\frac{t_{k+1}}{2} + \frac{t_k}{2} \right) \right) (B(t_{k+1}) - B(t_k)). \tag{3.32}$$

When we use the product " $\circ$ " called the Stratonovich-type product, the integral is calculated by Eq. (3.32). We can use $[f(B(t_{k+1})) + f(B(t_k))]/2$ instead of $f(B(t_{k+1}/2 + t_k/2))$ to calculate the integral. For example, the integral of $f(B(s)) = B(s) - B(0)$ is given by

$$\begin{aligned}
&\int_{s=0}^{s=t} (B(s) - B(0)) \circ dB(s) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[\frac{B(t_{k+1}) + B(t_k)}{2} - B(0) \right] (B(t_{k+1}) - B(t_k)) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[\frac{B(t_{k+1})^2 - B(t_k)^2}{2} - B(0)(B(t_{k+1}) - B(t_k)) \right] \\
&= \frac{(B(t) - B(0))^2}{2}.
\end{aligned} \tag{3.33}$$

More generally, we can define the integral for the stochastic process as

$$\int_{s=0}^{s=t} f(B) \times_h dB(s) \equiv \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f(B(ht_{k+1} + (1-h)t_k))(B(t_{k+1}) - B(t_k)), \tag{3.34}$$

where h $(0 \leq h \leq 1)$ is a constant. When $h = 0$ $(h = 1/2)$, $\times_h$ becomes $\cdot$ $(\circ)$ and used for the Itô (Stratonovich) integral. Since we can regard $f(ht_{k+1} + (1-h)t_k)$ as $hf(t_{k+1}) + (1-h)f(t_k)$, the integral of $f(B(s)) = B(s) - B(0)$ is given by

$$
\begin{aligned}
&\int_{s=0}^{s=t} (B(s) - B(0)) \times_h dB(s) \\
=& \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[(hB(t_{k+1}) + (1-h)B(t_k)) - B(0) \right] (B(t_{k+1}) - B(t_k)) \\
=& \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[\frac{B(t_{k+1})^2 - B(t_k)^2}{2} - \frac{(1-2h)(B(t_{k+1}) - B(t_k))}{2} - B(0)(B(t_{k+1}) - B(t_k)) \right] \\
=& \frac{(B(t) - B(0))^2}{2} - \frac{(1-2h)t}{2}
\end{aligned}
\tag{3.35}
$$

When we set $h = 0$ or $h = 1/2$, we find that Eq. (3.35) becomes the same as Eq. (3.31) or Eq. (3.33), respectively.

Using Eq. (3.25), we define the quantity $\xi(t)$ as

$$
\xi(t) \equiv \frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{B(t + \Delta t) - B(t)}{\Delta t}. \tag{3.36}
$$

From Eqs. (3.26) and (3.36), we obtain

$$
\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'). \tag{3.37}
$$

Then, we introduce the stochastic process $X(t)$ generated by the Wiener process $B(t)$ from the stochastic differential equation given by

$$
dX(t) = a(X(t), t) + b(X(t), t) \cdot dB(t), \tag{3.38}
$$

where we define $dX(t) \equiv X(t + dt) - X(t) = \lim_{\Delta t \to 0}[X(t + \Delta t) - X(t)]$. We first consider the case of the Itô-type product used at the product of $b(x, t)$ and $dB(t)$ in Eq. (3.38) although we can use other products such as the Stratonovich or more general types. Using Eq. (3.36), we obtain

$$
\dot{X}(t) = a(X(t), t) + b(X(t), t) \cdot \xi, \tag{3.39}
$$

where we defined $\dot{X}(t) \equiv \lim_{\Delta t \to 0}[X(t + \Delta t) - X(t)]/\Delta t$. In the physics, Eq. (3.39) is called the Langevin equation. Because of $\langle dX(t) \rangle = a(X(t), t)dt$, $\langle dX(t)^2 \rangle = b(X(t), t)dt$, and Itô's lemma, the quantity $df(X(t)) \equiv f(X(t + dt)) - f(X(t))$ is given by

$$
\begin{aligned}
df(X(t)) =& \partial_X f(X(t)) dX(t) + \frac{1}{2} \partial_X^2 f(X(t)) (dX(t))^2 \\
=& \left[a(X(t), t) + \frac{b(X(t), t)^2 [\partial_X^2 f(X(t))]}{2} \right] dt + b(X(t), t) [\partial_X f(X(t))] \cdot dB(t),
\end{aligned}
\tag{3.40}
$$

where we define $\partial_r f(r) \equiv df(r)/dr$.

Instead of Eq. (3.38), we can introduce another stochastic differential equation with the Stratonovich-type product $\circ$ as

$$
dX(t) = a(X(t), t) + b(X(t), t) \circ dB(t). \tag{3.41}
$$

We here change the Stratonovich-type product in Eq. (3.38) to the Itô type. From the definition of the Stratonovich integral in Eq. (3.32), we obtain

$$
\begin{aligned}
f(B(s),s)\circ dB(s) =& \frac{1}{2}f(B(s),s)\cdot dB(s)+\frac{1}{2}f(B+dB,s+ds)\cdot dB(s)\\
=& f(B,s)\cdot dB(s)+\frac{1}{2}\frac{\partial f(B,s)}{\partial B}\cdot dB(s).
\end{aligned}
\tag{3.42}
$$

Then, we obtain

$$
dX(t)=\left[a(X(t),t)+\frac{1}{2}\frac{\partial b(X(t),t)}{\partial X(t)}\right]dt+b(X(t),t)\cdot dB(t), \tag{3.43}
$$

using Itô's lemma and Eq. (3.40). Moreover, we can introduce another stochastic differential equation with the general product using $\times_h$ defined in Eq. (3.34) as

$$
dX(t)=a(X(t),t)+b(X(t),t)\times_h dB(t). \tag{3.44}
$$

Similar to the change from the Stratonovich-type product to the Itô one in Eq. (3.42), we can change the stochastic differential equation with the general product $\times_h$ to the Itô one and obtain

$$
f(B(s),s)\times_h dB(s) = f(B,s)\cdot dB(s)+h\frac{\partial f(B,s)}{\partial B}\cdot dB(s). \tag{3.45}
$$

Thus, Eq. (3.44) can be rewritten as

$$
dX(t)=\left[a(X(t),t)+h\frac{\partial b(X(t),t)}{\partial X(t)}\right]dt+b(X(t),t)\cdot dB(t). \tag{3.46}
$$

3.3.2 Multidimensional Wiener process

We can extend the one dimensional Wiener process $B(t)$ to the N dimensional one $\boldsymbol{B}=(B_1,\cdots,B_N)$. The process has the following two properties:

(1). $B_i(s)$ $(i=1,\cdots,N)$ is the stochastic process and continuous for any s.

(2). For $s=t_0,t_1,\cdots,t_k,\cdots,t_n=t,\cdots$, we consider the increments $\{B_i(t_{k+1})-B_i(t_k)\}_{k=0,1,\cdots}$ $(i=1,2,\cdots,N)$. The increments of B_i and B_j $(j\neq i)$ are independent. Moreover, the increments $B_i(t_{k+1})-B_i(t_k)$ and $B_i(t_{k'+1})-B_i(t_{k'})$ are also independent when $k\neq k'$. Then, the increments depend on the Gaussian distribution with mean 0 and variance Δt, and the transition probability from $\boldsymbol{B}(t_k)$ to $\boldsymbol{B}(t_{k+1})$ is given by

$$
W(\boldsymbol{B}(t_{k+1}),t_{k+1}|\boldsymbol{B}(t_k),t_k)=\frac{1}{\sqrt{(2\pi\Delta t)^N}}\exp\left\{-\frac{|\boldsymbol{B}(t_{k+1})-\boldsymbol{B}(t_k)|^2}{2\Delta t}\right\}. \tag{3.47}
$$

Similar to the one-dimensional case, for the infinitesimal time increment dt, we define the infinitesimal increment of $\boldsymbol{B}$ as

$$
d\boldsymbol{B}(t)\equiv\boldsymbol{B}(t+dt)-\boldsymbol{B}(t)\equiv\lim_{\Delta t\to 0}[\boldsymbol{B}(t+\Delta t)-\boldsymbol{B}(t)]. \tag{3.48}
$$

We can also derive the Itô's lemma in N-dimensional Wiener process as

$$
dB_i(t)dB_j(t')=\delta_{ij}\delta(t-t')dt,\quad dB_i dt=0,\quad (dt)^2=0. \tag{3.49}
$$

Using Eq. (3.36), we define the quantity $\boldsymbol{\xi}(t) = (\xi_1(t), \cdots, \xi_N(t))$ as

$$\boldsymbol{\xi}(t) \equiv \frac{d\boldsymbol{B}(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{\boldsymbol{B}(t+\Delta t) - \boldsymbol{B}(t)}{\Delta t}. \tag{3.50}$$

From Eq. (3.49) and (3.50), we obtain

$$\langle \xi_i(t) \rangle = 0, \quad , \langle \xi_i(t)\xi_j(s) \rangle = \delta_{ij}\delta(t-s). \tag{3.51}$$

Similar to the one-dimensional Wiener process, we can define the Itô integral, Stratonovich integral, and integral with $\times_h$ as

$$\int_{s=0}^{s=t} f(\boldsymbol{B}) \cdot dB_i(s) \equiv \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\boldsymbol{B}(t_k), t_k)(B_i(t_{k+1}) - B_i(t_k)), \tag{3.52}$$

$$\int_{s=0}^{s=t} f(\boldsymbol{B}) \circ dB_i(s) \equiv \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\boldsymbol{B}(t_{k+1}/2 + t_k/2))(B_i(t_{k+1}) - B_i(t_k)), \tag{3.53}$$

$$\int_{s=0}^{s=t} f(\boldsymbol{B}) \times_h dB_i(s) \equiv \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\boldsymbol{B}(ht_{k+1} + (1-h)t_k))(B_i(t_{k+1}) - B_i(t_k)). \tag{3.54}$$

By using the Itô-type product, we can introduce the stochastic differential equation

$$dX_i(t) = a_i(\boldsymbol{X}, t)dt + \sum_{j=1}^{N} b_{ij}(\boldsymbol{X}, t) \cdot dB_j(t) \quad (1 \le i \le N). \tag{3.55}$$

Then, we can derive the N-dimensional Langevin equation as

$$\dot{X}_i = a_i(\boldsymbol{X}, t) + \sum_{j=1}^{N} b_{ij}(\boldsymbol{X}, t) \cdot \xi_j(t) \quad (1 \le i \le N). \tag{3.56}$$

Similar to the one-dimensional case, we consider the quantity $f(\boldsymbol{X})$. Because of Eqs. (3.49) and (3.55), we obtain [75–77]

$$\begin{aligned} df(\boldsymbol{X}(t)) &= \sum_{i=1}^{N} [\partial_{X_i} f(\boldsymbol{X}(t))]dX_i + \sum_{i,j=1}^{N} \frac{1}{2}[\partial_{X_i}\partial_{X_j} f(\boldsymbol{X}(t))]dX_i dX_j \\ &= \sum_{i=1}^{N} \left(a_i(\boldsymbol{X}, t)[\partial_{X_i} f(\boldsymbol{X}(t))] + \sum_{j=1}^{N} \frac{G_{ij}(\boldsymbol{X}, t)^2 [\partial_{X_i}^2 f(\boldsymbol{X}, t)]}{2} \right) dt \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(\boldsymbol{X}, t)[\partial_{X_i} f(\boldsymbol{X}, t)] \cdot dB_j, \end{aligned} \tag{3.57}$$

where we define the matrix $b \equiv \{b_{ij}\}_{1 \le i,j \le N}$ and

$$G_{ij} \equiv \sum_{k=1}^{N} b_{ik} b_{jk} = (bb^\top)_{ij}. \tag{3.58}$$

Similar to the one-dimensional case, we can introduce the stochastic differential equation with $\times_h$ as

$$dX_i(t) = a_i(\boldsymbol{X},t)dt + \sum_{j=1}^{N} b_{ij}(\boldsymbol{X},t) \times_h dB_j(t) \quad (1 \le i \le N). \tag{3.59}$$

By the Taylor expansion and Itô's lemma in Eq. (3.49), we can obtain

$$f(\boldsymbol{B}(s)) \times_h dB_i(s) = f(\boldsymbol{B}(s)) \cdot dB_i(s) + h \left(\sum_{j=1}^{N} \frac{\partial f(\boldsymbol{B}(s))}{\partial B_j} \right) \cdot dB_i. \tag{3.60}$$

Thus, we can rewrite Eq. (3.59) by using the Itô product as

$$dX_i(t) = \left[a_i(\boldsymbol{X},t) + h \left(\sum_{j=1}^{N} \frac{\partial f(\boldsymbol{X})}{\partial X_j} \right) \right] dt + \sum_{j=1}^{N} b_{ij}(\boldsymbol{X},t) \cdot dB_j(t) \quad (1 \le i \le N). \tag{3.61}$$

3.4 Fokker-Planck equation

We show that we can obtain the Fokker-Planck which is a partial differential equation describing the dynamics of the probability distribution $p(\boldsymbol{X},t)$ when the stochastic process is the Wiener process [75]. By using the delta function $\delta(\boldsymbol{X}) \equiv \delta(X_1)\cdots\delta(X_N)$, we obtain

$$W(\boldsymbol{X}',t+\Delta t|\boldsymbol{X},t) = \int d\boldsymbol{Y} \delta(\boldsymbol{Y}-\boldsymbol{X}') W(\boldsymbol{Y},t+\Delta t|\boldsymbol{X},t). \tag{3.62}$$

Using the Taylor expansion of δ function, [1] we obtain

$$\begin{aligned}
&\delta(\boldsymbol{Y}-\boldsymbol{X}') = \delta(\boldsymbol{Y}-\boldsymbol{X}+\boldsymbol{X}-\boldsymbol{X}') \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(Y_{j_1}-X_{j_1})\cdots(Y_{j_n}-X_{j_n}) \frac{\partial^n}{\partial X_{j_1}\cdots\partial X_{j_n}} \delta(\boldsymbol{X}-\boldsymbol{X}') \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial X'_{j_1}\cdots\partial X'_{j_n}} (Y_{j_1}-X_{j_1})\cdots(Y_{j_n}-X_{j_n}) \delta(\boldsymbol{X}-\boldsymbol{X}').
\end{aligned} \tag{3.63}$$

By substituting Eq. (3.63) into Eq. (3.62), we obtain

$$\begin{aligned}
&W(\boldsymbol{X}',t+\Delta t|\boldsymbol{X},t) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial X'_{j_1}\cdots\partial X'_{j_n}} \int d\boldsymbol{Y} (Y_{j_1}-X_{j_1})\cdots(Y_{j_n}-X_{j_n}) W(\boldsymbol{Y},t+\Delta t|\boldsymbol{X},t) \delta(\boldsymbol{X}-\boldsymbol{X}') \\
&= \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial X'_{j_1}\cdots\partial X'_{j_n}} M_{j_1,\cdots,j_n}(\boldsymbol{X},t,\Delta t) \right] \delta(\boldsymbol{X}'-\boldsymbol{X}),
\end{aligned} \tag{3.64}$$

where we define

$$M_{j_1,\cdots,j_n}(\boldsymbol{X},t,\Delta t) \equiv \int d\boldsymbol{Y} (Y_{j_1}-X_{j_1})\cdots(Y_{j_n}-X_{j_n}) W(\boldsymbol{Y},t+\Delta t|\boldsymbol{X},t). \tag{3.65}$$

[1] Considering the Fourier transformation, we can expand the δ function.

We assume that all $M_{j_1,\cdots,j_n}(\boldsymbol{X},t,\Delta t)$ $(n \geq 1)$ can be expanded into a Taylor series with respect to Δt:

$$\frac{1}{n!}M_{j_1,\cdots,j_n}(\boldsymbol{X},t,\Delta t) = D_{j_1,\cdots,j_n}(\boldsymbol{X},t)\Delta t + O((\Delta t)^2). \tag{3.66}$$

Note that the term of $(\Delta t)^0$ vanishes because of Eq. (3.6). Using Eqs. (3.5), (3.64), and (3.66), we obtain

$$p(\boldsymbol{X}',t+\Delta t) = p(\boldsymbol{X}',t) + \sum_{n=1}^{\infty}(-1)^n \frac{\partial^n}{\partial X'_{j_1}\cdots\partial X'_{j_n}} D_{j_1,\cdots,j_n}(\boldsymbol{X}',t)p(\boldsymbol{X}',t)\Delta t + O((\Delta t)^2). \tag{3.67}$$

This is called the Kramers-Moyal expansion for N variables [78]. In the vanishing limit of Δt in Eq. (3.67), we derive the equation describing the evolution of $p(\boldsymbol{X},t)$ as

$$\frac{\partial p(\boldsymbol{X},t)}{\partial t} = \sum_{n=1}^{\infty}(-1)^n \frac{\partial^n}{\partial X_{j_1}\cdots\partial X_{j_n}} D_{j_1,\cdots,j_n}(\boldsymbol{X},t)p(\boldsymbol{X},t). \tag{3.68}$$

In the Wiener process, we can show that $M_{j_1,\cdots,j_n}$ $(n \geq 3)$ satisfy [2]

$$M_{j_1,\cdots,j_n}(\boldsymbol{X},t,\Delta t) = 0 \quad (n \geq 3). \tag{3.69}$$

This is satisfied in any value of Δt. Then, since $D_{j_1,\cdots,j_n}$ also vanishes when n satisfies $n \geq 3$, Eq. (3.68) can be rewritten as

$$\frac{\partial p(\boldsymbol{X},t)}{\partial t} = -\sum_{i,j}\left(\frac{\partial}{\partial X_i}D_i(\boldsymbol{X},t) - \frac{\partial^2}{\partial X_i \partial X_j}D_{ij}(\boldsymbol{X},t)\right)p(\boldsymbol{X},t), \tag{3.70}$$

where i and j in the sum in Eq. (3.70) go from 1 to N. Eq. (3.70) is called the Fokker-Planck equation. We can define the probability current as

$$j_i(\boldsymbol{X},t) \equiv \sum_j \left(D_i(\boldsymbol{X},t) - \frac{\partial}{\partial X_j}D_{ij}(\boldsymbol{X},t)\right)p(\boldsymbol{X},t). \tag{3.71}$$

Then, since we can rewrite Eq. (3.70) as

$$\frac{\partial p(\boldsymbol{X},t)}{\partial t} + \nabla\cdot\boldsymbol{j} = 0 \quad \boldsymbol{j} = (j_1,\cdots,j_N), \tag{3.72}$$

the Fokker-Planck equation is interpreted as the continuous equation for the probability distribution.

3.5 Equivalence of Langevin equation and Fokker-Planck equation

3.5.1 From Langevin equation to Fokker-Planck equation

We consider the N-dimensional stochastic process $\boldsymbol{X}(t) = (X_1(t),\cdots,X_N(t))$ and obtain the Fokker-Planck equation from the Langevin equation. The N dimensional Langevin equation with Itô-type product is given by Eq. (3.56). Because of $df(\boldsymbol{X}(t)) = f(\boldsymbol{X}(t+$

[2] Note that we may use the Pawula's theorem to show Eq. (3.69) [79].

$dt)) - f(\boldsymbol{X}(t))$, the function $f(\boldsymbol{X})$ satisfies

$$\langle f(\boldsymbol{X}) \rangle_{t+dt} = \langle f(\boldsymbol{X}) + df(\boldsymbol{X})) \rangle_t \,, \tag{3.73}$$

where $\langle \cdots \rangle_t$ is the statistical average at t. Using Eq. (3.57), we obtain

$$\begin{aligned} &\langle f(\boldsymbol{X}) \rangle_{t+dt} - \langle f(\boldsymbol{X}) \rangle_t \\ =&\sum_{i=1}^{N} \left\langle \left(a_i(\boldsymbol{X},t)[\partial_{X_i} f(\boldsymbol{X})] + \sum_{j=1}^{N} \frac{G_{ij}(\boldsymbol{X},t)[\partial_{X_i}\partial_{X_j} f(\boldsymbol{X})]}{2} \right) dt + \sum_{j=1}^{N} b_{ij}(\boldsymbol{X})[\partial_{X_i} f(\boldsymbol{X})] \cdot dB_j \right\rangle_t . \end{aligned} \tag{3.74}$$

Because of $\langle dB_i \rangle_t = 0$, we obtain

$$\frac{d}{dt} \langle f(\boldsymbol{X}) \rangle_t = \sum_{i=1}^{N} \left\langle a_i(\boldsymbol{X},t)[\partial_{X_i} f(\boldsymbol{X})] + \sum_{j=1}^{N} \frac{G_{ij}(\boldsymbol{X},t)[\partial_{X_i}\partial_{X_j} f(\boldsymbol{X})]}{2} \right\rangle_t . \tag{3.75}$$

The left-hand side of Eq. (3.75) can be rewritten as

$$\frac{d}{dt} \langle f(\boldsymbol{X}) \rangle_t = \frac{d}{dt} \int d\boldsymbol{X} \; f(\boldsymbol{X}) p(\boldsymbol{X},t) = \int d\boldsymbol{X} \; f(\boldsymbol{X}) \frac{\partial p(\boldsymbol{X},t)}{\partial t}. \tag{3.76}$$

We assume that the probability distribution vanishes at the boundary. Integrating the right-hand side of Eq. (3.75) by parts, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \left\langle a_i(\boldsymbol{X},t)[\partial_{X_i} f(\boldsymbol{X})] + \sum_{j=1}^{N} \frac{G_{ij}(\boldsymbol{X},t)[\partial_{X_i}\partial_{X_j} f(\boldsymbol{X})]}{2} \right\rangle_t \\ =&\sum_{i,j} \int d\boldsymbol{X} \; f(\boldsymbol{X}) \left[-\frac{\partial}{\partial X_i} a_i(\boldsymbol{X},t) p(\boldsymbol{X},t) + \frac{1}{2} \frac{\partial^2}{\partial X_i \partial X_j} G_{ij}(\boldsymbol{X},t) p(\boldsymbol{X},t) \right] . \end{aligned} \tag{3.77}$$

Since $f(\boldsymbol{X})$ is the arbitrary function of $\boldsymbol{X}$, from Eqs. (3.75), (3.76), and (3.77), we derive the partial differential equation of $p(\boldsymbol{X},t)$ as

$$\frac{\partial p(\boldsymbol{X},t)}{\partial t} = \sum_{i,j} \left[-\frac{\partial}{\partial X_i} a_i(\boldsymbol{X},t) + \frac{1}{2} \frac{\partial}{\partial X_i \partial X_j} G_{ij}(\boldsymbol{X},t) \right] p(\boldsymbol{X},t). \tag{3.78}$$

This is the same as the Fokker-Plank equation in Eq. (3.70), and we find that a_i and G_{ij} satisfies

$$a_i = D_i, \quad ,G_{ij} = 2D_{ij}, \tag{3.79}$$

comparing Eqs. (3.70) and (3.78).

3.5.2 From Fokker-Planck equation to Langevin equation

We show the derivation of the Langevin equation in Eq. (3.56) from the Fokker-Planck equation in Eq. (3.70). First, we express $W(\boldsymbol{X}', t+\Delta t | \boldsymbol{X}, t)$ by the Fokker-Planck equation.

When Δt is small, we obtain

$$\begin{aligned}
p(\boldsymbol{X}', t+\Delta t) &= p(\boldsymbol{X}', t) + \frac{\partial p(\boldsymbol{X}', t)}{\partial t}\Delta t + O((\Delta t)^2) \\
&= \left[1 - \sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}', t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}', t)\right)\Delta t\right] p(\boldsymbol{X}', t) + O((\Delta t)^2) \\
&\simeq \exp\left[-\sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}', t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}', t)\right)\Delta t\right] p(\boldsymbol{X}', t) \\
&= \int d\boldsymbol{X} \exp\left[-\sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}', t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}', t)\right)\Delta t\right] \delta(\boldsymbol{X}-\boldsymbol{X}')p(\boldsymbol{X}, t),
\end{aligned} \tag{3.80}$$

where we ignored the higher order terms of Δt and used Eq. (3.70). Since we can replace from $\boldsymbol{X}'$ to $\boldsymbol{X}'$ in D_i and D_{ij} in Eq. (3.80) by using the relation:

$$f(\boldsymbol{X})\delta(\boldsymbol{X}-\boldsymbol{X}') = f(\boldsymbol{X}')\delta(\boldsymbol{X}-\boldsymbol{X}'), \tag{3.81}$$

comparing Eqs. (3.5) and (3.80), we obtain

$$\begin{aligned}
W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t) &= \exp\left[-\sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}', t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}', t)\right)\Delta t\right] \delta(\boldsymbol{X}-\boldsymbol{X}') \\
&= \exp\left[-\sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}, t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}, t)\right)\Delta t\right] \delta(\boldsymbol{X}-\boldsymbol{X}').
\end{aligned} \tag{3.82}$$

By using the Fourier transformation, delta function is given by

$$\delta(\boldsymbol{X}-\boldsymbol{X}') = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} d\boldsymbol{k} \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{X}-\boldsymbol{X}')\right]. \tag{3.83}$$

From Eqs. (3.82) and (3.83), we obtain

$$\begin{aligned}
&W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t) \\
&= \exp\left[-\sum_{i,j}\left(\frac{\partial}{\partial X'_i}D_i(\boldsymbol{X}, t) - \frac{\partial^2}{\partial X'_i \partial X'_j}D_{ij}(\boldsymbol{X}, t)\right)\Delta t\right] \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} d\boldsymbol{k} \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{X}-\boldsymbol{X}')\right] \\
&= \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} d\boldsymbol{k} \exp\left[\sum_{i,j}\left(-ik_iD_i(\boldsymbol{X}, t)\Delta t - k_ik_jD_{ij}(\boldsymbol{X}, t)\Delta t + ik_i(X_i - X'_i)\right)\right] \\
&= \frac{1}{\sqrt{(4\pi\Delta t)^N \det\{D(\boldsymbol{X}, t)\}}} \\
&\quad \times \exp\left[-\sum_{i,j}\frac{[X'_i - X_i - D_i(\boldsymbol{X}, t)\Delta t]{D^{-1}}_{ij}(\boldsymbol{X}, t)[X'_j - X_j - D_j(\boldsymbol{X}, t)\Delta t]}{4\Delta t}\right],
\end{aligned} \tag{3.84}$$

where we define the vector $\boldsymbol{D}$ and matrix D as

$$\boldsymbol{D} \equiv \{D_i\}_{1\le i\le N} \qquad D \equiv \{D_{ij}\}_{1\le i,j\le N}. \tag{3.85}$$

From Eqs. (3.65) and (3.66), the matrix D satisfies

$$D_{ij} = D_{ji}. \tag{3.86}$$

Moreover, since D is a positive-semidefinite real symmetric matrix, it has the Cholesky decomposition form [80–82] given by

$$D(\boldsymbol{X}, t) = d(\boldsymbol{X}, t)d^{\top}(\boldsymbol{X}, t), \tag{3.87}$$

where d is a $N \times N$ real lower triangular matrix with positive diagonal elements. Therefore, we can rewrite $W(\boldsymbol{X}, t+\Delta t|\boldsymbol{X}', t)$ in Eq. (3.84) as

$$\begin{aligned} W(\boldsymbol{X}', t+\Delta t|\boldsymbol{X}, t) = & \frac{1}{\sqrt{(4\pi\Delta t)^N (\det\{d(\boldsymbol{X}, t)\})^2}} \\ & \times \exp\left[-\frac{|d^{-1}(\boldsymbol{X}, t)(\boldsymbol{X}' - \boldsymbol{X} - \boldsymbol{D}(\boldsymbol{X}, t)\Delta t)|^2}{4\Delta t}\right]. \end{aligned} \tag{3.88}$$

From Eq. (3.88), we can find that the process produced by the Fokker-Planck equation in Eq. (3.70) satisfies the Wiener process defined in Eq. (3.47). To derive of the Langevin equation from the transition rate in Eq. (3.88), we use the stochastic variable $\boldsymbol{B} = (B_1, \cdots, B_N)$ satisfying the transition probability in Eq. (3.47) as

$$W(\boldsymbol{B}', t+\Delta t|\boldsymbol{B}, t) = \frac{1}{\sqrt{(2\pi\Delta t)^N}} \exp\left\{-\frac{|\boldsymbol{B}' - \boldsymbol{B}|^2}{2\Delta t}\right\}, \tag{3.89}$$

where we defined $\boldsymbol{B} \equiv \boldsymbol{B}(t)$ and $\boldsymbol{B}' \equiv \boldsymbol{B}(t+\Delta t)$. When we transform the variables $\boldsymbol{B}$ and $\boldsymbol{B}'$ into $\boldsymbol{Y} \equiv \sqrt{2}d(\boldsymbol{X}, t)\boldsymbol{B}$ and $\boldsymbol{Y}' \equiv \sqrt{2}d(\boldsymbol{X}, t)\boldsymbol{B}'$, respectively, Eq. (3.89) can be rewritten as

$$W(\boldsymbol{Y}', t+\Delta t|\boldsymbol{Y}, t) = \frac{1}{\sqrt{(4\pi\Delta t)^N (\det d)^2}} \exp\left\{-\frac{|d^{-1}(\boldsymbol{Y}' - \boldsymbol{Y})|^2}{4\Delta t}\right\}. \tag{3.90}$$

When we regard $\boldsymbol{X}$, $\boldsymbol{B}$, and $\boldsymbol{Y}$ as the stochastic variables and compare Eqs. (3.88) and (3.90), we obtain

$$\begin{aligned} X_i(t+\Delta t) - X_i(t) - D_i(\boldsymbol{X}, t)\Delta t &= Y_i' - Y_i \\ &= \sum_{j=1}^{N} \sqrt{2} d_{ij}(\boldsymbol{X}, t)[B_j(t+\Delta t) - B_j(t)]. \end{aligned} \tag{3.91}$$

In the limit of $\Delta t \to dt$, we can rewrite Eq. (3.91) as

$$dX_i = D_i(\boldsymbol{X}, t)dt + \sum_{j=1}^{N} \sqrt{2} d_{ij}(\boldsymbol{X}, t) \cdot dB_j. \tag{3.92}$$

Note that we should use the Itô-type product in the right-hand side of Eq. (3.91) since $d_{ij}(\boldsymbol{X}, t)$ depends only on $\boldsymbol{X}(t)$ and t. This is consistent with the discussion in Chap 3.5.1.

By comparing Eqs. (3.55) and (3.92), we obtain

$$a_i = D_i, \quad b_{ij} = \sqrt{2} d_{ij} \tag{3.93}$$

From Eqs. (3.58), (3.87), and (3.93), we find that Eq. (3.79) is satisfied. Thus, we obtain the Langevin equation

$$\dot{X}_i = D_i(\boldsymbol{X}, t) + \sum_{j=1}^{N} \sqrt{2} d_{ij}(\boldsymbol{X}, t) \cdot \xi_j(t). \tag{3.94}$$

Note that $\boldsymbol{\xi}(t) \equiv (\xi_1(t), \cdots, \xi_N(t))$ satisfies

$$\begin{aligned} \langle \xi_i(t) \rangle =& 0, \\ \langle \xi_i(t) \xi_j(s) \rangle =& \delta_{ij} \delta(t-s). \end{aligned} \qquad (i, j = 1, 2, \cdots, N) \tag{3.95}$$

As we will show in Chap. 5, the Langevin equation can be regarded as the equation motion of the Brownian particle.

We showed that the master equation, Langevin equation, and Fokker-Planck equation are derived from the Chapman-Kolmogorov equation characterizing the Markov process. Especially, since the transition rate of the master equation does not depend on the form of the transition probability, the master equation can describe any Markov process. Thus, the results proved in the system described by the master equation hold for the system described by the Langevin equation or Fokker-Planck equation. Moreover, we showed that the Langevin equation and the Fokker-Planck equation are equivalent. Thus, we can choose which of them to use depending on the situation.

Chapter 4

Review: Trade-off relation between efficiency and power

4.1 Setting

We review Ref. [29] to show the trade-off relation between efficiency and power of heat engines described by the Markov process. We consider that the heat engine interacts with n heat baths with inverse temperature $\beta_1, \cdots, \beta_n$ and is operated by the agent. By using the temperature of the heat bath T_k $(k = 1, 2, \cdots, n)$, the inverse temperature satisfies $\beta_k = 1/T_k$. We assume that the heat baths are equilibrium at any time.

In the heat engine, N particles whose dynamics are determined by the classical mechanics are enclosed. We often use the word "system" to mean the heat engine and the other word "total system" to mean the heat bath and heat engine, respectively. The particles may interact each other, collide, and trapped by a potential. Let m_i, $\boldsymbol{x}_i$, and $\boldsymbol{v}_i$ $(i = 1, \cdots, N)$ denote the mass, position, and velocity of the i-th particle, respectively. Then, we can consider $\omega \equiv (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N; \boldsymbol{v}_1, \cdots, \boldsymbol{v}_N)$ as the point of the $6N$-dimensional phase space. We introduce a set of parameter $\lambda(t)$ characterizing the heat engine. When we consider a piston as an example of the heat engine, $\lambda(t)$ represents the volume of the piston and which of the heat bath interacting, etc at t. Then, we consider the energy of the system depending on parameter λ as

$$E^{\lambda}(\omega) \equiv \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_i|^2 + V^{\lambda}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N), \tag{4.1}$$

where $V^{\lambda}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N)$ is the potential and does not depend on the velocity of the particles.

For the derivation of the trade-off relation, we decompose the whole $6N$-dimensional phase space into small $6N$-dimensional parallelepiped. We assume that the length of one side of the parallelepiped is at most ε which is sufficiently small. We can use ω denoting the center of the small parallelepiped to distinguish the parallelepiped. Then, we introduce the probability $p_\omega(t)$ where the system is in the parallelepiped whose center is ω at time t. Using $p_\omega(t)$, we define the entropy $S(t)$ of the system [1, 28, 83] as

$$S(t) \equiv -\sum_{\omega} p_\omega(t) \ln p_\omega(t). \tag{4.2}$$

Since ε is sufficiently small, the energy defined in Eq. (4.1) of all ω in the same parallelepiped are almost same. Thus, we define the energy which the system is in the parallelepiped labeled by ω as

$$E_{\omega}^{\lambda} \equiv \sum_{j=1}^{N} \frac{1}{2} m_j |\boldsymbol{v}_j|^2 + V^{\lambda}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N). \tag{4.3}$$

Using the probability $p_{\omega}(t)$, we can define the internal energy as

$$E(t) \equiv \sum_{\omega} E_{\omega}^{\lambda} p_{\omega}(t). \tag{4.4}$$

We introduce the discretized master equation in Eq. (3.17) in Sec. 3.2 to describe the time evolution of $p_{\omega}(t)$ as

$$\frac{d}{dt} p_{\omega}(t) = \sum_{\omega'} R_{\omega' \to \omega}^{\lambda(t)} p_{\omega'}(t), \tag{4.5}$$

where $R_{\omega' \to \omega}^{\lambda(t)}$ is the transition rate and depends on $\lambda(t)$ and satisfies Eq. (3.18). Since the system interacts with the n heat baths and composed of N particles, $R_{\omega' \to \omega}^{\lambda(t)}$ can be written as

$$R_{\omega' \to \omega}^{\lambda(t)} = R_{\omega' \to \omega}^{0,\lambda(t)} + \sum_{j=1}^{N} \sum_{k=1}^{n} R_{\omega' \to \omega}^{j,k,\lambda(t)}, \tag{4.6}$$

where j and k are the index of the particle and heat bath, respectively. $R_{\omega' \to \omega}^{0,\lambda(t)} \Delta t$ is the transition probability from ω' to ω in the time interval Δt caused by the deterministic time evolution of the system, which is described by the classical mechanics. $R_{\omega' \to \omega}^{j,k,\lambda(t)} \Delta t$ is the transition probability from ω' to ω in the time interval Δt because of the interaction between the j-th particle and the k-th heat bath. When the system is isolated, in other words, any heat bath does not interact with the system, $R_{\omega' \to \omega}^{\lambda(t)}$ becomes $R_{\omega' \to \omega}^{0,\lambda(t)}$. Then, when $\lambda(t)$ does not change, $R_{\omega' \to \omega}^{0,\lambda(t)}$ does not depend on time, and the internal energy conserves and satisfy

$$\left. \frac{d}{dt} E(t) \right|_{\lambda(t) = \lambda_0 = \text{const.}} = \sum_{\omega, \omega'} E_{\omega}^{\lambda_0} R_{\omega' \to \omega}^{0,\lambda_0} p_{\omega'}(t) = 0. \tag{4.7}$$

Note that Eq. (4.7) is satisfied in any λ_0. Moreover, when the system is equilibrium and isolated, the probability distribution is expected to be uniform because of the principle of equal weight. Thus, we assume that $R_{\omega' \to \omega}^{0,\lambda(t)}$ satisfies

$$\sum_{\omega'} R_{\omega' \to \omega}^{0,\lambda(t)} = 0. \tag{4.8}$$

When the system connects with the heat bath with β_k and is equilibrium, the probability distribution becomes the Boltzmann distribution. Then, since it does not change in time, we can assume

$$\sum_{\omega'} R_{\omega' \to \omega}^{j,k,\lambda(t)} e^{-\beta_k E_{\omega'}^{\lambda}} = 0 \tag{4.9}$$

To define the heat and work, we consider the time derivative of Eq. (4.4):

$$\frac{d}{dt} E(t) = \sum_{\omega} \frac{d\lambda(t)}{dt} \frac{dE_{\omega}^{\lambda(t)}}{d\lambda(t)} p_{\omega}(t) + \sum_{\omega} E_{\omega}^{\lambda(t)} \frac{dp_{\omega}(t)}{dt}. \tag{4.10}$$

The first term of the right-hand side of Eq. (4.10) is interpreted as the work because the internal energy change is caused by the agent who changes $\lambda(t)$. Then, the second term of the right-hand side of Eq. (4.10) is interpreted as the heat current flowing from the heat baths to the system. Thus, the heat current is defined as

$$\begin{aligned}\dot{Q}(t) &\equiv \sum_{\omega} E_{\omega}^{\lambda(t)} \frac{dp_{\omega}(t)}{dt} \\ &= \sum_{\omega,\omega'} E_{\omega}^{\lambda(t)} R_{\omega'\to\omega}^{\lambda(t)} p_{\omega'}(t) \\ &= \sum_{j=1}^{N}\sum_{k=1}^{n}\sum_{\omega,\omega'} E_{\omega}^{\lambda(t)} R_{\omega'\to\omega}^{j,k,\lambda(t)} p_{\omega'}(t) \\ &= \sum_{k=1}^{n} \dot{Q}_k(t), \end{aligned} \tag{4.11}$$

where we used Eqs. (4.5), (4.6), and (4.7) and introduced $\dot{Q}_k$ as a heat current flowing from the k-th heat bath:

$$\begin{aligned}\dot{Q}_k(t) &\equiv \sum_{j=1}^{N} \dot{Q}_{j,k}(t), \\ \dot{Q}_{j,k}(t) &\equiv \sum_{\omega,\omega'} E_{\omega}^{\lambda(t)} R_{\omega'\to\omega}^{j,k,\lambda(t)} p_{\omega}(t). \end{aligned} \tag{4.12}$$

4.2 Inequality for heat current

We consider the entropy production of the total system and show the inequality for the heat current and the entropy production. When the heat flows, the entropy of the heat baths changes. Since the probability of the system changes simultaneously because of the flowing heat, the entropy of the system also changes. By using Eq. (4.2), the entropy change of the system is given by $\dot{S}(t)$. Then, considering the total entropy change rate of the system and heat baths, we can introduce the entropy production rate or the total system as

$$\dot{\Sigma}_{\mathrm{tot}}(t) \equiv \dot{S} - \sum_{k=1}^{n} \beta_k J_k(t), \tag{4.13}$$

where the first and second terms represent the entropy change of the system and heat baths, respectively. From Eqs. (4.2), (4.5), (4.6), and (4.11), we can rewrite Eq. (4.13) as

$$\begin{aligned}\dot{\Sigma}_{\mathrm{tot}} &= \sum_{j=1}^{N}\sum_{k=1}^{n}\sum_{\omega,\omega'} R_{\omega'\to\omega}^{j,k,\lambda(t)} p_{\omega'}(t) \left(-\log p_{\omega'}(t) - \beta_k E_{\omega}^{\lambda(t)}\right) \\ &= \sum_{j=1}^{N}\sum_{k=1}^{n} \dot{\Sigma}_{j,k}, \end{aligned} \tag{4.14}$$

where we defined

$$\dot{\Sigma}_{j,k} \equiv \sum_{\omega,\omega'} R_{\omega'\to\omega}^{j,k,\lambda(t)} p_{\omega'}(t) \left(-\log p_{\omega}(t) - \beta_k E_{\omega}^{\lambda(t)}\right). \tag{4.15}$$

For the derivation of the inequality for the heat current, we introduce the quantity

$$\tilde{R}^{j,k,\lambda}_{\omega\to\omega'} \equiv e^{\beta_k(E^\lambda_\omega - E^\lambda_{\omega'})} R^{j,k,\lambda(t)}_{\omega'\to\omega}. \tag{4.16}$$

Because of Eq. (4.9), $\tilde{R}^{j,k,\lambda}_{\omega\to\omega'}$ satisfies

$$\sum_{\omega'} \tilde{R}^{j,k,\lambda}_{\omega\to\omega'} = 0. \tag{4.17}$$

Using Eqs. (3.18) and (4.9), we obtain

$$\begin{aligned}
\sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t)(-\beta_k E^{\lambda(t)}_\omega) &= \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \left(-\beta_k E^{\lambda(t)}_{\omega'} + \log \frac{R^{j,k,\lambda(t)}_{\omega'\to\omega}}{\tilde{R}^{j,k,\lambda}_{\omega'\to\omega}} \right) \\
&= \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \left(\log \frac{R^{j,k,\lambda(t)}_{\omega'\to\omega}}{\tilde{R}^{j,k,\lambda}_{\omega'\to\omega}} \right).
\end{aligned} \tag{4.18}$$

Then, we can rewrite $\dot{\Sigma}_{j,k}$ in Eq. (4.15) as

$$\begin{aligned}
\dot{\Sigma}_{j,k} &= \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \left(-\log p_\omega(t) - \beta_k E^{\lambda(t)}_{\omega'} + \log \frac{R^{j,k,\lambda(t)}_{\omega'\to\omega}}{\tilde{R}^{j,k,\lambda}_{\omega'\to\omega}} \right) \\
&= \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \left(-\log p_\omega(t) + \log \frac{R^{j,k,\lambda(t)}_{\omega'\to\omega}}{\tilde{R}^{j,k,\lambda}_{\omega'\to\omega}} \right) + \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \log p_{\omega'}(t) \\
&= \sum_{\omega,\omega'} R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \log \frac{R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t)}{\tilde{R}^{j,k,\lambda}_{\omega'\to\omega} p_\omega(t)} \\
&= \sum_{\omega,\omega'} s(R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t), \tilde{R}^{j,k,\lambda}_{\omega'\to\omega} p_\omega(t))
\end{aligned} \tag{4.19}$$

using Eq. (3.18) at the second equality, and we defined $s(a,b) \equiv a\log(a/b) + b - a$. Since $s(a,b)$ satisfies

$$s(a,b) \geq \frac{8}{9}\frac{(a-b)^2}{a+b} \geq 0, \tag{4.20}$$

we obtain the inequality for $\dot{\Sigma}_{j,k}$ given by

$$\dot{\Sigma}_{j,k} \geq \sum_{\omega,\omega'} \frac{8}{9} \frac{(\tilde{A}^{j,k,-}_{\omega'\to\omega})^2}{\tilde{A}^{j,k,+}_{\omega'\to\omega}}, \tag{4.21}$$

where we defined

$$\tilde{A}^{j,k,\pm}_{\omega'\to\omega} \equiv R^{j,k,\lambda(t)}_{\omega'\to\omega} p_{\omega'}(t) \pm \tilde{R}^{j,k,\lambda}_{\omega'\to\omega} p_\omega(t). \tag{4.22}$$

By using Eq. (4.17), $J_{j,k}(t)$ in Eq. (4.12) satisfies

$$\begin{aligned}
\dot{Q}_{j,k}(t) &\equiv \sum_{\omega,\omega'} E^{\lambda(t)}_\omega R^{j,k,\lambda(t)}_{\omega'\to\omega} p_\omega(t) - \sum_{\omega,\omega'} E^{\lambda(t)}_\omega \tilde{R}^{j,k,\lambda(t)}_{\omega\to\omega'} p_\omega(t) \\
&= \sum_{\omega,\omega'} E^{\lambda(t)}_\omega \sqrt{\tilde{A}^{j,k,+}_{\omega'\to\omega}} \frac{\tilde{A}^{j,k,-}_{\omega'\to\omega}}{\sqrt{\tilde{A}^{j,k,+}_{\omega'\to\omega}}}
\end{aligned} \tag{4.23}$$

Using the Cauchy–Schwarz inequality and Eq. (4.21), we obtain

$$|\dot{Q}_{j,k}(t)| \leq \sqrt{\Theta_{j,k}(t)\dot{\Sigma}_{j,k}(t)} \tag{4.24}$$

with

$$\Theta_{j,k}(t) = \frac{9}{8}\sum_{\omega,\omega'}(E_\omega^{\lambda(t)})^2 \tilde{A}^{j,k,+}_{\omega'\to\omega}. \tag{4.25}$$

Then, we obtain the inequality

$$\begin{aligned}
(\dot{Q})^2 &= \left(\sum_{j=1}^{N}\sum_{k=1}^{n}\dot{Q}_{j,k}\right)^2 \\
&\leq \sum_{j=1}^{N}\sum_{k=1}^{n}\left(\dot{Q}_{j,k}\right)^2 \\
&\leq \sum_{j=1}^{N}\sum_{k=1}^{n}\left(\sqrt{\Theta_{j,k}(t)\dot{\Sigma}_{j,k}(t)}\right)^2 \\
&\leq \left(\sum_{j=1}^{N}\sum_{k=1}^{n}\Theta_{j,k}(t)\right)\left(\sum_{j=1}^{N}\sum_{k=1}^{n}\dot{\Sigma}_{j,k}(t)\right) \\
&= \Theta(t)\dot{\Sigma}_{\mathrm{tot}},
\end{aligned} \tag{4.26}$$

where we defined

$$\Theta(t) \equiv \sum_{j=1}^{N}\sum_{k=1}^{n}\Theta_{j,k}(t). \tag{4.27}$$

4.3 Derivation of trade-off relation

By using Eq. (4.26), we show the trade-off relation between the efficiency and power of the heat engines. We consider the heat engine operating between the two heat baths with the inverse temperatures β_h and β_c ($> \beta_h$). Let Q_h and Q_c denote the heat flowing from the heat baths with β_h and β_c to the system, respectively. When we introduce Δt_{cyc} as the cycle time, the internal energy and entropy changes of the system should satisfy

$$E(0) = E(\Delta t_{\mathrm{cyc}}), \quad S(0) = S(\Delta t_{\mathrm{cyc}}). \tag{4.28}$$

By using Eqs. (4.10) and (4.28), we derive the output work as

$$W \equiv \int_0^{\Delta t_{\mathrm{cyc}}} dt\ \dot{Q} = Q_h + Q_h. \tag{4.29}$$

In the heat engine, W and Q_h should be positive, and Q_c should be negative. Then, we can define the efficiency η and power P as

$$\eta \equiv \frac{W}{Q_h} = 1 + \frac{Q_c}{Q_h} = 1 - \frac{|Q_c|}{Q_h}, \quad P \equiv \frac{W}{\Delta t_{\mathrm{cyc}}}. \tag{4.30}$$

Carnot showed that the efficiency has the upper bound defined by the temperature of the heat baths called Carnot efficiency given by [2]

$$\eta_{\mathrm{C}} \equiv 1 - \frac{T_c}{T_h} = 1 - \frac{\beta_h}{\beta_c}. \tag{4.31}$$

By using Eq. (4.13), we also derive the total entropy production per cycle as

$$\Sigma_{\mathrm{tot}} \equiv \int_0^{\Delta t_{\mathrm{cyc}}} dt\ \dot{\Sigma}_{\mathrm{tot}} = -\beta_h Q_h - \beta_c Q_c = \beta_c Q_h \left(\eta_{\mathrm{C}} - \eta\right). \tag{4.32}$$

From Eq. (4.26), we obtain

$$\begin{aligned} Q_h^2 &\leq (|Q_h| + |Q_c|)^2 \\ &\leq \left(\int_0^{\Delta t_{\mathrm{cyc}}} dt\ |\dot{Q}| \right)^2 \\ &\leq \int_0^{\Delta t_{\mathrm{cyc}}} dt\ \sqrt{\Theta(t)\dot{\Sigma}_{\mathrm{tot}}} \\ &\leq \Delta t_{\mathrm{cyc}} \bar{\Theta} \Sigma_{\mathrm{tot}}, \end{aligned} \tag{4.33}$$

with

$$\bar{\Theta} \equiv \frac{1}{\Delta t_{\mathrm{cyc}}} \int_0^{\Delta t_{\mathrm{cyc}}} dt\ \Theta(t). \tag{4.34}$$

Using Eq. (4.29), (4.30), and (4.33), we derive the trade-off relation between the efficiency and power as

$$P \leq \bar{\Theta} \beta_c \eta (\eta_{\mathrm{C}} - \eta). \tag{4.35}$$

We consider the case that $\bar{\Theta}$ is finite. From the inequality in Eq. (4.35), we find that the power vanishes when the efficiency approaches the Carnot efficiency. Thus, it means that the compatibility of the Carnot efficiency and finite power is forbidden. However, if $\bar{\Theta}$ diverges, we may achieve the Carnot efficiency in the finite-power heat engine. In chap. 6 and 7, we study the detail of $\bar{\Theta}$ in the underdamped Brownian Carnot cycle and show that the compatibility of the Carnot efficiency and finite power is possible by considering the relaxation times of the Brownian particle. In the next chapter, we consider the Brownian motion and stochastic thermodynamics to construct the Brownian Carnot cycle.

Chapter 5

Stochastic thermodynamics

We consider the Brownian particle and introduce the thermodynamic quantities to describe the dynamics of the particle to construct the Brownian Carnot cycle. Since the Brownian particle is immersed in the medium, it moves randomly because of the random force from the medium. In this chapter, we introduce the Langevin equation to describe the random dynamics called the Brownian motion. We first consider the free Brownian motion as the simplest case to reveal the character of the random force. After that, we consider the particle trapped by the potential and introduce the thermodynamic quantities. Moreover, we introduce the Fokker-Planck equation which is equivalent to the Langevin equation to describe the evolution of the probability distribution of the particle.

5.1 Free Brownian motion

We first consider the free Brownian particle which is not trapped by the potential and immersed in the medium with temperature T [35]. We assume that the Brownian particle is large enough that its dynamics is described by the classical mechanics, but small enough that its fluctuation is non-negligible. As we assumed that the dynamics of the particle is subjected by the classical mechanics, its state is described by the position x and velocity v, which are the stochastic variables. Since the force which arises between the particle and medium is divided into the friction force and the random force, the Langevin equation given by

$$\begin{aligned}\dot{x} =& v, \\ m\dot{v} =& -\gamma v + \sqrt{D}\xi(t),\end{aligned} \tag{5.1}$$

where γ and D are the friction and diffusion coefficients, respectively. $\xi(t)$ is the Gaussian white noise satisfying

$$\langle \xi(t) \rangle = 0, \tag{5.2}$$

$$\left\langle \xi(t)\xi(t') \right\rangle = \delta(t-t'), \tag{5.3}$$

where $\langle \cdots \rangle$ is the statistical average. To decide D, we solve the second equation in Eq. (5.1) formally for the initial condition that v has the value v_0 at $t = 0$. For this condition, we obtain

$$v(t) = v_0 \exp\left\{-\frac{\gamma}{m}t\right\} + \frac{\sqrt{D}}{m}\int_0^t dt'\xi(t') \exp\left\{-\frac{\gamma}{m}(t-t')\right\}. \tag{5.4}$$

From Eq. (5.4), we obtain

$$\langle v(t_1)\rangle = v_0 \exp\left\{-\frac{\gamma}{m}t_1\right\}, \tag{5.5}$$

$$\begin{aligned}
&\langle v(t_1)v(t_2)\rangle \\
&= v_0^2 \exp\left\{-\frac{\gamma}{m}(t_1+t_2)\right\} + \frac{D}{m^2}\int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \delta(t_1-t_2)\exp\left\{-\frac{\gamma}{m}(t_1+t_2-t_1'-t_2')\right\} \\
&= v_0^2 \exp\left\{-\frac{\gamma}{m}(t_1+t_2)\right\} + \frac{D}{2m\gamma}\left(\exp\left\{-\frac{\gamma}{m}(|t_1-t_2|)\right\} - \exp\left\{-\frac{\gamma}{m}(t_1+t_2)\right\}\right). \qquad (5.6)
\end{aligned}$$

The right-hand side of Eq. (5.5) and the first term of the right-hand side of Eq. (5.6) approach zero when $t_1, t_2 \gg m/\gamma$ is satisfied. This means that the effect of the initial condition becomes smaller when t_1 and t_2 become larger. Thus, we find that m/γ is an important time scale for the Brownian motion and define it as a relaxation time of the velocity of the Brownian particle as

$$\tau_v \equiv \frac{m}{\gamma}. \tag{5.7}$$

Note that this is only determined by the particle and medium. When we consider $t_1, t_2 \gg \tau_v$, the system is almost equilibrium, and Eq. (5.6) is approximated by

$$\langle v(t_1)v(t_2)\rangle = \frac{D}{2m\gamma}\exp\left\{-\frac{\gamma}{m}(|t_1-t_2|)\right\}. \tag{5.8}$$

When $t_1 = t_2$ is satisfied, we obtain

$$\left\langle v(t)^2\right\rangle = \frac{D}{2m\gamma}. \tag{5.9}$$

Moreover, because of the law of equipartition of energy in the statistical mechanics [1], the average of the kinetic energy of the particle in equilibrium state should satisfy

$$\frac{1}{2}m\left\langle v^2\right\rangle = \frac{1}{2}k_{\mathrm{B}}T, \tag{5.10}$$

where k_{B} is the Boltzmann constant. Comparing Eqs. (5.9) and (5.10), we obtain

$$D = 2\gamma k_{\mathrm{B}}T. \tag{5.11}$$

This is called the Einstein relation. We consider the mean square displacement $\left\langle [x(t)-x(0)]^2\right\rangle$. When the time t satisfying $t \gg m/\gamma$, using Eq. (5.6), we obtain the mean square displacement as

$$\left\langle [x(t)-x(0)]^2\right\rangle = \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2)\rangle = \frac{D}{\gamma^2}|t|. \tag{5.12}$$

As shown in Eq. (5.12), the mean square displacement is proportional to the time, and this is one of the character of the free Brownian motion.

5.2 Brownian particle trapped by potential

We consider the Brownian particle trapped by the potential $V(x,t)$. Since the force associated with the potential acts on the particle, the Langevin equations describing the

Brownian motion are given by

$$\dot{x} =v,$$
$$m\dot{v} =-\gamma v-\frac{\partial V(x,t)}{\partial x}+\sqrt{2\gamma k_{\mathrm{B}}T}\xi(t). \tag{5.13}$$

We assume that we can change the functional form of $V(x,t)$ arbitrarily. We first consider one of the important stochastic processes called the Ornstein-Uhlenbeck process. Since the mass of the particle is sufficiently small, $\tau_v = m/\gamma$ also becomes small. Then, the duration of observation Δt becomes sufficiently larger than τ_v [16, 24, 84]. For example, the order of Δt and τ_v used in the experiment of the Brownian particle in Ref. [24] are seconds and milliseconds, respectively. When $\Delta t \gg \tau_v$ is satisfied, the inertial effect of the Brownian can be neglected, Eq. (5.13) can be rewritten as

$$\gamma\dot{x} = -\frac{\partial V(x,t)}{\partial x}+\sqrt{2\gamma k_{\mathrm{B}}T}\xi(t). \tag{5.14}$$

Eq. (5.14) is the differential equation of x only, and it is called the overdamped Langevin equation. In contrast to Eq. (5.14), Eq. (5.13) expressed by x and v is called the underdamped Langevin equations.

5.2.1 Ornstein-Uhlenbeck process

We here consider the overdamped dynamics of the particle trapped by the time-independent harmonic potential

$$V(x) = \frac{1}{2}\lambda x^2. \tag{5.15}$$

The overdamped Langevin equation of the particle in this potential is given by

$$\gamma\dot{x} = -\lambda x+\sqrt{2\gamma k_{\mathrm{B}}T}\xi(t). \tag{5.16}$$

Similar to the Langevin equation for the free Brownian particle in Eq. (5.1), we can solve Eq. (5.16) and obtain

$$x(t) = x_0\exp\left\{-\frac{\lambda}{\gamma}t\right\}+\sqrt{\frac{2k_{\mathrm{B}}T}{\gamma}}\int_0^t dt'\xi(t')\exp\left\{-\frac{\lambda}{\gamma}(t-t')\right\}, \tag{5.17}$$

where x_0 is the initial condition of $x(t)$. Similar to the relaxation time of the velocity in the free Brownian motion in Eq. (5.7), the relaxation time of this process is given by

$$\tau_x \equiv \frac{\lambda}{\gamma}. \tag{5.18}$$

Moreover, from the discussion of Sec. 3, we can derive the transition probability $W(x,t|x_0,0)$ corresponding to Eq. (5.16) as

$$W(x,t|x_0,0) = \sqrt{\frac{\lambda}{2\pi k_{\mathrm{B}}T(1-e^{-2t/\tau_x})}}\exp\left[-\frac{\lambda(x-x_0e^{-2t/\tau_x})^2}{2k_{\mathrm{B}}T(1-e^{-2t/\tau_x})}\right]. \tag{5.19}$$

The stochastic process with the above transition probability is called the Ornstein-Uhlenbeck process [68, 85].

We introduce the probability distribution for x at t as $p(x,t)$. When we choose the initial condition as the delta function $\delta(x-x_0)$, $p(x,t)$ is given by

$$p(x,t) = \sqrt{\frac{\lambda}{2\pi k_{\rm B}T(1-e^{-2t/\tau_x})}} \exp\left[-\frac{\lambda(x-x_0e^{-2t/\tau})^2}{2k_{\rm B}T(1-e^{-2t/\tau_x})}\right]. \tag{5.20}$$

When $t \gg \tau_x$ is satisfied, Eq. (5.20) is approximated by the Boltzmann distribution:

$$p^{\rm eq}(x) = \sqrt{\frac{\lambda}{2\pi k_{\rm B}T}} \exp\left[-\frac{\lambda x^2}{2k_{\rm B}T}\right]. \tag{5.21}$$

5.2.2 Definition of heat and work

We define the heat and work as forms of the energy change of the Brownian particle trapped by the potential $V(x,t)$ and consider its thermodynamics. In thermodynamics, when the system is contact with the heat bath, the heat is the energy change due to the interaction between the system and heat bath. We assume that the heat bath is equilibrium at any time. The first and third terms in the right-hand side of the second equation in Eq. (5.13) are the forces where the heat bath acts on the particle. When the particle moves infinitesimal displacement dx due to the forces, we can define the work done to the particle from the heat bath as the infinitesimal heat given by

$$dq(t) \equiv \left[-\gamma v(t) + \sqrt{2\gamma k_{\rm B}T}\xi(t)\right] \circ dx(t), \tag{5.22}$$

where $\circ$ is the Stratonovich product. We discuss the reason why we use the Stratonovich product to define the heat in the Appendix A. By taking the statistical average of $dq(t)$, we obtain the averaged infinitesimal heat dQ flowing to the particle:

$$dQ \equiv \langle dq \rangle = \left\langle \left[-\gamma v(t) + \sqrt{2\gamma k_{\rm B}T}\xi(t)\right] \circ dx(t) \right\rangle. \tag{5.23}$$

Hereafter, we call dQ the heat for simplicity. The mechanical energy of the Brownian particle is given by

$$e(x,v,t) \equiv \frac{1}{2}mv^2 + V(x,t). \tag{5.24}$$

From $e(x,v,t)$, we can define the internal energy of the particle as

$$E(t) = \langle e(x,v,t) \rangle = \frac{1}{2}m\left\langle v^2 \right\rangle + \langle V(x,t) \rangle. \tag{5.25}$$

Using Eqs. (5.13) and (5.22), we can derive the infinitesimal change of the mechanical energy in Eq. (5.24) as

$$\begin{aligned} de(x,v,t) &= \left(m\dot{v}(t) + \frac{\partial V(x,t)}{\partial x}\right) \circ dx(t) + \frac{\partial V(x,t)}{\partial t}dt \\ &= \left[-\gamma v(t) + \sqrt{2\gamma k_{\rm B}T}\xi(t)\right] \circ dx(t) + \frac{\partial V(x,t)}{\partial t}dt = dq(t) + \frac{\partial V(x,t)}{\partial t}dt. \end{aligned} \tag{5.26}$$

The second term in the right-hand side of Eq. (5.26) is the energy change due to the change of functional form of the potential. As we assumed that we can change it arbitrarily below Eq. (5.13), the energy change $(\partial V(x,t)/\partial t)dt$ is caused by us. Thus, since we can interpret it as the work done to the particle by us, we can define the infinitesimal work as

$$dw(x,t) \equiv \frac{\partial V(x,t)}{\partial t} dt. \tag{5.27}$$

Then, Eq. (5.26) is rewritten as

$$de = dq + dw. \tag{5.28}$$

By taking the statistical average of $dw(t)$, we derive the averaged infinitesimal work dW done to the particle as

$$dW(x,t) \equiv \left\langle \frac{\partial V(x,t)}{\partial t} dt \right\rangle. \tag{5.29}$$

From Eqs. (5.23), (5.25), (5.26), and (5.29), we obtain

$$dE = dQ + dW. \tag{5.30}$$

We find that Eqs. (5.28) and (5.30) are the first law of thermodynamics for the infinitesimal energy change in the Brownian motion.

5.2.3 Heat and work in Fokker-Planck system

We define the averaged heat and work by using the Fokker-Planck equation. The equation corresponding to Eq. (5.13) is derived from the discussion in Sec. 3.5.1 and given by

$$\frac{\partial}{\partial t} p(x,v,t) = -\frac{\partial}{\partial x} j_x(x,v,t) - \frac{\partial}{\partial v} j_v(x,v,t), \tag{5.31}$$

where we defined the probability currents as

$$j_x(x,v,t) \equiv vp(x,v,t), \tag{5.32}$$

$$j_v(x,v,t) \equiv -\left[\frac{\gamma}{m} v + \frac{1}{m}\frac{\partial V(x,t)}{\partial x} + \frac{\gamma k_{\mathrm{B}} T}{m^2}\frac{\partial}{\partial v}\right] p(x,v,t). \tag{5.33}$$

The internal energy can be defined as the average of the mechanical energy of the particle:

$$E(t) \equiv \int\int dxdv\ e(x,v,t)p(x,v,t) = \int\int dxdv\ \left[\frac{1}{2}mv^2 + V(x,t)\right] p(x,v,t). \tag{5.34}$$

From Eq. (5.34), the time derivative of the internal energy is given by

$$\begin{aligned} \frac{dE(t)}{dt} &= \int\int dxdv\ \frac{\partial e(x,v,t)}{\partial t} p(x,v,t) + \int\int dxdv\ e(x,v,t)\frac{\partial p(x,v,t)}{\partial t} \\ &= \int\int dxdv\ \frac{\partial V(x,t)}{\partial t} p(x,v,t) + \int\int dxdv\ \left[\frac{1}{2}mv^2 + V(x,t)\right]\frac{\partial p(x,v,t)}{\partial t}, \end{aligned} \tag{5.35}$$

where we use Eq. (5.24). Since the first term of the right-hand side of Eq. (5.35) is the averaged energy change, similar to Eq. (5.29), we can interpret it as the averaged work done to the particle per unit time:

$$\dot{W}(t) \equiv \int\int dxdv\ \frac{\partial V(x,t)}{\partial t} p(x,v,t). \tag{5.36}$$

Because of the first law of thermodynamics, the second term of the right-hand side of Eq. (5.35) can be interpreted as the heat current flowing from the heat bath to the system:

$$\dot{Q} \equiv \int\int dxdv\ e(x,v,t)\frac{\partial p(x,v,t)}{\partial t} = \int\int dxdv\ \left[\frac{1}{2}mv^2 + V(x,t)\right]\frac{\partial p(x,v,t)}{\partial t}. \tag{5.37}$$

When the thermodynamics process lasts for $0 \leq t \leq t_0$, the output work W and heat Q in this process are defined as

$$W \equiv -\int_0^{t_0} dt\dot{W}(t) \tag{5.38}$$

$$Q \equiv \int_0^{t_0} dt\dot{Q}(t) \tag{5.39}$$

where we used Eqs. (5.36) and (5.37). Since the internal energy change in this process is given by $E(t_0) - E(0)$, the output work W in Eq. (5.38) and heat Q in Eq. (5.39) satisfy

$$E(t_0) - E(0) = Q - W. \tag{5.40}$$

Moreover, since the probability distribution $p(x,v,t)$ is introduced, we can define the entropy of the system as

$$S(t) \equiv -\int\int dxdv\ p(x,v,t)\ln\{p(x,v,t)\}. \tag{5.41}$$

Since the heat bath is assumed to be equilibrium, the entropy change rate of the heat bath is given by $-\dot{Q}/T$. Thus, the entropy production rate of the total system is defined as the sum of the entropy change rate of the particle and heat bath:

$$\dot{\Sigma}_{\mathrm{tot}} \equiv \dot{S} - \frac{\dot{Q}}{T}. \tag{5.42}$$

where we used Eqs. (5.37) and (5.41). Thus, the entropy production in this process is given by

$$\Sigma_{\mathrm{tot}} \equiv \int_0^{t_0} dt\ \dot{\Sigma}_{\mathrm{tot}} = \Delta S - \int_0^{t_0} dt\ \frac{\dot{Q}}{T}, \tag{5.43}$$

where we define the entropy change of the system in this process as

$$\Delta S \equiv S(t_0) - S(0), \tag{5.44}$$

using Eq. (5.41).

5.3 Entropy production in Langevin system

We consider the time reversal in the thermodynamic process lasting for $0 \leq t \leq t_0$ and rewrite the entropy production in the system described by the Langevin equations [86–88]. We first introduce the trajectory of the stochastic variables $\boldsymbol{X} = (X_1, X_2, \cdots, X_N)$ in the phase space as $\vec{\boldsymbol{X}} \equiv \boldsymbol{X}(t)$ for $0 \leq t \leq t_0$. For example, when we consider the Brownian motion, the elements of $\boldsymbol{X}$ can be regarded as the set of the position and velocity of the particle. When we consider the time-reversal system, the infinitesimal time increment dt becomes $-dt$. Then, although the sign of the position is unchanged, that of the velocity is inverted. Thus, the elements of $\boldsymbol{X} = (X_1, X_2, \cdots, X_N)$ may be odd or even under the time-reversal. As the variables in the time-reversal system, we introduce $\boldsymbol{\epsilon X} \equiv (\epsilon_1 X_1, \epsilon_2 X_2, \cdots, \epsilon_N X_N)$ where $\epsilon_i = \pm 1$ for even and odd variables X_i, respectively. To summarize, $\boldsymbol{X}$ and dt change to $\boldsymbol{\epsilon X}$ and $(-dt)$ under the time reversal, respectively. For the simplicity, we only consider the case that the Langevin equation is given by

$$dX_i = a_i(\boldsymbol{X}, t)dt + b_i(\boldsymbol{X}, t) \cdot dB_i, \tag{5.45}$$

using the Itô-type product. We assume that $a_i(\boldsymbol{X}, t)$ in Eq. (5.45) is divided into the reversible part $a_i^{\mathrm{rev}}(\boldsymbol{X}, t)$ and irreversible part $a_i^{\mathrm{irr}}(\boldsymbol{X}, t)$:

$$dX_i = a_i^{\mathrm{rev}}(\boldsymbol{X}, t)dt + a_i^{\mathrm{irr}}(\boldsymbol{X}, t)dt + b_i(\boldsymbol{X}, t) \cdot dB_i, \tag{5.46}$$

where $a_i^{\mathrm{rev}}(\boldsymbol{X}, t)$ and $a_i^{\mathrm{irr}}(\boldsymbol{X}, t)$ are given by

$$a_i^{\mathrm{rev}}(\boldsymbol{X}, t) = \frac{1}{2}\left[a_i(\boldsymbol{X}, t) - \epsilon_i a_i(\boldsymbol{\epsilon X}, t)\right], \tag{5.47}$$

$$a_i^{\mathrm{irr}}(\boldsymbol{X}, t) = \frac{1}{2}\left[a_i(\boldsymbol{X}, t) + \epsilon_i a_i(\boldsymbol{\epsilon X}, t)\right]. \tag{5.48}$$

Moreover, $b_i(\boldsymbol{X}, t)$ is unchanged under the time reversal since it is defined by the second moment of X_i. From Sec. 3.5.1, we derive the Fokker-Planck equation corresponding to Eq. (5.45) as

$$\begin{aligned}\frac{\partial}{\partial t}p(\boldsymbol{X}, t) &= -\sum_{i=1}^{N} \frac{\partial}{\partial X_i} j_i(\boldsymbol{X}, t) \\ &= -\sum_{i=1}^{N} \frac{\partial}{\partial X_i}\left[a_i(\boldsymbol{X}, t) - \frac{\partial}{\partial X_i} D_i(\boldsymbol{X}, t)\right] p(\boldsymbol{X}, t),\end{aligned} \tag{5.49}$$

where we defined

$$j_i(\boldsymbol{X}, t) \equiv \left[a_i(\boldsymbol{X}, t) - \frac{\partial}{\partial X_i} D_i(\boldsymbol{X}, t)\right] p(\boldsymbol{X}, t), \tag{5.50}$$

$$D_i(\boldsymbol{X}, t) = \frac{1}{2} b_i(\boldsymbol{X}, t)^2. \tag{5.51}$$

Then, by using Eq. (5.46), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}p(\boldsymbol{X}, t) =& \sum_{i=1}^{N} \frac{\partial}{\partial X_i}[a_i^{\mathrm{rev}} p(\boldsymbol{X}, t)] - \sum_{i=1}^{N} \frac{\partial}{\partial X_i}\left[a_i^{\mathrm{irr}}(\boldsymbol{X}, t) - \frac{\partial}{\partial X_i} D_i(\boldsymbol{X}, t)\right] p(\boldsymbol{X}, t) \\ =& -\sum_{i=1}^{N} \frac{\partial}{\partial X_i}[j_i^{\mathrm{rev}}(\boldsymbol{X}, t) + j_i^{\mathrm{irr}}(\boldsymbol{X}, t)],\end{aligned} \tag{5.52}$$

where we defined

$$\begin{aligned}j_i^{\mathrm{rev}} \equiv& a_i^{\mathrm{rev}} p(\boldsymbol{X}, t), \\ j_i^{\mathrm{irr}} \equiv& a_i^{\mathrm{irr}} p(\boldsymbol{X}, t) - \frac{\partial}{\partial X_i} D_i(\boldsymbol{X}, t) p(\boldsymbol{X}, t).\end{aligned} \tag{5.53}$$

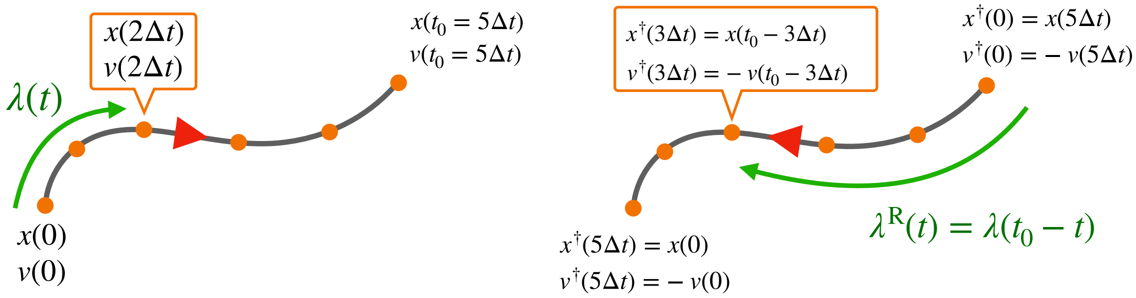

Figure 5.1: Schematic illustration of the trajectory $\boldsymbol{X}(t) = (\boldsymbol{x}(t), \boldsymbol{v}(t))$ $(0 \leq t \leq 5\Delta t = t_0)$ and its time-reversal trajectory $\boldsymbol{X}^\dagger(t) = (\boldsymbol{x}^\dagger(t), \boldsymbol{v}^\dagger(t))$ of the particle moving in a plane. The left illustration denotes the trajectory $\boldsymbol{X}(t)$, and the right one denotes its time-reversal trajectory $\boldsymbol{X}^\dagger(t)$. In the illustration, the gray lines denote the trajectories of the position of the particle. The orange points denote the position of the particle at $t = k\Delta t$ $(k = 0, 1, \cdots, 5)$. The red triangles denote the direction of the velocity. In the illustration of $\boldsymbol{X}^\dagger(t)$, the direction of the red arrow is in the opposite direction of that of $\boldsymbol{X}(t)$. The green arrow in the left illustration denotes the time evolution of the protocol $\lambda(t)$, and that in the right illustration denotes the time evolution of the time-reversal protocol $\lambda^{\mathrm{R}}(t)$. We find that $\boldsymbol{x}^\dagger(t) = \boldsymbol{x}(t_0 - t)$ and $\boldsymbol{v}^\dagger(t) = -\boldsymbol{v}(t_0 - t)$ are satisfied under the time reversal.

j_i^{rev} (j_i^{irr}) is a reversible (irreversible) probability current corresponding to X_i.

To derive the entropy production, we introduce $\mathcal{P}[\vec{\boldsymbol{X}}]$ as the probability distribution of the trajectory $\vec{\boldsymbol{X}} \equiv \boldsymbol{X}(t)$ for $0 \leq t \leq t_0$, with the initial probability distribution $p(\boldsymbol{X}(0), 0)$ describing the state of the system. We also introduce $\mathcal{P}^{\mathrm{R}}[\vec{\boldsymbol{X}}^\dagger]$ denoting the probability distribution of the time-reversed trajectory $\vec{\boldsymbol{X}}^\dagger \equiv \boldsymbol{\epsilon}\boldsymbol{X}(t_0 - t)$ with reversed protocol, which is how to change the potential and temperature of the heat bath. (see Fig. 5.1). [1] We introduce the quantity, which is a function of the trajectory $\vec{\boldsymbol{X}}$, given by

$$\mathcal{A}[\vec{\boldsymbol{X}}] \equiv \ln \frac{\mathcal{P}[\vec{\boldsymbol{X}}]}{\mathcal{P}^{\mathrm{R}}[\vec{\boldsymbol{X}}^\dagger]}. \tag{5.55}$$

Since the Jacobian of the transformation $\vec{\boldsymbol{X}} \to \vec{\boldsymbol{X}}^\dagger$ is unity, we obtain

$$\left\langle \exp\left[\mathcal{A}[\vec{\boldsymbol{X}}]\right]\right\rangle \equiv \int d\vec{\boldsymbol{X}} \exp\left[\mathcal{A}[\vec{\boldsymbol{X}}]\right] \mathcal{P}[\vec{\boldsymbol{X}}] = \int d\vec{\boldsymbol{X}}^\dagger \mathcal{P}^{\mathrm{R}}[\vec{\boldsymbol{X}}^\dagger] = 1. \tag{5.56}$$

Note that domain of integration for $\vec{\boldsymbol{X}}$ in Eq. (5.56) is over all possible trajectories. Moreover, the probability distribution at the final state should satisfy

$$p(\boldsymbol{X}(t_0), t_0) = p^{\mathrm{R}}(\boldsymbol{\epsilon}\boldsymbol{X}(t_0), t_0). \tag{5.57}$$

As discussed in Ref. [89], the entropy production of the total system along the trajectory

[1] Keeping the integral fluctuation theorem [19,89] in mind, we have some options of trajectories instead of $\boldsymbol{X}^\dagger(t) = \boldsymbol{\epsilon}\boldsymbol{X}(t_0 - t)$ for $0 \leq t \leq t_0$ as follows:

$$\boldsymbol{X}^R(t) \equiv \boldsymbol{X}(t_0 - t), \quad \boldsymbol{X}^T(t) = \epsilon\boldsymbol{X}(t). \tag{5.54}$$

More detailed discussion related to the trajectories and entropy change is held in Ref. [86].

relates to the time reversal and is defined by

$$\begin{aligned}\Delta s_{\rm tot}[\vec{\boldsymbol{X}}] \equiv& \ \ln \mathcal{P}[\vec{\boldsymbol{X}}] - \ln \mathcal{P}^{\rm R}[\vec{\boldsymbol{X}}^{\dagger}] \\ =& \ -\ln \frac{p(\boldsymbol{X}(t_0), t_0)}{p(\boldsymbol{X}(0), 0)} + \ln \frac{\mathcal{P}[\boldsymbol{X}(t_0)|\boldsymbol{X}(0)]}{\mathcal{P}^{\rm R}[\boldsymbol{\epsilon X}(0)|\boldsymbol{\epsilon X}(t_0)]} \\ =& \ \Delta s + \Delta s_{\rm b}, \end{aligned} \tag{5.58}$$

where we use Eq. (5.57), and Δs and $\Delta s_{\rm b}$ are the entropy change of the system and heat bath along the trajectory, respectively. Although $\Delta s_{\rm tot}$, Δs, and $\Delta s_{\rm b}$ depend on the trajectory, we often interested only in their statistical average over all the trajectories. When we take the statistical average, we obtain

$$\Sigma_{\rm tot} = \Delta S + \Delta S_{\rm b}, \tag{5.59}$$

$$\Sigma_{\rm tot} \equiv \langle \Delta s_{\rm tot} \rangle\,, \quad \Delta S \equiv \langle \Delta s \rangle\,, \quad \Delta S_{\rm b} \equiv \langle \Delta s_{\rm tot} \rangle\,. \tag{5.60}$$

For simplicity, we hereafter refer to the averaged entropy changes as the entropy changes. The same applies to the other physical quantities.

Using the transition probability corresponding to the Langevin equations in Eq. (5.45), we express $\Sigma_{\rm tot}$ by the reversible and irreversible probability currents in Eq. (5.53). We first consider the infinitesimal increment of the entropy production:

$$d(\Delta s_{\rm tot})[\boldsymbol{X}(t+dt), \boldsymbol{X}(t)] \equiv -\ln \frac{p(\boldsymbol{X}(t+dt), t+dt)}{p(\boldsymbol{X}(t), t)} + \ln \frac{\mathcal{P}(\boldsymbol{X}', t+dt|\boldsymbol{X}, t)}{\mathcal{P}^{\rm R}(\boldsymbol{\epsilon X}, t+dt|\boldsymbol{\epsilon X}', t)}, \tag{5.61}$$

where we defined $\boldsymbol{X} \equiv \boldsymbol{X}(t)$ and $\boldsymbol{X}' \equiv \boldsymbol{X}(t+dt)$, and dt is infinitesimal time increment. Since $\mathcal{P}(\boldsymbol{X}', t+dt|\boldsymbol{X}, t)$ is expressed by $W(\boldsymbol{X}', t+dt|\boldsymbol{X}, t)$ in Eq. (3.88) in Chap. 3, we obtain

$$\begin{aligned}\mathcal{P}(\boldsymbol{X}', t+dt|\boldsymbol{X}, t) &= W(\boldsymbol{X}', t+dt|\boldsymbol{X}, t) \\ &= \prod_i \sqrt{\frac{1}{4\pi D_i(\boldsymbol{X}, t)dt}} \exp\left[-\frac{(X_i' - X_i - a_i(\boldsymbol{X}, t)dt)^2}{4D_i(\boldsymbol{X}, t)dt}\right].\end{aligned} \tag{5.62}$$

To consider the transition probability $\mathcal{P}(\boldsymbol{\epsilon X}', t+dt|\boldsymbol{\epsilon X}, t)$ in the time-reversal trajectory, we derive the time-reversal Langevin equation from Eq. (5.45) as

$$\epsilon_i dX_i = [-\epsilon_i a_i^{\rm rev}(\boldsymbol{\epsilon X}', t) + \epsilon_i a_i^{\rm irr}(\boldsymbol{\epsilon X}', t)]dt + b_i(\boldsymbol{\epsilon X}', t) \cdot dB_i. \tag{5.63}$$

Thus, by considering the variable transformation from $\boldsymbol{\epsilon X}'$ to $\boldsymbol{\epsilon X}$ in $a_i^{\rm rev}$, $a_i^{\rm irr}$, and b_i, we can derive the transition probability $\mathcal{P}(\boldsymbol{\epsilon X}', t+dt|\boldsymbol{\epsilon X}, t)$ as

$$\begin{aligned}&\mathcal{P}^{\rm R}(\boldsymbol{\epsilon X}, t+dt|\boldsymbol{\epsilon X}', t) = W(\boldsymbol{\epsilon X}, t+dt|\boldsymbol{\epsilon X}', t) \\ &= \prod_i \sqrt{\frac{1}{4\pi D_i(\boldsymbol{\epsilon X}, t)dt}} \exp\left[-\frac{(\epsilon_i X_i - \epsilon_i X_i' - \epsilon_i[-a_i^{\rm rev}(\boldsymbol{\epsilon X}, t) + a_i^{\rm irr}(\boldsymbol{\epsilon X}, t)]dt)^2}{4D_i(\boldsymbol{\epsilon X}, t)dt}\right. \\ &\qquad \left. -dt\left(\frac{\partial \epsilon_i a_i^{\rm irr}(\boldsymbol{\epsilon X}, t)}{\partial(\epsilon_i X_i)} - \frac{\partial \epsilon_i a_i^{\rm rev}(\boldsymbol{\epsilon X}, t)}{\partial(\epsilon_i X_i)}\right) + dt\frac{\partial D_i(\boldsymbol{\epsilon X}, t)}{\partial(\epsilon_i X_i)}\right].\end{aligned} \tag{5.64}$$

Using Eqs. (5.53), (5.61), (5.62), and (5.64), we obtain

$$\begin{aligned}\dot{\Sigma}_{\text{tot}} \equiv& \frac{d\left\langle \Delta s_{\text{tot}}\right\rangle}{dt} \\ =& \sum_i \int d\boldsymbol{X} \frac{1}{p(\boldsymbol{X},t)D_i(\boldsymbol{X},t)} \left[p(\boldsymbol{X},t)a_i^{\text{irr}}(\boldsymbol{X},t) - D_i(\boldsymbol{X},t)\frac{\partial p(\boldsymbol{X},t)}{\partial X_i} - p(\boldsymbol{X},t)\frac{\partial D_i(\boldsymbol{X},t)}{\partial X_i}\right]^2 \\ =& \sum_i \int d\boldsymbol{X} \frac{j_i^{\text{irr}}(\boldsymbol{X},t)^2}{p(\boldsymbol{X},t)D_i(\boldsymbol{X},t)}, \end{aligned} \tag{5.65}$$

where we assume that the probability density and currents vanish at the boundaries. Thus, the entropy production in the thermodynamic process from $t = 0$ to $t = t_0$ is given by

$$\Sigma_{\text{tot}} \equiv \int_0^{t_0} dt\ \dot{\Sigma}_{\text{tot}} = \sum_i \int_0^{t_0} dt\ \int d\boldsymbol{X} \frac{j_i^{\text{irr}}(\boldsymbol{X},t)^2}{p(\boldsymbol{X},t)D_i(\boldsymbol{X},t)}. \tag{5.66}$$

5.4 Trade-off relation between efficiency and power in the underdamped Brownian heat engine

5.4.1 Inequality of entropy production rate

We show the trade-off relation between the efficiency and power in the underdamped Brownian heat engine based on Ref. [25]. First, we show that when we define $R(t)$ as the average of the physical quantity $r(\boldsymbol{X},t)$, there is an inequality between the current of $R(t)$ and the entropy production rate of the total system. In the underdamped system discussed in Sec. 5.2, we show that the heat current in Eq. (5.37) flowing from the heat bath to the particle satisfies the inequality involving the entropy production rate. We consider the Brownian heat engine and show the trade-off relation between the efficiency and power by using the inequality.

When we define the average of the physical quantity $r(\boldsymbol{X},t)$ as

$$R(t) \equiv \int d\boldsymbol{X} r(\boldsymbol{X},t)p(\boldsymbol{X},t), \tag{5.67}$$

we can derive its time derivative as

$$\begin{aligned}\dot{R}(t) \equiv& \int d\boldsymbol{X}\frac{\partial r(\boldsymbol{X},t)}{\partial t}p(\boldsymbol{X},t) + \int d\boldsymbol{X} r(\boldsymbol{X},t)\frac{\partial p(\boldsymbol{X},t)}{\partial t} \\ =& \left\langle \frac{\partial r(\boldsymbol{X},t)}{\partial t}\right\rangle - \sum_i \int d\boldsymbol{X} r(\boldsymbol{X},t)\frac{\partial j_i(\boldsymbol{X},t)}{\partial X_i} \\ =& \left\langle \frac{\partial r(\boldsymbol{X},t)}{\partial t}\right\rangle + \sum_i \int d\boldsymbol{X}\frac{\partial r(\boldsymbol{X},t)}{\partial X_i} j_i(\boldsymbol{X},t), \end{aligned} \tag{5.68}$$

using the Fokker-Planck equation in Eq. (5.52). The second term of the right-hand-side of Eq. (5.68) is a change rate of $R(t)$ expressed by the probability currents. Note that when $r(\boldsymbol{X},t)$ is the mechanical energy of the system, with a similar consideration to Sec. 5.2.3, the first and second terms of the right-hand-side of Eq. (5.68) are the work done to the system per unit time and the heat current, respectively. Thus, we can define current of physical quantity $R(t)$ as

$$J_R(t) \equiv \sum_i \int d\boldsymbol{X} \frac{\partial r(\boldsymbol{X},t)}{\partial X_i} j_i(\boldsymbol{X},t). \tag{5.69}$$

Since the probability current j_i can be divided into the reversible part j_i^{rev} and irreversible part j_i^{irr}, the current in Eq. (5.69) is also divided into two parts as

$$\begin{aligned} &J_R \equiv J_R^{\mathrm{rev}} + J_R^{\mathrm{irr}}, \\ &J_R^{\mathrm{rev}} \equiv \sum_i \int d\boldsymbol{X} \frac{\partial R}{\partial X_i} j_i^{\mathrm{rev}}, \quad J_R^{\mathrm{irr}} \equiv \sum_i \int d\boldsymbol{X} \frac{\partial R}{\partial X_i} j_i^{\mathrm{irr}}. \end{aligned} \tag{5.70}$$

The entropy production rate of the total system is defined in Eq. (5.65). We show that there is an inequality between the irreversible part of the current of physical quantity in Eq. (5.70) and entropy production rate in Eq. (5.65). Using the Cauchy-Schwarz inequality for Eq. (5.69), we can obtain

$$\left(\int d\boldsymbol{X} \frac{\partial R}{\partial X_i} j_i^{\mathrm{irr}}\right)^2 \leq \left[\int d\boldsymbol{X} \left(\frac{\partial R}{\partial X_i}\right)^2 D_i p\right]^2 \left[\int d\boldsymbol{X} \frac{(j_i^{\mathrm{irr}})^2}{p D_i}\right]. \tag{5.71}$$

Thus, we derive the inequality as

$$(J_R^{\mathrm{irr}})^2 = \left(\sum_i \int d\boldsymbol{X} \frac{\partial R}{\partial X_i} j_i^{\mathrm{irr}}\right)^2 \leq \left(\sum_i \int d\boldsymbol{X} \left(\frac{\partial R}{\partial X_i}\right)^2 D_i p\right)^2 \dot{\Sigma}_{\mathrm{tot}}. \tag{5.72}$$

We consider the underdamped Brownian motion described by Eq. (5.13) and consider the heat engine. In the heat engine, we show the trade-off relation between the efficiency and power by using Eq. (5.72). We first divide the probability currents Eqs. (5.32) and (5.33) into the reversible parts, j_x^{rev} and j_v^{rev}, and the irreversible parts, j_x^{irr} and j_v^{irr}, as

$$\begin{aligned} j_x(x,v,t) =& j_x^{\mathrm{rev}}(x,v,t) + j_x^{\mathrm{irr}}(x,v,t), \\ j_v(x,v,t) =& j_v^{\mathrm{rev}}(x,v,t) + j_v^{\mathrm{irr}}(x,v,t), \end{aligned} \tag{5.73}$$

where

$$\begin{aligned} j_x^{\mathrm{rev}}(x,v,t) \equiv& vp(x,v,t), \quad j_v^{\mathrm{rev}}(x,v,t) \equiv -\frac{1}{m}\frac{\partial V(x,t)}{\partial x} p(x,v,t), \\ j_x^{\mathrm{irr}}(x,v,t) \equiv& 0, \quad j_v^{\mathrm{irr}}(x,v,t) \equiv \left(-\frac{\gamma}{m} v - \frac{\gamma T(t)}{m^2}\frac{\partial}{\partial v}\right) p(x,v,t). \end{aligned} \tag{5.74}$$

Then, we can rewrite the heat current in Eq. (5.37) as

$$\dot{Q} = \int dx \int dv \, \left[\frac{1}{2} m v^2 + V(x,t)\right] \frac{\partial p}{\partial t} = \int dx \int dv \left[m v j_v + \frac{\partial V(x,t)}{\partial x} j_x\right], \tag{5.75}$$

where we used the Fokker-Planck equation in Eq. (5.52) and assumed that the probability currents at the boundary vanish. Moreover, the last equality is derived from the integration by parts. By using Eqs. (5.73), (5.74), and (5.75), the heat current can be rewritten as

$$\dot{Q}(t) = \int dx \int dv \; m v j_v^{\mathrm{irr}}(x,v,t). \tag{5.76}$$

Applying Eq. (5.65) to the underdamped system, we derive the entropy production rate of the total system as

$$\dot{\Sigma}(t) = \int dx \int dv \, \frac{m^2 (j_v^{\mathrm{irr}}(x,v,t))^2}{\gamma T(t) p(x,v,t)}, \tag{5.77}$$

using Eq. (5.74). Similar to Eq. (5.72), by using the Cauchy-Schwarz inequality, we derive the inequality between the heat current in Eq. (5.76) and the entropy production rate in Eq. (5.77) as

$$\dot{Q}^2 \leq \left(\int dx \int dv \, \gamma T v^2 p \right) \left(\int dx \int dv \, \frac{m^2 (j_v^{\mathrm{irr}})^2}{\gamma T p} \right) = \gamma T \sigma_v \dot{\Sigma}, \tag{5.78}$$

or, equivalently,

$$|\dot{Q}| \leq \sqrt{\gamma T \sigma_v \dot{\Sigma}}. \tag{5.79}$$

5.4.2 Trade-off relation in the Brownian heat engine

We introduce the heat engine with the cycle time Δt_{cyc} by using the underdamped Brownian particle discussed above. When the cycle operates between the heat baths with temperature T_h and T_c ($< T_h$), the Carnot efficiency is given by Eq. (1.1). For convenience, we introduce a function $\phi(t)$ to describe the time evolution of the temperature as

$$\frac{1}{T(t)} = \frac{1}{T_c} - \left(\frac{1}{T_c} - \frac{1}{T_h} \right) \phi(t) = \frac{1}{T_c} \left[1 - \eta_{\mathrm{C}} \phi(t) \right]. \tag{5.80}$$

Since the internal energy change $E(\Delta t_{\mathrm{cyc}}) - E(0)$ should vanish after the cycle operates, the output work W and heat Q satisfy

$$W = Q, \tag{5.81}$$

where we used Eq. (5.40). After one cycle, the entropy change in Eq. (5.44) of the system also vanish. Then, the entropy production of the total system in Eq. (5.43) per cycle satisfies

$$\Sigma_{\mathrm{tot}} = - \int_0^{\Delta t_{\mathrm{cyc}}} dt \frac{\dot{Q}(t)}{T(t)} = - \frac{1}{T_c} \int_0^{\Delta t_{\mathrm{cyc}}} dt \dot{W} + \frac{\eta_{\mathrm{C}}}{T_c} \int_0^{\Delta t_{\mathrm{cyc}}} dt \phi(t) \dot{Q} \tag{5.82}$$

When we consider the Carnot cycle, $\phi(t) = 1$ is satisfied in the hot isothermal process with T_h, and $\phi(t) = 0$ is satisfied in the cold isothermal process with T_c. Moreover, the heat current does not flow in the adiabatic processes. Thus, we can interpret

$$Q_h \equiv \int_0^{\Delta t_{\mathrm{cyc}}} dt \phi(t) \dot{Q} \tag{5.83}$$

as the heat flowing from the hot heat bath to the system. Using Eq. (5.83), we can define the efficiency as

$$\eta \equiv \frac{W}{Q_h}. \tag{5.84}$$

Then, from Eq. (5.82), we obtain

$$\Sigma_{\rm tot} = -\frac{W}{T_c} + \frac{\eta_{\rm C}}{T_c}Q_h = \frac{Q_h}{T_c}(\eta_{\rm C} - \eta). \tag{5.85}$$

Because of Eq. (5.85), the efficiency η approaches the Carnot efficiency $\eta_{\rm C}$ when the entropy production $\Sigma_{\rm tot}$ vanishes. Moreover, the power of this heat engine is given by

$$P \equiv \frac{W}{\Delta t_{\rm cyc}}. \tag{5.86}$$

By using Eq. (5.79) and Cauchy-Schwarz inequality, we can show the inequality between Q_h and $\Sigma_{\rm tot}$ as

$$\begin{aligned}(Q_h)^2 &= \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi(t)\ \dot{Q}(t)\right)^2 \leq \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi(t)\sqrt{\gamma T(t)\sigma_v \dot{\Sigma}}\right)^2 \\ &\leq \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi^2(t)\gamma T(t)\sigma_v\right)\left(\int_0^{\Delta t_{\rm cyc}} dt\ \dot{\Sigma}\right) = \Delta t_{\rm cyc} T_c^2 \chi \Sigma_{\rm tot},\end{aligned} \tag{5.87}$$

where

$$\chi \equiv \frac{1}{\Delta t_{\rm cyc} T_c^2}\int_0^{\Delta t_{\rm cyc}} dt\ \phi^2(t)\gamma T(t)\sigma_v = \frac{\gamma}{\Delta t_{\rm cyc} T_c}\int_0^{\Delta t_{\rm cyc}} dt\ \frac{\phi^2(t)}{1-\eta_{\rm C}\phi(t)}\sigma_v(t). \tag{5.88}$$

The work W and Q_h should be positive in the heat engine. Then, using Eqs. (5.85), (5.86), and (5.87), we can show the trade-off relation between the efficiency and power in the underdamped Brownian heat engine as

$$P = \frac{W}{\Delta t_{\rm cyc}} = \frac{W}{Q_h}\frac{1}{Q_h}\frac{Q_h^2}{\Delta t_{\rm cyc}} \leq \eta\frac{1}{Q_h}T_c^2\chi\Sigma = \chi T_c \eta(\eta_{\rm C} - \eta). \tag{5.89}$$

From Eq. (5.89), we find that the power vanishes when the efficiency approaches the Carnot efficiency when χ is finite. However, if χ diverges and η approaches $\eta_{\rm C}$ simultaneously, we may achieve compatibility of the Carnot efficiency and finite power.

Chapter 6

Compatibility of the Carnot efficiency and finite power in the small temperature-difference regime

6.1 Introduction

The Carnot cycle is one of the most important models of heat engines, which operates between hot and cold heat baths with constant temperatures T_h and T_c ($< T_h$). Moreover, the cycle is composed of two isothermal processes and two adiabatic processes. Carnot demonstrated that the efficiency of any heat engine is limited by the upper bound called the Carnot efficiency [2]:

$$\eta_{\mathrm{C}} \equiv 1 - \frac{T_c}{T_h}. \tag{6.1}$$

It is known that we can reach the Carnot efficiency by the reversible cycle, where the heat engine always remains at equilibrium and is typically operated quasistatically, which implies that the engine spends an infinitely long time per cycle. Moreover, power, defined as output work per unit time, is another important quantity for evaluating the performance of heat engines. When we operate the heat engines quasistatically, power vanishes. Thus, several studies have been devoted to investigating the feasibility of finite-power heat engines with Carnot efficiency [3–5, 5–16].

However, as shown in Chap. 4, the trade-off relation between power P and efficiency η recently proved in general heat engines described by the Markov process [29–31]. The trade-off relation is given by

$$P \leq A\eta(\eta_{\mathrm{C}} - \eta), \tag{6.2}$$

where A is a positive quantity depending on the heat engine details. Based on this relation, the power should vanish as the efficiency approaches the Carnot efficiency. Similar trade-off relations to Eq. (6.2) have been obtained in various heat engine models [25, 32–34]. In particular, Dechant and Sasa derived a specific expression of A for stochastic heat engines described by the Langevin equation [25].

Recently, Holubec and Ryabov reported that the Carnot efficiency could be obtained in a general class of finite-power Carnot cycles in the vanishing limit of the relaxation times [23]. Although this result seems to contradict the trade-off relation in Eq. (6.2), they pointed out the possibility that A in Eq. (6.2) diverges in the vanishing limit of the relaxation times, and the Carnot efficiency and finite power are compatible without breaking the trade-off relation in Eq. (6.2). Thus, it may be interesting to study how the efficiency and power depend on the relaxation times in more detail by using a specific model.

The Brownian Carnot cycle with instantaneous adiabatic processes and a time-dependent harmonic potential is a simple model, which is easy to analyze and is frequently used to study the efficiency and power [17, 22, 23, 90]. However, it is pointed out that the instantaneous adiabatic process in the overdamped Brownian Carnot cycle inevitably causes a heat leakage [17, 90, 91]. In the overdamped dynamics, the inertial effect of the Brownian particle is disregarded, and the system is only described by its position. Nevertheless, heat leakage is related to the kinetic energy of the particle, as seen below. When the overdamped limit is considered in the underdamped dynamics, the averaged kinetic energy of the Brownian particle is equal to $k_B T/2$ in the isothermal process with temperature T, where k_B is the Boltzmann constant. Then, after the instantaneous adiabatic processes in the above cycle, the kinetic energy relaxes toward the temperature of the subsequent isothermal process, and an additional heat proportional to the temperature difference flows. This heat leakage decreases the efficiency of the cycle. Thus, we must consider the underdamped dynamics to evaluate the effect of the heat leakage on the efficiency and power of the Brownian Carnot cycle with the instantaneous adiabatic processes.

The rest of this chapter is organized as follows. In Sec. 6.2, we introduce the Brownian particle trapped by the harmonic potential and describe it by the underdamped Langevin equation. We also introduce the isothermal process and instantaneous adiabatic process in this section. In Sec. 6.3, we construct the Carnot cycle using the Brownian particle. In Sec. 6.4, we present the results of numerical simulations of the underdamped Brownian Carnot cycle when we vary the temperature difference and the relaxation times of the system. From these results, we demonstrate that the efficiency of our cycle approaches the Carnot efficiency while maintaining finite power as the relaxation times vanish in the small temperature-difference regime. In Sec. 6.5, we explain the results of the numerical simulations in Sec. 6.4 based on the trade-off relation in Eq. (6.2). Section 6.6 presents the summary and discussion.

6.2 Model

6.2.1 Underdamped system

We consider the underdamped Brownian particle in contact with the heat bath with the time-dependent temperature $T(t)$ and trapped by a harmonic potential

$$V(x,t) = \frac{1}{2}\lambda(t)x^2, \tag{6.3}$$

where the stiffness $\lambda(t)$ depends on the time t. In this chapter, we only consider the case that the temperature is independent of the time. We can describe the dynamics of the Brownian particle by the underdamped Langevin equations given by

$$\dot{x} = v, \tag{6.4}$$

$$m\dot{v} = -\gamma v - \lambda x + \sqrt{2\gamma k_{\mathrm{B}} T}\xi, \tag{6.5}$$

where x, v, and m are the position, velocity, and mass of the particle, respectively. In the following, we set the Boltzmann constant $k_B = 1$ for simplicity. γ is the friction constant and is assumed to be independent of the temperature $T(t)$. The Gaussian white noise $\xi(t)$ satisfies $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$, where $\langle \cdots \rangle$ denotes the statistical average. The dot denotes the time derivative or a quantity per unit time. We introduce the distribution function $p(x, v, t)$ to describe the state of the system at time t. The time evolution of $p(x, v, t)$ can be described by the Kramers equation [75] corresponding to Eqs. (6.4) and (6.5),

$$\begin{aligned}\frac{\partial}{\partial t}p(x,v,t) &= -\frac{\partial}{\partial x}(vp(x,v,t)) + \frac{\partial}{\partial v}\left[\frac{\gamma}{m}v + \frac{\lambda}{m}x + \frac{\gamma T}{m^2}\frac{\partial}{\partial v}\right]p(x,v,t) \\ &= -\frac{\partial}{\partial x}j_x(x,v,t) - \frac{\partial}{\partial v}j_v(x,v,t),\end{aligned} \tag{6.6}$$

where $j_x(x, v, t)$ and $j_v(x, v, t)$ are the probability currents defined as follows:

$$j_x(x,v,t) \equiv vp(x,v,t), \tag{6.7}$$

$$j_v(x,v,t) \equiv -\left[\frac{\gamma}{m}v + \frac{\lambda}{m}x + \frac{\gamma T}{m^2}\frac{\partial}{\partial v}\right]p(x,v,t). \tag{6.8}$$

The relaxation times of position τ_x and velocity τ_v of the Brownian particle are defined as

$$\tau_x(t) \equiv \frac{\gamma}{\lambda(t)}, \tag{6.9}$$

$$\tau_v \equiv \frac{m}{\gamma}, \tag{6.10}$$

where $\tau_x(t)$ depends on time through the stiffness $\lambda(t)$. Note that we fix γ and change τ_x and τ_v by changing m and λ. Here, we define $\sigma_x(t) \equiv \langle x^2 \rangle$, $\sigma_v(t) \equiv \langle v^2 \rangle$, and $\sigma_{xv}(t) \equiv \langle xv \rangle$. Then, assuming that the probability distribution $p(x, v, t)$ is Gaussian, we obtain

$$p(x,v,t) = \frac{1}{\sqrt{4\pi^2\Phi}}\exp\left\{-\frac{\sigma_x v^2 + \sigma_v x^2 - 2\sigma_{xv}xv}{2\Phi}\right\}, \tag{6.11}$$

where we defined the quantity $\Phi(t)$ as

$$\Phi(t) \equiv \sigma_x(t)\sigma_v(t) - \sigma_{xv}(t)^2. \tag{6.12}$$

From the Cauchy-Schwarz inequality, Φ should satisfy

$$\Phi(t) \geq 0. \tag{6.13}$$

Then, from Eqs. (6.11) and (6.12), we can see that the state of the Brownian particle is described by only the three variables $\sigma_x(t)$, $\sigma_v(t)$, and $\sigma_{xv}(t)$. From Eq. (6.6), we can derive the time evolution equations of σ_x, σ_v, and σ_{xv} [22] as

$$\dot{\sigma}_x = 2\sigma_{xv}, \tag{6.14}$$

$$\dot{\sigma}_v = \frac{2\gamma T}{m^2} - \frac{2\gamma}{m}\sigma_v - \frac{2\lambda}{m}\sigma_{xv}, \tag{6.15}$$

$$\dot{\sigma}_{xv} = \sigma_v - \frac{\lambda}{m}\sigma_x - \frac{\gamma}{m}\sigma_{xv}. \tag{6.16}$$

We can use Eqs. (6.14)–(6.16) to describe the time evolution of the system instead of Eq. (6.6). Under the Gaussian distribution in Eq. (6.11), the internal energy $E(t)$ and entropy $S(t)$ of the particle are given by

$$E(t) \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ p(x,v,t)\ \left[\frac{1}{2}mv^2 + \frac{1}{2}\lambda(t)x^2\right] = \frac{1}{2}m\sigma_v(t) + \frac{1}{2}\lambda(t)\sigma_x(t), \quad (6.17)$$

$$S(t) \equiv -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ p(x,v,t)\ln\{p(x,v,t)\} = \frac{1}{2}\ln\Phi(t) + \ln(2\pi) + 1. \quad (6.18)$$

6.2.2 Isothermal process

We define the heat and work during a time interval $t_i < t < t_f$ in an isothermal process. In this process, the Brownian particle interacts with the heat bath at a constant temperature T. We assume that the stiffness $\lambda(t)$ changes smoothly in this process. The heat current $\dot{Q}$ flowing from the heat bath to the Brownian particle is defined as the statistical average of the work performed by the force from the heat bath to the Brownian particle (see Chap. 4 of Ref. [28]),

$$\dot{Q}(t) \equiv \left\langle \left(-\gamma v + \sqrt{2\gamma T}\xi(t)\right) \circ v \right\rangle, \quad (6.19)$$

where $\circ$ represents the Stratonovich-type product. Using Eqs. (6.4) and (6.5), we derive the heat current $\dot{Q}(t)$ as follows:

$$\dot{Q}(t) = \frac{1}{2}\lambda(t)\dot{\sigma}_x(t) + \frac{1}{2}m\dot{\sigma}_v(t). \quad (6.20)$$

Thus, we obtain the heat Q flowing in this interval as

$$Q = \int_{t_i}^{t_f} dt\ \left(\frac{1}{2}\lambda\dot{\sigma}_x\right) + \int_{t_i}^{t_f} dt\ \left(\frac{1}{2}m\dot{\sigma}_v\right) = Q^{\mathrm{o}} + \Delta K, \quad (6.21)$$

where

$$Q^{\mathrm{o}} \equiv \int_{t_i}^{t_f} dt\ \left(\frac{1}{2}\lambda\dot{\sigma}_x\right), \quad (6.22)$$

$$\Delta K \equiv \frac{1}{2}m\sigma_v(t_f) - \frac{1}{2}m\sigma_v(t_i). \quad (6.23)$$

Here, Q^{o} represents the heat related to the potential change, and ΔK is the difference between the initial and final (averaged) kinetic energies of the Brownian particle. In the overdamped system [17], Q^{o} is regarded as the heat instead of Q in Eq. (6.21). However, in the underdamped system under consideration, the heat also includes the kinetic part ΔK.

The output work during this interval is defined as follows:

$$\begin{aligned} W \equiv &-\int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ p(x,v,t)\frac{\partial V(x,t)}{\partial t} = -\frac{1}{2}\int_{t_i}^{t_f} dt\ \dot{\lambda}\sigma_x \\ &= Q - \Delta E, \end{aligned} \quad (6.24)$$

where we used Eqs. (6.17) and (6.21) for the derivation from the middle to the last equality, and defined $\Delta E \equiv E(t_f) - E(t_i)$. The last equality in Eq. (6.24) represents the first law of thermodynamics.

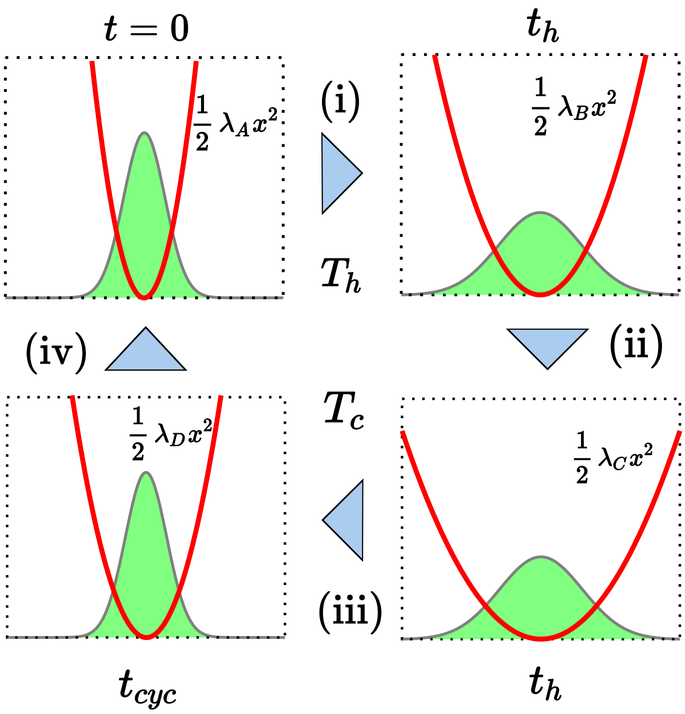

Figure 6.1: Schematic illustration of the Brownian Carnot cycle. In each box, the bottom horizontal line denotes the position coordinate x, and the boundary curve of the green filled area denotes the probability distribution of x. The red solid line corresponds to the harmonic potential. This cycle is composed of (i) hot isothermal process, (ii) instantaneous adiabatic process, (iii) cold isothermal process, and (iv) instantaneous adiabatic process.

6.2.3 Instantaneous adiabatic process

As an adiabatic process connecting the end of the isothermal process with temperature T_i to the beginning of the next isothermal process with temperature T_f, we use instantaneous changes in the potential and heat bath at $t = t_0$, which we regard as the final time of the isothermal process with temperature T_i [17]. In this process, the stiffness $\lambda(t)$ jumps from λ_i to λ_f, and we instantaneously switch the temperature of the heat bath from T_i to T_f, maintaining the probability distribution unchanged. Because this process is instantaneous, no heat exchange occurs, and the output work $W^{\rm ad}_{i\to f}$ is equal to the negative value of the internal energy change $\Delta E^{\rm ad}_{i\to f}$ due to the first law of thermodynamics as

$$W^{\rm ad}_{i\to f} = -\Delta E^{\rm ad}_{i\to f} = -\frac{1}{2}(\lambda_f - \lambda_i)\sigma_x(t_0). \tag{6.25}$$

6.3 Carnot cycle

We construct a Carnot cycle operating between the two heat baths with the temperatures T_h and T_c (see Fig. 6.1) by combining the isothermal processes and the instantaneous adiabatic processes introduced in Sec. 6.2.

First, we define a protocol of a finite-time Carnot cycle with stiffness $\lambda(t)$ as follows: The hot isothermal process with temperature T_h lasts for $0 < t < \Delta t_h$, and the stiffness λ varies from λ_A to λ_B [Fig. 6.1(i)]. In the following instantaneous adiabatic process, we switch the stiffness from λ_B to λ_C and the temperature of the heat bath from T_h to

T_c at $t = \Delta t_h$, [Fig. 6.1(ii)]. The cold isothermal process with temperature T_c lasts for $\Delta t_h < t < \Delta t_h + \Delta t_c$, and the stiffness λ varies from λ_C to λ_D [Fig. 6.1(iii)]. In the last instantaneous adiabatic process, we switch the stiffness from λ_D to λ_A and the temperature of the heat bath from T_c to T_h at $t = \Delta t_{\rm cyc}$, Fig. 6.1(iv), where $\Delta t_{\rm cyc} \equiv \Delta t_h + \Delta t_c$ is the cycle time, which is assumed nonzero. The final state of the Brownian particle in the cold (hot) isothermal process should agree with the initial state in the hot (cold) isothermal process.

We assume that the stiffness $\lambda(t)$ can be expressed as follows:

$$\lambda(t) = \Lambda(s) \quad \left(s \equiv \frac{t}{\Delta t_{\rm cyc}}\right), \tag{6.26}$$

using the scaling function $\Lambda(s)$ ($0 \leq s \leq 1$). Under this assumption, we can change the time scale of the protocol maintaining the protocol form unchanged, by selecting another value of $\Delta t_{\rm cyc}$. We also assume that $\Delta t_h/\Delta t_{\rm cyc}$ and $\Delta t_c/\Delta t_{\rm cyc}$ are finitely fixed for any value of $\Delta t_{\rm cyc}$. Furthermore, we assume that $\lambda(t_2)/\lambda(t_1)$ is finite at any time t_2 and t_1, where they are in the same isothermal process. We use this assumption to show that the heat current after the relaxation at the beginning of the isothermal processes is noninfinite in the Appendix C. Note that the word "finite" may situationally be used considering two meanings, "nonzero" (e.g., "finite power") or "noninfinite" (e.g., "finite time"). In this chapter, however, we refer to "nonzero and noninfinite" by "finite" except for the two examples above.

To consider the quasistatic Carnot cycle corresponding to the above finite-time Carnot cycle, we must consider the limit of $\Delta t_{\rm cyc} \to \infty$ and use the stiffness $\lambda^{\rm qs}(t)$ related to the finite-time stiffness through Eq. (6.26). Here, the index "qs" of $X^{\rm qs}$ denotes the physical quantity X evaluated in the quasistatic limit.

6.3.1 Quasistatic Carnot cycle: Quasistatic efficiency

We formulate the efficiency of the quasistatic Carnot cycle. To this end, we need to quantify the heat leakage caused by the adiabatic process. As the adiabatic processes are instantaneous, the initial distributions of the quasistatic isothermal processes do not agree with the equilibrium distributions at the temperature of the heat bath. Thus, a relaxation at the beginning of the isothermal processes exists, and in general, the relaxation is irreversible. After the relaxation in the quasistatic isothermal process with temperature T, the time derivative of the variables satisfies

$$\dot{\sigma}_x^{\rm qs}(t) = 0, \quad \dot{\sigma}_v^{\rm qs}(t) = 0, \quad \dot{\sigma}_{xv}^{\rm qs}(t) = 0. \tag{6.27}$$

Subsequently, from Eqs. (6.14)–(6.16), we obtain those values as follows:

$$\sigma_x^{\rm qs}(t) = \frac{T}{\lambda^{\rm qs}(t)}, \quad \sigma_v^{\rm qs}(t) = \frac{T}{m}, \quad \sigma_{xv}^{\rm qs}(t) = 0, \tag{6.28}$$

and the distribution in Eq. (6.11) in the quasistatic limit agrees with the Boltzmann distribution

$$p^{\rm qs}(x, v, t) = \sqrt{\frac{m\lambda^{\rm qs}(t)}{4\pi^2 T^2}} \exp\left\{-\frac{\lambda^{\rm qs}(t)x^2 + mv^2}{2T}\right\}. \tag{6.29}$$

After the relaxation in each quasistatic isothermal process, the system is in equilibrium with the heat bath and satisfies Eq. (6.28). Using Eqs. (6.18) and (6.28), we derive the quasistatic entropy as follows:

$$
\begin{aligned}
S^{\mathrm{qs}}(t) =& \frac{1}{2}\ln\sigma_x^{\mathrm{qs}}(t) + \frac{1}{2}\ln\sigma_v^{\mathrm{qs}}(t) + \ln(2\pi) + 1 \\
=& \frac{1}{2}\ln\left(\frac{T}{\lambda(t)}\right) + \frac{1}{2}\ln\left(\frac{T}{m}\right) + \ln(2\pi) + 1.
\end{aligned} \tag{6.30}
$$

As mentioned above, the quasistatic isothermal processes are composed of the relaxation part and the part after the relaxation. Because the instantaneous adiabatic process [Fig. 6.1(iv)] just before the quasistatic hot isothermal process [Fig. 6.1(i)] does not change the probability distribution, the initial distribution agrees with the final distribution in the quasistatic cold isothermal process. Thus, the variables σ_x^{qs}, σ_v^{qs}, and $\sigma_{xv}^{\mathrm{qs}}$ begin the quasistatic hot isothermal process with the following values:

$$
\sigma_x^{\mathrm{qs}} = \frac{T_c}{\lambda_D^{\mathrm{qs}}}, \quad \sigma_v^{\mathrm{qs}} = \frac{T_c}{m}, \quad \sigma_{xv}^{\mathrm{qs}} = 0, \tag{6.31}
$$

where we used Eq. (6.28). In the relaxation at the beginning of this process, the stiffness almost remains λ_A^{qs}, and the variables relax to

$$
\sigma_x^{\mathrm{qs}} = \frac{T_h}{\lambda_A^{\mathrm{qs}}}, \quad \sigma_v^{\mathrm{qs}} = \frac{T_h}{m}, \quad \sigma_{xv}^{\mathrm{qs}} = 0, \tag{6.32}
$$

owing to Eq. (6.28).

From Eqs. (6.31) and (6.32), the kinetic energy is $m\sigma_v/2 = T_c/2$ in the initial state and changes to $T_h/2$ during the relaxation. The kinetic energy remains $T_h/2$ after the relaxation because the system is in equilibrium with the heat bath at temperature T_h during the quasistatic hot isothermal process. Thus, a change in the kinetic energy in Eq. (6.23) in the quasistatic hot isothermal process is given by

$$
\Delta K_h^{\mathrm{qs}} = \frac{\Delta T}{2}, \tag{6.33}
$$

where $\Delta T \equiv T_h - T_c$. We can also derive the heat related to the potential change during the relaxation $Q_h^{\mathrm{rel,o,qs}}$ as follows. As the stiffness remains λ_A^{qs} during the relaxation, $Q_h^{\mathrm{rel,o,qs}}$ is derived as

$$
Q_h^{\mathrm{rel,o,qs}} = \int_{T_c/\lambda_D^{\mathrm{qs}}}^{T_h/\lambda_A^{\mathrm{qs}}} \frac{1}{2}\lambda_A^{\mathrm{qs}} d\sigma_x = \frac{1}{2}\lambda_A^{\mathrm{qs}}\left(\frac{T_h}{\lambda_A^{\mathrm{qs}}} - \frac{T_c}{\lambda_D^{\mathrm{qs}}}\right), \tag{6.34}
$$

using Eq. (6.22). The entropy change of the Brownian particle in this relaxation is given by

$$
\Delta S_h^{\mathrm{rel,qs}} \equiv \frac{1}{2}\ln\left(\frac{T_h}{\lambda_A^{\mathrm{qs}}}\frac{\lambda_D^{\mathrm{qs}}}{T_c}\right) + \frac{1}{2}\ln\left(\frac{T_h}{T_c}\right), \tag{6.35}
$$

where we used Eqs. (6.30)–(6.32).

After the relaxation in the quasistatic hot isothermal process, the probability distribution maintains the Boltzmann distribution in Eq. (6.29) with $T = T_h$, and σ_v does not change. Therefore, the final state of the process should satisfy

$$
\sigma_x^{\mathrm{qs}} = \frac{T_h}{\lambda_B^{\mathrm{qs}}} \quad \sigma_v^{\mathrm{qs}} = \frac{T_h}{m}, \quad \sigma_{xv}^{\mathrm{qs}} = 0, \tag{6.36}
$$

where we used Eq. (6.28). Because the second term on the right-hand side of Eq. (6.30) does not change in the quasistatic hot isothermal process, we derive the entropy change $\Delta S_h^{\mathrm{iso,qs}}$ after the relaxation in this process as follows:

$$\Delta S_h^{\mathrm{iso,qs}} \equiv \frac{1}{2} \ln \left(\frac{\lambda_A^{\mathrm{qs}}}{\lambda_B^{\mathrm{qs}}} \right). \tag{6.37}$$

Note that the quantities with the index "*iso*" do not include the contribution from the relaxation. Thus, the heat supplied to the Brownian particle after the relaxation in this process is given by

$$T_h \Delta S_h^{\mathrm{iso,qs}} = \frac{T_h}{2} \ln \left(\frac{\lambda_A^{\mathrm{qs}}}{\lambda_B^{\mathrm{qs}}} \right). \tag{6.38}$$

The heat related to the potential change in the quasistatic hot isothermal process is

$$Q_h^{\mathrm{o,qs}} = T_h \Delta S_h^{\mathrm{iso,qs}} + Q_h^{\mathrm{rel,o,qs}}. \tag{6.39}$$

Therefore, by using Eq. (6.33), the heat flowing in the quasistatic hot isothermal process is given by

$$\begin{aligned} Q_h^{\mathrm{qs}} =& Q_h^{\mathrm{o,qs}} + \Delta K_h^{\mathrm{qs}} = T_h \Delta S_h^{\mathrm{iso,qs}} + Q_h^{\mathrm{rel,o,qs}} + \frac{1}{2}\Delta T \\ =& T_h \Delta S_h^{\mathrm{iso,qs}} + Q_h^{\mathrm{rel,qs}}, \end{aligned} \tag{6.40}$$

where $Q_h^{\mathrm{rel,qs}}$ denotes the heat flowing during the relaxation at the beginning of this process, as

$$Q_h^{\mathrm{rel,qs}} \equiv Q_h^{\mathrm{rel,o,qs}} + \frac{1}{2}\Delta T. \tag{6.41}$$

From Eq. (6.24), the work in this process is given by

$$W_h^{\mathrm{qs}} = Q_h^{\mathrm{qs}} - \Delta E_h^{\mathrm{qs}}, \tag{6.42}$$

where ΔE_h^{qs} represents the internal energy change in this process.

After the instantaneous adiabatic process [Fig. 6.1(ii)], the quasistatic cold isothermal process [Fig. 6.1(iii)] begins with the variables in Eq. (6.36), and the variables relax to

$$\sigma_x^{\mathrm{qs}} = \frac{T_c}{\lambda_C^{\mathrm{qs}}} \quad \sigma_v^{\mathrm{qs}} = \frac{T_c}{m}, \quad \sigma_{xv}^{\mathrm{qs}} = 0, \tag{6.43}$$

where we used Eq. (6.28). Similar to the quasistatic hot isothermal process, the change in the kinetic energy in Eq. (6.23) satisfies

$$\Delta K_c^{\mathrm{qs}} = -\frac{\Delta T}{2}. \tag{6.44}$$

We also define the heat related to the potential change during the relaxation in the quasistatic cold isothermal process as

$$Q_c^{\mathrm{rel,o,qs}} \equiv \frac{1}{2} \lambda_C^{\mathrm{qs}} \left(\frac{T_c}{\lambda_C^{\mathrm{qs}}} - \frac{T_h}{\lambda_B^{\mathrm{qs}}} \right). \tag{6.45}$$

Then, the flowing heat and the entropy change of the particle during this relaxation are given by

$$Q_c^{\text{rel,qs}} \equiv \frac{1}{2}\lambda_C^{\text{qs}}\left(\frac{T_c}{\lambda_C^{\text{qs}}} - \frac{T_h}{\lambda_B^{\text{qs}}}\right) - \frac{1}{2}\Delta T, \tag{6.46}$$

$$\Delta S_c^{\text{rel,qs}} \equiv \frac{1}{2}\ln\left(\frac{T_c}{\lambda_C^{\text{qs}}}\frac{\lambda_B^{\text{qs}}}{T_h}\right) + \frac{1}{2}\ln\left(\frac{T_c}{T_h}\right), \tag{6.47}$$

similarly to Eqs. (6.35) and (6.41), where we used Eqs. (6.22), (6.30), (6.36), and (6.43)–(6.45).

After the relaxation, the variables change to the state in Eq. (6.31). Then, the entropy change after the relaxation in the quasistatic cold isothermal process is given by

$$\Delta S_c^{\text{iso,qs}} \equiv \frac{1}{2}\ln\left(\frac{\lambda_C^{\text{qs}}}{\lambda_D^{\text{qs}}}\right). \tag{6.48}$$

The heat related to the potential change in the quasistatic cold isothermal process is

$$Q_c^{\text{o,qs}} = T_c\Delta S_c^{\text{iso,qs}} + Q_h^{\text{rel,o,qs}}, \tag{6.49}$$

where we used Eqs. (6.45) and (6.48). Thus, the heat flowing in the quasistatic cold isothermal process is given by

$$\begin{aligned} Q_c^{\text{qs}} =& Q_c^{\text{o,qs}} + \Delta K_c^{\text{qs}} \\ =& T_c\Delta S_c^{\text{iso,qs}} + Q_c^{\text{rel,qs}}, \end{aligned} \tag{6.50}$$

where we used Eqs. (6.44)–(6.49). From Eq. (6.24), the work in this process is given by

$$W_c^{\text{qs}} = Q_c^{\text{qs}} - \Delta E_c^{\text{qs}}, \tag{6.51}$$

where ΔE_c^{qs} is the internal energy change in this process. After the quasistatic cold isothermal process [Fig. 6.1(iii)], the system proceeds to the instantaneous adiabatic process [Fig. 6.1(iv)] and returns to the initial state of the quasistatic hot isothermal process.

Subsequently, we consider the efficiency of the quasistatic Carnot cycle. As the cycle closes, the entropy change in the particle per cycle vanishes as

$$\Delta S_h^{\text{rel,qs}} + \Delta S_h^{\text{iso,qs}} + \Delta S_c^{\text{rel,qs}} + \Delta S_c^{\text{iso,qs}} = 0, \tag{6.52}$$

where we used Eqs. (6.35), (6.37), (6.47), and (6.48). Because the internal energy change in the particle per cycle vanishes, we derive the work per cycle from the first law of thermodynamics as

$$W^{\text{qs}} = Q_h^{\text{qs}} + Q_c^{\text{qs}}, \tag{6.53}$$

using Eqs. (6.42) and (6.51). In our quasistatic cycle, the entropy production per cycle Σ^{qs}, by which we imply the total entropy production per cycle including the particle and heat baths, is obtained as follows:

$$\Sigma^{\text{qs}} \equiv -\frac{Q_h^{\text{qs}}}{T_h} - \frac{Q_c^{\text{qs}}}{T_c}. \tag{6.54}$$

Because an entropy change in the particle per cycle vanishes, as seen from Eq. (6.52), the entropy production per cycle Σ^{qs} is expressed only by the entropy change of the heat baths. Using Eqs. (6.40), (6.53), and (6.54), we can derive the quasistatic efficiency as

$$\eta^{\rm qs} \equiv \frac{W^{\rm qs}}{Q_h^{\rm qs}} = \eta_{\rm C} - \frac{T_c \Sigma^{\rm qs}}{Q_h^{\rm qs}}. \tag{6.55}$$

From Eq. (6.55), $\Sigma^{\rm qs}$ should vanish to obtain $\eta_{\rm C}$. Using Eqs. (6.40), (6.50), and (6.52), we can rewrite $\Sigma^{\rm qs}$ in Eq. (6.54) as

$$\begin{aligned}\Sigma^{\rm qs} =& -\frac{T_h \Delta S_h^{\rm iso,qs} + Q_h^{\rm rel,qs}}{T_h} - \frac{T_c \Delta S_c^{\rm iso,qs} + Q_c^{\rm rel,qs}}{T_c} \\ =& \Delta S_h^{\rm rel,qs} - \frac{Q_h^{\rm rel,qs}}{T_h} + \Delta S_c^{\rm rel,qs} - \frac{Q_c^{\rm rel,qs}}{T_c} \\ =& \frac{1}{2}\left(-\ln\left(\frac{T_c\lambda_A^{\rm qs}}{T_h\lambda_D^{\rm qs}}\right) + \frac{T_c\lambda_A^{\rm qs}}{T_h\lambda_D^{\rm qs}} - 1\right) + \frac{1}{2}\left(-\ln\left(\frac{T_h\lambda_C^{\rm qs}}{T_c\lambda_B^{\rm qs}}\right) + \frac{T_h\lambda_C^{\rm qs}}{T_c\lambda_B^{\rm qs}} - 1\right) + \frac{(\Delta T)^2}{2T_hT_c},\end{aligned} \tag{6.56}$$

where we used Eqs. (6.35), (6.41), (6.46), and (6.47) at the last equality. The first and second terms on the right-hand side of Eq. (6.56), derived from $Q_{h,c}^{\rm rel,o,qs}$ in Eqs. (6.34) and (6.45) and the first term of $\Delta S_{h,c}^{\rm rel,qs}$ in Eqs. (6.35) and (6.47), denote the entropy production related to the potential energy in the relaxation in the hot and cold isothermal processes, respectively. The last term of Eq. (6.56) comes from the heat related to the kinetic energy. To achieve the Carnot efficiency, the entropy production should vanish, as shown in Eq. (6.55). In the overdamped Brownian Carnot cycle with the instantaneous adiabatic process in previous studies [16, 17, 23], the Carnot efficiency is obtained in the quasistatic limit. In the overdamped cycle, $(\Delta T)^2/(2T_hT_c)$ in Eq. (6.56) does not exist because σ_v is not considered. Thus, the entropy production in the overdamped cycle is given by

$$\Sigma^{\rm o,qs} \equiv f\left(\frac{T_c\lambda_A^{\rm qs}}{T_h\lambda_D^{\rm qs}}\right) + f\left(\frac{T_h\lambda_C^{\rm qs}}{T_c\lambda_B^{\rm qs}}\right), \tag{6.57}$$

where f is defined as

$$f(u) \equiv -\ln u + u - 1, \tag{6.58}$$

where $f(u)$ is a downwardly convex function with the minimum value of $f(1) = 0$. Thus, for the entropy production $\Sigma^{\rm o,qs}$ to vanish, the following condition is derived:

$$\frac{T_h}{\lambda_A^{\rm qs}} = \frac{T_c}{\lambda_D^{\rm qs}}, \quad \frac{T_h}{\lambda_B^{\rm qs}} = \frac{T_c}{\lambda_C^{\rm qs}}. \tag{6.59}$$

This condition was adopted in the previous studies on the overdamped Brownian Carnot cycle [16, 17, 23] in the quasistatic limit. We impose this condition on our underdamped cycle to reduce entropy production. Then, we obtain

$$\Delta S_h^{\rm rel,qs} + \Delta S_c^{\rm rel,qs} = 0, \tag{6.60}$$

using Eqs. (6.35) and (6.47). Thus, from Eq. (6.52), we derive

$$\Delta S_h^{\rm iso,qs} = -\Delta S_c^{\rm iso,qs} \equiv \Delta S^{\rm qs}. \tag{6.61}$$

In addition, because $Q_h^{\rm rel,o,qs}$ in Eq. (6.34) and $Q_c^{\rm rel,o,qs}$ in (6.45) vanish, we obtain

$$Q_h^{\mathrm{o,qs}} = T_h \Delta S^{\mathrm{qs}}, \quad Q_c^{\mathrm{o,qs}} = -T_c \Delta S^{\mathrm{qs}}, \tag{6.62}$$

$$Q_h^{\mathrm{rel,qs}} = -Q_c^{\mathrm{rel,qs}} = \frac{1}{2}\Delta T, \tag{6.63}$$

using Eqs. (6.39), (6.41), (6.46), and (6.49). The heat in Eqs. (6.40) and (6.50) can also be rewritten as follows:

$$Q_h^{\mathrm{qs}} = T_h \Delta S^{\mathrm{qs}} + \frac{1}{2}\Delta T, \quad Q_c^{\mathrm{qs}} = -T_c \Delta S^{\mathrm{qs}} - \frac{1}{2}\Delta T. \tag{6.64}$$

Using Eqs. (6.53) and (6.64), We can rewrite the work in Eq. (6.53) and the efficiency in Eq. (6.55) as follows:

$$W^{\mathrm{qs}} = \Delta T \Delta S^{\mathrm{qs}}, \tag{6.65}$$

$$\eta^{\mathrm{qs}} \equiv \frac{W^{\mathrm{qs}}}{Q_h^{\mathrm{qs}}} = \frac{\Delta T \Delta S^{\mathrm{qs}}}{T_h \Delta S^{\mathrm{qs}} + \frac{1}{2}\Delta T} < \eta_{\mathrm{C}}. \tag{6.66}$$

Despite considering the quasistatic limit of our Carnot cycle, however, the quasistatic efficiency η^{qs} is smaller than the Carnot efficiency because of the heat leakage $\Delta T/2$ in the denominator in Eq. (6.66), which is derived from a kinetic energy change in the particle due to the relaxation.

Here, we consider the small temperature-difference regime $\Delta T \to 0$ and assume that $\Delta S^{\mathrm{qs}} = O(1) > 0$. Then, we obtain $\Delta T \Delta S^{\mathrm{qs}} = O(\Delta T)$. As the contribution of the heat leakage to η^{qs} in Eq. (6.66) can be of a higher order of ΔT in the small temperature-difference regime, η^{qs} is approximated by the Carnot efficiency as

$$\eta^{\mathrm{qs}} = \frac{\Delta T \Delta S^{\mathrm{qs}}}{T_h \Delta S^{\mathrm{qs}}} + O[(\Delta T)^2] = \eta_{\mathrm{C}} + O[(\Delta T)^2]. \tag{6.67}$$

6.3.2 Finite-time Carnot cycle: Efficiency and power

In the following, we formulate the efficiency and power of the finite-time Carnot cycle. We assume that Eq. (6.59) is satisfied in the quasistatic limit of this cycle. When we use the protocol in Eq. (6.26), we obtain

$$\lambda_i^{\mathrm{qs}} = \lambda_i \quad (i = A, B, C, D), \tag{6.68}$$

and we can remove the index "*qs*" in Eq. (6.59). In general, finite-time processes are irreversible, and the work and heat of the finite-time isothermal processes are different from those of quasistatic processes. Thus, we express the work and heat in our finite-time cycle by using those in the quasistatic limit and the differences between the finite-time and quasistatic quantities. Below, we mainly consider the finite-time Carnot cycle. Thus, when we deal with a finite-time isothermal process or a finite-time cycle, we simply refer to them as an isothermal process or a cycle, respectively. Using Eq. (6.68), we can rewrite the entropy changes in Eqs. (6.37) and (6.48) in terms of the stiffness $\lambda(t)$ as

$$\Delta S_h^{\mathrm{iso,qs}} = \frac{1}{2}\ln\left(\frac{\lambda_A^{\mathrm{qs}}}{\lambda_B^{\mathrm{qs}}}\right) = \frac{1}{2}\ln\left(\frac{\lambda_A}{\lambda_B}\right) = \Delta S^{\mathrm{qs}}, \tag{6.69}$$

$$\Delta S_c^{\mathrm{iso,qs}} = \frac{1}{2}\ln\left(\frac{\lambda_C^{\mathrm{qs}}}{\lambda_D^{\mathrm{qs}}}\right) = \frac{1}{2}\ln\left(\frac{\lambda_C}{\lambda_D}\right) = -\Delta S^{\mathrm{qs}}. \tag{6.70}$$

From Eq. (6.21), we derive the heat flowing from the hot heat bath to the Brownian particle in the hot isothermal process as

$$Q_h = Q_h^{\rm o} + \Delta K_h, \tag{6.71}$$

where

$$\begin{aligned} Q_h^{\rm o} &= \frac{1}{2}\int_0^{\Delta t_h} dt\ \lambda\dot{\sigma}_x, \\ \Delta K_h &= \frac{1}{2}m\sigma_v(\Delta t_h) - \frac{1}{2}m\sigma_v(0). \end{aligned} \tag{6.72}$$

Note that $Q_h^{\rm o}$ and ΔK_h become $Q_h^{o,qs} = T_h\Delta S^{\rm qs}$ in Eq. (6.62) and $\Delta K_h^{\rm qs} = \Delta T/2$ in Eq. (6.33), respectively, under the condition of Eq. (6.59) in the quasistatic limit, as discussed in Sec. 6.3.1. Moreover, we find that Q_h^o and ΔK_h differ from $T_h\Delta S^{\rm qs}$ and $\Delta T/2$ because the process is not quasistatic. Here, we define the irreversible work $W_h^{\rm irr}$ to measure the difference between $Q_h^{\rm o}$ and $T_h\Delta S^{\rm qs}$ as

$$W_h^{\rm irr} \equiv T_h\Delta S^{\rm qs} - Q_h^{\rm o}. \tag{6.73}$$

Then, the heat in the hot isothermal process in Eq. (6.71) can be rewritten as follows:

$$Q_h = T_h\Delta S^{\rm qs} - W_h^{\rm irr} + \Delta K_h, \tag{6.74}$$

using Eqs. (6.71) and (6.73). Moreover, using Eqs. (6.24) and (6.74), we obtain the output work in the hot isothermal process as

$$W_h = T_h\Delta S^{\rm qs} - W_h^{\rm irr} + \Delta K_h - \Delta E_h, \tag{6.75}$$

where ΔE_h represents the internal energy change in this process. The reason that we call $W_h^{\rm irr}$ the irreversible work will be clarified later when we consider the output work per cycle.

The heat in Eq. (6.21) in the cold isothermal process is given by

$$Q_c = Q_c^{\rm o} + \Delta K_c, \tag{6.76}$$

where

$$Q_c^{\rm o} = \frac{1}{2}\int_{\Delta t_h}^{\Delta t_{\rm cyc}} dt\ \lambda\dot{\sigma}_x, \tag{6.77}$$

$$\Delta K_c = \frac{1}{2}m\sigma_v(\Delta t_{\rm cyc}) - \frac{1}{2}m\sigma_v(\Delta t_h) = -\Delta K_h. \tag{6.78}$$

Similar to $Q_h^{\rm o}$ and ΔK_h, $Q_c^{\rm o}$ becomes $-T_c\Delta S^{\rm qs}$ and ΔK_c becomes $-\Delta T/2$ under the condition of Eq. (6.59) in the quasistatic limit. In the same way as the hot isothermal process, we can define the irreversible work $W_c^{\rm irr}$ in this process and rewrite the heat in Eq. (6.76) as follows:

$$W_c^{\rm irr} \equiv -T_c\Delta S^{\rm qs} - Q_c^{\rm o}, \tag{6.79}$$

$$Q_c = -T_c\Delta S^{\rm qs} - W_c^{\rm irr} + \Delta K_c. \tag{6.80}$$

Using Eqs. (6.24) and (6.80), we derive the output work in the cold isothermal process as

$$W_c = -T_c\Delta S^{\mathrm{qs}} - W_c^{\mathrm{irr}} + \Delta K_c - \Delta E_c, \tag{6.81}$$

where ΔE_c represents the internal energy change in this process.

As the cycle closes, the internal energy change per cycle in the particle vanishes. From the first law of thermodynamics, we derive the output work per cycle as

$$W = Q_h + Q_c = \Delta T\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} - W_c^{\mathrm{irr}}, \tag{6.82}$$

using Eqs. (6.75), (6.78), and (6.81). As mentioned above, the irreversible works arise from the irreversibility of the isothermal processes. If the irreversible works in Eq. (6.82) vanish, the work will be the same as W^{qs} in Eq. (6.65). Thus, we call $W_{h,c}^{\mathrm{irr}}$ the irreversible works as the difference between W in Eq. (6.82) and W^{qs}. Using Eqs. (6.74) and (6.82), we obtain the efficiency η and power P of the Carnot cycle as follows:

$$\eta \equiv \frac{W}{Q_h} = \frac{\Delta T\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} - W_c^{\mathrm{irr}}}{T_h\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} + \Delta K_h}, \tag{6.83}$$

$$P \equiv \frac{W}{\Delta t_{\mathrm{cyc}}} = \frac{\Delta T\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} - W_c^{\mathrm{irr}}}{\Delta t_{\mathrm{cyc}}}. \tag{6.84}$$

6.3.3 Small relaxation-times regime

We consider the Carnot cycle in the regime where the relaxation times τ_v and $\tau_x(t)$ $(0 \leq t \leq \Delta t_{\mathrm{cyc}})$ are sufficiently small, which is of our main interest. From Eq. (C.15) in the Appendix C, the kinetic energy in this regime is approximated by

$$\frac{1}{2}m\sigma_v(0) = \frac{1}{2}m\sigma_v(\Delta t_{\mathrm{cyc}}) \simeq \frac{1}{2}T_c, \quad \frac{1}{2}m\sigma_v(\Delta t_h) \simeq \frac{1}{2}T_h. \tag{6.85}$$

Thus, the kinetic energy change in the isothermal processes is given by

$$\Delta K_h = -\Delta K_c \simeq \frac{\Delta T}{2}, \tag{6.86}$$

similarly to the quasistatic case, where we used Eq. (6.78). From Eqs. (6.74), (6.80), and (6.86), the heat in the isothermal processes can be evaluated as follows:

$$Q_h \simeq T_h\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} + \frac{\Delta T}{2}, \tag{6.87}$$

$$Q_c \simeq -T_c\Delta S^{\mathrm{qs}} - W_c^{\mathrm{irr}} - \frac{\Delta T}{2}. \tag{6.88}$$

From Eq. (6.83), the efficiency in the small relaxation-times regime is given by

$$\eta \simeq \frac{\Delta T\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} - W_c^{\mathrm{irr}}}{T_h\Delta S^{\mathrm{qs}} - W_h^{\mathrm{irr}} + \frac{\Delta T}{2}}. \tag{6.89}$$

Holubec and Ryabov pointed out the possibility of obtaining Carnot efficiency in a general class of finite-power Carnot cycle in the vanishing limit of the relaxation times [16,23]. In our underdamped Brownian Carnot cycle, we have to consider the heat leakage [$\Delta T/2$ in the denominator in Eq. (6.89)] because the kinetic energy cannot be neglected. Thus, it may be impossible to achieve the Carnot efficiency in our finite-power Carnot cycle.

Nevertheless, if W_h^{irr} and W_c^{irr} vanish in the vanishing limit of the relaxation times, the efficiency will reach the quasistatic efficiency in Eq. (6.66), and we can achieve the Carnot efficiency as seen from Eq. (6.67) in the small temperature-difference regime. Subsequently, we study how the efficiency and power depend on the relaxation times and temperature difference in Sec. 6.4.

6.4 Numerical simulations

In this section, we show the results of efficiency and power obtained through the numerical simulations of the proposed Brownian Carnot cycle as varying the relaxation times and temperature difference. In these simulations, we solved Eqs. (6.14)–(6.16) numerically by using the fourth-order Runge-Kutta method. The specific protocol $\lambda(t)$ for our simulations is given by

$$\lambda(t) = \begin{cases} \dfrac{T_h}{\sigma_a(1+b_1\frac{t}{\Delta t_h})^2} & (0 \le t \le \Delta t_h) \\ \dfrac{T_c}{\sigma_b(1+b_2\frac{t-\Delta t_h}{\Delta t_c})^2} & (\Delta t_h \le t \le \Delta t_{\text{cyc}}), \end{cases} \tag{6.90}$$

where σ_a and σ_b $(> \sigma_a)$ are positive constants, and we defined $b_1 \equiv \sqrt{\sigma_b/\sigma_a} - 1$ and $b_2 \equiv \sqrt{\sigma_a/\sigma_b} - 1$. This protocol is inspired by the optimal protocol in the overdamped Brownian Carnot cycle [16, 17] and satisfies Eq. (6.59) assigned to the protocol. This protocol also satisfies the scaling condition in Eq. (6.26). For all the simulations, we fixed $\sigma_b/\sigma_a = 2.0$, $T_c = 1.0$, $\Delta t_h = \Delta t_c = 1.0$, and $\gamma = 1.0$ and varied the temperature difference ΔT, or equivalently, the temperature T_h. We calculated the heat in Eqs. (6.71) and (6.76) and the work $W = Q_h + Q_c$ in Eq. (6.82) from the solution of Eqs. (6.14)–(6.16). Using the heat and work, we also numerically calculated the efficiency $\eta = W/Q_h$ using Eq. (6.83) and power $P = W/\Delta t_{\text{cyc}}$ using Eq. (6.84). Before starting to measure the thermodynamic quantities, we waited until the system settled down to a steady cycle. Moreover, when we take the limit $m \to 0$, the relaxation time of velocity $\tau_v = m/\gamma$ vanishes. By a simple calculation from Eqs. (6.9) and (6.90), we find that τ_x satisfies

$$\frac{\gamma\sigma_a}{T_h} \le \tau_x(t) \le \frac{\gamma\sigma_b}{T_c}. \tag{6.91}$$

Thus, the smaller σ_a and σ_b are, the smaller τ_x is. When we take the limit $\sigma_a, \sigma_b \to 0$ while maintaining σ_b/σ_a finite, $\tau_x(t)$ vanishes and $\lambda(t) = \gamma/\tau_x(t)$ from Eq. (6.9) diverges. Because $\tau_x(0) \propto \sigma_a$ and $\tau_v \propto m$ are satisfied, we varied the mass m and the parameter σ_a to vary the relaxation times. Note that in the numerical simulations, we selected a time step smaller than the relaxation times. Specifically, we set the time step as $\min(m, \sigma_a) \times 10^{-2}$ because of $\tau_x(0) \propto \sigma_a$ and $\tau_v \propto m$.

To evaluate the efficiency in Eq. (6.83) obtained numerically, we compared it with the quasistatic efficiency η^{qs} in Eq. (6.66). Because η_{C} in Eq. (6.1) is proportional to ΔT, the ratio of η^{qs} in Eq. (6.67) to η_{C} in the small temperature-difference regime satisfies

$$\frac{\eta^{\text{qs}}}{\eta_{\text{C}}} = 1 - O(\Delta T). \tag{6.92}$$

Similarly, we evaluate the power in Eq. (6.84) by using a criterion P^* defined as follows:

$$P^* \equiv \frac{W^{\text{qs}}}{\Delta t_{\text{cyc}}} = \frac{\Delta T \Delta S^{\text{qs}}}{\Delta t_{\text{cyc}}}, \tag{6.93}$$

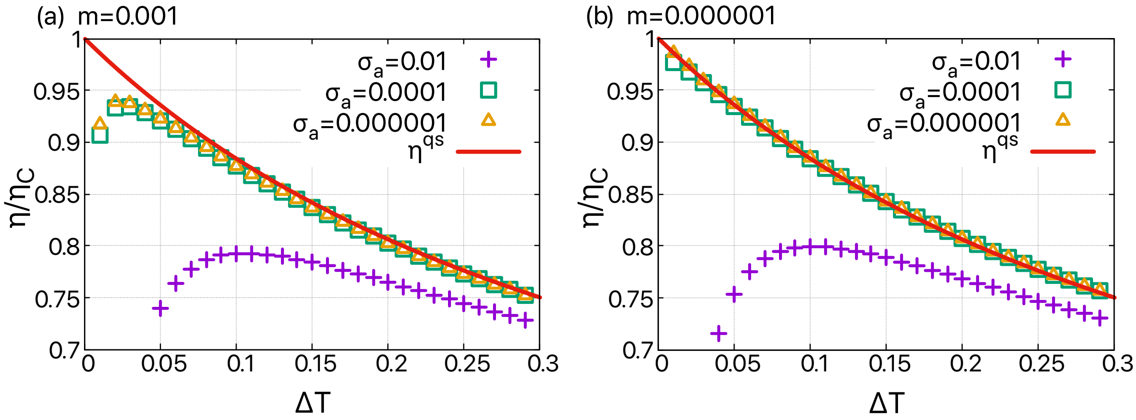

Figure 6.2: The ratio of the efficiency in Eq. (6.83) to the Carnot efficiency in our cycle with the protocol in Eq. (6.90) when τ_x varies at (a) $\tau_v = 10^{-3}$ and (b) $\tau_v = 10^{-6}$. Because the parameter σ_a is proportional to $\tau_x(0)$ in the protocol in Eq. (6.90), we vary σ_a to make τ_x small. Similarly, we vary the mass m because it is proportional to τ_v. In these simulations, we set $\sigma_a = 10^{-2}$ (purple plus), $\sigma_a = 10^{-4}$ (green square), and $\sigma_a = 10^{-6}$ (orange triangle). The red solid line corresponds to the ratio of η^{qs} in Eq. (6.66) to the Carnot efficiency. The efficiency appears to approach the Carnot efficiency in the vanishing limit of σ_a (or τ_x), m (or τ_v), and ΔT.

where W^{qs} is the quasistatic work in Eq. (6.53). Here, we regard the power as finite when the power in Eq. (6.84) is the same order as P^*.

Figure 6.2 shows the ratio of the efficiency of the proposed cycle with the protocol in Eq. (6.90) to the Carnot efficiency. We can see that the efficiency approaches η^{qs} with $\tau_x,\ \tau_v \to 0$. Considering Eqs. (6.66) and (6.89), we can expect that the irreversible works disappear. Thus, the efficiency can be regarded as the Carnot efficiency in the small relaxation-times and small temperature-difference regime.

Figure 6.3 shows the ratio of the power to P^* in Eq. (6.93), corresponding to Fig. 6.2. At any ΔT, we can see that the power approaches P^* as $\tau_x,\ \tau_v \to 0$. As the power in Eq. (6.84) is defined using the work in Eq. (6.82), the ratio of P to P^* is the same as the ratio of W to W^{qs} in Eq. (6.53). When the power P approaches P^*, the work W approaches W^{qs}. This implies that the irreversible works vanish. Because the power is of the same order as P^* from Fig. 6.3, we can consider the power to be finite. Therefore, Figs. 6.2 and 6.3 imply that the Carnot efficiency and finite power are compatible in the vanishing limit of the relaxation times in the small temperature-difference regime.

6.5 Theoretical analysis

This section analytically shows that it is possible to achieve the Carnot efficiency in our cycle in the vanishing limit of the relaxation times in the small temperature-difference regime without breaking the trade-off relation in Eq. (6.2), as implied in the numerical results in Sec. 6.4.

In general, the efficiency decreases when the entropy production increases, as shown in Eq. (6.97). As the adiabatic processes have no entropy production because no heat exchange is present, we have only to consider the entropy production in the isothermal processes. In the small relaxation-times regime, the efficiency in Eq. (6.83) is approximated by that in Eq. (6.89). If $W_{h,c}^{\text{irr}} \to 0$ is satisfied in the vanishing limit of the relaxation times, the efficiency in Eq. (6.89) approaches the quasistatic efficiency in Eq. (6.66).

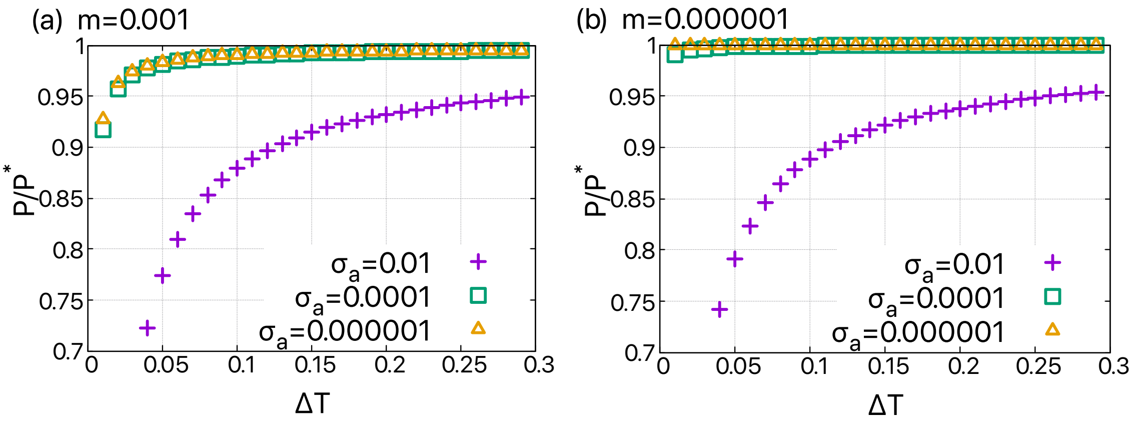

Figure 6.3: The ratio of the power in Eq. (6.84) to P^* in Eq. (6.93) in the proposed cycle corresponding to Figs. 6.2(a) and 6.2(b). The power appears to approach P^* in Eq. (6.93) in the vanishing limit of σ_a (or τ_x), m (or τ_v), and ΔT.

As seen in Eq. (6.67), it is expected that the contribution of the heat leakage to the efficiency can be neglected in the small temperature-difference regime. Thus, the efficiency in Eq. (6.83) approaches the Carnot efficiency in the small relaxation-times and small temperature-difference regime, and the power in Eq. (6.84) also approaches P^* in Eq. (6.93) simultaneously.

The numerical results imply that the irreversible works vanish in the vanishing limit of the relaxation times τ_x and τ_v. To derive a similar conclusion analytically, we first show that the irreversible works relate to the entropy production given by

$$\Sigma \equiv -\frac{Q_h}{T_h} - \frac{Q_c}{T_c}. \tag{6.94}$$

Similarly to Σ^{qs} in Eq. (6.54), the entropy production Σ is expressed only by an entropy change in the heat baths. In the small relaxation-times regime, we can express Σ in Eq. (6.94) as follows:

$$\Sigma \simeq \frac{-\frac{\Delta T}{2} - T_h \Delta S^{\mathrm{qs}} + W_h^{\mathrm{irr}}}{T_h} + \frac{\frac{\Delta T}{2} + T_c \Delta S^{\mathrm{qs}} + W_c^{\mathrm{irr}}}{T_c} = \frac{W_h^{\mathrm{irr}}}{T_h} + \frac{W_c^{\mathrm{irr}}}{T_c} + \frac{(\Delta T)^2}{2T_h T_c}, \tag{6.95}$$

using Eqs. (6.87) and (6.88). The last term on the right-hand side of Eq. (6.95) comes from the heat leakage due to the instantaneous adiabatic processes. From Eq. (6.95), the entropy production can be regarded as zero in the small temperature-difference regime when the irreversible works vanish. In general, the entropy production in Eq. (6.94) can also be rewritten as

$$\Sigma = \frac{Q_h}{T_c}(\eta_{\mathrm{C}} - \eta), \tag{6.96}$$

where we used Eqs. (6.1) and (6.83). This equation shows that the efficiency approaches the Carnot efficiency when the entropy production vanishes. Thus, by using Eqs. (6.95) and (6.96), we obtain the efficiency as

$$\eta = \eta_{\mathrm{C}} - \frac{T_c \Sigma}{Q_h} \simeq \eta_{\mathrm{C}} - \frac{T_c}{Q_h}\left(\frac{W_h^{\mathrm{irr}}}{T_h} + \frac{W_c^{\mathrm{irr}}}{T_c}\right) + O[(\Delta T)^2], \tag{6.97}$$

in the small relaxation-times regime. Here, the contribution of the heat leakage to the efficiency is $O[(\Delta T)^2]$, and it is negligible in the small temperature-difference regime.

We consider the trade-off relation in Eq. (6.2) to discuss the compatibility of the Carnot efficiency and finite power in our Brownian Carnot cycle. Using Eq. (6.96), we can rewrite Eq. (6.2) as

$$P \leq \frac{\eta T_c}{Q_h} A\Sigma \tag{6.98}$$

in terms of the entropy production Σ. When the quantity $A\Sigma$ is nonzero in the vanishing limit of the entropy production Σ, implying that A should diverge, the finite power may be allowed. In fact, when the entropy production Σ vanishes in the small temperature-difference regime, the irreversible works should vanish because of Eq. (6.95). Then, the power in Eq. (6.84) approaches P^* in Eq. (6.93), which implies that the power is regarded as finite. Thus, we find the expression $A\Sigma$ in our cycle below.

In our Carnot cycle, by using a function $\phi(t)$, the time evolution of the temperature is described as

$$\begin{aligned} \frac{1}{T(t)} =& \frac{1}{T_c} - \left(\frac{1}{T_c} - \frac{1}{T_h}\right)\phi(t) \\ =& \frac{1}{T_c}\left[1 - \eta_C \phi(t)\right]. \end{aligned} \tag{6.99}$$

$$\phi(t) \equiv \begin{cases} 1 & (0 < t < \Delta t_h) \\ 0 & (\Delta t_h < t < \Delta t_{\mathrm{cyc}}). \end{cases} \tag{6.100}$$

As shown in Sec. 5.4.2, the entropy production rate of the total system is given by the irreversible probability current. Applying Eq. (67) from Ref. [25], we can divide the probability currents in Eqs. (6.7) and (6.8) into the reversible parts, j_x^{rev} and j_v^{rev}, and the irreversible parts, j_x^{irr} and j_v^{irr}, as

$$\begin{aligned} j_x(x,v,t) =& j_x^{\mathrm{rev}}(x,v,t) + j_x^{\mathrm{irr}}(x,v,t), \\ j_v(x,v,t) =& j_v^{\mathrm{rev}}(x,v,t) + j_v^{\mathrm{irr}}(x,v,t), \end{aligned} \tag{6.101}$$

where

$$\begin{aligned} j_x^{\mathrm{rev}}(x,v,t) \equiv& v p(x,v,t), \qquad j_x^{\mathrm{irr}}(x,v,t) \equiv 0, \\ j_v^{\mathrm{rev}}(x,v,t) \equiv& -\frac{\lambda(t)}{m} x p(x,v,t), \\ j_v^{\mathrm{irr}}(x,v,t) \equiv& \left(-\frac{\gamma}{m} v - \frac{\gamma T(t)}{m^2}\frac{\partial}{\partial v}\right) p(x,v,t). \end{aligned} \tag{6.102}$$

Then, the heat current and entropy production rate is given by

$$\dot{Q}(t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \; m v j_v^{\mathrm{irr}}(x,v,t). \tag{6.103}$$

$$\dot{\Sigma}(t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \; \frac{m^2 (j_v^{\mathrm{irr}}(x,v,t))^2}{\gamma T(t) p(x,v,t)}. \tag{6.104}$$

We derive Q_h and $\Sigma_{\rm tot}$

$$Q_h = \int_0^{\Delta t_h} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ mvj_v^{\rm irr} = \int_0^{\Delta t_{\rm cyc}} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ \phi(t) mvj_v^{\rm irr}, \quad (6.105)$$

$$\Sigma = \int_0^{\Delta t_{\rm cyc}} dt\ \dot{\Sigma}(t) = \int_0^{\Delta t_{\rm cyc}} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ \frac{m^2 (j_v^{\rm irr}(x,v,t))^2}{\gamma T(t) p(x,v,t)}. \quad (6.106)$$

using Eqs. (5.75), (6.100), (6.103), and (6.100). As shown in Sec. 5.4.2, because $\gamma T(t)\sigma_v$ and $\dot{\Sigma}$ are positive, we can derive the following bound for the heat in Eq. (6.105):

$$\begin{aligned}(Q_h)^2 &= \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi(t)\ \dot{Q}(t) \right)^2 \\ &\le \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi(t) \sqrt{\gamma T(t) \sigma_v \dot{\Sigma}} \right)^2 \\ &\le \left(\int_0^{\Delta t_{\rm cyc}} dt\ \phi^2(t) \gamma T(t) \sigma_v \right) \left(\int_0^{\Delta t_{\rm cyc}} dt\ \dot{\Sigma} \right) = \Delta t_{\rm cyc} T_c^2 \chi \Sigma,\end{aligned} \quad (6.107)$$

where

$$\chi \equiv \frac{\gamma}{\Delta t_{\rm cyc} T_c} \int_0^{\Delta t_{\rm cyc}} dt\ \frac{\phi^2(t)}{1 - \eta_{\rm C} \phi(t)} \sigma_v(t), \quad (6.108)$$

and we used Eq. (5.79). Using Eqs. (6.96) and (6.107), we can derive the trade-off relation in our cycle as

$$\begin{aligned}P =& \frac{W}{\Delta t_{\rm cyc}} = \frac{W}{Q_h} \frac{1}{Q_h} \frac{Q_h^2}{\Delta t_{\rm cyc}} \\ \le& \eta \frac{1}{Q_h} T_c^2 \chi \Sigma \\ =& \chi T_c \eta (\eta_{\rm C} - \eta).\end{aligned} \quad (6.109)$$

By comparing Eqs. (6.98) and (6.109), we obtain $A = T_c \chi$. We will show that in the limit of $\tau_x, \tau_v \to 0$, the entropy production Σ vanishes and χ diverges while $\chi\Sigma$ maintains positive. For this purpose, we rewrite Eq. (6.104) as follows. In our model (Sec. 6.2), the probability distribution was assumed to be the Gaussian distribution shown in Eq. (6.11). Thus, we can differentiate the distribution function $p(x, v, t)$ with respect to v as

$$\frac{\partial p}{\partial v} = \frac{\sigma_{xv} x - \sigma_x v}{\sigma_x \sigma_v - {\sigma_{xv}}^2} p. \quad (6.110)$$

We can rewrite the entropy production rate in Eq. (6.104) by using the variables σ_x, σ_v, and σ_{xv} and derive the expression of $\dot{\Sigma}$ under the assumption of the Gaussian distribution as

$$\begin{aligned}\dot{\Sigma}(t) =& \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\ \frac{m^2}{\gamma T p} \left\{ \left(\frac{\gamma}{m} v + \frac{\gamma T}{m^2} \frac{\partial}{\partial v} \right) p \right\}^2 \\ =& \frac{m^2}{\gamma T} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \left\{ \frac{\gamma}{m} v + \frac{\gamma T}{m^2} \frac{\sigma_{xv} x - \sigma_x v}{\sigma_x \sigma_v - {\sigma_{xv}}^2} \right\}^2 p \\ =& \frac{\frac{\gamma}{m} (T - m\sigma_v)^2 + (2T - m\sigma_v) \gamma \frac{{\sigma_{xv}}^2}{\sigma_x}}{T \left(m\sigma_v - \tau_v \gamma \frac{{\sigma_{xv}}^2}{\sigma_x} \right)},\end{aligned} \quad (6.111)$$

where we used Eqs. (6.9), (6.10), (6.15), (6.20), (6.102), and (6.110). Using Eqs. (6.15) and (6.20), we obtain

$$\dot{Q} = \frac{\gamma}{m}(T - m\sigma_v). \tag{6.112}$$

Thus, Eq. (6.111) can be rewritten as

$$\dot{\Sigma}(t) = \frac{\tau_v \dot{Q}^2 + (2T - m\sigma_v)\gamma \frac{{\sigma_{xv}}^2}{\sigma_x}}{T\left(m\sigma_v - \tau_v \gamma \frac{{\sigma_{xv}}^2}{\sigma_x}\right)}. \tag{6.113}$$

Integrating Eq. (6.113) with respect to time, we derive the entropy production per cycle Σ in our cycle as

$$\Sigma = \int_0^{\Delta t_{\mathrm{cyc}}} dt \frac{\tau_v \dot{Q}^2(t) + [2T(t) - m\sigma_v(t)]\gamma \frac{{\sigma_{xv}}^2(t)}{\sigma_x(t)}}{T(t)\left(m\sigma_v(t) - \tau_v \gamma \frac{{\sigma_{xv}}^2(t)}{\sigma_x(t)}\right)}. \tag{6.114}$$

6.5.1 Small relaxation-times regime

We evaluate the entropy production in Eq. (6.114) in the small relaxation-times regime. In the hot isothermal process, the process can be divided into the relaxation part and the part after the relaxation. Because the relaxation time of the system at the beginning of the hot isothermal process is given by $\tau_0 \equiv \max(\tau_x(0), \tau_v)$, the entropy production in the hot isothermal process Σ_h is divided as

$$\Sigma_h \equiv \int_0^{\Delta t_h} dt\ \dot{\Sigma} = \int_0^{\tau_0} dt\ \dot{\Sigma} + \int_{\tau_0}^{\Delta t_h} dt\ \dot{\Sigma}, \tag{6.115}$$

where the first and second terms in Eq. (6.115) represent the entropy production in the relaxation and after the relaxation, respectively. We first evaluate the entropy production after the relaxation. From Eqs. (C.15) and (C.17) in the Appendix C, the variables σ_x, σ_v, and σ_{xv} after the relaxation satisfy

$$\sigma_x \simeq \frac{T}{\lambda}, \quad \sigma_v \simeq \frac{T}{m}, \quad \sigma_{xv} \simeq -\frac{T}{2\lambda^2}\frac{d\lambda}{dt}, \tag{6.116}$$

where T is T_h or T_c. Then, we can obtain

$$\gamma \frac{{\sigma_{xv}}^2(t)}{\sigma_x(t)} \simeq \frac{\tau_x(t)T}{4}\left(\frac{d}{dt}\ln\lambda(t)\right)^2. \tag{6.117}$$

The heat current $\dot{Q}(t)$ in Eq. (6.20) is represented as

$$\dot{Q}(t) \simeq \frac{1}{2}\lambda(t)\dot{\sigma}_x(t) \simeq -\frac{T}{2}\left(\frac{d}{dt}\ln\lambda\right), \tag{6.118}$$

where we used Eq. (6.116), and $\dot{Q}$ is noninfinite because $d(\ln\lambda)/dt$ is noninfinite. Note that we obtain

$$\begin{aligned}\frac{d}{ds}\ln\Lambda =& \Delta t_{\mathrm{cyc}}\frac{d}{dt}\ln\lambda,\\ \frac{dQ}{ds} =& \Delta t_{\mathrm{cyc}}\frac{dQ}{dt} = -\frac{T}{2}\left(\frac{d}{ds}\ln\Lambda\right),\end{aligned} \tag{6.119}$$

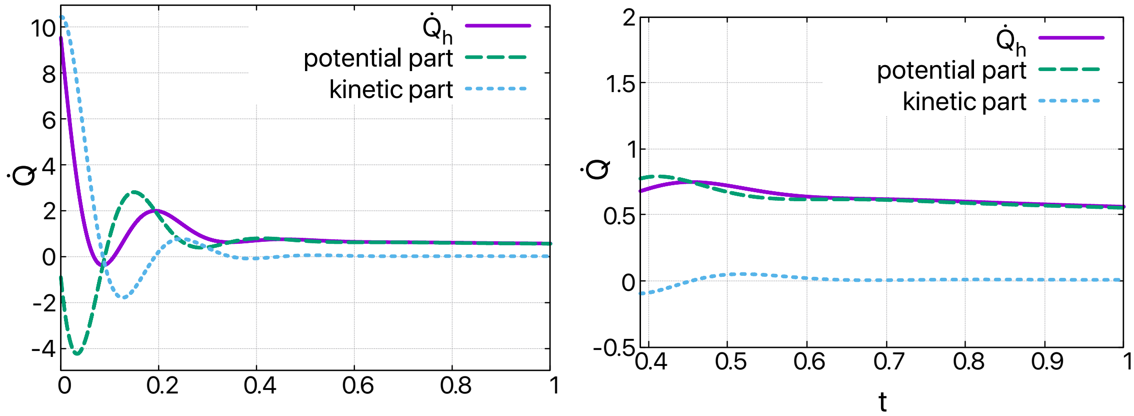

Figure 6.4: Time evolution of $\dot{Q}_h(t)$ (purple solid line), its potential part $\lambda\dot{\sigma}_x/2$ (green dashed line), and its kinetic part $m\dot{\sigma}_v/2$ (sky-blue dotted line) in the hot isothermal process. We can see a relaxation at the beginning of the process. The lower figure is an enlargement view of a part of the upper figure, which shows that $\dot{Q}_h(t) \simeq \lambda(t)\dot{\sigma}_x(t)/2$ and $m\dot{\sigma}_v(t) \simeq 0$ are satisfied. In this simulation, we used $\lambda(t)$ in Eq. (6.90) and set $T_h = 2.0$, $T_c = 1.0$, $t_h = t_c = 1.0$, $m = 0.1$, $\sigma_a = 0.1$, $\gamma = 1.0$, and $\sigma_b/\sigma_a = 2.0$.

using s and Eqs. (6.26) and (6.118). Because $d(\ln\lambda)/dt$ is noninfinite, $d(\ln\Lambda)/ds$ and dQ/ds are also noninfinite after the relaxation when Δt_{cyc} is finite. Figure 6.4 shows a time evolution of the heat current $\dot{Q}_h$, its potential part $\lambda\dot{\sigma}_x/2$, and its kinetic part $m\dot{\sigma}_v/2$ in the hot isothermal process with the protocol in Eq. (6.90). In this simulation, we used the same parameters as in Sec. 6.4. From the figure, we can see a relaxation at the beginning of the process. As implied in Eq. (6.118), the heat current $\dot{Q}_h$ is almost equal to its potential part $\lambda\dot{\sigma}_x/2$, and the kinetic part $m\dot{\sigma}_v/2$ almost vanishes after the relaxation.

Using Eqs. (6.113), (6.116), and (6.117), the entropy production rate after the relaxation is given by

$$\dot{\Sigma}(t) \simeq \frac{1}{\Delta t_{\mathrm{cyc}} T} \frac{\frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T - \frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}, \tag{6.120}$$

where we used $s = t/\Delta t_{\mathrm{cyc}}$ to compare the cycle time Δt_{cyc} and the relaxation times τ_x and τ_v. Then, we derive the entropy production after the relaxation in the hot isothermal process as

$$\int_{\tau_0}^{\Delta t_h} dt\,\dot{\Sigma} \simeq \frac{1}{T_h}\int_{\tau_0/\Delta t_{\mathrm{cyc}}}^{\Delta t_h/\Delta t_{\mathrm{cyc}}} ds\, \frac{\frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_h^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_h - \frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_h}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}. \tag{6.121}$$

To consider the entropy production in the relaxation, we rewrite $\dot{\Sigma}$ in Eq. (6.113) by using the heat current in Eq. (6.20) and the time derivative of the entropy in Eq. (6.18) as follows:

$$\dot{\Sigma}(t) = \dot{S}(t) - \frac{\dot{Q}(t)}{T(t)}. \tag{6.122}$$

Because the temperature of the heat bath is constant, we derive the entropy production in the relaxation in the hot isothermal process as

$$\int_0^{\tau_0} dt\ \dot{\Sigma} = S(\tau_0) - S(0) - \frac{Q_h^{rel}}{T_h}, \tag{6.123}$$

where Q_h^{rel} is the heat flowing in this relaxation. In the small relaxation-times regime, the relaxation is very fast (see the Appendix C), and the stiffness is regarded to be unchanged in the relaxation. From Eq. (C.15), σ_x is also unchanged during the relaxation under the condition of Eq. (6.59). Thus, the heat related to the potential change in Eq. (6.22) in the relaxation vanishes. By using Eqs. (6.21) and (6.86), Q_h^{rel} is evaluated as

$$Q_h^{rel} \simeq \frac{\Delta T}{2}. \tag{6.124}$$

In addition because $d(\ln\Lambda)/ds$ is noninfinite, as shown in the Appendix C, we can approximate the entropy in Eq. (6.18) after the relaxation by

$$S(t) \simeq \frac{1}{2}\ln(T^2(t)) + \frac{1}{2}\ln\left(\frac{4\pi^2}{m\lambda(t)}\right) + 1, \tag{6.125}$$

where we used the approximation

$$m\lambda(\sigma_x\sigma_v - {\sigma_{xv}}^2) \simeq T^2 - \frac{\tau_x}{\Delta t_{\rm cyc}}\frac{\tau_v}{\Delta t_{\rm cyc}}\frac{T^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2 \simeq T^2, \tag{6.126}$$

from Eqs. (6.26) and (6.116). The initial state of the hot isothermal process is given by the final state of the cold isothermal process as

$$\sigma_x \simeq \frac{T_c}{\lambda_D}, \quad \sigma_v \simeq \frac{T_c}{m}, \quad \sigma_{xv} \simeq -\frac{T_c}{2\lambda_D^2}\left.\frac{d\lambda}{dt}\right|_{t=\Delta t_{\rm cyc}-0}, \tag{6.127}$$

from Eq. (6.116). Because the stiffness remains λ_A in the relaxation, the variables relax to the following values:

$$\sigma_x \simeq \frac{T_h}{\lambda_A}, \quad \sigma_v \simeq \frac{T_h}{m}, \quad \sigma_{xv} \simeq -\frac{T_h}{2\lambda_A^2}\left.\frac{d\lambda}{dt}\right|_{t=0+0}, \tag{6.128}$$

from Eq. (6.116). Using Eqs. (6.125)–(6.128), the difference between $S(0)$ and $S(\tau_0)$ can be approximated by

$$S(\tau_0) - S(0) \simeq \frac{1}{2}\ln(T_h^2) - \frac{1}{2}\ln(T_c^2) + \frac{1}{2}\ln\left(\frac{\lambda_D}{\lambda_A}\right). \tag{6.129}$$

We can then evaluate the entropy production in the relaxation in Eq. (6.123) as

$$\int_0^{\tau_0} dt\ \dot{\Sigma} \simeq \frac{1}{2}\ln(T_h^2) - \frac{1}{2}\ln(T_c^2) + \frac{1}{2}\ln\left(\frac{\lambda_D}{\lambda_A}\right) - \frac{\Delta T}{2T_h} = \frac{1}{2}\ln\left(\frac{T_h}{T_c}\right) - \frac{\Delta T}{2T_h}, \tag{6.130}$$

using Eqs. (6.59), (6.68), (6.124), and (6.129). Thus, by using Eqs. (6.121) and (6.130),

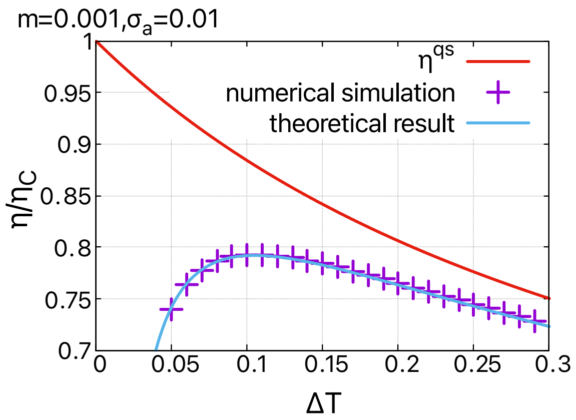

Figure 6.5: The ratio of the efficiency to the Carnot efficiency derived from the numerical simulations in Fig. 6.2 in Sec. 6.4 (purple plus) and theoretical analysis (sky-blue solid line). We set $m = 10^{-3}$ and $\sigma_a = 10^{-2}$. Although the relaxation times corresponding to these parameters are not very small among the parameters used in Fig. 6.2, the theoretical result and numerical simulations show a good agreement. We have confirmed a better agreement with smaller parameters (data not shown).

the entropy production in the hot isothermal process in Eq. (6.115) is given by

$$\Sigma_h \simeq \frac{1}{2}\ln\left(\frac{T_h}{T_c}\right) - \frac{\Delta T}{2T_h} + \frac{1}{T_h}\int_{\tau_0/\Delta t_{\rm cyc}}^{\Delta t_h/\Delta t_{\rm cyc}} ds\, \frac{\frac{\tau_v}{\Delta t_{\rm cyc}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_h^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_h - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_h}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}. \quad (6.131)$$

Similarly, the entropy production in the cold isothermal process Σ_c is given by

$$\begin{aligned}\Sigma_c \equiv \int_{\Delta t_h}^{\Delta t_{\rm cyc}} dt\,\dot{\Sigma} &\simeq \frac{1}{2}\ln\left(\frac{T_c}{T_h}\right) + \frac{\Delta T}{2T_c} \\ &+ \frac{1}{T_c}\int_{(\Delta t_h+\tau_1)/\Delta t_{\rm cyc}}^{1} ds\, \frac{\frac{\tau_v}{\Delta t_{\rm cyc}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_c^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_c - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_c}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2},\end{aligned} \quad (6.132)$$

where $\tau_1 \equiv \max(\tau_x(\Delta t_h + 0), \tau_v)$ is the relaxation time at the beginning of the cold isothermal process. Because no entropy production is present in the adiabatic processes, the entropy production Σ per cycle in the small relaxation-times regime is given by

$$\begin{aligned}\Sigma =& \Sigma_h + \Sigma_c \\ \simeq& \frac{1}{T_h}\int_{\tau_0/\Delta t_{\rm cyc}}^{\Delta t_h/\Delta t_{\rm cyc}} ds\, \frac{\frac{\tau_v}{\Delta t_{\rm cyc}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_h^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_h - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_h}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2} \\ &+ \frac{1}{T_c}\int_{(\Delta t_h+\tau_1)/\Delta t_{\rm cyc}}^{1} ds\, \frac{\frac{\tau_v}{\Delta t_{\rm cyc}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_c^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_c - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_c}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2} \\ &+ \frac{(\Delta T)^2}{2T_hT_c},\end{aligned} \quad (6.133)$$

using Eqs. (6.131) and (6.132).

Comparing Eqs. (6.95) and (6.133), we can derive the expression of the irreversible works as

$$W_h^{\mathrm{irr}} = \int_{\tau_0/\Delta t_{\mathrm{cyc}}}^{\Delta t_h/\Delta t_{\mathrm{cyc}}} ds \, \frac{\frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_h^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_h - \frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_h}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}, \tag{6.134}$$

$$W_c^{\mathrm{irr}} = \int_{(\Delta t_h+\tau_1)/\Delta t_{\mathrm{cyc}}}^{1} ds \, \frac{\frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\left(\frac{dQ(s)}{ds}\right)^2 + \frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_c^2}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_c - \frac{\tau_v}{\Delta t_{\mathrm{cyc}}}\frac{\tau_x}{\Delta t_{\mathrm{cyc}}}\frac{T_c}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2}. \tag{6.135}$$

As shown in the Appendix C, dQ/ds and $d(\ln\Lambda)/ds$ are noninfinite after the relaxation. Thus, the entropy production rate in Eq. (6.120) after the relaxation vanishes in the vanishing limit of the relaxation times. From Eqs. (6.134) and (6.135), it turns out that the integrand of $W_{h,c}^{\mathrm{irr}}$, which is $T_{h,c}\dot{\Sigma}$, vanishes at any s in the vanishing limit of the relaxation times, and the irreversible works also vanish. Therefore, we can confirm that the efficiency in Eq. (6.89) approaches the quasistatic efficiency in Eq. (6.66) in this limit, theoretically explaining the results of the numerical simulations. Figure 6.5 compares the efficiency obtained from the numerical simulations in Fig. 6.2 and the efficiency derived from the theoretical analysis in the small relaxation-times regime. Here, the efficiency of the theoretical analysis was derived by calculating the irreversible works in Eqs. (6.134) and (6.135) and substituting them into Eq. (6.89). Note that we used Eq. (6.118) to calculate dQ/ds in Eqs. (6.134) and (6.135). We can see that the theoretical result and numerical simulations show a good agreement.

We provide a qualitative explanation for the behavior of the efficiency in Figs. 6.2 and 6.5, as below. We consider the case that the relaxation times are small but finite. Then, from the above discussion, W_h^{irr} and W_c^{irr} are positive and small. When ΔT is large, $\Delta T \Delta S^{\mathrm{qs}}$ in the numerator of Eq. (6.89) is sufficiently larger than $W_{h,c}^{\mathrm{irr}}$ since we use the protocol satisfying $\Delta S^{\mathrm{qs}} = O(1)$ in the numerical simulation. Since T_h is larger than ΔT, $T_h \Delta S^{\mathrm{qs}}$ in the denominator of Eq. (6.89) is also sufficiently larger than $W_{h,c}^{\mathrm{irr}}$. Thus, the efficiency should mainly depend on T_h, ΔT, and ΔS^{qs} as shown in Eq. (6.89). Although the efficiency is smaller than the Carnot efficiency because of $\Delta T/2$ due to the heat leakage in the denominator of Eq. (6.89), the heat leakage becomes small and the efficiency increases toward the Carnot efficiency as ΔT becomes small. At the same time, however, the irreversible works can be comparable to $\Delta T \Delta S^{\mathrm{qs}}$. From Eq. (6.90), the stiffness in each isothermal process depends only on the corresponding temperature. Since dQ/ds in Eqs. (6.134) and (6.135) is evaluated by the protocol as shown in Eq. (6.118), $W_{h,c}^{\mathrm{irr}}$ depend only on the temperature of each isothermal process, but do not depend on ΔT in the lowest order of ΔT. Thus, the irreversible works maintain finite even when ΔT vanishes. Then, $\Delta T \Delta S^{\mathrm{qs}}$ in Eq.(6.89) approaches zero while $W_{h,c}^{\mathrm{irr}}$ are positively finite. Thus, the efficiency turns from increase to decrease as ΔT becomes small and takes the maximum for a specific value of ΔT as shown in Figs. 6.2 and 6.5.

By using $s = t/\Delta t_{\mathrm{cyc}}$, the quantity $\phi(t)$ in Eq. (6.100) can be expressed as

$$\phi(s) = \begin{cases} 1 & (0 < s < \Delta t_h/\Delta t_{\mathrm{cyc}}) \\ 0 & (\Delta t_h/\Delta t_{\mathrm{cyc}} < s < 1). \end{cases} \tag{6.136}$$

Thus, χ in Eq. (6.108) is rewritten by using the relaxation time of the velocity as

$$\chi = \frac{1}{\Delta t_{\mathrm{cyc}}}\frac{\Delta t_{\mathrm{cyc}}}{\tau_v}\left(\frac{1}{T_c}\int_0^1 ds \, \frac{m\sigma_v \phi^2}{1-\eta_{\mathrm{C}}\phi}\right) = \frac{C}{\Delta t_{\mathrm{cyc}}}\frac{\Delta t_{\mathrm{cyc}}}{\tau_v}, \tag{6.137}$$

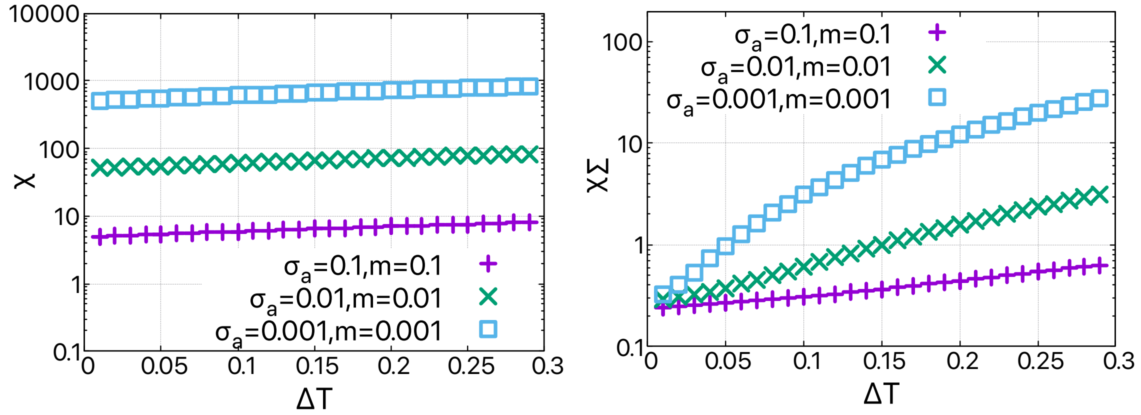

Figure 6.6: The quantities χ in Eq. (6.108) and $\chi\Sigma$ when τ_x and τ_v are varied. Because the parameter σ_a is proportional to $\tau_x(0)$ in the protocol in Eq. (6.90), we vary σ_a to make τ_x be small. Similarly, we vary the mass m because it is proportional to τ_v. In these simulations, we used $(\sigma_a = 0.1, m = 0.1)$ (purple plus), $(\sigma_a = 0.01, m = 0.01)$ (green cross), and $(\sigma_a = 0.001, m = 0.001)$ (sky-blue square). We can see that χ diverges at each ΔT when we consider the limit of $\sigma_a, m \to 0$ $(\tau_x, \tau_v \to 0)$. In addition, we can also see that the values of $\chi\Sigma$ are positively finite for the vanishing limit of ΔT for any relaxation times.

where C is a positive constant given by

$$C \equiv \frac{1}{T_c}\int_0^1 ds\, \frac{m\sigma_v\phi^2}{1-\eta_{\rm C}\phi}. \tag{6.138}$$

In the relaxation at the beginning of each isothermal process, $m\sigma_v$ is positively finite. After the relaxation, $m\sigma_v$ is approximated by the temperature of the heat bath. Thus, C is positively finite. From Eq. (6.137), χ turns out to diverge in the limit of $\tau_v/\Delta t_{\rm cyc} \to 0$ when $\Delta t_{\rm cyc}$ is finite. Although $\tau_v/\Delta t_{\rm cyc} \to 0$ is satisfied even when $\Delta t_{\rm cyc}$ diverges and τ_v is maintained finite, we do not consider that case because it is in the quasistatic limit. Using Eqs. (6.133) and (6.137), we can obtain $\chi\Sigma$ as follows:

$$\begin{aligned}\chi\Sigma \simeq & \frac{C}{\Delta t_{\rm cyc}T_h}\int_{\tau_0/\Delta t_{\rm cyc}}^{\Delta t_h/\Delta t_{\rm cyc}} ds\, \frac{\left(\frac{dQ}{ds}\right)^2 + \frac{\tau_x T_h^2}{4\tau_v}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_h - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_h}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2} \\ & + \frac{C}{\Delta t_{\rm cyc}T_c}\int_{(\Delta t_h+\tau_1)/\Delta t_{\rm cyc}}^{1} ds\, \frac{\left(\frac{dQ}{ds}\right)^2 + \frac{\tau_x T_c^2}{4\tau_v}\left(\frac{d}{ds}\ln\Lambda\right)^2}{T_c - \frac{\tau_v}{\Delta t_{\rm cyc}}\frac{\tau_x}{\Delta t_{\rm cyc}}\frac{T_c}{4}\left(\frac{d}{ds}\ln\Lambda\right)^2} \\ & + \frac{C}{\tau_v}\frac{(\Delta T)^2}{2T_hT_c}.\end{aligned} \tag{6.139}$$

Here, we consider the vanishing limit of the relaxation times in the small temperature-difference regime and evaluate the efficiency and power in this limit. As seen in Eq. (6.97), the efficiency approaches the Carnot efficiency when Σ vanishes. Moreover, we evaluate Σ in the vanishing limit of τ_x and τ_v in the small temperature-difference regime. In this limit, we can show that dQ/ds and $d(\ln\Lambda)/ds$ in Eq. (6.133) do not diverge after the relaxation (see the Appendix C). Thus, when the relaxation times vanish at any time after the relaxation, the entropy production rate always vanishes from Eq. (6.120), and the first and second terms on the right-hand side of Eq. (6.133) also vanish. In addition, when ΔT is small, the third term in Eq. (6.133), which is due to the relaxation, is $O[(\Delta T)^2]$ and can be ignored. Therefore, the entropy production per cycle in Eq. (6.133) should be $O[(\Delta T)^2]$, and the efficiency can be regarded as the Carnot efficiency because of the reasoning presented below Eq. (6.97). Then, because dQ/ds and $d(\ln \Lambda)/ds$ are always noninfinite, the first and second terms on the right-hand side of Eq. (6.139) are positively finite in the vanishing limit of τ_x and τ_v. Even when ΔT is small, $\chi\Sigma$ is positive, and the right-hand side of the trade-off relation in Eq. (6.109) is positive. Therefore, the finite power may be allowed even when Σ vanishes. In the above limit, because the irreversible works in Eqs. (6.134) and (6.135) vanish, the power in Eq. (6.84) approaches P^* in Eq. (6.93), which implies that the power is finite. Therefore, the Carnot efficiency is achievable in the finite-power Brownian Carnot cycle without breaking the trade-off relation in Eq. (6.109).

In Fig. 6.6, we numerically confirmed that χ increases and $\chi\Sigma$ remains positively finite in the limit of $\Delta T \to 0$ when we consider smaller relaxation times. We can expect χ to diverge while maintaining $\chi\Sigma$ positively finite in the vanishing limit of the relaxation times in the limit of $\Delta T \to 0$. This result implies that Σ vanishes while maintaining $\chi\Sigma$ positively finite, and we can expect that Σ vanishes and χ diverges simultaneously in the vanishing limit of the relaxation times.

6.6 Summary and discussion of this chapter

Motivated by the previous study [23], we studied the relaxation-times dependence of the efficiency and power in a Brownian Carnot cycle with the instantaneous adiabatic processes and time-dependent harmonic potential, described by the underdamped Langevin equation. In this system, we numerically showed that the Carnot efficiency is compatible with finite power in the vanishing limit of the relaxation times in the small temperature-difference regime. We analytically showed that the present results are consistent with the trade-off relation between efficiency and power, which was proved for more general systems in [25, 29, 30]. By expressing the trade-off relation using the entropy production in terms of the relaxation times of the system, we demonstrated that such compatibility is possible by both the diverging constant and the vanishing entropy production in the trade-off relation in the vanishing limit of the relaxation times.

In the numerical simulation results in Sec. 6.4, we used a specific protocol. However, we can use other protocols satisfying the following three conditions to achieve the Carnot efficiency and finite power in the small temperature-difference regime. The first condition is that the protocol should satisfy the condition in Eq. (6.59). For such a protocol, the heat leakage in the relaxation at the beginning of the isothermal processes is $O(\Delta T)$. Thus, heat leakage can be neglected in the small temperature-difference regime, compared with the heat flowing in the isothermal processes. The second condition is that the stiffness is expressed by using a scaling function as in Eq. (6.26). The third condition of the protocols is that the stiffness diverges at any time of time. This is satisfied by the vanishing relaxation time of position, and it is one of the necessary conditions for the entropy production rate vanishing after the relaxation, as we showed in Sec. 6.5. When the entropy production rate at any time vanishes, irreversible works also vanish, which allows us to derive the compatibility of the Carnot efficiency and finite power in the small temperature-difference regime.

Note that we showed that achieving both the Carnot efficiency and finite power is possible in the small temperature-difference regime without breaking the trade-off relation in Eq. (6.98) of the proposed cycle. In the linear irreversible thermodynamics, which can describe the heat engines operating in the small temperature-difference regime, the currents of the systems are described by the linear combination of affinities, and their coefficients are called the Onsager coefficients. When these coefficients have the reciprocity resulting from the time-reversal symmetry of the systems, a previous study [7] showed that the compatibility of the Carnot efficiency with finite power is forbidden. The same study also showed that the compatibility can be allowed in the systems without time-reversal symmetry. However, in some studies related to the concrete systems without time-reversal symmetry [8–13], the compatibility has not been found thus far. On the other hand, there is a possibility of the compatibility of the Carnot efficiency and finite power when the Onsager coefficients with reciprocity show diverging behaviors (cf. Eq. (7) in Ref. [31]). The Onsager coefficients of our Carnot cycle can be obtained in the same way as Ref. [18], which have reciprocity. In the vanishing limit of the relaxation times, we can show the divergence of these Onsager coefficients. Although the effect of the asymmetric limit of the non-diagonal Onsager coefficients on the linear irreversible heat engines realizing the Carnot efficiency at finite power was studied in Ref. [7], this case is different from our case where all of the Onsager coefficients show the diverging behaviors.

Furthermore, another study reported the compatibility of the Carnot efficiency with finite power using a time-delayed system within the linear response theory [92]. Because the time-delayed systems are not described by the Markovian dynamics, the trade-off relation in Eq. (6.2) may not be applied to them. Thus, there may be a possibility to achieve the Carnot efficiency in finite-power non-Markovian heat engines. In this paper, however, we showed that achieving both the Carnot efficiency and finite power is possible in a Markovian heat engine.

Although we have used the instantaneous adiabatic process, the other type of adiabatic process can be used for the study of the Brownian Carnot cycle [23, 24, 27, 91]. In this adiabatic process, the system contacts with a heat bath with varying temperature that maintains vanishing heat flow between the system and the heat bath on average. While the Brownian Carnot cycle utilizing this adiabatic process does not suffer from the heat leakage, mathematical treatment may become more difficult. In Chap. 7, we discuss the Brownian Carnot cycle with the finite-time adiabatic process.

Chapter 7

Achieving Carnot efficiency in finite-power Brownian Carnot cycle with arbitrary temperature difference

7.1 Introduction

In Chap. 6, we used an underdamped Brownian Carnot cycle, where the instantaneous change of the potential and temperature of the heat bath is used as an adiabatic process [17, 22, 23, 90]. We theoretically and numerically showed the compatibility of the Carnot efficiency and finite power in the vanishing limit of the relaxation times in the small temperature-difference regime. This result was obtained by explicitly expressing the relaxation-times dependence of A and η in Eq. (6.2). In the large temperature-difference regime, however, we cannot achieve the compatibility even in the vanishing limit of the relaxation times. Just after the instantaneous adiabatic processes, there exists a relaxation of the kinetic energy of the particle. The heat flowing in the relaxation is regarded as the inevitable heat leakage that reduces the efficiency. Since the heat leakage is proportional to the temperature difference, we cannot neglect it for a large temperature difference. Thus, the compatibility of the Carnot efficiency and finite power has not been established yet without the restriction of the small temperature difference.

In this chapter, we will show that the compatibility of the Carnot efficiency and finite power is achievable in the vanishing limit of the relaxation times in an underdamped Brownian Carnot cycle with arbitrary temperature difference,, where finite-time adiabatic processes [27, 91] are introduced instead of the instantaneous ones. To be more specific, we assume that in this cycle, the Brownian particle is in contact with the heat bath with time-dependent temperature $T(t)$ and trapped by a time-dependent harmonic potential in Eq. (6.3). Then, a finite-time adiabatic process can be realized by carefully controlling both of the temperature of the heat bath and stiffness to prevent the heat flowing at any time during the process. Similar to Chap. 6, note that the word "finite" in this paper means "nonzero" and "noninfinite." For example, the finite-time adiabatic process means the adiabatic process where the time taken for this process is not zero and not infinite. Remarkably, this carefully-controlled adiabatic process can eliminate the heat leakages that exist in the instantaneous adiabatic process because of the continuous nature of the process. From the detailed calculations, we can show that A in Eq. (6.2) in our cycle diverges while making the entropy production per cycle vanish, under the fixed cycle time in the vanishing limit of the relaxation times for arbitrary temperature difference. Therefore, we can establish the compatibility of the Carnot efficiency and finite power within the framework of the trade-off relation in Eq. (6.2).

This chapter is organized as follows. In Sec. 7.2, we introduce the thermodynamic processes, where the Brownian particle is in contact with the heat bath at any time and introduced in Sec. 6.2, and consider them in the small relaxation-times regime. In Sec. 7.3, we consider the finite-time adiabatic processes. In Sec. 7.4, we construct the Carnot cycle by using the isothermal and finite-time adiabatic processes. In Sec. 7.5, we theoretically show that the compatibility of the Carnot efficiency and finite power is achievable without breaking the trade-off relation in Eq. (6.2). In Sec. 7.6, we present the results of numerical simulations of our cycle when we vary the temperature difference and relaxation times of the system. In Sec. 7.7, we summarize this chapter.

7.2 Thermodynamic process

We consider a Brownian particle in Sec. 6.2.1 and introduce thermodynamic process lasting for $t_i \le t \le t_f$. In this chapter, the temperature may depend on the time and we define $X_i \equiv X(t_i)$, $X_f \equiv X(t_f)$, and

$$\Delta X \equiv X_f - X_i \tag{7.1}$$

for any physical quantity $X(t)$. Then, the time taken for this process is given by

$$\Delta t \equiv t_f - t_i. \tag{7.2}$$

In the thermodynamic process, we operate $T(t)$ and $\lambda(t)$. Generally, they are functions of t including their initial and final values and Δt. We call these functions protocol. We also call the thermodynamic process in contact with the heat bath with the constant temperature the isothermal process, where we we can choose $\lambda(t)$ independently of $T(t)$. On the other hand, in the finite-time adiabatic process, $T(t)$ and $\lambda(t)$ cannot be chosen independently once Δt is fixed (see Sec. 7.3). As we will see, when we give Δt, we have to control $T(t)$ and $\lambda(t)$ to satisfy the restrictions in that process.

To define the work and heat, we consider the internal-energy change rate given by

$$\dot{E}(t) = \int dx \int dv \ \frac{\partial p(x,v,t)}{\partial t} \left[\frac{1}{2}mv^2 + \frac{1}{2}\lambda(t)x^2\right] + \int dx \int dv \ p(x,v,t) \ \left(\frac{1}{2}\dot{\lambda}(t)x^2\right), \tag{7.3}$$

using Eq. (6.17). Similar to the isothermal process discussed in Sec. 6.2.2, when the temperature depends on the time, we can define the work and heat per unit time.

$$\dot{W}(t) \equiv -\int dx \int dv \ p(x,v,t) \ \left(\frac{1}{2}\dot{\lambda}(t)x^2\right) = -\frac{1}{2}\dot{\lambda}(t)\sigma_x(t), \tag{7.4}$$

$$\begin{aligned}\dot{Q}(t) \equiv& \int dx \int dv \ \frac{\partial p(x,v,t)}{\partial t} \left[\frac{1}{2}mv^2 + \frac{1}{2}\lambda(t)x^2\right] \\ =&\frac{1}{2}m\dot{\sigma}_v(t) + \frac{1}{2}\lambda(t)\dot{\sigma}_x(t) \\ =&\frac{\gamma}{m}[T(t) - m\sigma_v(t)],\end{aligned} \tag{7.5}$$

using the Eqs. (6.14), (6.15), and (7.3). Then, from Eqs. (7.4) and (7.5), we obtain the output work W and heat Q in this process as

$$W \equiv -\int_{t_i}^{t_f} dt\ \frac{1}{2}\dot{\lambda}(t)\sigma_x(t), \tag{7.6}$$

$$Q \equiv \int_{t_i}^{t_f} dt\ \frac{\gamma}{m}[T(t) - m\sigma_v(t)] = W + \Delta E. \tag{7.7}$$

Using Eqs. (6.14)-(6.16), (6.18), and (7.5), we can derive the entropy production rate $\dot{\Sigma}$ of the total system including the heat bath and particle as

$$\begin{aligned}\dot{\Sigma}(t) \equiv& \dot{S}(t) - \frac{\dot{Q}(t)}{T(t)} = \frac{\frac{\gamma}{m}(T - m\sigma_v)^2 + (2T - m\sigma_v)\gamma\frac{{\sigma_{xv}}^2}{\sigma_x}}{T\left(m\sigma_v - m\frac{{\sigma_{xv}}^2}{\sigma_x}\right)} \\ =& \frac{m}{\gamma}\frac{\dot{Q}^2}{Tm\sigma_v} + \frac{\gamma}{m}\frac{T{\sigma_{xv}}^2}{m\sigma_v\left(\sigma_x\sigma_v - {\sigma_{xv}}^2\right)} \geq 0,\end{aligned} \tag{7.8}$$

where the last inequality comes from Eq. (6.13). From Eq. (7.8), we obtain the entropy production of the total system in this process as

$$\Sigma = \int_{t_i}^{t_f} dt\ \dot{\Sigma}(t) = \Delta S - \int_{t_i}^{t_f} dt\ \frac{\dot{Q}(t)}{T(t)}, \tag{7.9}$$

where $\Delta S \equiv S_f - S_i$ is the entropy change of the particle in this process.

7.2.1 Thermodynamic process in the small relaxation-times regime

We consider the thermodynamic process mentioned in Sec. 7.2, in the small relaxation-times regime where the relaxation times in Eqs. (6.9) and (6.10) are sufficiently smaller than $\Delta t = t_f - t_i$ at any time. Below, we refer to "small relaxation time" when $\tau_j/\Delta t \ll 1$ ($j = x, v$) is satisfied. We assume that Δt is finite in this regime. For convenience, we introduce a normalized time

$$s \equiv \frac{t - t_i}{\Delta t} \quad (0 \leq s \leq 1). \tag{7.10}$$

In the small relaxation-times regime in this process, we assume that $T(t)$ and $\lambda(t)$ ($t_i \leq t \leq t_f$) depend on time and vary smoothly and slowly, that is, on the time scale sufficiently longer than the relaxation times τ_x and τ_v. Moreover, we also assume that $T(t')/T(t)$ and $\lambda(t')/\lambda(t)$ are finite at any times t and t' ($t_i \leq t' \leq t_f$). We can rewrite $\lambda(t')/\lambda(t)$ as

$$\frac{\lambda(t')}{\lambda(t)} = \frac{\tau_x(t)}{\tau_x(t')}, \tag{7.11}$$

by using Eq. (6.9). From Eq. (7.11), we find that the assumption above Eq. (7.11) means that when we make $\tau_x(t)$ small, $\tau_x(t')$ for any t' in the same process also becomes small. From the Taylor expansion, we have

$$\frac{T(t + \delta t)}{T(t)} \simeq 1 + \delta t\frac{d}{dt}\ln T(t), \tag{7.12}$$

$$\frac{\lambda(t + \delta t)}{\lambda(t)} \simeq 1 + \delta t\frac{d}{dt}\ln \lambda(t), \tag{7.13}$$

where δt is assumed to be sufficiently smaller than Δt. Since, from the assumption above Eq. (7.11), the left-hand side of Eqs. (7.12) and (7.13) are finite, $d(\ln T)/dt$ and $d(\ln\lambda)/dt$ do not diverge even when the relaxation time of position vanishes, in other words, $\lambda(t)$ in Eq. (6.9) diverges. As shown in Appendix C, the variables σ_x, σ_v, and σ_{xv} in the small relaxation-times regime can be approximated by

$$\sigma_x \simeq \frac{T}{\lambda}, \quad \sigma_v \simeq \frac{T}{m}, \quad \sigma_{xv} \simeq \frac{1}{2}\frac{d}{dt}\left(\frac{T}{\lambda}\right). \tag{7.14}$$

We can rewrite σ_{xv} in Eq. (7.14) as

$$\sigma_{xv} \simeq \frac{T}{2\lambda}\frac{d}{dt}\ln\left(\frac{T}{\lambda}\right) = \frac{\tau_x T}{2\gamma}\frac{d}{dt}\ln\left(\frac{T}{\lambda}\right), \tag{7.15}$$

using Eq. (6.9). Because $d(\ln T)/dt$ and $d(\ln\lambda)/dt$ do not diverge, σ_{xv} vanishes when τ_x vanishes. Then, since we can neglect ${\sigma_{xv}}^2$ compared with $\sigma_x\sigma_v$, Φ in Eq. (6.12) is approximated by

$$\Phi \simeq \frac{T^2}{m\lambda}. \tag{7.16}$$

We consider the heat current in Eq. (7.5) in the small relaxation-times regime. In the vanishing limit of the relaxation times, $\dot{Q}$ in Eq. (7.5) appears to diverge since $\gamma/m = 1/\tau_v$ diverges, which is the coefficient of $T - m\sigma_v$ in Eq. (7.5). However, since $m\sigma_v$ approaches T in the above limit from Eq. (7.14), $\dot{Q}$ may become finite. In fact, we can evaluate $\dot{Q}$ using Eq. (7.14) and the second line of Eq. (7.5). By using Eq. (**??**), we obtain

$$\lambda(t)\dot{\sigma}_x(t) \simeq T\left(\frac{d}{dt}\ln\frac{T}{\lambda}\right), \quad m\dot{\sigma}_v \simeq \dot{T} = T\frac{d}{dt}\ln T. \tag{7.17}$$

Then, the heat current $\dot{Q}(t)$ in Eq. (7.5) is approximated by

$$\dot{Q}(t) = \frac{1}{2}m\dot{\sigma}_v + \frac{1}{2}\lambda\dot{\sigma}_x \simeq \frac{T}{2}\left(\frac{d}{dt}\ln\frac{T^2}{\lambda}\right). \tag{7.18}$$

Since $d(\ln T)/dt$ and $d(\ln\lambda)/dt$ do not diverge even in the vanishing limit of the relaxation times, $\dot{Q}$ does not diverge. Similarly, the entropy production rate in Eq. (7.8) is approximated by

$$\dot{\Sigma}(t) \simeq \frac{1}{T}\frac{\tau_v\dot{Q}^2 + \frac{\tau_x T^2}{4}\left(\frac{d}{dt}\ln\frac{T}{\lambda}\right)^2}{T - \frac{\tau_v\tau_x T}{4}\left(\frac{d}{dt}\ln\frac{T}{\lambda}\right)^2} = \frac{1}{\Delta t}\frac{1}{T}\frac{\frac{\tau_v}{\Delta t}\left(\frac{dQ}{ds}\right)^2 + \frac{\tau_x T}{4\Delta t}\left(\frac{d}{ds}\ln\frac{T}{\lambda}\right)^2}{T - \frac{\tau_v}{\Delta t}\frac{\tau_x}{\Delta t}\frac{T}{4}\left(\frac{d}{ds}\ln\frac{T}{\lambda}\right)^2}, \tag{7.19}$$

where we used Eqs. (6.9), (6.10), (7.10), and (6.116). Since $\dot{Q}$ and $d\ln(T/\lambda)/dt$ do not diverge, and Δt is finite, we can see that dQ/ds and $d\ln(T/\lambda)/ds$ do not diverge. Thus, we can see that the entropy production rate $\dot{\Sigma}$ vanishes in the vanishing limit of τ_x and τ_v. The entropy production in this process in the small relaxation-times regime is given by

$$\Sigma \simeq \int_0^1 ds\frac{1}{T}\frac{\frac{\tau_v}{\Delta t}\left(\frac{dQ}{ds}\right)^2 + \frac{\tau_x T^2}{4\Delta t}\left(\frac{d}{ds}\ln\frac{T}{\lambda}\right)^2}{T - \frac{\tau_v}{\Delta t}\frac{\tau_x}{\Delta t}\frac{T}{4}\left(\frac{d}{ds}\ln\frac{T}{\lambda}\right)^2}, \tag{7.20}$$

Since $\dot{\Sigma}$ vanishes in the vanishing limit of the relaxation times, the entropy production Σ

also vanishes.

7.3 Finite-time adiabatic process

To construct the Carnot cycle, we consider the adiabatic process connecting the end of the isothermal process with the temperature T_i and the beginning of the other isothermal process with the temperature T_f. We introduce the finite-time adiabatic process where the heat current in Eq. (7.5) vanishes at any time. In such a process, we should control the temperature of the heat bath so as to satisfy

$$T(t) = m\sigma_v(t). \tag{7.21}$$

Then, using Eqs. (6.15) and (7.21), we find that $\lambda(t)$ should satisfy

$$\lambda(t) = -\frac{\dot{T}(t)}{2\sigma_{xv}(t)}. \tag{7.22}$$

Since the heat Q in Eq. (7.7) also vanishes in this process, we derive the relation between the output work and the internal energy change in this process as

$$W = -\Delta E. \tag{7.23}$$

Moreover, since $\dot{Q}$ vanishes, the entropy change of the particle in this process satisfies

$$\Delta S = \Sigma \geq 0, \tag{7.24}$$

where we used Eq. (7.9).

Next, we consider how to choose the protocol in the finite-time adiabatic process. We cannot choose Δt, T, and λ independently since they have to satisfy the restriction in Eq. (7.22). We consider the case that the relaxation times is much smaller than Δt, because the entropy production of the total system vanishes in the vanishing limit of the relaxation times, as mentioned below Eq. (7.20). For the entropy production of the total system to vanish in the vanishing limit of the relaxation times, Δt should be much larger than the relaxation times. Thus, we specify how to choose the protocol when we give a finite Δt.

In the finite-time adiabatic process with Δt given, σ_x, σ_v, σ_{xv}, T, and λ should satisfy Eqs. (6.14)–(6.16), and (7.21). In Appendix D, we derive the protocol of the finite-time adiabatic process from Eqs. (6.14)–(6.16), and (7.21) when we give the time evolution of $\sigma_x(t)$. Then, $T(t)$ and $\lambda(t)$ are expressed by $\sigma_x(t)$. To determine $\sigma_x(t)$, we have to give the initial and final values and how to connect them. We assume that we can arbitrarily connect these values as long as $\sigma_x(t)$ is smooth. Keeping the small relaxation-times regime of our interest in mind, we impose the following condition on the initial and final values of $\sigma_x(t)$. Since the finite-time adiabatic process continuously connects the isothermal processes, we assume that T_i and λ_i (T_f and λ_f) in the finite-time adiabatic process are the same as those at the end (beginning) of the isothermal process with T_i (T_f). In the small relaxation-times regime, the isothermal process should satisfy Eq. (7.14) at any time. Thus, the initial and final values of $\sigma_x(t)$ should satisfy

$$\sigma_{xi} \simeq \frac{T_i}{\lambda_i}, \quad \sigma_{xf} \simeq \frac{T_f}{\lambda_f}. \tag{7.25}$$

Below, we only consider the finite-time adiabatic process satisfying the condition in Eq. (7.25). σ_{xi} and σ_{xf} are changeable only through λ_i and λ_f since T_i and T_f are assumed to be given in the Carnot cycle. Moreover, when we construct the Carnot cycle in the small relaxation-times regime in Sec. 7.4, we assume that we can determine the initial and final values of $\lambda(t)$ only in the hot isothermal process among the four processes, which specifically correspond to λ_1 and λ_2 in Fig. 7.1, respectively. Thus, we can give λ_i (λ_f) in the finite-time adiabatic process connecting the end of the hot (cold) isothermal process and the beginning of the cold (hot) isothermal process, where $\lambda_i = \lambda_2$ ($\lambda_f = \lambda_1$) in Fig. 7.1. This means that we can give only one of σ_{xi} and σ_{xf} arbitrarily in the finite-time adiabatic process. The other is determined by the condition for Δt as shown below.

The previous study [91] considered the finite-time adiabatic process in the overdamped regime of the present model and revealed how Δt depends on the time evolution of the protocol and the state of the particle. We here develop a similar discussion to Ref. [91] and reveal restriction among Δt and the five variables ($T(t)$ and $\lambda(t)$ as the protocol and $\sigma_x(t)$, $\sigma_v(t)$, and $\sigma_{xv}(t)$ representing the state of the Brownian particle) in our underdamped system. From Eqs. (6.12), (6.14)–(6.16), and (7.21), we obtain

$$\begin{aligned}\frac{d\Phi}{dt} =& \frac{d}{dt}(\sigma_x\sigma_v - {\sigma_{xv}}^2)\\ =& 2\sigma_{xv}\frac{T}{m} + \sigma_x\left(\frac{2\gamma T}{m^2} - \frac{2\gamma}{m}\frac{T}{m} - \frac{2\lambda}{m}\sigma_{xv}\right) - 2\sigma_{xv}\left(\frac{T}{m} - \frac{\lambda}{m}\sigma_x - \frac{\gamma}{m}\sigma_{xv}\right)\\ =& \frac{2\gamma}{m}{\sigma_{xv}}^2 = \frac{\gamma}{2m}\left(\frac{d\sigma_x}{dt}\right)^2.\end{aligned} \tag{7.26}$$

We derive $\Delta\Phi$ in the finite-time adiabatic process as

$$\Delta\Phi = \int_{t_i}^{t_f} dt\ \frac{d\Phi}{dt} = \frac{1}{2}\frac{\gamma}{m}\int_{t_i}^{t_f} dt\ \left(\frac{d\sigma_x}{dt}\right)^2 = \frac{1}{2}\frac{\gamma}{m}\frac{1}{\Delta t}\int_0^1 ds\ \left(\frac{d\sigma_x}{ds}\right)^2, \tag{7.27}$$

using Eq. (7.10). Thus, we obtain Δt as

$$\Delta t = \frac{1}{2}\frac{\gamma}{m}\frac{1}{\Delta\Phi}\int_0^1 ds\ \left(\frac{d\sigma_x}{ds}\right)^2. \tag{7.28}$$

When Φ at the beginning and end of the finite-time adiabatic process satisfies Eq. (7.16), Eq. (7.28) can be rewritten as

$$\Delta t \simeq \frac{\gamma}{2}\frac{1}{T_f^2/\lambda_f - T_i^2/\lambda_i}\int_0^1 ds\ \left(\frac{d\sigma_x}{ds}\right)^2. \tag{7.29}$$

Giving the finite Δt and $\sigma_x(t)$ including an undetermined value σ_{xf} (σ_{xi}), we show how to determine λ_f (λ_i) to satisfy Eq. (7.29). Since we can give only one of λ_i and λ_f, we consider each case. We first consider the case of giving λ_i. In the small relaxation-times regime, we obtain $\dot{\sigma}_x(t) = 2\sigma_{xv}(t) = O(\tau_x(t))$ from Eqs. (6.14) and (7.15). Moreover, since $\lambda_i/\lambda(t)$ is finite for any t from the assumption above Eq. (7.11), we obtain $O(\tau_x(t)) = O(\tau_{xi})$. Thus, the order of $d\sigma_x(s)/ds = \Delta t\dot{\sigma}_x(t)$ is $O(\tau_{xi})$ for any t, and the order of the integral in Eq. (7.29) is $O(\tau_{xi}^2)$. To make Eq. (7.29) consistent with the finite Δt in the vanishing limit of τ_{xi}, we impose the condition for λ_i and λ_f as

$$\frac{T_f^2}{\lambda_f} - \frac{T_i^2}{\lambda_i} = \frac{\alpha}{\gamma}\tau_{xi}^2 = \frac{\alpha\gamma}{\lambda_i^2}, \tag{7.30}$$

where α is a finite constant to be determined. Then, from Eq. (7.29), we obtain

$$\Delta t \simeq \frac{\lambda_i^2}{2\alpha}\int_0^1 ds\,\left(\frac{d\sigma_x}{ds}\right)^2. \tag{7.31}$$

Because $\lambda_i = \gamma/\tau_{xi}$ and the order of the integral in Eq. (7.31) is $O(\tau_{xi}^2)$, the product of λ_i^2 and the integral remains finite even when we make τ_{xi} small. When we give the finite Δt, α in Eq. (7.31) should be positively finite. Noting that $\sigma_x(t)$ connecting σ_{xi} and σ_{xf} is given, where σ_{xi} is determined by the given λ_i from Eq. (7.25) and σ_{xf} is to be determined, the result of the integral in Eq. (7.31) becomes the function of σ_{xf}, which can be rewritten in terms of λ_f, using Eq. (7.25). Solving Eqs. (7.30) and (7.31), we can formally obtain α and λ_f.

We can see that the entropy production of the total system is given by

$$\Sigma = \Delta S \simeq \frac{1}{2}\ln\left(\frac{T_f^2}{\lambda_f}\frac{\lambda_i}{T_i^2}\right) \simeq \frac{1}{2}\ln\left(1+\frac{\alpha\tau_{xi}}{T_i^2}\right), \tag{7.32}$$

from Eqs. (6.18), (7.16), (7.24), and (7.30). Therefore, the entropy production vanishes in the vanishing limit of the relaxation times ($\tau_{xi} \to 0$).

Next, we consider the case of giving λ_f. Then, we impose the condition instead of Eq. (7.30) as

$$\frac{T_f^2}{\lambda_f} - \frac{T_i^2}{\lambda_i} = \frac{\alpha'}{\gamma}\tau_{xf}^2 = \frac{\alpha'\gamma}{\lambda_f^2}, \tag{7.33}$$

where α' is a finite constant to be determined. By the similar discussion to the case of giving λ_i, we find that α' is positively finite, and obtain

$$\Delta t \simeq \frac{\lambda_f^2}{2\alpha'}\int_0^1 ds\,\left(\frac{d\sigma_x}{ds}\right)^2, \tag{7.34}$$

$$\Sigma \simeq -\frac{1}{2}\ln\left(1-\frac{\alpha'\tau_{xf}}{T_f^2}\right). \tag{7.35}$$

When we give Δt, we can obtain α' and λ_i by solving Eqs. (7.33) and (7.34). From Eq. (7.35), we can see that the entropy production vanishes in the vanishing limit of the relaxation times ($\tau_{xf} \to 0$).

7.4 Carnot cycle in the small relaxation-times regime

We construct a Carnot cycle in the small relaxation-times regime by connecting the hot and cold isothermal processes with the temperature T_h and T_c ($< T_h$) by the finite-time adiabatic processes (Fig. 7.1). We assume that $T(t)$ and $\lambda(t)$ are smooth during each process and continuous in the whole cycle including all the switchings between the processes. Note that $m\sigma_v(t) = T(t)$ holds in the finite-time adiabatic processes from Eq. (7.21), but that equality may not hold in the isothermal processes. This may be inconsistent with

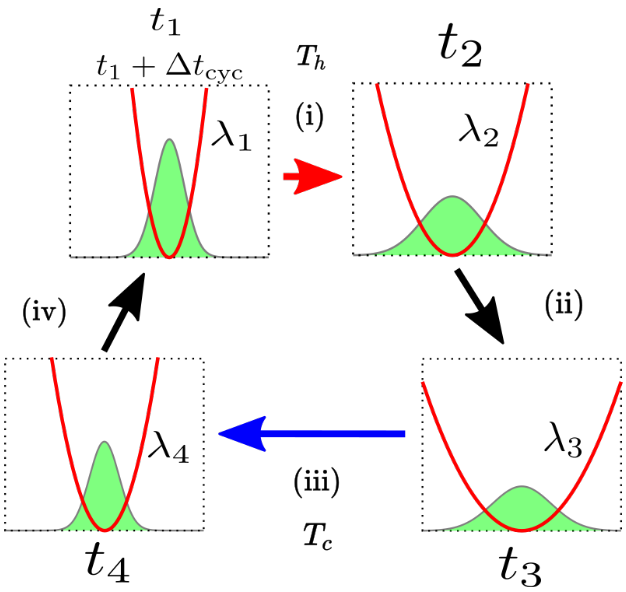

Figure 7.1: Schematic illustration of the Brownian Carnot cycle. In each box, the bottom horizontal line denotes the position coordinate x, and the boundary curve of the green filled area denotes the probability distribution of x. The red solid line corresponds to the harmonic potential with λ_i $(i = 1, 2, 3, 4)$. This cycle is composed of (i) the hot isothermal process, (ii) the finite-time adiabatic process, (iii) the cold isothermal process, and (iv) the finite-time adiabatic process.

the continuity assumption of $T(t)$ at the switchings between the isothermal and finite-time adiabatic processes, which means that the finite-time adiabatic processes may not be realized under the continuous $T(t)$. In the small relaxation-times regime, however, since the approximate equality $\sigma_v \simeq T/m$ in Eq. (7.14) is satisfied in the isothermal processes, we can regard Eq. (7.21) as being approximately satisfied at the beginning and end of the finite-time adiabatic processes. That is, the consistency with the continuity assumption of $T(t)$ recovers in the small relaxation-times regime. Thus, since we are interested in whether the compatibility of the Carnot efficiency and finite power is possible in the vanishing limit of the relaxation times, we consider only the Carnot cycle in the small relaxation-times regime.

We use the following protocol: (i) The hot isothermal process lasts for $t_1 \leq t \leq t_2$. The temperature of the heat bath satisfies $T(t) = T_h$, and the stiffness $\lambda(t)$ changes from λ_1 to λ_2. (ii) The finite-time adiabatic process connecting the end of the hot isothermal process and the beginning of the cold one lasts for $t_2 \leq t \leq t_3$. The temperature of the heat bath changes from T_h to T_c, and the stiffness changes from λ_2 to λ_3. (iii) The cold isothermal process lasts for $t_3 \leq t \leq t_4$. The temperature of the heat bath satisfies $T(t) = T_c$, and the stiffness $\lambda(t)$ changes from λ_3 to λ_4. (iv) The finite-time adiabatic process connecting the end of the cold isothermal process and the beginning of the hot one lasts for $t_4 \leq t \leq t_1 + \Delta t_{\rm cyc}$, where $\Delta t_{\rm cyc}$ is the cycle time. The temperature of the heat bath changes from T_c to T_h, and the stiffness changes from λ_4 to λ_1. In the above protocol, we assume that we can choose λ_1 and λ_2 arbitrarily and also assume that we choose the time taken for each process to be finite. When λ_2 is given, the initial value of τ_x of the finite-time adiabatic process (ii) is determined from Eq. (6.9). Then, since Δt in this process is assumed to be finite, the condition of the finite-time adiabatic process in Eq. (7.30) should be satisfied because of the discussion in Sec. 7.3. Applying Eq. (7.30) to the finite-time adiabatic process (ii), we impose the condition as

$$\frac{T_c^2}{\lambda_3} - \frac{T_h^2}{\lambda_2} = \alpha_{h\to c}\frac{\gamma}{\lambda_2^2}, \tag{7.36}$$

where $\alpha_{h\to c}$ is a finite positive constant. Here, the indexes "$h \to c$" and "$c \to h$" denote the quantities in the finite-time adiabatic processes (ii) and (iv), respectively. When we give $\sigma_x(t)$ and Δt in this process, we obtain the equation for $\alpha_{h\to c}$ and λ_3 from Eq. (7.31). By solving Eqs. (7.31) and (7.36) simultaneously, we can obtain $\alpha_{h\to c}$ and λ_3. In the finite-time adiabatic process (iv), the final value of τ_x is given since λ_1 is determined. Then, since Δt in this process is assumed to be finite, the condition in Eq. (7.33) should be satisfied. Applying Eq. (7.33) to the finite-time adiabatic process (iv), we impose the condition as

$$\frac{T_h^2}{\lambda_1} - \frac{T_c^2}{\lambda_4} = \alpha'_{c\to h}\frac{\gamma}{\lambda_1^2}, \tag{7.37}$$

where $\alpha'_{c\to h}$ is a finite positive constant. When we give $\sigma_x(t)$ and Δt in this process, we can obtain λ_4 and $\alpha'_{c\to h}$ by solving Eqs. (7.34) and (7.37) simultaneously. For convenience, we introduce the function $\phi(t)$ to describe the time evolution of the temperature $T(t)$ as

$$T(t) = \frac{T_h T_c}{T_h + (T_c - T_h)\,\phi(t)}. \tag{7.38}$$

In our Carnot cycle, since we assume that $T(t)$ is continuous, $\phi(t)$ should also be continuous, satisfying

$$\begin{cases} \phi(t) = 1 & (t_1 \le t \le t_2) \\ 0 \le \phi(t) \le 1 & (t_2 \le t \le t_3) \\ \phi(t) = 0 & (t_3 \le t \le t_4) \\ 0 \le \phi(t) \le 1 & (t_4 \le t \le t_1 + \Delta t_{\rm cyc}). \end{cases} \tag{7.39}$$

7.4.1 Construction of the Carnot cycle

In the hot isothermal process (i), the time taken for this process is given by

$$\Delta t_h \equiv t_2 - t_1. \tag{7.40}$$

We choose Δt_h as a finite value. When the entropy of the particle changes from S_1 to S_2 in this process, the entropy change ΔS_h in this process is given by

$$\Delta S_h \equiv S_2 - S_1. \tag{7.41}$$

Substituting $T(t) = T_h$ into Eq. (7.9), the heat Q_h flowing from the heat bath to the particle in this process is expressed by the entropy change of the particle ΔS_h and entropy production of the total system Σ_h in this process as

$$Q_h = T_h \Delta S_h - T_h \Sigma_h. \tag{7.42}$$

Since the entropy production Σ_h is nonnegative as seen from Eq. (7.8), we derive the inequality

$$T_h \Delta S_h \ge Q_h. \tag{7.43}$$

$Q_h > 0$ should be satisfied since the heat should flow from the heat bath to the particle in the hot isothermal process in the Carnot cycle useful as a heat engine. Therefore, $T_h \Delta S_h$ as the upper bound of Q_h in Eq. (7.43) has to be positive, and we obtain a necessary condition for the hot isothermal process:

$$S_2 > S_1. \tag{7.44}$$

Since we consider the small relaxation-times regime, we can derive a condition for λ_1 and λ_2 from Eq. (7.44). From Eqs. (6.18) and (7.16), the entropy change in the hot isothermal process in this regime can be approximated by

$$\Delta S_h \simeq \frac{1}{2} \ln \left(\frac{\lambda_1}{\lambda_2} \right). \tag{7.45}$$

Because of $\Delta S_h > 0$, λ_1 and λ_2 should satisfy

$$\lambda_1 > \lambda_2. \tag{7.46}$$

Moreover, λ_1/λ_2 is finite because of the assumption above Eq. (7.11). Then, because of Eqs. (7.45) and (7.46), ΔS_h in the small relaxation-times regime is positively finite.

In the finite-time adiabatic process (ii), the entropy change of the particle $\Delta S_{h\to c}$ is equal to the entropy production of the total system $\Sigma_{h\to c}$ because of Eq. (7.24). When the entropy of the particle changes from S_2 to S_3, we obtain

$$\Delta S_{h\to c} \equiv S_3 - S_2 = \Sigma_{h\to c}. \tag{7.47}$$

From the nonnegatively of $\Sigma_{h\to c}$ in Eq. (7.8), the relation

$$S_3 \geq S_2 \tag{7.48}$$

should be satisfied. The time taken for this process is given by

$$\Delta t_{h\to c} \equiv t_3 - t_2. \tag{7.49}$$

We choose $\Delta t_{h\to c}$ as a finite value.

In the isothermal process (iii), we can repeat the discussion similar to the isothermal process (i). In this process, the entropy of the particle changes from S_3 to S_4. Then, the time Δt_c taken for this process, the entropy change ΔS_c, and the heat Q_c flowing from the heat bath to the particle in this process are given by

$$\Delta t_c \equiv t_4 - t_3, \tag{7.50}$$
$$\Delta S_c \equiv S_4 - S_3, \tag{7.51}$$
$$Q_c = T_c \Delta S_c - T_c \Sigma_c, \tag{7.52}$$

where Σ_c is the entropy production of the total system in this process. We choose Δt_c as a finite value. Note that we cannot determine the sign of ΔS_c at this point by considering the sign of Q_c unlike the case of the hot isothermal process (i). We will determine the sign of ΔS_c later by considering the sum of the entropy change of the particle during one cycle.

In the finite-time adiabatic process (iv), the entropy change of the particle $\Delta S_{c\to h}$ is equal to the entropy production of the total system $\Sigma_{c\to h}$ because of Eq. (7.24). For the

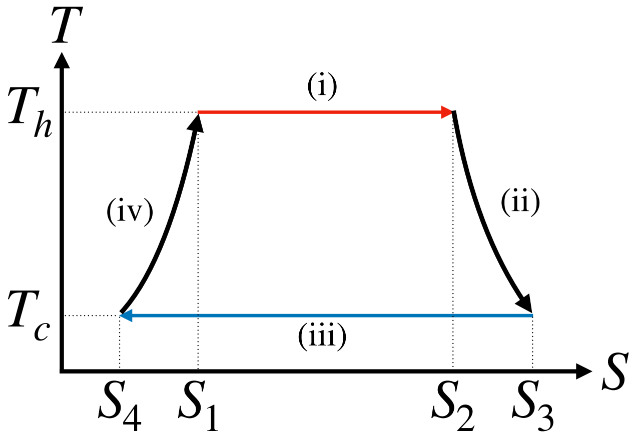

Figure 7.2: The entropy of the Brownian particle and temperature of the heat bath in our Carnot cycle with the finite-time adiabatic processes. From the restriction of the finite-time adiabatic processes, the entropy at the ends of the isothermal processes should satisfy $S_2 \leq S_3$ and $S_4 \leq S_1$.

cycle to close, the entropy of the particle should change from S_4 to S_1, which leads to

$$\Delta S_{c\to h} \equiv S_1 - S_4 = \Sigma_{c\to h}. \tag{7.53}$$

Since the entropy production $\Sigma_{c\to h}$ is nonnegative, S_4 and S_1 should satisfy

$$S_1 \geq S_4. \tag{7.54}$$

The time taken for this process is given by

$$\Delta t_{c\to h} \equiv (t_1 + \Delta t_{\mathrm{cyc}}) - t_4. \tag{7.55}$$

We choose $\Delta t_{c\to h}$ as a finite value.

After one cycle, the system returns to the initial state. Then, from Eqs. (7.44), (7.48), and (7.54), we obtain the trapezoid-like T-S diagram of our cycle in Fig. 7.2 from Eqs. (7.44), (7.48), and (7.54). The sum of the entropy change of the particle satisfies

$$\Delta S_h + \Delta S_{h\to c} + \Delta S_c + \Delta S_{c\to h} = 0. \tag{7.56}$$

Using Eqs. (7.47), (7.53), and (7.56), we obtain

$$\Delta S_c = -(\Delta S_h + \Sigma_{h\to c} + \Sigma_{c\to h}). \tag{7.57}$$

Using Eqs. (6.18) and (7.16), we find that the entropy change of the particle in the cold isothermal process in the small relaxation-times regime can be approximated by

$$\Delta S_c \simeq \frac{1}{2} \ln\left(\frac{\lambda_3}{\lambda_4}\right) < 0, \tag{7.58}$$

where the last inequality comes from $\Delta S_h + \Sigma_{h\to c} + \Sigma_{c\to h} > 0$ and Eq. (7.57). Then, we obtain

$$\lambda_3 < \lambda_4. \tag{7.59}$$

From the first law of the thermodynamics, we derive the output work per cycle as

$$
\begin{aligned}
W =& Q_h + Q_c \\
=& T_h \Delta S_h + T_c \Delta S_c - T_h \Sigma_h - T_c \Sigma_c \\
=& (T_h - T_c)\Delta S_h - T_h \Sigma_h - T_c(\Sigma_{h\to c} + \Sigma_c + \Sigma_{c\to h}),
\end{aligned} \tag{7.60}
$$

using Eqs. (7.42), (7.52), and (7.57). Since the entropy production of the total system in each process is nonnegative, the work W has the upper bound W_0 as

$$
W_0 \equiv (T_h - T_c)\Delta S_h \geq W. \tag{7.61}
$$

Since $T_h - T_c$ and ΔS_h are finite, W_0 is also finite. When the entropy production in each process vanishes, we obtain $W = W_0$ from Eq. (7.60). By using Eqs. (7.40), (7.49), (7.50), and (7.55), the time taken for each process satisfies

$$
\Delta t_{\mathrm{cyc}} = \Delta t_h + \Delta t_{h\to c} + \Delta t_c + \Delta t_{c\to h}. \tag{7.62}
$$

Since the time taken for each process is finite, Δt_{cyc} is finite. Using Eqs. (6.1), (7.43), and (7.61), we obtain the conditions for the efficiency η and power P of our cycle as

$$
\eta \equiv \frac{W}{Q_h} \leq \frac{W_0}{T_h \Delta S_h} = \frac{(T_h - T_c)\Delta S_h}{T_h \Delta S_h} = \eta_{\mathrm{C}} \tag{7.63}
$$

$$
P \equiv \frac{W}{\Delta t_{\mathrm{cyc}}} \leq \frac{W_0}{\Delta t_{\mathrm{cyc}}} = \frac{(T_h - T_c)\Delta S_h}{\Delta t_{\mathrm{cyc}}} \equiv P_0, \tag{7.64}
$$

where P_0 is the power when $W = W_0$ is satisfied. When the entropy productions Σ_h, Σ_c, $\Sigma_{h\to c}$, and $\Sigma_{c\to h}$ vanish, Q_h and W approach $T_h \Delta S_h$ in Eq. (7.43) and W_0 in Eq. (7.61), respectively. Then, the efficiency approaches the Carnot efficiency η_{C} and the power approaches P_0.

7.5 Theoretical analysis

7.5.1 Trade-off relation

We show the trade-off relation between the efficiency η and power P in our cycle. From Eq. (7.8), we obtain the following inequality:

$$
\dot{\Sigma} \geq \frac{\dot{Q}^2}{\gamma T \sigma_v}, \tag{7.65}
$$

or equivalently,

$$
|\dot{Q}| \leq \sqrt{\gamma T \sigma_v \dot{\Sigma}}. \tag{7.66}
$$

Since the finite-time adiabatic processes satisfy $\dot{Q} = 0$, Q_h can be written as

$$
Q_h = \int_{t_1}^{t_2} dt\ \dot{Q} = \int_{t_1}^{t_1+\Delta t_{\mathrm{cyc}}} dt\ \phi(t)\dot{Q}(t), \tag{7.67}
$$

where we used Eq. (7.39). Then, by using the Cauchy-Schwarz inequality, we derive the inequality for Q_h as

$$Q_h^2 = \left(\int_{t_1}^{t_1+\Delta t_{\rm cyc}} dt \ \phi \dot{Q} \right)^2 \leq \Delta t_{\rm cyc} T_c^2 \chi \Sigma_{\rm cyc}, \tag{7.68}$$

using Eq. (7.66), where $\Sigma_{\rm cyc}$ is the entropy production of the total system per cycle and χ is defined as

$$\chi \equiv \frac{\gamma}{T_c^2 \Delta t_{\rm cyc}} \int_{t_1}^{t_1+\Delta t_{\rm cyc}} dt \ \phi(t)^2 T(t) \sigma_v(t). \tag{7.69}$$

Multiplying both sides of Eq. (7.68) by $W/(\Delta t_{\rm cyc} Q_h^2)$, we derive the inequality for the power in Eq. (7.64) as

$$P \leq \frac{T_c^2 \eta}{Q_h} \chi \Sigma_{\rm cyc}. \tag{7.70}$$

using Eq. (7.63). Since the entropy change of the particle vanishes after one cycle, the entropy production of the total system per cycle is equal to that of the heat bath. Thus, the entropy production per cycle relates to the efficiency as

$$\Sigma_{\rm cyc} = -\frac{Q_h}{T_h} - \frac{Q_c}{T_c} = \frac{Q_h}{T_c}(\eta_{\rm C} - \eta), \tag{7.71}$$

using Eqs. (6.1), (7.60), and (7.63). From Eq. (7.71), it turns out that the efficiency becomes the Carnot efficiency when $\Sigma_{\rm cyc}$ vanishes. By using Eqs. (7.70) and (7.71), we can derive the trade-off relation between the efficiency and power:

$$P \leq \chi T_c \eta (\eta_{\rm C} - \eta). \tag{7.72}$$

This inequality is the same as Eq. (6.109) in Sec. 6.5, where we applied the method based on Sec. 5.4.2.

7.5.2 Compatibility of the Carnot efficiency and finite power

We show the compatibility of the Carnot efficiency and finite power in the vanishing limit of the relaxation times in our cycle. If the entropy productions of the total system in all the processes vanish, the entropy production per cycle vanishes. Then, from Eq. (7.71), we can achieve the Carnot efficiency because of Eq. (7.71), and the power also approaches a finite P_0 in Eq. (7.64). Thus, we evaluate the entropy production in each process in the small relaxation-times regime.

From Eq. (7.20), the entropy productions in the isothermal processes are given by

$$\Sigma_h \simeq \frac{1}{T_h} \int_0^1 ds_h \frac{\frac{\tau_v}{\Delta t_h} \left(\frac{dQ}{ds_h} \right)^2 + \frac{\tau_x}{\Delta t_h} \frac{T_h^2}{4} \left(\frac{d}{ds_h} \ln \frac{T_h}{\lambda} \right)^2}{T_h - \frac{\tau_x}{\Delta t_h} \frac{\tau_v}{\Delta t_h} \frac{T_h}{4} \left(\frac{d}{ds_h} \ln \frac{T_h}{\lambda} \right)^2}, \tag{7.73}$$

$$\Sigma_c \simeq \frac{1}{T_c} \int_0^1 ds_c \frac{\frac{\tau_v}{\Delta t_c} \left(\frac{dQ}{ds_c} \right)^2 + \frac{\tau_x}{\Delta t_c} \frac{T_c^2}{4} \left(\frac{d}{ds_c} \ln \frac{T_c}{\lambda} \right)^2}{T_c - \frac{\tau_x}{\Delta t_c} \frac{\tau_v}{\Delta t_c} \frac{T_c}{4} \left(\frac{d}{ds_c} \ln \frac{T_c}{\lambda} \right)^2}, \tag{7.74}$$

where s_h and s_c are the normalized times in the corresponding isothermal processes:

$$s_h \equiv \frac{t-t_1}{\Delta t_h}, \quad s_c \equiv \frac{t-t_3}{\Delta t_c}. \tag{7.75}$$

In the isothermal processes, λ is not constant because of Eqs. (7.46) and (7.59), while T is constant.s Then, from Eq. (7.18), we find that dQ/ds_h in Eq. (7.73) and dQ/ds_c in Eq. (7.74) are finite when λ changes smoothly except when λ takes extremal values. Thus, in the vanishing limit of the relaxation times, the integrand in Eqs. (7.73) and (7.74) vanishes, and Σ_h and Σ_c also vanish.

Since we choose $\Delta t_{h\to c}$ and $\Delta t_{c\to h}$ as finite values, $\alpha_{h\to c}$ and $\alpha'_{c\to h}$ in Eqs. (7.36) and (7.37) are positively finite because of the discussion below Eq. (7.31). By applying Eqs. (7.32) and (7.35) to the present finite-time adiabatic processes (ii) and (iv), respectively, we derive the entropy productions of the total system in these processes as

$$\Sigma_{h\to c} \simeq \frac{1}{2}\ln\left(1+\frac{\alpha_{h\to c}\tau_x(t_2)}{T_h^2}\right), \tag{7.76}$$

$$\Sigma_{c\to h} \simeq -\frac{1}{2}\ln\left(1-\frac{\alpha'_{c\to h}\tau_x(t_1)}{T_h^2}\right). \tag{7.77}$$

These entropy productions vanish in the vanishing limit of the relaxation times.

Note that there may exist an entropy production at the switchings between the isothermal and finite-time adiabatic processes. As shown in Appendix E, this entropy production is caused by the discontinuity of the time derivative of $T(t)$ and $\lambda(t)$ at the switchings, although we assume that $T(t)$ and $\lambda(t)$ are continuous. However, we can also show that this entropy production is negligible in the small relaxation-times regime.

Using Eqs. (7.73), (7.74), (7.76), and (7.77), we obtain the entropy production of the total system per cycle:

$$\begin{aligned}\Sigma_{\rm cyc} =&\Sigma_h+\Sigma_c+\Sigma_{h\to c}+\Sigma_{c\to h}\\ \simeq&\frac{1}{T_h}\int_0^1 ds_h \frac{\frac{\tau_v}{\Delta t_h}\left(\frac{dQ}{ds_h}\right)^2+\frac{\tau_x}{\Delta t_h}\frac{T_h^2}{4}\left(\frac{d}{ds_h}\ln\frac{T_h}{\lambda}\right)^2}{T_h-\frac{\tau_x}{\Delta t_h}\frac{\tau_v}{\Delta t_h}\frac{T_h}{4}\left(\frac{d}{ds_h}\ln\frac{T_h}{\lambda}\right)^2}\\ &+\frac{1}{T_c}\int_0^1 ds_c \frac{\frac{\tau_v}{\Delta t_c}\left(\frac{dQ}{ds_c}\right)^2+\frac{\tau_x}{\Delta t_c}\frac{T_c^2}{4}\left(\frac{d}{ds_c}\ln\frac{T_c}{\lambda}\right)^2}{T_c-\frac{\tau_x}{\Delta t_c}\frac{\tau_v}{\Delta t_c}\frac{T_c}{4}\left(\frac{d}{ds_c}\ln\frac{T_c}{\lambda}\right)^2}\\ &+\frac{1}{2}\ln\left(1+\frac{\alpha_{h\to c}\tau_x(t_2)}{T_h^2}\right)-\frac{1}{2}\ln\left(1-\frac{\alpha'_{c\to h}\tau_x(t_1)}{T_h^2}\right).\end{aligned} \tag{7.78}$$

From the discussion below Eqs. (7.75) and (7.77), the entropy productions in all the processes vanish in the vanishing limit of the relaxation times, and $\Sigma_{\rm cyc}$ also vanishes in this limit. Then, the heat Q_h in Eq. (7.42) and work W in Eq. (7.60) become $T_h\Delta S_h$ in Eq. (7.43) and W_0 in Eq. (7.61), respectively. Thus, the efficiency in Eq. (7.63) approaches the Carnot efficiency, and the power in Eq. (7.64) approaches P_0. Since $\Delta t_{\rm cyc}$ in Eq. (7.62) and W_0 are finite, P_0 is finite. Although this may seem to be inconsistent with the trade-off relation in Eq. (7.72), we below show that there is no inconsistency.

In the small relaxation-times regime, since σ_v is approximated by Eq. (7.14), χ in

Eq. (7.69) is approximated by

$$\chi \simeq \frac{1}{\tau_v \Delta t_{\rm cyc}} \left(\int_{t_1}^{t_1+\Delta t_{\rm cyc}} dt\ \phi^2 T^2 \right) = \frac{1}{\tau_v} \left(\int_0^1 ds_{\rm cyc}\ \phi^2 T^2 \right) = \frac{C}{\tau_v}, \tag{7.79}$$

where $s_{\rm cyc} \equiv (t - t_1)/\Delta t_{\rm cyc}$, and C is a constant defined as

$$C \equiv \int_0^1 ds_{\rm cyc}\ \phi^2 T^2. \tag{7.80}$$

Since T is finite and ϕ satisfies Eq. (7.39), C is finite. From Eq. (7.79), χ turns out to diverge in the vanishing limit of the relaxation times.

Next, we consider the right-hand side of Eq. (7.70). In the small relaxation-times regime, $\chi\Sigma_{\rm cyc}$ in Eq. (7.70) is approximated by

$$\begin{aligned}
\chi\Sigma_{\rm cyc} \simeq & \frac{C}{T_h} \int_0^1 ds_h \frac{\frac{1}{\Delta t_h}\left(\frac{dQ}{ds_h}\right)^2 + \frac{\tau_x}{\tau_v \Delta t_h}\frac{T_h^2}{4}\left(\frac{d}{ds_h}\ln\frac{T_h}{\lambda}\right)^2}{T_h - \frac{\tau_x}{\Delta t_h}\frac{\tau_v}{\Delta t_h}\frac{T_h}{4}\left(\frac{d}{ds_h}\ln\frac{T_h}{\lambda}\right)^2} \\
& + \frac{C}{T_c} \int_0^1 ds_c \frac{\frac{1}{\Delta t_c}\left(\frac{dQ}{ds_c}\right)^2 + \frac{\tau_x}{\tau_v \Delta t_c}\frac{T_c^2}{4}\left(\frac{d}{ds_c}\ln\frac{T_c}{\lambda}\right)^2}{T_h - \frac{\tau_x}{\Delta t_c}\frac{\tau_v}{\Delta t_c}\frac{T_c}{4}\left(\frac{d}{ds_c}\ln\frac{T_c}{\lambda}\right)^2} \\
& + \frac{C}{2\tau_v}\ln\left(1 + \frac{\alpha_{h\to c}\tau_x(t_2)}{T_h^2}\right) - \frac{C}{2\tau_v}\ln\left(1 - \frac{\alpha'_{c\to h}\tau_x(t_1)}{T_h^2}\right)
\end{aligned} \tag{7.81}$$

from Eqs. (7.78) and (7.79). As shown below Eq. (**??**), dQ/ds_h and dQ/ds_c in the isothermal processes are finite except when λ takes extremal values. Then, the first term of the numerator of the integrand in the first and second terms of Eq. (7.81) is positively finite even in the vanishing limit of the relaxation times since Δt_h and Δt_c are finite. Thus, the first and second terms in Eq. (7.81) do not vanish in the vanishing limit of the relaxation times, which means that $\chi\Sigma_{\rm cyc}$ does not vanish in this limit, while $\Sigma_{\rm cyc}$ vanishes and the efficiency approaches the Carnot efficiency because of Eq. (7.71). Therefore, the right hand-side of the trade-off relation in Eq. (7.70) does not vanish, which means that the finite power is allowed. In fact, the power also approaches the finite P_0 in this limit. Thus, the compatibility of the Carnot efficiency and finite power is achievable by taking the vanishing limit of the relaxation times in our Brownian Carnot cycle with arbitrary temperature difference.

7.6 Numerical simulation

We show numerical results of the compatibility of the Carnot efficiency and finite power in the vanishing limit of the relaxation times. In this simulation, we solved Eqs. (6.14)–(6.16) numerically by the fourth-order Runge-Kutta method. Our specific protocol in the

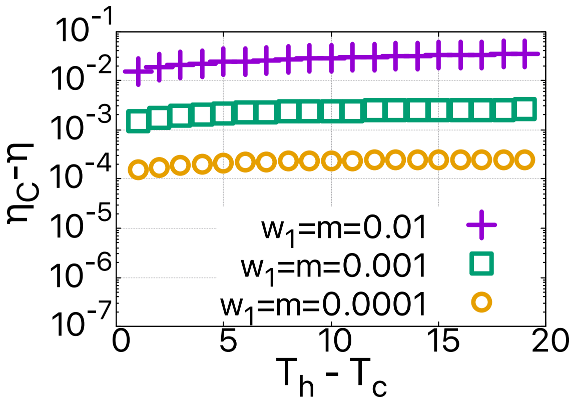

Figure 7.3: The difference between the Carnot efficiency and our efficiency measured with the protocol in Eqs. (7.82), (7.90), and (7.91) when we vary the relaxation times $\tau_x(t_1)$ and τ_v, which are proportional to w_1 and m, respectively. In these simulations, we set $w_1 = m = 10^{-2}$ (purple plus), $w_1 = m = 10^{-3}$ (green square), and $w_1 = m = 10^{-4}$ (orange circle). The efficiency appears to approach the Carnot efficiency in the vanishing limit of w_1 (or τ_x) and m (or τ_v).

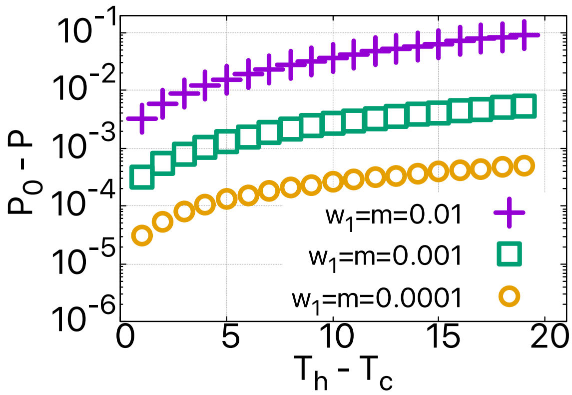

Figure 7.4: The power in Eq. (7.64) of our cycle, corresponding to Fig. 7.3. The power approaches P_0 in Eq. (7.102) in the vanishing limit of w_1 (or τ_x) and m (or τ_v) for any temperature difference.

isothermal processes is given by

$$\begin{aligned} T(t) =& T_h \quad (t_1 \le t \le t_2), \\ T(t) =& T_c \quad (t_3 \le t \le t_4), \\ \lambda_h(t) =& \frac{T_h}{w_1\left[1 + (\sqrt{w_2/w_1} - 1)\frac{t-t_1}{\Delta t_h}\right]^2} \quad (t_1 \le t \le t_2), \\ \lambda_c(t) =& \frac{T_c}{w_3\left[1 + (\sqrt{w_4/w_3} - 1)\frac{t-t_3}{\Delta t_c}\right]^2} \quad (t_3 \le t \le t_4), \end{aligned} \tag{7.82}$$

where $\lambda_h(t)$ and $\lambda_c(t)$ are the time evolution of $\lambda(t)$ in the isothermal process with temperature $T_{h,c}$, and w_j $(j = 1,2,3,4)$ are positive parameters. This protocol is inspired by the optimal protocol in the overdamped Brownian Carnot cycle with the instantaneous adiabatic processes [16,17] and used in our previous study [26]. From Eqs. (6.9) and (7.82),

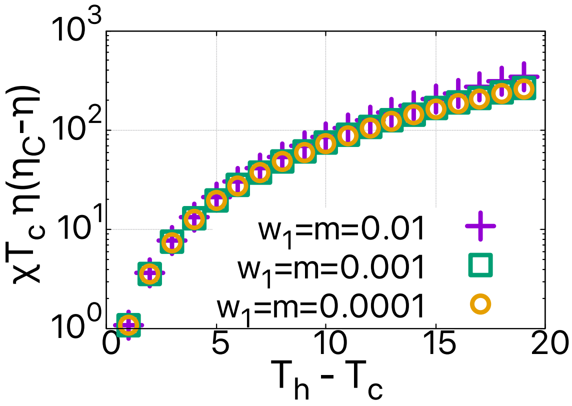

Figure 7.5: The right-hand side of Eq. (7.72) corresponding to Fig. 7.3. Although the efficiency approaches the Carnot efficiency in the limit of $w_1, m \to 0$ ($\tau_x, \tau_v \to 0$) in Fig. 7.3, $\chi T_c \eta(\eta_\mathrm{C} - \eta)$ remains almost unchanged. This means that χ diverges in this limit.

we obtain

$$w_j = \frac{T(t_j)}{\lambda_j} = \frac{T(t_j)}{\gamma}\tau_x(t_j). \tag{7.83}$$

Note that, using Eqs. (7.14) and (7.83), we obtain

$$\sigma_x(t_j) \simeq w_j \tag{7.84}$$

in the small relaxation-times regime. From Eqs. (7.46) and (7.83), $w_2/w_1 > 1$ should be satisfied. For all the simulations, we fixed $w_2/w_1 = 2.0$, since we can choose w_1 and w_2 arbitrarily, corresponding to the assumption that we can choose λ_1 and λ_2 arbitrarily, as mentioned above Eq. (7.36). Note that w_2/w_1 should be finite since λ_1/λ_2 is finite as shown below Eq. (7.46). We also fixed $T_c = 1.0$, $\Delta t_h = \Delta t_c = 1.0$, and $\gamma = 1.0$ and varied w_1, m, and the temperature difference $T_h - T_c$ (or equivalently, the temperature T_h). Note that Eqs. (7.83) and (7.84) satisfy the condition in Eq. (7.25).

We have to consider the finite-time adiabatic processes to determine w_3 and w_4. By using Eqs. (7.36), (7.37), and (7.83), we obtain

$$w_3 = \frac{T_h}{T_c}w_2 + \frac{\gamma\alpha_{h\to c}}{T_h^2 T_c}w_2^2, \tag{7.85}$$

$$w_4 = \frac{T_h}{T_c}w_1 - \frac{\gamma\alpha'_{c\to h}}{T_h^2 T_c}w_1^2, \tag{7.86}$$

where $\alpha_{h\to c}$ and $\alpha'_{c\to h}$ are constants. Below, we explain how to determine w_3, w_4, $\alpha_{h\to c}$, and $\alpha'_{c\to h}$ from Eqs. (7.31), (7.34), (7.85), and (7.86) when $\Delta t_{h\to c}$ and $\Delta t_{c\to h}$ are given. First, we give $\sigma_x(t)$ in the finite-time adiabatic processes to obtain the protocol. From Eq. (7.84), we give $\sigma_{x,h\to c}(t)$ satisfying $\sigma_{xi,h\to c} = w_2$ and $\sigma_{xf,h\to c} = w_3$. Although $\sigma_{xi,h\to c}$ is given since we can give w_2, $\sigma_{xf,h\to c}$ is undetermined. Similarly, we give $\sigma_{x,c\to h}(t)$ satisfying $\sigma_{xi,c\to h} = w_4$ and $\sigma_{xf,c\to h} = w_1$, where $\sigma_{xf,c\to h}$ is given and $\sigma_{xi,c\to h}$ is undetermined. Moreover, we assume $\Delta t_{h\to c} = \Delta t_{c\to h} = 1.0$. Using $\sigma_x(t)$, $\Delta t_{h\to c}$, and $\Delta t_{c\to h}$, we can obtain the equations for w_3, w_4, $\alpha_{h\to c}$, and $\alpha'_{c\to h}$ from Eqs. (7.31) and (7.34) in our cycle. Then, solving those equations together with Eqs. (7.85) and (7.86), we can determine w_3, w_4, $\alpha_{h\to c}$, and $\alpha'_{c\to h}$. Note that we can choose $\Delta t_{h\to c}$ and $\Delta t_{c\to h}$ arbitrarily as long as

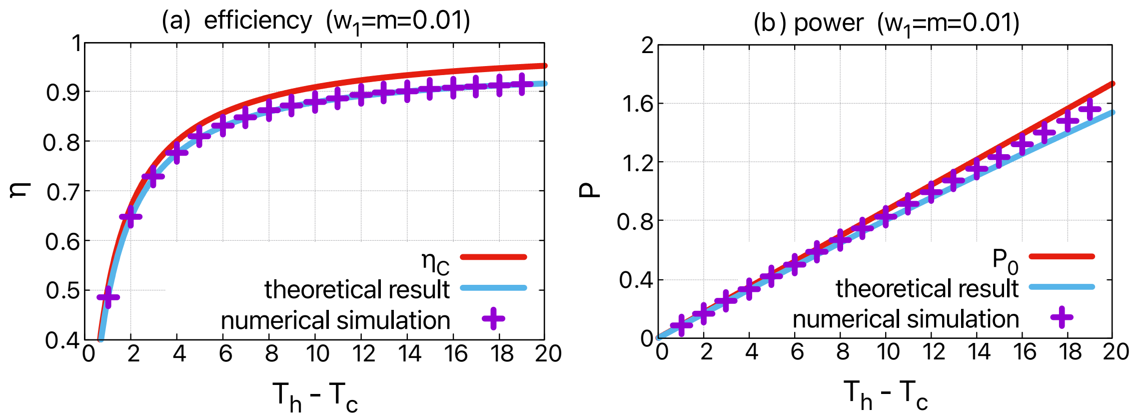

Figure 7.6: (a) Efficiency and (b) power derived from the numerical simulations in Figs. 7.3 and 7.4 (purple plus) and theoretical analysis (sky-blue solid line). We set $w_1 = m = 10^{-2}$. Although the relaxation times corresponding to these parameters are not very small among the parameters used in Fig. 7.3, the theoretical results and numerical simulations show a good agreement. We have confirmed a better agreement with smaller parameters (data not shown).

they are finite, although we set $\Delta t_{h\to c} = \Delta t_{c\to h} = 1.0$ in our simulation for simplicity.

In the finite-time adiabatic processes, we give

$$\sigma_x(t) = \begin{cases} w_{h\to c}(t) & (t_2 \le t \le t_3) \\ w_{c\to h}(t) & (t_4 \le t \le t_1 + \Delta t_{\rm cyc}), \end{cases} \tag{7.87}$$

$$w_{h\to c}(t) \equiv w_2 + (w_3 - w_2)(t - t_2)/\Delta t_{h\to c}, \tag{7.88}$$
$$w_{c\to h}(t) \equiv w_4 + (w_1 - w_4)(t - t_4)/\Delta t_{c\to h} \tag{7.89}$$

to obtain the protocol. Then, from Eqs. (D.3) and (D.5) in Appendix D, we find that the protocol is obtained as

$$\begin{aligned} T_{h\to c}(t) &= \left(1 - \frac{w_3 \frac{t-t_2}{\Delta t_{h\to c}}}{w_{h\to c}(t)}\right) T_h + \frac{w_3 \frac{t-t_2}{\Delta t_{h\to c}}}{w_{h\to c}(t)} T_c \quad (t_2 \le t \le t_3), \\ T_{c\to h}(t) &= \left(1 - \frac{w_1 \frac{t-t_4}{\Delta t_{c\to h}}}{w_{c\to h}(t)}\right) T_c + \frac{w_1 \frac{t-t_4}{\Delta t_{c\to h}}}{w_{c\to h}(t)} T_h \quad (t_4 \le t \le t_1 + \Delta t_{\rm cyc}), \end{aligned} \tag{7.90}$$

$$\begin{aligned} \lambda_{h\to c}(t) &= \frac{2T_{h\to c}(t) - \gamma \frac{(w_3 - w_2)}{\Delta t_{h\to c}}}{2w_{h\to c}(t)} \quad (t_2 \le t \le t_3), \\ \lambda_{c\to h}(t) &= \frac{2T_{c\to h}(t) - \gamma \frac{(w_1 - w_4)}{\Delta t_{c\to h}}}{2w_{c\to h}(t)} \quad (t_4 \le t \le t_1 + \Delta t_{\rm cyc}). \end{aligned} \tag{7.91}$$

Note that the approximate equality in Eq. (7.84) becomes equality only in the vanishing limit of the relaxation times. Although we derived the protocol in the adiabatic processes by regarding $\sigma_x(t_j)$ as $w_j (j = 1, 2, 3, 4)$ in Eqs. (7.87)–(7.89), the equality in Eq. (7.84) is not satisfied exactly in the simulation. Thus, the time evolution of $\sigma_x(t)$ realized by solving Eqs. (6.14)–(6.16) with the protocol in Eqs. (7.90) and (7.91) is not exactly the same as the given $\sigma_x(t)$ in Eq. (7.87). However, their difference becomes small when the relaxation times are sufficiently small. Moreover, because the domains of $w_{h\to c}$ and $w_{c\to h}$ are $t_2 \le t \le t_3$ and $t_4 \le t \le t_1 + \Delta t_{\rm cyc}$, respectively, they satisfy

$$0 \le \frac{w_3^{\frac{t-t_2}{\Delta t_{h\to c}}}}{w_{h\to c}(t)},\ \frac{w_1^{\frac{t-t_4}{\Delta t_{c\to h}}}}{w_{c\to h}(t)} \le 1. \tag{7.92}$$

Thus, $T_c \le T_{h\to c}(t), T_{c\to h}(t) \le T_h$ is satisfied because of Eq. (7.90). Then, $T_{h\to c}(t)$ and $T_{c\to h}(t)$ are finite at any time even in the vanishing limit of w_j ($j = 1, 2, 3, 4$). By using Eq. (7.87), we can calculate the integral in Eqs. (7.31) and (7.34). Then, $\Delta t_{h\to c}$ and $\Delta t_{c\to h}$ in the small relaxation-times regime satisfy

$$\Delta t_{h\to c} \simeq \frac{T_h^2}{2\alpha_{h\to c}}\left(\frac{w_3}{w_2} - 1\right)^2, \tag{7.93}$$

$$\Delta t_{c\to h} \simeq \frac{T_h^2}{2\alpha'_{c\to h}}\left(\frac{w_4}{w_1} - 1\right)^2. \tag{7.94}$$

From Eqs. (7.85) and (7.86), we obtain

$$\frac{w_3}{w_2} = \frac{T_h}{T_c} + \frac{\gamma\alpha_{h\to c}}{T_h^2 T_c} w_2 \simeq \frac{T_h}{T_c}, \tag{7.95}$$

$$\frac{w_4}{w_1} = \frac{T_h}{T_c} - \frac{\gamma\alpha'_{c\to h}}{T_h^2 T_c} w_1 \simeq \frac{T_h}{T_c}, \tag{7.96}$$

where the last approximate equalities hold because w_j in Eq.(7.83) is sufficiently small in the small relaxation-times regime and $\alpha_{h\to c}$ and $\alpha'_{c\to h}$ should be finite for any value of the relaxation times. Then, Eqs. (7.93) and (7.94) become

$$\Delta t_{h\to c} \simeq \frac{T_h^2}{2\alpha_{h\to c}}\left(\frac{T_h}{T_c} - 1\right)^2, \tag{7.97}$$

$$\Delta t_{c\to h} \simeq \frac{T_h^2}{2\alpha'_{c\to h}}\left(\frac{T_h}{T_c} - 1\right)^2. \tag{7.98}$$

Since we choose $\Delta t_{h\to c} = \Delta t_{c\to h} = 1.0$, $\alpha_{h\to c}$ and $\alpha'_{c\to h}$ are given by

$$\alpha_{h\to c} \simeq \frac{T_h^2}{2}\left(\frac{T_h}{T_c} - 1\right)^2, \tag{7.99}$$

$$\alpha'_{c\to h} \simeq \frac{T_h^2}{2}\left(\frac{T_h}{T_c} - 1\right)^2. \tag{7.100}$$

From Eqs. (7.85), (7.86), (7.99), and (7.100), we obtain w_3 and w_4.

By numerically calculating the integrals in Eqs. (7.6) and (7.7) from the solution to Eqs. (6.14)–(6.16), we obtained the heat Q_h and Q_c in Eqs. (7.42) and (7.52) and the work W in Eq. (7.60). Using the heat and work, we also calculated the efficiency $\eta = W/Q_h$ in Eq. (7.63) and power $P = W/\Delta t_{\rm cyc}$ in Eq. (7.64). In this simulation, we choose the initial condition as $\sigma_x(t_1) = w_1$, $\sigma_v(t_1) = T_h/m$, and $\sigma_{xv}(t_1) = 0$. Before starting to measure the thermodynamic quantities, we waited until the system settled down to a steady cycle. Therefore, our results are insensitive to the initial condition. When we consider the small relaxation-times regime in the present protocol, we should take the limit of $w_1 \to 0$ and the limit of $m \to 0$ for the following reasons. In the limit of $w_1 \to 0$, we can see that all w_j vanish from $w_2/w_1 = 2.0$ and Eqs. (7.85) and (7.86). Then, $w_{h\to c}(t)$ in Eq. (7.88) and $w_{c\to h}(t)$ in Eq. (7.89) vanish at any time. Since $T(t)$ is finite at any time in the limit of $w_1 \to 0$ as shown below Eq. (7.92), $\lambda(t)$ in Eqs. (7.82) and (7.91) diverges and the relaxation time of position $\tau_x(t)$ in Eq. (6.9) vanishes at any time. Moreover, in the limit of $m \to 0$, the relaxation time of velocity τ_v in Eq. (6.10) vanishes. Note that in the numerical simulations, we selected a time step smaller than the relaxation times. Specifically, we set the time step of the Runge-Kutta method as $\min(w_1, m) \times 10^{-2}$ because of $\tau_x(t_1) = \gamma w_1/T_h$ and $\tau_v = m/\gamma$.

We here confirm that the present protocol satisfies the assumption of finite $T(t')/T(t)$ and $\lambda(t')/\lambda(t)$, as mentioned above Eq. (7.11), where t and t' are any times in a process. Since $T_c \leq T(t) \leq T_h$ is satisfied at any time, $T(t')/T(t)$ is finite in all the processes. Moreover, from Eqs. (7.95) and (7.96) and the finite w_2/w_1, we find that w_3/w_1 and w_4/w_1 are also finite. Then, using Eqs. (7.82), (7.88), (7.89), and (7.91), we can confirm that $\lambda(t')/\lambda(t)$ is finite in all the processes even in the vanishing limit of the relaxation times.

Figure 7.3 shows the difference between the Carnot efficiency and our efficiency measured with the protocol in Eqs. (7.82), (7.90), and (7.91). In Chap. 6, the Carnot efficiency is achievable only in the small temperature-difference regime even when we take the vanishing limit of the relaxation times. In contrast to that, we can see that the efficiency approaches the Carnot efficiency in this limit even when the temperature difference is large.

Figure 7.4 shows the difference between P_0 in Eq. (7.64) and P corresponding to Fig. 7.3. In the small relaxation-times regime, we obtain

$$\Delta S_h \simeq \frac{1}{2} \ln \left(\frac{w_2}{w_1} \right) = \frac{1}{2} \ln 2, \tag{7.101}$$

using $w_2/w_1 = 2.0$ and Eqs. (7.45) and (7.83). In the vanishing limit of the relaxation times, since the approximate equality in Eq. (7.14) becomes equality, the approximate equality in Eq. (7.101) also becomes equality. Then, we obtain $W_0 = (T_h - T_c) \ln 2/2$ because of Eq. (7.61). Since we use $\Delta t_h = \Delta t_c = \Delta t_{h\to c} = \Delta t_{c\to h} = 1.0$, we have $\Delta t_{\rm cyc} = 4.0$ in our simulation. Thus, we obtain

$$P_0 = \frac{\ln 2}{8}(T_h - T_c), \tag{7.102}$$

from Eq. (7.64). From Fig. 7.4, we can see that the power approaches P_0 in the vanishing limit of the relaxation times for arbitrary temperature difference. Thus, the power turns out to be finite.

Figure 7.5 shows the upper bound of the power in Eq. (7.72). We can see that the upper bound of the power remains finite even when the efficiency approaches the Carnot efficiency as shown in Fig. 7.3. This means that χ in Eq. (7.69) diverges in the vanishing limit of the relaxation times, and also means that the finite power is allowed.

In Fig. 7.6, we compare the results of the numerical simulation with the theoretical analysis derived in Sec. 7.5 for the efficiency and power. To obtain the theoretical results in Fig. 7.6, we calculated the entropy productions in Eqs. (7.73), (7.74), (7.76), and (7.77). Note that we used Eq. (7.18) to calculate $\dot{Q}$ in Eqs. (7.73) and (7.74) from the time derivative of the protocol in Eq. (7.82). Moreover, we set $\Delta S_h = \ln 2/2$ in the theoretical analysis as in the numerical simulation. Then, we derived the heat in the hot isothermal process and work from Eqs. (7.42) and (7.60). Using the heat and work, we can obtain the efficiency in Eq. (7.63) and power in Eq. (7.64). The numerical simulation and theoretical analysis in Fig. 7.6 show a good agreement. Thus, this result shows the validity of the theoretical analysis in Sec. 7.5. From Figs. 7.3, 7.4, and 7.5, we can see that the compatibility of the Carnot efficiency with finite power is achievable without breaking the trade-off relation.

7.7 Summary of this chapter

We studied the relaxation-times dependence of the efficiency and power in an underdamped Brownian Carnot cycle with the finite-time adiabatic processes [27,91] and time-dependent harmonic potential. We showed that the compatibility of the Carnot efficiency and finite power is achievable in the vanishing limit of the relaxation times in our cycle. In Chap. 6, we showed that the compatibility of the Carnot efficiency and finite power is possible only in the small temperature-difference regime in the Brownian Carnot cycle with the instantaneous adiabatic processes. In this chapter, we considered the finite-time adiabatic processes and represented the entropy production in terms of the relaxation times in the small relaxation-times regime. Then, the entropy production vanishes in the vanishing limit of the relaxation times. We constructed the Carnot cycle with the finite-time adiabatic processes in the small relaxation-times regime. By the theoretical analysis of our cycle, we derived the trade-off relation and showed that in the vanishing limit of the relaxation times, the entropy production per cycle vanishes, in other words, the efficiency approaches the Carnot efficiency. Then, we also showed that the finite power is achievable without breaking the trade-off relation in Eq. (7.72). Moreover, we confirmed that our theoretical analysis agrees with the results of our numerical simulation. We finally note that we can use other protocols satisfying the assumption above Eq. (7.11) and continuity at the switchings between the processes instead of the present protocol in our simulation.

Appendix

A Reason of using the Stratonovich-type product in the definition of the heat

We explain why we use the Stratonovich-type product to define the heat current in Eq. (5.22). For the simplicity, we consider the heat in the free Brownian motion. We assume that $x(t)$ is the one-dimensional stochastic process produced by the Langevin equation:

$$\begin{aligned}\dot{x} =& v, \\ m\dot{v} =& -\gamma v + \sqrt{2\gamma k_{\mathrm{B}} T}\xi(t).\end{aligned} \tag{A.1}$$

In the stochastic process, there are some definition of the integral such as Itô integral and Stratonovich integral. They are generalized to the integral used by the $\times_h$

$$\int_{s=0}^{s=t} f(x) \times_h dx(s) \equiv \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f(x(ht_{k+1} + (1-h)t_k))(x(t_{k+1}) - x(t_k)). \tag{A.2}$$

When the Brownian particle is described by the Langevin equation in Eq. (A.1), by using $\times_h$, we can define the heat flowing from the heat bath to the particle between $s = 0$ and $s = t$ as

$$\begin{aligned} q_h \equiv& \int_{s=0}^{s=t} \left[-\gamma v(s) + \sqrt{2\gamma k_{\mathrm{B}} T}\xi(s)\right] \times_h dx(s) \\ =& \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \Big[-\gamma v(t_k)^2 + v(t_k)\sqrt{2\gamma k_{\mathrm{B}} T}\xi(t_k) \qquad \text{(A.3)} \\ & + h v(t_k)\Delta t \left(-\gamma \dot{v}(t_k) + \frac{1}{\Delta t}\sqrt{2\gamma k_{\mathrm{B}} T}(\xi(t_{k+1}) - \xi(t_k))\right)\Big] \Delta t. \qquad \text{(A.4)}\end{aligned}$$

Since $v(t_k)$ satisfies

$$v(t_k) = \sum_{l=0}^{k-1} (v_{l+1} - v_l) = \frac{1}{m}\sum_{l=0}^{k-1} [-\gamma v(t_l) + \sqrt{2\gamma k_{\mathrm{B}} T}\xi(t_l)], \tag{A.5}$$

Eq. (A.3) is rewritten as

$$q_h = \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \left[-\gamma v(t_k)^2 - h\frac{2\gamma k_{\mathrm{B}} T}{m}\right] \Delta t \tag{A.6}$$

where we use the Itô's lemma. Because of the heat, the kinetic energy of the particle changes. Since the mechanical energy change in each step is given by $mv(t_{k+1})^2/2 - mv(t_k)^2/2$, the mechanical energy change between $s=0$ and $s=t$ is given by

$$\Delta e = \lim_{\Delta t\to 0}\sum_{k=0}^{n-1}\left[\frac{mv(t_{k+1})^2}{2} - \frac{mv(t_k)^2}{2}\right] = \lim_{\Delta t\to 0}\sum_{k=0}^{n-1}\left[-\frac{\gamma}{m}v(t_k)^2 + \frac{\gamma k_{\mathrm{B}}T}{m}\right], \tag{A.7}$$

where we used Eq. (A.5). From the physical requirements, the total energy should conserve in the total system, and $q_h = \Delta e$ should be satisfied. Comparing Eqs. (A.3) and (A.7), we obtain $h = 1/2$, and the product $\times_h$ becomes the Stratonovich product $\circ$.

B Review: Entropy production along a trajectory

We consider a Brownian motion governed by the overdamped Langevin equation

$$\dot{x} = \mu F(x, \lambda(t)) + \sqrt{2D}\xi, \tag{B.1}$$

where μ, D, and λ are the mobility, diffusion constant, and protocol, respectively. Note that they satisfy the Einstein's relation $T = \mu D$, where T is the temperature of the heat bath. We assume that the force $F(x, \lambda(t))$ is divided into the force from the potential $-\partial V(x, \lambda(t)))/\partial x$ and the force applied to the particle directly $f(x, \lambda(t))$. ξ is the Gaussian white noise satisfying $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. The Fokker-Planck equation corresponding to Eq. (B.1) is given by

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} = -\frac{\partial}{\partial x}\left(\mu F(x,\lambda) - D\frac{\partial}{\partial x}\right)p(x,t). \tag{B.2}$$

When the trajectory is given by $x(t)$, we can define the trajectory-dependent entropy for the system as

$$s(t) \equiv -\ln p(x(t), t). \tag{B.3}$$

Then, the averaged entropy of the system is defined as

$$S(t) \equiv -\int dx\ p(x,t)\ln\{p(x,t)\} \equiv \langle s(t)\rangle\,. \tag{B.4}$$

Note that $p(x,t)$ in Eq. (B.3) is the solution of Eq. (B.2). From Eq. (B.3), the entropy change rate of the system along the trajectory is given by

$$\begin{aligned}\dot{s}(t) = &-\frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial t}\bigg|_{x(t)} - \frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial x}\bigg|_{x(t)}\dot{x} \\ = &-\frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial t}\bigg|_{x(t)} + \frac{1}{p(x,t)}\frac{j(x,t)}{Dp(x,t)}\bigg|_{x(t)}\dot{x} - \frac{\mu F(x,\lambda)}{D}\bigg|_{x(t)}\dot{x},\end{aligned} \tag{B.5}$$

where we used Eqs. (B.2) and (B.3). In overdamped dynamics, the force applied from the heat bath to the particle is given by $-\mu\dot{x} + \sqrt{2D}\xi$. Thus, the heat current flowing from the heat bath to the particle is defined as

$$\dot{q} \equiv (-\mu\dot{x} + \sqrt{2D}\xi)\circ\dot{x} = -F(x,\lambda)\circ\dot{x}. \tag{B.6}$$

Since the heat bath is assumed to be equilibrium, the entropy change rate of the heat bath is defined by

$$\dot{s}_b \equiv -\frac{\dot{q}}{T} = \frac{\mu F(x,\lambda)\circ\dot{x}}{D}. \tag{B.7}$$

By using Eqs. (B.5) and (B.7), the trajectory-dependent entropy production is given by

$$\dot{s}_{\rm tot} \equiv \dot{s}_b + \dot{s}(t) = -\left.\frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial t}\right|_{x(t)} + \left.\frac{1}{p(x,t)}\frac{j(x,t)}{Dp(x,t)}\right|_{x(t)}\dot{x}. \tag{B.8}$$

When the evolution of the probability distribution $p(x,t)$ is described by the Fokker-Planck equation in Eq. (B.2), from Eq. (3.82), we obtain

$$\begin{aligned} p(x',t+\Delta t) &= p(x',t) + \frac{\partial p(x',t)}{\partial t}\Delta t + O(\Delta t^2) \\ &\simeq \int dx \left(1 - \frac{1}{p(x,t)}\frac{\partial j(x,t)}{\partial x'}\Delta t\right)\delta(x-x')p(x,t), \end{aligned} \tag{B.9}$$

where Δt is sufficiently small. From Eq. (B.9), we derive the transition probability as

$$W(x',t+\Delta t|x,t) = \left(1 - \frac{1}{p(x,t)}\frac{\partial j(x,t)}{\partial x'}\Delta t\right)\delta(x-x'). \tag{B.10}$$

Thus, we obtain

$$\langle x'|x,t\rangle = \int dx'\; x' W(x',t+\Delta t|x,t) = x - \frac{j(x,t)}{p(x,t)}\Delta t \tag{B.11}$$

Then, $\langle \dot{x}|x,t\rangle$ is derived as

$$\langle \dot{x}|x,t\rangle = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle x'-x|x,t\rangle = \frac{j(x,t)}{p(x,t)}. \tag{B.12}$$

Substituting Eq. (B.12) into Eq. (B.13), we obtain

$$\dot{s}_{\rm tot} \equiv \dot{s}_{\rm b} + \dot{s}(t) = -\left.\frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial t}\right|_{x(t)} + \left.\frac{1}{p(x,t)}\frac{j(x,t)^2}{Dp(x,t)^2}\right|_{x(t)}. \tag{B.13}$$

Considering the average on $\dot{s}_{\rm tot}$ over all the trajectories, we derive the averaged entropy production of the total system as

$$\dot{\Sigma}_{\rm tot} \equiv \langle \dot{s}_{\rm tot}\rangle = \int dx \frac{j(x,t)^2}{Dp(x,t)} \geq 0 \tag{B.14}$$

The averaged entropy change of the heat bath is given by

$$\dot{S}_{\rm b} \equiv \langle \dot{s}_{\rm b}\rangle = \int dx \frac{F(x,t)j(x,t)}{T}, \tag{B.15}$$

where we used Eqs. (B.7) and (B.12) and the Einstein's relation. From Eqs. (B.13), (B.14), and (B.15), the averaged entropy change of the system is given by

$$\dot{S} = \langle \dot{s}\rangle = \dot{\Sigma}_{\rm tot} - \dot{S}_{\rm b} \tag{B.16}$$

We obtain the fluctuation theorems by considering the time reversal. We assume that the domain of t is given by $0 \leq t \leq t_0$. Under the time reversal, $\lambda(t)$ and $x(t)$ become $\lambda^\dagger(t) \equiv \lambda(t_0 - t)$ and $x^\dagger(t) \equiv x(t_0 - t)$, respectively. When we give the initial and final values $x_0 = x(0) = x^\dagger(t_0) = x^\dagger_{t_0}$ and $x_{t_0} = x(t_0) = x^\dagger(0) = x^\dagger_0$, by considering the transition probability corresponding to Eq. (B.1), we obtain [93]

$$\ln \frac{p[x(t)|x_0]}{p^\dagger[x^\dagger(t)|x^\dagger_0]} = \int_0^t \frac{F(x,t)\circ \dot{x}}{T} dt = \Delta s_{\mathrm{b}}. \tag{B.17}$$

We define the initial probability of the time-reversed trajectory as $p_1(x^\dagger_0) \equiv p_1(x_t)$. When we define the quantity as

$$R[x(t), \lambda(t); p_0, p_1] \equiv \ln \frac{p[x(t)|x_0]p_0}{p^\dagger[x^\dagger(t)|x^\dagger_0]p_1(x^\dagger_0)} = \Delta s_{\mathrm{b}} + \ln \frac{p_0(x)}{p_1(x_t)}, \tag{B.18}$$

we find that it satisfies

$$\left\langle e^{-R} \right\rangle \equiv \sum_{x(t),x_0} p[x(t)|x_0]p_0(x_0)e^{-R} = \sum_{x(t),x_0} p^\dagger[x^\dagger(t)|x^\dagger_0]p_1(x^\dagger_0) = 1. \tag{B.19}$$

Moreover, from Eq. (B.3), we derive the entropy change along the trajectory as

$$\ln \frac{p_0(x)}{p_1(x_t)} = \Delta s. \tag{B.20}$$

Then, since R in Eq. (B.18) becomes the entropy change of the total system along the trajectory $-\Delta s_{\mathrm{tot}}$, we find that Eq. (B.19) becomes

$$\left\langle e^{-\Delta s_{\mathrm{tot}}} \right\rangle = 1. \tag{B.21}$$

Using the Jensen's inequality, we derive the second law of thermodynamics as

$$\Sigma_{\mathrm{tot}} = \langle \Delta s_{\mathrm{tot}} \rangle \geq 0. \tag{B.22}$$

In the above discussion, we obtained the entropy production of the along the trajectory by considering the time reversal. We generalize it to the system described by the overdamped coupled Langevin equations and has the discrete n-states. We introduce the transition rate $w_{m\to n}$ from the state m to the state n. Then, the evolution of the probability distribution is described by the master equation given by

$$\frac{d}{dt} p_n(t) = \sum_{m\neq n} [w_{m\to n}p_m - w_{n\to m}p_n]. \tag{B.23}$$

We assume that the system evolves along the stochastic trajectory $n(t)$ and its state changes from n_j^- to n_j^+ at t_j. Similar to Eq. (B.3), the entropy of the system along the trajectory is defined as

$$s(t) \equiv -\ln p_{n(t)}(t). \tag{B.24}$$

From Eq. (B.23), its time derivative is given by

$$\dot{s}(t) = -\frac{1}{p_n(t)} \frac{\partial p_n(t)}{\partial t} - \sum_j \delta(t - t_j) \ln \frac{p_{n_j^+}(t_j)}{p_{n_j^-}(t_j)} \tag{B.25}$$

The second term is the entropy change of the system due to the jump at t_j. Since the entropy change of the heat bath along trajectory in Eq. (B.17) is written only by the transition probability of the trajectory and that of the time-reversal trajectory, we can derive it by using the transition rate as

$$\dot{s}_{\mathrm{b}}(t) \equiv -\sum_j \delta(t-t_j) \ln \frac{w_{n_j^- \to n_j^+}}{w_{n_j^+ \to n_j^-}}. \tag{B.26}$$

Using Eqs. (B.25)and and (B.26), we define the entropy production of the total system along the trajectory as

$$\dot{s}_{\mathrm{tot}} \equiv \dot{s} + \dot{s}_{\mathrm{b}} = -\frac{1}{p_n(t)} \frac{\partial p_n(t)}{\partial t} - \sum_j \delta(t-t_j) \ln \frac{p_{n_j^+}(t_j) w_{n_j^- \to n_j^+}}{p_{n_j^-}(t_j) w_{n_j^+ \to n_j^-}}. \tag{B.27}$$

Thus, we derive the averaged $\dot{s}$, $\dot{s}_{\mathrm{b}}$, and $\dot{s}_{\mathrm{tot}}$ as

$$\dot{S}(t) \equiv \langle \dot{s} \rangle = \sum_{n,k} p_n w_{k\to n} \ln \frac{p_n}{p_k}, \tag{B.28}$$

$$\dot{S}_{\mathrm{b}}(t) \equiv \langle \dot{s}_{\mathrm{b}} \rangle = \sum_{n,k} p_n w_{k\to n} \ln \frac{w_{n_j^- \to n_j^+}}{w_{n_j^+ \to n_j^-}}, \tag{B.29}$$

$$\dot{S}_{\mathrm{tot}} \equiv \langle \dot{s}_{\mathrm{tot}} \rangle = \sum_{n,k} p_n w_{k\to n} \ln \frac{p_{n_j^+}(t_j) w_{n_j^- \to n_j^+}}{p_{n_j^-}(t_j) w_{n_j^+ \to n_j^-}}. \tag{B.30}$$

C Derivation of Eqs. (6.116) and (7.14)

We show that the variables σ_x, σ_v, and σ_{xv} behave like Eqs. (6.116) and (7.14) in the small relaxation-times regime when the temperature $T(t)$ and stiffness $\lambda(t)$ satisfy the assumption above Eq. (7.11). For the above purpose, we first show that the variables σ_x, σ_v, and σ_{xv} relax toward Eqs. (6.116) and (7.14) when the temperature T and stiffness λ are constant. After that, we consider the case that the temperature $T(t)$ and stiffness $\lambda(t)$ depend on time and show that these variables satisfy Eqs. (6.116) and (7.14).

We assume that a thermodynamic process lasts for $t_i \le t \le t_f$ and we thus have $\Delta t = t_f - t_i$ in Eq. (7.2). The temperature T and stiffness λ are assumed to be constant. When we set $\sigma_x(t_i) = \sigma_{x0}$, $\sigma_v(t_i) = \sigma_{v0}$, and $\sigma_{xv}(t_i) = \sigma_{xv0}$ as an initial condition, we can solve Eqs. (6.14)–(6.16) using the Laplace transform [90], and we can obtain σ_x and σ_v as follows:

$$\sigma_x(t) = \frac{T}{\lambda} + \frac{m}{\lambda} D_1 e^{-\frac{\gamma}{m}(t-t_i)} + \frac{(\gamma + m\omega^*)^2}{4\lambda^2} D_2 e^{-\left(\frac{\gamma}{m}-\omega^*\right)(t-t_i)} + \frac{(\gamma - m\omega^*)^2}{4\lambda^2} D_3 e^{-\left(\frac{\gamma}{m}+\omega^*\right)(t-t_i)}, \tag{C.1}$$

$$\sigma_v(t) = \frac{T}{m} + D_1 e^{-\frac{\gamma}{m}(t-t_i)} + D_2 e^{-\left(\frac{\gamma}{m}-\omega^*\right)(t-t_i)} + D_3 e^{-\left(\frac{\gamma}{m}+\omega^*\right)(t-t_i)}, \tag{C.2}$$

where

$$\omega^* \equiv \frac{\gamma}{m}\sqrt{1 - 4\frac{m\lambda}{\gamma^2}}, \tag{C.3}$$

$$D_1 \equiv \frac{\lambda}{m\omega^{*2}}\left(4\frac{T}{m} - 2\sigma_{v0} - 2\frac{\lambda}{m}\sigma_{x0} - 2\frac{\gamma}{m}\sigma_{xv0}\right), \tag{C.4}$$

$$D_2 \equiv -\frac{1}{2\omega^{*2}}\left[\frac{\gamma T}{m^2}\left(\frac{\gamma}{m} - \omega^*\right) + \left(2\frac{\lambda}{m} - \frac{\gamma^2}{m^2} + \frac{\gamma}{m}\omega^*\right)\sigma_{v0} - 2\frac{\lambda^2}{m^2}\sigma_{x0} + 2\frac{\lambda}{m}\left(-\frac{\gamma}{m} + \omega^*\right)\sigma_{xv0}\right], \tag{C.5}$$

$$D_3 \equiv -\frac{1}{2\omega^{*2}}\left[\frac{\gamma T}{m^2}\left(\frac{\gamma}{m} + \omega^*\right) + \left(2\frac{\lambda}{m} - \frac{\gamma^2}{m^2} - \frac{\gamma}{m}\omega^*\right)\sigma_{v0} - 2\frac{\lambda^2}{m^2}\sigma_{x0} + 2\frac{\lambda}{m}\left(-\frac{\gamma}{m} - \omega^*\right)\sigma_{xv0}\right]. \tag{C.6}$$

We can also derive σ_{xv} using Eqs. (6.14) and (C.1). Note that since the exponential terms in Eqs. (C.1) and (C.2) vanish as $t \to \infty$, σ_x and σ_v relax to time-independent T/λ and T/m, respectively. Using τ_x in Eq. (6.9) and τ_v in Eq. (6.10), we can rewrite Eq. (C.3) as

$$\omega^* = \frac{1}{\tau_v}\sqrt{1 - 4\frac{\tau_v}{\tau_x}}. \tag{C.7}$$

Then, the exponential functions in Eqs. (C.1) and (C.2) are represented by

$$e^{-\frac{\gamma}{m}(t-t_i)} = e^{-(t-t_i)/\tau_v}, \tag{C.8}$$

$$e^{-\left(\frac{\gamma}{m}-\omega^*\right)(t-t_i)} = e^{-\left(1-\sqrt{1-4\frac{\tau_v}{\tau_x}}\right)(t-t_i)/\tau_v}, \tag{C.9}$$

$$e^{-\left(\frac{\gamma}{m}+\omega^*\right)(t-t_i)} = e^{-\left(1+\sqrt{1-4\frac{\tau_v}{\tau_x}}\right)(t-t_i)/\tau_v}. \tag{C.10}$$

By considering the magnitude relationship between τ_x and τ_v, we show that the relaxation time of the system is evaluated as $\max(\tau_x, \tau_v)$. When $\tau_x \leq 4\tau_v$, ω^* in Eq. (C.7) becomes purely imaginary. Thus, we can consider that the exponential terms in Eqs. (C.1) and (C.2), which are expressed by Eqs. (C.8)–(C.10), are sufficiently smaller than the first terms of Eqs. (C.1) and (C.2) when

$$t - t_i \gg \tau_v \tag{C.11}$$

is satisfied. Therefore, we can regard the relaxation time of the system as τ_v. On the other hand, the case of $\tau_x > 4\tau_v$ is as follows. Since the exponential function of the second terms in Eqs. (C.1) and (C.2) is expressed by the relaxation times as in Eq. (C.8), it becomes sufficiently smaller than the first terms when Eq. (C.11) is satisfied. Moreover, because the fourth terms in Eqs. (C.1) and (C.2) are expressed by Eq. (C.10) and $1+\sqrt{1-4\tau_v/\tau_x} > 1$, those terms are also sufficiently smaller than the first terms when Eq. (C.11) is satisfied. When $4\tau_v/\tau_x$ becomes small, however, $1 - \sqrt{1-4\tau_v/\tau_x}$ in the exponent of Eq. (C.9), by which the third terms in Eqs. (C.1) and (C.2) are expressed, approaches 0 and we have to reconsider the case. When $\tau_x \gg \tau_v$, the exponent of Eq. (C.9) is approximated by

$$-\frac{1}{\tau_v}\left(1 - \sqrt{1 - 4\frac{\tau_v}{\tau_x}}\right)(t - t_i) \simeq -\frac{2}{\tau_x}(t - t_i), \tag{C.12}$$

which makes Eq. (C.9) vanish when

$$t - t_i \gg \tau_x \tag{C.13}$$

is satisfied. Then, the third terms of Eqs. (C.1) and (C.2) are sufficiently smaller than the first terms. When $\tau_x \gg \tau_v$, the time for the third terms in Eqs. (C.1) and (C.2) to vanish is longer than the time for the second and fourth terms to vanish. Therefore, the relaxation time of the system is evaluated as τ_x. In summary, the relaxation time of the system is represented by

$$\tau \equiv \max(\tau_x, \tau_v). \tag{C.14}$$

Therefore, we can see that when $t - t_i \gg \tau$ is satisfied, σ_x and σ_v are approximated by

$$\sigma_x \simeq \frac{T}{\lambda}, \quad \sigma_v \simeq \frac{T}{m} \tag{C.15}$$

from Eqs. (C.1) and (C.2). When σ_x and σ_v are changing toward Eq. (C.15), we consider that the system is in the relaxation. Moreover, when $t - t_i \gg \tau$ is satisfied, we use the phrase "after the relaxation". Since the initial condition is included only in D_1, D_2, and D_3 in Eqs. (C.4)–(C.6), we find that the variables σ_x and σ_v relax to the values determined by T, m, and λ even when we choose other initial conditions. In the limit of $\tau \to 0$, σ_x and σ_v satisfy Eq. (C.15) when $t - t_i > 0$.

From Eq. (6.16), we can obtain σ_{xv} by differentiating Eq. (C.1) with respect to time. Because T and λ are assumed to be constant, the time derivative of the first term in Eq. (C.1) vanishes. Moreover, we can neglect the exponential terms after the relaxation. After the relaxation, we can see that the time derivative of the remaining terms in Eq. (C.1) also vanishes. Thus, σ_{xv} vanishes after the relaxation.

Subsequently, we consider the thermodynamic process where the temperature T and stiffness λ depend on time. In the statement above Eq. (7.11), we assumed that T and λ vary smoothly and slowly. Then, in the small relaxation-times regime, we can expect that the fast relaxation dynamics rapidly vanishes and only the slow dynamics remains accompanying the change of T and λ. Therefore, as the resulting approximate dynamics, we obtain the same expression as Eq. (C.15), by replacing the constant T and λ with the time-dependent variables. Then, we obtain the time derivative of σ_x and σ_v after the relaxation in the process as

$$\dot{\sigma}_x \simeq \frac{d}{dt}\left(\frac{T}{\lambda}\right) = \frac{T}{\lambda}\left(\frac{d}{dt}\ln\frac{T}{\lambda}\right), \quad \dot{\sigma}_v \simeq \frac{\dot{T}}{m}. \tag{C.16}$$

From Eqs. (6.14) and (C.16), σ_{xv} becomes

$$\sigma_{xv}(t) = \frac{1}{2}\dot{\sigma}_x \simeq \frac{T(t)}{2\lambda(t)}\left(\frac{d}{dt}\ln\frac{T(t)}{\lambda(t)}\right). \tag{C.17}$$

Therefore, we obtain the results in Eqs. (6.116), (7.14), and (7.15) in the small relaxation-times regime. In Appendix E, we mention that we may need to reconsider these results at the switching between the processes.

D Derivation of the protocol in the finite-time adiabatic process

We derive the protocol of the finite-time adiabatic process by giving a finite Δt and the time evolution of $\sigma_x(t)$. We assume that the finite-time adiabatic process lasts for $t_i \leq t \leq t_f$, and the temperature changes from T_i to T_f. In the finite-time adiabatic process, we need to specify the time evolution of the five variables ($T(t)$ and $\lambda(t)$ in the protocol and σ_x, σ_v, and σ_{xv} to representing the state of the Brownian particle). In the finite-time adiabatic process in Sec. 7.3, however, there are only four equations in Eqs. (6.14)–(6.16) and (7.21). Therefore, we have insufficient number of the equations. However, we can obtain the closed equations if we give the time evolution of one of the five variables.

As we show below, we obtain the protocol by giving the time evolution of $\sigma_x(t)$. To determine the time evolution of σ_v, σ_{xv}, T, and λ, we solve Eqs. (6.14)–(6.16) and (7.21). From the given σ_x and Eq. (6.14), we obtain σ_{xv}. Using Eqs. (6.14) and (7.21), we can rewrite Eqs. (6.15) and (6.16) as

$$\dot{T} = -\lambda\dot{\sigma}_x, \tag{D.1}$$

$$m\ddot{\sigma}_x = 2T - 2\lambda\sigma_x - \gamma\dot{\sigma}_x. \tag{D.2}$$

From Eq. (D.1), we obtain $\lambda(t)$ as

$$\lambda(t) = -\frac{\dot{T}(t)}{\dot{\sigma}_x(t)}. \tag{D.3}$$

Substituting this into Eq. (D.2), we derive the differential equation of T as

$$\dot{T} + \left(\frac{d}{dt}\ln\sigma_x\right)T = \frac{1}{2}\left(\frac{d}{dt}\ln\sigma_x\right)\left(m\ddot{\sigma}_x + \gamma\dot{\sigma}_x\right). \tag{D.4}$$

By solving Eq. (D.4), we can derive the time evolution of $T(t)$ as

$$T(t) = \frac{1}{2\sigma_x(t)}\left(\gamma\int_{t_i}^{t}\dot{\sigma}_x(t)^2 dt + 2T_i\sigma_{xi} + \frac{m}{2}\dot{\sigma}_x(t)^2 - \frac{m}{2}\dot{\sigma}_{xi}{}^2\right), \tag{D.5}$$

using the initial condition $T(t_i) = T_i$. Then, from Eq. (7.21), we obtain σ_v. Note that $\Delta\Phi$ in the finite-time adiabatic process satisfies

$$\Delta\Phi = \frac{\sigma_{xf}T_f}{m} - \frac{\sigma_{xi}T_i}{m} - \frac{1}{4}\left(\dot{\sigma}_{xf}{}^2 - \dot{\sigma}_{xi}{}^2\right) = \frac{1}{2}\frac{\gamma}{m}\int_{t_i}^{t_f}\dot{\sigma}_x(t)^2 dt, \tag{D.6}$$

where we used Eqs. (6.12), (6.14), (7.21), and (7.27). From Eqs. (D.5) and (D.6), we can confirm that $T(t)$ satisfies $T(t_f) = T_f$. Substituting $T(t)$ in Eq. (D.5) and the given $\sigma_x(t)$ into Eq. (D.3), we can obtain the time evolution of $\lambda(t)$.

E Entropy production at the switchings

We show that the entropy production at the switchings between the isothermal and finite-time adiabatic processes can be neglected in the small relaxation-times regime. In Sec. 7.4, although we assumed that T and λ in our Carnot cycle are continuous at the switchings, we do not assume that the time derivative of T and λ are continuous. If σ_{xv} always satisfies Eq. (7.14) even in the vicinity of the switchings and the time derivative of T and λ are discontinuous at the switchings, σ_{xv} becomes discontinuous. However, the variables σ_x, σ_v, and σ_{xv} should be continuous because their time evolution is described by the differential equations in Eqs. (6.14)–(6.16). Thus, we may consider that the variables do not satisfy Eq. (7.14) just after the switchings and relax to Eq. (7.14). This means that there exists a relaxation just after the switchings.

From Eq. (7.71), the entropy production per cycle should vanish to achieve the Carnot efficiency. Thus, the entropy production in the relaxation after the switchings may af-

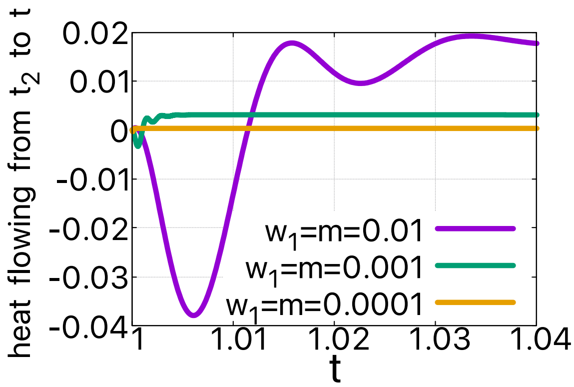

Figure E.1: Heat flowing between t_2 and t in the finite-time adiabatic process (ii) in Fig. 7.1. In this simulation, we used the protocol in Eqs. (7.82), (7.90), and (7.91) in Sec. 7.6. We chose $T_h = 10.0$, $T_c = 1.0$, and the other parameters used in the numerical simulation in Sec. 7.6. We here set $t_2 = 1.0$. Thus, the finite-time adiabatic process (ii) lasts for $1.0 \leq t \leq 2.0$. We find that although the relaxation after the switching at $t = t_2$ exists, the heat flowing in the relaxation becomes smaller when the relaxation times become smaller.

fect the efficiency. We evaluate the entropy production in the total system due to that relaxation in the small relaxation-times regime and show that it can be neglected. Since the variables σ_x, σ_v, and σ_{xv} may not satisfy Eqs. (7.14) and (7.15) in the relaxation, we cannot rewrite the entropy production rate by using the relaxation times as shown in Eq. (7.19). However, since the variables just before the switchings and after the relaxation satisfy Eqs. (7.14) and (7.15), we can evaluate the entropy of the particle and heat flowing in the relaxation as shown below. Then, by using Eq. (7.9), we evaluate the entropy production.

Although we here focus on the switching from the hot isothermal process to the finite-time adiabatic process, corresponding to $t = t_2$ in Fig. 7.1, the similar discussion is available in the other switchings. At that switching, the temperature and stiffness satisfy $T = T_h$ and $\lambda = \lambda_2$, respectively. When we assume that T and λ vary smoothly and slowly, as in the statement above Eq. (7.11), we can expect that T and λ remain unchanged in the relaxation. Then, σ_x and σ_v just before the switching and after the relaxation are the same because of Eqs. (7.14) and (7.15). Thus, from Eqs. (6.18) and (7.16), the entropy change of the particle in this relaxation satisfies

$$\Delta S^{\mathrm{rel}} \simeq 0, \tag{E.1}$$

in the small relaxation-times regime because of Eqs. (6.18) and (7.16), where the index "rel" means the quantity in this relaxation. Moreover, since T and λ are regarded as unchanged in the relaxation, we can see from Eq. (7.14) that each of σ_x and σ_v just before the relaxation takes the same value as that after the relaxation. Therefore, we can evaluate the heat Q^{rel} flowing in this relaxation in the small relaxation-times regime as

$$Q^{\mathrm{rel}} \simeq \frac{1}{2} m (\Delta\sigma_v)^{\mathrm{rel}} + \frac{1}{2}\lambda_2 (\Delta\sigma_x)^{\mathrm{rel}} \simeq 0 \tag{E.2}$$

from Eqs. (7.5) and (7.7). Figure E.1 shows that the relaxation exists after the switching at $t = t_2$ in the finite-time adiabatic process (ii) in Fig. 7.1 with the protocol in Sec. 7.6. However, we can see that the heat flowing in the relaxation becomes smaller when the relaxation times become smaller. Thus, we can neglect Q^{rel} in Eq. (E.2) in the small relaxation-times regime. Since we can regard T as T_h in the relaxation, we derive the entropy production Σ^{rel} of the total system in this relaxation by using Eqs. (7.9), (E.1), and (E.2) as

$$\Sigma^{\mathrm{rel}} \simeq \Delta S^{\mathrm{rel}} - \frac{Q^{\mathrm{rel}}}{T_h} \simeq 0. \tag{E.3}$$

Thus, the entropy production Σ^{rel} can be neglected in the small relaxation-times regime.

F Numerical simulation of Langevin system

We show Heun's method to solve the Langevin equation numerically. For simplicity, we consider the one-dimensional Langevin equation:

$$dX(t) = a(X,t)dt + b(X,t) \circ dB(t), \quad X(t_0) = X_0, \tag{F.1}$$

using the Stratonovich-type product. When X'_n is the approximate solution at $t = t_n$, we obtain the difference equation for X'_n as

$$X'_{n+1} = X'_n + \frac{1}{2}(a_1 + a_2)\Delta t + \frac{1}{2}(b_1 + b_2)\xi_n \sqrt{\Delta t}, \tag{F.2}$$

$$a_1 \equiv a(X'_n, t), \quad a_2 \equiv a(X'_n + a_1\Delta t + \xi_n\sqrt{\Delta t}, t + \Delta t), \tag{F.3}$$

$$b_1 \equiv b(X'_n, t), \quad b_2 \equiv b(X'_n + a_1\Delta t + \xi_n\sqrt{\Delta t}, t + \Delta t) \tag{F.4}$$

from Eq. (F.1), where Δt is a time step and $\xi(t)$ is the Gaussian white noise satisfying $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = \delta(t - s)$. The Box–Muller's method is often used to generate the Gaussian white noise from a uniformly distributed random numbers [80]. Because of the Itô's lemma in Eq. (3.26), we express a increment of the Wiener process as $\xi_n\sqrt{\Delta t}$. Similar to Sec. 5.2.3, we can calculate the heat and work in the underdamped Langevin system [28].

Figure F.1 shows the numerical results of the (a) Efficiency and (b) power in the Brownian Carnot cycle with the finite-time adiabatic process derived from solving Eqs. (6.14)–(6.16) and Langevin equations in Eqs. (6.4) and (6.5). Although the numerical result of the Langevin equations fluctuates, it shows a good agreement with the numerical result of Eqs. (6.14)–(6.16).

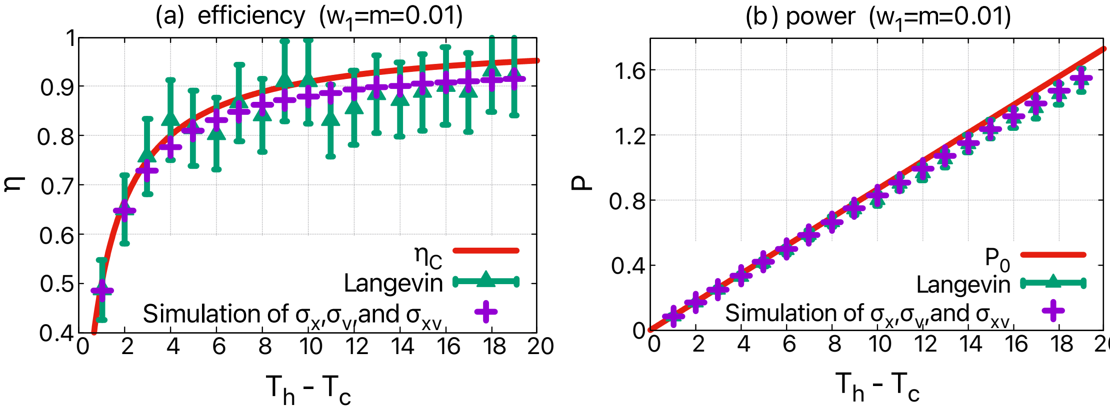

Figure F.1: The numerical results of the (a) Efficiency and (b) power in the Brownian Carnot cycle with the finite-time adiabatic process derived from solving Eqs. (6.14)–(6.16) (purple plus) and Langevin equations in Eqs. (6.4) and (6.5) (green triangle). We used the protocol in Sec. 7.6 and set $w_1 = m = 10^{-2}$. The red lines in the left and right figures show the Carnot efficiency in Eq. (1.1) and P_0 in Eq. (7.64), respectively. In the Langevin simulation, we chose the time step as $\Delta t = 10^{-5}$ and calculated the average values of 10000 times for the work and heat to obtain the efficiency and power. We used the standard error to obtain the error bars.

Bibliography

[1] H. B. Callen. *Thermodynamics and an Introduction to Thermostatistics.* Wiley, New York, 2nd edition, 1985.

[2] S. Carnot. *Reflections on the Motive Power of Fire and on Machines Fitted to Develop that Power.* Paris: Bachelier, 1824.

[3] M. Polettini and M. Esposito. Carnot efficiency at divergent power output. *Europhys. Lett.*, 118(4):40003, may 2017.

[4] N. Shiraishi. Stationary engines in and beyond the linear response regime at the carnot efficiency. *Phys. Rev. E*, 95:052128, May 2017.

[5] T. Hondou and K. Sekimoto. Unattainability of carnot efficiency in the brownian heat engine. *Phys. Rev. E*, 62:6021–6025, Nov 2000.

[6] M. Campisi and R. Fazio. The power of a critical heat engine. *Nat. Comm.*, 7(1):11895, 2016.

[7] G. Benenti, K. Saito, and G. Casati. Thermodynamic bounds on efficiency for systems with broken time-reversal symmetry. *Phys. Rev. Lett.*, 106:230602, Jun 2011.

[8] K. Brandner, K. Saito, and U. Seifert. Strong bounds on onsager coefficients and efficiency for three-terminal thermoelectric transport in a magnetic field. *Phys. Rev. Lett.*, 110:070603, Feb 2013.

[9] V. Balachandran, G. Benenti, and G. Casati. Efficiency of three-terminal thermoelectric transport under broken time-reversal symmetry. *Phys. Rev. B*, 87:165419, Apr 2013.

[10] K. Yamamoto, O. Entin-Wohlman, A. Aharony, and N. Hatano. Efficiency bounds on thermoelectric transport in magnetic fields: The role of inelastic processes. *Phys. Rev. B*, 94:121402(R), Sep 2016.

[11] J. Stark, K. Brandner, K. Saito, and U. Seifert. Classical nernst engine. *Phys. Rev. Lett.*, 112:140601, Apr 2014.

[12] R. Sánchez, B. Sothmann, and A. N. Jordan. Chiral thermoelectrics with quantum hall edge states. *Phys. Rev. Lett.*, 114:146801, Apr 2015.

[13] B. Sothmann, R. Sánchez, and A. N. Jordan. Quantum nernst engines. *EPL (Europhysics Letters)*, 107(4):47003, aug 2014.

[14] Y.-H. Ma, D. Xu, H. Dong, and C.-P. Sun. Universal constraint for efficiency and power of a low-dissipation heat engine. *Phys. Rev. E*, 98:042112, Oct 2018.

[15] P. Abiuso and M. Perarnau-Llobet. Optimal cycles for low-dissipation heat engines. *Phys. Rev. Lett.*, 124:110606, Mar 2020.

[16] V. Holubec and A. Ryabov. Diverging, but negligible power at carnot efficiency: Theory and experiment. *Phys. Rev. E*, 96:062107, Dec 2017.

[17] T. Schmiedl and U. Seifert. Efficiency at maximum power: An analytically solvable model for stochastic heat engines. *Europhys. Lett.*, 81(2):20003, dec 2007.

[18] Y. Izumida and K. Okuda. Onsager coefficients of a brownian carnot cycle. *Eur. Phys. J. B*, 77:499–504, October 2010.

[19] U. Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. *Rep. Prog. Phys.*, 75(12):126001, nov 2012.

[20] A. Rosas, C. V. Broeck, and K. Lindenberg. Onsager coefficients for a brownian particle in space-periodic and time-periodic potentials. *Journal of Physics A: Mathematical and Theoretical*, 49(48):484001, nov 2016.

[21] O. Raz, Y. Subaş ı, and R. Pugatch. Geometric heat engines featuring power that grows with efficiency. *Phys. Rev. Lett.*, 116:160601, Apr 2016.

[22] A. Dechant, N. Kiesel, and E. Lutz. Underdamped stochastic heat engine at maximum efficiency. *Europhys. Lett.*, 119(5):50003, sep 2017.

[23] V. Holubec and A. Ryabov. Cycling tames power fluctuations near optimum efficiency. *Phys. Rev. Lett.*, 121:120601, Sep 2018.

[24] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, and R. A. Rica. Adiabatic processes realized with a trapped brownian particle. *Phys. Rev. Lett.*, 114:120601, Mar 2015.

[25] A. Dechant and S. Sasa. Entropic bounds on currents in langevin systems. *Phys. Rev. E*, 97:062101, Jun 2018.

[26] K. Miura, Y. Izumida, and K. Okuda. Compatibility of carnot efficiency with finite power in an underdamped brownian carnot cycle in small temperature-difference regime. *Phys. Rev. E*, 103:042125, Apr 2021.

[27] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados. Building an irreversible carnot-like heat engine with an overdamped harmonic oscillator. *J. Stat. Mech.*, 2020(9):093207, sep 2020.

[28] K. Sekimoto. *Stochastic Energetics (Lecture Notes in Physics)*. Springer, 2010.

[29] N. Shiraishi, K. Saito, and H. Tasaki. Universal trade-off relation between power and efficiency for heat engines. *Phys. Rev. Lett.*, 117:190601, Oct 2016.

[30] N. Shiraishi and H. Tajima. Efficiency versus speed in quantum heat engines: Rigorous constraint from lieb-robinson bound. *Phys. Rev. E*, 96:022138, Aug 2017.

[31] N. Shiraishi. <research note > from the thermodynamics of fluctuating systems to the laws of heat engines: The universal relationship between power and efficiency. *BusseiKenkyu*, 7(2):072213, 11 2018 (Japanese).

[32] P. Pietzonka and U. Seifert. Universal trade-off between power, efficiency, and constancy in steady-state heat engines. *Phys. Rev. Lett.*, 120:190602, May 2018.

[33] T. Koyuk, U. Seifert, and P. Pietzonka. A generalization of the thermodynamic uncertainty relation to periodically driven systems. *J. Phys. A: Math. Theor.*, 52(2):02LT02, dec 2018.

[34] A. Dechant. Multidimensional thermodynamic uncertainty relations. *Journal of Physics A: Mathematical and Theoretical*, 52(3):035001, dec 2018.

[35] M. Toda R. Kubo and N. Hashitsume. *Statistical Physics II: Nonequilibrium Statistical Mechanics*. Springer, Berlin ; New York, 2nd edition edition, June 1998.

[36] L. Onsager. Reciprocal relations in irreversible processes. i. *Phys. Rev.*, 37:405–426, Feb 1931.

[37] L. Onsager. Reciprocal relations in irreversible processes. ii. *Phys. Rev.*, 38:2265–2279, Dec 1931.

[38] Y. Oono and M. Paniconi. Steady State Thermodynamics. *Progress of Theoretical Physics Supplement*, 130:29–44, 01 1998.

[39] H. Hayakawa. *Nonequilibrium statistical mechanics*. Saiensu Sha, 2007 (Japanese).

[40] Y. Izumida and K. Okuda. Onsager coefficients of a finite-time carnot cycle. *Phys. Rev. E*, 80:021121, Aug 2009.

[41] B. Sothmann and M. Büttiker. Magnon-driven quantum-dot heat engine. *EPL (Europhysics Letters)*, 99(2):27001, jul 2012.

[42] O. Entin-Wohlman and A. Aharony. Three-terminal thermoelectric transport under broken time-reversal symmetry. *Phys. Rev. B*, 85:085401, Feb 2012.

[43] M. Horvat, T. Prosen, Giuliano Benenti, and G. Casati. Railway switch transport model. *Phys. Rev. E*, 86:052102, Nov 2012.

[44] D. M. Kennes, D. Schuricht, and V. Meden. Efficiency and power of a thermoelectric quantum dot device. *EPL (Europhysics Letters)*, 102(5):57003, jun 2013.

[45] J. Meair and P. Jacquod. Scattering theory of nonlinear thermoelectricity in quantum coherent conductors. *Journal of Physics: Condensed Matter*, 25(8):082201, jan 2013.

[46] S.-Y. Hwang, D. Sánchez, M. Lee, and R. López. Magnetic-field asymmetry of nonlinear thermoelectric and heat transport. *New Journal of Physics*, 15(10):105012, oct 2013.

[47] M. F. Ludovico and L. Arrachea. Pumping charge with ac magnetic fluxes and the dynamical breakdown of onsager symmetry. *Phys. Rev. B*, 87:115408, Mar 2013.

[48] Y. Apertet, H. Ouerdane, C. Goupil, and Ph. Lecoeur. From local force-flux relationships to internal dissipations and their impact on heat engine performance: The illustrative case of a thermoelectric generator. *Phys. Rev. E*, 88:022137, Aug 2013.

[49] G. Benenti, G. Casati, and J. Wang. Conservation laws and thermodynamic efficiencies. *Phys. Rev. Lett.*, 110:070604, Feb 2013.

[50] F. Mazza, R. Bosisio, G. Benenti, V. Giovannetti, R. Fazio, and F. Taddei. Thermoelectric efficiency of three-terminal quantum thermal machines. *New Journal of Physics*, 16(8):085001, aug 2014.

[51] G. Benenti, G. Casati, and C. Mejía-Monasterio. Thermoelectric efficiency in momentum-conserving systems. *New Journal of Physics*, 16(1):015014, jan 2014.

[52] J. Ren, J. Fransson, and J.-X. Zhu. Nanoscale spin seebeck rectifier: Controlling thermal spin transport across insulating magnetic junctions with localized spin. *Phys. Rev. B*, 89:214407, Jun 2014.

[53] J. Matthews, F. Battista, D. Sánchez, P. Samuelsson, and H. Linke. Experimental verification of reciprocity relations in quantum thermoelectric transport. *Phys. Rev. B*, 90:165428, Oct 2014.

[54] J.-H. Jiang. Thermodynamic bounds and general properties of optimal efficiency and power in linear responses. *Phys. Rev. E*, 90:042126, Oct 2014.

[55] O. Entin-Wohlman, J.-H. Jiang, and Y. Imry. Efficiency and dissipation in a two-terminal thermoelectric junction, emphasizing small dissipation. *Phys. Rev. E*, 89:012123, Jan 2014.

[56] F. Plastina, A. Alecce, T. J. G. Apollaro, G. Falcone, G. Francica, F. Galve, N. Lo Gullo, and R. Zambrini. Irreversible work and inner friction in quantum thermodynamic processes. *Phys. Rev. Lett.*, 113:260601, Dec 2014.

[57] K. Proesmans and C. V. Broeck. Onsager coefficients in periodically driven systems. *Phys. Rev. Lett.*, 115:090601, Aug 2015.

[58] M. Polettini, G. Verley, and M. Esposito. Efficiency statistics at all times: Carnot limit at finite power. *Phys. Rev. Lett.*, 114:050601, Feb 2015.

[59] K. Brandner, K. Saito, and U. Seifert. Thermodynamics of micro- and nano-systems driven by periodic temperature variations. *Phys. Rev. X*, 5:031019, Aug 2015.

[60] J.-H. Jiang, B. K. Agarwalla, and D. Segal. Efficiency statistics and bounds for systems with broken time-reversal symmetry. *Phys. Rev. Lett.*, 115:040601, Jul 2015.

[61] R. S. Whitney. Quantum coherent three-terminal thermoelectrics: Maximum efficiency at given power output. *Entropy*, 18(6), 2016.

[62] K. Brandner and U. Seifert. Periodic thermodynamics of open quantum systems. *Phys. Rev. E*, 93:062134, Jun 2016.

[63] L. D. Landau and E. M. Lifshitz. *Statistical Physics*. Elsevier, Amsterdam, 3rd edition, 1980. bibtex: landa;b;sp80.

[64] S.R. de Groot and P. Mazur. *Non-equilibrium thermodynamics*. North-Holland Pub. Co., 1962.

[65] K. M. Blundell S. J. Blundell. *Concepts in thermal physics*. Oxford University Press, Oxford, 2nd ed. edition, 2010.

[66] S. R. De Groot, L. Jansen, and P. Mazur. Non-equilibrium thermodynamics and liquid helium ii. *Physica*, 16(9):691–708, 1950.

[67] M. F. Weber and E. Frey. Master equations and the theory of stochastic path integrals. *Reports on Progress in Physics*, 80(4):046601, mar 2017.

[68] N. G. Van Kampen. *Stochastic Processes in Physics and Chemistry.* North Holland, 3rd edition, 2007.

[69] C. W. Gardiner. *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences.* Springer-Verlag, 2nd edition, 1996.

[70] K. Ito. *Probability theory.* Iwanami Shoten, 1953 (Japanese).

[71] N. Funaki. *Stochastic differential equation.* Iwanami Shoten, 1997 (Japanese).

[72] E. T. Jaynes. *Probability theory: The logic of science.* Cambridge University Press, Cambridge, 2003.

[73] M. Chaichian and A. Demichev. *Path Integrals in Physics: Volume I Stochastic Processes and Quantum Mechanics.* Institute of physics series in mathematical and computational physics. Taylor & Francis, 2001.

[74] M. Chaichian and A. Demichev. *Path Integrals in Physics: Volume II Quantum Field Theory, Statistical Physics and other Modern Applications.* CRC Press, 2018.

[75] H. Risken. *The Fokker-Planck Equation.* Springer, New York, 2nd edition, 1996.

[76] I. Lubashevsky. Towards multi-dimensional nonlinear langevin equation. *Proceedings of the ISCIE International Symposium on Stochastic Systems Theory and its Applications*, 2014:278–283, 2014.

[77] M. Ding, Z. Tu, and X. Xing. Covariant formulation of nonlinear langevin theory with multiplicative gaussian white noises. *Phys. Rev. Research*, 2:033381, Sep 2020.

[78] H.A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7(4):284–304, 1940.

[79] R. Pawula. Generalizations and extensions of the fokker- planck-kolmogorov equations. *IEEE Transactions on Information Theory*, 13(1):33–41, 1967.

[80] W. T. Vetterling W. H. Press, S. A. Teukolsky and B. P. Flannery. *Numerical Recipes 3rd Edition: The Art of Scientific Computing.* Cambridge University Press, 3 edition, September 2007.

[81] T. Yamamoto. *Basics of Matrix Analysis : Advanced Linear Algebra.* Saiensu Sha, 2010 (Japanese).

[82] T. Yamamoto. *Introduction to Numerical Analysis.* Saiensu Sha, 2nd edition, 1995 (Japanese).

[83] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing).* Wiley-Interscience, USA, 2006.

[84] É. Roldán, I. A. Martínez, L. Dinis, and R. A. Rica. Measuring kinetic energy changes in the mesoscale with low acquisition rates. *Applied Physics Letters*, 104(23):234103, 2014.

[85] G. E. Uhlenbeck and L. S. Ornstein. On the theory of the brownian motion. *Phys. Rev.*, 36:823–841, Sep 1930.

[86] R. E. Spinney and I. J. Ford. Entropy production in full phase space for continuous stochastic dynamics. *Phys. Rev. E*, 85:051113, May 2012.

[87] F. D. Nobre G. A. Casas and E. M. F. Curado. Entropy production and nonlinear fokker-planck equations. *Phys. Rev. E*, 86:061136, Dec 2012.

[88] D. M. Busiello and A. Maritan. Entropy production in master equations and fokker–planck equations: facing the coarse-graining and recovering the information loss. *Journal of Statistical Mechanics: Theory and Experiment*, 2019(10):104013, oct 2019.

[89] U. Seifert. Entropy production along a stochastic trajectory and an integral fluctuation theorem. *Phys. Rev. Lett.*, 95:040602, Jul 2005.

[90] D. Arold, A. Dechant, and E. Lutz. Heat leakage in overdamped harmonic systems. *Phys. Rev. E*, 97:022131, Feb 2018.

[91] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados. Finite-time adiabatic processes: Derivation and speed limit. *Phys. Rev. E*, 101:032129, Mar 2020.

[92] M. V. S. Bonança. Approaching carnot efficiency at maximum power in linear response regime. *Journal of Statistical Mechanics: Theory and Experiment*, 2019(12):123203, dec 2019.

[93] J. Kurchan. Fluctuation theorem for stochastic dynamics. *Journal of Physics A: Mathematical and General*, 31(16):3719–3729, apr 1998.