Spatio and temporal dynamics of solutions for reaction-diffusion equations with nonlocal effect

非局所効果を持つ反応拡散方程式における解の時空間ダイナミクスについて

Hiroshi ISHII

Graduate School of Science, Hokkaido University

Department of Mathematics

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Hiroshi ISHII

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Abstract

The pattern formation problem is one of the most fascinating and essential problems in the natural sciences. In recent years, the study of pattern formation has been theoretically analyzed using reaction-diffusion equations with nonlocal effect described by convolution with the appropriate integral kernel as mathematical models in various fields such as biology, material science, and medicine. Therefore, the mathematical analysis of the behavior of solutions in reaction-diffusion equations with nonlocal effect is becoming more and more important with each passing year. This thesis focuses on developing new analytical methods for theoretically considering the spatio and temporal dynamics of solutions and their applications to understand how nonlocal effect affects the pattern formation process.

First, this thesis introduces the reaction-diffusion equation and mathematical models with nonlocal effect in Section 1. Then, it explains the effectiveness of mathematical modeling with nonlocal effect and its relationship with the reaction-diffusion equation. After that, as the first result, a new method to show the existence of traveling wave solutions is described, and its applications are presented in Section 2. In addition, as the second result, a method for analyzing weak interactions between localized patterns, such as stationary and traveling wave solutions, is explained, and its applications are given in Section 3. Finally, as the third result, we consider the asymptotic behavior of the zero points of solutions to the diffusion equation and the fractional diffusion equation in Section 4, with the motivation to analyze the nonlocal effect on the spatial propagation mechanism of substances.

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1 Background and motivation

1.1 Reaction-diffusion equations

The understanding of the formation mechanism of spontaneous spatio-temporal patterns are one of the most attractive and essential topics in the natural sciences. In several fields such as biology and chemistry, theoretical analysis using reaction-diffusion equations has been conducted as a study of pattern formation problems. Here, we introduce the scalar equation as an example of the reaction-diffusion equations:

$$u_t = du_{xx} + f(u), \tag{1.1}$$

where $u = u(t, x) \in \mathbb{R}$ is some variable representing the density, concentration, or order parameter at time $t > 0$ and position x. Each subscript of u represents a partial derivative, $d > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. du_{xx} is called the diffusion term, where d represents the diffusion coefficient, and the nonlinear term $f(u)$ is called the reaction term. (1.1) appear as mathematical models of developmental phenomena of biological species [30, 49], phase separation phenomena [28, 29] and so on.

On the other hand, in biology and chemistry, mathematical models of multi-component reaction-diffusion equations have been proposed to describe the interaction of multiple proteins, chemicals, and species. Here, we introduce the two-component reaction-diffusion equation:

$$\begin{cases} u_t = d_u u_{xx} + f(u, v), \\ v_t = d_v v_{xx} + g(u, v), \end{cases} \tag{1.2}$$

where $d_u, d_v > 0$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are nonlinear functions. (1.2) appears as mathematical models of chemical reaction systems [32, 34], propagation mechanisms of nerve pulses [10], and epidermal pattern formation in organisms [51]. In particular, one of the important properties for pattern formation problems that does not appear in scalar reaction-diffusion equations is the Turing instability [51, 62]. It is a mechanism that destabilizes a constant stationary solution and generates a spontaneous pattern in space. Let us consider the Activator-Inhibitor system as a concrete example introduced by [62]. Let $u(t, x)$ be the activator and $v(t, x)$ be the inhibitor. The Activator-Inhibitor system is a reaction-diffusion equation that can be linearized around an equilibrium as follows:

$$\begin{cases} u_t = d_u u_{xx} + c_1 u - c_2 v, \\ v_t = d_v v_{xx} + c_3 u - c_4 v, \end{cases} \tag{1.3}$$

where $c_j > 0$ $(j = 1, 2, 3, 4)$. Then, it is known that if the trivial solution $(u, v) = (0, 0)$ of (1.3) is stable in the sense of ODE and $d_u \ll d_v$, a spontaneous spatially periodic pattern will appear, and this phenomenon is an example of Turing instability.

As an extension of the reaction-diffusion equations, the reaction-diffusion equations with nonlocal effect have been proposed in recent years. The nonlocal effect is often described by a convolution with a suitable integral kernel. In the remainder of this section, the mathematical models with nonlocal effect are introduced. Finally, the purpose and outline of this thesis are explained.

1.2 Nonlocal diffusion equations

The diffusion equation is derived under the assumption that the diffusion effect of a substance is local, and relates to the propagation phenomena of many micro-particles in Brownian motion. Recently, the nonlocal diffusion equation, which is described below, has been proposed to capture the diffusion effect in the broader framework [3]:

$$u_t = K * u - u, \quad t > 0,\ x \in \mathbb{R}^m, \tag{1.4}$$

where $(K * u)(t,x) := \int_{\mathbb{R}^m} K(x-y)u(y)dy$. In this subsection, we assume that $K \in L^1(\mathbb{R}^m) \cap C(\mathbb{R}^m)$ is nonnegative and radial, and satisfies $\int_{\mathbb{R}^m} K(y)dy = 1$. $K(x-y)$ represents the probability density of a substance moving from position y to position x in unit time, and $(K * u)(t,x)$ is the sum of the concentrations of substances moving from all positions to position x in unit time. Furthermore, in connection with the nonlocal diffusion equation, we also define the fractional diffusion equation, which is known as a diffusion equation with nonlocal effect:

$$u_t = -(-\Delta)^{s/2}u, \quad t > 0,\ x \in \mathbb{R}^m. \tag{1.5}$$

for $s \in (0,2)$, where

$$(-\Delta)^{s/2}u(t,x) := C_{m,s}\int_{\mathbb{R}^m} \frac{u(t,x)-u(t,y)}{|x-y|^{m+s}}dy, \quad C_{m,s} := \frac{\Gamma((m+s)/2)}{2^{-s}\pi^{m/2}|\Gamma(-s/2)|}$$

We will refer to $K * u - u$ and $-(-\Delta)^{s/2}u$ as nonlocal diffusion and fractional diffusion, respectively, while normal diffusion u_{xx} is referred to as local diffusion. The properties of the local diffusion equation and the fractional diffusion equation will be explained in Section 4.

Let us discuss the mathematical relationship between the local diffusion, the fractional diffusion and the nonlocal diffusion. We define the Fourier transform and inverse transform as follows:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^m} f(x)e^{-i\langle x,\xi\rangle}dx, \quad \check{g}(x) = \mathcal{F}^{-1}[g](\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} g(\xi)e^{i\langle x,\xi\rangle}d\xi$$

for $x, \xi \in \mathbb{R}^m$ and $f, g \in L^1(\mathbb{R}^m)$. Here, we note that $(-\Delta)^{s/2}$ can be defined as the pseudo-differential operator:

$$\mathcal{F}[(-\Delta)^{s/2}u](t,\xi) = |\xi|^s\hat{u}(t,\xi)$$

for $\xi \in \mathbb{R}^m$. By using Fourier transform, we define the fundamental solutions $G^s(t,x)$ of the fractional diffusion equations $(0 < s < 2)$ and the local diffusion equation $(s = 2)$ as

$$\hat{G^s}(t,\xi) := e^{-t|\xi|^s}.$$

As examples, $G^2(t,x) = \dfrac{1}{(4t\pi)^{m/2}}e^{-|x|^2/4t}$ and $G^1(t,x) = \dfrac{B_m t}{(t^2+|x|^2)^{(m+1)/2}}$, where $B_m := \Gamma(\frac{m+1}{2})\pi^{-(m+1)/2}$.

With the above preparations, let us explain the asymptotic behavior of solutions to the nonlocal diffusion equation based on the result in [3]. Suppose that $K(x)$ satisfies

$$\hat{K}(\xi) = 1 - d_s|\xi|^s + o(|\xi|^s) \quad (|\xi| \to 0) \tag{1.6}$$

for some $d_s > 0$ and $s \in (0, 2]$. Then, for any nonnegative $u(0, \cdot) \in L^1(\mathbb{R}^m)$ satisfying $\hat{u}(0, \cdot) \in L^1(\mathbb{R}^m)$, there exists a unique solution $u(t, x)$ to the nonlocal diffusion equation such that

$$\hat{u}(t, \xi) = e^{t(\hat{K}(\xi)-1)}\hat{u}(0, \xi).$$

Furthermore, the asymptotic behavior of the solution is given by

$$\lim_{t\to+\infty} \max_{x\in\mathbb{R}^m} \left| t^{m/s}u(t, xt^{1/s}) - \|u(0, \cdot)\|_{L^1}G^s\,(d_s, x) \right| = 0.$$

Here, we note that $s = 2$ and $d_s = \dfrac{1}{2m}\int_{\mathbb{R}^m} |x|^2K(x)dx$ hold in the condition (1.6) if $K(x)$ satisfies

$$\int_{\mathbb{R}^m} |x|K(x)dx < \infty, \quad \int_{\mathbb{R}^m} |x|^2K(x)dx < \infty.$$

From the above asymptotic behavior results, it can be seen that the behavior of solutions to the nonlocal diffusion equation asymptotically behaves like the diffusion equation or the fractional diffusion equation, depending on the properties of the integral kernel. Therefore, the nonlocal diffusion equations are beneficial in applications because it can capture various diffusion processes in a broader framework than the conventional equations by providing appropriate integral kernels.

In recent years, the reaction-diffusion equation, in which a nonlocal diffusion term replaces the diffusion term, has been actively studied. For example, in the case of scalar equations, the following equation is analyzed:

$$u_t = K * u - u + f(u).$$

Such equations often appear as mathematical models of dispersion phenomena of living organisms [38] and phase separation phenomena [5]. The analytical results of these steady-state solutions and traveling wave solutions are presented in Section 2 and Section 3.

1.3 Pattern formation problem

As mentioned in Subsection 1.1, the reaction-diffusion equations are known to exhibit a variety of spatio-temporal patterns. However, for complex spatio-temporal patterns to appear, there must be more than two components. In the case of mathematical models with nonlocal effect, it has been reported that complex spatio-temporal patterns can appear even with scalar equations [25, 50, 60]. In this subsection, we introduce the Kernel based Turing model (KT model) proposed by [50]:

$$u_t = \chi(K * u) - \alpha u, \quad t > 0,\ x \in \mathbb{R}^m \tag{1.7}$$

where $K \in C(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ is a non-zero radial function, $\alpha := \int_{\mathbb{R}^m} K(x)dx$ and

$$\chi(u) = \begin{cases} u_M & (u > u_M), \\ u & (u \in [0, u_M]), \\ 0 & (u < 0), \end{cases}$$

in which u_M is a positive constant. In the KT model, it has been reported that complex spatial patterns appear depending on the shape of the integral kernels. The mechanism of the pattern formation is Turing instability. Now, since $\chi(u)$ is piecewise linear, let us consider the equation (1.7) without χ. When the integral kernel is nonnegative, the solution decays as in the nonlocal diffusion equation, but solution may have a destabilizing wavenumber in the case of a sign-changing integral kernel. For example, in the case of $m = 1$, let us consider

$$K(x) = \frac{1}{\sqrt{\pi}} \left(e^{-x^2} - \frac{1}{2} e^{-x^2/4} \right). \tag{1.8}$$

Then, we have

$$\hat{K}(\xi) = e^{-\xi^2/4} - e^{-\xi^2} > 0 = \hat{K}(0) = \alpha \ (\xi \neq 0).$$

We note that for any $u(0, \cdot) \in L^1(\mathbb{R}^m)$ satisfying $u(\hat{0, \cdot}) \in L^1(\mathbb{R}^m)$, the solution of the equation (1.7) without χ can be represented as

$$\hat{u}(t, \xi) = e^{t(\hat{K}(\xi) - \alpha)} \hat{u}(0, \xi).$$

Thus, in the case of (1.8), we can see that a spatially spontaneous periodic pattern appears because of the destabilizing wavenumber.

To analyze scalar mathematical models with nonlocal effects such as the KT model, a method of approximating and analyzing them with multi-component reaction-diffusion equations has been proposed [58, 59]. Recently, in [25], a method has been proposed to derive a single mathematical model with nonlocal effect from the multi-component linear reaction-diffusion equations such as (1.4). This method allows us to consider the pattern formation problem for a given reaction-diffusion network expressed in the integral kernel.

1.4 Continuation method to spatially discrete mathematical model

Nonlocal effect also appears when approximating spatially discrete mathematical models [26]. In this subsection, we will take the discrete diffusion equation as an example:

$$u_t = \frac{u(t, x - l) - 2u(t, x) + u(t, x + l)}{l^2}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.9}$$

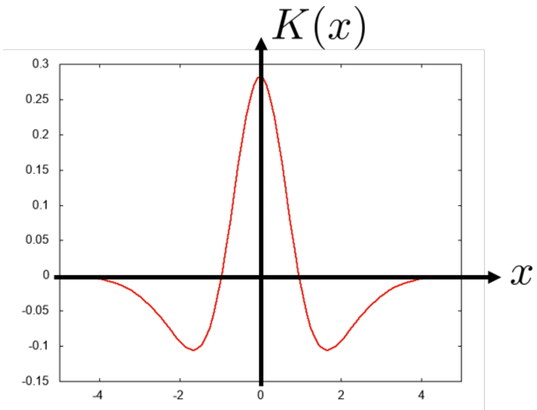

Figure 1: The graph of the integral kernel (1.8).

where l is a positive constant that represents the size of a cell. Such discrete diffusion term often appears in mathematical model [14]. Since discrete diffusion poses many technical difficulties in analysis, continuation is often used. One of the most common continuum methods is to take the limit as $l \to +\infty$ assuming the cell size is small enough. Using such a method, the right-hand side of (1.9) is represented as

$$\frac{u(t,x-l)-2u(t,x)+u(t,x+l)}{l^2} \to u_{xx}(t,x)$$

by taking a limit as $l \to +0$. However, the method is often not appropriate for some mathematical models because the information about the cell shape is lost.

To overcome such a problem, a method has recently been proposed to reconsider the far-field effect as a convolution with a delta function and approximate it by a mollifier to make it continuous [26]. In the case of (1.9), we first write the equation by the delta function as follows:

$$u_t = \frac{\delta_l * u - 2u + \delta_{-l} * u}{l^2},$$

where $\delta(x)$ is the delta function and $\delta_a(x) = \delta(x-a)$ for $a \in \mathbb{R}$. Let $\varphi \in C^\infty(\mathbb{R})$ be a positive mollifier and $\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. In this case, it is well known that the following limit holds in the sense of distribution:

$$\lim_{\varepsilon \to +0} \varphi_\varepsilon(x) = \delta(x).$$

Therefore, we expect that (1.9) can be formally approximated by the following mathematical model with nonlocal effect:

$$u_t = K * u - \frac{2}{l^2}u,$$

where $K(x) = \dfrac{1}{l^2}\{\varphi_\varepsilon(x-l)+\varphi_\varepsilon(x+l)\}$. This method allows for continuity while retaining information on cell shape.

In [26], more general spatially discrete mathematical models are made continuous by nonlocal effect based on the above idea, and the consistency with the discrete mathematical model is checked by numerical simulations. Furthermore, the error evaluation of each solution is performed in the case of nonlinear scalar equations. It is shown that for a finite time, the approximation is possible in the sense of L^2 norm as long as ε is small enough.

1.5 Mathematical models with nonlocal effect

Mathematical models with nonlocal effect appear in many other fields. In the phase transition of solid material, $u(t, x)$ is regarded as an order parameter representing the state at time t and position x. For example, suppose that $u = \pm 1$ represent the state of two different orientations of a perfect crystal denoting by S_A and S_B. Then, the following Helmholtz free-energy functional is derived [5]:

$$E(u) = \frac{1}{4}\int\int_{\mathbb{R}^2} K(x-y)(u(x)-u(y))^2 dxdy + \int_{\mathbb{R}} W(u(x))dx,$$

where W is a double-well potential with minima at ± 1 and $K \in L^1(\mathbb{R})$ represents as $K(x) = K^{AA}(x) + K^{BB}(x) - 2K^{AB}(x)$, in which $K^{a,b}(x-y)$ is the energy of interaction between the orientation S_a and S_b at positions x and y. The equation of the gradient flow of $E(u)$ is given by

$$u_t = K * u - \alpha u - W'(u),$$

where $\alpha = \int_{\mathbb{R}} K(x)dx$. We note that in this model, the integral kernel does not have to be nonnegative as in the nonlocal diffusion equation, and the integral kernel may have a negative part.

Mathematical models with nonlocal effect also appear in the field of neural science. In [2], the formation of pulse waves generated during firing phenomena in the brain was explained by the following model:

$$u_t = K * g(u) - u,$$

where $g(u)$ is a monotonically non-decreasing function. Then, $u(t, x)$, $g(u)$ and $K(x-y)$ represent the average membrane potential of the neurons located at time $t > 0$ and position x, the pulse emission rate and the degree of stimulation of a neuron at x from a pulse produced by a neuron at y, respectively. From the perspective of mathematical analysis, the case of

$$g(u) = \begin{cases} 1 & (u \geq u_0) \\ 0 & (u < u_0) \end{cases}$$

for $u_0 > 0$ has been analyzed extensively, and studies on the existence and the stability of pulse solutions and the interaction between pulse solutions have been reported [2, 10, 11, 35, 36, 37]. In this case, $K * g(u)$ can be replaced by an indefinite integral of K by assuming the shape of the solution, such as its height and width, and the equation can be used to analyze the existence and interaction of pulse solutions.

In addition, mathematical models with nonlocal effect have been proposed to describe the process of suture formation in the skull [57, 72] and peristalsis and its dysfunction in the esophagus [54].

1.6 The purpose of the thesis

Mathematical models with nonlocal effect are effective in reproducing phenomena, and the analysis of the pattern formation mechanism has become increasingly important in recent years. In particular, localized patterns, such as stationary and traveling wave solutions with relatively simple structures, are very important for understanding the pattern formation of mathematical models. However, there are many technical problems in mathematical analysis, such as difficulty in analyzing the local properties of solutions due to the nonlocality of the equations, and in some cases, the equations cannot apply the comparison principle and other methods for comparing solutions. Therefore, it isn't easy to analyze the basic properties of localized patterns, such as existence and stability.

This thesis focuses on the development of new analytical methods for considering the spatio and temporal dynamics of solutions and their applications, to theoretically understand how nonlocal effect affects the pattern formation process.

1.7 Outline of the thesis

The structure of this thesis is as follows: In Section 2, we discuss the existence of traveling wave solutions for semilinear scalar equations with a sign-changing integral kernel. After explaining the main result and its proof, we discuss the properties of traveling wave solutions by numerical simulations. This thesis also presents the results of constructing a new comparison principle for scalar equations with a sign-changing integral kernel. In Section 3, we analyze the time evolution of spatial patterns that can be approximated by the superposition of multiple localized patterns. We first show that for the general reaction-diffusion equation with nonlocal effect, the solution can be approximated by the superposition of localized patterns if the distances between the localized patterns are sufficiently large. Furthermore, we explain that the positions of each localized pattern can be derived from ordinary differential equations. After that, we analyze the time evolution of spatial patterns for a specific mathematical model. Finally, we consider the behavior of zero points for the diffusion equation and the fractional diffusion equation in Section 4. To understand the asymptotic shape of the solutions, we focus on the asymptotic behavior of the zero points, and explain the results and differences in each case.

As a reminder, the notation and symbols introduced in each section are used only in that section.

2 Existence of traveling wave solutions to nonlocal semilinear scalar equation with sigh-changing integral kernel

The contents from Subsection 2.3 to Subsection 2.7 are based on the paper [23] and master's thesis of the author [45]. These results are based on the joint research with Professor Shin-Ichiro Ei, Professor Jong-Shenq Guo, and Professor Chin-Chin Wu. The results in Subsection 2.8 are newly obtained in this thesis.

2.1 Introduction

Mathematical models with nonlocal effect appear in various fields, as explained in the previous section. When the integral kernel is nonnegative, mathematical analyses of these mathematical models have reported many results concerning the existence and stability of nontrivial stationary solutions and traveling wave solutions. The analysis methods to the reaction-diffusion equations can apply many equations with the nonnegative integral kernel, such as the comparison principle. However, for equations with a sign-changing integral kernel, only exceptional cases have been treated. One of the reasons for this is that even a scalar equation has inherent properties of the solution equivalent to those of the multi-component reaction-diffusion equation, such as Turing instability.

The motivation of this study is to develop an analytical method for equations with a sign-changing integral kernel. We consider the scalar equation which has a diffusion term, a nonlocal term, and a simple nonlinear term. The aim is to show a traveling wave solution, which is one of the localization patterns. In the next subsection, we describe the problem setup and introduce previous studies on the problem. After that, we state the main result and give its proof based on the results of [23, 45]. The results of rigorous mathematical analysis are presented, followed by a discussion of the behavior of the solutions by numerical calculations.

In this thesis, super-sub solutions are newly defined for the equation with a sign-changing kernel. Furthermore, a comparison principle for solutions of evolution equations with a sign-changing integral kernel is developed. Finally, we present the result and give a summary.

2.2 Setting

In this section, we consider the following semilinear scalar equation with nonlocal effect:

$$u_t = du_{xx} + (K * u - \alpha u) + f(u),\ t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

where d is a non-negative constant.

Throughout of this section, we assume that the integral kernel $K(x)$ satisfies

$$K(x) = K(-x) \not\equiv 0\ (x \in \mathbb{R}),\ K \in C(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \alpha := \int_{\mathbb{R}} K(x)dx \geq 0 \tag{K1}$$

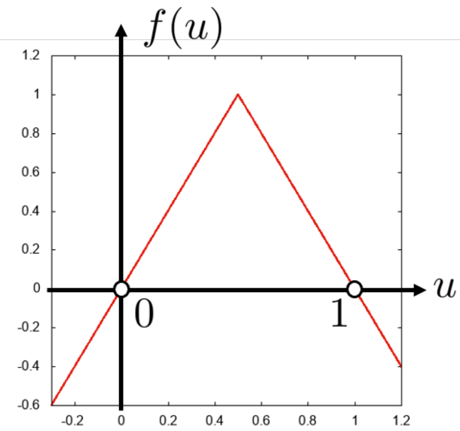

Figure 2: The graph of typical example of the nonlinear function (2.2).

and

$$\forall\lambda > 0, \quad \int_{\mathbb{R}} |K(x)|e^{\lambda x} dx < \infty. \tag{K2}$$

Furthermore, we suppose that the nonlinear term $f(u)$ satisfies

$$\begin{cases} f \in \mathrm{Lip}_{loc}(\mathbb{R}), \ f(0) = f(1) = 0, \\ f > 0 \text{ in } (0,1), \ f < 0 \text{ in } (1,\infty), \ f'(0) > 0. \end{cases} \tag{N1}$$

Typical examples are $K(x) = \dfrac{1}{\sqrt{\pi}}\left(e^{-x^2} - \dfrac{1}{2}e^{-x^2/4}\right)$ as in Fig. 1 and

$$f(u) = \begin{cases} 2 - 2u, & u \in \left(\frac{1}{2}, +\infty\right), \\ 2u, & u \in \left(-\infty, \frac{1}{2}\right] \end{cases} \tag{2.2}$$

as in Fig. 2.

In this section, we first consider the existence of traveling wave solutions. Here, a solution u to (2.1) is called a traveling wave solution if there exist a constant c (the wave speed) and a sufficiently smooth function ϕ (the wave profile) such that $u(t,x) = \phi(x+ct)$. Setting the moving coordinate $\xi := x + ct$, the wave profile ϕ satisfies

$$c\phi' = d\phi'' + (K * \phi - \alpha\phi) + f(\phi), \ \xi \in \mathbb{R}. \tag{TW1}$$

Moreover, we focus on the traveling wave solutions connecting the state $u = 0$ and $u = 1$:

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1. \tag{TW2}$$

The equation (2.1) has been studied as an equation that replaces the local diffusion of the reaction-diffusion equation with the nonlocal diffusion. The existence of traveling wave solutions satisfying (TW1) and (TW2) has been investigated when $d = 0$ the integral kernel is nonnegative [18, 61, 69]. The main argument of the proof is based on the comparison principle. The detailed idea is to construct a monotone sequence of functions from a monotone upper solution, and to show that there exists a monotone solution as its limit. To show that the limiting monotone solution is not trivial, lower solution is constructed well.

On the other hand, the equation (2.1) appears as a mathematical model in neural science [56] and material science [5, 8], where the integral kernel may change sign in order to describe the interaction between substances. In this cases, the comparison principle does not hold in general for the equation (2.1), and similar methods cannot be used. Because of this, the existence of traveling wave solutions was not even known.

2.3 Main results

Before stating the main results, we prepare some notations and assumptions. At first, we set

$$K^{\pm}(x) := \max\{\pm K(x), 0\}.$$

We remark that $K = K^{+} - K^{-}$ and $|K| = K^{+} + K^{-}$. Let us define two quantities as follows:

$$c_Q := \inf_{\lambda\in(0,\hat{\lambda})} \frac{Q(\lambda)}{\lambda}, \quad Q(\lambda) := d\lambda^2 + \int_{\mathbb{R}} K(y)e^{-\lambda y}dy - \alpha + f'(0), \tag{2.3}$$

$$c_R := \inf_{\lambda\in(0,\infty)} \frac{R(\lambda)}{\lambda}, \quad R(\lambda) := d\lambda^2 + \int_{\mathbb{R}} |K(y)|e^{-\lambda y}dy - \alpha + f'(0). \tag{2.4}$$

Here, $\hat{\lambda}$ is the smallest positive zero of $Q(\lambda)$ if it exists, otherwise, we set $\hat{\lambda} = +\infty$. From the definition, $c_R > c_Q$ holds.

Remark 2.1. *There is a case $c_Q = 0$ when K has small negative parts. For example, we set*

$$K(x) = (1+\varepsilon)J(x) - \frac{\varepsilon}{2}(J(x-1) + J(x+1)),$$

where $\varepsilon > 0$, $J \in C_0(\mathbb{R})$ is nonnegative even function satisfying $\mathrm{supp}J(x) \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\int_{\mathbb{R}} J(y)dy = 1$. In this case, $\alpha = 1$ holds. For convenience, we put $f'(0) = 1$ and $d = 0$. Then, we obtain

$$Q(\lambda) = \int_{\mathbb{R}} K(y)e^{-\lambda y}dy = \{1 + \varepsilon - \varepsilon\cosh(\lambda)\} \int_{\mathbb{R}} J(y)e^{-\lambda y}dy.$$

Since $\int_{\mathbb{R}} J(y)e^{-\lambda y}dy$ is positive for all $\lambda > 0$, we immediately have $\hat{\lambda} = \cosh^{-1}\left(\frac{1+\varepsilon}{\varepsilon}\right) \in (0,\infty)$. Thus, we obtain $c_Q = 0$.

Next, we assume that the nonlinear term $f(u)$ satisfies (N1) and the following condition:

$$\exists u^- \leq 0,\ \exists u^+ \geq 1 \text{ s.t. } \forall u \in [u^-,\ u^+],\ |f(u)| \leq f'(0)|u|. \tag{N2}$$

We give the following two assumptions:

Assumption 2.2. *f satisfies* (N1) *and* (N2)*. There exists $a \in (u^-, 0)$ such that*

$$f(a) = 0,\ f(u) < 0 \text{ in } (a, 0),\ f(u) > 0 \text{ in } (-\infty, a). \tag{2.5}$$

And there is a constant $\eta \in (0, 1)$ satisfying

$$f(u) = f'(0)u \quad \text{for } u \in [0, \eta]. \tag{2.6}$$

Moreover, K^- satisfies

$$\int_{\mathbb{R}} K^-(y)dy \leq \min\left\{\frac{-f(\delta)}{\delta - \gamma}, \frac{f(\gamma)}{\delta - \gamma}\right\} \tag{2.7}$$

for some $\delta \in (1, u^+)$ and $\gamma \in (u^-, a)$.

Example 2.3. *Let us consider the case that*

$$f(u) = \begin{cases} 1 - u, & u \in \left(\frac{1}{2}, +\infty\right), \\ u, & u \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ -1 - u, & u \in \left(-\infty, -\frac{1}{2}\right). \end{cases} \tag{2.8}$$

Then, $a = -1$, $\eta = \frac{1}{2}$ and $f'(0) = 1$ holds. f satisfies (N2) *for any $u^- \in (0, \infty)$ and $u^+ \in (1, \infty)$. Furthermore, since we have*

$$\sup_{\delta > 1} \frac{-f(\delta)}{\delta - \gamma} = 1,\ \forall \gamma < -1, \quad \sup_{\gamma < -1} \frac{f(\gamma)}{\delta - \gamma} = 1,\ \forall \delta > 1,$$

if the integral kernel $K(x)$ satisfies

$$\int_{\mathbb{R}} K^-(y)dy < 1,$$

then Assumption 2.2 holds.

Assumption 2.4. *f satisfies* (N1) *and* (N2)*. Also, f satisfies* (2.6) *and $f'(0) > \alpha$. Furthermore, K^- satisfies*

$$\int_{\mathbb{R}} K^-(y)dy \leq \min\left\{\frac{-f(\delta)}{\delta}, \frac{(f'(0) - \alpha)\eta}{\delta}\right\} \tag{2.9}$$

for some $\delta \in (1, u^+)$.

Example 2.5. *Let f be* (2.8) *and let $\alpha = 0$. Then, since we have*

$$\frac{-f(\delta)}{\delta} = \frac{\delta - 1}{\delta}, \quad \frac{(f'(0) - \alpha)\eta}{\delta} = \frac{1}{2\delta}$$

by setting $\delta = \frac{3}{2}$, if the integral kernel $K(x)$ satisfies

$$\int_{\mathbb{R}} K^+(y)dy = \int_{\mathbb{R}} K^-(y)dy \leq \frac{1}{3},$$

then Assumption 2.4 holds.

Let us introduce the first main result.

Theorem 2.6. *Let Assumption 2.2 be enforced. Then, for all $c > c_R$, there exists $\phi(x) \not\equiv 0$ satisfying* (TW1) *and $\phi(-\infty) = 0$.*

Theorem 2.6 only mentions the existence of nontrivial traveling wave solutions satisfying $\phi(-\infty) = 0$. Whether the wave profile is positive or satisfies $\phi(+\infty) = 1$ is not well understood.

Before stating next main result, we define

$$C_b^k(\mathbb{R}) := \{g \in C^k(\mathbb{R}) \mid ||g^{(j)}||_\infty < \infty,\ j = 0, 1, \ldots, k\}, \quad \|g\|_\infty := \sup_{\xi \in \mathbb{R}} |g(\xi)|.$$

In particular, $C_b(\mathbb{R}) := C_b^0(\mathbb{R})$. Then, the following result holds.

Theorem 2.7. *Let Assumption 2.4 be enforced. Then, for all $c > c_R$, there exists $\phi > 0$ satisfying* (TW1), *$\phi(-\infty) = 0$ and*

$$\liminf_{\xi \to +\infty} \phi(\xi) > 0. \tag{2.10}$$

Especially, if $c > \max\{c_R, c_K\}$ and $\phi \in C_b^2(\mathbb{R})$, then $\phi(+\infty) = 1$ holds. Here, c_K is a constant define as

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

Remark 2.8. *As we will show in Proposition 2.20 below, we obtain $\phi \in C_b^2(\mathbb{R})$ if $d > 0$, or if $d > 0$ and $f \in C^1(\mathbb{R})$.*

To prove the existence of traveling wave solutions, the upper-lower solutions are newly defined to be natural extensions of the case that the integral kernel is nonnegative. After that, we construct to show the existence of traveling wave solutions by using Schauder's fixed point theorem.

2.4 Construction of upper-lower solutions

We first introduce the new notion of upper-lower solutions of (TW1).

Definition 2.9. *For $c > 0$, a pair of continuous function $\{\overline{\phi}, \underline{\phi}\}$ are upper-lower solutions of* (TW1) *if*

$$\begin{aligned} c\overline{\phi}'(\xi) &\geq d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)),\ \forall\, \xi \in \mathbb{R} \setminus A, \\ c\underline{\phi}'(\xi) &\leq d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)),\ \forall\, \xi \in \mathbb{R} \setminus A, \end{aligned}$$

holds for some finite set $A \subset \mathbb{R}$.

Each differential-integral inequality is not described by a single function due to the fact that the integral kernel has a negative part, which makes it very difficult to construct upper-lower solutions, and this is the technical difficulty of this study.

Next, we construct upper-lower solutions of (TW1). For any $c > c_R$, we can take $\lambda_1 \in (0, \hat{\lambda})$ satisfying the following conditions:

$$Q(\lambda_1) = c\lambda_1, \quad Q(\lambda) > c\lambda \ \text{ for all } \lambda \in [0, \lambda_1). \tag{2.11}$$

From the definition of c_R, there are two roots $\lambda_2 < \lambda_3$ of $R(\lambda) = c\lambda$ satisfying

$$R(\lambda) < c\lambda,\ \forall\, \lambda \in (\lambda_2, \lambda_3); \quad R(\lambda) > c\lambda,\ \forall\, \lambda \in [0, \lambda_2) \cup (\lambda_3, \infty). \tag{2.12}$$

Here, we note that $\lambda_2 > \lambda_1$.

Now, we fix $\nu > 1$ satisfying $\nu\lambda_1 \in (\lambda_2, \lambda_3)$. For $h > 1$, we consider the function

$$\rho(\xi) := e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi},\ \xi_0 := \frac{-\ln h}{(\nu - 1)\lambda_1},\ \xi_M := \frac{-\ln(h\nu)}{(\nu - 1)\lambda_1} \tag{2.13}$$

Then, we have

$$\rho(\xi) \begin{cases} > 0 & (\xi < \xi_0) \\ = 0 & (\xi = \xi_0) \\ < 0 & (\xi > \xi_0) \end{cases}$$

Moreover,

$$\rho(\xi) \leq \rho(\xi_M) = C(\nu) h^{-1/(\nu-1)},\ \forall\, \xi \in \mathbb{R} \tag{2.14}$$

holds, where $C = C(\nu) := \nu^{-1/(\nu-1)}(1 - 1/\nu)$. Now, we fix h satisfying $\rho(\xi_M) \leq \eta$. Here, η is a constant defined in (2.6). For convenience, we set

$$\begin{aligned} N_1(\xi) &:= -c\overline{\phi}'(\xi) + d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)), \\ N_2(\xi) &:= -c\underline{\phi}'(\xi) + d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)). \end{aligned}$$

From Definition 2.9, if there are functions $\{\overline{\phi}, \underline{\phi}\}$ satisfying $N_1(\xi) \leq 0$ and $N_2(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ expect for finite points, then these functions are upper-lower solutions of (TW1). Let us define

$$I^{\pm}(\lambda) := \int_{\mathbb{R}} K^{\pm}(y) e^{-\lambda y} dy.$$

We remark that $I^+(\lambda),\ I^-(\lambda) < \infty$ holds, since K satisfies (K2).

Lemma 2.10. *Let Assumption 2.2 be enforced. We set*

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}, \delta\}, \quad \underline{\phi}(\xi) = \max\{e^{\lambda_1 \xi} - he^{\nu\lambda_1 \xi}, \gamma\} \tag{2.15}$$

Then, for all $c > c_R$, $\{\overline{\phi}, \underline{\phi}\}$ are upper-lower solutions of (TW1).

Proof. We define ξ_i $(i = 1, 2)$ as

$$e^{\lambda_1 \xi_1} + he^{\nu\lambda_1 \xi_1} = \delta, \quad e^{\lambda_1 \xi_2} - he^{\nu\lambda_1 \xi_2} = \gamma.$$

Then, the functions defined by (2.15) are represented as

$$\overline{\phi}(\xi) = \begin{cases} e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}, & \xi \leq \xi_1, \\ \delta, & \xi \geq \xi_1, \end{cases} \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda_1 \xi} - he^{\nu\lambda_1 \xi}, & \xi \leq \xi_2, \\ \gamma, & \xi \geq \xi_2. \end{cases}$$

We first prove $N_1(\xi) \leq 0$. When $\xi < \xi_1$, $\overline{\phi}(\xi) = e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}$ holds. By using $\overline{\phi}(\xi) \leq e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi} \leq \delta \leq u^+$, $\underline{\phi}(\xi) \geq e^{\lambda_1 \xi} - he^{\nu\lambda_1 \xi}$ for $\xi \in \mathbb{R}$, (N2), (2.11) and (2.12), we obtain

$$\begin{aligned} N_1(\xi) \;\leq\; & -c(\lambda_1 e^{\lambda_1 \xi} + h\nu\lambda_1 e^{\nu\lambda_1 \xi}) + d\{\lambda_1^2 e^{\lambda_1 \xi} + h(\nu\lambda_1)^2 e^{\nu\lambda_1 \xi}\} \\ & + \int_{\mathbb{R}} K^+(y)[e^{\lambda_1(\xi - y)} + he^{\nu\lambda_1(\xi-y)}]dy \\ & - \int_{\mathbb{R}} K^-(y)[e^{\lambda_1(\xi - y)} - he^{\nu\lambda_1(\xi-y)}]dy - \alpha(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) + f(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) \\ = \;& e^{\lambda_1 \xi}\left\{-c\lambda_1 + d\lambda_1^2 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\ & + he^{\nu\lambda_1 \xi}\left\{-c\nu\lambda_1 + d(\nu\lambda_1)^2 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) \\ = \;& e^{\lambda_1 \xi}\left\{-c\lambda_1 + Q(\lambda_1) - f'(0)\right\} \\ & + he^{\nu\lambda_1 \xi}\left\{-c\nu\lambda_1 + R(\nu\lambda_1) - f'(0)\right\} + f(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) \\ < \;& f(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) - f'(0)(e^{\lambda_1 \xi} + he^{\nu\lambda_1 \xi}) \leq 0. \end{aligned}$$

In the case $\xi > \xi_1$, we know $\overline{\phi}(\xi) = \delta$. Since we have $\overline{\phi}(\xi) \leq \delta$ and $\underline{\phi}(\xi) \geq \gamma$ for $\xi \in \mathbb{R}$ and (2.7), we deduce

$$\begin{aligned} N_1(\xi) \;\leq\; & 0 + 0 + \delta \int_{\mathbb{R}} K^+(y)dy - \gamma \int_{\mathbb{R}} K^-(y)dy - \alpha\delta + f(\delta) \\ = \;& (\delta - \gamma) \int_{\mathbb{R}} K^-(y)dy + f(\delta) \leq 0. \end{aligned}$$

Thus, $N_1 \leq 0$ holds on $\mathbb{R} \setminus \{\xi_1\}$.

Next, we show $N_2(\xi) \geq 0$. When $\xi < \xi_2$, we know that $\underline{\phi}(\xi) = e^{\lambda_1 \xi} - he^{\nu\lambda_1 \xi}$. Then, by using (2.11) and (2.12), we obtain

$$\begin{aligned}
N_2(\xi) &\geq -c(\lambda_1 e^{\lambda_1\xi} - h\nu\lambda_1 e^{\nu\lambda_1\xi}) + d\{\lambda_1^2 e^{\lambda_1\xi} - h(\nu\lambda_1)^2 e^{\nu\lambda_1\xi}\} \\
&\quad + \int_{\mathbb{R}} K^+(y)[e^{\lambda_1(\xi-y)} - he^{\nu\lambda_1(\xi-y)}]dy \\
&\quad - \int_{\mathbb{R}} K^-(y)[e^{\lambda_1(\xi-y)} + he^{\nu\lambda_1(\xi-y)}]dy - \alpha(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\
&= e^{\lambda_1\xi}\left\{-c\lambda_1 + d\lambda_1^2 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\
&\quad -he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + d(\nu\lambda_1)^2 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\
&= e^{\lambda_1\xi}\left\{-c\lambda_1 + Q(\lambda_1) - f'(0)\right\} \\
&\quad -he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + R(\nu\lambda_1) - f'(0)\right\} + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\
&> f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}).
\end{aligned}$$

In the case that $e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \leq 0$, we obtain $f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \geq 0$ from (N2) and $u^- \leq \gamma \leq e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \leq 0$. On the other hand, when $e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \geq 0$ holds, we have $\rho(\xi) \leq \eta$ from the choice of h. Thus, by using (2.6), we deduce $f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) = 0$. Thus, we obtain $N_2(\xi) \geq 0$ for $\xi < \xi_2$.

Finally, since $\underline{\phi}(\xi) = \gamma$ for $\xi > \xi_2$, we have

$$\begin{aligned}
N_2(\xi) &\geq 0 + 0 + \gamma \int_{\mathbb{R}} K^+(y)dy - \delta \int_{\mathbb{R}} K^-(y)dy - \alpha\gamma + f(\gamma) \\
&= -(\delta - \gamma)\int_{\mathbb{R}} K^-(y)dy + f(\gamma) \geq 0
\end{aligned}$$

from (2.7). Thus, $N_2 \geq 0$ holds on $\mathbb{R} \setminus \{\xi_2\}$.

From the above calculations, we obtain the desired assertion. □

Next, we introduce the upper-lower solutions under the condition of Assumption 2.4. λ_1, ν are same as the above argument. Only h is changed here so that $\rho(\xi_M) = \eta$ is satisfied. Then, the following lemma is obtained.

Lemma 2.11. *Let Assumption 2.4 be enforced. We set*

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}, \delta\}, \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}, & \xi \leq \xi_M, \\ \eta, & \xi \geq \xi_M \end{cases} \tag{2.16}$$

Then, for all $c > c_R$, $\{\overline{\phi}, \underline{\phi}\}$ are upper-lower solutions of (TW1).

Proof. Let $\xi_1 \in \mathbb{R}$ be the constant defined in the proof of Lemma 2.10. In the case $\xi < \xi_1$, we obtain $N_1(\xi) \leq 0$ from the same calculation to the proof of Lemma 2.10.

When $\xi > \xi_1$, we have $\overline{\phi}(\xi) = \delta$ holds. By using $\overline{\phi}(\xi) \leq \delta$, $\underline{\phi}(\xi) \geq 0$ for $\xi \in \mathbb{R}$ and (2.9), we deduce

$$\begin{aligned}
N_1(\xi) &\leq 0 + \delta \int_{\mathbb{R}} K^+(y)dy - \alpha\delta + f(\delta) \\
&= \delta \int_{\mathbb{R}} K^-(y)dy + f(\delta) \leq 0.
\end{aligned}$$

Hence, $N_1 \le 0$ holds on $\mathbb{R} \setminus \{\xi_1\}$.

Next, let us show $N_2(\xi) \ge 0$. In the case $\xi < \xi_M$, we know that $\underline{\phi}(\xi) = e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi} \in (0, \eta]$. Then, from the same calculation to the proof of Lemma 2.10 and the conditions (2.11), (2.12) and (2.6), we obtain

$$\begin{aligned}
N_2(\xi) \ &\ge \ -c(\lambda_1 e^{\lambda_1 \xi} - h\nu\lambda_1 e^{\nu\lambda_1 \xi}) + d\{\lambda_1^2 e^{\lambda_1 \xi} - h(\nu\lambda_1)^2 e^{\nu\lambda_1 \xi}\} + e^{\lambda_1 \xi} I^+(\lambda_1) - h e^{\nu\lambda_1 \xi} I^+(\nu\lambda_1) \\
&\qquad -e^{\lambda_1 \xi} I^-(\lambda_1) - h e^{\nu\lambda_1 \xi} I^-(\nu\lambda_1) - \alpha(e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}) + f(e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}) \\
&= \ e^{\lambda_1 \xi}\left\{-c\lambda_1 + d\lambda_1^2 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\
&\qquad -h e^{\nu\lambda_1 \xi}\left\{-c\nu\lambda_1 + d(\nu\lambda_1)^2 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}) \\
&> \ f(e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}) - f'(0)(e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}) = 0.
\end{aligned}$$

Finally, since $\underline{\phi}(\xi) = \eta$ holds when $\xi > \xi_M$, we have

$$\begin{aligned}
N_2(\xi) \ &\ge \ 0 + 0 + 0 - \delta \int_{\mathbb{R}} K^-(y)dy - \alpha\eta + f(\eta) \\
&= \ -\delta \int_{\mathbb{R}} K^-(y)dy - \alpha\eta + f'(0)\eta \ge 0
\end{aligned}$$

by using (2.9). Thus, $N_2 \ge 0$ holds on $\mathbb{R} \setminus \{\xi_M\}$.

From the above calculations, we obtain the desired assertion. $\square$

2.5 Construction of invariant sets

2.5.1 The case $d = 0$

Let $\kappa := \sup\left\{\frac{|f(u)-f(v)|}{|u-v|} \mid u, v \in [u^-, u^+],\ u \ne v\right\}$. We introduce the integral operator

$$P_0^c[z](\xi) \ := \ \frac{1}{c}\int_{-\infty}^{\xi} e^{-\mu(\xi - y)} G[z](y)dy,$$

where $\mu := (\alpha + \kappa)/c$ and $G[z](\xi) := (K * z)(\xi) + \kappa z(\xi) + f(z(\xi))$.

By differentiating $P_0^c[z]$, we obtain

$$\frac{d}{d\xi} P_0^c[z](\xi) = \frac{1}{c}\left\{(K * z)(\xi) + \kappa z(\xi) + f(z(\xi))\right\} - \frac{\alpha + \kappa}{c} P_0^c[z](\xi).$$

Thus, we can find that ϕ is a fixed point of P_0^c if and only if (c, ϕ) satisfies (TW1).

In the following, we assume that there are upper-lower solutions $\{\overline{\phi}, \underline{\phi}\}$ satisfying

$$(1)\ \underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [u^-, u^+], \quad (2)\ \underline{\phi}(\xi) \le \overline{\phi}(\xi)\ (\xi \in \mathbb{R}).$$

Let $\Gamma := \{\psi \in C(\mathbb{R}) \mid \underline{\phi}(\xi) \le \psi(\xi) \le \overline{\phi}(\xi)\}$. Then, we obtain the following lemma:

Lemma 2.12. $\Gamma \subset C(\mathbb{R})$ *is a invariant set of* P_0^c.

Proof. Let us show that $P_0^c[z] \in \Gamma$ for any $z \in \Gamma$. Now, we have

$$G[z](y) \leq (K^+ * \overline{\phi})(y) - (K^- * \underline{\phi})(y) + \kappa\overline{\phi}(y) + f(\overline{\phi}(y)) \leq c\overline{\phi}'(y) + c\mu\overline{\phi}(y),$$

for almost all $y \in \mathbb{R}$, because $\kappa z_1 + f(z_1) \leq \kappa z_2 + f(z_2)$ holds for any $z_1, z_2 \in \mathbb{R}$ satisfying $u^- \leq z_1 \leq z_2 \leq u^+$. From this, we obtain

$$\begin{aligned} P_0^c[z](\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-\mu(\xi-y)} G[z](y)dy \\ &\leq \int_{-\infty}^{\xi} e^{-\mu(\xi-y)}[\overline{\phi}'(y) + \mu\overline{\phi}(y)]dy = \overline{\phi}(\xi). \end{aligned}$$

Similarly, $P_0^c[z](\xi) \geq \underline{\phi}(\xi)$ holds. It is obvious that $P_0^c[z] \in C(\mathbb{R})$, and then we have $P_0^c[z] \in \Gamma$. □

Next, let us prove that $P_0^c : \Gamma \to \Gamma$ is a compact operator with respect to a suitable weighted norm. For sufficiently small $\theta > 0$, we define

$$B_\theta(\mathbb{R}) := \{z \in C(\mathbb{R}) \mid \|z\|_\theta < \infty\}, \quad \|z\|_\theta = \sup_{\xi\in\mathbb{R}} |z(\xi)|e^{-\theta|\xi|}. \tag{2.17}$$

We remark that $(B_\theta(\mathbb{R}), \|z\|_\theta)$ is a Banach space and Γ is a closed convex subset of $B_\theta(\mathbb{R})$.

Lemma 2.13. *Let $\theta \in (0, \mu)$. Then, $P_0^c : \Gamma \to \Gamma$ is continuous with respect to $\|\cdot\|_\theta$.*

Proof. We first prove that $G : \Gamma \to B_\theta(\mathbb{R})$ is continuous. For any $z_1,\ z_2 \in \Gamma$, we have

$$\begin{aligned} |G[z_1](\xi) - G[z_2](\xi)|e^{-\theta|\xi|} &\leq |K| * |z_1 - z_2|(\xi)e^{-\theta|\xi|} + \kappa\|z_1 - z_2\|_\theta + \|f(z_1) - f(z_2)\|_\theta \\ &\leq |K| * |z_1 - z_2|(\xi)e^{-\theta|\xi|} + 2\kappa\|z_1 - z_2\|_\theta. \end{aligned}$$

Since $|\xi - y| - |y| \leq |\xi|$ $(\xi,\ y \in \mathbb{R})$ holds, we deduce

$$\begin{aligned} |K| * |z_1 - z_2|(\xi)e^{-\theta|\xi|} &= \int_{\mathbb{R}} |K(y)|\ |z_1 - z_2|(\xi - y)e^{-\theta|\xi|}dy \\ &\leq \int_{\mathbb{R}} |K(y)|e^{\theta|y|}\ |z_1 - z_2|(\xi - y)e^{-\theta|\xi-y|}dy \\ &\leq \left(\int_{\mathbb{R}} |K(y)|e^{\theta|y|}dy\right) \|z_1 - z_2\|_\theta. \end{aligned}$$

Hence, we obtain

$$\|G[z_1](\xi) - G[z_2](\xi)\|_\theta \leq \left(\int_{\mathbb{R}} |K(y)|e^{\theta|y|}dy + 2\kappa\right) \|z_1 - z_2\|_\theta. \tag{2.18}$$

This means that $G : \Gamma \to B_\theta(\mathbb{R})$ is continous.

We next show that $P_0^c : \Gamma \to \Gamma$ is continuous. For all $z_1,\ z_2 \in \Gamma$, we obtain

$$
\begin{aligned}
|P_0^c[z_1](\xi) - P_0^c[z_2](\xi)|e^{-\theta|\xi|} &\leq \frac{1}{c}\int_{-\infty}^{\xi} e^{-\mu(\xi-y)}|G[z_1]-G[z_2]|(y)e^{-\theta|\xi|}dy \\
&\leq \frac{\int_{\mathbb{R}}|K(y)|e^{\theta|y|}dy + 2\kappa}{c}\|z_1-z_2\|_\theta \int_{-\infty}^{\xi} e^{-\mu(\xi-y)}e^{-\theta(|\xi|-|y|)}dy
\end{aligned}
$$

by using (2.18). Here, when $\xi \leq 0$, we have

$$
\int_{-\infty}^{\xi} e^{-\mu(\xi-y)}e^{-\theta(|\xi|-|y|)}dy = \int_{-\infty}^{\xi} e^{-(\mu-\theta)(\xi-y)}dy = \frac{1}{\mu-\theta}.
$$

Furthermore, in the case that $\xi > 0$, we deduce

$$
\begin{aligned}
\int_{-\infty}^{\xi} e^{-\mu(\xi-y)}e^{-\theta(|\xi|-|y|)}dy &= e^{-(\mu+\theta)\xi}\left(\int_{-\infty}^{0} e^{(\mu-\theta)y}dy + \int_{0}^{\xi} e^{(\mu+\theta)y}dy\right) \\
&= \frac{1}{\mu+\theta} + e^{-(\mu+\theta)\xi}\left(\frac{1}{\mu-\theta} - \frac{1}{\mu+\theta}\right) \leq \frac{1}{\mu-\theta}.
\end{aligned}
$$

Thus,

$$
\|P_0^c[z_1] - P_0^c[z_2]\|_\theta \leq \frac{\int_{\mathbb{R}}|K(y)|e^{\theta|y|}dy + 2\kappa}{c(\mu-\theta)}\|z_1-z_2\|_\theta
$$

holds. This inequality yields the conclusion of the lemma. □

Lemma 2.14. *Let $\theta \in (0,\mu)$. Then, $P_0^c : \Gamma \to \Gamma$ is compact with respect to $\|\cdot\|_\theta$.*

Proof. First, it is obvious that $P_0^c\Gamma$ is uniformly bounded, because $P_0^c\Gamma \subset \Gamma$ holds from Lemma 2.12. Thus, let us prove that $P_0^c\Gamma$ is equicontinous. By setting $M = \max\{\|\overline{\phi}\|_\infty, \|\underline{\phi}\|_\infty\}$, for any $z \in \Gamma$, we have

$$
c\left|\frac{d}{d\xi}P_0^c[z](\xi)\right| \leq \left(\int_{\mathbb{R}}|K(y)|dy + \mu + \kappa\right)M + \sup_{u\in[-M,M]}|f(u)|.
$$

Thus, we immediately deduce that $P_0^c\Gamma \subset \Gamma$ is equicontinous.

We next prove that $P_0^c : \Gamma \to \Gamma$ is a compact operator with respect to the weighted norm $\|\cdot\|_\theta$. Namely, we show that for any sequence $\{z_k\}_{k=1}^\infty \subset \Gamma$, $\{P[z_k]\}_{k=1}^\infty \subset \Gamma$ has a convergence subsequence with respect to the norm $\|\cdot\|_\theta$. Now, let us define $\tilde{P}_n : \Gamma \to \Gamma$ as

$$
\tilde{P}_n[z](\xi) := \begin{cases} P_0^c[z](n) & \xi > n, \\ P_0^c[z](\xi) & \xi \in [-n,n], \\ P_0^c[z](-n) & \xi < -n \end{cases}
$$

for $n \in \mathbb{N}$. Then, $\tilde{P}_n : \Gamma \to \Gamma$ is a compact operator with respect to the norm $\|\cdot\|_\infty$ from the Ascoli-Arzelá theorem. Hence, for any sequence $\{z_k\}_{k=1}^\infty \subset \Gamma$, there exist a subsequence $\{z_{k_1(j)}\}_{j=1}^\infty \subset \{z_k\}_{k=1}^\infty$ and $\psi_1 \in \Gamma$ such that

$$\sup_{\xi\in[-1,1]} |\tilde{P}_1[z_{k_1(j)}](\xi) - \psi_1(\xi)| \to 0 \ (j \to \infty).$$

Thus, there is $\varphi_1 \in \{z_{k_1(j)}\}_{j=1}^\infty$ satisfying

$$\sup_{\xi\in[-1,1]} |\tilde{P}_1[\varphi_1](\xi) - \psi_1(\xi)| \leq 1.$$

Similarly, there exist $\{z_{k_2(j)}\}_{j=1}^\infty \subset \left(\{z_{k_1(j)}\}_{j=1}^\infty \backslash \{\varphi_1\}\right)$ and $\psi_2 \in \Gamma$ such that

$$\sup_{\xi\in[-2,2]} |\tilde{P}_2[z_{k_2(j)}](\xi) - \psi_2(\xi)| \to 0 \ (j \to \infty), \quad \psi_2(\xi) = \psi_1(\xi) \ (\xi \in [-1,1]).$$

Thus, there is $\varphi_2 \in \{z_{k_2(j)}\}_{j=1}^\infty$ satisfying

$$\sup_{\xi\in[-2,2]} |\tilde{P}_2[\varphi_2](\xi) - \psi_2(\xi)| \leq \frac{1}{2}.$$

By applying the above argument repeatedly, for any $n \geq 2$, there exist $\varphi_n \in \{z_k\}_{k=1}^\infty$ and $\psi_n \in \Gamma$ such that

$$\sup_{\xi\in[-n,n]} |\tilde{P}_n[\varphi_n](\xi) - \psi_n(\xi)| \leq \frac{1}{n}, \quad \varphi_n \not\equiv \varphi_{n-1}, \quad \psi_n(\xi) = \psi_{n-1}(\xi) \ (\xi \in [-n+1, n-1]).$$

From the construction method, $\{\psi_n\}_{n=1}^\infty$ converges pointwise to some ψ satisfying

$$\psi(\xi) = \psi_n(\xi) \ (\xi \in [-n,n]), \ \psi \in \Gamma.$$

Hence, we obtain

$$\begin{aligned} \|P_0^c[\varphi_n] - \psi\|_\theta &\leq \sup_{|\xi|>n} |P_0^c[\varphi_n](\xi) - \psi_n(\xi)|e^{-\theta|\xi|} + \sup_{|\xi|\leq n} |P_0^c[\varphi_n](\xi) - \psi(\xi)|e^{-\theta|\xi|} \\ &\leq 2Me^{-\theta n} + \sup_{|\xi|\leq n} |\tilde{P}_n[\varphi_n](\xi) - \psi_n(\xi)| \\ &\leq 2Me^{-\theta n} + \frac{1}{n} \to 0 \ (n \to \infty), \end{aligned}$$

because $|z| \leq M$ holds for all $z \in \Gamma$. This means that $P_0^c : \Gamma \to \Gamma$ is a compact operator with respect to the weighted norm $\|\cdot\|_\theta$. $\square$

Thus, the following proposition is obtained by applying Schaudar's fixed point theorem:

Proposition 2.15. *For $c > 0$, we assume that there are upper-lower solutions $\{\overline{\phi}, \underline{\phi}\}$ of* (TW1) *satisfying the following conditions:*

$$(1)\ \underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [u^-, u^+], \quad (2)\ \underline{\phi}(\xi) \le \overline{\phi}(\xi)\ (\xi \in \mathbb{R}).$$

Then, there exists a solution ϕ of (TW1) *such that $\underline{\phi} \le \phi \le \overline{\phi}$ holds on $\mathbb{R}$.*

The upper-lower solutions constructed in Subsection 2.3 satisfies the assumptions in the statement of Proposition 2.15. Hence, in the case that $d = 0$, we obtain Theorem 2.6 and the first part of Theorem 2.7 from Lemma 2.10, Lemma 2.11 and Proposition 2.15.

2.5.2 The case $d > 0$

Next, we consider the case $d > 0$. For $c > 0$, we introduce the integral operator

$$P_d^c[z](\xi) \ := \ \frac{1}{d(\mu^+ - \mu^-)} \left[\int_\xi^{+\infty} e^{\mu^+(\xi - y)} + \int_{-\infty}^{\xi} e^{\mu^-(\xi - y)} \right] G[z](y) dy,$$

where $\mu^+ > 0 > \mu^-$ are roots of $d\mu^2 - c\mu - (\alpha + \kappa) = 0$. By differentiating $P_d^c[z]$, we obtain

$$c\frac{d}{d\xi} P_d^c[z](\xi) = d\frac{d^2}{d\xi^2} P_d^c[z](\xi) - (\alpha + \kappa) P_d^c[z](\xi) + \{(K * z)(\xi) + \kappa z(\xi) + f(z(\xi))\}.$$

This means that ϕ is a fixed point of P_d^c if and only if (c, ϕ) satisfies (TW1).

Here, we assume that there are upper-lower solutions $\{\overline{\phi}, \underline{\phi}\}$ satisfying

$$\begin{array}{ll} (1) & \underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [u^-, u^+], \\ (2) & \underline{\phi}(\xi) \le \overline{\phi}(\xi)\ (\forall \xi \in \mathbb{R}), \\ (3) & \overline{\phi}'(z-) \ge \overline{\phi}'(z+),\ \underline{\phi}'(z-) \le \underline{\phi}'(z+),\ (\forall z \in \mathcal{A}), \end{array}$$

where $\overline{\phi}(z\pm) := \lim_{\xi \to z \pm 0} \overline{\phi}(\xi)$, $\underline{\phi}(z\pm) := \lim_{\xi \to z \pm 0} \underline{\phi}(\xi)$ and $\mathcal{A}$ is a finite set given by Definition 2.9.

Lemma 2.16. *$\Gamma \subset C(\mathbb{R})$ is a invariant set of P_d^c.*

Proof. We prove that $P_d^c[z] \in \Gamma$ for any $z \in \Gamma$. Since $\mathcal{A}$ is a finite set, we denote the elements as $\xi_1 > \xi_2 > \cdots > \xi_N$. Setting $\xi_0 = \infty$ and $\xi_{N+1} = -\infty$, let us consider the case that $\xi \in (\xi_{k+1}, \xi_k)$. Since we have

$$\begin{aligned} G[z](y) &\le (K^+ * \overline{\phi})(y) - (K^- * \underline{\phi})(y) + \kappa\overline{\phi}(y) + f(\overline{\phi}(y)) \\ &\le -d\overline{\phi}''(y) + c\overline{\phi}'(y) + (\alpha + \kappa)\overline{\phi}(y) \end{aligned}$$

for almost all $y \in \mathbb{R}$,

$$
\begin{aligned}
P_d^c[z](\xi) &= \frac{1}{d(\mu^+ - \mu^-)} \left[\int_\xi^{+\infty} e^{\mu^+(\xi-y)} + \int_{-\infty}^{\xi} e^{\mu^-(\xi-y)} \right] G[z](y) dy, \\
&\leq \frac{1}{d(\mu^+ - \mu^-)} \left[\int_\xi^{+\infty} e^{\mu^+(\xi-y)} + \int_{-\infty}^{\xi} e^{\mu^-(\xi-y)} \right] [-d\overline{\phi}''(y) + c\overline{\phi}'(y) + (\alpha+\kappa)\overline{\phi}(y)] dy, \\
&\leq \overline{\phi}(\xi) + \frac{1}{d(\mu^+ - \mu^-)} \left\{ \sum_{j=1}^{k} e^{\mu^+(\xi-\xi_j)} (\overline{\phi}'(\xi_j+) - \overline{\phi}'(\xi_j-)) \right. \\
&\qquad\qquad \left. + \sum_{j=k+1}^{N} e^{\mu^-(\xi-\xi_j)} (\overline{\phi}'(\xi_j+) - \overline{\phi}'(\xi_j-)) \right\}, \\
&\leq \overline{\phi}(\xi)
\end{aligned}
$$

holds. Similarly, we obtain $P_d^c[z](\xi) \geq \underline{\phi}(\xi)$. It is obvious that $P_d^c[z] \in C(\mathbb{R})$. Thus, we deduce $P_d^c[z] \in \Gamma$. □

The followings are obtained in the same line of the case that $d = 0$. We omit the proof here.

Lemma 2.17. *Let $\theta \in (0, -\mu^-)$. Then $P_d^c : \Gamma \to \Gamma$ is continuous with respect to $\|\cdot\|_\theta$.*

Lemma 2.18. *Let $\theta \in (0, -\mu^-)$. Then, $P_d^c : \Gamma \to \Gamma$ is compact with respect to $\|\cdot\|_\theta$.*

Thus, we also obtain the following proposition:

Proposition 2.19. *For $c > 0$, we assume that there are upper-lower solutions $\{\overline{\phi}, \underline{\phi}\}$ of* (TW1) *satisfying the following conditions:*

$$
\begin{aligned}
&(1) \quad \underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [u^-, u^+], \\
&(2) \quad \underline{\phi}(\xi) \leq \overline{\phi}(\xi)\ (\forall \xi \in \mathbb{R}), \\
&(3) \quad \overline{\phi}'(z-) \geq \overline{\phi}'(z+),\ \underline{\phi}'(z-) \leq \underline{\phi}'(z+),\ (\forall z \in \mathcal{A}).
\end{aligned}
$$

Then, there exists a solution ϕ of (TW1) *such that $\underline{\phi} \leq \phi \leq \overline{\phi}$ holds on $\mathbb{R}$.*

The upper-lower solutions constructed in Subsection 2.3 satisfies the assumptions in the statement of Proposition 2.19. Hence, in the case that $d > 0$, we obtain Theorem 2.6 and the first part of Theorem 2.7 from Lemma 2.10, Lemma 2.11 and Proposition 2.19.

2.6 Regularity of wave profiles and asymptotic profiles

We first give the result of the regularity of wave profiles.

Proposition 2.20. *Let* (c, ϕ) *be a solution of* (TW1) *satisfying* $c > 0$, $\phi \in C_b(\mathbb{R})$ *and* $\phi : \mathbb{R} \to [u^-, u^+]$.
(1) *Suppose* $d > 0$. *Then,* $\phi \in C_b^2(\mathbb{R})$ *holds if* $f \in \mathrm{Lip}_{loc}(\mathbb{R})$. *In particular, if* $f \in C^k(\mathbb{R})$ *for some* $k \in \mathbb{N}$, *then* $\phi \in C_b^{k+2}(\mathbb{R})$.
(2) *Suppose* $d = 0$. *Then,* $\phi \in C_b^1(\mathbb{R})$ *holds if* $f \in \mathrm{Lip}_{loc}(\mathbb{R})$ *In particular, if* $f \in C^k(\mathbb{R})$ *for some* $k \in \mathbb{N}$, *then* $\phi \in C_b^{k+1}(\mathbb{R})$

Proof. We give only the proof of (1), because the basic approach is same.

Assume that $d > 0$ and $f \in \mathrm{Lip}_{loc}(\mathbb{R})$. Since ϕ is a fixed point of the integral operator P_d^c,

$$\phi(\xi) = \frac{1}{d(\mu^+ - \mu^-)} \left[\int_\xi^{+\infty} e^{\mu^+(\xi - y)} + \int_{-\infty}^\xi e^{\mu^-(\xi - y)} \right] G[\phi](y) dy$$

holds. Thus, we obtain $\phi \in C^1(\mathbb{R})$ and

$$\phi'(\xi) = \frac{1}{d(\mu^+ - \mu^-)} \left[\mu^+ \int_\xi^{+\infty} e^{\mu^+(\xi - y)} + \mu^- \int_{-\infty}^\xi e^{\mu^-(\xi - y)} \right] G[\phi](y) dy.$$

Here, since we know that $G[\phi] \in C_b(\mathbb{R})$ if $\phi \in C_b(\mathbb{R})$, we have

$$\begin{aligned} \|\phi'\|_\infty &\leq \frac{1}{d(\mu^+ - \mu^-)} \left[\mu^+ \int_\xi^{+\infty} e^{\mu^+(\xi - y)} - \mu^- \int_{-\infty}^\xi e^{\mu^-(\xi - y)} \right] |G[\phi](y)| dy \\ &\leq \frac{\|G[\phi]\|_\infty}{d(\mu^+ - \mu^-)} \left[\mu^+ \int_\xi^{+\infty} e^{\mu^+(\xi - y)} dy - \mu^- \int_{-\infty}^\xi e^{\mu^-(\xi - y)} dy \right] \\ &= \frac{2\|G[\phi]\|_\infty}{d(\mu^+ - \mu^-)}, \end{aligned}$$

and then $\phi \in C_b^1(\mathbb{R})$ holds. Furthermore, we immediately deduce $\phi \in C_b^2(\mathbb{R})$ from the fact that ϕ satisfies (TW1).

When $f \in C^k(\mathbb{R})$, we have $\phi \in C_b^{k+2}(\mathbb{R})$ by differentiating (TW1) with respect to ξ. □

In the case that the integral kernel has negative parts, the equilibrium $u = 1$ of the equation (2.1) is not always stable in the sense of PDE. Hence, to obtain the limit of the right hand side tail of the wave profile ϕ, we perform the analysis based on the L^2 estimate with reference to [1]. We first obtain the following lemma.

Lemma 2.21. *Let* $(c, \phi) \in \mathbb{R} \times C_b^2(\mathbb{R})$ *be a solution of* (TW1). *Assume that* $c > c_K$. *Then, we have* $\phi' \in L^2(\mathbb{R})$ *and* $\phi'(+\infty) = 0$.

Proof. Let

$$F(u) := \int_0^u f(s) ds, \ M := ||\phi||_\infty, \ M' := ||\phi'||_\infty,$$

$$M_f := \max_{u \in [-M, M]} |F(u)|, \ m_j := \int_\mathbb{R} z^j |K(z)| dz \ (j = 0, 1, 2)$$

Then, we remark that $c_K = \sqrt{m_0 m_2}$. Multiplying ϕ' by (TW1) and then integrating from $-q < 0$ to $p > 0$, we obtain

$$
\begin{aligned}
0 \le c\int_{-q}^{p} (\phi')^2(\xi)d\xi &= \int_{-q}^{p} \{\phi'[d\phi'' + (K*\phi - \alpha\phi) + f(\phi)]\}(\xi)d\xi \\
&= \int_{-q}^{p} \left[\frac{d}{2}\{(\phi')^2\}' + \phi'(K*\phi - \alpha\phi) + (F(\phi))'\right](\xi)d\xi \\
&\le \left(\int_{-q}^{p} (\phi')^2(\xi)d\xi\right)^{1/2} \left(\int_{-q}^{p} \{(K*\phi - \alpha\phi)(\xi)\}^2 d\xi\right)^{1/2} + dM' + 2M_f.
\end{aligned}
$$

For $\xi \in \mathbb{R}$, we have

$$
\begin{aligned}
(K*\phi - \alpha\phi)(\xi) &= \int_{\mathbb{R}} K(\xi - y)(\phi(y) - \phi(\xi))dy \\
&= \int_{\mathbb{R}}\int_0^1 K(\xi - y)(y - \xi)\phi'(\xi + s(y - \xi))dsdy.
\end{aligned}
$$

From the Cauchy-Schwarz inequality, we deduce

$$
\begin{aligned}
&\{(K*\phi - \alpha\phi)(\xi)\}^2 \\
\le\ & \left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)(y-\xi)\phi'(\xi + s(y-\xi))|dsdy\right)^2 \\
\le\ & \left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)|(y-\xi)^2 dsdy\right)\left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)|[\phi'(\xi + s(y-\xi))]^2 dsdy\right) \\
=\ & m_2\left(\int_{\mathbb{R}}\int_0^1 |K(z)|(\phi'(\xi + sz))^2 dsdz\right).
\end{aligned}
$$

Thus, it follows

$$
\begin{aligned}
\int_{-q}^{p} \{[(K*\phi) - \alpha\phi](\xi)\}^2 d\xi &\le m_2 \int_{-q}^{p}\int_{\mathbb{R}}\int_0^1 |K(z)|[\phi'(\xi + sz)]^2 dsdzd\xi \\
&= m_2 \int_{\mathbb{R}}\int_0^1 |K(z)| \int_{-q}^{p} [\phi'(\xi + sz)]^2 d\xi dsdz \\
&= m_2 \int_{\mathbb{R}}\int_0^1 |K(z)| \int_{-q+sz}^{p+sz} [\phi'(\xi)]^2 d\xi dsdz.
\end{aligned}
$$

Here, by using $|\phi'| \le M'$, the following inequality is obtained:

$$
\begin{aligned}
\int_{-q+sz}^{p+sz} [\phi'(\xi)]^2 d\xi &= \int_{-q+sz}^{-q} [\phi'(\xi)]^2 d\xi + \int_{-q}^{p} [\phi'(\xi)]^2 d\xi + \int_{p}^{p+sz} [\phi'(\xi)]^2 d\xi \\
&\le \int_{-q}^{p} [\phi'(\xi)]^2 d\xi + 2(M')^2 s|z|, \ \forall\, s \in [0,1],\ z \in \mathbb{R}.
\end{aligned}
$$

Thus, we deduce

$$\int_{-q}^{p} \{[(K*\phi) - \alpha\phi](\xi)\}^2 d\xi \le m_2 \left\{ m_0 \int_{-q}^{p} (\phi')^2 + (M')^2 m_1 \right\}$$

holds. Therefore, we obtain

$$c\int_{-q}^{p} (\phi')^2 \quad \le \quad \sqrt{m_2} \left\{ m_0 \left(\int_{-q}^{p} (\phi')^2 \right)^2 + (M')^2 m_1 \left(\int_{-q}^{p} (\phi')^2 \right) \right\}^{1/2} + dM' + 2M_f.$$

If $c > c_K = \sqrt{m_0 m_2}$, then $\{\int_{-q}^{p} (\phi')^2 \mid p > 0,\, q > 0\}$ is uniformly bounded. Thus, $\phi' \in L^2(\mathbb{R})$ holds. Since ϕ' is uniform continuous on $\mathbb{R}$, we obtain $\phi'(+\infty) = 0$. □

Proposition 2.22. *Let* $(c, \phi) \in \mathbb{R} \times C_b^2(\mathbb{R})$ *be a solution of* (TW1). *Define the set* $\mathcal{B} \subset \mathbb{R}$ *as*

$$\mathcal{B} := \left\{ u \in \left[\inf_{\xi \in \mathbb{R}} \phi(\xi),\ \sup_{\xi \in \mathbb{R}} \phi(\xi) \right] \mid f(u) = 0 \right\}.$$

If $\mathcal{B}$ *is a discrete set, then for any wave profile* ϕ *satisfying* $c > c_K$*, there exists* $\phi(+\infty)$ *such that* $\phi(+\infty) \in \mathcal{B}$*.*

Proof. Let X be the set of all accumulation points of ϕ as $\xi \to +\infty$. We note that X is not empty, because ϕ is bounded and continuous, We fix an arbitrary $l \in X$. Then, there exists a sequence $\{\xi_n\}$ satisfying $\xi_n \to +\infty$ and $\phi(\xi_n) \to l$ $(n \to \infty)$. By setting $\psi_n(\xi) := \phi(\xi + \xi_n)$, we have

$$c\psi_n'(\xi) = d\psi_n''(\xi) + (K*\psi_n)(\xi) - \alpha\psi_n(\xi) + f(\psi_n(\xi)),\ \xi \in \mathbb{R}.$$

We remark that for any $L > 0$ and $1 < p < \infty$, $\{\psi_n\}$ is bounded on $W^{2,p}([-L, L])$. From the Sobolev's embedding theorem, there exists a subsequence $\{\psi_{n(k)}\} \subset \{\psi_n\}$ such that $\psi_{n(k)} \to \psi$ $(k \to \infty)$ on $C_{loc}^1(\mathbb{R})$ strongly and on $W_{loc}^{2,p}(\mathbb{R})$ weakly. Therefore, ψ satisfies

$$c\psi'(\xi) = d\psi''(\xi) + (K*\psi)(\xi) - \alpha\psi(\xi) + f(\psi(\xi)),\ \xi \in \mathbb{R},$$

From Lemma 2.21, we have

$$\psi'(\xi) = \lim_{k \to \infty} \phi'(\xi + \xi_{n(k)}) = 0,\ \forall\, \xi \in \mathbb{R}$$

and thus obtain $\psi(\xi) \equiv l$. Thus, $l \in \mathcal{B}$ and $X \subset \mathcal{B}$ hold. Since ϕ is continuous, X is connect. Thus, X is a discrete set because $\mathcal{B}$ is a discrete set. Therefore, $\phi(+\infty)$ exists and belongs to $\mathcal{B}$. The proposition is proved. □

From Proposition2.22, it is shown that sufficiently fast traveling wave solutions converge to an equilibrium at infinity. In particular, in the case of Theorem 2.7, the fact that $\liminf_{\xi \to +\infty} \phi(\xi)$ is positive immediately shows that $\phi(+\infty)$ exists and $\phi(+\infty) = 1$. Thus, the latter part of Theorem 2.7 is also shown.

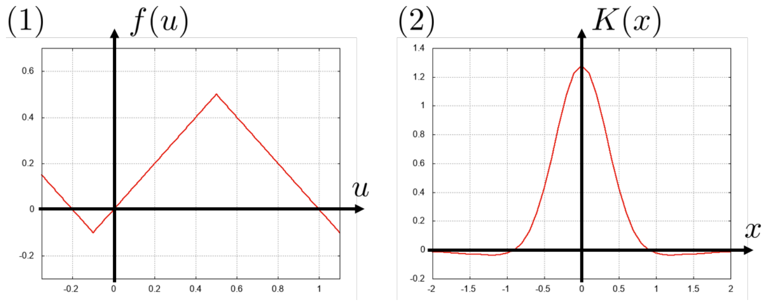

Figure 3: (1) The graph of the nonlinear term (2.19) when $a = 0.2$. (2) The graph of the integral kernel (2.20) when $B_1 = 2.4$, $b_1 = 4.0$, $B_2 = 0.4/3.0$, $b_2 = 4.0/9.0$. (This figure is taken from my master's thesis [45].)

2.7 Numerical simulations

In this subsection, we introduce the results of numerical simulation to the equation (2.1). To observe the traveling wave solutions obtained in main results, we first consider the case that the nonlinear term $f(u)$ is given by

$$f(u) = \begin{cases} 1 - u, & u \in \left(\frac{1}{2}, +\infty\right), \\ u, & u \in \left[-\frac{a}{2}, \frac{1}{2}\right], \\ -a - u, & u \in \left(-\infty, -\frac{a}{2}\right), \end{cases} \tag{2.19}$$

where $a > 0$. Moreover, we give the integral kernel $K(x)$ as

$$K(x) = \frac{1}{\sqrt{\pi}}(B_1 e^{-b_1 x^2} - B_2 e^{-b_2 x^2}), \tag{2.20}$$

where B_1, B_2 and b_1, b_2 are positive constants. In this case, we have $\alpha = B_1/\sqrt{b_1} - B_2/\sqrt{b_2}$. The interval of numerical simulations is denoted by $\Omega := (0, L)$ with $L > 0$.

Since $K * u$ is a nonlocal term, boundary conditions need to be extended outside the domain. Thus, to observe the traveling wave solutions satisfying (TW2), we impose the condition

$$u(t, x) = \begin{cases} 1, & x \geq L, \\ 0, & x \leq 0 \end{cases} \tag{2.21}$$

to the numerical simulation.

We first observe the traveling wave solution (Fig. 4), when the initial function is given by

$$u(0, x) = \begin{cases} 1, & x > L_0, \\ e^{\mu(x - L_0)}, & x \leq L_0 \end{cases} \tag{2.22}$$

in which $L_0 \in (0, L)$. From the result of numerical simulations, we expect that the traveling wave solutions are stable in some sense. Next, the result of the numerical

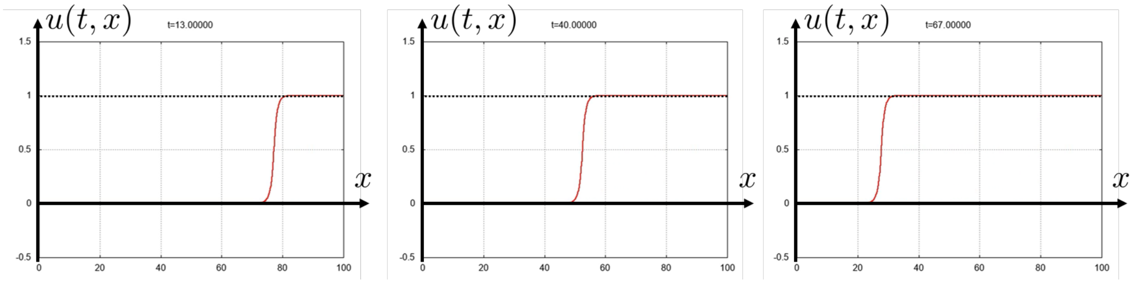

Figure 4: The result of numerical simulation when the initial function is (2.22). Here, we give $d = 0.01$ and $\mu = 1.0$. From left to right: $t = 13.0$, $t = 40.0$, and $t = 67.0$. (This figure is taken from my master's thesis [45].)

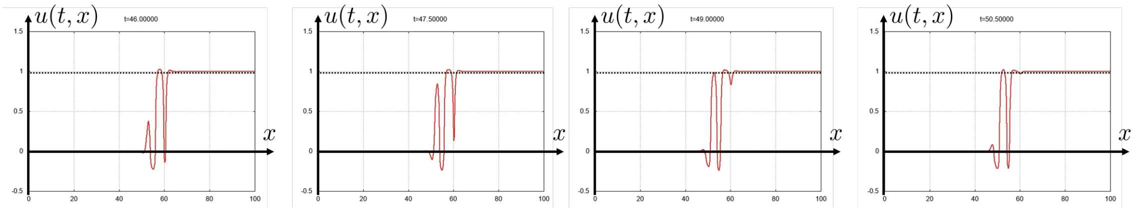

Figure 5: The result of numerical simulation when the initial function is (2.23). Here, we give $d = 0.01$. From left to right: $t = 46.0$, $t = 47.5$, $t = 49.0$, and $t = 50.5$. (This figure is taken from my master's thesis [45].)

simulation is Fig. 5, when the intial function is given by

$$u(0, x) = \begin{cases} 1, & x > L_0, \\ 0, & x \leq L_0. \end{cases} \tag{2.23}$$

Then, periodic traveling wave solutions, where the shape of the solution changes periodically, can be observed instead of traveling wave solutions. From this numerical result, it can be deduced that the equation (2.1) has a sign-changing solution when the integral kernel changes sign. Furthermore, when the nonlinear term is changed, a traveling wave solution with a peak near $u = 1$ appears, as seen in Figure 6. This suggests the possibility that there are situations where non-monotonic traveling wave solutions exist. From these numerical simulations, we can understand that the behavior of the solutions to the equation (2.1) is diverse when $K(x)$ changes sign.

2.8 Comparison principle in the case of sign-changing integral kernel

The comparison principle is a very useful theory that can evaluate time-evolving solutions from above and below by constructing super-sub solutions. However, it is known that the comparison principle in the usual sense does not hold in general for the equation (2.1) with a sign-changing integral kernel because of the possibility of Turing instability.

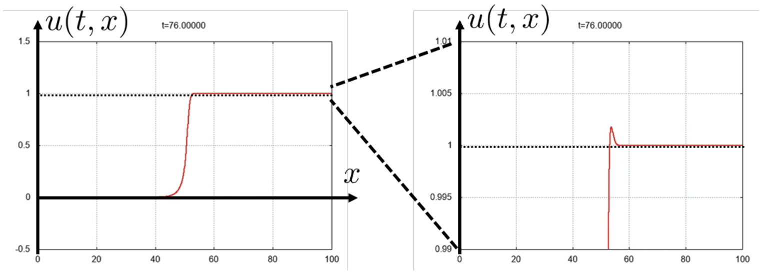

Figure 6: The result of numerical simulation when the initial function is (2.22). Here, we give $d = 0.01$ and $f(u) = u(1-u)(u+0.25)$. The left figure shows the profile at $t = 76.0$, and the right figure is an image magnified near $u = 1$ in the left figure. (This figure is taken from my master's thesis [45].)

Therefore, as a final result of this section, we introduce a new definition of super-sub solution based on the definition of upper-lower solution introduced in [23, 45], and show that a different form of the comparison principle is possible. In particular, only the case $d = 0$ is intrinsically important, and if it can be introduced in this case, it can be extended to the case $d > 0$. Therefore, this subsection deals only with the case $d = 0$.

Let us now introduce a new definition of super-sub solutions.

Definition 2.23. *Let $T \in (0, +\infty)$. A pair of function $\{\overline{u}, \underline{u}\} \subset C^1([0,T]; L^\infty(\mathbb{R}))$ are super-sub solutions of* (2.1) *with $d = 0$ if for any $t \in [0, T]$,*

$$\begin{aligned} \overline{u}_t &\geq (K^+ * \overline{u}) - (K^- * \underline{u}) - \alpha\overline{u} + f(\overline{u}), \\ c\underline{u}_t &\leq (K^+ * \underline{u}) - (K^- * \underline{u}) - \alpha\underline{u} + f(\underline{u}) \end{aligned}$$

holds for almost all $x \in \mathbb{R}$.

Remark 2.24. *In the usual comparison principle, a super-sub solution is defined so that each is a function satisfying a single inequality, and it is common to define a super-sub solution such that a solution to the equation* (2.1) *with $d = 0$ is both a super solution and a sub solution [18, 69]. However, in this definition, the super-sub solutions are not defined by a single inequality for each of them because the integral kernel has a negative part, and the solution of the equation* (2.1) *with $d = 0$ is not always a super solution or a sub solution.*

Remark 2.25. *We note that for $c > 0$, $(\overline{u}(t,x), \underline{u}(t,x)) = (\overline{\phi}(x+ct), \underline{\phi}(x+ct))$ is upper-lower solutions of* (2.1) *with $d = 0$, where $\{\overline{\phi}, \underline{\phi}\}$ are upper-lower solutions of* (TW1) *with $d = 0$.*

Then, the following theorem is obtained.

Theorem 2.26. *Let $T \in (0, +\infty)$ and $u \in C^1([0,T]; L^\infty(\mathbb{R}))$ be a solution of* (2.1) *with $d = 0$. Let $\{\overline{u}, \underline{u}\} \subset C^1([0,T]; L^\infty(\mathbb{R}))$ be super-sub solutions of* (2.1) *with $d = 0$. If $\underline{u}(0,x) \leq u(0,x) \leq \overline{u}(0,x)$ holds almost all $x \in \mathbb{R}$, then for any $t \in [0,T]$, $\underline{u}(t,x) \leq u(t,x) \leq \overline{u}(t,x)$ holds almost all $x \in \mathbb{R}$.*

Proof. The proof is based on the method of [69] and uses the theory of ordinary differential equations on $L^\infty(\mathbb{R})$.

Let $\overline{v}, \underline{v} \in C^1([0,T]; L^\infty(\mathbb{R}))$ be

$$\overline{v}(t,x) := e^{\zeta t}(\overline{u} - u)(t,x), \quad \underline{v}(t,x) := e^{\zeta t}(u - \underline{u})(t,x),$$

where ζ is a positive constant determined later. Since $u(t,x)$ is a solution of (2.1) with $d = 0$, we have

$$\begin{aligned}
\overline{v}_t &= K^+ * \overline{v} + K^- * \underline{v} + (\zeta - \alpha)\overline{v} + e^{\zeta t}(f(u + e^{-\zeta t}\overline{v}) - f(u)) + e^{\zeta t} a_1(t,x) \\
&=: F_1(t,x,\overline{v},\underline{v}), \\
\underline{v}_t &= K^+ * \underline{v} + K^- * \overline{v} + (\zeta - \alpha)\underline{v} + e^{\zeta t}(f(u) - f(u - e^{-\zeta t}\underline{v})) + e^{\zeta t} a_2(t,x) \\
&=: F_2(t,x,\overline{v},\underline{v}),
\end{aligned}$$

where

$$\begin{aligned}
a_1(t,x) &= \overline{u}_t - (K^+ * \overline{u} - K^- * \underline{u} - \alpha\overline{u} + f(\overline{u})), \\
a_2(t,x) &= -\{\underline{u}_t - (K^+ * \underline{u} - K^- * \overline{u} - \alpha\underline{u} + f(\underline{u})).
\end{aligned}$$

We remark that for any $t \in (0,T]$, a_1 and a_2 are non-negative for almost all $x \in \mathbb{R}$. Let $\overline{w}, \underline{w} \in C^1([0,T]; L^\infty(\mathbb{R}))$ be solutions of integral equations

$$\begin{aligned}
\overline{w}(t,x) &= \overline{v}(0,x) + \int_0^t \max\{F_1(s,x,\overline{w}(s,x),\underline{w}(s,x)), 0\} ds, \\
\underline{w}(t,x) &= \underline{v}(0,x) + \int_0^t \max\{F_2(s,x,\overline{w}(s,x),\underline{w}(s,x)), 0\} ds.
\end{aligned}$$

Thus, for any $t \in (0,T]$, we obtain

$$\overline{w}(t,x) \geq \overline{v}(0,x) \geq 0, \quad \underline{w}(t,x) \geq \underline{v}(0,x) \geq 0$$

for almost all $x \in \mathbb{R}$. Now, we deduce that $F_j(t,x,\overline{w}(t,x),\underline{w}(t,x)) \geq 0$ $(j = 1,2)$ hold for any $t \in (0,T]$ and almost all $x \in \mathbb{R}$ by giving ζ sufficiently large such that

$$(\zeta - \alpha)\overline{v} + e^{\zeta t}(f(u + e^{-\zeta t}\overline{v}) - f(u)) \geq 0$$

and

$$(\zeta - \alpha)\underline{v} + e^{\zeta t}(f(u) - f(u - e^{-\zeta t}\underline{v})) \geq 0$$

hold for any $t \in (0,T]$ and almost all $x \in \mathbb{R}$. This means that $\overline{v}(t,x) \equiv \overline{w}(t,x)$ and $\underline{v}(t,x) \equiv \underline{w}(t,x)$ hold for any $t \in [0,T]$ and almost all x, because they satisfy same equations. Therefore, the theorem is proved. □

Theorem 2.26 implies that if we can construct an upper-lower solution of (TW1) with $d = 0$ that satisfies the assumptions of Proposition 2.15, then a time-evolving solution like a traveling wave solution will appear when the initial values are given in between.

Remark 2.27. *In the case that $d > 0$, a comparison principle of a similar form can also be constructed based on the argument [7, 12]. However, if we want to show based on their proof, we need to introduce a moving coordinate into the equation* (2.1) *and discuss it to make $\{\overline{u}(t,x), \underline{u}(t,x)\} = \{\overline{\phi}(x+ct), \underline{\phi}(x+ct)$ a super-sub solution of* (2.1) *using the upper-lower solutions $\{\overline{u}, \underline{u}\}$ of* (TW1)*. The results are not presented in this thesis, as it would complicate the discussion a bit.*

2.9 Summary

In this study, we have developed a method to show the existence of traveling wave solutions for nonlocal semilinear scalar equations with a sign-changing integral kernel. We were able to show the existence of traveling wave solutions in a particular case. These ideas can be applied to other mathematical models with a sign-changing integral kernel. For example, the problem of showing the existence of traveling wave solutions can be attributed to the problem of constructing upper-lower solutions. In this study, we considered a time evolution equation with a monostable nonlinear term, but it is also expected to be applied to mathematical models in which the nonlinear term is bistable [8, 60] or has a convolution term with a nonlinear term [2, 71]. Furthermore, it was shown that traveling wave solutions with sufficiently large speeds converge to a constant at infinity, regardless of the stability of an equilibrium. In [1], which was used as a reference in the proof, it was applied to the traveling wave solution in the equation containing the convolution in the nonlinear term, and the evaluation depended on the L^2 norm of the derivative of the traveling wave solution. In the present study, we did not include any nonlinear effects on the nonlocal terms, and thus obtained relatively clean evaluations that do not depend on the quantitative properties of wave profoles.

In addition, numerical calculations showed that two characteristic spatio-temporal patterns emerged depending on the initial value: traveling wave solution and periodic traveling wave solution. From this result, it is expected that the characteristic pattern appears stable in some sense, depending on the initial value. The comparison principle based on the newly defined super-sub solution may reveal the stability of the traveling wave solution in some sense, but this is a challenge for the future.

3 Weak interaction of localized patterns for reaction-diffusion equations with nonlocal effect

The contents from Subsection 3.2 to Subsection 3.4 are based on the paper [24] which is joint research with Professor Shin-Ichiro Ei. The contents in Subsection 3.5 and Subsection 3.6 are introduced the extended results of [24]. The results are new in this thesis.

3.1 Background and motivation

Pattern formation problems for reaction-diffusion equations with nonlocal effects are challenging to analyze because the nonlocality of the equations makes it impossible to immediately apply ordinary methods for reaction-diffusion equations such as phase-plane analysis and variational methods. For the analysis of the reaction-diffusion equation with nonlocal effect, there have been many works [6, 7, 8, 9, 12, 13, 16, 18, 23, 58, 73] by using monotone semiflow, implicit function theorem, fixed point theorem, and so on. In particular, the existence and the stability of pulse solutions and front solutions as localized patterns have been extensively investigated [6, 7, 8, 9, 12, 13, 16]. A single pulse or a single front solution was constructed together to consider their stability in their works. From the perspective of pattern formation, it is essential to analyze not only the existence and stability of localized patterns, but also the temporal changes caused by the interaction of localized patterns. In the case of reaction-diffusion equations, Ei [22] has proposed a method to analyze the weak interaction between localized patterns, but the case with nonlocal effect is still an unexplored topic. Mainly, how nonlocal effects affect weak interactions is an exciting topic, and its analysis is essential from the perspective of pattern formation problems.

The purpose of this study is to consider weak interactions between localized patterns for reaction-diffusion equations with nonlocal effect. We first introduce the problem setup and then explain the main results and applications based on the results of [24]. In particular, it is considered in [24] the case where the nonlocal effect is given by a linear convolution such as $K * u$.

Later, in this thesis, we developed the theory of [24] to the case where the nonlocal effect is a nonlinear convolution such as $f(K * g(u))$. The results are also explained. After describing some applications, we summarize our work.

3.2 Setting

In this thesis, we treat the following multi-component reaction-diffusion equations with nonlocal effect:

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + \boldsymbol{K} * \boldsymbol{u} + F(\boldsymbol{u}),\ t > 0,\ x \in \mathbb{R}, \tag{3.1}$$

where $\boldsymbol{u} = \boldsymbol{u}(t,x) = {}^t(u_1(t,x), u_2(t,x), \cdots, u_n(t,x)) \in \mathbb{R}^n$, $D = diag(d_1, d_2, ..., d_n)$ $(d_j \geq 0)$, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function, $\boldsymbol{K} = \boldsymbol{K}(x) \in \mathbb{R}^{n\times n}$,

$$
\begin{aligned}
(\boldsymbol{K} * \boldsymbol{u})(t,x) &:= \begin{pmatrix} K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,n} \\ K_{2,1} & K_{2,2} & \cdots & \cdots & K_{2,n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ K_{n,1} & K_{n,2} & \cdots & \cdots & K_{n,n} \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} (t,x) \\
&= \begin{pmatrix} \sum_{k=1}^{n} (K_{1,k} * u_k)(t,x) \\ \sum_{k=1}^{n} (K_{2,k} * u_k)(t,x) \\ \vdots \\ \vdots \\ \sum_{k=1}^{n} (K_{n,k} * u_k)(t,x) \end{pmatrix},
\end{aligned}
$$

the $*$ denotes the convolution with respect to the spatial variable, in which the integral kernels $K_{j,k}$ are the functions satisfying

$$
\begin{cases} K_{j,k} \in C(\mathbb{R}) \cap L^1(\mathbb{R}), \ K_{j,k}(x) = K_{j,k}(-x) \ (x \in \mathbb{R}), \\ \forall \lambda \in \mathbb{R}, \quad \int_{\mathbb{R}} |K_{j,k}(y)| e^{\lambda y} dy < \infty. \end{cases} \tag{3.2}
$$

A typical example of $K_{j,k}$ is $K_{j,k}(x) = e^{-x^2}$. The first purpose of this thesis is to give a mathematical criteria for the interaction between multiple pulse or front solutions for (3.1).

We set

$$
A(\lambda) := \begin{pmatrix} \tilde{K}_{1,1}(\lambda) & \tilde{K}_{1,2}(\lambda) & \cdots & \cdots & \tilde{K}_{1,n}(\lambda) \\ \tilde{K}_{2,1}(\lambda) & \tilde{K}_{2,2}(\lambda) & \cdots & \cdots & \tilde{K}_{2,n}(\lambda) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \tilde{K}_{n,1}(\lambda) & \tilde{K}_{n,2}(\lambda) & \cdots & \cdots & \tilde{K}_{n,n}(\lambda) \end{pmatrix},
$$

where $\tilde{K}_{j,k}(\lambda) := \int_{\mathbb{R}} K_{j,k}(y) e^{\lambda y} dy$ for $j,\ k = 1, 2, \ldots, N$.

In this study, we impose the following assumptions on (3.1)

Assumption 3.1. *We suppose for* (3.1) *that:*

H1) [Existence of stable equilibria]
There exist linearly stable equilibria P^- and P^+ in the ODE

$$
\boldsymbol{u}_t = A(0)\boldsymbol{u} + F(\boldsymbol{u}).
$$

H2) [Existence of traveling wave solutions]
There exist a constant θ, positive constants α, β and a function $P(z)$ satisfying the equation

$$\begin{cases} \mathbf{0} = DP_{zz} - \theta P_z + \boldsymbol{K} * P + F(P) \quad (z \in \mathbb{R}), \\ |P(z) - P^+| \leq O(e^{-\alpha z}) \ (z \to +\infty), \\ |P(z) - P^-| \leq O(e^{\beta z}) \ (z \to -\infty). \end{cases} \tag{3.3}$$

H3) [Linearized stability of traveling wave solutions]
Let a differential operator L be

$$L\boldsymbol{v} = D\boldsymbol{v}_{zz} - \theta \boldsymbol{v}_z + \boldsymbol{K} * \boldsymbol{v} + F'(P(z))\boldsymbol{v},$$

for $\boldsymbol{v} \in \{H^2(\mathbb{R})\}^n$, where the domain $\mathcal{D}(L)$ is defined as

$$\mathcal{D}(L) = \{\boldsymbol{v} = {}^t(v_1, v_2, \cdots, v_n) \in \{L^2(\mathbb{R})\}^n \mid \theta \boldsymbol{v} \in \{H^1(\mathbb{R})\}^n, \ D\boldsymbol{v} \in \{H^2(\mathbb{R})\}^n\}.$$

This means that

$$v_j \in \begin{cases} H^2(\mathbb{R}) & (if \ d_j > 0), \\ H^1(\mathbb{R}) & (if \ d_j = 0 \ and \ \theta \neq 0), \\ L^2(\mathbb{R}) & (if \ d_j = 0 \ and \ \theta = 0) \end{cases}$$

for any $j = 1, 2, \ldots, n$ if $\boldsymbol{v} \in \mathcal{D}(L)$. Then, the spectrum $\Sigma(L)$ of L is given by $\Sigma(L) = \Sigma_0 \cup \{0\}$, where 0 is a simple eigenvalue with an eigenfunction P_z and there exists a positive constant $\rho_0 > 0$ such that $\Sigma_0 \subset \{z \in \mathbb{C} \mid \Re(z) < -\rho_0\}$. Here, $\Re(z)$ denotes the real part of z.

We call $P(z)$ satisfying the Hypothesis 3.1 for a constant θ "(linearly) stable traveling wave solution with velocity θ". Many models of reaction-diffusion systems and nonlocal equations have linearly stable traveling wave solutions in this sense [7, 19, 28, 29, 48, 70, 73].

Transforming (3.1) by $z := x + \theta t$, we have

$$\boldsymbol{u}_t = D\boldsymbol{u}_{zz} - \theta \boldsymbol{u}_z + \boldsymbol{K} * \boldsymbol{u} + F(\boldsymbol{u}) =: \mathcal{L}(\boldsymbol{u}). \tag{3.4}$$

We note that the stable traveling wave solution $P(z)$ is a stable stationary solution of (3.4). Throughout this thesis, we call $P(z)$ "pulse solution" when $P^- = P^+$ and "front solution" when $P^- \neq P^+$, respectively.

The purpose of this thesis is to give a general criterion for (3.1) to analyze their interaction together with applications under the above assumptions about the existence and the stability of a single traveling wave solution.

3.3 Main results in the case of linear nonlocal effect

We present results on the weak interaction of the pulse solutions and the front solutions. The proofs are given in Appendix A.

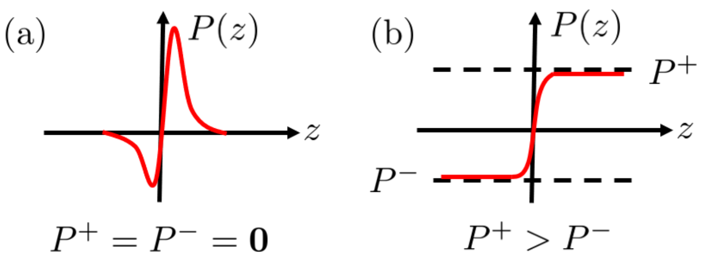

Figure 7: The images of localized patterns. (a) Pulse solution when $P^+ = P^- = \mathbf{0}$. (b) Front solution when $n = 1$ and $P^+ > P^-$. (This figure is taken from [24].)

3.3.1 Interaction of pulse solutions

Let us consider the interaction of pulse solutions. Suppose that $P(z)$ is a stable pulse solution of (3.1) with velocity θ. Then, we can assume that $P^- = P^+ = \mathbf{0} = {}^t(0, 0, \cdots, 0) \in \mathbb{R}^n$ without loss of generality. Fixing an arbitrarily natural number N, we consider the interaction of $N+1$ pulse solutions. We define

$$P(z; \boldsymbol{h}) := P(z) + P(z - z_1) + \cdots + P(z - z_N),$$

where $\boldsymbol{h} = (h_1, h_2, ..., h_N)$ for $h_j > 0$, $z_0 = 0$ and

$$z_j = z_j(\boldsymbol{h}) = z_{j-1} + h_j \quad (j = 1, 2, ..., N).$$

Define the set

$$\mathcal{M}(h^*) = \{\Xi(l)P(z; \boldsymbol{h}) \mid l \in \mathbb{R},\ \min \boldsymbol{h} > h^*\},$$

where $\Xi(l)$ is the translation operator defined as $(\Xi(l)\boldsymbol{v})(z) = \boldsymbol{v}(z - l)$ for $\boldsymbol{v} \in \{L^2(\mathbb{R})\}^n$.

Moreover, we set the quantity

$$\delta(\boldsymbol{h}) = \sup_{z \in \mathbb{R}} |\mathcal{L}(P(z; \boldsymbol{h}))|.$$

We note that $\delta(\boldsymbol{h})$ is sufficiently small as long as $\min \boldsymbol{h}$ is large enough. In fact, $\delta(\boldsymbol{h})$ satisfies $\delta(\boldsymbol{h}) \to 0$ as $\min \boldsymbol{h} \to +\infty$, since $\mathcal{L}(P(z - z_j)) = \mathbf{0}$ and $\mathcal{L}(\mathbf{0}) = \mathbf{0}$ for $j = 0, 1, \ldots, N$.

Furthermore, define functions

$$H_j(\boldsymbol{h}) = \langle\, \mathcal{L}(P(\cdot + z_j; \boldsymbol{h})), \Phi^*(\cdot)\, \rangle_{L^2}$$

for $j = 0, 1, ..., N$, where Φ^* is an eigenfunction corresponding to 0 eigenvalue of the adjoint operator L^* of L and normalized by $\langle\, P_z, \Phi^*\, \rangle_{L^2} = 1$. We note that the domain $\mathcal{D}(L^*)$ is equal to $\mathcal{D}(L)$ and Φ^* satisfies

$$L^*\Phi^* := D\Phi^*_{zz} + \theta\Phi^*_z + {}^t\boldsymbol{K} * \Phi^* + {}^tF'(P(z))\Phi^* = 0. \tag{3.5}$$

By applying the same line of argument in [22] based on the theory of infinite dimensional dynamical systems, we obtain the following results.

Theorem 3.2. [22] *There exist positive constants* h^*, C_0 *and a neighborhood* $U = U(h^*)$ *of* $M(h^*)$ *in* $\{H^2(\mathbb{R})\}^n$ *such that if* $\boldsymbol{u}(0, \cdot) \in U$, *then there exist functions* $l(t) \in \mathbb{R}$ *and* $\boldsymbol{h}(t) \in \mathbb{R}^N$ *such that*

$$\|\boldsymbol{u}(t, \cdot) - \Xi(l(t))P(\cdot; \boldsymbol{h}(t))\|_\infty \leq C_0 \delta(\boldsymbol{h}(t)) \tag{3.6}$$

holds as long as $\min \boldsymbol{h}(t) > h^*$, *where* $\boldsymbol{u}(t, z)$ *is a solution of* (3.4) *and* $\|\cdot\|_\infty$ *is the sup-norm on* $\mathbb{R}$. *Functions* $l(t) \in \mathbb{R}$ *and* $\boldsymbol{h}(t) \in \mathbb{R}^N$ *satisfy*

$$\dot{\boldsymbol{h}} = \boldsymbol{H}(\boldsymbol{h}) + O(\delta^2), \tag{3.7}$$

$$\dot{l} = -H_0(\boldsymbol{h}) + O(\delta^2), \tag{3.8}$$

where $\delta = \delta(\boldsymbol{h}(t))$ *and* $\boldsymbol{H} = (H_0 - H_1, H_1 - H_2, \cdots, H_{N-1} - H_N)$.

Theorem 3.3. [22] *Suppose all of the elements* d_j *of* D *are positive. Then, there exist positive constants* C_0, C_1 *and* h^* *such that if*

$$\dot{\boldsymbol{h}} = \boldsymbol{H}(\boldsymbol{h}) \tag{3.9}$$

has an equilblium $\overline{\boldsymbol{h}}$ *satisfying* $\min \overline{\boldsymbol{h}} > h^*$ *and the set of eigenvalues* $\Sigma(\boldsymbol{H}'(\overline{\boldsymbol{h}})) \subset \{z \in \mathbb{C} \mid \Re(z) < -C_0\delta(\overline{\boldsymbol{h}})\}$, *there exists a stable traveling wave solution* $\overline{P}(z + \overline{\theta}t)$ *of* (3.1) *such that*

$$\|\overline{P}(\cdot) - P(\cdot; \overline{\boldsymbol{h}})\|_\infty \leq C_1 \delta(\overline{\boldsymbol{h}})$$

and $\overline{\theta} = H_0(\overline{\boldsymbol{h}}) + O(\delta^2(\overline{\boldsymbol{h}}))$. *Here,* $\boldsymbol{H}'(\overline{\boldsymbol{h}})$ *denotes the linearized matrix of* $\boldsymbol{H}$ *with respect to* $\overline{\boldsymbol{h}}$.

If (3.9) *has an equilblium* $\underline{\boldsymbol{h}}$ *such that* $\min \underline{\boldsymbol{h}} > h^*$ *and the set of eigenvalues* $\Sigma(H'(\underline{\boldsymbol{h}})) \subset \{z \in \mathbb{C} \mid \Re(z) < -C_0\delta(\underline{\boldsymbol{h}})\} \cup \{z \in \mathbb{C} \mid \Re(z) > C_0\delta(\underline{\boldsymbol{h}})\}$ *and at least one eigenvalue of* $H'(\underline{\boldsymbol{h}})$ *is in* $\{z \in \mathbb{C} \mid \Re(z) > C_0\delta(\underline{\boldsymbol{h}})\}$, *there exists an unstable traveling wave solution* $\underline{P}(z + \underline{\theta}t)$ *of* (3.1) *such that*

$$\|\underline{P}(\cdot) - P(\cdot; \underline{\boldsymbol{h}})\|_\infty \leq C_1 \delta(\underline{\boldsymbol{h}})$$

and $\underline{\theta} = H_0(\underline{\boldsymbol{h}}) + O(\delta^2(\underline{\boldsymbol{h}}))$.

In [22], he constructed an attractive local invariant manifold giving the dynamics of interacting localized patterns in the case of multi-component reaction-diffusion equations. In its proof, the integral manifold theory was used. The proof of [22] can be also applied to multi-component reaction diffusion equations with perturbations given by bounded operators in $\{L^2(\mathbb{R})\}^n$. Now, the nonlocal term $\boldsymbol{K} * \boldsymbol{u}$ is a bounded operator on $\{L^2(\mathbb{R})\}^n$. Therefore, we can extend theorems in [22].

From Theorem 3.2, when the distances between localized patterns are sufficiently large, the motion of localized patterns can be reduced to the equation (3.7) for the distances between them. However, it is difficult to analyze $H_j(\boldsymbol{h})$ directly. When the pulse solution $P(z)$ converges to $\boldsymbol{0}$ in an exponentially monotone way, $H_j(\boldsymbol{h})$ can be represented by the explicit form approximately.

Theorem 3.4. *Suppose $P(z)$ converges to $\mathbf{0}$ satisfying*

$$\begin{aligned} P(z) &= e^{-\alpha z}(\boldsymbol{a}^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ P(z) &= e^{\beta z}(\boldsymbol{a}^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constant vectors $\boldsymbol{a}^\pm \in \mathbb{R}^n$, and suppose $\Phi^*(z)$ also converges to $\mathbf{0}$ in an exponentially monotone way such that

$$\begin{aligned} \Phi^*(z) &= e^{-\beta z}(\boldsymbol{b}^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ \Phi^*(z) &= e^{\alpha z}(\boldsymbol{b}^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for non-zero constant vectors $\boldsymbol{b}^\pm \in \mathbb{R}^n$. Then, functions $H_j(\boldsymbol{h})$ are represented by

$$H_j(\boldsymbol{h}) = (M_\beta e^{-\beta h_{j+1}} + M_\alpha e^{-\alpha h_j})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) \ (j = 1, 2, \cdots, N-1), \tag{3.10}$$

$$H_0(\boldsymbol{h}) = M_\beta e^{-\beta h_1}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), \tag{3.11}$$

$$H_N(\boldsymbol{h}) = M_\alpha e^{-\alpha h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), \tag{3.12}$$

for a constant $\gamma' > 0$ and the constants M_α, M_β are given by

$$M_\alpha = \langle (2\alpha D + \theta I + A'(\alpha))\boldsymbol{a}^+, \boldsymbol{b}^- \rangle, \tag{3.13}$$

$$M_\beta = \langle (2\beta D - \theta I + A'(\beta))\boldsymbol{a}^-, \boldsymbol{b}^+ \rangle, \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$, $I \in \mathbb{R}^{n \times n}$ is the identity matrix and $A'(\lambda) \in \mathbb{R}^{n \times n}$ is the function with respect to λ defined by

$$A'(\lambda) := \begin{pmatrix} \tilde{K}'_{1,1}(\lambda) & \tilde{K}'_{1,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{1,n}(\lambda) \\ \tilde{K}'_{2,1}(\lambda) & \tilde{K}'_{2,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{2,n}(\lambda) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \tilde{K}'_{n,1}(\lambda) & \tilde{K}'_{n,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{n,n}(\lambda) \end{pmatrix}$$

in which

$$\tilde{K}'_{j,k}(\lambda) := \int_{\mathbb{R}} y K_{j,k}(y) e^{\lambda y} dy, \ (j, k = 1, 2, \ldots n).$$

Remark 3.5. *Given a function $G(z) : \mathbb{R} \to \mathbb{R}^n$, we write*

$$G(z) = e^{-\alpha z}(\boldsymbol{a} + O(e^{-\gamma z})) \quad (z \to +\infty)$$

for some positive constants α, γ and a nonzero constant vector $\boldsymbol{a} \in \mathbb{R}^n$ if there exist a positive real number C_0 and a real number C_1 such that

$$|e^{\alpha z} G(z) - \boldsymbol{a}| \le C_0 e^{-\gamma z} \quad (\forall z \ge C_1).$$

We also write

$$G(z) = e^{\alpha z}(\boldsymbol{a} + O(e^{\gamma z})) \quad (z \to -\infty)$$

if there exist a positive real number C_0 and a real number C_1 such that

$$|e^{-\alpha z} G(z) - \boldsymbol{a}| \le C_0 e^{\gamma z} \quad (\forall z \le C_1).$$

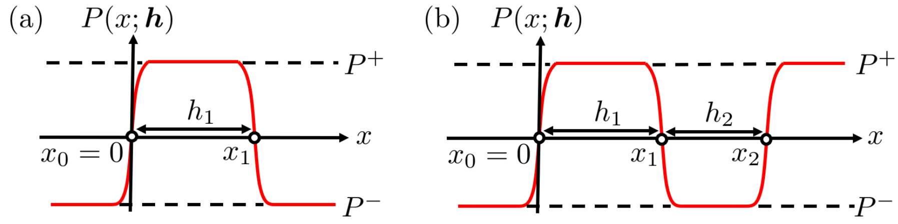

Figure 8: The image of $P(z;\boldsymbol{h})$. (a) $(N^+,N^-)=(1,1)$. (b) $(N^+,N^-)=(2,1)$. (This figure is taken from [24].)

3.3.2 Interaction of front solutions

Next, let us consider the interaction of front solutions. We consider only the case of the velocity $\theta=0$. We use x as the space variable instead of z because $x=z$ in this case. Basically, we use the same notations as in the previous subsection with $\theta=0$.

Suppose that $P(x)$ is a stable front solution of (3.1) with $\theta=0$. We note that $P(-x)$ is also a stable front solution of (3.1) connecting from P^+ to P^-. We define the number of front solutions as $N+1=N^++N^-$, where N^+ and N^- are the numbers of front solutions of the shapes $P(x)$ and $P(-x)$, respectively. We note that either $N^+=N^-$ or $N^+-1=N^-$ holds. Then, $N+1$ front solutions $P(x;\boldsymbol{h})$ are defined as

$$\begin{aligned} P(x;\boldsymbol{h}) &= P(x)+P(-(x-x_1))+P(x-x_2)+\cdots \\ &\qquad +P((-1)^N(x-x_N))-\{N^+P^+ +(N^- -1)P^-\} \end{aligned}$$

if $N^+=N^-$,

$$\begin{aligned} P(x;\boldsymbol{h}) &= P(x)+P(-(x-x_1))+P(x-x_2)+\cdots \\ &\qquad +P((-1)^N(x-x_N))-\{(N^+-1)P^+ +N^-P^-\} \end{aligned}$$

if $N^+-1=N^-$, where $\boldsymbol{h}=(h_1,h_2,\cdots,h_N)\in\mathbb{R}^N$, $x_j=\sum_{k=1}^{k=j}h_k$ for $j=1,2,\cdots,N$ and $x_0=0$. Moreover, we define functions $H_j(\boldsymbol{h})$ $(j=0,1,\cdots,N)$ by

$$H_j(\boldsymbol{h})=\langle\mathcal{L}(P(x+x_j;\boldsymbol{h})),\Phi^*((-1)^jx)\rangle_{L^2}.$$

By applying the same line of argument in [22], we can also obtain the following result.

Theorem 3.6. [22] *Theorems 3.2 and 3.3 hold in the same statements but*

$$\dot{h}_j = (-1)^{j+1}(H_{j-1}(\boldsymbol{h})+H_j(\boldsymbol{h}))+O(\delta^2)\ (j=1,2,\cdots,N), \tag{3.15}$$
$$\dot{l} = -H_0(\boldsymbol{h})+O(\delta^2), \tag{3.16}$$

and

$$\boldsymbol{H}=(H_0+H_1,-(H_1+H_2),\cdots,(-1)^{N+1}(H_{N-1}+H_N)).$$

Same as Theorem 3.4, $H_j(\boldsymbol{h})$ can be represented by the explicit form approximately, when the front solution $P(x)$ converges to $P^{\pm}$ in an exponentially monotone way.

Theorem 3.7. *Suppose $P(x)$ converges to $P^{\pm}$ as*

$$\begin{aligned} P(x) - P^{+} &= e^{-\alpha x}(\boldsymbol{a}^{+} + O(e^{-\gamma x})) \quad (x \to +\infty), \\ P(x) - P^{-} &= e^{\beta x}(\boldsymbol{a}^{-} + O(e^{\gamma x})) \quad (x \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constant vectors $\boldsymbol{a}^{\pm} \in \mathbb{R}^n$, and suppose $\Phi^*(x)$ converges to $\mathbf{0}$ in an exponentially monotone way such that

$$\begin{aligned} \Phi^*(x) &= e^{-\alpha x}(\boldsymbol{b}^{+} + O(e^{-\gamma x})) \quad (x \to +\infty), \\ \Phi^*(x) &= e^{\beta x}(\boldsymbol{b}^{-} + O(e^{\gamma x})) \quad (x \to \infty) \end{aligned}$$

for non-zero constant vectors $\boldsymbol{b}^{\pm} \in \mathbb{R}^n$. Then, functions $H_j(\boldsymbol{h})$ are represented by

$$\begin{aligned} H_{2j-1}(\boldsymbol{h}) &= (M^{+}e^{-\alpha h_{2j-1}} - M^{-}e^{-\beta h_{2j}})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) && (3.17) \\ & \qquad (j = 1, 2, \cdots, N^{+}), \\ H_{2j}(\boldsymbol{h}) &= (M^{+}e^{-\alpha h_{2j+1}} - M^{-}e^{-\beta h_{2j}})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) && (3.18) \\ & \qquad (j = 1, 2, \cdots, N^{-}), \\ H_0(\boldsymbol{h}) &= M^{+}e^{-\alpha h_1}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), && (3.19) \\ H_N(\boldsymbol{h}) &= \begin{cases} M^{+}e^{-\alpha h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})) & (if\ N^{+} = N^{-}), \\ M^{-}e^{-\beta h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})) & (if\ N^{+} - 1 = N^{-}) \end{cases} && (3.20) \end{aligned}$$

for a constant $\gamma' > 0$ and the constants $M^{\pm}$ are given by

$$\begin{aligned} M^{+} &= \langle (2\alpha D + A'(\alpha))\boldsymbol{a}^{+}, \boldsymbol{b}^{+} \rangle, && (3.21) \\ M^{-} &= \langle (2\beta D + A'(\beta))\boldsymbol{a}^{-}, \boldsymbol{b}^{-} \rangle. && (3.22) \end{aligned}$$

3.4 Applications I

In this subsection, we analyze the interaction of two stationary front solutions to the nonlocal scalar equation

$$u_t = du_{xx} + K * u + f(u), \tag{3.23}$$

where $d > 0$, $K \in C^1(\mathbb{R})$, $K' \in L^1(\mathbb{R})$, $\kappa := \int_{\mathbb{R}} K(y)dy$ and $g(u) := f(u) + \kappa u$ is a smooth function satisfying

$$\begin{cases} g(\pm 1) = g(a) = 0, g'(\pm 1) < 0 < g'(a),\ \int_{-1}^{1} g(u)du = 0, \\ g < 0 \text{ in } (-1, a) \cup (1, \infty),\ g > 0 \text{ in } (-\infty, 0) \cup (a, 1), \\ g' \geq 0 \text{ in } [r_1, r_2],\ g' \leq 0 \text{ in } [-1, 1] \backslash [r_1, r_2] \end{cases}$$

for some constants $a,\ r_1,\ r_2 \in (-1, 1)$ with $r_1 < r_2$. A typical example of g is $g(u) = u(1 - u^2)$.

3.4.1 Case of a nonnegative integral kernel

We first consider the interaction of two stationary front solutions when K is nonnegative. In this case, there exists a strictly increasing stable stationary front solution to the equation (3.23) such that satisfying $P(\pm\infty) = \pm 1$ [7, 13, 73]. Furthermore, when K satisfies

$$\forall \lambda > 0, \quad A(\lambda) = \tilde{K}(\lambda) = \int_{\mathbb{R}} K(y) e^{\lambda y} dy < \infty,$$

it is reported that $P(x)$ converges to ± 1 in an exponentially monotone way by [73]. Thus, we assume that $P(x)$ converges to 1 as

$$P(x) - 1 = e^{-\alpha x}(a^{+} + O(e^{-\gamma x})) \ (x \to +\infty) \tag{3.24}$$

for some positive constant α and a non-zero constant a^{+}. Then, α is a positive solution of

$$G(\lambda) := d\lambda^2 + A(\lambda) + f'(1) = 0.$$

From the non-negativity of K, we immediately deduce that $G(\lambda)$ is strictly monotonically increasing function for $\lambda > 0$. Hence, we have $G'(\lambda) = 2d\lambda + A'(\lambda) > 0$ for $\lambda > 0$. Moreover, since Φ^* is give by

$$\Phi^*(x) \quad = \quad \frac{1}{\|P_x\|_{L^2}^2} P_x(x) \to -\frac{\alpha a^{+}}{\|P_x\|_{L^2}^2} e^{-\alpha x} \ (x \to +\infty),$$

we obtain

$$M^{+} = \frac{-\alpha (a^{+})^2}{\|P_x\|_{L^2}^2} G'(\alpha) < 0. \tag{3.25}$$

Thus, the equations of l and h are given as

$$\begin{cases} \dot{l} = -H_0(h) + O(\delta^2) \sim -M^{+} e^{-\alpha h} > 0, \\ \dot{h} = H_0(h) + H_1(h) + O(\delta^2) \sim 2M^{+} e^{-\alpha h} < 0. \end{cases}$$

This means the attractivity of two front solutions (Figure 9).

3.4.2 Interaction of very slow front solutions

We next focus on the interaction of two front solutions with very slow wave speed when K is nonnegative. Let us consider the equation (3.23) with small perturbation like

$$u_t = du_{xx} + K * u + f(u) + \varepsilon f_1(u, K_1 * u), \tag{3.26}$$

where ε is a sufficiently small constant, $K_1 \in C(\mathbb{R}) \cap L^1(\mathbb{R})$, and $f_1(u, v) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function satisfying

$$f(\pm 1, \pm \int_{\mathbb{R}} K_1(y) dy) = 0.$$

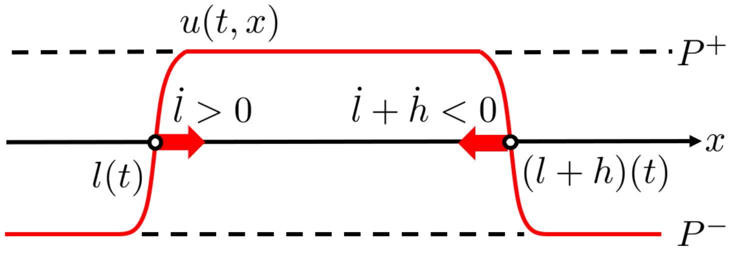

Figure 9: The image of the interaction of two standing fronts, when the kernel is a non-negative function. (This figure is taken from [24].)

Let us consider the solution of (3.26) with the initial function $u(0,x)$ close to $P(x - l(0), h(0))$ for sufficiently large $h(0)$. By a quite similar proof in [22, 27], when ε is sufficiently small, we can show that the solution $u(t,x)$ is close to $P(x - l(t), h(t))$ and

$$\begin{cases} \dot{l} = -M^+ e^{-\alpha h} - \varepsilon C_f + O(\delta^2 + \varepsilon^2), \\ \dot{h} = 2M^+ e^{-\alpha h} + 2\varepsilon C_f + O(\delta^2 + \varepsilon^2) \end{cases} \tag{3.27}$$

holds as long as $h(t)$ is sufficiently large, where M^+ is the constant given by (3.25) and $C_f = \dfrac{1}{\|P_x\|^2} \langle\, f_1(P, K_1 * P), P_x \,\rangle_{L^2}$. In the case that $\varepsilon C_f > 0$, two front solutions moves attractively. When $\varepsilon C_f < 0$, we can prove that (3.27) has an unstable equilibrium. Thus, in this case, we can find an unstable stationary solution by a quite similar way to the proof of Theorem 3.6.

As example, we give $f_1(u,v) = v$ and an odd function $K_1(x)$ satisfying

$$K_1 < 0 \text{ in } (0, \infty).$$

Then, we have $C_f = \dfrac{1}{\|P_x\|^2} \langle\, K_1 * P, P_x \,\rangle_{L^2}$.

Since $P(x)$ is monotonically increasing, we obtain

$$(K_1 * P)(x) = \int_{\mathbb{R}} K_1(y) P(x - y) dy = \int_0^\infty K_1(y)(P(x-y) - P(x+y)) dy > 0$$

and $C_f > 0$. From this, we can show that the existence of an unstable stationary solution for (3.26) if ε is a sufficiently small negative constant.

3.4.3 Case of a sign-changing integral kernel

There are few results on front solutions to the equation (3.23) when the integral kernel has a negative part. When $d = 0$, the existence of stationary front solutions to (3.23) was proved in [8] using variational method under the assumptions that K satisfies $\kappa > 0$, $\hat{K}(\xi) = \int_{\mathbb{R}} K(x) e^{-ix\xi} dx \le \hat{K}(0) = \kappa$ for all $\xi \in \mathbb{R}$ and some conditions. However, to the best of our knowledge, there are no results on linearized stability.

On the other hand, J. Siebert and E. Schöll [60] numerically suggested the existence of a front solution with an oscillating tail when the integral kernel has a negative part. From the result, we expect that there are cases when the front solutions of (3.23) with sign changing integral kernels have oscillatory tails. However, we were unfortunately unable to reveal the stability of front solutions in the case of sign changing integral kernels. We leave it as an open problem.

In the following, we suppose that the existence of a single stationary front solution for (3.23) with a sign-changing integral kernel and analyze formally the interaction of two stationary front solutions.

At first, let us consider the interaction of two stationary front solutions with exponentially decaying oscillatory tails. Assume that there is a stable stationary front solution of (3.23) with an oscillatory tail satisfying

$$P(x) - 1 \to \Re(e^{\lambda^+ x} a^+ (1 + O(e^{-\gamma x}))) \ (x \to +\infty),$$

where $a^+ \in \mathbb{C}\backslash\{0\}$ and $\lambda^+ = -\alpha + i\nu^+$ for constants $\alpha > 0$ and $\nu^+ \neq 0$. Applying Theorem 3.6, the equation for the distance h between front solutions is represented as

$$\dot{h} = H_0 + H_1.$$

We know that $H_0(h) = H_1(h)$ from the definition of $H_j(h)$ $(j = 0, 1)$. Moreover, we can calculate as

$$\begin{aligned} H_0(h) &= \frac{1}{\|P_x\|^2} \langle\, \mathcal{L}(P(\cdot; h)), P_x \,\rangle_{L^2} \\ &= \frac{1}{\|P_x\|^2} \langle\, f(P(\cdot; h)) - f(P(\cdot)) - f(P(-(\cdot - h))) + f(1), P_x \,\rangle_{L^2} . \end{aligned}$$

by using the fact that $P(\pm(x - h))$ are front solutions to (3.23) for any constant $h \in \mathbb{R}$ and $g(1) = \kappa + f(1) = 0$. From a quite similar argument to Subsection 4.5 in [22], we obtain

$$H_0(h) = H_1(h) = \Re(M^+ e^{\lambda^+ h}(1 + O(e^{-\gamma' h})))$$

for a constant $\gamma' > 0$ as long as h is sufficiently large, where

$$M^+ = \frac{a^+}{\|P_x\|^2} \int_{-\infty}^{\infty} e^{-\lambda^+ x} P_x(x)\{f'(P(x)) - f'(1)\}dx.$$

We note that the constant M^+ is well-defined because the integral is represented by the Fourier transformation due to the form of λ^+. Let $M^+ = A^+ + iB^+$. Then,

$$H_0(h) = H_1(h) \sim e^{-\alpha h}(A^+ \cos(\nu^+ h) + B^+ \sin(\nu^+ h))$$

holds. Hence, the equation of h is given as

$$\dot{h} = H_0 + H_1 + O(\delta^2(h)) \sim 2e^{-\alpha h}(A^+ \cos(\nu^+ h) + B^+ \sin(\nu^+ h)) \tag{3.28}$$

for sufficiently large h. We can immediately understand that stable and unstable equilibria appear alternatively in (3.28). Thus, we can easily give the proof of the existence and the stability for multiple front solutions from Theorem 3.6 if there is a stable stationary front solution with oscillatory tails satisfying Assumption 3.1.

Next, we analyze the interaction of two stationary front solutions with exponentially and monotonically decaying tails. Assume that there is a stable stationary front solution of (3.23) satisfying (3.24). By same calculation in Subsection 3.4.1, the equation of h is give as

$$\dot{h} \sim 2M^{+}e^{-\alpha h},$$

where

$$M^{+} = \frac{-\alpha(a^{+})^2}{\|P_x\|_{L^2}^2}G'(\alpha).$$

To check the sign of M^{+}, we focus on $G'(\alpha)$. We remark that α satisfies $G(\alpha) = 0$. When the integral kernel is sign-changing, $G(\lambda)$ is not always monotonically increasing. For example, when

$$K(x) = \frac{\varepsilon}{\sqrt{4\pi}}\left\{\frac{1}{\sqrt{q_1}}e^{-\frac{x^2}{4q_1}} - \frac{1}{\sqrt{q_2}}e^{-\frac{x^2}{4q_2}}\right\} \ (\varepsilon, q_1, q_2 > 0), \tag{3.29}$$

we have

$$A(\lambda) = \varepsilon\left\{e^{q_1\lambda^2} - e^{q_2\lambda^2}\right\}.$$

Thus, we obtain

$$G(\lambda) = d\lambda^2 + \varepsilon\left\{e^{q_1\lambda^2} - e^{q_2\lambda^2}\right\} + f'(1).$$

By setting $d = 1.0,\ \varepsilon = 0.01,\ q_1 = 1.0,\ q_2 = 2.0,\ f'(1) = -1$, we are able to observe that there are two positive solutions α_1 and α_2 of $G(\lambda) = 0$ (Figure 10 (b)), where α_1 and α_2 denote the first and second positive root of $G(\lambda) = 0$, respectively. If $\alpha = \alpha_1$, then

$$M^{+} = \frac{-\alpha(a^{+})^2}{\|P_x\|_{L^2}^2}G'(\alpha) < 0$$

holds by $G'(\alpha_1) > 0$, which means the attractivity of two front solutions. In the case that $\alpha = \alpha_2$, we see

$$M^{+} = \frac{-\alpha(a^{+})^2}{\|P_x\|_{L^2}^2}G'(\alpha) > 0$$

by $G'(\alpha_2) < 0$. This implies the repulsiveness of two front solutions.

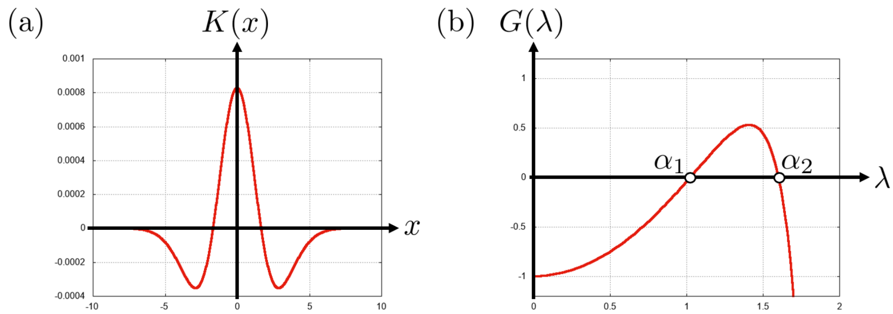

Figure 10: (a) The graph of (3.29) when $\varepsilon = 0.01,\ q_1 = 1.0,\ q_2 = 2.0$. (b) The graph of $G(\lambda)$ when $d = 1.0,\ f'(1) = -1$ and the integral kernel is same as Figure 10 (a). (This figure is taken from [24].)

By numerical simulations, α_1, α_2 can be calculated approximately as $\alpha_1 = 1.0264\ldots$ and $\alpha_2 = 1.6022\ldots$. When we solve (3.23) numerically on the interval $(0, 40)$ in the same parameters as Figure 10, the stable standing front solution is observed (Figure 11 (a)). In fact, we expect that the numerical solution converges to 1 in an exponentially monotone way with the decay exponent $\alpha = 1.0260\ldots$ at $x = 32.0$, because Figure 11 (b) is the graph of $\log|u(t,x)-1|$ at the place where $u(t,x)$ is close to 1 looks like linear. Here, the decay exponent at $x = a$ is calculated as

$$\alpha \sim -\frac{\partial}{\partial x}(\log|u(t,x)-1|)\bigg|_{x=a} \sim -\frac{\log|u(t,a+\eta)-1| - \log|u(t,a)-1|}{\eta},$$

in which η is a sufficiently small constant. Thus, we think that the front solution with the exponential decay rate α_1 is a stable one. Therefore, we expect that two stable front solutions are interacting attractively in this example.

In general, we expect that the decay rate α is given by $\alpha = \min\{\lambda > 0 \mid G(\lambda) = 0\}$ if there exists a stable stationary front solution of (3.23) satisfying (3.24). Under this expectation, $G'(\alpha) \geq 0$ always holds by the property of $G(0) = g'(1) < 0$. Thus, we find that the attractive motion will generally appear and suspect that the repulsive motion will not in most case.

3.5 The case of scalar equations with the nonliner nonlocal term

We also explain the results for the more general case of nonlocal effect:

$$u_t = du_{xx} + f(u, K * g(u)),\ t > 0,\ x \in \mathbb{R}, \tag{3.30}$$

where, $u = u(t,x) \in \mathbb{R}$, $d > 0$, $f(u,v) : \mathbb{R}^2 \to \mathbb{R}$ and $g(u) : \mathbb{R} \to \mathbb{R}$ are smooth nonlinear functions. Assume that K satisfies (3.2).

We impose the following similar assumption to the equation (3.30):

Assumption 3.8. *The equation* (3.30) *satisfies the followings:*

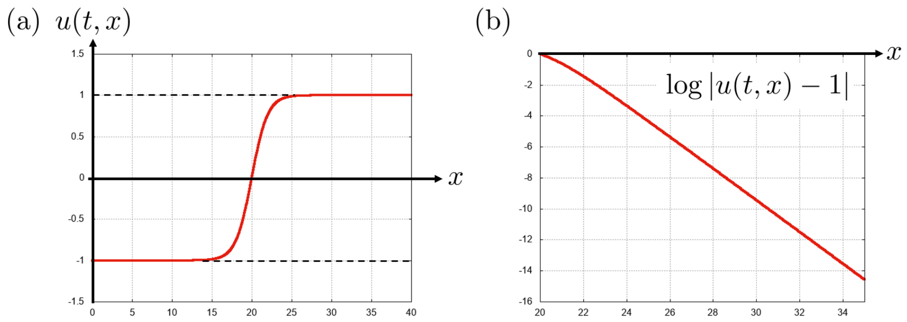

Figure 11: (a) The numerical solution of (3.23) on the interval $(0, 40)$ when $t = 100.0$, where $f(u) = \frac{1}{2}u(1-u^2)$ and the other parameters are same as that in Figure 10. (b) The graph of $\log|u(t, x) - 1|$ on the interval $(20, 35)$ when $t = 100.0$, where $u(t, x)$ is the numerical solution of (3.23). (This figure is taken from [24].)

(H1) The equation (3.30) *has linearly stable equilibria* $P^{\pm}$.

(H2) There exist a constant θ, *positive constants* α, β *and a function* $P(z)$ *such that*

$$\begin{cases} 0 = dP_{zz} - \theta P_z + f(P, K * g(P)), \ z \in \mathbb{R}, \\ |P(x) - P^+| \leq O(e^{-\alpha z}) \ (z \to +\infty), \\ |P(x) - P^-| \leq O(e^{\beta z}) \ (z \to -\infty). \end{cases} \tag{3.31}$$

(H3) Let L *be a linear operator around the stationary front solution* $P(z)$, *that is, for* $v \in H^2(\mathbb{R})$, *we define*

$$Lv := dv_{zz} - \theta v_z + \eta(z)v + \zeta(z)[K * (g'(P)v)],$$

where $\eta(x) := f_u(P(z), K * g(P(z)))$, $\zeta(z) := f_v(P(z), K * g(P(z)))$. *Then, the spectrum* $\Sigma(L)$ *of* L *is given by* $\Sigma(L) = \Sigma_0 \cup \{0\}$, *where 0 is a simple eigenvalue with an eigenfunction* P_z *and there exists a positive constant* $\rho_0 > 0$ *such that* $\Sigma_0 \subset \{z \in \mathbb{C} \mid \Re(z) < -\rho_0\}$.

Other notations are given in the same way as in the previous subsection. In this case, the following theorem is obtained in the same way.

Theorem 3.9. [22] *Let Assumption 3.8 be enforced.*

(i) When $P^+ = P^-$, *Theorem 3.2 and Theorem 3.3 hold in the same statements;*

(ii) When $P^+ > P^-$ *and* $\theta = 0$, *Theorem 3.6 holds in the same statements;*

Furthermore, if $\min \boldsymbol{h}$ is sufficiently large, $H_j(\boldsymbol{h})$ for the pulse solution can be approximated as follows.

Theorem 3.10. *Assume that $P^+ = P^- = 0$. Suppose $P(z)$ converges to $P^\pm$ as*

$$\begin{aligned} P(z) &= e^{-\alpha z}(a^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ P(z) &= e^{\beta z}(a^- + O(e^{\gamma x})) \quad (z \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constants $a^\pm \in \mathbb{R}$, and suppose $\Phi^*(z)$ converges to 0 in an exponentially monotone way such that

$$\begin{aligned} \Phi^*(z) &= e^{-\alpha z}(b^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ \Phi^*(z) &= e^{\beta z}(b^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for non-zero constants $b^\pm \in \mathbb{R}$. Then, functions $H_j(\boldsymbol{h})$ are represented by (3.10), (3.11) and (3.12) for a constant $\gamma' > 0$ and the constants M_α, M_β are given by

$$\begin{aligned} M_\alpha &= [2\alpha d + \theta + f_v(0, \tilde{K}(0)g(0))g'(0)\tilde{K}'(\alpha)]a^+b^-, && (3.32) \\ M_\beta &= [2\beta d - \theta + f_v(0, \tilde{K}(0)g(0))g'(0)\tilde{K}'(\beta)]a^-b^+, && (3.33) \end{aligned}$$

In the case of the stationary front solution, we obtain the following.

Theorem 3.11. *Assume that $P^+ > P^-$ and $\theta = 0$. Suppose $P(x)$ converges to $P^\pm$ as*

$$\begin{aligned} P(x) - P^+ &= e^{-\alpha x}(a^+ + O(e^{-\gamma x})) \quad (x \to +\infty), \\ P(x) - P^- &= e^{\beta x}(a^- + O(e^{\gamma x})) \quad (x \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constants $a^\pm \in \mathbb{R}$, and suppose $\Phi^*(x)$ converges to 0 in an exponentially monotone way such that

$$\begin{aligned} \Phi^*(x) &= e^{-\alpha x}(b^+ + O(e^{-\gamma x})) \quad (x \to +\infty), \\ \Phi^*(x) &= e^{\beta x}(b^- + O(e^{\gamma x})) \quad (x \to -\infty) \end{aligned}$$

for non-zero constants $b^\pm \in \mathbb{R}$. Then, functions $H_j(\boldsymbol{h})$ are represented by (3.17), (3.18), (3.19) and (3.20) for a constant $\gamma' > 0$ and the constants $M^\pm$ are given by

$$\begin{aligned} M^+ &= [2\alpha d + f_v(P^+, \tilde{K}(0)g(P^+))g'(P^+)\tilde{K}'(\alpha)]a^+b^+, && (3.34) \\ M^- &= [2\beta d + f_v(P^-, \tilde{K}(0)g(P^-))g'(P^-)\tilde{K}'(\beta)]a^-b^-. && (3.35) \end{aligned}$$

We omit the proof because the discussion is similar to the case of a linear convolution.

3.6 Applications II

In this subsection, we present applications for interaction between two localized patterns to mathematical models with nonlinear nonlocal effect. The applications here are not all mathematically rigorous, but are partly formal arguments. A mathematically rigorous proof is a future work.

3.6.1 Interaction of stationary solutions with nonngative kernel

In this section, we consider the interaction between two stationary front solutions for the equation

$$u_t = du_{xx} + f(K * g(u)) - u, \quad t > 0,\ x \in \mathbb{R}, \tag{3.36}$$

where $d > 0$, $K \in C^1(\mathbb{R})$ is nonnegative and satisfies $\|K\|_{L^1} \neq 0$, $f(u)$ and $g(u)$ are monotonically non-decreasing smooth functions. (3.36) are proposed mathematical models with the local diffusion of phase separations [52] when $g(u) = u$, and neural fields [2] when $f(u) = u$.

It is known that if (H1) in Assumption 3.8 and appropriate assumptions on K, f, and g are given, then traveling wave solutions to (3.36) exist when $P^+ > P^-$ and the profiles are strictly monotone [13]. Furthermore, linearized stability is also analyzed by [7] when $\mathcal{D}(L)$ is continuous and converges to zero at infinity, and by [73] when $\mathcal{D}(L) = H^2(\mathbb{R})$ and $g(u) = u$. In addition, it has been investigated in [73] that the traveling wave solutions decay exponentially to $P^\pm$ at infinity, at least when $g(u) = u$. Although not all results have been obtained in general form, we shall proceed to discussion under the hypothesis that (3.36) satisfies Assumption 3.8 with $\theta = 0$.

From now on, we assume that a linearly stable stationary front solution exists in (3.36). Then, the linearized operator is given by

$$Lv = dv_{xx} + f'(K * g(P))(K * (g'(P)v)) - v$$

for $v \in H^2(\mathbb{R})$. In this case, it is shown by [63] that if $\inf_{x\in\mathbb{R}} f'(K * P(x))$ and $\inf_{x\in\mathbb{R}} g'(P(x)))$ are positive and $K' \in L^1(\mathbb{R})$, then there exists an eigenfunction $\Phi^* > 0$ corresponding to the eigenvalue 0 of L^* that satisfies $\langle\, P_x, \Phi^* \,\rangle = 1$. Henceforth, Φ^* is assumed to be a positive value function. It is not clear whether the exponential decay occurs at infinity or not, but this is a future issue that should be properly considered. However, we will assume exponential decay for the sake of discussion.

Let us consider the interaction between two stationary front solutions. In this case, we obtain an approximation to the equation

$$\begin{cases} \dot{l} = -H_0(h) + O(\delta^2) \sim -M^+ e^{-\alpha h}, \\ \dot{h} = H_0(h) + H_1(h) + O(\delta^2) \sim 2M^+ e^{-\alpha h} \end{cases}$$

in the same way as in Application I, where

$$M^+ \;=\; [2\alpha d + f'(\tilde{K}(0)g(P^+))g'(P^+)\tilde{K}'(\alpha)]a^+ b^+.$$

Here, from the monotonicity of $P(x)$ and the positivity of Φ^*, we expect $a^+ < 0 < b^+$ and deduce $M^+ < 0$. This conjecture means that the two front solutions interact in moving attractively.

Although formal, we found that an attractive interaction is expected when the integral kernel is nonnegative. On the other hand, the case where the integral kernel has a negative part is interesting and should be considered, but in this case it is difficult to even show the existence of localized patterns, and there are few results for (3.36). While deviating from the assumptions of this study, localized patterns have been analyzed as a mathematical model of the neural field when $d = 0$ and $f(u) = u$,

$$g(u) = \begin{cases} 1 & (u \geq u_0) \\ 0 & (u < u_0) \end{cases}$$

for $u_0 > 0$, the existence and stability of stationary pulse solutions have been analyzed, and their weak interactions also have been formally analyzed [2, 10, 11, 35, 36, 37]. It is well known that in this case the stationary pulse solution can be expressed as an indefinite integral of the integral kernel. In particular, the ordinary differential system of pulse solution positions was derived by [11], and it was suggested that the time evolution of each position can be represented by a stationary pulse solution. In other words, the interaction between the pulse solutions is expected to change depending on the shape of the integral kernel.

We expect the interaction to change depending on the shape of the integral kernel even under the assumptions of this study. However, due to technical problems, the properties of the localization pattern when the integral kernel has a negative part are not well understood analytically. Therefore, I would like to continue my research with the hope that future developments in mathematical analysis techniques will reveal such structures.

3.6.2 Mathematical model for describing emergence and evolution of biological species

Here, we consider the following reaction-diffusion equation with nonlocal effect [65, 66, 67, 68]:

$$u_t = du_{xx} + au^2(1 - K_\varepsilon * u) - \sigma u, \quad t > 0,\ x \in \mathbb{R}, \tag{3.37}$$

where a, d, σ are positive constants and $K_\varepsilon(x) := K(\varepsilon x)$ for $\varepsilon > 0$. (3.37) is proposed by [67] as a mathematical model for describing emergence and evolution of biological species.

In [68], they considered the case that

$$K(x) = \begin{cases} h & (|x| < 1), \\ 0 & (|x| \geq 1) \end{cases} \tag{3.38}$$

for $h > 0$ and proved the existence of stationary pulse solutions for (3.37) if ε is sufficiently small. Furthermore, Volpert [66] suggested the existence of stable pulse solutions by using numerical simulations.

First, we present their results here and extend the result on the existence of pulse solutions based on the argument in [68]. Let us consider the stationary problems to (3.37)

$$0 = dv'' + v^2(1 - K_\varepsilon * v) - \sigma v, \quad x \in \mathbb{R}. \tag{3.39}$$

If $v \in L^1(\mathbb{R})$, then we get

$$(K_\varepsilon * v)(x) = \int_{\mathbb{R}} K(\varepsilon y)v(x-y)dy \to K(0)\int_{\mathbb{R}} v(y)dy =: K(0)\langle\, v\,\rangle$$

by taking the limit as $\varepsilon \to +0$, so (3.39) is attributed to

$$0 = dv'' + av_\varepsilon^2(1 - K(0)\langle\, v\,\rangle) - \sigma v, \quad x \in \mathbb{R}. \tag{3.40}$$

Then, the following lemma holds.

Lemma 3.12. *Suppose that* $0 < K(0) < \dfrac{a}{24\sqrt{d\sigma}}$. *Then,*

$$v_1(x) := p_1\mathrm{sech}^2(qx), \quad v_2(x) := p_2\mathrm{sech}^2(qx)$$

are two positive solutions of (3.40)*, where* $q = \sqrt{\dfrac{\sigma}{4d}}$ *and* $p_1 > p_2 > 0$ *are roots of* $\dfrac{2K(0)}{q}p^2 - p + \dfrac{3\sigma}{2a} = 0$.

From Lemma 3.12, we expect that there are two stationary pulse solutions for (3.39) by using the implicit function theorem when ε is sufficiently small. To apply the implicit function theorem based on the above conjecture, the right side of (3.39) was analyzed to be continuous around $\varepsilon = 0$ and bounded in the weighted Hölder space $C_\mu^{2+r}(\mathbb{R})$ with the weight function $\mu(x) := 1 + x^2$ in [68]. In addition, when $P(x) = v_1$ or $P(x) = v_2$, they considered linearized eigenvalue problem to (3.40) in the suitable subset of $C_\mu^{2+r}(\mathbb{R})$:

$$L_0 w := dw'' + 2aP(x)(1 - \langle\, P\,\rangle)w - aP^2\langle\, w\,\rangle - \sigma w = \lambda w.$$

It was shown that L_0 has a simple eigenvalue $\lambda = 0$ and the corresponding eigenfunction is $P'(x)$ in the appropriate space. Furthermore, they also considered the properties of eigenfunction corresponding to $\lambda = 0$ for the adjoint operator L_0^* of L_0 in the suitable space:

$$L_0^* w := dw'' + 2aP(x)(1 - \langle\, P\,\rangle)w - a\langle\, P^2 w\,\rangle - \sigma w = 0.$$

As a result, they proved that $L_0^* w = 0$ has a unique solution $w = P'(x)$ except for constant multiples. With the above preparations, the following result is obtained by applying the implicit function theorem.

Proposition 3.13. *Suppose that* $K(x)$ *satisfies* (3.2) *and* $0 < K(0) < \dfrac{a}{24\sqrt{d\sigma}}$. *Furthermore, assume that* $K \in C^r(\mathbb{R})$ *for some* $r \in (0,1)$. *Then, for sufficiently small* $\varepsilon > 0$*, there exist two non-trivial stationary even pulse solutions of* (3.39) *converges to* 0 *at* $\pm\infty$.

Remark 3.14. *If* $K(x)$ *is a even function, satisfies* $K \equiv 0$ *on* $(R, +\infty)$ *for some* $R > 0$ *and belongs to* $C^r([-R,R])$ *for some* $r \in (0,1)$*, then the conclusion of Proposition 3.13 also holds by same argument. Taking this into consideration, the conditions of Proposition 3.13 can be put in a form that also includes* (3.38)

In the following, at least one of the pulse solutions obtained by Proposition 3.13 is assumed to be linearly stable, and to converge to 0 in an exponentially monotone way at $\pm\infty$. In the rest, we formally discuss the interaction between the two stationary pulse solutions. We can assume that $\alpha = \beta$, $a^+ = a^-$ and $b^+ = -b^-$ hold in Theorem 3.10 because $P(x)$ is even. From Theorem 3.10 with $f(u, v) = a^2u(1 - v) - \sigma u$ and $g(u) = u$, we obtain an approximation to the equation

$$\begin{cases} \dot{l} = -H_0(h) + O(\delta^2) \sim -M_\alpha e^{-\alpha h}, \\ \dot{h} = H_0(h) - H_1(h) + O(\delta^2) \sim 2M_\alpha e^{-\alpha h}, \end{cases}$$

where $M_\alpha = 2\alpha d a^+ b^+$. Based on the idea of the proof of Proposition 3.13, we expect

$$P(x) \simeq v_j(x), \quad \Phi^*(x) \simeq \frac{1}{\|v_j'\|_{L^2}^2} v_j'(x)$$

for $j = 1, 2$ to hold in some sense as long as $\varepsilon > 0$ is sufficiently small. Thus, assuming that $a^+ > 0 > b^+$ holds if $\varepsilon > 0$ is sufficiently small, we have $M_\alpha < 0$. This consideration means that the two pulse solutions interact in moving attractively.

3.7 Summary

This study introduced a method to analyze weak interactions between localized patterns for reaction-diffusion equations with nonlocal effect. These methods can be applied to the analysis of various mathematical models, and the results are general. However, information on the shape of the eigenfunction corresponding to the 0 eigenvalue in the adjoint operator of the linearized operator is necessary to apply the results of this study, and their analysis is very difficult. To avoid that analysis in this thesis. we have also analyzed formally the weak interaction for the equations (3.26), (3.36), (3.37) as concrete applications. In the case of (3.26), we showed that when the tail of the front solution oscillates, both repulsive and attractive behaviors can be seen, and also conjectured that when the tail is monotonic, only the attractive behavior can be seen regardless of the shape of the integral kernels. Since the analysis of the weak interaction in this study focused on the behavior in the situation where the integral kernel was fully decayed, it is expected that the behavior would have been attractive in the case of a monotonic tail independent of the integral kernel.

In connection with this research, we are currently analyzing the behavior of solutions to the following equation:

$$u_t = \varepsilon^2 u_{xx} + \frac{1}{2}u(1 - u^2) + \eta(K * u + \iota), \quad t > 0, \ x \in \mathbb{R}, \tag{3.41}$$

where $u = u(t, x) \in \mathbb{R}$, $\iota \in \mathbb{R}$ and ε, η are sufficiently small constants. This problem is the Allen-Cahn equation when $\eta = 0$ and is related to the stationary problem of the three-component Fitz-Hugh Nagumo equation [20]

$$
\begin{cases}
u_t = \varepsilon^2 u_{xx} + \dfrac{1}{2}u(1-u^2) + \eta(a_1 v + a_2 w + \iota),\\
\tau_v v_t = d_v v_{xx} + u - v,\\
\tau_w w_t = d_w w_{xx} + u - v
\end{cases}
$$

when $\eta = \varepsilon$, where $a_1, a_2 \in \mathbb{R}$ and $d_v,\ d_w, \tau_v,\ \tau_w$ are positive constants.

If $\eta = 0$, then the equation (3.41) has a stationary front solution $p(x) = \tanh(\frac{x}{2\varepsilon})$. Thus, when η is sufficiently small, We can show that a superposition of the front solutions can approximate the solution by considering $\eta(K * u + \gamma)$ as a perturbation term, and also derive the ordinary differential equation of the distance between the front solutions. Furthermore, when ε is sufficiently small, the stationary front solution can be approximated as the Heaviside step function, and the derived ordinary differential equation can be approximated so that the indefinite integral of the integral kernel appears explicitly. Through such analysis, it can be formally shown that the spatial pattern changes in time depending on the shape of the integral kernel depending on the relationship between η and ε. However, a mathematically rigorous proof of η in a form that includes the ε dependence has not yet been given, and this is one of the studies currently underway. I'll do my best to analyze it, so please stay tuned for future developments.

4 Movement of zero points of solutions to the diffusion equation and the fractional diffusion equation

The content of Subsection 4.2 is based on the papers [46]. The results in Subsection 4.3 are newly introduced in this thesis.

4.1 Background and motivation

The analysis of the time evolution of the shape of the solution is one of the most critical issues in the study of parabolic equations such as reaction-diffusion equations. In the case of reaction-diffusion equations, when an initial function is given that is slightly perturbed near an equilibrium, its dynamics is expected to be approximated by the solution of an equation linearized near the equilibrium, as long as the solution is close enough to the equilibrium. With appropriate variable transformations, the linearized equation often results in a partial differential equation describing the diffusion phenomenon, such as the diffusion equation and the fractional diffusion equation. Therefore, to consider the effect of spatial propagation due to nonlocal effect near an equilibrium, we analyze the linear partial differential equation describing the diffusion phenomenon in this section.

The shape of the solution can be understood to a certain extent if the behavior of the solution's zero points, maximum and minimum points, and extreme points can be understood. In general, how the solutions of parabolic equations behave asymptotically is a subject of interest and has been studied. The asymptotic shape of the solution can be analyzed in different ways depending on whether the domain is bounded or unbounded [41]. For example, let us consider the case of the diffusion equation.

$$u_t = u_{xx}$$

In the case of bounded domains, the eigenvalues of the Laplace operator are discrete under appropriate boundary conditions. The solution of the diffusion equation can be expressed by a Fourier series expansion using the eigenfunctions of the Laplace operator. Therefore, by analyzing the shape of the eigenfunction corresponding to the smallest non-zero eigenvalue, it is possible to understand the asymptotic shape of the solution.

On the other hand, in the case of unbounded domains, the same method as in the case of bounded domains is not applicable. Therefore, the method of analysis by writing down the solution using the fundamental solution and the method of analysis using the conserved quantity using the solution are used. With these analytical methods, the behavior of the maximum point is well analyzed for linear parabolic equations, and results are obtained for particular unbounded domains such as whole space, half-space, and regions outside the unit circle [15, 47, 42, 43]. To understand the asymptotic shape of the solution, information such as the critical points and the zero points of the solution is also essential, but these have not yet been fully analyzed.

This section first considers the Cauchy problem for the diffusion equation in the whole space and the behavior of the zero points of its solution. We present results on the behavior of the zero points of the diffusion equation based on the results of [46].

After that, we analyze the behavior of the zero points in the case of the fractional diffusion equation, which is one of the diffusion equations with nonlocal effect, and compare the results with those of the diffusion equation. Note that in the case of the fractional diffusion equation, this is the first result to be rigorously analyzed in this thesis.

4.2 The case of the diffusion equation

4.2.1 Setting and previous studies

Let us consider a Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \quad (t > 0,\ x \in \mathbb{R}), \\ u(0, x) = u_0(x) \quad (x \in \mathbb{R}), \end{cases} \tag{4.1}$$

where $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Throughout this section, we assume that $\|u_0\|_{L^1} \neq 0$.

Let us analyze the movement of the zero level set $Z(t) := \{x \in \mathbb{R} \mid u(t, x) = 0\}$, where $u(t, x)$ is the explicitly represented solution

$$u(t, x) = (G(t) * u_0)(x) = \int_{\mathbb{R}} G(t, x - y) u_0(y) dy \quad (t > 0,\ x \in \mathbb{R}) \tag{4.2}$$

of (4.1). Here, $G(t, x)$ is the heat kernel defined as

$$G(t, x) := \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \tag{4.3}$$

There are several results on the time variation and asymptotic behavior of $Z(t)$. In the case of parabolic equations, Angenent showed that $Z(t)$ is discrete [4]. It is also known that the number of elements in $Z(t)$ is non-increasing with time [21, 53].

In the case of the diffusion equation, the asymptotic behavior of the zero points (elements of $Z(t)$) has also been analyzed. Mizoguchi evaluated the upper bound of $Z(t)$ in the case that $u_0(x)$ is sign-changing at finite time, and showed that $Z(t) \subset [-C_0 t, C_0 t]$ holds with some $C_0 > 0$ when enough time has passed [55]. Moreover, it has been reported that there are $u_0(x)$ and $x^*(t) \in Z(t)$ such that $x^*(t) > C_1 t$ holds for sufficiently large $t > 0$ with some $C_1 > 0$. Chung [17] reported the asymptotic behavior of zero points when $u_0(x)$ belongs to $L^1(\mathbb{R}, 1 + |x|^{k+1})$ for some $k \in \mathbb{N}$ and satisfies

$$\int_{\mathbb{R}} y^j u_0(y) dy = 0 \ (j = 0, 1, \ldots, k - 1), \quad \int_{\mathbb{R}} y^k u_0(y) dy \neq 0. \tag{4.4}$$

Then, there exist $x_j^*(t) \in Z(t)$ $(j = 1, 2, \ldots, k)$ for sufficiently large $t > 0$ such that

$$\lim_{t \to +\infty} \frac{x_j^*(t)}{2\sqrt{t}} = h_j,$$

where h_j $(j = 1, 2, \ldots, k)$ are mutually different zero points of the Hermite polynomial $H_k(x)$ with the order k defined as

$$H_k(x) := (-1)^k e^{x^2} \frac{d^k}{dx^k}[e^{-x^2}].$$

The proof is based on the Hermite polynomial approximation based on the asymptotic expansion for the heat equation. Especially, if $u_0(x)$ belongs to $L^1(\mathbb{R}, 1 + |x|^{k+1})$ for some $k \in \mathbb{N}$ and satisfies (4.4), then it is well-known that $(4t)^{(k+1)/2}u(t, 2\sqrt{t}x)$ converges uniformly to

$$\frac{e^{-x^2}}{\sqrt{\pi}k!} \left(\int_{\mathbb{R}} y^k u_0(y) dy \right) H_k\left(x\right)$$

as $t \to +\infty$ (see [17] and the references therein).

We expect from these results that the elements of $Z(t)$ have various asymptotic profiles. However, there is no result for asymptotic profiles without the condition (4.4). Furthermore, even though there is a case that the asymptotic behavior of an element of $Z(t)$ is $O(t)$ as $t \to +\infty$, the characterization of its coefficient has not been given. Therefore, the purpose of this subsection is to give asymptotic profiles of the elements of $Z(t)$ in detail.

4.2.2 Main results

Suppose that $u_0(x)$ satisfies

$$\forall \lambda \in \mathbb{R}, \quad \int_{\mathbb{R}} e^{\lambda y} |u_0(y)| dy < \infty. \tag{H1}$$

Define the bilateral Laplace transform of $u_0(x)$ and the zero level set on $\mathbb{R}$ as

$$U_0(\eta) := \int_{\mathbb{R}} e^{\eta y} u_0(y) dy, \quad \mathcal{N}(U_0) := \{\eta \in \mathbb{R} \mid U_0(\eta) = 0\}.$$

When $\eta_0 \in \mathcal{N}(F)$ satisfies

$$\begin{aligned} U_0^{(j)}(\eta_0) &= \int_{\mathbb{R}} y^j e^{\eta y} u_0(y) dy = 0 \quad (j = 0, \ldots, k-1), \\ U_0^{(k)}(\eta_0) &= \int_{\mathbb{R}} y^k e^{\eta y} u_0(y) dy \neq 0 \end{aligned}$$

for some $k \in \mathbb{N}$, we say that $\eta_0 \in \mathcal{N}(U_0)$ has multiplicity k. $U_0(\eta)$ is analytic on $\mathbb{C}$ and $U_0 \not\equiv 0$, and $\mathcal{N}(U_0)$ is thus discrete. Moreover, for any $\eta_0 \in \mathcal{N}(U_0)$, there exists $k \in \mathbb{N}$ such that η_0 has multiplicity k.

We then obtain the following results:

Theorem 4.1. *Suppose that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies* (H1)*. Assume that $\mathcal{N}(U_0) \neq \emptyset$. Then, for all $\eta_0 \in \mathcal{N}(U_0)$ with multiplicity $k \in \mathbb{N}$, there exist $T > 0$ and $x_j^*(t) \in Z(t)$ $(j = 1, \dots, k)$ for $t > T$ satisfying*

$$x_j^*(t) = 2t\eta_0 + 2\sqrt{t}h_j + \frac{U_0^{(k+1)}(\eta_0)}{(k+1)U_0^{(k)}(\eta_0)} + o(1) \quad (t \to +\infty)$$

for $j = 1, \dots, k$, where h_j $(j = 1, \dots, k)$ are mutually different zero points of the Hermite polynomial $H_k(x)$ with the order k.

Remark 4.2. *We can compose $x_j^*(t) \in Z(t)$ $(j = 1, \dots, k)$ obtained in Theorem 4.1 as $x_j^* \in C(T, \infty)$ $(j = 1, \dots, k)$ by applying the argument and results of [4].*

From Theorem 4.1, for any $\eta_0 \in U_0(F)$ and sufficiently large $t > 0$, there exists $x^*(t) \in Z(t)$ such that

$$\lim_{t \to +\infty} \frac{x^*(t)}{2t} = \eta_0.$$

Thus, if $\eta_0 \neq 0$, then $x^*(t)$ diverges with order t as time passes. In the case that $0 \in \mathcal{N}(U_0)$ with multiplicity $k > 1$, for sufficiently large $t > 0$, there exists $x^*(t) \in Z(t)$ such that

$$\lim_{t \to +\infty} \frac{x^*(t)}{2\sqrt{t}} = h,$$

where h is a zero point of $H_k(x)$. When $0 \in \mathcal{N}(U_0)$ with multiplicity k and k is odd, for sufficiently large $t > 0$, there is $x^*(t) \in Z(t)$ satisfying

$$\lim_{t \to +\infty} x^*(t) = \frac{U^{(k+1)}(0)_0}{(k+1)U_0^{(k)}(0)} = \frac{\displaystyle\int_{\mathbb{R}} y^{k+1} u_0(y)dy}{(k+1)\displaystyle\int_{\mathbb{R}} y^k u_0(y)dy}.$$

Hence, by analyzing the zero points of $U_0(\eta)$ and their multiplicity, we can understand the long time behavior of elements of $Z(t)$

We next give the result of the asymptotic behavior of an element of $Z(t)$.

Theorem 4.3. *Suppose that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies* (H1)*. Assume that there exist $T > 0$ and $x^*(t) \in Z(t)$ for $t > T$ satisfying $x^* \in C(T, \infty)$ and*

$$\limsup_{t \to +\infty} \left| \frac{x^*(t)}{2t} \right| < \infty. \tag{4.5}$$

Then, $\theta := \lim_{t \to +\infty} \frac{x^*(t)}{2t}$ exists and belongs to $\mathcal{N}(U_0)$.

Finally, we mention the case that $\mathcal{N}(F) = \emptyset$. Before stating the result, we introduce a notation. For a function f on $\mathbb{R}$ with $f(x) \not\equiv 0$, let $z(f)$ be the number of sign changes; i.e. the supremum of j such that

$$f(x_i)f(x_{i+1}) < 0, \quad i = 1, 2, \ldots, j$$

for some $-\infty < x_1 < x_2 < \ldots < x_{j+1} < +\infty$.

If $u_0(x)$ satisfies $z(u_0) < +\infty$, then (4.5) in the statement of Theorem 4.3 always holds by Theorem 1.1 in [55]. We thus deduce the following result from a proof by contradiction.

Corollary 4.4. *Suppose that* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfies* (H1). *Assume that* $\mathcal{N}(U_0) = \emptyset$ *and* $z(u_0) < \infty$. *There then exists* $T > 0$ *such that* $Z(t) = \emptyset$ *for any* $t > T$. *Furthermore, for all* $x \in \mathbb{R}$ *and* $t > T$, $u(t, x)$ *has the same sign as* $\int_{\mathbb{R}} u_0(y)dy$.

The proof of Theorem 4.1 is based on the asymptotic expansion of the solution using the fundamental solution. However, since the self-similarity of the heat kernel is $O(\sqrt{t})$, it is difficult to find the zero points that diverge at $O(t)$ by simply applying the asymptotic expansion. For this reason, we introduced a moving coordinate and rewrote the equation well. We then applied asymptotic expansion to the equation, and succeeded in showing the existence of a zero point that diverges at $O(t)$. For the proof of Theorem 4.3, since the solution can be written as a convolution of the fundamental solution, we can transform the solution well and take the limit at $t \to +\infty$.

The detail of the proofs is explained in Appendix B.1.

4.3 The case of the fractional diffusion equation

We next consider a Cauchy problem to the fractional diffusion euqation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{s/2} u = 0 \quad (t > 0,\ x \in \mathbb{R}), \\ u(0, x) = u_0(x) \quad (x \in \mathbb{R}), \end{cases} \tag{4.6}$$

where $s \in (0, 2)$ and we recall

$$(-\Delta)^{s/2} u(t, x) := C_s \int_{\mathbb{R}} \frac{u(t, x) - u(t, y)}{|x - y|^{1+s}} dy, \quad C_s := \frac{\Gamma((1 + s)/2)}{2^{-s}\sqrt{\pi}|\Gamma(-s/2)|}.$$

Here, we always assume that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported and satisfies $\|u_0\|_{L^1} \neq 0$ in this subsection. Furthermore, from the pseudo-differential operator:

$$\mathcal{F}[(-\Delta)^{s/2} u](t, \xi) = |\xi|^s \hat{u}(t, \xi)$$

for $\xi \in \mathbb{R}^N$, we define the fundamental solution $G^s(t, x)$ as

$$\hat{G}^s(t, \xi) := e^{-t|\xi|^s}.$$

The following properties of the basic solution are known.

Lemma 4.5 ([33]). *For $s \in (0,2)$, the following statements hold:*

(1) $G^s \in C^\infty((0,+\infty)\times\mathbb{R})$;

(2) $G^s(t,x) = t^{-1/s}G^s(1,x/t^{1/s})$;

(3) $G^s(1,x)$ *is positive, even and monotone decreasing with respect to* x;

(4) For any $k \in \mathbb{N}\cup\{0\}$, $\lim_{|x|\to+\infty} |x|^{1+s+k}|\partial_x^k G^s(1,x)| = C_0(s,k)$ *holds, where* $C_0(s,k)$ *is a positive constant defined by*

$$\begin{aligned} C_0(s,0) &:= \frac{2^{(1+2s)/2}s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right), \\ C_0(s,k) &:= C_0(s,k-1)(s+k) \quad (k\geq 1); \end{aligned}$$

As a difference from the basic solution $G^2(t,x) := \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$ of the diffusion equation, $G^2(1,x)$ decays exponentially at infinity in the case of the diffusion equation, while $G^s(1,x)$ ($s \in (0,2)$) decays algebraically in the case of the fractional diffusion equation from Lemma 4.5 (4). Furthermore, unlike the case of the diffusion equation, $\partial_x^k G^s(1,x)$ ($s\in(0,2)$) decay faster as $k \in \mathbb{N}\cup\{0\}$ increases. This difference allows us to derive an evaluation that is not possible in the case of the diffusion equation. It will be discussed later.

Let us analyze the movement of the zero level set $Z^s(t)$, where $u(t,x)$ is the explicitly represented solution

$$u(t,x) = (G^s(t)*f)(x) = \int_{\mathbb{R}} G^s(t,x-y)f(y)dy \quad (t>0,\ x\in\mathbb{R}) \tag{4.7}$$

of (4.6).

First, we introduce the main result.

Theorem 4.6. *Let $s \in (0,2)$. Suppose that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported. Then, there is a positive constant $\tilde{C}$ such that $Z^s(t) \subset [-\tilde{C}t^{1/s}, \tilde{C}t^{1/s}]$ for sufficiently large $t>0$.*

In the case of the diffusion equation, there exists a zero point whose asymptotic behavior is $O(t)$, whereas in the case of fractional diffusion equation, the upper bound of the zero level set is $O(t^{1/s})$. In other words, even if $s \to 2-0$, the upper bound is discontinuous.

Finally, we present results on the asymptotic behavior of the zero point.

Theorem 4.7. *Let $s \in (0,2)$. Suppose that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported. Let $k \in \mathbb{N}\cup\{0\}$ satisfying*

$$\int y^j u_0(y)dy = 0 \quad (j=0,1,\dots,k-1), \quad \int y^k u_0(y)dy \neq 0. \tag{4.8}$$

Then, the following statements hold:

(1) If $k = 0$, then for sufficiently large $t > 0$, we have $Z(t) = \emptyset$;

(2) When $k > 0$, let us define $\mathcal{N}_k(G^s) := \{x \in \mathbb{R} \mid \partial_x^k G^s(1,x) = 0,\ \partial_x^{k+1} G^s(1,x) \neq 0\}$. Then, for each $g \in \mathcal{N}_k(G^s)$, there exist $T > 0$ and $x^* \in Z^s(t)$ for $t > T$ satisfying

$$\lim_{t \to +\infty} \frac{x_j^*(t)}{t^{1/s}} = g.$$

Remark 4.8. *$\mathcal{N}_k(G^s)$ is discrete, because $\partial_x^k G^s(1,x)$ for $k \in \mathbb{N} \cup \{0\}$ can be represented as*

$$\partial_x^k G^s(1,x) = \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi)^k e^{-|\xi|^s + ix\xi} d\xi$$

and it is analytic by extending the spatial variable x to $\mathbb{C}$. We can show that for any $k \in \mathbb{N}$, $\partial_x^k G^s(1,x)$ has at least one or more zero points, but it is not certain that the zero points are simple. However, as we will see later, there is a special case in which all zero points are simple.

When $\int_{\mathbb{R}} u_0(y)dy \neq 0$, the zero point may still exist after a long enough time in the case of the diffusion equation, but in the case of the fractional diffusion equation, the zero point will disappear after a finite time. On the other hand, when $\int_{\mathbb{R}} u_0(y)dy = 0$, there is a zero point that tends to asymptote according to the self-similarity scale for the fractional diffusion equation as well as for the diffusion equation. However, since the basic solutions of the fractional diffusion equation cannot generally be written down in a concrete form like a Gaussian kernel without the case that $s = 1$, the properties of the derivatives were not well understood, so the set $\mathcal{N}_k(G^s)$ was introduced.

In the case that $s = 1$, we know that $G^1(1,x) = \dfrac{1}{\pi} \dfrac{1}{1+x^2}$. So, we immediately deduce that

$$\mathcal{N}_k(G^s) = \left\{ \tan\left(\frac{(2j-k-1)\pi}{2(k+1)} \right) \mid j = 1, 2, \ldots, k \right\},$$

because it is well-known that

$$\frac{d^k}{dx^k} \left[\frac{1}{1+x^2} \right] = k! \cos^{k+1}(\tan^{-1} x) \sin\left((k+1) \tan^{-1} x + \frac{(k+1)\pi}{2} \right)$$

holds for all $k \in \mathbb{N}$. Thus, the following corollary is obtained.

Corollary 4.9. *Suppose that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported. Let $s = 1$ and $k \in \mathbb{N} \cup \{0\}$ satisfying (4.8). If $k > 0$, then there exist $T > 0$ and $x_j^* \in Z^s(t)$ $(j = 1, \ldots, k)$ for $t > T$ satisfying*

$$\lim_{t \to +\infty} \frac{x_j^*(t)}{t} = \tan\left(\frac{(2j-k-1)\pi}{2(k+1)} \right)$$

for $j = 1, \ldots, k$

The proofs of main results are in Appendix B.2.

4.3.1 Key Lemma

Here, we introduce the key lemma needed to obtain the main result.

Lemma 4.10. *Let $s \in (0,2)$. Suppose that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported. Then, for all $n \in \mathbb{N} \cup \{0\}$, there is $C > 0$ such that*

$$\frac{1}{\{G^s(t,x)\}^A} \left| u(t,x) - \sum_{j=0}^{n} \frac{(-1)^j}{j!} \left(\int_{\mathbb{R}} y^j u_0(y) dy \right) \partial_x^j G^s(t,x) \right| \le C t^{-B} \tag{4.9}$$

holds for any $t > 1$ and $x \in \mathbb{R}$, where $A = \dfrac{1+s+n}{1+s}$, $B = \dfrac{1}{s}(n+2-A) = \dfrac{1}{s(1+s)}\{1+s+sn\}$.

Remark 4.11. *The inequality (4.9) is derived in the paper [44] when $A = 0$ and $B = (n+2)/s$, and in the paper [64] when $n = 0$ and $A = 1$, $B = 1/s$. Note that Lemma 4.10 is an extension of those results.*

As mentioned in the paper [64], Lemma 4.10 is not expected to be obtained in the case of the diffusion equation. In fact, if we consider

$$\frac{1}{G^s(t,x)} \left| u(t,x) - \left(\int u_0(y) dy \right) G^s(t,x) \right|$$

for $s \in (0,2]$ and the solution $u(t,x) = G^s(t, x-h)$ of the diffusion equation ($s = 2$) or the fractional diffusion equation ($s \in (0,2)$) for $h > 0$, then in the case of the diffusion equation ($s = 2$),

$$\frac{1}{G^s(t,x)} \left| u(t,x) - \left(\int u_0(y) dy \right) G^s(t,x) \right| = |e^{(2xh - h^2)/4t} - 1|$$

holds and is unbounded with respect to $x \in \mathbb{R}$ for all $t > 0$. On the other hand, in the case of the fractional diffusion equation ($s \in (0,2)$),

$$\frac{1}{G^s(t,x)} \left| u(t,x) - \left(\int u_0(y) dy \right) G^s(t,x) \right| = \left| \frac{G^s(t,x-h)}{G^s(t,x)} - 1 \right|$$

is bounded because

$$\lim_{|x| \to +\infty} \frac{G^s(t,x-h)}{G^s(t,x)} = 1$$

holds for Lemma 4.5. From above consideration, it can be expected that the decay of the basic solution at infinity is largely determined whether the inequality (4.9) holds or not.

The proof of Lemma 4.10 is also in Appendix B.2.

Remark 4.12. *We note that using the results of [44] and [64], we can derive at least the result of Theorem 4.7. However, it is not obvious whether the theorem 4.6 immediately follows, therefore in this study, we derive a new evaluation of higher-order asymptotic expansions such as Lemma 4.10.*

4.4 Summary

In this section, the behavior of the zero points of the solutions in the diffusion equation and the fractional diffusion equation is discussed to investigate the effect of spatial propagation. As a result, we clarified some properties different from those of the diffusion equation, such as the upper bound of the zero level set and the condition that the zero level set becomes empty at a finite time. These properties imply that similar behavior occurs when the solution of a nonlinear parabolic equation is near an equilibrium.

Note that the analysis method of the zero level set can be extended to the Euclidean space $\mathbb{R}^m$. For the case of the diffusion equation, the treatment is given in [46], and for the case of the fractional diffusion equation, we only need to extend Lemma 4.10. An analytical method using asymptotic expansion is effective even in the case of $\mathbb{R}^m$. To simplify the content of this thesis, I will refrain from giving a detailed introduction.

One of the future studies is to analyze the behavior of solutions near the equilibrium of the reaction-diffusion equation

$$u_t = u_{xx} + f(u).$$

Mizoguchi [55] discussed the upper bound of the zero level set when $f(u) = |u|^{p-1}u$ for $p > 1$, and reported that it has similar properties to the diffusion equation if $p > 3$ and the decay of the global solution is sufficiently fast. If the equilibrium is stable in some sense, the behavior of the level set at the equilibrium is expected to be similar to that of the zero level set in the diffusion equation, but this will be clarified in future studies. The similar conjecture can be made for the case of nonlinear fractional diffusion equations,

$$u_t + (-\Delta)^{s/2}u = f(u),$$

and clarification of the conjecture is also a future problem.

Another future work is to analyze the behavior of the zero level set of nonlocal diffusion equations

$$u_t = K * u - u, \quad t > 0, \ x \in \mathbb{R},$$

where $K \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ is non-negative and even, and satisfies $\|K\|_{L^1} = 1$. In the case of the diffusion equation and the fractional should-diffusion equation, we were able to analyze the asymptotic behavior of the zero level set by applying the asymptotic expansion using the fundamental solution, as described in Appendix B. Asymptotic expansions of solutions using the basic solution have been reported for the case of nonlocal diffusion equations [3, 39, 40], but the fundamental solution $G_K(t, x)$ of the nonlocal diffusion equations is defined as

$$\hat{G_K}(t, \xi) = e^{-t} + e^{-t}(e^{t\hat{K}(\xi)} - 1),$$

which makes self-similarity and evaluation of decay in the spatial direction technically difficult. We expect the behavior of the zero level set to behave like a diffusion equation when the integral kernel decays fast enough at infinity and like a fractional power diffusion equation when the integral kernel algebraically decays slowly. This conjecture is also an important issue to be solved in the future, and we look forward to furthering research on it.

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A Proofs of main results in Section 3

A.1 Proof of Theorem 3.4

From the same calculation in [22], we gain (3.10), (3.11) and (3.12), where

$$\begin{aligned} M_\alpha &= \int_{\mathbb{R}} e^{-\alpha z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^+, \Phi^*(z)\rangle dz, \\ M_\beta &= \int_{\mathbb{R}} e^{\beta z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^-, \Phi^*(z)\rangle dz. \end{aligned}$$

First, we consider M_α. Since positive constants α, β and non-zero vectors $\boldsymbol{a}^\pm \in \mathbb{R}^n$ satisfy

$$\begin{cases} \mathbf{0} = \alpha^2 D\boldsymbol{a}^+ + \alpha\theta\boldsymbol{a}^+ + A(\alpha)\boldsymbol{a}^+ + F'(\mathbf{0})\boldsymbol{a}^+, \\ \mathbf{0} = \beta^2 D\boldsymbol{a}^- - \beta\theta\boldsymbol{a}^- + A(\beta)\boldsymbol{a}^- + F'(\mathbf{0})\boldsymbol{a}^-, \end{cases}$$

we obtain

$$\begin{aligned} \langle F'(\mathbf{0})\boldsymbol{a}^+, \Phi^*(z)\rangle &= -\langle \{\alpha^2 D + \alpha\theta I + A(\alpha)\}\boldsymbol{a}^+, \Phi^*(z)\rangle \\ &= -\langle \boldsymbol{a}^+, \{\alpha^2 D + \alpha\theta I + {}^t A(\alpha)\}\Phi^*(z)\rangle. \end{aligned}$$

From the equation (3.5), we have

$$\begin{aligned} \langle F'(P(z))\boldsymbol{a}^+, \Phi^*(z)\rangle &= \langle \boldsymbol{a}^+, {}^t F'(P(z))\Phi^*(z)\rangle \\ &= -\langle \boldsymbol{a}^+, D\Phi^*_{zz} + \theta\Phi^*_z + {}^t\boldsymbol{K} * \Phi^*\rangle. \end{aligned}$$

Therefore, M_α is represented as

$$\begin{aligned} M_\alpha = &\int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, D\{\alpha^2\Phi^*(z) - \Phi^*_{zz}(z)\}\rangle dz + \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \theta\{\alpha\Phi^*(z) - \Phi^*_z(z)\}\rangle dz \\ &+ \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \{{}^t A(\alpha)\Phi^*(z) - {}^t\boldsymbol{K} * \Phi^*\}\rangle dz =: I_1 + I_2 + I_3. \end{aligned}$$

Since we know that

$$\lim_{z\to+\infty} e^{\beta z} D\Phi^*_z(z) = -\beta D\boldsymbol{b}^+, \quad \lim_{z\to-\infty} e^{-\alpha z} D\Phi^*_z(z) = \alpha D\boldsymbol{b}^-,$$

we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, D\{\alpha^2\Phi^*(z) - \Phi^*_{zz}(z)\}\rangle dz \\ &= -\int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha z}\langle \boldsymbol{a}^+, D\Phi^*_z(z) + \alpha D\Phi^*(z)\rangle\right] dz \\ &= 2\alpha\langle \boldsymbol{a}^+, D\boldsymbol{b}^-\rangle = 2\alpha\langle D\boldsymbol{a}^+, \boldsymbol{b}^-\rangle. \end{aligned}$$

Similary, we obtain

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \theta\{\alpha\Phi^*(z) - \Phi_z^*(z)\}\rangle dz \\ &= -\theta\int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha z}\langle \boldsymbol{a}^+, \Phi^*(z)\rangle\right] dz \\ &= \theta\langle \boldsymbol{a}^+, \boldsymbol{b}^-\rangle. \end{aligned}$$

Finally, to compute I_3, we consider $\int_{\mathbb{R}} e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi_k^*(z) - (K_{k,j} * \varphi_k^*)(z)\}dz$ for $j, k = 1, 2, \cdots, n$, where $\Phi^* = {}^t(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*)$. Since $\tilde{K}_{k,j}$ is an even function, the integrand can be rewritten as

$$\begin{aligned} &e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi_k^*(z) - (K_{k,j} * \varphi_k^*)(z)\} \\ &\quad = e^{-\alpha z}\{\tilde{K}_{k,j}(-\alpha)\varphi_k^*(z) - (K_{k,j} * \varphi_k^*)(z)\} \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y)\left\{e^{-\alpha(z+y)}\varphi_k^*(z) - e^{-\alpha z}\varphi_k^*(z-y)\right\} dy \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y)\int_0^y \frac{d}{ds}\left[e^{-\alpha(z+s)}\varphi_k^*(z-y+s)\right] dsdy. \end{aligned}$$

Notice that

$$\frac{d}{ds}\left[e^{-\alpha(z+s)}\varphi_k^*(z-y+s)\right] = \frac{d}{dz}\left[e^{-\alpha(z+s)}\varphi_k^*(z-y+s)\right],$$

we obtain

$$\begin{aligned} &\int_{\mathbb{R}} e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi_k^*(z) - (K_{k,j} * \varphi_k^*)(z)\}dz \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y)\int_0^y\int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha(z+s)}\varphi_k^*(z-y+s)\right] dzdsdy \\ &\quad = -b_k^-\int_{\mathbb{R}} yK_{k,j}(y)e^{-\alpha y}dy = b_k^-\tilde{K}'_{k,j}(\alpha), \end{aligned}$$

where $\boldsymbol{b}^- = {}^t(b_1^-, b_2^-, \ldots, b_n^-)$. Therefore,

$$I_3 = \int_{\mathbb{R}} e^{-\alpha z}\left\langle \boldsymbol{a}^+, \{{}^tA(\alpha)\Phi^*(z) - {}^t\boldsymbol{K} * \Phi^*\}\right\rangle dz = \langle \boldsymbol{a}^+, {}^tA'(\alpha)\boldsymbol{b}^-\rangle = \langle A'(\alpha)\boldsymbol{a}^+, \boldsymbol{b}^-\rangle.$$

From above calculation, we gain (3.13). We also obtain (3.14) by the same argument.

A.2 Proof of Theorem 3.7

From the same calculation in [22], we can gain (3.17), (3.18), (3.19) and (3.20), where

$$\begin{aligned} M^+ &= \int_{\mathbb{R}} e^{\alpha x}\langle\{F'(P(x)) - F'(P^+)\}\boldsymbol{a}^+, \Phi^*(x)\rangle dx, \\ M^- &= \int_{\mathbb{R}} e^{-\beta x}\langle\{F'(P(x)) - F'(P^-)\}\boldsymbol{a}^-, \Phi^*(x)\rangle dx. \end{aligned}$$

By the argument similar to the proof of Theorem 3.4, we obtain (3.21) and (3.22).

B Proofs of main results in Section 4

B.1 Proofs of main results in the case of the diffusion equation

Let $u(t,x)$ be the solution (4.2) to (4.1). Throughout this subsection, suppose that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying (H1).

B.1.1 Proof of Theorem 4.1

Let us explain the proof of Theorem 4.1. In characterizing the coefficient of $O(t)$, we consider the behavior of $u(t, x+2t\eta_0)$ for all $\eta_0 \in \mathcal{N}(U_0)$ and sufficiently large $t > 0$. This argument is important to obtaining asymptotic profiles and is an important difference from previous studies.

We apply the following result to prove Theorem 4.1.

Theorem B.1. *[17, Theorem 2.2] Let u be the solution (4.2) to the heat equation (4.1) with initial data $u_0 \in L^1(\mathbb{R}; 1+|x|^{n+1})$ for some nonnegative integer n. There is then a positive constant $C = C(n, u_0)$ such that*

$$\left| u(t,x) - G(t,x) \sum_{j=0}^{n} \frac{\int_{\mathbb{R}} y^j u_0(y)dy}{(4t)^{j/2} j!} H_j\left(\frac{x}{2\sqrt{t}}\right) \right| \le C t^{-n/2-1}$$

for all $x \in \mathbb{R}$ and $t > 0$.

Let $\eta_0 \in \mathbb{R}$. By setting, $v(t,x) := e^{\eta_0 x + \eta_0^2 t} u(t, x+2t\eta_0)$. Then, $v(t,x)$ is the solution to the heat equation and the initial data $v_0(x) := e^{\eta_0 x} u_0(x)$ satisfies (H1), and we can thus apply Theorem B.1 to $v(t,x)$ for any $n \in \mathbb{N}$. Using $U_0^{(j)}(\eta_0) = \displaystyle\int_{\mathbb{R}} y^j e^{\eta_0 y} u_0(y) dy$ and Theorem B.1, the following lemma is obtained.

Lemma B.2. *Let u be the solution (4.2) to the heat equation (4.1) with initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying (H1). Suppose that $\eta_0 \in \mathbb{R}$. Then, for any integer $n \ge 0$, there is a positive constant $C = C(n, u_0)$ such that*

$$\left| e^{\eta_0 x + \eta_0^2 t} u(t, x+2t\eta_0) - G(t,x) \sum_{j=0}^{n} \frac{U_0^{(j)}(\eta_0)}{(4t)^{j/2} j!} H_j\left(\frac{x}{2\sqrt{t}}\right) \right| \le C t^{-n/2-1}$$

for any $x \in \mathbb{R}$ and $t > 0$.

Proof of Theorem 4.1. Suppose that $\mathcal{N}(U_0) \neq \emptyset$. Let $\eta_0 \in \mathcal{N}(U_0)$ with the multiplicity $k \in \mathbb{N}$. We deduce from Lemma B.2 with $n = k$ that $(4t)^{(k+1)/2} v(t, 2\sqrt{t}x)$ converges uniformly to

$$\frac{1}{\sqrt{\pi}} e^{-x^2} \frac{U_0^{(k)}(\eta_0)}{k!} H_k(x)$$

as $t \to +\infty$. Thus, for each zero point h of the Hermite polynomial $H_k(x)$, there exist $T > 0$ and $z(t)$ for $t > T$ such that $v(t, z(t)) = 0$ for any $t > T$ and

$$\lim_{t\to+\infty} \frac{z(t)}{2\sqrt{t}} = h.$$

We next deduce the coefficient of $O(1)$. Let h be a zero point of the Hermite polynomial $H_k(x)$. We consider $w(t,x) = (4t)^{k/2+1}v(t, x + 2\sqrt{t}h)$. Using Lemma B.2 with $n = k + 1$, we obtain

$$\left| v(t,x) - G(t,x) \sum_{j=k}^{k+1} \frac{U_0^{(j)}(\eta_0)}{(4t)^{j/2} j!} H_j \left(\frac{x}{2\sqrt{t}} \right) \right| \leq Ct^{-(k+1)/2-1}$$

for all $x \in \mathbb{R}$. Multiplying $(4t)^{k/2+1}$ and replacing x by $x + 2\sqrt{t}h$, the above inequality is rewritten as

$$\left| w(t,x) - K(t, x + 2\sqrt{t}h) \sum_{j=0}^{1} \frac{U_0^{(j+k)}(\eta_0)}{(4t)^{(j-1)/2}(j+k)!} H_{j+k} \left(\frac{x}{2\sqrt{t}} + h \right) \right| \leq Ct^{-1/2},$$

where $K(t,x) := \dfrac{1}{\sqrt{\pi}} e^{-x^2/4t}$. We remark that $K(t, x + 2\sqrt{t}h)$ converges pointwise to $\dfrac{1}{\sqrt{\pi}} e^{-h^2}$ as $t \to +\infty$. We have

$$\lim_{t\to+\infty} \sum_{j=0}^{1} (4t)^{(1-j)/2} \frac{U_0^{(j+k)}(\eta_0)}{(j+k)!} H_{j+k} \left(\frac{x}{2\sqrt{t}} + h \right)$$
$$= \frac{H_{k+1}(h)}{(k+1)!} \{ -(k+1)U_0^{(k)}(\eta_0)x + U_0^{(k+1)}(\eta_0) \}$$

for any $x \in \mathbb{R}$ from the fact that $H_k'(h) = 2hH_k(h) - H_{k+1}(h) = -H_{k+1}(h) \neq 0$, and $w(t,x)$ converges thus pointwise to

$$\frac{e^{-h^2} H_{k+1}(h)}{(k+1)!} \{ -(k+1)U_0^{(k)}(\eta_0)x + U_0^{(k+1)}(\eta_0) \}$$

as $t \to +\infty$. On this basis, there exist $\tilde{T} > 0$ and $\tilde{z}(t)$ for $t > \tilde{T}$ such that $w(t, \tilde{z}(t)) = 0$ for any $t > \tilde{T}$ and

$$\lim_{t\to+\infty} \tilde{z}(t) = \frac{U_0^{(k+1)}(\eta_0)}{(k+1)U_0^{(k)}(\eta_0)}.$$

Therefore, Theorem 4.1 is proved because

$$w(t,x) = (4t)^{k/2+1}v(t, x + 2\sqrt{t}h) = (4t)^{k/2+1} e^{\eta_0 x + t\eta_0^2} u(t, x + 2t\eta_0 + 2\sqrt{t}h).$$

□

B.1.2 Proof of Theorem 4.3

To prove Theorem 4.3, we prepare a notation and lemma. $u(t,x)$ is now expressed as

$$u(t,x) = G(t,x)Q\left(t, \frac{x}{2t}\right),$$

where $P(t,\eta)$ is defined by

$$Q(t,\eta) := \int_{\mathbb{R}} e^{\eta y - y^2/4t} u_0(y)dy.$$

We remark that $Q(t,\eta)$ is well-defined on $(0,\infty)\times\mathbb{R}$ and belongs to $C^\infty((0,\infty)\times\mathbb{R})$, because we know that $u(t,x)$ belongs to $C^\infty((0,\infty)\times\mathbb{R})$ [31]. $u(t,x)$ has the same sign as $Q\left(t, \frac{x}{2t}\right)$, and we thus analyze $Q(t,\eta)$ in the following. We deduce the following lemma from the dominated convergence theorem and (H1).

Lemma B.3. *For any $M > 0$, $Q(t,\eta)$ converges uniformly to $U_0(\eta)$ on $[-M,M]$ as $t \to +\infty$.*

Proof of Theorem 4.3. Define

$$\overline{\theta} := \limsup_{t\to+\infty} \frac{x^*(t)}{2t}, \quad \underline{\theta} := \liminf_{t\to+\infty} \frac{x^*(t)}{2t}.$$

Fix $\theta_0 \in [\underline{\theta}, \overline{\theta}]$. Because $x^*(t) \in C(T,\infty)$, there exists $\{t_j\}_{j\in\mathbb{N}} \subset (T,\infty)$ such that $t_j \to +\infty$ and $\eta_j = x^*(t_j)/2t_j \to \theta_0$ hold as $j \to +\infty$. From the assumption that $x^*(t) \in Z(t)$ for $t > T$, we have $Q(t_j,\eta_j) = 0$ for all $j \in \mathbb{N}$. Using Lemma B.3 and taking a limit as $k \to +\infty$, we obtain $U_0(\theta_0) = 0$. This means that $\theta = \overline{\theta} = \underline{\theta}$ and $\theta \in \mathcal{N}(U_0)$, because $\mathcal{N}(U_0)$ is discrete. □

B.2 Proofs of main results in the case of the fractional diffusion equation

B.2.1 Proof of Lemma 4.10

We fix $n \in \mathbb{N}\cup\{0\}$ arbitrary, From Taylor's theorem, we have

$$\begin{aligned}
&\left| u(t,x) - \sum_{j=0}^{n} \frac{(-1)^j}{j!} \left(\int_{\mathbb{R}} y^j u_0(y)dy\right) \partial_x^j G^s(t,x) \right| \\
&\quad \le \int_{\mathbb{R}} \left| \left\{ G^s(t,x-y) - \sum_{j=0}^{n} \frac{(-y)^j}{j!} \partial_x^j G^s(t,x) \right\} u_0(y) \right| dy \\
&\quad \le \frac{1}{n!} \int_{\mathbb{R}} \int_0^1 |\partial_x^{n+1} G^s(t,x-\theta y) y^{n+1} u_0(y)| (1-\theta)^n d\theta dy
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t > 0$. Since we know that $\partial_x^j G^s(t,x) = t^{-(j+1)/s}\partial_x^j G^s(1, xt^{-1/s})$, we obtain

$$\begin{aligned}&\left|u(t,x) - \sum_{j=0}^{n} \frac{(-1)^j}{j!}\left(\int_{\mathbb{R}} y^j u_0(y)dy\right)\partial_x^j G^s(t,x)\right|\\ &\quad \le \frac{1}{t^{(n+2)/s}n!}\int_{\mathbb{R}}\int_0^1 |\partial_x^{n+1}G^s(1, X-\theta Y)y^{n+1}u_0(y)|(1-\theta)^n d\theta dy\end{aligned}$$

for all $x \in \mathbb{R}$ and $t > 0$. where $X = xt^{-1/s}$ and $Y = yt^{-1/s}$. Here, we remark that $G^s(t,x) = t^{-1/s}G^s(1,X)$. Then,

$$\begin{aligned}&\frac{1}{\{G^s(t,x)\}^A}\left|u(t,x) - \sum_{j=0}^{n} \frac{(-1)^j}{j!}\left(\int_{\mathbb{R}} y^j u_0(y)dy\right)\partial_x^j G^s(t,x)\right|\\ &\quad \le \frac{t^{-B}}{n!}\int_{\mathbb{R}}\int_0^1 \left|\frac{\partial_x^{n+1}G^s(1, X-\theta Y)}{\{G^s(1,X)\}^A}y^{n+1}u_0(y)\right|(1-\theta)^n d\theta dy\end{aligned}$$

holds. We fix $R > 0$ satisfying $\mathrm{supp} u_0 \subset [-R, R]$. We deduce that $Y \in [-R, R]$ for all $t > 1$ and

$$\begin{aligned}\lim_{|X|\to+\infty} |X|^{2+s+n}|\partial_x^{n+1}G^s(1,X)| &= C_0(s, n+1),\\ \lim_{|X|\to+\infty} |X|^{1+s+n}\{G^s(1,X)\}^A &= \lim_{|X|\to+\infty}\{|X|^{1+s}G^s(1,X)\}^A = C_0(s,0)^A\end{aligned}$$

from Lemma 4.5, we thus obtain

$$\lim_{|X|\to+\infty}\left|\frac{\partial_x^{n+1}G^s(1,X-\theta Y)}{\{G^s(1,X)\}^A}\right| = \lim_{|X|\to+\infty}\frac{|X|^{1+s+n}}{|X-\theta Y|^{2+s+n}}\frac{|X-\theta Y|^{2+s+n}|\partial_x^{n+1}G^s(1,X-\theta Y)|}{\{|X|^{1+s}G^s(1,X)\}^A} = 0$$

for all $(\theta, y) \in [0,1] \times [-R, R]$, and

$$\sup_{X\in\mathbb{R}, Y\in[-R,R], \theta\in[0,1]}\left|\frac{\partial_x^{k+1}G^s(1,X-\theta Y)}{\{G^s(1,X)\}^A}\right| < +\infty.$$

holds. Therefore, for all $x \in \mathbb{R}$ and $t > 1$,

$$\frac{1}{\{G^s(t,x)\}^A}\left|u(t,x) - \sum_{j=0}^{n} \frac{(-1)^j}{j!}\left(\int_{\mathbb{R}} y^j u_0(y)dy\right)\partial_x^j G^s(t,x)\right| \le Ct^{-B}$$

holds with some $C > 0$ depending on $s,\ n,\ u_0$.

B.2.2 Proof of Theorem 4.6

Using Lemma 4.10, we analyze the zero level set $Z^s(t)$ of the solution (4.7).

Suppose that $u_0 \in L^\infty(\mathbb{R})$ is compactly supported. We remark that there is $k \in \mathbb{N} \cup \{0\}$ satisfying (4.8). In fact, if we consider the Laplace transform $U_0(\eta)$ $(\eta \in \mathbb{R})$ of $u_0(x)$, then by its analyticity, when it satisfies

$$U_0^{(k)}(0) = \int_{\mathbb{R}} x^k u_0(x)dx = 0$$

for all $k \in \mathbb{N} \cup \{0\}$, $U_0 \equiv 0$ holds, i.e. we have $u_0 \equiv 0$.

We fix $k \in \mathbb{N} \cup \{0\}$ satisfying (4.8). By applying Lemma 4.10 with $n = k$, we have

$$\frac{1}{\{G^s(t,x)\}^A} \left| u(t,x) - \frac{(-1)^k}{k!} \left(\int_{\mathbb{R}} y^k u_0(y)dy \right) \partial_x^{k_0} G^s(t,x) \right| \le Ct^{-B}$$

for all $x \in \mathbb{R}$ and $t > 1$. Here, we fix $x^*(t) \in Z^s(t)$ arbitrary for $t > 1$. Then,

$$\frac{1}{k!\{G^s(t,x^*(t))\}^A} \left| \left(\int_{\mathbb{R}} y^k u_0(y)dy \right) \partial_x^k G^s(t,x^*(t)) \right| \le Ct^{-B}$$

holds for all $t > 1$. Using $\partial_x^j G^s(t,x) = t^{-(j+1)/s} \partial_x^j G^s(1, xt^{-1/s})$ for $j \in \mathbb{N} \cup \{0\}$, The inequality is reformulated as

$$\frac{1}{k!\{G^s(1,x^*(t)t^{-1/s})\}^A} \left| \left(\int_{\mathbb{R}} y^k u_0(y)dy \right) \partial_x^k G^s(1,x^*(t)t^{-1/s}) \right| \le Ct^{-1/s} \qquad \text{(B.1)}$$

Since we deduce that

$$\lim_{|X|\to+\infty} \frac{|\partial_x^k G^s(1,X)|}{\{G^s(1,X)\}^A} = \lim_{|X|\to+\infty} \frac{|X^{1+2s+k} \partial_x^k G^s(1,X)|}{\{|X|^{1+2s} G^s(1,X)\}^A} = \frac{C_0(s,k)}{\{C_0(s,0)\}^A} > 0,$$

the inequality (B.1) implies that there exists $\tilde{C} > 0$ such that $|x^*(t)t^{-1/s}| < \tilde{C}$ holds for sufficiently large $t > 0$. Thus, the theorem is proved.

B.2.3 Proof of Theorem 4.7

(i) By using Lemma (4.10) with $n = 0$, we have

$$\left| \frac{u(t,x)}{G^s(t,x)} - \int u_0(y)dy \right| \le Ct^{-1/2s}.$$

If there exists $x^*(t) \in Z^s(t)$ for $t > 1$, then $u_0(x)$ satisfies

$$\left| \int u_0(y)dy \right| \le Ct^{-1/s}.$$

However, $\int_{\mathbb{R}} u_0(y)dy \neq 0$ from the assumption. Thus, $Z^s(t) = \emptyset$ holds for sufficiently large $t > 0$.

(ii) We fix an arbitrary $g \in \mathcal{N}_k(G^s)$. Then, there is $\theta > 0$ such that

$$\partial_x^k G(1, g+\theta)\partial_x^k G(1, g-\theta) < 0 \textit{ and } \mathcal{N}_k(G^s) \cap (g-\theta, g+\theta) = \{g\}.$$

Using Lemma (4.10) with $n = k$ and multiplying the inequality (4.9) by $\{G^s(t,x)\}^A$,

$$\left| u(t,x) - \frac{(-1)^k}{t^{(1+k)/s} k!} \left(\int_{\mathbb{R}} y^k u_0(y) dy \right) \partial_x^k G^s(1, xt^{-1/s}) \right| \le Ct^{-(k+2)/2s} \{G^s(1, xt^{-1/s})\}^A$$

holds for all $(t,x) \in (1,+\infty)\mathbb{R}$. This means that $t^{(1+k)/s} u(t, xt^{1/s})$ converges unifomly to

$$\frac{(-1)^k}{k!} \left(\int_{\mathbb{R}} y^k u_0(y) dy \right) \partial_x^k G^s(1,x)$$

as $t \to +\infty$. Thus, for sufficiently large $t > 0$, there exists $x^* \in Z^s(t)$ such that

$$\lim_{t\to+\infty} \frac{x^*(t)}{t^{1/s}} = g.$$

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