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博士学位論文

On the pulse dynamics for reaction-diffusion systems on one-dimensional domains with various boundary conditions
(1次元領域上の反応拡散系に現れるパルスの様々な境界条件下におけるダイナミクスについて)

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Contents

1 Introduction of this Thesis

This thesis deals with reaction-diffusion systems on one-dimensional domains under Neumann, and periodic boundary conditions. Moreover, This thesis deals with reaction-diffusion systems on metric graphs composed of multiple half-lines $(0, \infty)$ joined at the origin under Kirchhoff's boundary conditions, especially H-shaped metric graphs and metric graphs with circles. I describe constructing a new theory using the pulse interaction theory([8], [9], [10]) for the above reaction-diffusion systems on one-dimensional domains with various boundary conditions.

Reaction-diffusion systems have been widely used to study problems related to spatiotemporal pattern formations in biological and chemical phenomena. There have been many studies of these problems in one-dimensional space, and various solutions have been clarified. Among them, the pulse or front-type solutions, which are localized in pulse or front shapes, are typical in reaction-diffusion systems. Their analysis is also essential in natural phenomena. For example, many researcher(e.g. [3], [7], [17], [22], [39]) conducted research to a traveling pulse solution for the FitzHugh-Nagumo equation([15], [29]). As a result, we could see how electrical signals propagate along the axon of a nerve from the viewpoint of pulse dynamics. Another example is the Fisher-KPP equation, which describes the density distribution of biological species([26], [33]).The Fisher-KPP equation was also studied by many researchers(e.g. [26], [33]). As a result, it is now possible to treat the increase or decrease in the density distribution of species from the viewpoint of front dynamics. Against this background, studies on pulse/front dynamics have been intensively conducted, and these have been rigorously studied and rapidly developed. In particular, the following Allen-Cahn

$$\partial_t u = \epsilon^2 \partial_{xx} u + \frac{1}{2} u(1 - u^2),\ t > 0, x \in \mathbb{R}, \tag{1.0.1}$$

where $0 < \epsilon \ll 1$ has been studied intensively, and pioneering results on the front dynamics of the Allen-Cahn equation have been derived ([2], [11], [14], [27]). From this result, it is shown that if $u(0, x)$ is close enough to $\tanh\left(\frac{x-l_1(0)}{2\epsilon}\right) + \tanh\left(\frac{-x+l_2(0)}{2\epsilon}\right)$ with $l_1(0) \ll l_2(0)$, then $u(t, x)$ will stay near $\tanh\left(\frac{x-l_1(t)}{2\epsilon}\right) + \tanh\left(\frac{-x+l_2(t)}{2\epsilon}\right)$ as long as $l_1(t) \ll l_2(t)$, and the motion is essentially governed by

$$\begin{cases} \dot{l}_1 = 12\epsilon e^{-\frac{1}{\epsilon}h}, \\ \dot{l}_2 = -12\epsilon e^{-\frac{1}{\epsilon}h}, \end{cases} \tag{1.0.2}$$

where $h := l_2(t) - l_1(t)$. Here $\tanh\left(\frac{x-l_1}{2\epsilon}\right)$ and $\tanh\left(\frac{-x+l_1}{2\epsilon}\right)$ are translation and reflection of the stable stationary solution $\tanh\left(\frac{x}{2\epsilon}\right)$ of (1.0.1). Thus, we see $\tanh\left(\frac{x-l_1(t)}{2\epsilon}\right)$ and $\tanh\left(\frac{-x+l_2(t)}{2\epsilon}\right)$ are attracted to each other in time from (1.0.2). Subsequently, methods for the analysis of the dynamics using equations of motion such as (1.0.2) were extended to a reaction-diffusion system on $\mathbb{R}$:

$$\partial_t \boldsymbol{U} = D\partial_{xx}\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}),\ t > 0, x \in \mathbb{R}, \tag{1.0.3}$$

where $\boldsymbol{U} \in \mathbb{R}^N, D := diag\{d_1, \ldots, d_N\}$ and $\boldsymbol{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function. [8] has allowed us to analyze the dynamics with equations of motion for starting from solution $\boldsymbol{S}^*(x)$ of (1.0.3). Here we fix $r \in \mathbb{N}$ and $\boldsymbol{S}^*(x)$ is sufficiently close to $\sum_{j=1}^{r} \boldsymbol{S}(x-\bar{l}_j)$ with $\bar{l}_j \gg 1\,(j = 1, \ldots, r)$ satisfying $\min\{\bar{l}_2-\bar{l}_1, \ldots, \bar{l}_r-\bar{l}_{r-1}\} \gg 1$, and has r peaks (in the front case, r transition layers), where $\boldsymbol{S}(x)$ is a stable pulse (front)-type stationary solution of (1.0.3)(we call $\boldsymbol{S}^*(x)$ in this paper r-layered pulse/front-type stationary solution). Later on, [9] extended the analysis to the pulse/front dynamics for the reaction-diffusion systems on $(0, \infty)$ with boundary conditions, including the Neumann and Dirichlet boundary condition. As a result, [8] and [9] have become powerful methods for investigating pulse/front dynamics for reaction-diffusion systems.

At the same time, [8] has made pioneering works on the stability analysis to $\boldsymbol{S}^*(x)$ of (1.0.3). Moreover, ([9]) has made pioneering works on the stability analysis to pulse/front-type stationary solutions for reaction-diffusion systems on $(0, \infty)$ with boundary conditions including the Neumann and Dirichlet boundary conditions. They have influenced later studies on linearized eigenvalue problems and stability analysis for reaction-diffusion systems.

In fact, by applying the results of [8] and [9], Ei-Shimatani-Wakasa([12]) have studied the existence of $\boldsymbol{S}^*(x)$ for reaction-diffusion systems on $(0, K)$ with the Neumann boundary condition or the periodic boundary condition for $K \gg 1$. Moreover, they obtained some results for the linearized eigenvalue problems associated with the above $\boldsymbol{S}^*(x)$. As a result, though it has been possible to obtain concrete expressions for eigenvalues and eigenfunctions for linearized eigenvalue problems associated with r-layered front-stationary solutions only for one-dimensional scalar reaction-diffusion equations([35]-[38]), [12] has been extended to linearized eigenvalue problems for general type of reaction-diffusion systems. From this result, it is now possible to obtain eigenvalues that determine the stability for r-layered pulse/front-type stationary solutions satisfying the Neumann or periodic boundary conditions, and eigenfunctions associated with those eigenvalues can be obtained concretely for pulse solutions such that an nerve impulse of Fitz-Hugh Nagume system. Thus, a new theory has recently been established to investigate linearized eigenvalue problems, especially in pulse-type stationary solutions for reaction-diffusion systems.

On the other hand, many researchers have attracted much attention to the following reaction-diffusion systems on $\Omega := \bigcup_{j=1}^{R} \Omega_j$ in recent years, where $\Omega_j := \{x\boldsymbol{e}_j \in \mathbb{R}^2 \mid x > 0\}$ $(j = 1, \ldots, R)$ for $R \in \mathbb{N}$ with $R \geq 3$, and $\boldsymbol{e}_j\,(j = 1, \ldots, R)$ denote the unit directional vectors of Ω_j satisfying $\boldsymbol{e}_i \neq \pm\boldsymbol{e}_j(i \neq j)$. We denote the restriction of $\boldsymbol{u}$ to Ω_j as $\boldsymbol{u}_j(x) := \boldsymbol{u}(x\boldsymbol{e}_j)$ for a function $\boldsymbol{u}$ on Ω. Then we consider

$$\begin{cases} \partial_t \boldsymbol{U}_j = D\partial_{xx}\boldsymbol{U}_j + \boldsymbol{F}(\boldsymbol{U}_j), t > 0, x > 0\,(j = 1, \ldots, R), \\ \displaystyle\sum_{j=1}^{R} \partial_x \boldsymbol{U}_j(t, +0) = \boldsymbol{0}, \boldsymbol{U}_1(t, +0) = \cdots = \boldsymbol{U}_R(t, +0), t > 0, \end{cases} \tag{1.0.4}$$

where $\boldsymbol{U}_j(t, x) := \boldsymbol{U}(t, x\boldsymbol{e}_j)$ and $\boldsymbol{U}_j \in \mathbb{R}^N$. Here the boundary condition of (1.0.4) at the junction point $\boldsymbol{O} := (0, 0) \in \mathbb{R}^2$ is called the Kirchhoff boundary condition.

The dynamics and stability analysis for (1.0.4) are currently being actively studied. The problem for reaction-diffusion equations on various metric graphs including Ω has been studied intensively in recent years because those problems have proved to be significant from the viewpoint of applications. For example, [6] considered metric graphs as a branching channel geometry and mathematically analyzed the density of species in terms of front dynamics for the Fisher-KPP type equations. Moreover, The results are consistent with the transition of biological density in rivers, suggesting that metric graphs has an essential role in applied mathematics. Another result, many other results on frontal dynamics have been reported in recent years, such as frontal traveling wave solutions, stationary solutions, and stability analysis(see e.g. [23],[24],[25]). In addition, some results have been reported recently on pulse-type stationary solutions for reaction-diffusion systems on metric graphs(see e.g. [18], [19]). However, there are still many theoretical issues to be solved for pulse dynamics, such as how pulse traveling wave solutions for neural equations on a metric graph. Based on this situation, we have also studied pulse dynamics for reaction-diffusion systems on Ω based on [8] and [9]. As a result, we analyzed the pulse dynamics for the reaction-diffusion systems on Ω under some assumptions([10]). Later, By [10], I obtain some results that the pulse/front dynamics for reaction-diffusion systems on an H-shaped metric graph (Chapter 5) and the front dynamics for reaction-diffusion equations on metric graphs with a circle (Chapter 6).

In this thesis, I describe pulse/front dynamics and linearized eigenvalue problems of systems associated with reaction-diffusion systems on domains with several boundary conditions by extending [8] and [9].

This doctoral dissertation consists of six chapters, excluding the acknowledgments and references. Chapter 1 is an introduction to this doctoral thesis. Chapter 2 is the assumptions and Preliminaries used in all latter chapters. Chapter 3 is illustrated in linearized eigenvalue problems for reaction-diffusion systems under the Neumann or the periodic boundary conditions. Chapter 4 introduces the results of [10] in preparation for Chapters 5 and 6. Chapter 5 is illustrated in the author's results on the pulse dynamics for a reaction-diffusion system on an H-shaped metric graph consisting of two connected star-shaped regions. Chapter 6 is shown the author's results on the front dynamics for reaction-diffusion systems on a loop-edge-metric graph connected by a half-line and a circle.

2 Preliminaries

Throughout this thesis, we assume the following.

2.1 A pulse-type stationary solution

we consider one-dimensional reaction-diffusion systems:

$$\partial_t \boldsymbol{U} = D\partial_{xx}\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}),\ t > 0,\ x \in \mathbb{R}, \tag{2.1.1}$$

where $\boldsymbol{U} \in \mathbb{R}^N, D := diag\{d_1, \ldots, d_N\}$ and $\boldsymbol{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function. As in [10], we define the right hand side of (2.1.1) by $\mathcal{F}(\boldsymbol{U}) := D\partial_{xx}\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U})$. Then we make the exactly the same following assumptions as in (H1)-(H3) of [10] for (2.1.1)((7) in [10]).

(H1) $\mathbf{0} := {}^t(0, \ldots, 0) \in \mathbb{R}^N$ is a linearly stable equilibrium of (2.1.1)((7) in [10]). That is, the spectrum $\Sigma(L_0)$ of L_0 satisfies $\Sigma(L_0) \subset \{z \in \mathbb{C};\, Re(z) < -\rho_0\}$ for $\rho_0 > 0$, where $L_0 := D\partial_{xx} + \boldsymbol{F}'(\mathbf{0})$.

(H2) (2.1.1)((7) in [10]) has a linearly stable stationary pulse solution, say $\boldsymbol{S}(x)$, that is, there exists $\boldsymbol{S}(x)$ satisfying $\mathcal{F}(\boldsymbol{S}(x)) = \mathbf{0}, \boldsymbol{S}(x) \to \mathbf{0}$ as $|x| \to \infty$ and $\Sigma(L) = \Sigma_1 \cup \{0\}$ with a simple eigenvalue 0, where $L := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}(x))$, $\Sigma(L)$ is the spectrum of L and Σ_1 is a set satisfying $\Sigma_1 \subset \{z \in \mathbb{C};\, Re(z) < -\rho_0\}$.

Remark 2.1 In this thesis, we call $\boldsymbol{S}(x)$ satisfying (H2) as a pulse-type stationary solution.

(H3) $\boldsymbol{S}(x)$ is an even function and there exist a positive constant $\gamma > \alpha$ and a non-zero vector $\boldsymbol{a} \in \mathbb{R}^N$ such that

$$\boldsymbol{S}(x) = \mathrm{e}^{-\alpha|x|}\boldsymbol{a} + O(\mathrm{e}^{-\gamma|x|}) \quad (|x| \to \infty).$$

By (H2), (H3), we note there exists an eigenfunction Φ^* of the adjoint operator L^* of L satisfying $L^*\Phi^* = 0$, $\Phi^*(x)$ as an odd function and $\Phi^*(x)$ is uniquely determined by the normalization $\langle \boldsymbol{U}_x, \Phi^* \rangle_{L^2} = 1$. Then we assume the next condition for Φ^* as follows:

(A1) There exists non-zero vector $\boldsymbol{a}^* \in \mathbb{R}^N$ such that $\Phi^*(x) \to \pm \mathrm{e}^{-\alpha|x|}\boldsymbol{a}^*$ as $x \to \pm\infty$.

2.2 A front-type stationary solution

We make exactly the same following assumptions as in (H1)'-(H3)' of [10] for (2.1.1)((7) in [10]).

(H1)' $\boldsymbol{S}_\pm \in \mathbb{R}^N$ is a linearly stable equilibria of (2.1.1)((7) in [10]).That is, the spectrum $\Sigma(L_\pm)$ of $L_\pm$ satisfy $\Sigma(L_\pm) \subset \{z \in \mathbb{C}; Re(z) < -\rho_0\}$ for $\rho_0 > 0$, where $L_\pm := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}_\pm)$.

Remark 2.2. $\boldsymbol{S}_\pm$ for (H1)' satisfy $\boldsymbol{S}_- \neq \boldsymbol{S}_+$.

(H2)' (2.1.1)((7) in [10]) has a linearly stable stationary front solution, say $\boldsymbol{S}(x)$, that is, there exists $\boldsymbol{S}(x)$ satisfying $\mathcal{F}(\boldsymbol{S}(x)) = \boldsymbol{0}, \boldsymbol{S}(x) \to \boldsymbol{S}_\pm$ as $x \to \pm\infty$ and $\Sigma(L) = \Sigma_1 \cup \{0\}$ with simple eigenvalue 0, where $L := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}(x))$, $\Sigma(L)$ is the spectrum of L and Σ_1 is a set satisfying $\Sigma_1 \subset \{z \in \mathbb{C}; Re(z) < -\rho_0\}$.

Remark 2.3. In this thesis, we call $\boldsymbol{S}(x)$ satisfying (H2)' as a front-type stationary solution. If $\boldsymbol{S}_+ = \boldsymbol{S}_-$ holds, $\boldsymbol{S}(x)$ is a pulse-type stationary solution.

(H3)' There exist positive constants $\gamma_\pm > \alpha_\pm$ and non-zero vectors $\boldsymbol{a}_\pm \in \mathbb{R}^N$ such that

$$\boldsymbol{S}(x) = \boldsymbol{S}_\pm + \mathrm{e}^{-\alpha_\pm|x|}\boldsymbol{a}_\pm + O(\mathrm{e}^{-\gamma_\pm|x|}) \quad (x \to \pm\infty).$$

By (H2)' and (H3)', there exists an eigenfunction Φ^* of the adjoint operator L^* of L satisfying $L^*\Phi^* = 0$ and $\Phi^*(x)$ is uniquely determined by the normalization $\langle \boldsymbol{S}_x, \Phi^* \rangle_{L^2} = 1$. Then we assume the next condition for Φ^* as follows:

(A2) There exist non-zero vectors $\boldsymbol{a}^*_\pm \in \mathbb{R}^N$ such that $\Phi^*_\pm(x) \to \mathrm{e}^{-\alpha_\pm|x|}\boldsymbol{a}^*_\pm$ as $x \to \pm\infty$.

3 Linearized eigenvalues problems

3.1 Introduction

Linearized eigenvalue problems are problems to investigate the stability of stationary solutions for reaction-diffusion systems. Analytical methods for eigenvalues and eigenfunctions have been studied intensively for a long time. Sturm-Liouville theory and the SLEP method ([31], [32]), which were developed in the course of these studies, are still powerful tools for linearized eigenvalue problems. However, there are still few general methods to analyze linearized eigenvalue problems. Against this background, many researchers have studied this problem. In the process, studies of the one-dimensional Allen-Cahn equation with the Neumann boundary condition have been developed.

$$\begin{cases} \partial_t u = \epsilon^2 \partial_{xx} u + u(1-u^2), \ \ t > 0, \, x \in (0,1), \\ \partial_x u(t,0) = \partial_x u(t,1) = 0, \end{cases} \tag{3.1.1}$$

where $0 < \epsilon \ll 1$. (3.1.1) was rigorously studied by [4]. Fix $r \in \mathbb{N}$. By many studies, we see that there exists a stationary solution $u(x;\epsilon)$ of (3.1.1) satisfying being sufficiently close to $\sum_{j=1}^{r}(-1)^j \tanh\left(\frac{x-z_j}{\sqrt{2}\epsilon}\right)$ $(j = 1, \ldots, r)$, where $z_j = \frac{2j-1}{2r}$. Here $r \in \mathbb{N}$ denote the number of a stable front-type stationary solution $S(x)$ of (3.1.1) and $u(x;\epsilon)$ is a function of x with a parameter ϵ. Furthermore, starting from (3.1.1), various researches for 3.1.1 was performed(e.g. [1], [2], [14]). Thereafter, Wakasa-Yotsutani([35], [36], [37], [38]) analyzed next linearized eigenvalue problem for $u(x;\epsilon)$:

$$\begin{cases} \epsilon^2 \partial_{xx}\varphi(x) + F'(u(x;\epsilon))\varphi(x) + \lambda\phi(x) = 0, \quad x \in (0,1), \\ \partial_x \varphi(0) = \partial_x \varphi(1) = 0, \end{cases} \tag{3.1.2}$$

where $0 < \epsilon \ll 1$ and $F(u) = u(1-u^2)$ or $F(u) = \sin u$. [35], [36], [37], [38] reported that all eigenvalues λ_n $(n = 0, 1, 2, 3, \ldots)$ of (3.1.2) and eigenfunctions $\varphi_n(x)$ associated with λ_n hold the following. In 3.1, $\sim$ means asymptotic equivalence, and we denote $\lambda_n = \lambda_n^{r,\epsilon}$, $\varphi_n = \varphi_n^{r,\epsilon}$ in the following (1) and (2). In addtion, The following results (1) and (2) are rigorously proved in [35], [36], [37], [38].

(1) When $f(u) = u(1-u^2)$

We consider $u(x;\epsilon) \sim \sum_{j=1}^{r}(-1)^l \tanh\left(\frac{x-z_j}{\sqrt{2}\epsilon}\right)$.The following(I)-(III) hold up([37], [38]).

(I) For $0 \le n < r$,

$$\begin{cases} \lambda_n^{r,\epsilon} \sim -96\cos^2\dfrac{n\pi}{2r}\exp\left(-\dfrac{\sqrt{2}}{r\epsilon}\right), \\ \varphi_n^{r,\epsilon}(x) \sim \mathrm{sech}^2\left(\dfrac{x-z_j}{\sqrt{2}\epsilon}\right)\cos n\pi z_j \end{cases}$$

in a neighborhood of z_j $(j = 1, \ldots, r)$ as $\epsilon \to 0$.

(II) For $r \le n < 2r$,

$$\begin{cases} \lambda_n^{r,\epsilon} \sim \dfrac{3}{2} - 12\cos\dfrac{(n-r)\pi}{r}\exp\left(-\dfrac{1}{\sqrt{2}r\epsilon}\right), \\ \varphi_n^{r,\epsilon}(x) \sim 2(-1)^j\tanh\left(\dfrac{x-z_j}{\sqrt{2}\epsilon}\right)\mathrm{sech}\left(\dfrac{x-z_j}{\sqrt{2}\epsilon}\right)\cos(n-r)\pi z_j \end{cases}$$

in a neighborhood of z_j $(j = 1, \ldots, r)$ as $\epsilon \to 0$.

(III) For $n \ge 2r$,

$$\begin{cases} \lambda_n^{r,\epsilon} \sim 2 + (n-2r)^2\pi^2\epsilon^2, \\ \varphi_n^{r,\epsilon}(x) \sim \left(\dfrac{3}{2}\tanh^2\left(\dfrac{x-z_j}{\sqrt{2}\epsilon}\right) - \dfrac{1}{2}\right)\cos(n-2r)\pi x \end{cases}$$

in a neighborhood of z_j $(j = 1, \ldots, r)$ as $\epsilon \to 0$.

(2) When $f(u) = \sin u$

We consider $u(x;\epsilon) \sim \sum_{j=1}^{r}(-1)^j 2\mathrm{Arcsin}\left\{\tanh\left(\frac{x-z_j}{\epsilon}\right)\right\}$ $(j = 1, \ldots, r)$. The following(I)-(II) hold up([35], [36], [37], [38]).

(I) For $0 \le n < r$,

$$\begin{cases} \lambda_n^{r,\epsilon} \sim -16\cos^2\dfrac{n\pi}{2r}\exp\left(\dfrac{-1}{r\epsilon}\right), \\ \varphi_n^{r,\epsilon}(x) \sim \mathrm{sech}\left(\dfrac{x-z_j}{\epsilon}\right)\cos n\pi z_j \end{cases}$$

in a neighborhood of z_j $(j = 1, \ldots, r)$ as $\epsilon \to 0$.

(II) For $n \geq r$,

$$\begin{cases} \lambda_n^{r,\epsilon} \sim 1 + (n-r)^2\pi^2\epsilon^2, \\ \varphi_n^{r,\epsilon}(x) \sim (-1)^j \tanh\left(\dfrac{x - z_j}{\epsilon}\right)\cos(n-r)\pi x \end{cases}$$

in a neighborhood of z_j ($j = 1, \ldots, r$) as $\epsilon \to 0$.

[35]-[38] are complete analyses for (3.1.2), and have been famous for fully the analyzed example for linearized eigenvalues problems.

Among them, it is interesting that when $0 \leq n < r$, the leading terms of eigenvalues and their eigenfunctions of (3.1.2) have the above same expressions((1)-(I) and (2)-(I)). Both (1)-(I) and (2)-(I) suggest that at least the leading terms of eigenvalues and their eigenfunctions of $0 \leq n < r$ with linearized eigenvalue problems may be a universal property independent of the nonlinear term in reaction-diffusion systems.

On the other hand, linearized eigenvalue problems associated with r-layered pulse stationary solutions for reaction-diffusion systems with the Neumann boundary condition still need to be solved. But recently, Ei-Shimatani-Wakasa has considered the linearized eigenvalue problem associated with r-layered pulse stationary solutions $\boldsymbol{S}^*$ for reaction-diffusion systems with the Neumann boundary condition by [8] and [9]. Moreover, they have analyzed some results in eigenvalues and eigenfunctions([12]). Here $\boldsymbol{S}^*$ satisfies

$$\begin{cases} \mathbf{0} = D\partial_{xx}\boldsymbol{S}^{\boldsymbol{*}} + \boldsymbol{F}(\boldsymbol{S}^*), \quad x \in (0, K), \\ \partial_x \boldsymbol{S}^*(0) = \partial_x \boldsymbol{S}^*(K) = \mathbf{0} \end{cases}$$

for $K \gg 1$ and is sufficiently close to $\sum_{j=1}^{r} \boldsymbol{S}(x - l_j^*)$ with $l_j^* \gg 1$ ($j = 1, \ldots, r$) satisfying $\min\{l_2^* - l_1^*, \ldots, l_r^* - l_{r-1}^*\} \gg 1$, and $\boldsymbol{S}(x)$ is a stable pulse (front)-type stationary solution of (2.1.1). r denotes the number of $\boldsymbol{S}(x)$.

In this chapter, I describe the formula of eigenvalues and eigenfunctions on the linearized eigenvalue problem for reaction-diffusion systems with the Neumann or the periodic boundary conditions ([12]). [12] indicates that the universality in the leading terms suggested in [35]-[38] holds not only for a single equation but also for a wide range of reaction-diffusion systems in general.

3.2 In the case of the Neumann boundary condition

In this subsection, we deal with the following system

$$\begin{cases} \partial_t \boldsymbol{U} = D\partial_{xx}\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}) & (t,x) \in (0,\infty)\times(0,K), \\ \partial_x \boldsymbol{U}(t,+0) = \partial_x \boldsymbol{U}(t,K) = \boldsymbol{0} \end{cases} \tag{3.2.3}$$

for $K \gg 1$, where $D := diag\{d_1,\ldots,d_N\}$ and $F : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function.

Hereafter we use notions $\delta^*(\boldsymbol{l}) := e^{-\alpha h^*(\boldsymbol{l})}$, $\delta^*_+(\boldsymbol{l}) := e^{-\alpha_+ h^*(\boldsymbol{l})}$, and $h^*(\boldsymbol{l}) := \min\{2l_1, l_2 - l_1, \ldots, l_r - l_{r-1}, 2(K-l_r)\}$.

3.2.1 Main results for r-layered pulse-stationary solutions

Assume (H1),(H2),(H3),(A1). By [8] and [9], we have the following theorem.

Theorem 3.1([8], [9]). Suppose that there exist some sufficiently large positive constants $\overline{h}$, K_0 such that $K_0 > \overline{h}r$. Moreover suppose for any $K > K_0$, there exists $\boldsymbol{l}_0 = {}^t(l^0_1, l^0_2, \ldots, l^0_r)$ such that $0 < l^0_1 < \cdots < l^0_r < K$ with $h^*(\boldsymbol{l}_0) > \overline{h}$ and the initial value $\boldsymbol{U}(0,x)$ of (3.2.3) is sufficiently close to $\sum_{j=1}^r \boldsymbol{S}(x - l^0_j)$. Then there exists $\boldsymbol{l}(t) = {}^t(l_1(t), l_2(t), \ldots, l_r(t))$ with $0 < l_j(t) < K$ $(j = 1, 2, \ldots, r)$ such that the solution $\boldsymbol{U}(t,x)$ of (3.2.3) satisfies

$$\sup_{x\in[0,K]} |\boldsymbol{U}(t,x) - \boldsymbol{U}^*(x;\boldsymbol{l}(t))| \to \boldsymbol{0}\,(t\to\infty)$$

as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$, where $\boldsymbol{U}^*(x;\boldsymbol{l}) := \sum_{j=1}^r \boldsymbol{S}(x-l_j) + \boldsymbol{\sigma}^*(x;\boldsymbol{l})$ and $\boldsymbol{\sigma}^*(x;\boldsymbol{l})$ is a C^1 function with respect to $\boldsymbol{l}$ with $\boldsymbol{\sigma}^*(x;\boldsymbol{l}) := {}^t(\sigma^*_1(x;\boldsymbol{l}), \ldots, \sigma^*_r(x;\boldsymbol{l}))$ satisfying

$$||\boldsymbol{\sigma}^*(x;\boldsymbol{l})||_{L^\infty(0,K)} \le O(\delta^*(\boldsymbol{l})).$$

Moreover $\boldsymbol{l}(t)$ satisfies

$$\begin{cases} \dfrac{dl_1}{dt} = M_0(\mathrm{e}^{-2\alpha l_1} - \mathrm{e}^{-\alpha(l_2-l_1)}) + H^*_1(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_j}{dt} = M_0(\mathrm{e}^{-\alpha(l_j-l_{j-1})} - \mathrm{e}^{-\alpha(l_{j+1}-l_j)}) + H^*_j(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_r}{dt} = M_0(\mathrm{e}^{-\alpha(l_r-l_{r-1})} - \mathrm{e}^{-2\alpha(K-l_r)}) + H^*_r(\boldsymbol{l}) \end{cases}$$

with $|H^*_j(\boldsymbol{l})| \le O((\delta^*(\boldsymbol{l}))^2)$ as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$, where $M_0 := 2\alpha\langle D\boldsymbol{a}, \boldsymbol{a}^*\rangle$.

We define $H_j(\boldsymbol{l}) := M_0(\mathrm{e}^{-\alpha(l_j - l_{j-1})} - \mathrm{e}^{-\alpha(l_{j+1}-l_j)})$ and $\boldsymbol{H}(\boldsymbol{l}) := {}^t(H_1(\boldsymbol{l}), \ldots, H_r(\boldsymbol{l}))$ for $\boldsymbol{l} = {}^t(l_1, \ldots, l_r)$, where $l_0 := -l_1$ and $l_{r+1} := 2K - l_r$. Furthermore let $\tilde{\boldsymbol{l}} := {}^t(\tilde{l}_1, \ldots, \tilde{l}_r)$ be the equilibrium point such that $\boldsymbol{H}(\tilde{\boldsymbol{l}}) = 0$ and $h(\tilde{\boldsymbol{l}}) > \overline{h}$.

$$\dot{\boldsymbol{l}} = \boldsymbol{H}(\boldsymbol{l}) \tag{3.2.4}$$

and let $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$ be the linearized matrix of $\boldsymbol{H}(\tilde{\boldsymbol{l}})$ with respect to $\tilde{\boldsymbol{l}}$.

Corollary 3.1 ([8], [9]). For any $K > K_0$, there exists an equilibrium of (3.2.4) $\tilde{\boldsymbol{l}} = {}^t(\bar{l}_1, \ldots, \tilde{l}_r)$ such that $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}})) \subset \left\{z \in \mathbb{C}\,;\ |z| \geq O(\delta^*(\bar{\boldsymbol{l}})\right\}$ holds. Then there exists the stationary solution $\boldsymbol{S}^*(x)$ of (3.2.3) such that

$$\sup_{x\in[0\,,\,K]} \left|\boldsymbol{S}^*(x) - \boldsymbol{U}^*(x; \tilde{\boldsymbol{l}})\right| \leq O(\delta^*(\tilde{\boldsymbol{l}})),$$

where $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}}))$ is the set of eigenvalues for $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$.

Remark 3.1. $\boldsymbol{S}^*(x)$ stated in Corollary 3.1 is called an r-layered pulse stationary solution.

We consider the eigenvalue problem

$$\begin{cases} L_{\mathcal{N}}(\boldsymbol{S}^*)\boldsymbol{\Phi}(x) = \lambda\boldsymbol{\Phi}(x),\ x \in (0, K), \\ \partial_x\boldsymbol{\Phi}(0) = \partial_x\boldsymbol{\Phi}(K) = \boldsymbol{0}, \end{cases} \tag{3.2.5}$$

where $L_{\mathcal{N}}(\boldsymbol{S}^*) := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}^*)$ on $H^2(0, K)$ with the Neumann boundary condition. Then the following holds for (3.2.5).

Theorem 3.2. Suppose there exist sufficiently large positive constants K_0 and $\overline{h}$ satisfying $K_0 > \overline{h}r$. Then for any $K > K_0$, there exist positive constants ρ_1 and C_1 such that $\rho_1 \leq C_1 e^{-\alpha\frac{K}{r}}$ holds and $L_{\mathcal{N}}(\boldsymbol{S}^*)$ has r eigenvalues $\lambda_n\,(n = 0, 1, \ldots, r-1)$ in $B(0\,; \rho_1)$, while the other spectrum λ of $L_{\mathcal{N}}(\boldsymbol{S}^*)$ satisfy $Re\{\lambda\} < -\beta_1$ for a positive constant β_1, where $B(0\,; \epsilon) := \{z \in \mathbb{R}\,; |z| \leq \epsilon, \epsilon > 0\}$. Furthermore, excluding constant doubling, r eigenvalues $\lambda_n\,(n = 0, 1, \ldots, r-1)$ of $L_{\mathcal{N}}(\boldsymbol{S}^*)$ and eigenfunctions $\boldsymbol{\Phi}_n(x)\,(n = 0, 1, \ldots, r-1)$ associated with $\lambda_n\,(n = 0, 1, \ldots, r-1)$ satisfy

$$\lambda_n = -4M_0\alpha\mathrm{e}^{-\alpha\frac{K}{r}}\cos^2\left(\frac{n\pi}{2r}\right)(1 + O(\mathrm{e}^{-\alpha\frac{K}{r}})),$$

$$\sup_{x\in[0\,.\,K]} \left|\boldsymbol{\Phi}_n(x) - \sum_{j=1}^{r}(-1)^{j+1}\cos\left(\frac{n\pi}{K}\,\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha\frac{K}{r}}),$$

where $\bar{l}_j := \dfrac{(2j-1)K}{2r}\,(j = 1, \ldots, r)$.

Remark 3.2. From Theorem 3.2, we know that signs of $\lambda_n\,(n=0,1,\ldots,r-1)$ are determined by M_0. If $M_0>0\,(<0)$, then $\lambda_n<0\,(>0)\,(n=0,1,\ldots,r-1)$. Therefore, if $M_0>0\,(<0)$, we see that $\boldsymbol{S}^*$ stated in Corollary 3.2. is stable (unstable).

3.2.2 Main results for r-layered front-stationary solutions

In this subsection, we additionary assume the following to (H2)'.

(H2)" $\boldsymbol{S}(x)=-\boldsymbol{S}(-x)$. Then $\alpha^+=\alpha^-, \boldsymbol{S}_+=-\boldsymbol{S}_-, \boldsymbol{a}_+=-\boldsymbol{a}_-$ and $\boldsymbol{a}_+^*=\boldsymbol{a}_-^*$.

We assume (H1)' - (H3)' and (A2). By the same arguments as for [8], [9], the following holds.

Theorem 3.3([8], [9]). Suppose that there exists some sufficiently large positive constants $\overline{h}$, K_0 such that $K_0>\overline{h}r$. Moreover suppose for any $K>K_0$, there exists $\boldsymbol{l}_0={}^t(l_1^0,\,l_2^0,\,\ldots,l_r^0)$ such that $0<l_1^0<\cdots<l_r^0<K$ with $h^*(\boldsymbol{l}_0)>\overline{h}$ and the initial value $\boldsymbol{U}(0,x)$ of (3.2.3) is sufficiently close to $\sum_{j=1}^r \boldsymbol{S}((-1)^j(x-l_j^0))+\tilde{\boldsymbol{S}}$. Then there exists $\boldsymbol{l}(t)={}^t(l_1(t),\,l_2(t),\,\ldots,l_r(t))$ with $0<l_j(t)<K\,(j=1,\,2,\ldots,\,r)$ such that the solution $\boldsymbol{U}(t,x)$ of (3.2.3) satisfies

$$\sup_{x\in[0,K]}|\boldsymbol{U}(t,x)-\boldsymbol{U}^*(x;\boldsymbol{l}(t))|\to\boldsymbol{0}\,(t\to\infty)$$

as long as $h^*(\boldsymbol{l}(t))>\overline{h}$, where $\boldsymbol{U}^*(x;\boldsymbol{l})=\sum_{j=1}^r \boldsymbol{S}((-1)^j(x-l_j))+\tilde{\boldsymbol{S}}+\boldsymbol{\sigma}^*(x;\boldsymbol{l})$ and $\boldsymbol{\sigma}^*(x;\boldsymbol{l})$ is a C^1 function with respect to $\boldsymbol{l}$ with $\boldsymbol{\sigma}^*(x;\boldsymbol{l}):={}^t(\sigma_1^*(x;\boldsymbol{l}),\ldots,\sigma_r^*(x;\boldsymbol{l}))$ satisfying

$$||\boldsymbol{\sigma}^*(x;\boldsymbol{l})||_{L^\infty(0,K)}\le O(\delta_+^*(\boldsymbol{l})).$$

Moreover $\boldsymbol{l}(t)$ satisfies

$$\begin{cases} \dfrac{dl_1}{dt}=M_+(\mathrm{e}^{-2\alpha_+l_1}-\mathrm{e}^{-\alpha_+(l_2-l_1)})+H_1^*(\boldsymbol{l}),\\ \vdots\\ \dfrac{dl_j}{dt}=M_+(\mathrm{e}^{-\alpha_+(l_j-l_{j-1})}-\mathrm{e}^{-\alpha_+(l_{j+1}-l_j)})+H_j^*(\boldsymbol{l}),\\ \vdots\\ \dfrac{dl_r}{dt}=M_+(\mathrm{e}^{-\alpha_+(l_r-l_{r-1})}-\mathrm{e}^{-2\alpha_+(K-l_r)})+H_r^*(\boldsymbol{l}) \end{cases}$$

with $|H_j^*(\boldsymbol{l})|\le O((\delta_+^*(\boldsymbol{l}))^2)$ as long as $h^*(\boldsymbol{l}(t))>\overline{h}$. Here $M_+:=2\alpha_+\langle D\boldsymbol{a}_+,\boldsymbol{a}_+^*\rangle$ and $\tilde{\boldsymbol{S}}:=\dfrac{\boldsymbol{S}_++(-1)^r\boldsymbol{S}_+}{2}$.

We define $H_j(\boldsymbol{l}) := M_+(\mathrm{e}^{-\alpha_+(l_j-l_{j-1})} - \mathrm{e}^{-\alpha_+(l_{j+1}-l_j)})$ and $\boldsymbol{H}(\boldsymbol{l}) := {}^t(H_1(\boldsymbol{l}),\dots,H_r(\boldsymbol{l}))$ for $\boldsymbol{l} = {}^t(l_1,\dots,l_r)$, where $l_0 := -l_1$ and $l_{r+1} := 2K - l_r$. Furthermore let $\tilde{\boldsymbol{l}} := {}^t(\tilde{l}_1,\dots,\tilde{l}_r)$ be the equilibrium point such that $\boldsymbol{H}(\tilde{\boldsymbol{l}}) = 0$ and $h(\tilde{\boldsymbol{l}}) > \overline{h}$.

$$\dot{\boldsymbol{l}} = \boldsymbol{H}(\boldsymbol{l}) \tag{3.2.6}$$

and let $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$ be the linearized matrix of $\boldsymbol{H}(\tilde{\boldsymbol{l}})$ with respect to $\tilde{\boldsymbol{l}}$.

Corollary 3.2 ([8],[9]). For any $K > K_0$, there exists an equilibrium of (3.2.6) $\tilde{\boldsymbol{l}} = {}^t(\tilde{l}_1,\dots,\tilde{l}_r)$ such that $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}})) \subset \left\{z \in \mathbb{C}\,;\ |z| \geq O(\delta_+^*(\tilde{\boldsymbol{l}})\right\}$ holds. Then there exists a stationary solution $\boldsymbol{S}^*(x)$ of (3.2.3) such that

$$\sup_{x\in[0\,,\,K]}\left|\boldsymbol{S}^*(x) - \boldsymbol{U}^*(x;\tilde{\boldsymbol{l}})\right| \leq O(\delta_+^*(\tilde{\boldsymbol{l}})),$$

where $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}}))$ is the set of eigenvalues for $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$.

Remark 3.3. $\boldsymbol{S}^*(x)$ statedin Corollary 3.2 is called an r-layered front stationary solution.

We consider

$$\begin{cases} L_{\mathcal{N}}(\boldsymbol{S}^*)\boldsymbol{\Phi}(x) = \lambda\boldsymbol{\Phi}(x),\ x \in (0, K), \\ \partial_x\boldsymbol{\Phi}(0) = \partial_x\boldsymbol{\Phi}(K) = 0. \end{cases}$$

Then the following holds.

Theorem 3.4. Suppose there exist sufficiently large positive constants K_0 and $\overline{h}$ satisfying $K_0 > \overline{h}r$. Then for any $K > K_0$, there exist positive constants ρ_1 and C_1 such that $\rho_1 \leq C_1 e^{-\alpha_+\frac{K}{r}}$ holds and $L_{\mathcal{N}}(\boldsymbol{S}^*)$ has r eigenvalues $\lambda_n, (n = 0, 1, \dots, r-1)$ in $B(0\,;\rho_1)$, while the other spectrum λ of $L_{\mathcal{N}}(\boldsymbol{S}^*)$ satisfy $Re\{\lambda\} < -\beta_1$ for a positive constant β_1,where $B(0\,;\epsilon) := \{z \in \mathbb{R}\,; |z| \leq \epsilon, \epsilon > 0\}$. Furthermore, excluding constant doubling, r eigenvalues $\lambda_n\,(n = 0, 1, \dots, r-1)$ of $L_{\mathcal{N}}(\boldsymbol{S}^*)$ and eigenfunctions $\boldsymbol{\Phi}_n(x)\,(n = 0, 1, \dots, r-1)$ associated with $\lambda_n\,(n = 0, 1, \dots, r-1)$ satisfy

$$\lambda_n = -4M_+\alpha_+\mathrm{e}^{-\alpha_+\frac{K}{r}}\cos^2\left(\frac{n\pi}{2r}\right)(1 + O(\mathrm{e}^{-\alpha_+\frac{K}{r}})),$$

$$\sup_{x\in[0\,.\,K]}\left|\boldsymbol{\Phi}_n(x) - \sum_{j=1}^{r}\cos\left(\frac{n\pi}{K}\,\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha_+\frac{K}{r}}),$$

where $\bar{l}_j := \dfrac{(2j-1)K}{2r}\,(j = 1, \dots, r)$.

Remark 3.4. From Theorem 3.4, we know that signs of $\lambda_n\,(n=0,1,\dots,r-1)$ are determined by M_+. If $M_+>0\,(<0)$, then $\lambda_n<0\,(>0)\,(n=0,1,\dots,r-1)$. Therefore, if $M_+>0\,(<0)$, we also know that $\boldsymbol{S}^*$ stated in Corollary 3.2. is stable (unstable).

3.2.3 The application to the Allen-Cahn-equation

Fix $r=3$. Let us consider the Allen-Cahn equation

$$\begin{cases} 0=\epsilon^2\partial_{xx}S^*+F(S^*),\ t>0,\ x\in(0,1),\\ \partial_x S^*(0)=\partial_x S^*(1)=0, \end{cases}$$

and its linearized eigenvalue problem associated with $S^*(x)$ stated in Corollary 3.2

$$\begin{cases} \epsilon^2\partial_{xx}\Phi(x)+F'(S^*(x))\Phi(x)=\lambda\Phi(x), x\in(0,1),\\ \partial_x\Phi(0)=\partial_x\Phi(1)=0, \end{cases} \tag{3.2.7}$$

where $0<\epsilon\ll 1$ and $F(u)=u(1-u^2)$ or $F(u)=\sin u$.

We denote by $U\sim V$ when $\sup\limits_{x\in[0,1]}|U(x;\epsilon)-V(x;\epsilon)|\le O(e^{-\frac{\alpha}{\epsilon r}})\,(\epsilon\to 0)$ in 3.2.3.

<u>In the case of $F(u)=u(1-u^2)$.</u>

First, we note that the Allen-Cahn equation on $\mathbb{R}$ has the following stable front-type stationary solution

$$S(x)=\tanh\left(\frac{x}{\sqrt{2}\epsilon}\right)$$

connectiong $S_\pm=\pm1$([13]). By exactly the same calculation as for [8], [9], we obtain $a_+=-2,\ \alpha_+=\dfrac{\sqrt{2}}{\epsilon}, a_+^*=3$. Therefore, we see

$$M_+=2\alpha_+\left\langle Da_+,a_+^*\right\rangle=2\cdot\frac{\sqrt{2}}{\epsilon}\cdot(-2)\cdot(\epsilon^2\cdot 3)=-12\sqrt{2}\epsilon<0.$$

From the above, $\lambda_n\,(n=0,1,2)$ satisfy

$$\lambda_n\sim 96\exp\left(-\frac{\sqrt{2}}{3\epsilon}\right)\cos^2\frac{n\,\pi}{6}\,(n=0,1,2)$$

and there exists $\rho_0 > 0$ such that $Re\{\lambda_n\} < -\rho_0\,(n = 3, 4, 5, \ldots)$ by Theorem 3.4. Moreover, eigenfunctions $\mathbf{\Phi}_n(x)\,(n = 0, 1, \ldots 2)$ associated with $\lambda_n\,(n = 0, 1, 2)$ satisfy

$$\mathbf{\Phi}_0(x) \sim \sum_{j=1}^{3} \mathrm{sech}^2\left(\frac{x - \bar{l}_j}{\sqrt{2}\epsilon}\right),$$

$$\mathbf{\Phi}_1(x) \sim \sum_{j=1}^{3} \cos\left\{\pi \bar{l}_j\right\} \mathrm{sech}^2\left(\frac{x - \bar{l}_j}{\sqrt{2}\epsilon}\right),$$

$$\mathbf{\Phi}_2(x) \sim \sum_{j=1}^{3} \cos\left\{2\pi \bar{l}_j\right\} \mathrm{sech}^2\left(\frac{x - \bar{l}_j}{\sqrt{2}\epsilon}\right),$$

where $\bar{l}_1 = \dfrac{1}{6}$, $\bar{l}_2 = \dfrac{1}{2}$, and $\bar{l}_3 = \dfrac{5}{6}$ by Theorem 3.4 (Fig 1). This result is consistent with [37], [38].

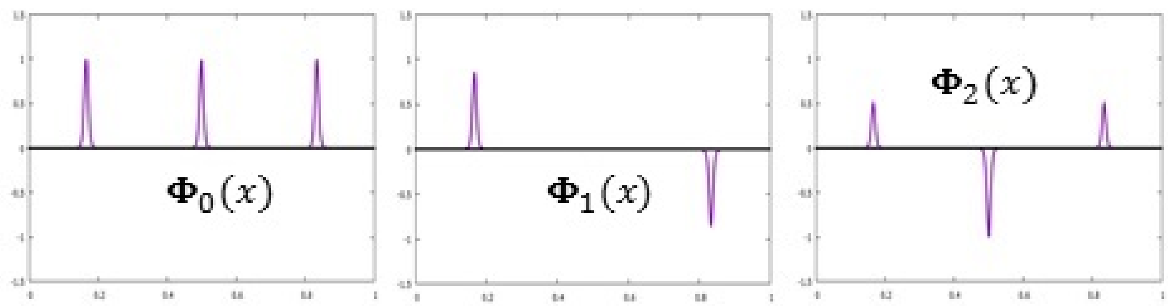

Figure 1: Profiles of $\Phi_0(x), \Phi_1(x), \Phi_2(x)$ with the Neumann boundary conditions(the Allen-Cahn equation).

<u>In the case of $F(u) = \sin u$.</u>

First, we note that the Allen-Cahn equation on $\mathbb{R}$ has the following stable front-type stationary solution.

$$S(x) = 2\mathrm{Arcsin}\left\{\tanh\left(\frac{x}{\epsilon}\right)\right\}$$

(e.g.[37], [38]). By exactly the same calculation as in [12], we obtain $a_+ = -4$, $\alpha_+ = \dfrac{1}{\epsilon}$, $a_+^* = \dfrac{1}{2}$. Therefore, we see

$$M_+ = 2\alpha_+ \left\langle Da_+, a_+^* \right\rangle = -4\epsilon < 0.$$

From the above, $\lambda_n\,(n=0,1,2)$ satisfy

$$\lambda_n \sim 16\exp\left(-\frac{1}{3\epsilon}\right)\cos^2\frac{n\,\pi}{6}\,(n=0,1,2)$$

and there exists $\rho_0>0$ such that $Re\{\lambda_n\}<-\rho_0\,(n=3,4,5,\ldots)$ by Theorem 3.4. Moreover eigenfunctions $\mathbf{\Phi}_n(x)\,(n=0,1,2)$ associated with $\lambda_n\,(n=0,1,2)$ satisfy

$$\mathbf{\Phi}_0(x)\sim\sum_{j=1}^{3}\mathrm{sech}\left(\frac{x-\bar{l}_j}{\epsilon}\right),$$

$$\mathbf{\Phi}_1(x)\sim\sum_{j=1}^{3}\cos\left\{\pi\bar{l}_j\right\}\mathrm{sech}\left(\frac{x-\bar{l}_j}{\epsilon}\right),$$

$$\mathbf{\Phi}_2(x)\sim\sum_{j=1}^{3}\cos\left\{2\pi\bar{l}_j\right\}\mathrm{sech}\left(\frac{x-\bar{l}_j}{\epsilon}\right),$$

where $\bar{l}_1=\frac{1}{6}$, $\bar{l}_2=\frac{1}{2}$, $\bar{l}_3=\frac{5}{6}$ by Theorem 3.4. This result is also consistent with [35]-[38].

Remark 3.5. In (3.2.7) for $F(u)=u(1-u^2)$ or $F(u)=\sin u$, eigenvalues for $n=0,1,\ldots,r$ are represented by the product of the cosine functions and the exponential functions, and eigenfunctions associated with those eigenvalues are represented by the product of the cosine function and the hyperbolic function.

3.2.4 The application to the Gray-Scott-model

Fix $r=3$. As fixing the same parameter as [8], let consider the Gray-Scott-model

$$\begin{cases} 0=\partial_{xx}S_u^*-S_u^*(S_v^*)^2+\epsilon^2(1-S_u^*), & x\in(0,K),\\ 0=\epsilon^2\partial_{xx}S_v^*-\epsilon^{1/2}S_v^*+S_u^*(S_v^*)^2, \\ \partial_x S_u^*(0)=\partial_x S_u^*(K)=0, \\ \partial_x S_v^*(0)=\partial_x S_v^*(K)=0, \end{cases}$$

and its linearized eigenvalue problem associated with $\boldsymbol{S}^*(x)=(S_u^*(x),S_v^*(x))$ stated in Corollary 3.1

$$\begin{cases} L_{\mathcal{N}}(\boldsymbol{S}^*)\mathbf{\Phi}_n(x)=\lambda_n\mathbf{\Phi}_n(x), & x\in(0,K),\\ \partial_x\mathbf{\Phi}_n(0)=\partial_x\mathbf{\Phi}_n(K)=\mathbf{0}\,(n=0,1,\ldots), \end{cases} \tag{3.2.8}$$

where $0 < \epsilon \ll 1$ and $K \gg 1$.

We denote by $U \sim V$ when $\sup_{x\in[0,K]} |U(x) - V(x)| \le O(e^{-\alpha\frac{K}{r}})\,(K \to \infty)$ in 3.2.4. Note that results for eigenvalues and eigenfunctions for this linearized eigenvalue problem are entirely new.

First, we note that the Gray-Scott-model on $\mathbb{R}$ has the stable pulse-type solution $\boldsymbol{S}(x) = (S_u(x), S_v(x))$ as $\epsilon \to 0$(see [5]). By [8], we know that $\boldsymbol{S}(x)$ and $\Phi^*(x)$ satisfy

$$\boldsymbol{S}(x) \to e^{-\epsilon|x|}\boldsymbol{a} + \boldsymbol{U}^*,\ \ \Phi^*(x) \to e^{-\epsilon|x|}\boldsymbol{a}^*$$

as $x \to \pm\infty$. Here $\boldsymbol{a} = {}^t(-a, 0)$ and $\boldsymbol{a}^* = {}^t(-\epsilon^{3/4}a^*, 0), \boldsymbol{U}^* := {}^t(1,\,0)$ by certain positive constants a, a^*. Therefore, we also immediately see that $M_0 > 0$ by Theorem 3.2.

From the above, $\lambda_n\,(n = 0, 1, 2)$ and $\boldsymbol{\Phi}_n(x)\,(n = 0, 1, 2)$ of (3.2.8) satisfy

$$\text{(i) } \lambda_0 \sim -4M_0\epsilon \mathrm{e}^{\frac{-\epsilon K}{r}},\ \ \boldsymbol{\Phi}_0(x) \sim \begin{pmatrix} \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\partial_x S_u\left(x - \bar{l}_j\right) \\ \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\partial_x S_v\left(x - \bar{l}_j\right) \end{pmatrix}.$$

$$\text{(ii) } \lambda_1 \sim -3M_0\epsilon \mathrm{e}^{\frac{-\epsilon K}{r}},\ \ \boldsymbol{\Phi}_1(x) \sim \begin{pmatrix} \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\cos\left\{\frac{\pi}{K}\bar{l}_j\right\}\partial_x S_u\left(x - \bar{l}_j\right) \\ \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\cos\left\{\frac{\pi}{K}\bar{l}_j\right\}\partial_x S_v\left(x - \bar{l}_j\right) \end{pmatrix}.$$

$$\text{(iii) } \lambda_2 \sim -M_0\epsilon \mathrm{e}^{\frac{-\epsilon K}{r}},\ \ \boldsymbol{\Phi}_2(x) \sim \begin{pmatrix} \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\cos\left\{\frac{2\pi}{K}\bar{l}_j\right\}\partial_x S_u\left(x - \bar{l}_j\right) \\ \displaystyle\sum_{j=1}^{3}(-1)^{j+1}\cos\left\{\frac{2\pi}{K}\bar{l}_j\right\}\partial_x S_v\left(x - \bar{l}_j\right) \end{pmatrix}$$

by Theorem 3.2. Here $\bar{l}_1 = \dfrac{K}{6}$, $\bar{l}_2 = \dfrac{K}{2}$, $\bar{l}_3 = \dfrac{5K}{6}$ (Fig 2). Moreover, there exists $\rho_0 > 0$ such that $Re\,\{\lambda_n\} < -\rho_0$ for $n \ge 3$.

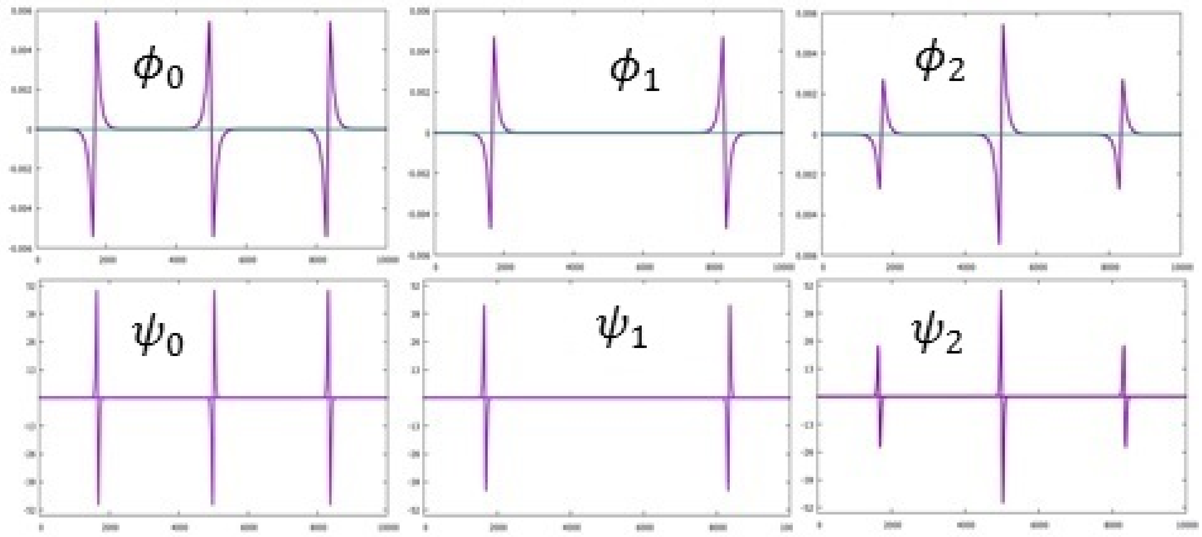

Figure 2: Profiles of $\mathbf{\Phi}_n(x) := (\phi_n(x), \psi_n(x)), n = 0, 1, 2$ with the Neumann boundary conditions (the Gray-Scott model)

3.3 In the case of the Periodic boundary condition

In this subsection, we deal with the following system

$$\begin{cases} \partial_t \boldsymbol{U} = D\partial_{xx}\boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}) \quad (t, x) \in (0, \infty) \times (0, K), \\ \boldsymbol{U}(t, +0) = \boldsymbol{U}(t, K), \\ \partial_x \boldsymbol{U}(t, +0) = \partial_x \boldsymbol{U}(t, K) \end{cases} \tag{3.3.9}$$

for $K \gg 1$, where $D := diag\{d_1, \ldots, d_N\}$ and $F : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function. We define $h^*(\boldsymbol{l}) := \min\{K - l_r + l_1, l_2 - l_1, \ldots, l_r - l_{r-1}\}$ and $\delta^*(\boldsymbol{l}) := e^{-\alpha\tilde{h}^*(\boldsymbol{l})}$. Furthermore, $\sim$ is defined exactly as in section 3.2.

Note that all the results in this section are new.

3.3.1 Main results for r-layered pulse-stationary solutions

Assume (H1),(H2),(H3),(A1). Then we have the following theorem by [8] and [9].

Theorem 3.5([8], [9]). Suppose that there exist sufficiently large positive constants $\overline{h}$, K_0 such that $K_0 > \overline{h}r$. Moreover suppose for any $K > K_0$, there exists $\boldsymbol{l}_0 = {}^t(l_1^0,\, l_2^0,\, \dots\,, l_r^0)$ such that $0 < l_1^0 < \cdots < l_r^0 < K$ with $h^*(\boldsymbol{l}_0) > \overline{h}$ and the initial value $\boldsymbol{U}(0,x)$ of (3.3.9) is sufficiently close to $\sum_{j=1}^r \boldsymbol{S}(x - l_j^0)$. Then there exists $\boldsymbol{l}(t) = {}^t(l_1(t),\, l_2(t),\, \dots\,, l_r(t))$ with $0 < l_j(t) < K$ ($j = 1, 2, \dots, r$) such that the solution $\boldsymbol{U}(t,x)$ of (3.3.9) satisfies

$$\sup_{x\in[0,K]} |\boldsymbol{U}(t,x) - \boldsymbol{U}^*(x;\boldsymbol{l}(t))| \to \boldsymbol{0}\,(t\to\infty)$$

as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$, where $\boldsymbol{U}^*(x;\boldsymbol{l}) = \sum_{j=1}^r \boldsymbol{S}(x-l_j) + \boldsymbol{\sigma}^*(x;\boldsymbol{l})$ and $\boldsymbol{\sigma}^*(x;\boldsymbol{l})$ is a C^1 function with respect to $\boldsymbol{l}$ with $\boldsymbol{\sigma}^*(x;\boldsymbol{l}) := {}^t(\sigma_1^*(x;\boldsymbol{l}),\dots,\sigma_r^*(x;\boldsymbol{l}))$ satisfying

$$||\boldsymbol{\sigma}^*(x;\boldsymbol{l})||_{L^\infty(0,K)} \le O(\delta^*(\boldsymbol{l})).$$

Moreover $\boldsymbol{l}(t)$ satisfies

$$\begin{cases} \dfrac{dl_1}{dt} = M_0(\mathrm{e}^{-\alpha(K-l_r+l_1)} - \mathrm{e}^{-\alpha(l_2-l_1)}) + H_1^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_j}{dt} = M_0(\mathrm{e}^{-\alpha(l_j-l_{j-1})} - \mathrm{e}^{-\alpha(l_{j+1}-l_j)}) + H_j^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_r}{dt} = M_0(\mathrm{e}^{-\alpha(l_r-l_{r-1})} - \mathrm{e}^{-\alpha(K-l_r+l_1)}) + H_r^*(\boldsymbol{l}) \end{cases}$$

with $|H_j^*(\boldsymbol{l})| \le O((\delta^*(\boldsymbol{l}))^2)$ as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$. Here $M_0 := 2\alpha\langle D\boldsymbol{a}, \boldsymbol{a}^*\rangle$.

We define $H_j(\boldsymbol{l}) := M_0(\mathrm{e}^{-\alpha(l_j-l_{j-1})} - \mathrm{e}^{-\alpha(l_{j+1}-l_j)})$ and $\boldsymbol{H}(\boldsymbol{l}) := {}^t(H_1(\boldsymbol{l}),\dots,H_r(\boldsymbol{l}))$ for $\boldsymbol{l} = {}^t(l_1,\dots\,,l_r)$, where $l_0 := -K + l_r$ and $l_{r+1} := K + l_1$. Furthermore Let $\tilde{\boldsymbol{l}} := {}^t(\tilde{l}_1,\dots,\tilde{l}_r)$ be the equilibrium point such that $\boldsymbol{H}(\tilde{\boldsymbol{l}}) = 0$ and $h(\tilde{\boldsymbol{l}}) > \overline{h}$.

$$\dot{\boldsymbol{l}} = \boldsymbol{H}(\boldsymbol{l}) \tag{3.3.10}$$

and let $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$ be the linearized matrix of $\boldsymbol{H}(\tilde{\boldsymbol{l}})$ with respect to $\tilde{\boldsymbol{l}}$.

Corollary 3.3 ([8], [9]). For any $K > K_0$, there exists an equilibrium of (3.3.10) $\bar{\boldsymbol{l}} = {}^t(\tilde{l}_1,\dots,\tilde{l}_r)$ such that $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}})) \subset \left\{z\in\mathbb{C};\ |z| \ge O(\delta^*(\tilde{\boldsymbol{l}}))\right\}$ holds. Then there exists a stationary solution $\boldsymbol{S}^*(x)$ of (3.3.9) such that

$$\sup_{x\in[0\,,\,K]} \left|\boldsymbol{S}^*(x) - \boldsymbol{U}^*(x;\tilde{\boldsymbol{l}})\right| \le O(\delta^*(\boldsymbol{l})),$$

where $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}}))$ is the set of eigenvalues for $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$.

Remark 3.6. The stationary solution $\boldsymbol{S}^*(x)$ stated in Corollary 3.3 is called an r-layered pulse stationary solution.

We consider

$$\begin{cases} L_{\mathcal{P}}(\boldsymbol{S}^*)\tilde{\boldsymbol{\Phi}}(x) = \tilde{\lambda}\tilde{\boldsymbol{\Phi}}(x),\ x \in (0, K), \\ \tilde{\boldsymbol{\Phi}}(0) = \tilde{\boldsymbol{\Phi}}(K), \\ \partial_x\tilde{\boldsymbol{\Phi}}(0) = \partial_x\tilde{\boldsymbol{\Phi}}(K), \end{cases}$$

where $L_{\mathcal{P}}(\boldsymbol{S}^*) := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}^*)$ on $H^2(0, K)$ with the periodic boundary condition. The following holds.

Theorem 3.6. We define $r_0 := \dfrac{r}{2} \in \mathbb{N}$. Suppose there exist sufficiently large positive constants K_0 and $\overline{h}$ satisfying $K_0 > \overline{h}r$. Then for any $K > K_0$, there exist positive constants ρ_1 and C_1 such that $\rho_1 \leq C_1 e^{-\alpha\frac{K}{r}}$ holds and $L_{\mathcal{P}}(\boldsymbol{S}^*)$ has r eigenvalues $\tilde{\lambda}_n, (n = 0, \ldots, r-1)$ in $B(0\,;\rho_1)$, while the other spectrum $\tilde{\lambda}$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ satisfy $Re\{\tilde{\lambda}\} < -\beta_1$ for a positive constant β_1, where $B(0\,;\epsilon) := \{z \in \mathbb{R}\,; |z| \leq \epsilon, \epsilon > 0\}$ Furthermore, excluding constant doubling, the r eigenvalues $\tilde{\lambda}_n\,(n = 0, 1, \ldots, r-1)$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ and eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x)\,(n = 0, 1, \ldots, r-1)$ associated wuth $\tilde{\lambda}_n\,(n = 0, 1, \ldots, r-1)$ satisfy the following conditions.

$$\tilde{\lambda}_n = -4M_0\alpha\mathrm{e}^{-\alpha\frac{K}{r}}\sin^2\left(\frac{n\pi}{r}\right)(1 + O(\mathrm{e}^{-\alpha\frac{K}{r}}))$$

and are semisimple. Furthermore, there exist eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x)$, $\tilde{\boldsymbol{\Psi}}_n(x)$ associated with $\tilde{\lambda}_n$ $(n = 0, 1, \ldots, r-1)$ satisfy following.

(a) When $n = 0$, excluding constant doubling,

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha\frac{K}{r}}).$$

(b) When $1 \leq n \leq r_0 - 1$, excluding constant doubling,

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}\sin\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha\frac{K}{r}}),$$

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Psi}}_n(x) - \sum_{j=1}^{r}\cos\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha\frac{K}{r}}).$$

(c) When $n = r_0$, excluding constant doubling,

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}(-1)^{j+1}\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha\frac{K}{r}}).$$

Here $\bar{l}_j := \dfrac{(2j-1)K}{2r}\,(j = 1,\ldots,r)$.

Theorem 3.7. We define $r_1 := \dfrac{r+1}{2} \in \mathbb{N}$. Suppose there exist sufficiently large positive constants K_0 and $\overline{h}$ satisfying $K_0 > \overline{h}r$. Then for any $K > K_0$, there exist positive constants ρ_1 and C_1 such that $\rho_1 \le C_1 e^{-\alpha\frac{K}{r}}$ holds and $L_{\mathcal{P}}(\boldsymbol{S}^*)$ has r eigenvalues $\tilde{\lambda}_n, (n = 0,\ldots,r-1)$ in $B(0\,;\rho_1)$, while the other spectrum $\tilde{\lambda}$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ satisfy $Re\{\tilde{\lambda}\} < -\beta_1$ for a positive constant β_1,where $B(0\,;\epsilon) := \{z \in \mathbb{R}\,; |z| \le \epsilon, \epsilon > 0\}$. Furthermore, excluding constant doubling, the r eigenvalues $\tilde{\lambda}_n\,(n = 0,1,\ldots,r-1)$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ and eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x)\,(n = 0,1,\ldots,r-1)$ associated wuth $\tilde{\lambda}_n\,(n = 0,1,\ldots,r-1)$ satisfy the following conditions.

$$\tilde{\lambda}_n = -4M_0\alpha \mathrm{e}^{-\alpha\frac{K}{r}} \sin^2\left(\frac{n\pi}{r}\right)(1 + O(\mathrm{e}^{-\alpha\frac{K}{r}}))$$

and are semisimple. Furthermore, there exist eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x), \tilde{\boldsymbol{\Psi}}_n(x)$ associated with $\tilde{\lambda}_n\ (n = 0,1,\ldots,r-1)$ satisfy following.

(a) When $n = 0$, excluding constant doubling,

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r} \partial_x \boldsymbol{S}(x-\bar{l}_j)\right| \le O(\mathrm{e}^{-\alpha\frac{K}{r}}).$$

(b) When $1 \le n \le r_1 - 1$, excluding constant doubling,

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r} \sin\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x \boldsymbol{S}(x-\bar{l}_j)\right| \le O(\mathrm{e}^{-\alpha\frac{K}{r}}),$$

$$\sup_{x\in[0\,,K]}\left|\tilde{\boldsymbol{\Psi}}_n(x) - \sum_{j=1}^{r} \cos\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x \boldsymbol{S}(x-\bar{l}_j)\right| \le O(\mathrm{e}^{-\alpha\frac{K}{r}}).$$

Here $\bar{l}_j := \dfrac{(2j-1)K}{2r}\,(j = 1,\ldots,r)$.

3.3.2 Main results for r-layered front-stationary solutions

Assume (H1)'-(H3)' and (A2). By [8], [9], we have the following theorem.

Theorem 3.8([8], [9]). Suppose that there exist sufficiently large positive constants $\overline{h}$, K_0 such that $K_0 > \overline{h}r$. Moreover suppose for any $K > K_0$, there exists $\boldsymbol{l}_0 = {}^t(l_1^0,\, l_2^0,\,\ldots,l_r^0)$ such that $0 < l_1^0 < \cdots < l_r^0 < K$ with $h^*(\boldsymbol{l}^0) > \overline{h}$ and the initial value $\boldsymbol{U}(0,x)$ of (3.3.9) is sufficiently close to $\sum_{j=1}^{r} \boldsymbol{S}((-1)^j(x - l_j^0))$. Then there exists $\boldsymbol{l}(t) = {}^t(l_1(t),\, l_2(t),\,\ldots,l_r(t))$ with $0 < l_j(t) < K\,(j = 1,\,2,\ldots,\,r)$ such that the solution $\boldsymbol{U}(t,x)$ of (3.3.9) satisfies

$$\sup_{x\in[0,K]} |\boldsymbol{U}(t,x) - \boldsymbol{U}^*(x;\boldsymbol{l}(t))| \to 0\,(t \to \infty)$$

as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$, where $\boldsymbol{U}^*(x;\boldsymbol{l}) = \sum_{j=1}^{r} \boldsymbol{S}((-1)^j(x-l_j)) + \boldsymbol{\sigma}^*(x;\boldsymbol{l})$ and $\boldsymbol{\sigma}^*(x;\boldsymbol{l})$ is a C^1 function with respect to $\boldsymbol{l}$ with $\boldsymbol{\sigma}^*(x;\boldsymbol{l}) := {}^t(\sigma_1^*(x;\boldsymbol{l}),\ldots,\sigma_r^*(x;\boldsymbol{l}))$ satisfying

$$||\boldsymbol{\sigma}^*(x;\boldsymbol{l})||_{L^\infty(0,K)} \le O(\delta_+^*(\boldsymbol{l})).$$

Moreover $\boldsymbol{l}(t)$ satisfies

$$\begin{cases} \dfrac{dl_1}{dt} = M_+(\mathrm{e}^{-\alpha_+(K-l_r+l_1)} - \mathrm{e}^{-\alpha_+(l_2-l_1)}) + H_1^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_j}{dt} = M_+(\mathrm{e}^{-\alpha_+(l_j-l_{j-1})} - \mathrm{e}^{-\alpha_+(l_{j+1}-l_j)}) + H_j^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_r}{dt} = M_+(\mathrm{e}^{-\alpha_+(l_r-l_{r-1})} - \mathrm{e}^{-\alpha_+(K-l_r+l_1)}) + H_r^*(\boldsymbol{l}) \end{cases}$$

with $|H_j^*(\boldsymbol{l})| \le O((\delta_+^*(\boldsymbol{l}))^2)$ as long as $h^*(\boldsymbol{l}(t)) > \overline{h}$. Here $M_+ := 2\alpha\langle D\boldsymbol{a}_+, \boldsymbol{a}_+^*\rangle$ and $\tilde{\boldsymbol{S}} := \dfrac{\boldsymbol{S}_+ + (-1)^r\boldsymbol{S}_+}{2}$

We define $H_j(\boldsymbol{l}) := M_+(\mathrm{e}^{-\alpha(l_j-l_{j-1})} - \mathrm{e}^{-\alpha(l_{j+1}-l_j)})$ and $\boldsymbol{H}(\boldsymbol{l}) := {}^t(H_1(\boldsymbol{l}),\ldots,H_r(\boldsymbol{l}))$ for $\boldsymbol{l} = {}^t(l_1,\ldots,l_r)$, where $l_0 := -K + l_r$ and $l_{r+1} := K + l_1$. Furthermore Let $\tilde{\boldsymbol{l}} := {}^t(\tilde{l}_1,\ldots,\tilde{l}_r)$ be the equilibrium point such that $\boldsymbol{H}(\tilde{\boldsymbol{l}}) = 0$ and $h(\tilde{\boldsymbol{l}}) > \overline{h}$.

$$\dot{\boldsymbol{l}} = \boldsymbol{H}(\boldsymbol{l}) \tag{3.3.11}$$

and let $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$ be the linearized matrix of $\boldsymbol{H}(\tilde{\boldsymbol{l}})$ with respect to $\tilde{\boldsymbol{l}}$.

Corollary 3.4 ([8], [9]). Suppose for any $K > K_0$, there exists an equilibrium of (3.3.11) $\tilde{\boldsymbol{l}} = {}^t(\tilde{l}_1,\ldots,\tilde{l}_r)$ such that $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}})) \subset \left\{z \in \mathbb{C};\ |z| \ge O(\delta_+^*(\tilde{\boldsymbol{l}}))\right\}$ holds. Then there exists a stationary solution $\boldsymbol{S}^*(x)$ of (3.3.9) such that

$$\sup_{x\in[0\,,K]} \left|\boldsymbol{S}^*(x) - \boldsymbol{U}^*(x;\tilde{\boldsymbol{l}})\right| \le O(\delta_+^*(\tilde{\boldsymbol{l}})),$$

where $\Sigma(\boldsymbol{H}'(\tilde{\boldsymbol{l}}))$ is the set of eigenvalues for $\boldsymbol{H}'(\tilde{\boldsymbol{l}})$.

Remark 3.7. $\boldsymbol{S}_r(x)$ stated in Corollary 3.4 is also called a r-layered front stationary solution.

In the above $\boldsymbol{S}^*(x)$, we consider

$$\begin{cases} L_{\mathcal{P}}(\boldsymbol{S}^*)\tilde{\boldsymbol{\Phi}}(x) = \tilde{\lambda}\tilde{\boldsymbol{\Phi}}(x),\ x \in (0, K), \\ \tilde{\boldsymbol{\Phi}}(0) = \tilde{\boldsymbol{\Phi}}(K), \\ \partial_x\tilde{\boldsymbol{\Phi}}(0) = \partial_x\tilde{\boldsymbol{\Phi}}(K), \end{cases}$$

The following holds.

Theorem 3.9. We define $r_0 := \dfrac{r}{2} \in \mathbb{N}$. There exist a sufficiently large positive constants K_0 and $\overline{h}$ satisfying $K_0 > \overline{h}r$. Then for any $K > K_0$, there exist a positive constants ρ_1 and C_1 such that $\rho_1 \leq C_1 e^{-\alpha_+\frac{K}{r}}$ holds and $L_{\mathcal{P}}(\boldsymbol{S}^*)$ has r eigenvalues $\tilde{\lambda}_n, (n = 0, \ldots, r-1)$ in $B(0\,;\rho_1)$, while the other spectrum $\tilde{\lambda}$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ satisfy $Re\{\tilde{\lambda}\} < -\beta_1$ for a positive constant β_1, where $B(0\,;\epsilon) := \{z \in \mathbb{R}\,; |z| \leq \epsilon, \epsilon > 0\}$. Furthermore, excluding constant doubling, r eigenvalues $\tilde{\lambda}_n\,(n = 0, 1, \ldots, r-1)$ of $L_{\mathcal{P}}(\boldsymbol{S}^*)$ and eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x)\,(n = 0, 1, \ldots, r-1)$ associated wuth $\tilde{\lambda}_n\,(n = 0, 1, \ldots, r-1)$ satisfy the following conditions.

$$\tilde{\lambda}_n = -4M_+\alpha_+\mathrm{e}^{-\alpha_+\frac{K}{r}}\sin^2\left(\frac{n\pi}{r}\right)(1 + O(\mathrm{e}^{-\alpha_+\frac{K}{r}}))$$

and are semisimple. Furthermore, there exist eigenfunctions $\tilde{\boldsymbol{\Phi}}_n(x), \tilde{\boldsymbol{\Psi}}_n(x)$ associated with $\tilde{\lambda}_n$, $(0 \leq n \leq r-1)$ satisfies following.

(a) When $n = 0$, excluding constant doubling,

$$\sup_{x\in[0\,,\,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}(-1)^{j+1}\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha_+\frac{K}{r}}).$$

(b) When $1 \leq n \leq r_0 - 1$, excluding constant doubling,

$$\sup_{x\in[0\,,\,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}(-1)^{j+1}\sin\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha_+\frac{K}{r}}),$$

$$\sup_{x\in[0\,,\,K]}\left|\tilde{\boldsymbol{\Psi}}_n(x) - \sum_{j=1}^{r}(-1)^{j+1}\cos\left(\frac{2n\pi}{K}\bar{l}_j\right)\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha_+\frac{K}{r}}).$$

(c) When $n = r_0$, excluding constant doubling,

$$\sup_{x\in[0\,,\,K]}\left|\tilde{\boldsymbol{\Phi}}_n(x) - \sum_{j=1}^{r}\partial_x\boldsymbol{S}(x - \bar{l}_j)\right| \leq O(\mathrm{e}^{-\alpha_+\frac{K}{r}}).$$

Here $\bar{l}_j := \dfrac{(2j-1)K}{2r}\,(j = 1, \ldots, r)$.

3.3.3 The application to the Allen-Cahn equation

Fix $r = 4$. Let us consider the Allen-Cahn equation

$$\begin{cases} 0 = \epsilon^2 \partial_{xx} S^* + F(S^*),\ t > 0,\ x \in (0,1) \\ S^*(0) = S^*(1),\ \partial_x S^*(0) = \partial_x S^*(1) \end{cases}$$

and its linearized eigenvalue problem associated with $S^*(x)$ stated in Corollary 3.4

$$\begin{cases} \epsilon^2 \partial_{xx} \tilde{\Phi}(x) + F'(S^*(x))\tilde{\Phi}(x) = \tilde{\lambda}\tilde{\Phi}(x), x \in (0,1), \\ \tilde{\Phi}(0) = \tilde{\Phi}(1),\ \partial_x \tilde{\Phi}(0) = \partial_x \tilde{\Phi}(1), \end{cases} \tag{3.3.12}$$

where $0 < \epsilon \ll 1$ and $F(u) = u(1-u^2)$ or $F(u) = \sin u$.
"$\sim$" is exactly the same as in subsection 3.2.3.

<u>In case of $F(u) = u - u^3$.</u>

Since M_+ is obtained exactly as in 3.2.3, eigenvalues of (3.3.12) satisfy

$$\tilde{\lambda}_n \sim 96\text{exp}\left(-\frac{\sqrt{2}}{4\epsilon}\right)\sin^2\frac{n\pi}{4} \qquad (n = 0,1,2,3)$$

and there exists $\rho_0 > 0$ such that $Re\left\{\tilde{\lambda}_n\right\} < -\rho_0\,(n = 4,5,6,\ldots)$ by Theorem 3.9. Moreover $\tilde{\Phi}_n\,(n = 0,1,2,3)$ associated with $\tilde{\lambda}_n\,(n = 0,1,2,3)$ satisfy

$$\begin{aligned}
\tilde{\Phi}_0(x) &\sim \sum_{j=1}^{4}(-1)^{j+1}\text{sech}^2\left(\frac{x-\tilde{l}_j}{\sqrt{2}\epsilon}\right), \\
\tilde{\Phi}_1(x) &\sim \sum_{j=1}^{4}(-1)^{j+1}\sin\left\{2\pi\bar{l}_j\right\}\text{sech}^2\left(\frac{x-\bar{l}_j}{\sqrt{2}\epsilon}\right), \\
\tilde{\Psi}_1(x) &\sim \sum_{j=1}^{4}(-1)^{j+1}\cos\left\{2\pi\bar{l}_j\right\}\text{sech}^2\left(\frac{x-\bar{l}_j}{\sqrt{2}\epsilon}\right), \\
\tilde{\Phi}_2(x) &\sim \sum_{j=1}^{4}\text{sech}^2\left(\frac{x-\bar{l}_j}{\sqrt{2}\epsilon}\right),
\end{aligned}$$

where $\tilde{l}_1 = \frac{1}{8}$, $\tilde{l}_2 = \frac{3}{8}$, $\tilde{l}_3 = \frac{5}{8}$, $\tilde{l}_4 = \frac{7}{8}$ by Theorem 3.9 (Fig 3).

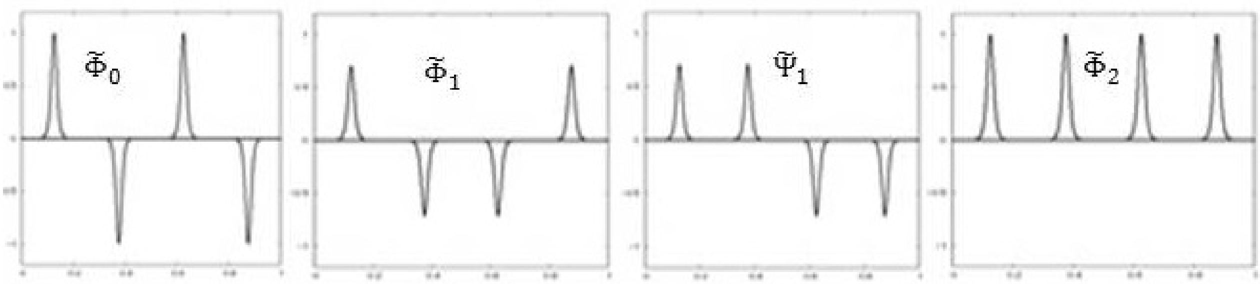

Figure 3: Profiles of $\tilde{\Phi}_0(x), \tilde{\Phi}_1(x), \tilde{\Psi}_1(x), \tilde{\Phi}_2(x)$ with the periodic boundary conditions (the Allen-Cahn equation)

<u>In case of $F(u) = \sin u$.</u>

Since M_+ is obtained exactly as in 3.2.3, eigenvalues of (3.3.12) satisfies

$$\tilde{\lambda}_n \sim 16\mathrm{exp}\left(-\frac{1}{4\epsilon}\right)\sin^2\frac{n\pi}{4} \qquad (n = 0, 1, 2, 3),$$

and there exists $\rho_0 > 0$ such that $Re\left\{\tilde{\lambda}_n\right\} < -\rho_0\,(n = 4, 5, 6, \ldots)$ by Theorem 3.9. Moreover $\tilde{\Phi}_n\,(n = 0, 1, 2, 3)$ associated with $\tilde{\lambda}_n\,(n = 0, 1, 2, 3)$ satisfy

$$\tilde{\Phi}_0(x) \sim \sum_{j=1}^{4}(-1)^{j+1}\mathrm{sech}\left(\frac{x - \tilde{l}_j}{\epsilon}\right),$$

$$\tilde{\Phi}_1(x) \sim \sum_{j=1}^{4}(-1)^{j+1}\sin\left\{2\pi\bar{l}_j\right\}\mathrm{sech}\left(\frac{x - \bar{l}_j}{\epsilon}\right),$$

$$\tilde{\Psi}_1(x) \sim \sum_{j=1}^{4}(-1)^{j+1}\cos\left\{2\pi\bar{l}_j\right\}\mathrm{sech}\left(\frac{x - \bar{l}_j}{\epsilon}\right),$$

$$\tilde{\Phi}_2(x) \sim \sum_{j=1}^{4}\mathrm{sech}\left(\frac{x - \bar{l}_j}{\epsilon}\right),$$

where $\tilde{l}_1 = \frac{1}{8}$, $\tilde{l}_2 = \frac{3}{8}$, $\tilde{l}_3 = \frac{5}{8}$, $\tilde{l}_4 = \frac{7}{8}$ by Theorem 3.9.

3.3.4 The application to the Gray-Scott-model

Fix $r = 3$. "$\sim$" is exactly the same as in subsection 3.2.4.
As taking the same parameter as [8], let us consider the Gray-Scott-model

$$\begin{cases} 0 = \partial_{xx}\tilde{S}_u^* - \tilde{S}_u^*(\tilde{S}_v^*)^2 + \epsilon^2(1-\tilde{S}_u^*), \quad x \in (0,K), \\ 0 = \epsilon^2\partial_{xx}\tilde{S}_v^* - \epsilon^{1/2}\tilde{S}_v^* + \tilde{S}_u^*(\tilde{S}_v^*)^2, \\ \tilde{S}_u^*(0) = \tilde{S}_u^*(K),\, \partial_x\tilde{S}_u^*(0) = \partial_x\tilde{S}_u^*(K), \\ \tilde{S}_v^*(0) = \tilde{S}_v^*(K),\, \partial_x\tilde{S}_v^*(0) = \partial_x\tilde{S}_v^*(K), \end{cases}$$

and its linearized eigenvalue problem associated with $\boldsymbol{S}^*(x) = (\tilde{S}_u^*(x), \tilde{S}_v^*(x))$ stated in Corollary 3.3

$$\begin{cases} L_{\mathcal{P}}(\boldsymbol{S}^*)\widetilde{\boldsymbol{\Phi}}_n(x) = \tilde{\lambda}_n\widetilde{\boldsymbol{\Phi}}_n(x), \quad x \in (0,K), \\ \widetilde{\boldsymbol{\Phi}}(0) = \widetilde{\boldsymbol{\Phi}}(K),\, \partial_x\widetilde{\boldsymbol{\Phi}}(0) = \partial_x\widetilde{\boldsymbol{\Phi}}(K) \qquad (n = 0,1,\ldots), \end{cases}$$

where $0 < \epsilon \ll 1$ and $K \gg 1$. Since M_0 is obtained exactly as in 3.2.4, there exists $\rho_0 > 0$ such that $Re\left\{\tilde{\lambda}_n\right\} < -\rho_0$ $(n = 3,4,5,\ldots)$. Eigenvalues $\tilde{\lambda}_n$ $(n = 0,1,2)$ and corresponding eigenfunctions $\tilde{\boldsymbol{\Phi}}_n$ $(n = 0,1,2)$ are expressed as follows by Theorem 3.7.

$$(i)\ \tilde{\lambda}_0 \sim 0,\ \ \tilde{\boldsymbol{\Phi}}_0(x) \sim \begin{pmatrix} \sum_{j=1}^{3}\partial_x S_u(x-\bar{l}_j) \\ \sum_{j=1}^{3}\partial_x S_v(x-\bar{l}_j) \end{pmatrix},$$

$$(ii)\ \tilde{\lambda}_1 \sim -3M_0\epsilon \mathrm{e}^{\frac{-\epsilon K}{r}},\ \ \tilde{\boldsymbol{\Phi}}_1(x) \sim \begin{pmatrix} \sum_{j=1}^{3}\sin\left(\frac{2\pi}{K}\bar{l}_j\right)\partial_x S_u(x-\bar{l}_j) \\ \sum_{j=1}^{3}\sin\left(\frac{2\pi}{K}\bar{l}_j\right)\partial_x S_v(x-\bar{l}_j) \end{pmatrix},$$

$$(iii)\ \tilde{\lambda}_2 \sim -3M_0\epsilon \mathrm{e}^{\frac{-\epsilon K}{r}},\ \ \tilde{\boldsymbol{\Psi}}_1(x) \sim \begin{pmatrix} \sum_{j=1}^{3}\cos\left(\frac{2\pi}{K}\bar{l}_j\right)\partial_x S_u(x-\bar{l}_j) \\ \sum_{j=1}^{3}\cos\left(\frac{2\pi}{K}\bar{l}_j\right)\partial_x S_v(x-\bar{l}_j) \end{pmatrix},$$

where $\bar{l}_j = (2j-1)\frac{K}{2r}$ $(j = 1,\ldots,3)$ (Fig 4).

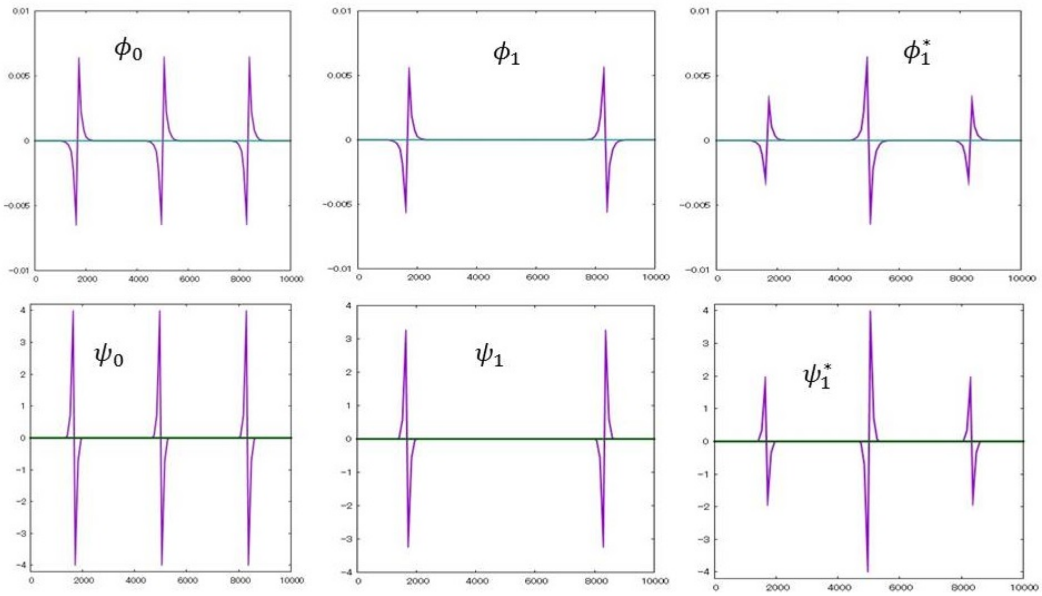

Figure 4: Profiles of $\tilde{\mathbf{\Phi}}_0(x), \tilde{\mathbf{\Phi}}_1(x), \tilde{\mathbf{\Psi}}_1(x)$ with the periodic boundary conditions(the Gray-Scott-model). Here, $\tilde{\mathbf{\Phi}}_n :=^t (\phi_n, \psi_n)\ (n = 0, 1)$, $\tilde{\mathbf{\Psi}}_1 :=^t (\phi_1^*, \psi_1^*)$.

3.4 Proof of theorem 3.2

In this section, we give the proof of Theorem3.2. For other theorems, it is the same as in Theorem 3.2. First, the following holds.

Proposition 3.1. Fix one arbitrary $K > K_0$. Then the equilibrium point $\bar{\boldsymbol{l}} = (\bar{l}_1, \ldots, \bar{l}_j, \ldots, \bar{l}_r)$ with $0 < \bar{l}_j < K$ for $\bar{\boldsymbol{H}}(\boldsymbol{l})$ is uniquely denoted by

$$\bar{\boldsymbol{l}} = \begin{pmatrix} \dfrac{K}{2r} \\ \vdots \\ (2j-1)\dfrac{K}{2r} \\ \vdots \\ (2r-1)\dfrac{K}{2r} \end{pmatrix}.$$

Furthermore, the linearized matrix $\boldsymbol{H}'(\bar{\boldsymbol{l}})$ is uniquely denoted by

$$\boldsymbol{H}'(\bar{\boldsymbol{l}}) = M_0\alpha \mathrm{e}^{-\alpha\frac{K}{r}} \begin{pmatrix} -3 & 1 & 0 & & & \\ 1 & -2 & 1 & & \text{\huge 0} & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\huge 0} & & 1 & -2 & 1 \\ & & & 0 & 1 & -3 \end{pmatrix}.$$

Proof. It is shown immediately by direct calculation. □

Proposition 3.2. Fix one arbitrary $K > K_0$. Then eigenvalues $\mu_n\,(n = 0, 1, \ldots, r-1)$ for $\boldsymbol{H}'(\bar{\boldsymbol{l}})$ satisfy

$$\mu_n = -4M_0\alpha \mathrm{e}^{-\alpha\frac{K}{r}} \cos^2\left(\frac{n\pi}{2r}\right). \tag{3.4.13}$$

Furthermore, eigenvectors $\boldsymbol{\phi}_n\,(n = 0, 1, \ldots, r-1)$ associated with $\mu_n\,(n = 0, 1, \ldots, r-1)$ satisfy

$$\boldsymbol{\phi}_n = {}^t\left(\cos\left(\frac{n\pi}{K}\bar{l}_1\right), \ldots, (-1)^{j+1}\cos\left(\frac{n\pi}{K}\bar{l}_j\right), \ldots, (-1)^{r+1}\cos\left(\frac{n\pi}{K}\bar{l}_r\right)\right) \tag{3.4.14}$$

excluding constant doubling. Here $\bar{l}_j = \dfrac{(2j-1)K}{2r}\,(j = 1, \ldots, r)$.

Proof. It can be shown immediately by substituting (3.4.13), (3.4.14) into $\boldsymbol{H}'(\bar{\boldsymbol{l}})\mu_n = \mu_n\boldsymbol{\phi}_n$. □

Based on the above, we provide proof.

From Theorem 3.1, the solution $\boldsymbol{U}(t,x)$ of (3.2.3) may be expressed by

$$\boldsymbol{U}(t,x) = \sum_{j=1}^{r} \boldsymbol{S}(x - l_j(t)) + \boldsymbol{\sigma}^*(x;\boldsymbol{l}), \tag{3.4.15}$$

where

$$\begin{cases} \dfrac{dl_1}{dt} = M_0(\mathrm{e}^{-2\alpha l_1(t)} - \mathrm{e}^{-\alpha(l_2(t)-l_1(t))}) + H_1^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_j}{dt} = M_0(\mathrm{e}^{-\alpha(l_j(t)-l_{j-1}(t))} - \mathrm{e}^{-\alpha(l_{j+1}(t)-l_j(t))}) + H_j^*(\boldsymbol{l}), \\ \quad\vdots \\ \dfrac{dl_r}{dt} = M_0(\mathrm{e}^{-\alpha(l_r(t)-l_{r-1}(t))} - \mathrm{e}^{-2\alpha(K-l_r(t))}) + H_r^*(\boldsymbol{l}) \end{cases} \tag{3.4.16}$$

with $|H_j^*(\boldsymbol{l})| \leq O((\delta^*(\boldsymbol{l}))^2)$ as long as $l_j(t) \gg 1\,(j = 1, \dots N)$.

Let $H_j(\boldsymbol{l}(t)) = \dfrac{dl_j}{dt}$ of (3.4.16). By (3.4.15), We know

$$\partial_t \boldsymbol{U} = -\sum_{j=1}^{r} H_j(\boldsymbol{l}(t))\partial_x \boldsymbol{S}(x - l_j(t)) + \partial_t \boldsymbol{\sigma}^*(x; \boldsymbol{l}(t))$$

holds. Since $H_j^*(\boldsymbol{l})$ and $\boldsymbol{\sigma}^*(\boldsymbol{l})$ are C^1 smooth with respect to $\boldsymbol{l}$, we can differentiate those functions by l_j. Hereafter we omit the higher order terms such as $\boldsymbol{\sigma}^*$ and H_j^* for simplicity, but we note that the following calculations are all rigorously justified by the property of $\boldsymbol{\sigma}^*$ and H_j^*.

We differentiate

$$\boldsymbol{W}(x; \boldsymbol{l}) := -\sum_{j=1}^{r} H_j(\boldsymbol{l})\partial_x \boldsymbol{S}(x - l_j)$$

by l_i $(i = 1,\, 2,\, \dots, r)$ and substitute $\bar{\boldsymbol{l}}$ for Proposition 3.1 into $\partial_{l_i}\boldsymbol{W}(x; \boldsymbol{l})$. We obtain

$$\partial_{l_i}\boldsymbol{W}(x; \bar{\boldsymbol{l}}) = -\sum_{j=1}^{r}\left\{\frac{\partial H_j(\bar{\boldsymbol{l}})}{\partial l_i}\right\}\partial_x \boldsymbol{S}(x - \bar{l}_j)$$

holds. Let the matrix $\boldsymbol{V}_0$ be

$$\boldsymbol{V}_0(x\,;\,\boldsymbol{l}) = (\partial_x \boldsymbol{S}(x - l_1), \dots, \partial_x \boldsymbol{S}(x - l_j), \dots, \partial_x \boldsymbol{S}(x - l_r)).$$

We have

$$\begin{aligned}\left(\partial_{l_1}\boldsymbol{W}(x; \bar{\boldsymbol{l}}), \dots, \partial_{l_j}\boldsymbol{W}(x; \bar{\boldsymbol{l}}), \dots, \partial_{l_r}\boldsymbol{W}(x; \bar{\boldsymbol{l}})\right) &= -\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\begin{pmatrix} \partial_1 H_1(\bar{\boldsymbol{l}}) \dots \partial_i H_1(\bar{\boldsymbol{l}}) & \dots \partial_r H_1(\bar{\boldsymbol{l}}) \\ \vdots \quad \ddots \quad \vdots & \ddots \quad \vdots \\ \partial_1 H_j(\bar{\boldsymbol{l}}) \dots \partial_i H_j(\bar{\boldsymbol{l}}) & \dots \partial_r H_j(\bar{\boldsymbol{l}}) \\ \vdots \quad \ddots \quad \vdots & \ddots \quad \vdots \\ \partial_1 H_r(\bar{\boldsymbol{l}}) \dots \partial_i H_r(\bar{\boldsymbol{l}}) & \dots \partial_r H_r(\bar{\boldsymbol{l}}) \end{pmatrix} \\ &= -\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\boldsymbol{H}'(\bar{\boldsymbol{l}}),\end{aligned}$$

where $\partial_i H_i(\bar{\boldsymbol{l}}) := \dfrac{\partial H_i(\bar{\boldsymbol{l}})}{\partial l_i}$. Moreover we have

$$\boldsymbol{W}(x; \boldsymbol{l}) = \boldsymbol{W}(x; \bar{\boldsymbol{l}}) + \frac{\partial \boldsymbol{W}(x; \bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l} - \bar{\boldsymbol{l}}) + \boldsymbol{G}_1(x; \boldsymbol{l}; \bar{\boldsymbol{l}}) = \frac{\partial \boldsymbol{W}(x; \bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l} - \bar{\boldsymbol{l}}) + \boldsymbol{G}_1(x; \boldsymbol{l}; \bar{\boldsymbol{l}}),$$

where $\boldsymbol{G}_1(x; \boldsymbol{l}; \bar{\boldsymbol{l}})$ is a function satisfying $\boldsymbol{G}_1(x; \boldsymbol{l}; \bar{\boldsymbol{l}}) = O\left(\left|\boldsymbol{l} - \bar{\boldsymbol{l}}\right|^2\right)$. Thus we see

$$\boldsymbol{W}(x;\boldsymbol{l})=\frac{\partial \boldsymbol{W}(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_1(x;\boldsymbol{l};\bar{\boldsymbol{l}})=-\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\boldsymbol{H}'(\bar{\boldsymbol{l}})(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_1(x;\boldsymbol{l};\bar{\boldsymbol{l}})$$

and

$$\boldsymbol{W}(x;\boldsymbol{l}(t))=-\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\boldsymbol{H}'(\bar{\boldsymbol{l}})(\boldsymbol{l}(t)-\bar{\boldsymbol{l}})+\boldsymbol{G}_1(x;\boldsymbol{l}(t);\bar{\boldsymbol{l}}) \tag{3.4.17}$$

where $\boldsymbol{l}=\boldsymbol{l}(t)$.

Next, we define

$$\tilde{\boldsymbol{S}}^*(x\,;\boldsymbol{l}):=\sum_{j=1}^{r}\boldsymbol{S}(x-l_j).$$

We see

$$\tilde{\boldsymbol{S}}^*(x\,;\boldsymbol{l})=\tilde{\boldsymbol{S}}^*(x\,;\bar{\boldsymbol{l}})+\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_2(x;\boldsymbol{l};\bar{\boldsymbol{l}}),$$

where $\boldsymbol{G}_2(x;\boldsymbol{l};\bar{\boldsymbol{l}})=O\left(\left|\boldsymbol{l}-\bar{\boldsymbol{l}}\right|^2\right)$. We have

$$\begin{aligned}
\mathcal{L}(\tilde{\boldsymbol{S}}^*(x\,;\boldsymbol{l}))&=\mathcal{L}\left(\tilde{\boldsymbol{S}}^*(x\,;\bar{\boldsymbol{l}})+\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_2(x;\boldsymbol{l};\bar{\boldsymbol{l}})\right)\\
&=D\partial_{xx}\left(\tilde{\boldsymbol{S}}^*(x\,;\bar{\boldsymbol{l}})+\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})\right)+\boldsymbol{F}(\tilde{\boldsymbol{S}}^*(x\,;\bar{\boldsymbol{l}}))\\
&+\boldsymbol{F}'(\tilde{\boldsymbol{S}}^*(x\,;\bar{\boldsymbol{l}}))\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_3(x;\boldsymbol{l};\bar{\boldsymbol{l}})\\
&=L(\tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}}))\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}-\bar{\boldsymbol{l}})+\boldsymbol{G}_3(x;\boldsymbol{l};\bar{\boldsymbol{l}}),
\end{aligned}$$

where $\mathcal{L}(\boldsymbol{U})=D\partial_{xx}+\boldsymbol{F}(\boldsymbol{U})$, and $\boldsymbol{G}_3(x;\boldsymbol{l};\bar{\boldsymbol{l}})$ is a function satisfying $\boldsymbol{G}_3(x;\boldsymbol{l};\bar{\boldsymbol{l}})=O\left(\left|\boldsymbol{l}-\bar{\boldsymbol{l}}\right|^2\right)$. Thus we obtain

$$\mathcal{L}(\tilde{\boldsymbol{S}}^*(x\,;\boldsymbol{l}(t)))=L_{\mathcal{N}}(\tilde{\boldsymbol{S}}^*)\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}(\boldsymbol{l}(t)-\bar{\boldsymbol{l}})+\boldsymbol{G}_3(x;\boldsymbol{l}(t);\bar{\boldsymbol{l}}),$$

where $\boldsymbol{l}=\boldsymbol{l}(t)$. By (3.4.17) and $\boldsymbol{U}_t=\mathcal{L}(\boldsymbol{U})$,

$$-\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\boldsymbol{H}'(\bar{\boldsymbol{l}})\underline{\boldsymbol{l}}(t)+\boldsymbol{G}_1^*(x;\underline{\boldsymbol{l}}(t))=L_{\mathcal{N}}(\tilde{\boldsymbol{S}}^*)\frac{\partial \tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial \boldsymbol{l}}\underline{\boldsymbol{l}}(t)+\boldsymbol{G}_3^*(x;\underline{\boldsymbol{l}}(t)) \tag{3.4.18}$$

holds for $0<x<K$, where $\underline{\boldsymbol{l}}(t):=\boldsymbol{l}(t)-\bar{\boldsymbol{l}}$ and $\boldsymbol{G}_i^*(x;\underline{\boldsymbol{l}}(t)):=\boldsymbol{G}_i(x;\underline{\boldsymbol{l}}(t)+\bar{\boldsymbol{l}};\bar{\boldsymbol{l}})$ $(i=1,3)$. We consider

$$\boldsymbol{l}(t)=\bar{\boldsymbol{l}}+\epsilon e^{\mu t}\phi+o(\epsilon)$$

with $\boldsymbol{l}(0) = \bar{\boldsymbol{l}} + \epsilon\phi + o(\epsilon)$, where $\epsilon \to 0$ and μ are eigenvalues for $\boldsymbol{H}'(\bar{l})$, and ϕ are eigenvectors associated with μ. Then we can derive

$$-\mu\{\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\phi\} + \widetilde{\boldsymbol{G}}_1^\epsilon(t,x) = -L_{\mathcal{N}}(\tilde{\boldsymbol{S}}^*)\{\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})\phi\} + \widetilde{\boldsymbol{G}}_3^\epsilon(t,x)$$

by (3.4.18) and $\dfrac{\partial\tilde{\boldsymbol{S}}^*(x;\bar{\boldsymbol{l}})}{\partial\boldsymbol{l}} = -\boldsymbol{V}_0(x\,;\bar{\boldsymbol{l}})$, where $\widetilde{\boldsymbol{G}}_i^\epsilon(t,x)\,(i = 1,3)$ is a function satisfying $|\widetilde{\boldsymbol{G}}_i^\epsilon(t,x)| = O(|\epsilon|)$. The rest can be proved, using Proposition 3.2, [30], [34] and Proposition 4.1([30], [34]) in [8].

Instead of Proposition 3.1 and Proposition 3.2, the following two hold for periodic boundary conditions. It is described only when r is even. We define $r_0 := \dfrac{r}{2}$.

Proposition 3.3. Fix one arbitrary $K > K_0$. The equilibrium point $\bar{\boldsymbol{l}}_{\mathcal{P}} = (\bar{l}_1,\ldots,\bar{l}_j,\ldots,\bar{l}_r)$ with $0 < \bar{l}_j < K$ for $\bar{\boldsymbol{H}}_{\mathcal{P}}(\boldsymbol{l})$ is uniquely denoted by

$$\bar{\boldsymbol{l}}_{\mathcal{P}}(\omega_0) = \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_0 + (j-1)\dfrac{K}{r} \\ \vdots \\ \omega_0 + (r-1)\dfrac{K}{r} \end{pmatrix}$$

for any $\omega_0 \in [0,K]$, where $\bar{\boldsymbol{l}}_{\mathcal{P}} = \bar{\boldsymbol{l}}_{\mathcal{P}}(\omega_0)$. Furthermore, the linearized matrix $\boldsymbol{H}'_{\mathcal{P}}(\bar{\boldsymbol{l}}_{\mathcal{P}})$ is uniquely denoted by

$$\boldsymbol{H}'_{\mathcal{P}}(\tilde{\boldsymbol{l}}_{\mathcal{P}}) = M_0\alpha\mathrm{e}^{-\alpha\frac{K}{r}} \begin{pmatrix} -2 & 1 & 0 & & & 1 \\ 1 & -2 & 1 & & \text{\Large 0} & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\Large 0} & & 1 & -2 & 1 \\ 1 & & & 0 & 1 & -2 \end{pmatrix}.$$

Proof. By quite a similar way to Proposition 3.1, we can show Proposition 3.3. □

Remark 3.8. We can take $\omega_0 = \dfrac{K}{2r}$ without loss of generality.

Since $\boldsymbol{H}'_{\mathcal{P}}(\tilde{\boldsymbol{l}}_{\mathcal{P}})$ is a circulant matrix, we see that the eigenvalues $\{\tilde{\mu}_n\}$ and the eigenvectors $\{\tilde{\phi}_n\}$ for $\boldsymbol{H}'_{\mathcal{P}}(\tilde{\boldsymbol{l}}_{\mathcal{P}})$ hold by the following(see e.g. [16]).

Proposition 3.4. Fix one arbitrary $K > K_0$. Then eigenvalues $\tilde{\mu}_n\,(n = 0, 1, \ldots, r-1)$ for $\boldsymbol{H}'_{\mathcal{P}}(\bar{\boldsymbol{l}}_{\mathcal{P}})$ satisfy

$$\tilde{\mu}_n = -4M_0\alpha \mathrm{e}^{-\alpha\frac{K}{r}} \sin^2\left(\frac{n\pi}{r}\right)$$

and are semisimple. Moreover eigenvectors $\tilde{\boldsymbol{\phi}}_n$ assosiated with $\tilde{\mu}_n\,(0 \leq n \leq r-1)$ are the following.
(a) If $n = 0$, $\tilde{\mu}_0 = 0$ is a simple eigenvalue and the eigenvector $\tilde{\boldsymbol{\phi}}_0$ satisfy

$$\tilde{\boldsymbol{\phi}}_0 = (1, 1, \ldots, 1)$$

holds.
(b) If $1 \leq n \leq r_0 - 1$, $\tilde{\mu}_n$ has a degree of overlap 2 and eigenvectors $\tilde{\boldsymbol{\phi}}_n$ satisfy

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}_n &= {}^t\left(\sin\left(\frac{2n\pi}{K}\tilde{l}_1\right), \ldots, \sin\left(\frac{2n\pi}{K}\tilde{l}_j\right), \ldots, \sin\left(\frac{2n\pi}{K}\tilde{l}_r\right)\right),\\
\tilde{\boldsymbol{\psi}}_n &= {}^t\left(\cos\left(\frac{2n\pi}{K}\tilde{l}_1\right), \ldots, \cos\left(\frac{2n\pi}{K}\tilde{l}_j\right), \ldots, \cos\left(\frac{2n\pi}{K}\tilde{l}_r\right)\right)
\end{aligned}$$

except for the constant doubling.
(c) If $n = r_0$, $\tilde{\mu}_{r_0}$ is a simple eigenvalue and the eigenvector $\tilde{\boldsymbol{\phi}}_{N_0}$ satisfy

$$\tilde{\boldsymbol{\phi}}_{r_0} = (1, -1, \ldots, 1, -1)$$

except for the constant doubling. Here $\bar{l}_j = \dfrac{(2j-1)K}{2r}\;(j = 1, \ldots, r)$.

If r is odd, remove (c) above for $r_0 := \dfrac{r+1}{2}$. We can prove the rest by the quite similar way to 3.4.

3.5 Concluding remarks

This chapter treats the linearized eigenvalue problem associated with r-layered pulse/front stationary solutions for reaction-diffusion systems with the Neumann or periodic boundary conditions. Moreover, it is found that the eigenvalues and eigenfunctions for $0 \leq n \leq r-1$ can be expressed in trigonometric functions. These eigenvalues and eigenfunctions for $0 \leq n \leq r-1$ are obtained in a form that fully contains the results of [35]-[38]. From now on, we will work on future issues, including the following:

(i) Linearized eigenvalue problem for the case where the front type stationary solution $\boldsymbol{S}(x)$ is not an odd function respecting to x .

(ii) Find the specific solution representation of the eigenvalues and eigenfunctions for $n \geq r$.

For (i), the analysis method is already known and will be studied in the future. For (ii), the approach method needs to be better understood. So this is a significant issue to be addressed in the future.

4 Existing results and the proof of "The dynamics of pulse solutions for reaction diffusion systems on a star shaped metric graph with the Kirchhoff's boundary condition"

4.1 Introduction

This chapter is only a preparation to introduce existing results and their proofs from [10] for Capters 5 and 6.

Exactly as in [10], we express (1.0.4) as

$$\partial_t \boldsymbol{U} = \Delta_\Omega \boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}) \tag{4.1.1}$$

on Ω with the Kirchhoff's boundary condition. Here $\Omega = \bigcup_{j=1}^{R} \Omega_j$ (Fig 5) for $R \in \mathbb{N}$ with $R \geq 3$ and $\Omega_j = \{x\boldsymbol{e}_j \in \mathbb{R}^2 \,|\, x > 0\}$, and $\Delta_\Omega := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ for $\boldsymbol{x} = (x, y) \in \Omega$. The same way as [10], Let us call Ω a star shaped metric graph.

Interesting results have been reported for the problem of (4.1.1). For example, [25] consider

$$\partial_t u_j = \partial_{xx} u_j + u_j(1 - u_j)\left(u_j - \frac{1}{2}\right), -\infty < t < \infty, x\boldsymbol{e}_0 \in \Omega_0, x\boldsymbol{e}_1 \in \Omega_1, x\boldsymbol{e}_2 \in \Omega_2 \tag{4.1.2}$$

with the Kirchhoff's boundary condition, where $\Omega_0 := \{x\boldsymbol{e}_0 \in \mathbb{R}^2 \,|\, x < 0\}$ and $\boldsymbol{e}_0$ is the unit directional vectors of Ω_0. The above equations considered on $\mathbb{R}$ have a front-type stationary solution $S(x) = \dfrac{1}{1 + \exp(1/\sqrt{2})}$, and have no front-type traveling wave solution. However, [25] led to the result that (4.1.2) has no stationary solution, and there exists a front traveling wave. [25] was extended to a more general scalar reaction-diffusion equation without a traveling wave solution. It has become important in studying traveling wave and stationary solutions for reaction-diffusion equations on metric graphs. Later on, [23] and [24] developed problems for scalar reaction-diffusion equations with traveling wave solutions on a metric graph composed of multiple semi-infinite intervals. At that time, many results for front traveling wave solutions were also reported by [23] and [24].

On the other hand, the problem for reaction-diffusion equations on metric graphs has also produced exciting results in terms of applications. For example, [6] considered a metric graph composed of multiple semi-infinite intervals to be a channel geometry. They performed a mathematical analysis for Fisher-KPP-type equations on this graph. Their results are consistent with the transition of organism density in natural rivers, suggesting that this graph is also a crucial application domain. In addition, [6] also has many important mathematical results for reaction-diffusion equations on metric graphs composed of multiple semi-infinite intervals.

From this point of view, numerical and mathematical analyses have been vigorously conducted for problems for reaction-diffusion systems on metric graphs in recent years(e.g. [18], [19], [20], [28]). However, until now, there have been no theoretical results for pulse-traveling wave solutions or pulse dynamics for reaction-diffusion systems on Ω, and this is one of the significant challenges. Analyzing these systems on metric graphs is a significant problem from pure and applied mathematical viewpoints. Against this background, we have considered pulse/front dynamics for general reaction-diffusion systems on Ω using [8] and [9]. As a result, we obtained some results for the pulse dynamics to reaction-diffusion systems on Ω([10]). In this chapter, we introduce existing results([10]) for the pulse dynamics to reaction-diffusion systems on Ω and their proofs from [10] in preparation for Chapters 5 and 6.

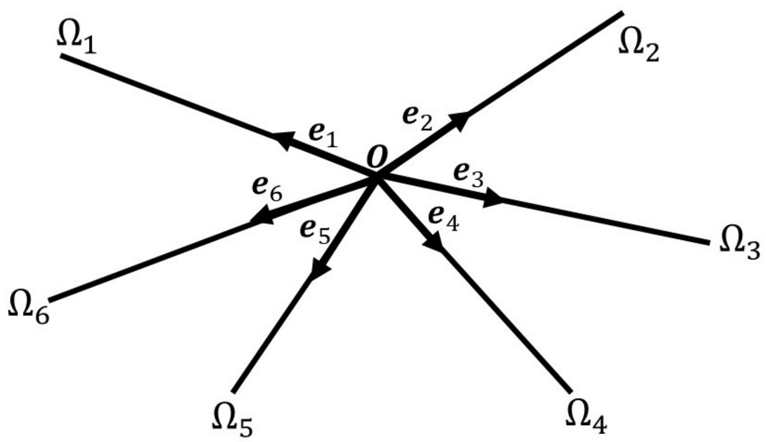

Figure 5: The star shaped metric graph in the case of $R = 6$

4.2 Existing results

In this section, I quote the exact same wording as in [10]. (B):$\sim$.(B) means that $\sim$ is quoted from B. We make the same assumptions as in Section 2 and define the following exactly the same as [10].

([10]): We denote $L^2_+ := \{L^2(\mathbb{R}_+)\}^N$ with the L^2-norm by $||\cdot||_{L^2_+}$ and the inner product L^2_+ by $\langle \boldsymbol{U}, \boldsymbol{V}\rangle_{L^2_+} := \int_0^{+\infty} \langle \boldsymbol{U}(x), \boldsymbol{V}(x)\rangle dx$ for $\boldsymbol{U}, \boldsymbol{V} \in L^2_+$. H^2_+, L^∞_+ are similarly defined by $H^2_+ := \{H^2(\mathbb{R}_+)\}^N$ and $L^\infty_+ := \{L^\infty(\mathbb{R}_+)\}^N$.

For $\Omega = \cup_{j=1}^R \Omega_j$ with $\Omega_j = \{x\boldsymbol{e}_j \in \mathbb{R}^2; x > 0\}$, we define $\{L^2(\Omega)\}^N$ by all $\boldsymbol{U}(\boldsymbol{x}) \in \mathbb{R}^N$ ($\boldsymbol{x} \in \Omega$) satisfying $\boldsymbol{U}_j \in L^2_+$ for $\boldsymbol{U}_j(x) = \boldsymbol{U}(x\boldsymbol{e}_j)$ together with the inner product $\langle \boldsymbol{U}, \boldsymbol{V}\rangle_{L^2(\Omega)} := \sum_{j=1}^R \int_0^{+\infty} \langle \boldsymbol{U}_j(x), \boldsymbol{V}_j(x)\rangle dx$ for $\boldsymbol{U}, \boldsymbol{V} \in \{L^2(\Omega)\}^N$, where $\boldsymbol{U}_j(x) = \boldsymbol{U}(x\boldsymbol{e}_j)$ and $\boldsymbol{V}_j(x) = \boldsymbol{V}(x\boldsymbol{e}_j)$. $\{H^2(\Omega)\}^N$ and $\{L^\infty(\Omega)\}^N$ are similarly defined by $\{H^2(\Omega)\}^N := \{\boldsymbol{U} \in \{L^2(\Omega)\}^N; \boldsymbol{U}_j \in H^2_+ \,(j = 1, \dots, R), \boldsymbol{U}_j(x) = \boldsymbol{U}(x\boldsymbol{e}_j)\}$ and
$\{L^\infty(\Omega)\}^N := \{\boldsymbol{U}; \boldsymbol{U}_j \in L^\infty_+ \,(j = 1, \dots, R), \boldsymbol{U}_j(x) = \boldsymbol{U}(x\boldsymbol{e}_j)\}$.([10])

4.2.1 The existing result of "Motion of pulse solutions"

We also assume the same assumptions (H1) - (H3) and (A1), but we make the following modifications exactly the same as [10].

([10]): (H1) means the followings: Let $\boldsymbol{w}_j(x;\lambda)(j = 1, 2, \dots, 2N)$ be the fundamental functions of the ODE

$$(\lambda - L_0)\boldsymbol{u} = \boldsymbol{0}, x \in \mathbb{R}. \tag{4.2.3}$$

Then $\boldsymbol{w}_j(x;\lambda) \in \mathbb{C}^N$ for $\lambda \in \mathbb{C}$ with $Re(\lambda) > -\rho$ are the forms of

$$\boldsymbol{w}_j(x;\lambda) = \mathrm{e}^{\pm\mu_j(\lambda)x}\boldsymbol{b}_j(x;\lambda),$$

for vector valued polynomials $\boldsymbol{b}_j(x;\lambda) \in \mathbb{C}^N$ of x and $Re(\mu_j(\lambda)) > 0$ with the normalization $|\boldsymbol{b}_j(0;\lambda)| = 1$. We assume $0 < Re(\mu_1(\lambda)) \le \cdots \le Re(\mu_N(\lambda))$ and put $\boldsymbol{w}_j(x;\lambda) := \mathrm{e}^{-\mu_j(\lambda)x}\boldsymbol{b}_j(x;\lambda)$ for $j = 1, \dots, N$ and $\boldsymbol{w}_j(x;\lambda) = \mathrm{e}^{\mu_j(\lambda)x}\boldsymbol{b}_j(x;\lambda)$ for $j = N+1, \dots 2N$. Here, we note that $\boldsymbol{b}_j(x;\lambda) = \boldsymbol{b}_{j+N}(x;\lambda)$ hold for $j = 1, \dots, N$ and that $\{\boldsymbol{b}_j(x;\lambda)\}_{j=1}^N$ are linearly independent for any $x \ge 0$ and $Re(\lambda) > -\rho_0$. In particular, we put $\alpha_j := \mu_j(0), \boldsymbol{a}_j(x) := \boldsymbol{b}_j(x;0)$ and $\boldsymbol{m}_j(x) := \boldsymbol{w}_j(x;0) = \mathrm{e}^{\pm\alpha_j x}\boldsymbol{a}_j(x)$.

Related to (H3), we assume

(H4) $\boldsymbol{m}_1(x) = \mathrm{e}^{-\alpha x}\boldsymbol{a}$ that is, $\alpha_1 = \alpha > 0$ and $\boldsymbol{a}_1(x) = \boldsymbol{a} \in \mathbb{R}^N$. Moreover, $\alpha < Re(\alpha_j)$ $(j = 2, \ldots, N)$ holds.([10])

Moreover, we consider the following settings and situations exactly the same as [10].

([10]): Quite similarly for $L_0^* := D\partial_{xx} + {}^t\boldsymbol{F}'(\boldsymbol{0})$, the adjoint system

$$(\lambda - L_0^*)\boldsymbol{u} = \boldsymbol{0}, x \in \mathbb{R} \tag{4.2.4}$$

of (4.2.3)((9) in [10]) has the same properties as follows: Let $\boldsymbol{w}_j^*(x;\lambda), (j = 1, 2, \ldots, 2N)$ be the fundamental functions of the ODE (4.2.4)((10) in [10]). Then, $\boldsymbol{w}_j^*(x;\lambda) \in \mathbb{C}^N$ for $\lambda \in \mathbb{C}$ with $Re(\lambda) > -\rho_0$ are the forms of

$$\boldsymbol{w}_j^*(x;\lambda) = \mathrm{e}^{\pm\overline{\mu_j(\lambda)}x}\boldsymbol{b}_j^*(x;\lambda)$$

for vector valued polynomials $\boldsymbol{b}_j^*(x;\lambda) \in \mathbb{C}^N$ of x and $Re(\overline{\mu_j(\lambda)}) > 0$ with the normalization $|\boldsymbol{b}_j^*(0;\lambda)| = 1$.We note that $0 < Re(\overline{\mu_1(\lambda)}) \leq \cdots \leq Re(\overline{\mu_N(\lambda)})$ hold and we put $\boldsymbol{w}_j^*(x;\lambda) := \mathrm{e}^{-\overline{\mu_j(\lambda)}x}\boldsymbol{b}_j^*(x;\lambda)$ for $j = 1, \ldots, N$ and $\boldsymbol{w}_j^*(x;\lambda) := \mathrm{e}^{\overline{\mu_j(\lambda)}x}\boldsymbol{b}_j^*(x;\lambda)$ for $j = N+1, \ldots 2N$. $\boldsymbol{b}_j^*(x;\lambda) = \boldsymbol{b}_{j+N}^*(x;\lambda), j = 1, \ldots, N$ hold for $j = 1, \ldots, N$ and $\{\boldsymbol{b}_j^*(x;\lambda)\}_{j=1}^N$ are linearly independent for any $x \geq 0$ and $Re(\lambda) > -\rho_0$. In particular, we put $\boldsymbol{a}_j^*(x) := \boldsymbol{b}_j^*(x;0)$ and $\boldsymbol{m}_j^*(x) := \boldsymbol{w}_j^*(x;0) = \mathrm{e}^{\pm\overline{\alpha_j}x}\boldsymbol{a}_j^*(x)$ with noting $\overline{\alpha_j} = \overline{\mu_j(0)}$.

Related (H3) and (H4), there exists $\boldsymbol{a}^* \in \mathbb{R}^N$ such that $\boldsymbol{m}_1^*(x) = \mathrm{e}^{-\alpha x}\boldsymbol{a}^*$, that is $\alpha_1 = \overline{\alpha_1} = \alpha > 0$ and $\boldsymbol{a}_1^*(x) = \boldsymbol{a}^* \in \mathbb{R}^N$.([10])

Quite similar to [10], we define $\boldsymbol{l} := {}^t(l_1, \ldots, l_r), H_j(\boldsymbol{l}) := M_0 e^{-\alpha l_j}\left[\dfrac{2}{R}\sum_{i=1}^{r} e^{-\alpha l_i} - e^{-\alpha l_j}\right]$ and $\boldsymbol{H}(\boldsymbol{l}) := {}^t(H_1(\boldsymbol{l}), \ldots, H_r(\boldsymbol{l}))$, where $M_0 := 2\alpha\langle D\boldsymbol{a}, \boldsymbol{a}^*\rangle$. We denote $\max\{l_1, \ldots, l_r\}$ and $\min\{l_1, \ldots, l_r\}$ by $\max \boldsymbol{l}$ and $\min \boldsymbol{l}$, respectively. Moreover, by quite a similar to [10], we also define

$$\boldsymbol{S}(\boldsymbol{x};\boldsymbol{l}) := \begin{cases} \boldsymbol{S}(x - l_j), \boldsymbol{x} = x\boldsymbol{e}_j \in \Omega_j \; (j = 1, \ldots, r), \\ \quad 0, \qquad\quad \boldsymbol{x} = x\boldsymbol{e}_j \in \Omega_j \; (j = r+1, \ldots, R), \end{cases}$$

where r be a positive integer with $1 \leq r \leq R$ and fix it arbitrarily.

Theorem 4.1([10], Theorem 2.2). Assume (H1)-(H4). Then there exists $l^* \gg 1$ such that for the initial date $\boldsymbol{U}(0, \boldsymbol{x})$ sufficiently close to $\boldsymbol{S}(\boldsymbol{x};\boldsymbol{l}_0)$ for $\min \boldsymbol{l}_0 > l^*$ in $\{H^2(\Omega)\}^N$, the solution $\boldsymbol{U}(t, \boldsymbol{x})$ of (4.1.1)((3) in [10]) satisfies

$$||\boldsymbol{U}(t, \cdot) - \boldsymbol{S}(\cdot;\boldsymbol{l}(t))||_{L^\infty(\Omega)} \leq O(e^{-\alpha l^*})$$

as long as $\min \boldsymbol{l}(t) > l^*$. Moreover, there exists $\widetilde{H}_j(\boldsymbol{l})$ such that $\widetilde{H}_j(\boldsymbol{l}) = H_j(\boldsymbol{l}) + e^{-\alpha l_j} O(e^{-\gamma \min \boldsymbol{l}})$ and each $l_j(t)$ satisfies

$$\frac{dl_j}{dt} = \widetilde{H}_j(\boldsymbol{l})\,(j = 1, \ldots, r)$$

as long as $\min \boldsymbol{l}(t) > l^*$ for $\gamma > \alpha$.

4.2.2 The existing result of "Motion of front solutions"

We consider the motion of front solutions for (2.1.1). We also assume the same assumptions (H1)' - (H3)' and (A2), but we make the following modifications exactly the same as [10].

([10]): (H1)' means the followings: Let $\boldsymbol{w}_j^{\pm}(x;\lambda)(j = 1, 2, \ldots, 2N)$ be the fundamental functions of the ODE

$$(L_{\pm} - \lambda)\boldsymbol{u} = \boldsymbol{0}, x \in \mathbb{R}.$$

Then $\boldsymbol{w}_j^{\pm}(x;\lambda) \in \mathbb{C}^N$ for $\lambda \in \mathbb{C}$ with $Re(\lambda) > -\rho$ are the forms of

$$\boldsymbol{w}_j^{\pm}(x;\lambda) = w_j^{\pm}(x;\lambda)\mathrm{e}^{\pm\mu_j^{\pm}(\lambda)x}\boldsymbol{b}_j^{\pm}(\lambda),$$

for polynomials $w_j^{\pm}(x;\lambda)$ of x,vector $\boldsymbol{b}_j^{\pm}(\lambda) \in \mathbb{C}^N$and $Re(\mu_j^{\pm}(\lambda)) > 0$ with the normalization $\boldsymbol{w}_j^{\pm}(0;\lambda) = 1$. We assume $0 < Re(\mu_1^{\pm}(\lambda)) \leq \cdots \leq Re(\mu_N^{\pm}(\lambda))$ and put $\boldsymbol{w}_j^{\pm}(x;\lambda) := w_j^{\pm}(x;\lambda)\mathrm{e}^{-\mu_j^{\pm}(\lambda)x}\boldsymbol{b}_j^{\pm}(\lambda)$ for $j = 1, \ldots, N$ and $\boldsymbol{w}_j^{\pm}(x;\lambda) = w_j^{\pm}(x;\lambda)\mathrm{e}^{\mu_j^{\pm}(\lambda)x}\boldsymbol{b}_j^{\pm}(\lambda)$ for $j = N+1, \ldots 2N$. In particular, we put $\alpha_j^{\pm} := \mu_j^{\pm}(0), m_j^{\pm}(x) := w_j^{\pm}(x;0), \boldsymbol{a}_j^{\pm} := \boldsymbol{b}_j^{\pm}(0)$ and $\boldsymbol{m}_j^{\pm}(x) := \boldsymbol{w}_j^{\pm}(x;0) = m_j^{\pm}(x)\mathrm{e}^{\pm\alpha_j^{\pm}(\lambda)x}\boldsymbol{a}_j^{\pm}(x)$.

Related to (H3)', we assume:

(H4)' $\boldsymbol{m}_1^{\pm}(x) = \mathrm{e}^{-\alpha_{\pm}x}\boldsymbol{a}^{\pm}$, that is, $m_1^{\pm}(x) = 1, \alpha_1^{\pm} = \alpha_{\pm} > 0$ and $\boldsymbol{a}_1^{\pm} = \boldsymbol{a}_{\pm} \in \mathbb{R}^N$.([10])

Moreover, we consider the following settings and situations exactly the same as [10].

Quite similar to [10], $\boldsymbol{l} := {}^t(l_1, \ldots, l_r), H_j^{\pm}(\boldsymbol{l}) := M_{\pm}e^{-\alpha_{\pm}l_j}\left[\frac{2}{R}\sum_{i=1}^{r} e^{-\alpha_{\pm}l_i} - e^{-\alpha_{\pm}l_j}\right]$ and $\boldsymbol{H}_{\pm}(\boldsymbol{l}) := {}^t(H_1^{\pm}(\boldsymbol{l}), \ldots, H_r^{\pm}(\boldsymbol{l}))$, where $M_{\pm} := \pm 2\alpha_{\pm}\langle D\boldsymbol{a}_{\pm}, \boldsymbol{a}_{\pm}^*\rangle$. We denote $\max\{l_1, \ldots, l_r\}$ and $\min\{l_1, \ldots, l_r\}$ by $\max \boldsymbol{l}$ and $\min \boldsymbol{l}$, respectively. Moreover, by quite a similar to [10], we also define

$$\boldsymbol{S}(\boldsymbol{x};\boldsymbol{l}) := \begin{cases} \boldsymbol{S}(x - l_j), & \boldsymbol{x} = x\boldsymbol{e}_j \in \Omega_j\,(j = 1, \ldots, r), \\ \boldsymbol{S}_-, & \boldsymbol{x} = x\boldsymbol{e}_j \in \Omega_j\,(j = r+1, \ldots, R). \end{cases}$$

Theorem 4.2([10] Theorem 2.3). Assume (H1)'-(H4)'. Then there exists $l^* \gg 1$ such that for the initial date $\boldsymbol{U}(0,\boldsymbol{x})$ sufficiently close to $\boldsymbol{S}(\boldsymbol{x};\boldsymbol{l}_0)$ for $\min \boldsymbol{l}_0 > l^*$ in $\{H^2(\Omega)\}^N$, the solution $\boldsymbol{U}(t,\boldsymbol{x})$ of (4.1.1)((3) in [10]) satisfies

$$||\boldsymbol{U}(t,\cdot) - \boldsymbol{S}(\cdot;\boldsymbol{l}(t))||_{L^\infty(\Omega)} \le O(e^{-\alpha_- l^*})$$

as long as $\min \boldsymbol{l}(t) > l^*$. Moreover, there exists $\widetilde{H}_j^-(\boldsymbol{l})$ such that $\widetilde{H}_j^-(\boldsymbol{l}) = H_j^-(\boldsymbol{l}) + e^{-\alpha_- l_j} O(e^{-\gamma \min \boldsymbol{l}})$ and each $l_j(t)$ satisfies

$$\frac{dl_j}{dt} = \widetilde{H}_j^-(\boldsymbol{l})\,(j = 1,\dots,r)$$

as long as $\min \boldsymbol{l}(t) > l^*$ for $\gamma > \alpha_-$.

4.3 The proof of Theorem 4.1

For self-completeness and in Chapters 5 and 6, I quote the proof of Theorem 4.1([10], Theorem 2.2) into this section. Theorem 4.2 ([10] Theorem 2.3) can be shown the quite a similar way to the proof of Theorem 4.1 ([10], Theorem 2.2) (see [10]). In 4.3, U and U_j is identical to $\boldsymbol{U}$ and $\boldsymbol{U}_j$ $(j = 1,\dots,r)$, respectively. S is identical to $\boldsymbol{S}$ of (H2).

Hereafter, I quote the same wording as "Proof of Theorem 2.2" in [10].

It suffices to show the proof of Theorem 4.1 ([10], Theorem 2.2) only in the case of $r < R$.

For fixed $\boldsymbol{l} = {}^t(l_1,\dots,l_r)$, we define the function $G(\boldsymbol{x};\boldsymbol{l})$ on Ω satisfying

$$\begin{cases} \boldsymbol{0} = D\partial_x^2 G_j + \boldsymbol{F}'(\boldsymbol{0})G_j, \\ \displaystyle\sum_{j=1}^{r}\{\partial_x S(-l_j) + \partial_x G_j(0)\} + \sum_{j=r+1}^{R} \partial_x G_j(0) = \boldsymbol{0}, \\ S(-l_1) + G_1(0) = \cdots = S(-l_r) + G_r(0) = G_{r+1}(0) = \cdots = G_R(0), \\ G_j(+\infty) = \boldsymbol{0}, \end{cases}, \tag{4.3.5}$$

where $G_j(x) = G_j(x;\boldsymbol{l}) := G(x\boldsymbol{e}_j;\boldsymbol{l})\,(j = 1,\dots,R)$. Since $\boldsymbol{m}_k(x)\,(k = 1,\dots,2N)$ are the fundamental functions of the ODE

$$\boldsymbol{0} = D\partial_x^2 \boldsymbol{m} + \boldsymbol{F}'(\boldsymbol{0})\boldsymbol{m},\ x \in \mathbb{R}$$

and $\boldsymbol{m}_k(+\infty) = \boldsymbol{0}$ hold for $k = 1,\dots,N, G(x)$ with $G(+\infty) = \boldsymbol{0}$ is expressed by a linear combination of $\boldsymbol{m}_k(x)\,(k = 1,\dots,N)$, that is, $G_j(x;\boldsymbol{l}) = \sum_{k=1}^{N} c_{jk}\boldsymbol{m}_k(x)$ and hence $G_j(x;\boldsymbol{l}) = c_{j1}(\boldsymbol{l})e^{-\alpha x}\boldsymbol{a} + O(e^{-\gamma x})$ holds for $\gamma > \alpha$ by (H4). Hereafter, we use the same symbol γ as a positive constant larger than α.

Lemma 4.1 ([10], Lemma 4.1). The coefficients $\{c_{jk}(\boldsymbol{l})\}$ of $G_j(x;\boldsymbol{l})$ are given by

$$\begin{cases} c_{j1}(\boldsymbol{l}) = C_j(\boldsymbol{l}) := \dfrac{2}{R}\displaystyle\sum_{k=1}^{r} e^{-\alpha l_k} - e^{-\alpha l_j},\ 1 \le j \le r, \\ c_{j1}(\boldsymbol{l}) = C_j(\boldsymbol{l}) := \dfrac{2}{R}\displaystyle\sum_{k=1}^{r} e^{-\alpha l_k}, \qquad r+1 \le j \le R, \\ c_{jk}(\boldsymbol{l}) = O(e^{-\gamma_1 \min \boldsymbol{l}}) \qquad 1 \le j \le R, 2 \le k \le N \end{cases} \tag{4.3.6}$$

as $\min \boldsymbol{l} \to +\infty$ for $\gamma_1 > \alpha$.

Proof. Since $S(x-l)$ has the asymptotic profile as $l \to +\infty$, $S(x-l) \to e^{\alpha(x-l)}\boldsymbol{a}$ together with $\partial_x S(x-l) \to \alpha e^{\alpha(x-l)}\boldsymbol{a}$, $S(-l) = e^{-\alpha l}\boldsymbol{a} + O(e^{-\gamma l})$ and $\partial_x S(-l) \to \alpha e^{\alpha(x-l)}\boldsymbol{a} + O(e^{-\gamma l})$ hold for $\gamma > \alpha$. Then substituting these profiles and $G_j(0;\boldsymbol{l}) = c_{j1}(\boldsymbol{l})\boldsymbol{a} + \boldsymbol{b}_j, \partial_x G_j(0;\boldsymbol{l}) = -\alpha c_{j1}(\boldsymbol{l})\boldsymbol{a} + \boldsymbol{b}'_j$ with $\boldsymbol{b}_j = \displaystyle\sum_{k=2}^{N} c_{jk}\boldsymbol{m}_k(0)$ and $\boldsymbol{b}'_j = \displaystyle\sum_{k=2}^{N} c_{jk}\partial_x \boldsymbol{m}_k(0)$ into the Kirchhoff boundary condition of (4.3.5)((25) in [10]), we have

$$\begin{cases} \displaystyle\sum_{j=1}^{r}\{\alpha e^{-\alpha l_j} - \alpha c_{j1}(\boldsymbol{l})\}\boldsymbol{a} - \sum_{j=r+1}^{R} \alpha c_{j1}(\boldsymbol{l})\boldsymbol{a} + \sum_{j=1}^{R}\{O(e^{-\gamma l_j}) + \boldsymbol{b}'_j\} + \sum_{j=r+1}^{R} \boldsymbol{b}'_j = \boldsymbol{0}, \\ e^{-\alpha l_1}\boldsymbol{a} + c_{11}(\boldsymbol{l})\boldsymbol{a} + O(e^{-\gamma l_1}) + \boldsymbol{b}_1 = \cdots = e^{-\alpha l_r}\boldsymbol{a} + c_{r1}(\boldsymbol{l})\boldsymbol{a} + O(e^{-\gamma l_r}) + \boldsymbol{b}_r \\ = c_{r+1,1}(\boldsymbol{l})\boldsymbol{a} + \boldsymbol{b}_{r+1} = \cdots = c_{R,1}(\boldsymbol{l})\boldsymbol{a} + \boldsymbol{b}_R. \end{cases} \tag{4.3.7}$$

Since $\{C_j(\boldsymbol{l})\}$ defined (4.3.6)((26) in [10]) satisfy

$$\begin{cases} \displaystyle\sum_{j=1}^{r}\{e^{-\alpha l_j} - C_j(\boldsymbol{l})\} - \sum_{j=r+1}^{R} C_j(\boldsymbol{l}) = 0, \\ e^{-\alpha l_1} + C_1(\boldsymbol{l}) = \cdots = e^{-\alpha l_r} + C_r(\boldsymbol{l}) = C_{r+1}(\boldsymbol{l}) = \cdots = C_R(\boldsymbol{l}), \end{cases}$$

(4.3.7)((27) in [10]) becomes by taking $c_{j1}(\boldsymbol{l}) = C_j(\boldsymbol{l})$,

$$\begin{cases} \displaystyle\sum_{j=1}^{r}\{O(e^{-\gamma l_j}) + \boldsymbol{b}'_j\} + \sum_{j=r+1}^{R} \boldsymbol{b}'_j = \boldsymbol{0}, \\ O(e^{-\gamma l_1}) + \boldsymbol{b}_1 = \cdots = O(e^{-\gamma l_r}) + \boldsymbol{b}_r = \boldsymbol{b}_{r+1} = \cdots = \boldsymbol{b}_R. \end{cases} \tag{4.3.8}$$

(4.3.8)((28) in [10]) means that $\boldsymbol{b}_j$ and $\boldsymbol{b}'_j$ can be taken in $O(e^{-\gamma \min \boldsymbol{l}})$ for $\gamma > \alpha$ and also $c_{jk}(\boldsymbol{l}) = O(e^{-\gamma_1 \min \boldsymbol{l}})$ $(1 \le j \le R, 2 \le k \le N)$ for some $\gamma_1 > \alpha$. $\square$

Thus we can get the function $G(\boldsymbol{x};\boldsymbol{l})$ satisfying (4.3.5)((25) in [10]) and $G_j(x;\boldsymbol{l}) = C_j(\boldsymbol{l})e^{-\alpha x}\boldsymbol{a} + O(e^{-\gamma_1(x+\min \boldsymbol{l})})$ for $\gamma_1 > \alpha$.

We express the solution $U(t,\boldsymbol{x})$ of (4.1.1)((3) in [10]) by $U(t,\boldsymbol{x}) = S(\boldsymbol{x};\boldsymbol{l}) + G(\boldsymbol{x};\boldsymbol{l}) + V(t,\boldsymbol{x})$. Then, the equation (1.0.4)((2) in [10]) or (4.1.1)((3) in [10]) becomes the equation of V

$$\begin{cases} \partial_t\{S(\cdot - l_j) + G_j + V_j\} = L(l_j)V_j + L(l_j)G_j + K_j(\boldsymbol{l}, V_j)(j = 1,\dots,r), \\ \partial_t\{G_j + V_j\} = L_0 V_j + L_0 G_j + K_j(\boldsymbol{l}, V_j)(j = r+1,\dots,R), \\ \sum_{j=1}^{R} \partial_x V_j(t,+0) = 0, V_1(t,+0) = \cdots = V_R(t,+0), \end{cases} \tag{4.3.9}$$

where $V_j(t,x) := V(t,x\boldsymbol{e}_j), G_j(x) = G_j(x;\boldsymbol{l}) := G(x\boldsymbol{e}_j;\boldsymbol{l}), L(l) := D\partial_x^2 + \boldsymbol{F}'(S(x - l)), L_0 := D\partial_x^2 + \boldsymbol{F}'(\boldsymbol{0})$. $K_j(\boldsymbol{l}, V_j)(j = 1,\dots,R)$ are functions satisfying $|K_j(\boldsymbol{l}, V_j)(x)| \le O(|G_j(x,\boldsymbol{l})|^2 + |V_j(t,x)|^2)$. Here we define the operators for $U \in \{H^2(\Omega)\}^N$ as follows:

$$\{\mathcal{L}(\boldsymbol{l})U\}(x\boldsymbol{e}_j) := \begin{cases} \{L(l_j)U_j\}(x) & (j = 1,\dots,r), \\ \{L_0 U_j\}(x) & (j = r+1,\dots,R) \end{cases}$$

with the Kirchhoff boundary condition $\sum_{j=1}^{R} \partial_x U_j(+0) = \boldsymbol{0}, U_1(+0) = \ldots = U_R(+0)$ and

$$\{\mathcal{L}_0 U\}(x\boldsymbol{e}_j) := \{L_0 U_j\}(x) \ (j = 1,\dots,R)$$

with $\sum_{j=1}^{R} \partial_x U_j(+0) = \boldsymbol{0}, U_1(+0) = \cdots = U_R(+0)$,where $U_j(x) := U(x\boldsymbol{e}_j)$.

Lemma ([10], Lemma 4.2). The spectral set $\sum(\mathcal{L}_0)$ satisfies $\sum(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C};\ Re(\lambda) < -\gamma_2\}$ for $\gamma_2 > 0$.

Proof. Let us consider the equation$(\mathcal{L}_0 - \lambda)U = \boldsymbol{f}$ for arbitrarily given $\boldsymbol{f} \in \{L^2(\Omega)\}^N$ and λ with $Re(\lambda) > -\rho_0$. Here define maps L_D, L_N:$\{L^2(\mathbb{R}_+)\}^N \to \{L^2(\mathbb{R})\}^N$

$$L_D[\boldsymbol{h}](x) := \begin{cases} \boldsymbol{h}(x), & x > 0, \\ -\boldsymbol{h}(-x), & x < 0, \end{cases} \quad L_N[\boldsymbol{h}](x) := \begin{cases} \boldsymbol{h}(x), & x > 0, \\ \boldsymbol{h}(-x), & x < 0 \end{cases}$$

for $\boldsymbol{h} \in \{L^2(\mathbb{R}_+)\}$.

For $\boldsymbol{f} \in \{L^2(\Omega)\}^N, \boldsymbol{f}_j(x) = \boldsymbol{f}(x\boldsymbol{e}_j)\,(j = 1,\dots,R)$ belong to $\{L^2(\mathbb{R}_+)\}^N$ and hence we can put $W_j^D(\lambda) = (\lambda - L_0)^{-1}L_D[\boldsymbol{f}_j], W_j^N(\lambda) = (\lambda - L_0)^{-1}L_N[\boldsymbol{f}_j] \in \{H^2(\mathbb{R})\}^N$ for $Re(\lambda) > -\rho_0$ by the assumption (H1). Since $(\lambda - L_0)(W_j^D - W_j^N) = \boldsymbol{0}$ on $\mathbb{R}^+$ and $W_j^D(x;\lambda) - W_j^N(x;\lambda) \to \boldsymbol{0}$ as $x \to +\infty, W_j^D - W_j^N$ is expressed by the linear combination of the fundamental functions as $W_j^D(x;\lambda) - W_j^N(x;\lambda) = \sum_{k=1}^{N} a_k(\lambda)\boldsymbol{w}_k(x;\lambda)$ for some $a_k(\lambda)$. In general, all of U in the forms of $U(x) = W^N(x;\lambda) + \sum_{k=1}^{N} a_k(x;\lambda)\boldsymbol{w}_k(x;\lambda)$ satisfies $(\lambda - L_0)U = \boldsymbol{f}$ on $\mathbb{R}_+$.

Let $B_0(\lambda)$ be the Dirichlet-Neumann map of $(\lambda - L_0)$ on $\mathbb{R}_+$ defined by $\partial_x \boldsymbol{m}(0) = B_0(\lambda)\boldsymbol{m}(0)$ for the function $\boldsymbol{m}(x)$ satisfying $(\lambda - L_0)\boldsymbol{m} = \boldsymbol{0}$ on $\mathbb{R}_+$ and $\boldsymbol{m}(+\infty) = \boldsymbol{0}$. In fact, it is defined as the linear map $B_0(\lambda) : \sum_{k=1}^{N} a_k \boldsymbol{b}_k(0;\lambda) \to \sum_{k=1}^{N} a_k\{\partial_x \boldsymbol{b}_k(0;\lambda) - \mu_k(\lambda)\boldsymbol{b}_k(0;\lambda)\}$. Now we define the function $U(\boldsymbol{x}) = U_0[\boldsymbol{f}](\boldsymbol{x})$ on Ω by

$$U(x\boldsymbol{e}_j) = U_j(x) := W_j^N(x;\lambda) + \boldsymbol{m}_j(x;\boldsymbol{m}^* - W_j^N(0;\lambda)),$$

where $\boldsymbol{m}^* := \frac{1}{R}\sum_{k=1}^{R} W_k^N(0;\lambda)$ and $\boldsymbol{m}(x;\boldsymbol{m}_0)$ is the function on $\mathbb{R}_+$ satisfying

$$\begin{cases} (\lambda - L_0)\boldsymbol{m} = \boldsymbol{0}, x \in \mathbb{R}_+, \\ \boldsymbol{m}(0) = \boldsymbol{m}_0, \boldsymbol{m}(+\infty) = \boldsymbol{0} \end{cases}$$

for given $\boldsymbol{m}_0 \in \mathbb{R}^N$, which is determined uniquely. We will show that $U(\boldsymbol{x})$ satisfies the Kirchhoff boundary condition as follows:

First, we note that $U_j(0) = W_j^N(0;\lambda) + (\boldsymbol{m}^* - W_j^N(0;\lambda)) = \boldsymbol{m}^*$ holds for $j = 1, \ldots, R$.

Next, since

$$\begin{aligned} \partial_x U_j(0) &= \partial_x W_j^N(0;\lambda) + \partial_x \boldsymbol{m}_j(0;\boldsymbol{m}^* - W_j^N(0;\lambda)) \\ &= \boldsymbol{0} + B_0(\lambda)\boldsymbol{m}_j(0;\boldsymbol{m}^* - W_j^N(0;\lambda)) \\ &= B_0(\lambda)(\boldsymbol{m}^* - W_j^N(0;\lambda)) \end{aligned}$$

holds, we see

$$\sum_{j=1}^{R} \partial_x U_j(0) = B_0(\lambda)\{R\boldsymbol{m}^* - \sum_{j=1}^{R} W_j^N(0;\lambda)\} = \boldsymbol{0}.$$

Thus, the resolvent $(\lambda - \mathcal{L}_0)^{-1}$ exists for $Re(\lambda) > -\rho_0$, which is given by $(\lambda - \mathcal{L}_0)^{-1}\boldsymbol{f} = U_0[\boldsymbol{f}]$. □

Lemma ([10], Lemma 4.3). The spectral set $\sum(\mathcal{L}(\boldsymbol{l}))$ consists of $\Sigma(\mathcal{L}(\boldsymbol{l})) = \Sigma_0(\boldsymbol{l}) \cup \Sigma_1(\boldsymbol{l})$ for sufficiently large $\min \boldsymbol{l}$, where $\Sigma_0(\boldsymbol{l}) \subset \{\lambda \in \mathbb{C};\ |\lambda| < -\gamma_3 e^{-\theta_0 \alpha \min \boldsymbol{l}}\}$ and $\Sigma_1(\boldsymbol{l}) \subset \{\lambda \in \mathbb{C};\ Re(\lambda) < -\gamma_4\}$ for positive constants γ_3, γ_4 and $0 < \theta_0 < 1$

Proof. Define the linear map $B(\boldsymbol{x};\boldsymbol{l})$ on Ω by

$$B(x\boldsymbol{e}_j;\boldsymbol{l}) = B_j(x;l_j) := \begin{cases} \boldsymbol{F}'(S(x-l_j)) - \boldsymbol{F}'(\boldsymbol{0}), j = 1,\dots,r, \\ \boldsymbol{O}, j = r+1,\dots,R, \end{cases}$$

where $\boldsymbol{O}$ denotes the zero matrix. Note that $|B_j(x;l_j)| \le O(e^{-\alpha|x-l_j|})$ and $\mathcal{L}(\boldsymbol{l}) = \mathcal{L}_0 + B(\cdot,\boldsymbol{l})$ hold. Fixing $\bar{l} > 0$ sufficiently large and taking $\min \boldsymbol{l} > l$, we put $x_k = x_k(\boldsymbol{l}) := \frac{k}{5}\min \boldsymbol{l}$ and intervals $I_1 := (x_2,x_3), I_2 := (x_1,x_4)$ on each Ω_j. Let $\{\chi_k(x)\}$ be the smooth cut-off functions satisfying $0 \le \chi_k(x) \le 1$,

$$\chi_1(x) = \begin{cases} 1, x \le x_2, \\ 0, x \ge x_3, \end{cases} \quad \chi_2(x) = \begin{cases} 0, x \le x_2, \\ 1, x \ge x_3, \end{cases} \quad \chi_1(x) + \chi_2(x) = 1$$

and

$$\chi_3(x) = \begin{cases} 1, x \in I_1, \\ 0, x \notin I_2. \end{cases}$$

Hereafter in this section, we consider the equation $(\lambda - \mathcal{L}(\boldsymbol{l}))U = \boldsymbol{f}$ for $\boldsymbol{f} \in \{L_2(\Omega)\}^N$ and $\lambda \in \Sigma_2(\boldsymbol{l};\theta)$, where $\Sigma_2(\boldsymbol{l};\theta) := \{\lambda \in \mathbb{C}; |\lambda| \ge \gamma_3 e^{-\theta\alpha \min \boldsymbol{l}}, Re(\lambda) \ge -\rho_1\}$ for $0 < \theta < 1$ and $0 < \rho_1 < \rho_0$. Then we can define the operator $D(\lambda)$ on Ω by

$$\{D(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j) := \begin{cases} \chi_1(x)\{(\lambda-\mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j) + \chi_2(x)\{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}(x), j = 1,\dots,r, \\ \{(\lambda-\mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j), j = r+1,\dots,R \end{cases}$$

for $\lambda \in \Sigma_2(\boldsymbol{l};\theta)$, where $\boldsymbol{f}_j(x) = \boldsymbol{f}(x\boldsymbol{e}_j)$. We represent the restriction of $D(\lambda)$ on Ω_j by $\{D_j(\lambda)\boldsymbol{f}\}(x) := \{D(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j)$. Here we note that $||(\lambda-\mathcal{L}_0)^{-1}|| \le \gamma$ and $||(\lambda - D(\lambda))^{-1}|| \le \dfrac{\gamma'}{|\lambda|}$ for $Re(\lambda) > -\rho_1$, where γ and γ' are positive constants independent of $\lambda \in \Sigma_2(\boldsymbol{l};\theta), \boldsymbol{f} \in \{L_2(\Omega)\}^N$ and $\boldsymbol{l}$ with $\min \boldsymbol{l} > \bar{l}$. The same notations γ, γ' and additional $\gamma'', \dots$ are used again below as general positive constants independent of them.

First, we note that for $x > x_3$, $\{(\lambda - L(l_j))D_j(\lambda)\boldsymbol{f}\}(x) = \boldsymbol{f}_j(x)\,(j = 1,\dots,r)$ and $\{(\lambda - L_0)D_j(\lambda)\boldsymbol{f}\}(x) = \boldsymbol{f}_j(x)\,(j = r+1,\dots,R)$ hold, which means

$$\{(\lambda - \mathcal{L}(\boldsymbol{l}))D(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j) = \boldsymbol{f}_j(x)\,(x > x_3). \tag{4.3.10}$$

For $0 < x < x_2, \{D_j(\lambda)\boldsymbol{f}\}(x) = \{(\lambda-\mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j)$ holds. Hence we see for $j = 1,\dots,r$,

$$\begin{aligned}\{(\lambda - L(l_j))D_j(\lambda)\boldsymbol{f}\}(x) &= \{(\lambda - L_0 - B_j(l_j))[\{(\lambda-\mathcal{L}_0)^{-1}\boldsymbol{f}\}(\cdot\boldsymbol{e}_j)](x) \\ &= \boldsymbol{f}_j(x) - B_j(x;l_j)\{(\lambda-\mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j) \\ &= \boldsymbol{f}_j(x) + \{O(e^{-\frac{3}{5}\alpha\min \boldsymbol{l}})\boldsymbol{f}\}(x\boldsymbol{e}_j)\end{aligned}$$

because of $|B_j(x;l_j)| \leq O(e^{-\alpha|x-l_j|}) \leq O(e^{-\alpha|x_2-l_j|}) \leq O(e^{-\frac{3}{5}\alpha \min \boldsymbol{l}})$ for $0 < x < x_2 = \frac{2}{5}\min \boldsymbol{l}$ and for $j = r+1, \ldots, R$,

$$
\begin{aligned}
\{(\lambda - L_0)D_j(\lambda)\boldsymbol{f}\}(x) &= \{(\lambda - L_0)[\{(\lambda - \mathcal{L}_0)^{-1}\boldsymbol{f}\}(\cdot\boldsymbol{e}_j)](x) \\
&= \boldsymbol{f}_j(x).
\end{aligned}
$$

Thus, we have

$$
\{(\lambda - \mathcal{L}(\boldsymbol{l}))D(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j) = \boldsymbol{f}_j(x) + (\Lambda_1(\boldsymbol{l})\boldsymbol{f})(x\boldsymbol{e}_j)(0 < x < x_2), \tag{4.3.11}
$$

where $\{\Lambda_1(\boldsymbol{l})\boldsymbol{f}\}(x\boldsymbol{e}_j) := \chi_1(x)B_j(x;l_j)\{(\lambda - \mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j)$ is an bounded operator on $\{L_2(\Omega)\}^N$ satisfying $||\Lambda_1(\boldsymbol{l})|| \leq O(e^{-\frac{3}{5}\alpha \min \boldsymbol{l}})$.

Finally, we give the estimation for $x_2 < x < x_3$. Let $U_j(x) = \{(\lambda - \mathcal{L}_0)^{-1}\boldsymbol{f}\}(x\boldsymbol{e}_j)$ and $V_j(x) = \{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}(x)$. Then we see

$$
\begin{aligned}
\{(\lambda - L_0)V_j\}(x) &= \{(\lambda - L(l_j) + B_j(\cdot;l_j))V_j\}(x) \\
&= L_N[\boldsymbol{f}_j](x) + B_j(x;l_j)\{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}(x) \\
&= \boldsymbol{f}_j(x) + B_j(x;l_j)\{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}(x)
\end{aligned}
$$

for $x_2 < x < x_3$. By the multiplication of $\chi_3(x)$, we can assume that the above equation holds for any $x > 0$, that is,

$$
\{(\lambda - L_0)V_j\}(x) = \boldsymbol{f}_j(x) + \chi_3(x)B_j(x;l_j)(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j](x) \tag{4.3.12}
$$

on $\mathbb{R}_+$. Since $|\chi_3(x)B_j(x;l_j)| \leq O(e^{-\alpha(l_j - x_3)}) \leq O(e^{-\frac{2}{5}\alpha \min \boldsymbol{l}})$ holds for $x > 0$, we have

$$
\begin{aligned}
||\chi_{j,3}B_j(\cdot;l_j)\{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}||_{L^2_+} &\leq \frac{\gamma' e^{-\frac{2}{5}\alpha \min \boldsymbol{l}}}{|\lambda|}||\boldsymbol{f}_j||_{L^2_+} \\
&\leq \frac{\gamma' e^{-\alpha\left(\frac{2}{5}\min \boldsymbol{l} - \theta \min \boldsymbol{l}\right)}}{\gamma_3}||\boldsymbol{f}_j||_{L^2_+} \leq \gamma e^{-\alpha\left(\frac{2}{5} - \theta\right)\min \boldsymbol{l}}||\boldsymbol{f}_j||_{L^2_+}
\end{aligned}
$$

for $\lambda \in \Sigma_2(\boldsymbol{l};\theta)$, where γ' and γ are positive constants.

Let $W_j(x) = W_j[\boldsymbol{f}_j](x) := (\lambda - L_0)^{-1}L_N[\boldsymbol{g}_j](x)$ with $\boldsymbol{g}_j(x) := \chi_3(x)B_j(x;l_j)\{(\lambda - L(l_j))^{-1}L_N[\boldsymbol{f}_j]\}(x)$. Then $||W_j[\boldsymbol{f}_j]||_{H^2(\mathbb{R}_+)} \leq \gamma' e^{-\alpha\left(\frac{2}{5} - \theta\right)\min l}||\boldsymbol{f}_j||_{L^2_+}$ holds for $\gamma' > 0$ together with $V_j - U_j - W_j \in \{L^2(\mathbb{R}_+)\}^N$ and $(\lambda - L_0)(V_j - U_j - W_j) = \boldsymbol{0}$. Hence $V_j(x) - U_j(x) - W_j(x)$ is represented by fundamental functions $\{\boldsymbol{w}_j\}$ in the form of

$$
V_j(x) - U_j(x) - W_j(x) = \sum_{j=1}^{N} a_j e^{-\mu_j(\lambda)x}\boldsymbol{b}_j(x;\lambda).
$$

Since $||V_j - U_j - W_j||_{L^\infty(\mathbb{R}_+)} \leq \gamma ||V_j - U_j - W_j||_{H^2(\mathbb{R}_+)} \leq \gamma' ||\boldsymbol{f}||_{L^2(\Omega)}$ hold for $\gamma > 0$ and $\gamma' > 0$ and also $Re(\mu_j(\lambda)) > \rho'$ for a positive constant ρ' by (H1), we have $||V_j - U_j - W_j||_{H^2(I_1)} \leq \gamma'' e^{-\rho' x_2} ||\boldsymbol{f}||_{L^2(\Omega)} = \gamma'' e^{-\frac{2}{5}\rho' \min \boldsymbol{l}} ||\boldsymbol{f}||_{L^2(\Omega)}$ for $\gamma'' > 0$. Then we see

$$\begin{aligned}
\{D_j(\lambda)\boldsymbol{f}\}(x) &= \chi_1(x) U_j(x) + \chi_2(x) V_j(x) \\
&= \{\chi_1(x) + \chi_2(x)\} U_j(x) + \chi_2(x)\{V_j(x) - U_j(x)\} \\
&= U_j(x) + \widetilde{W}_j(x),
\end{aligned}$$

where $\widetilde{W}_j(x) := \chi_2(x)\{V_j(x) - U_j(x)\}$. $\widetilde{W}_j(x)$ is estimated on I_1 as

$$\begin{aligned}
||\widetilde{W}_j||_{H^2(I_1)} &\leq ||\chi_2(V_j - U_j - W_j)||_{H^2(I_1)} + ||\chi_2 W_j||_{H^2(I_1)} \\
&\leq \gamma'' e^{-\frac{2}{5}\rho' \min \boldsymbol{l}} ||\boldsymbol{f}||_{L^2(\Omega)} + \gamma' e^{-\alpha\left(\frac{2}{5} - \theta\right) \min \boldsymbol{l}} ||\boldsymbol{f}_j||_{L^2_+} \\
&\leq \gamma''' e^{-\rho'' \min \boldsymbol{l}} ||\boldsymbol{f}_j||_{L^2(\Omega)}
\end{aligned}$$

for $\gamma''' > 0$ and $\rho'' > 0$ by taking $0 < \theta < \frac{2}{5}$. Therefore, it follows on I_1 that

$$\begin{aligned}
\{(\lambda - L(l_j)) D_j(\lambda)\boldsymbol{f}\}(x) &= \{(\lambda - L(l_j))(U_j + \widetilde{W}_j)\}(x) \\
&= \{(\lambda - L_0) U_j\}(x) + B_j(x; l_j) U_j(x) + \{(\lambda - L(l_j))\widetilde{W}_j\}(x) \\
&= \boldsymbol{f}_j(x) + \{\widetilde{B}_j(\lambda)\boldsymbol{f}\}(x)
\end{aligned}$$

for $j = 1, \ldots, r$, where $\{\widetilde{B}_j(\lambda)\boldsymbol{f}\}(x) := B_j(x; l_j) U_j(x) + \{(\lambda - L(l_j))\widetilde{W}_j\}(x)$ with

$$||\widetilde{B}_j(\lambda)\boldsymbol{f}||_{L^2(I_1)} \leq O(e^{-\frac{2}{5}\alpha \min \boldsymbol{l}} + e^{-\rho'' \min \boldsymbol{l}}) ||\boldsymbol{f}||_{L^2(\Omega)} \leq O(e^{-\gamma \min \boldsymbol{l}}) ||\boldsymbol{f}||_{L^2(\Omega)}$$

for a positive constant γ.

On the other hand, $(\lambda - L_0) D_j(\lambda)\boldsymbol{f} = \boldsymbol{f}_j$ is clear for $j = r+1, \ldots, R$. Thus, we have

$$\{(\lambda - \boldsymbol{\mathcal{L}}(\boldsymbol{l})) D(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j) = \boldsymbol{f}_j(x) + \{\Lambda_2(\lambda)\boldsymbol{f}\}(x\boldsymbol{e}_j)\,(x_2 < x < x_3) \tag{4.3.13}$$

with $||\Lambda_2(\lambda)\boldsymbol{f}||_{L^2(I_1)} \leq \gamma' e^{-\gamma \min l} ||\boldsymbol{f}||_{L^2(\Omega)}$ for a positive constant γ'.

(4.3.10)((30) in [10]), (4.3.11)((31) in [10]) and (4.3.13)((33) in [10]) lead the equation for the operator $\boldsymbol{\mathcal{L}}(\boldsymbol{l})$ on $\{L^2(\Omega)\}^N$

$$(\lambda - \boldsymbol{\mathcal{L}}(\boldsymbol{l})) D(\lambda)\boldsymbol{f} = \boldsymbol{f} + \Lambda_3(\lambda)\boldsymbol{f}$$

with $||\Lambda_3(\lambda)\boldsymbol{f}||_{L^2(\Omega)} \leq \gamma' e^{-\gamma \min l} ||\boldsymbol{f}||_{L^2(\Omega)}$ for a positive constant γ and γ'.Taking $\tilde{l}$ sufficiently large, we see that $Id + \Lambda_3(\lambda)$ is invertible for $\min \boldsymbol{l} > \tilde{l}$ and hence

$$(\lambda - \boldsymbol{\mathcal{L}}(\boldsymbol{l})) D(\lambda)(Id + \Lambda_3(\lambda))^{-1}\boldsymbol{f} = \boldsymbol{f}$$

holds, which implies that the resolvent $(\lambda - \boldsymbol{\mathcal{L}}(\boldsymbol{l}))^{-1}$ exists for $\lambda \in \Sigma_2(\boldsymbol{l}; \theta_0)$ with $0 < \theta_0 < \frac{2}{5}$. □

Remark (**Remark 4.4** in [10]). From the proof, the resolvent $(\lambda - \mathcal{L}(\boldsymbol{l}))^{-1}$ is given by $(\lambda - \mathcal{L}(\boldsymbol{l}))^{-1} = D(\lambda)(Id + \Lambda_3(\lambda))^{-1}$ and the estimate $||(\lambda - \mathcal{L}(\boldsymbol{l}))^{-1}|| \leq \gamma \left(1 + \dfrac{1}{|\lambda|}\right)$ is obtained.

Let $X := \{L^2(\Omega)\}^N$ and $Q_p(\boldsymbol{l}), R_p(\boldsymbol{l})$ be projections on X corresponding to the spectral sets $\Sigma_0(\boldsymbol{l})$ and $\Sigma_1(\boldsymbol{l})$ of the operator $\mathcal{L}(\boldsymbol{l})$, respectively. We also denote the projections on $\{L^2(\mathbb{R})\}^N$ corresponding to spectral sets $\{0\}$ and Σ_1 of L by Q_p and R_p, respectively, according to (H2). Then we note that the projection Q_p is given by $Q_pU = \langle U, \Phi^* \rangle_{L^2(\mathbb{R})} \partial_x S$ for $U \in \{L^2(\mathbb{R})\}^N$ by (H2) and (H3). Similarly, we give a projection $Q_p^+(l)$ on $L_+^2 = \{L^2(\mathbb{R}_+)\}^N$ by $\{Q_p^+(l)U\}(x) := \langle U, \Phi^*(\cdot; l)\rangle_{L_+^2} \partial_x S(x - l)$ for $U \in L_+^2$, where we define $\Phi^*(x; l) := \dfrac{1}{\langle \partial_x S(\cdot - l), \Phi^*(\cdot - l)\rangle_{L_+^2}} \Phi^*(x - l)$ so as to satisfy the normalization of $\langle \partial_x S(\cdot - l), \Phi^*(\cdot; l)\rangle_{L_+^2} = 1$.

Here, we note that the resolvent $(\lambda - \mathcal{L}(\boldsymbol{l}))^{-1}$ is given by $(\lambda - \mathcal{L}(\boldsymbol{l}))^{-1} = D(\lambda) + \Lambda_4(\lambda)$ with $||\Lambda_4(\lambda)||_X \leq \gamma e^{-\gamma' \min \boldsymbol{l}}$ for $\lambda \in \Sigma_2(\boldsymbol{l}; \theta_0)$ from the proof of Lemma ([10], Lemma 4.3), where γ andγ' are certain positive constants. Then by taking the Dunford integral around $\Sigma_0(\boldsymbol{l})$, we have

$$\{Q_p(\boldsymbol{l})U\}(x\boldsymbol{e}_j) = \begin{cases} \chi_2(x)\{Q_p(l_j)L_N[U_j]\}(x) + \{\Lambda_5(\boldsymbol{l})U\}(x\boldsymbol{e}_j), & j = 1, \ldots, r, \\ \{\Lambda_5(\boldsymbol{l})U\}(x\boldsymbol{e}_j), & j = r+1, \ldots, R \end{cases}$$

for $U \in X$, where $U_j(x) := U(x\boldsymbol{e}_j)$ and $\Lambda_5(\boldsymbol{l})$ is a bounded operator on X with $||\Lambda_5(\boldsymbol{l})||_X \leq O(e^{-\gamma_5 \min \boldsymbol{l}})$ for a positive constant γ_5. Hence putting

$$\{\widetilde{Q}_p(\boldsymbol{l})U\}(x\boldsymbol{e}_j) := \begin{cases} \{Q_p^+(l_j)U_j\}(x), & j = 1, \ldots, r, \\ \boldsymbol{0}, & j = r+1, \ldots, R \end{cases}$$

for $U \in X$, we see that $||Q_p(\boldsymbol{l}) - \widetilde{Q}_p(\boldsymbol{l})||_X \leq O(e^{-\gamma_5 \min \boldsymbol{l}}) \ll 1$ holds for sufficiently large $\min \boldsymbol{l}$, which implies that the spaces $\widetilde{E}(\boldsymbol{l}) := \tilde{Q}_p(\boldsymbol{l})X$ and $E(\boldsymbol{l}) := Q_p(\boldsymbol{l})X$ are homeomorphic and that there exist $\{\lambda_1(\boldsymbol{l}), \ldots, \lambda_r(\boldsymbol{l})\} \subset \mathbb{C}, \{\Phi_1(\boldsymbol{l}), \ldots, \Phi_r(\boldsymbol{l})\} \subset X$ satisfying $|\lambda_j(\boldsymbol{l})| \leq \gamma_3 e^{-\theta_0 \alpha \min \boldsymbol{l}}$ and

$$\Phi_j(\boldsymbol{l})(x\boldsymbol{e}_k) = \begin{cases} \chi_2(x)\partial_x S(x - l_k) + O(e^{-\gamma_5 \min \boldsymbol{l}}) = \partial_x S(x - l_k) + O(e^{-\gamma_5 \min \boldsymbol{l}}), & k = j, \\ O(e^{-\gamma_5 \min \boldsymbol{l}}), & k \neq j \end{cases}$$

such that $\Sigma_0(\boldsymbol{l}) = \{\lambda_1(\boldsymbol{l}), \ldots, \lambda_r(\boldsymbol{l})\}$ and $E(\boldsymbol{l}) = span\{\Phi_1(\boldsymbol{l}), \ldots, \Phi_r(\boldsymbol{l})\}$.

The adjoint operator $\mathcal{L}^*(\boldsymbol{l})$ also has the same properties as $\mathcal{L}(\boldsymbol{l})$, specifically, $\overline{\Sigma_0(\boldsymbol{l})} \subset$

$\Sigma(\mathcal{L}^*(\boldsymbol{l}))$ and there exist $\{\Phi_1^*(\boldsymbol{l}),\dots,\Phi_r^*(\boldsymbol{l})\}\subset X$ satisfying

$$\Phi_j^*(\boldsymbol{l})(x\boldsymbol{e}_k)=\begin{cases}\Phi^*(x-l_k)+O(e^{-\gamma_5\min\boldsymbol{l}}), & k=j,\\ O(e^{-\gamma_5\min\boldsymbol{l}}), & k\neq j\end{cases}$$

such that the corresponding eigenspace, say $E^*(\boldsymbol{l})$ to the spectral set $\overline{\Sigma_0(\boldsymbol{l})}$ is given by $E^*(\boldsymbol{l})=span\{\Phi_1^*(\boldsymbol{l}),\dots,\Phi_r^*(\boldsymbol{l})\}$. Let $E^\perp(\boldsymbol{l}):=R_p(\boldsymbol{l})X$. We note that $E^\perp(\boldsymbol{l})$ is expressed as $E^\perp(\boldsymbol{l})=\{V\in X;\langle V,\Phi^*(\boldsymbol{l})\rangle_X=0\,(j=1,\dots,r)\}$.

Lemma ([10], Lemma 4.5). $\{\Phi_1(\boldsymbol{l}),\dots,\Phi_r(\boldsymbol{l})\}$ and $\{\Phi_1^*(\boldsymbol{l}),\dots,\Phi_r^*(\boldsymbol{l})\}$ can be taken as

$$\Phi_j(\boldsymbol{l})(x\boldsymbol{e}_k)=\begin{cases}\partial_x S(x-l_j)+\alpha\left(\dfrac{2}{R}-1\right)e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a}+e^{-\alpha l_j}O(e^{-\gamma_7\min\boldsymbol{l}}+e^{-\gamma_6 x}), k=j,\\ \dfrac{2\alpha}{R}e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a}+e^{-\alpha l_j}O(e^{-\gamma_7\min\boldsymbol{l}}+e^{-\gamma_6 x}), k\neq j.\end{cases} \tag{4.3.14}$$

and

$$\Phi_j^*(\boldsymbol{l})(x\boldsymbol{e}_k)=\begin{cases}\Phi^*(x-l_j)-\alpha\left(\dfrac{2}{R}-1\right)e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a}^*+e^{-\alpha l_j}O(e^{-\gamma_7\min\boldsymbol{l}}+e^{-\gamma_6 x}), k=j,\\ -\dfrac{2\alpha}{R}e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a}^*+e^{-\alpha l_j}O(e^{-\gamma_7\min\boldsymbol{l}}+e^{-\gamma_6 x}), k\neq j.\end{cases} \tag{4.3.15}$$

for $\gamma_6>\alpha$ and $\gamma_7>0$. Here we denoted remainders by $O(e^{-\gamma_7\min\boldsymbol{l}}+e^{-\gamma_6 x})$ when the remainders are expressed as $e^{-\gamma_7\min\boldsymbol{l}}\widetilde{\Phi}(\boldsymbol{x})+e^{-\gamma_6 x}\widetilde{\Psi}(\boldsymbol{x})$ for some $\widetilde{\Phi},\widetilde{\Psi}\in\{L^2(\Omega)\}^N\cup\{L^\infty(\Omega)\}^N$.

Proof. We only show (4.3.14) ((34) in [10]). Define $\Phi_j(\boldsymbol{x})$ by

$$\Phi_{jk}(x):=\Phi_j(x\boldsymbol{e}_k)=\begin{cases}\partial_x S(x-l_j), & k=j,\\ \mathbf{0}, & k\neq j\end{cases}$$

and $\psi_j(\boldsymbol{x})$ by

$$\begin{cases}L_0\psi_{jk}=\mathbf{0}, x>0,\\ \{\partial_x^2 S(-l_j)+\partial_x\psi_{jj}(0)\}+\displaystyle\sum_{k\neq j}^{R}\partial_x\psi_{jk}(0)=\mathbf{0},\\ \partial_x S(-l_j)+\psi_{jj}(0)=\psi_{jk}(0)\,(k\neq j),\\ \psi_{jk}(\infty)=\mathbf{0},\end{cases}$$

where $\psi_{jk}(x) := \psi_j(x\boldsymbol{e}_k)$. Then by the quite similar way to the proof of Lemma 4.1 ([10], Lemma 4.1),

$$\begin{cases} \psi_{jj}(x) & = \alpha(\dfrac{2}{R} - 1)e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + O(e^{-(\alpha x + \gamma_6' l_j)} + e^{-(\gamma_6' x + \alpha l_j)}) \\ & = \alpha(\dfrac{2}{R} - 1)e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + e^{-\alpha l_j}e^{-\alpha x}O(e^{-\gamma_7' l_j} + e^{-\gamma_7' x}), j = k, \\ \psi_{jk}(x) & = \dfrac{2\alpha}{R}e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + O(e^{-(\alpha x + \gamma_6' l_j)} + e^{-(\gamma_6' x + \alpha l_j)}) \\ & = \dfrac{2\alpha}{R}e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + e^{-\alpha l_j}e^{-\alpha x}O(e^{-\gamma_7' l_j} + e^{-\gamma_7' x}), k \neq j, \end{cases}$$

hold for $\gamma_6' > \alpha$ and $\gamma_7' > 0$. Hence for $j = 1, \dots, r$, we see

$$\{\mathcal{L}(\boldsymbol{l})(\Phi_j + \psi_j)\}(x\boldsymbol{e}_k) = \begin{cases} \{L(l_k)(\partial_x S(\cdot - j_j) + \psi_{jj})\}(x) = \{L(l_j)\psi_{jj}\}(x), k = j, \\ \{L(l_j)\psi_{jk}\}(x), k = 1, \dots, r, k \neq j, \\ \{L_0\psi_{jk}\}(x) = \boldsymbol{0}, k = r+1, \dots, R \end{cases}$$

and therefore

$$\begin{aligned} \{L(l_k)\psi_{jk}\}(x) =& \{L_0\psi_{jk}\}(x) + B_k(x; l_k)\psi_{jk}(x) = \boldsymbol{0} + O(e^{-\alpha|x - l_k|} \cdot e^{-\alpha l_j}e^{-\alpha x}), \\ & ||L(l_k)\psi_{jk}||_{L^2_+} \leq O(e^{-\alpha l_j}e^{-\alpha l_k}\sqrt{l_k + 1}) \end{aligned}$$

hold, where $B_k(x; l_k) = B(x\boldsymbol{e}_k; \boldsymbol{l}) = \boldsymbol{F}'(S(x - l_k)) - \boldsymbol{F}'(\boldsymbol{0})$ in Lemma ([10], Lemma 4.3). Thus, we have $||\mathcal{L}(\boldsymbol{l})(\Phi_j + \psi_j)||_X \leq O(e^{-\alpha l_j}e^{-\gamma_8 \min \boldsymbol{l}})$ for $\gamma_8 > 0$, which means that $||R_p(\boldsymbol{l})(\Phi_j + \psi_j)||_{L^\infty(\Omega)} \leq O(e^{-\alpha l_j}e^{-\gamma_8 \min \boldsymbol{l}})$ and that we can take

$$\Phi_j(\boldsymbol{l}) = \Phi_j + \psi_j + O(e^{-\alpha l_j}e^{-\gamma_8 \min \boldsymbol{l}})(j = 1, \dots, r),$$

that is, for $j = 1, \dots, r$,

$$\Phi_j(\boldsymbol{l})(x\boldsymbol{e}_k) = \begin{cases} \partial_x S(x - l_j) + \alpha(\dfrac{2}{R} - 1)e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + e^{-\alpha l_j}O(e^{-\gamma_8 \min \boldsymbol{l}} + e^{-\gamma_6' x}), k = j, \\ \dfrac{2\alpha}{R}e^{-\alpha l_j}e^{-\alpha x}\boldsymbol{a} + e^{-\alpha l_j}O(e^{-\gamma_8 \min \boldsymbol{l}} + e^{-\gamma_6' x}), k \neq j. \end{cases}$$

Quite similarly, we have

$$\Phi_j^*(\boldsymbol{l}) = \Phi_j^* + \psi_j^* + O(e^{-\alpha l_j}e^{-\gamma_8 \min \boldsymbol{l}})\,(j = 1, \dots, r),$$

where $\Phi_j^*(\boldsymbol{x})$ is the function of

$$\Phi_{jk}^*(x) := \Phi_j^*(x\boldsymbol{e}_k) = \begin{cases} \Phi^*(x - l_j), k = j, \\ \boldsymbol{0}, \qquad k \neq j \end{cases}$$

and $\psi_j^*(\boldsymbol{x})$ by

$$\begin{cases} L_0^* \psi_{jk}^* = \mathbf{0}, x > 0, \\ \{\partial_x \Phi^*(-l_j) + \partial_x \psi_{jj}^*(0)\} + \sum_{k \neq j}^{R} \partial_x \psi_{jk}^*(0) = \mathbf{0}, \\ \Phi^*(-l_j) + \psi_{jj}^*(0) = \psi_{jk}^*(0)\,(k \neq j), \\ \psi_{jk}^*(+\infty) = \mathbf{0}, \end{cases}$$

where $\psi_{jk}^*(x) := \psi_j^*(x\boldsymbol{e}_k)$. They are calculated as

$$\Phi_j(\boldsymbol{l})(x\boldsymbol{e}_k) = \begin{cases} \Phi(x - l_j) - \alpha(\frac{2}{R} - 1)e^{-\alpha l_j} e^{-\alpha x} \boldsymbol{a}^* + e^{-\alpha l_j} O(e^{-\gamma_8 \min l} + e^{-\gamma_6' x}), k = j, \\ -\frac{2\alpha}{R} e^{-\alpha l_j} e^{-\alpha x} \boldsymbol{a}^* + e^{-\alpha l_j} O(e^{-\gamma_8 \min l} + e^{-\gamma_6' x}), k \neq j \end{cases}$$

for $j = 1, \ldots, r$.

Thus, the proof is completed.

Consider the solution $\mathcal{U}(t, \boldsymbol{x})$ of (1.0.4)((2) in [10]) or (4.1.1)((3) in [10]). Decomposing the solution $\mathcal{U}(t, \boldsymbol{x})$ by $\mathcal{U}(t, \boldsymbol{x}) = S(\boldsymbol{x}; \boldsymbol{l}(t)) + G(\boldsymbol{x}; \boldsymbol{l}(t)) + \mathcal{V}(t, \boldsymbol{x})$ with $\mathcal{V}(t) \in E^{\perp}(\boldsymbol{l}(t))$, we see (4.1.1)((3) in [10]) becomes (4.3.9)((29) in [10]), that is,

$$\partial_t\{S(\cdot; \boldsymbol{l}(t)) + G(\cdot; \boldsymbol{l}(t)) + \mathcal{V}(t, \cdot)\} = \mathcal{L}(\boldsymbol{l})\{G(\cdot; \boldsymbol{l}(t)) + \mathcal{V}(t, \cdot)\} + K(\boldsymbol{l}(t), \mathcal{V}(t, \cdot)), \quad (4.3.16)$$

where $K(\boldsymbol{l}, \mathcal{V})$ are functions satisfying $K_j(\boldsymbol{l}, \mathcal{V}) = K_j(\boldsymbol{l}, \mathcal{V}_j)$ on Ω_j and $|K_j(\boldsymbol{l}, \mathcal{V}_j)(x)| \leq O(|\mathcal{V}_j(t, x)|^2 + |G_j(x; \boldsymbol{l})|^2)\,(j = 1, \ldots, R)$ as in (4.3.9)((29) in [10]) for $\mathcal{V}_j = \mathcal{V}_j(t, x) := \mathcal{V}(t, x\boldsymbol{e}_j)$. Fixing $\boldsymbol{l}^* := (l^*, \ldots, l^*) \in \mathbb{R}^r$ for sufficiently large $l^* \gg 1$, we define the map $\Pi(\boldsymbol{l}) : E^{\perp}(\boldsymbol{l}^*) \to E^{\perp}(\boldsymbol{l})$ by $\Pi(\boldsymbol{l})W := V(1)$ for $W \in E^{\perp}(\boldsymbol{l}^*)$, where $V(\tau)$ is the solution of

$$\begin{cases} \frac{dV}{d\tau} = -\sum_{j,k=1}^{r} (l_j - l^*) \langle V, \partial_{l_j} \Phi_k^*(\theta(\tau)) \rangle_X \Phi_k(\theta(\tau)), \\ V(0) = W, \end{cases}$$

where $\theta(\tau) := (1-\tau)\boldsymbol{l}^* + \tau \boldsymbol{l}$. Then $\Pi(\boldsymbol{l})$ is a homeomorphism from $E^{\perp}(\boldsymbol{l}^*)$ to $E^{\perp}(\boldsymbol{l})$, which was proved in Lemma 4.1 of [8]([5] in [10]). Transforming $\mathcal{V}(t, \cdot) = \Pi(\boldsymbol{l}(t))\mathcal{W}(t, \cdot)$, we see (4.3.16)((36) in [10]) is

$$\langle \dot{\boldsymbol{l}}, \partial_l\{\boldsymbol{S}(\cdot; \boldsymbol{l}) + G(\cdot; \boldsymbol{l})\} + \partial_l \Pi(\boldsymbol{l})\mathcal{W} \rangle + \Pi(\boldsymbol{l})\partial_t \mathcal{W} = \mathcal{L}(\boldsymbol{l})\{G(\cdot; \boldsymbol{l}) + \Pi(\boldsymbol{l})\mathcal{W}\} + K(\boldsymbol{l}, \Pi(\boldsymbol{l})\mathcal{W}), \quad (4.3.17)$$

or equivalently

$$\begin{cases} \dot{\boldsymbol{l}} = \mathcal{H}(\boldsymbol{l}, \mathcal{W}), \\ \partial_t \mathcal{W} = \mathcal{A}(\boldsymbol{l})\mathcal{W} + J(\boldsymbol{l}, \mathcal{W}) \end{cases} \quad (4.3.18)$$

by operating the projections $Q_p(\boldsymbol{l})$ and $R_p(\boldsymbol{l})$ on (4.3.17)((37) in [10]), where $\mathcal{A}(\boldsymbol{l}) := \Pi^{-1}(\boldsymbol{l})\mathcal{L}(\boldsymbol{l})\Pi(\boldsymbol{l})$ and

$$J(\boldsymbol{l},\mathcal{W}) = \Pi^{-1}(\boldsymbol{l})R_p(\boldsymbol{l})\{\mathcal{L}(\boldsymbol{l})G(\boldsymbol{l}) + K(\boldsymbol{l},\Pi(l)\mathcal{W}) - \langle\mathcal{H}(\boldsymbol{l},\mathcal{W}),\partial_{\boldsymbol{l}}\{\boldsymbol{S}(\cdot;\boldsymbol{l}) + G(\cdot;\boldsymbol{l})\}\rangle\}.$$

Here $\langle\cdot,\cdot\rangle$ means the usual inner product of vectors in $\mathbb{R}^r$. We also note that $\langle\dot{\boldsymbol{l}},\partial_{\boldsymbol{l}}\Pi(\boldsymbol{l})\mathcal{W}\rangle \in E(\boldsymbol{l})$ from the definition of $\Pi(\boldsymbol{l})$ and hence $R_p(\boldsymbol{l})\langle\dot{\boldsymbol{l}},\partial_{\boldsymbol{l}}\Pi(\boldsymbol{l})\mathcal{W}\rangle = 0$ holds.

$\mathcal{H}(\boldsymbol{l},\mathcal{W})$ is obtained by taking the inner product of (4.3.17)((37) in [10]) with $\Phi_j^*(\boldsymbol{l})$ in X, which is

$$\langle\langle\dot{\boldsymbol{l}},\partial_{\boldsymbol{l}}\{S(\cdot;\boldsymbol{l}) + G(\cdot;\boldsymbol{l})\} + \partial_{\boldsymbol{l}}\Pi(\boldsymbol{l})\mathcal{W}\rangle,\Phi_j^*(\boldsymbol{l})\rangle_X = \langle\{\mathcal{L}(\boldsymbol{l})G(\cdot;\boldsymbol{l}) + K(\boldsymbol{l},\Pi(\boldsymbol{l})\mathcal{W})\},\Phi_j^*(\boldsymbol{l})\rangle_X. \tag{4.3.19}$$

Since

$$\begin{aligned}\langle\langle\dot{\boldsymbol{l}},\partial_{\boldsymbol{l}}S(\cdot;\boldsymbol{l})\rangle,\Phi_j^*(\boldsymbol{l})\rangle_X &= \sum_{k=1}^{r}\dot{l}_k\langle\partial_{l_k}S(\cdot;\boldsymbol{l}),\Phi_j^*(\boldsymbol{l})\rangle_X \\ &= -\sum_{k=1}^{r}\dot{l}_k\langle S(\cdot - l_k),\Phi_j^*(\boldsymbol{l})(\cdot\boldsymbol{e}_k)\rangle_{L^2_+} \\ &= -\dot{l}_j\{1 + O(e^{-\gamma_5\min\boldsymbol{l}})\} - \sum_{k\neq j}^{r}\dot{l}_k O(e^{-\gamma_5\min\boldsymbol{l}})\end{aligned}$$

holds, the matrix

$$\mathcal{B}(\boldsymbol{l},\mathcal{W}) = \{a_{jk}\}_{1\le j,k\le r} := \{\langle\partial_{l_k}\{\boldsymbol{S}(\cdot;\boldsymbol{l}) + G(\cdot;\boldsymbol{l}) + \Pi(\boldsymbol{l})\mathcal{W}\},\Phi_j^*(\boldsymbol{l})\rangle_X\}_{1\le j,k\le r}$$

is invertible for a sufficiently small $\mathcal{W}$. Since (4.3.19)((39) in [10]) is equivalently written as

$$\mathcal{B}(\boldsymbol{l},\mathcal{W})\dot{\boldsymbol{l}} = \boldsymbol{b}(\boldsymbol{l},\mathcal{W}),$$

where

$$\boldsymbol{b}(\boldsymbol{l},\mathcal{W}) := \begin{pmatrix}\langle\mathcal{L}(\boldsymbol{l})G(\cdot;\boldsymbol{l}) + K(\boldsymbol{l},\Pi(\boldsymbol{l})\mathcal{W}),\Phi_1^*(\boldsymbol{l})\rangle_X \\ \vdots \\ \langle\mathcal{L}(\boldsymbol{l})G(\cdot;\boldsymbol{l}) + K(\boldsymbol{l},\Pi(\boldsymbol{l})\mathcal{W}),\Phi_r^*(\boldsymbol{l})\rangle_X\end{pmatrix}$$

$\mathcal{H}(\boldsymbol{l},\mathcal{W})$ in (4.3.18)((38) in [10]) is given by $\mathcal{H}(\boldsymbol{l},\mathcal{W}) := \mathcal{B}^{-1}(\boldsymbol{l},\mathcal{W})\boldsymbol{b}(\boldsymbol{l},\mathcal{W})$. Let $\mathcal{H}(\boldsymbol{l},\mathcal{W}) = {}^t(\mathcal{H}_1(\boldsymbol{l},\mathcal{W}),\ldots,\mathcal{H}_r(\boldsymbol{l},\mathcal{W}))$ and $\boldsymbol{b}(\boldsymbol{l},\mathcal{W}) = {}^t(b_1(\boldsymbol{l},\mathcal{W}),\ldots,b_r(\boldsymbol{l},\mathcal{W}))$.

Lemma ([10], Lemma 4.6). There exists $\gamma_9 > \alpha$ such that

$$\mathcal{H}_j(\boldsymbol{l},\mathcal{W}) = \{M_0C_j(\boldsymbol{l}) + O(e^{-\gamma_9\min\boldsymbol{l}})\}e^{-\alpha l_j} + e^{-\alpha l_j}O(||G(\cdot;\boldsymbol{l})||^2_{L^\infty(\Omega)} + ||\mathcal{W}||^2_{L^\infty(\Omega)})$$

holds for $\min\boldsymbol{l} \ge l^*$ and a sufficiently small $\mathcal{W}$.

Proof. We express $\Phi_j^*(\boldsymbol{l})$ by $\Phi_j^*(\boldsymbol{l}) = \Phi_j^* + \psi_j^* + \eta_j^*$ $(j = 1, \ldots, r)$ according to Lemma ([10], Lemma 4.5), where $\Phi_j^*$and ψ_j^* were already defined in Lemma ([10], Lemma 4.5) and η_j^* are functions satisfying $||\eta_j^*||_{L^\infty(\Omega)} \leq O(e^{-\alpha l_j} e^{-\gamma_8 \min \boldsymbol{l}})$. By $\mathcal{L}^*(\boldsymbol{l}) = \mathcal{L}_0^* + {}^t B(\cdot; \boldsymbol{l})$ for $B(\boldsymbol{x}; \boldsymbol{l})$ in Lemma ([10], Lemma 4.3) and $\Phi^*(x - l) \to -e^{-\alpha|x-l|}\boldsymbol{a}^* = -e^{-\alpha(x-l)}\boldsymbol{a}^*$ as $l \to +\infty$, we have

$$
\begin{aligned}
\langle \mathcal{L}(\boldsymbol{l})G(\cdot, \boldsymbol{l}), \Phi_j^*(\boldsymbol{l})\rangle_X &= \sum_{k=1}^{r} \langle L(l_k)G_k(\cdot, \boldsymbol{l}), \Phi_j^*(\boldsymbol{l})(\cdot \boldsymbol{e}_k)\rangle_{L_+^2} + \sum_{k=r+1}^{R} \langle L_0 G_k(\cdot, \boldsymbol{l}), \Phi_j^*(\boldsymbol{l})(\cdot \boldsymbol{e}_k)\rangle_{L_+^2} \\
&= \langle L(l_j)G_j(\cdot, \boldsymbol{l}), \Phi_j^*(\boldsymbol{l})(\cdot \boldsymbol{e}_j)\rangle_{L_+^2} + \sum_{k \neq j}^{r} \langle L(l_j)G_k(\cdot, \boldsymbol{l}), \Phi_j^*(\boldsymbol{l})(\cdot \boldsymbol{e}_k)\rangle_{L_+^2} + \boldsymbol{0} \\
&= \langle L(l_j)G_j(\cdot, \boldsymbol{l}), \Phi_{jj}^* + \psi_{jj}^*)\rangle_{L_+^2} + \langle (L_0 + B_j(\cdot, l_j))G_j(\cdot, \boldsymbol{l}), \eta_{jj}^*)\rangle_{L_+^2} \\
&\qquad + \sum_{k \neq j}^{r} \langle \{L_0 + B_k(\cdot, l_k)\}G_k(\cdot; \boldsymbol{l}), \boldsymbol{0} + \psi_{jk}^* + \eta_{jk}^* \rangle_{L_+^2} \\
&= \langle L(l_j)G_j(\cdot, \boldsymbol{l}), \Phi_{jj}^* + \psi_{jj}^* \rangle_{L_+^2} \\
&\qquad + \langle B_j(\cdot; l_j)G_j(\cdot, \boldsymbol{l}), \eta_{jj}^* \rangle_{L_+^2} + \sum_{k \neq j}^{r} \langle B_k(\cdot, l_k)G_k(\cdot; \boldsymbol{l}), \psi_{jk}^* + \eta_{jk}^* \rangle_{L_+^2}.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
&\langle L(l_j)G_j(\cdot, \boldsymbol{l}), \Phi_{jj}^* + \psi_{jj}^* \rangle_{L_+^2} \\
&= [\langle D\partial_x G_j(x; \boldsymbol{l}), \Phi_{jj}^*(x) + \psi_{jj}^*(x)\rangle]_0^{+\infty} - [\langle DG_k(x; \boldsymbol{l}), \partial_x\{\Phi_{jj}^*(x) + \psi_{jj}^*(x)\}\rangle]_0^{+\infty} \\
&\qquad + \langle G_j(\cdot, \boldsymbol{l}), L^*(l_j)(\Phi_{jj}^* + \psi_{jj}^*)\rangle_{L_+^2} \\
&= \alpha C_j(\boldsymbol{l}) \cdot \{-e^{-\alpha l_j} - \alpha \left(\frac{2}{R} - 1\right) e^{-\alpha l_j}\} \langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle \\
&\qquad + C_j(\boldsymbol{l}) \cdot \{-\alpha e^{-\alpha l_j} + \alpha^2 \left(\frac{2}{R} - 1\right) e^{-\alpha l_j}\} \langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle + O(e^{-\alpha l_j} e^{-\gamma' \min l}) \\
&\qquad + \langle G_j(\cdot; \boldsymbol{l}), L^*(l_j)\Phi_{jj}^* \rangle_{L_+^2} + \langle G_j(\cdot; \boldsymbol{l}), (L_0^* + {}^t B_j(\cdot; l_j)\psi_{jj}^* \rangle_{L_+^2} \\
&= -2\alpha \langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle C_j(\boldsymbol{l}) e^{-\alpha l_j} + O(e^{-\alpha l_j} e^{-\gamma' \min \boldsymbol{l}}) + \langle G_j(\cdot; \boldsymbol{l}), {}^t B_j(\cdot; l_j)\psi_{jj}^* \rangle_{L_+^2}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\langle \mathcal{L}(\boldsymbol{l})G_j(\cdot,\boldsymbol{l}),\Phi^*_{jj}(\boldsymbol{l})\rangle_X\\
&\quad = -2\alpha\langle D\boldsymbol{a},\boldsymbol{a}^*\rangle C_j(\boldsymbol{l})e^{-\alpha l_j} + O(e^{-\alpha l_j}e^{-\gamma'\min\boldsymbol{l}}) + \sum_{k=1}^{r}\langle B_k(\cdot;l_k)G_k(\cdot;l_j),\psi^*_{jk}+\eta^*_{jk}\rangle_{L^2_+}\\
&\quad = -2\alpha\langle D\boldsymbol{a},\boldsymbol{a}^*\rangle C_j(\boldsymbol{l})e^{-\alpha l_j} + O(e^{-\alpha l_j}e^{-\gamma'\min\boldsymbol{l}})\\
&\qquad\quad + \sum_{k=1}^{r}\int_0^{+\infty} O(e^{-\alpha|x-l_k|}e^{-\alpha\min\boldsymbol{l}}e^{-\alpha x}(e^{-\alpha l_j}e^{-\alpha x}+e^{-\alpha l_j}e^{-\gamma_8\min\boldsymbol{l}}))dx\\
&\quad = -2\alpha\langle D\boldsymbol{a},\boldsymbol{a}^*\rangle C_j(\boldsymbol{l})e^{-\alpha l_j} + O(e^{-\alpha l_j}e^{-\gamma'\min\boldsymbol{l}})\\
&\qquad\quad + e^{-\alpha l_j}e^{-\alpha\min\boldsymbol{l}}\sum_{k=1}^{r}\int_0^{\infty} O(e^{-\alpha|x-l_k|}e^{-\alpha x}(e^{-\alpha x}+e^{-\gamma_8\min\boldsymbol{l}}))dx\\
&\quad = -2\alpha\langle D\boldsymbol{a},\boldsymbol{a}^*\rangle C_j(\boldsymbol{l})e^{-\alpha l_j} + O(e^{-\alpha l_j}e^{-\gamma'\min\boldsymbol{l}}) + (e^{-\alpha l_j}e^{-\alpha\min\boldsymbol{l}}\sum_{k=1}^{r}O(e^{-\alpha l_k}+e^{-\gamma_8\min\boldsymbol{l}}e^{-\alpha l_k}l_k)\\
&\quad = -2\alpha\langle D\boldsymbol{a},\boldsymbol{a}^*\rangle C_j(\boldsymbol{l})e^{-\alpha l_j} + O(e^{-\alpha l_j}e^{-\gamma''\min\boldsymbol{l}})\\
&\quad = \{-M_0C_j(\boldsymbol{l}) + O(e^{-\gamma''\min\boldsymbol{l}})\}e^{-\alpha l_j}
\end{aligned}
$$

for $\gamma',\gamma''>\alpha$.

The remainder of the proof of obvious from Lemma ([10], Lemma4.5).

□

Now, we denote by X^ω for $0<\omega<1$ the fractional powered space of X by $\mathcal{A}(\boldsymbol{l}^*)$ with the embedding to $\{L^\infty(\Omega)\}^N$. We define

$$
\delta(\boldsymbol{l}) := \begin{cases} e^{-\alpha\min\boldsymbol{l}} & (\min\boldsymbol{l}\ge l^*),\\ e^{-\alpha l^*} & (\min\boldsymbol{l}<l^*),\end{cases}\qquad \delta(\boldsymbol{l};\boldsymbol{l}') := \begin{cases} e^{-\alpha\min\{\boldsymbol{l},\boldsymbol{l}'\}} & (\min\{\boldsymbol{l},\boldsymbol{l}'\}\ge l^*),\\ e^{-\alpha l^*} & (\min\{\boldsymbol{l},\boldsymbol{l}'\}<l^*)\end{cases}
$$

and

$$
\begin{aligned}
\widetilde{W}(D_1,D_2) := \{&\mathcal{W}\in C(\mathbb{R}^r;E^\perp(\boldsymbol{l}^*)\cap X^\omega);\\
&||\mathcal{W}(\boldsymbol{l})||_\omega\le D_1\delta(\boldsymbol{l}), ||\mathcal{W}(\boldsymbol{l})-\mathcal{W}(\boldsymbol{l}')||_\omega \le D_2\delta(\boldsymbol{l},\boldsymbol{l}')|\boldsymbol{l}-\boldsymbol{l}'|\}.
\end{aligned}
$$

Here we note to extend the region $\{\min\boldsymbol{l}\ge l^*\}$ of $\boldsymbol{l}$ to the whole space $\mathbb{R}^r$ such that all estimates above hold by using an appropriate cut-off function such as $||G(\cdot;l)||_{L^\infty(\Omega)}\le O(\delta(\boldsymbol{l}))$ holds for $\boldsymbol{l}\in\mathbb{R}^r$. As the result, we assume that for $\boldsymbol{l}\in\mathbb{R}^r$ and $\mathcal{W}\in\widetilde{W}(D_1,D_2)$, the result in Lemma (Lemma 4.6 in [10]) and

$$
\begin{aligned}
&||J(\boldsymbol{l},\mathcal{W})||_X \le O(||G(\boldsymbol{l})||_{L^\infty(\Omega)} + ||G(\boldsymbol{l})||_{L^\infty(\Omega)}||\mathcal{W}||_{L^\infty(\Omega)} + ||\mathcal{W}||^2_{L^\infty(\Omega)} + |\mathcal{H}(\boldsymbol{l},\mathcal{W})|),\\
&|\mathcal{H}_j(\boldsymbol{l},\mathcal{W})-\mathcal{H}_j(\boldsymbol{l}',\mathcal{W}')| \le O(\delta(l_j,l_j')\cdot\delta(\boldsymbol{l},\boldsymbol{l}'))\{|\boldsymbol{l}-\boldsymbol{l}'| + ||\mathcal{W}-\mathcal{W}'||_{L^\infty(\Omega)}\},\\
&||J(\boldsymbol{l},\mathcal{W})-J(\boldsymbol{l}',\mathcal{W}')||_X \le O(\delta(\boldsymbol{l},\boldsymbol{l}'))\{|\boldsymbol{l}-\boldsymbol{l}'| + ||\mathcal{W}-\mathcal{W}'||_{L^\infty(\Omega)}\}
\end{aligned}
$$

hold. Then we see

$$||J(\boldsymbol{l}, \mathcal{W})||_X \leq O(\delta(\boldsymbol{l}) + (D_1 + D_1^2)\delta^2(\boldsymbol{l}))$$

for $\mathcal{W} \in \widetilde{W}(D_1, D_2)$.Remainders are quite a similar way to [8]([5] in [10]) and we can construct a function $\sigma^* \in \widetilde{W}(D_1, D_2)$ for appropriate D_1 and D_2 such that $\mathcal{M}^* := \{(\boldsymbol{l}, \sigma^*(\boldsymbol{l})); \boldsymbol{l} \in \mathbb{R}^r\}$ is an attractive invariant manifold of (4.3.18)((38) in [10]). It means that the function

$$U(t) = S(\boldsymbol{l}(t)) + G(\boldsymbol{l}(t)) + \Pi(\boldsymbol{l}(t))\sigma^*(\boldsymbol{l}(t))$$

with the solution

$$\begin{cases} \dfrac{d\boldsymbol{l}}{dt} = \mathcal{H}(\boldsymbol{l}, \sigma^*(\boldsymbol{l})), \\ \boldsymbol{l}(0) = \boldsymbol{l}_0 \in \mathbb{R}^r \end{cases}$$

is a solution of (4.1.1)((2) in [10]), (1.0.4)((3) in [10]). Lemma ([10], Lemma4.6) leads

$$\mathcal{H}_j(\boldsymbol{l}, \sigma^*(\boldsymbol{l})) = M_0 C_j(\boldsymbol{l}) e^{-\alpha l_j} + e^{-\alpha l_j}(e^{-\gamma_{10} \min \boldsymbol{l}}) = H_j(\boldsymbol{l}) + e^{-\alpha l_j}(e^{-\gamma_{10} \min \boldsymbol{l}})$$

for $\gamma_{10} > \alpha$. Thus the proof Theorem 4.1 ([10], Theorem 2.2) is completed by taking $\widetilde{\boldsymbol{H}}(\boldsymbol{l}) = \mathcal{H}(\boldsymbol{l}, \sigma^*(\boldsymbol{l}))$ for $\widetilde{\boldsymbol{H}}(\boldsymbol{l}) := {}^t(\widetilde{H}_1(\boldsymbol{l}), \ldots, \widetilde{H}_r(\boldsymbol{l}))$. □

5 The pulse dynamics for reaction-diffusion systems on an H-shaped metric graph

5.1 Introduction

Exactly as in Chapter 4, we define $\boldsymbol{e}_0 := (1,0) \in \mathbb{R}^2$ and $\boldsymbol{e}_k, \tilde{\boldsymbol{e}}_k, k \in \mathbb{N}$ are the unit direction vectors of Ω_k satisfying $\boldsymbol{e}_0 \neq \pm\boldsymbol{e}_k$, $\boldsymbol{e}_0 \neq \pm\tilde{\boldsymbol{e}}_k$ and $\boldsymbol{e}_i \neq \pm\boldsymbol{e}_j (i \neq j)$, $\tilde{\boldsymbol{e}}_i \neq \pm\tilde{\boldsymbol{e}}_j (i \neq j)$. Let an metric graph Ω be a graph satisfying

$$\Omega := \Omega_0 \cup \bigcup_{j=1}^{m_L} \Omega_j^L \cup \bigcup_{i=1}^{m_R} \Omega_i^R$$

for $m_L, m_R \in \mathbb{N}$ with $m_R, m_L \geq 2$, where $\Omega_0 := \{x\boldsymbol{e}_0 \in \mathbb{R}^2 \,|\, 0 < x < K\}$, $\Omega_j^L := \{x\boldsymbol{e}_j \in \mathbb{R}^2 \, (j = 1, \ldots, m_L) \,|\, x > 0\}$ and $\Omega_i^R := \{x\tilde{\boldsymbol{e}}_i + K\boldsymbol{e}_0 \in \mathbb{R}^2 \, (i = 1, \ldots, m_R) \,|\, x > 0\}$. Moreover we denote the restriction of $\boldsymbol{u}$ to Ω_0 as $\boldsymbol{u}_0(x) := \boldsymbol{u}(x\boldsymbol{e}_0)$, $\boldsymbol{u}$ to Ω_j^L as $\boldsymbol{u}_j(x) := \boldsymbol{u}(x\boldsymbol{e}_j)$, and $\boldsymbol{u}$ to Ω_i^R as $\widetilde{\boldsymbol{u}}_i(x) := \boldsymbol{u}(x\tilde{\boldsymbol{e}}_i + K\boldsymbol{e}_0)$ for a function $\boldsymbol{u}$ on Ω. In this section, we consider the following reaction-diffusion systems on Ω

$$\begin{cases} \partial_t \boldsymbol{U}_0 = D\partial_{xx}\boldsymbol{U}_0 + \boldsymbol{F}(\boldsymbol{U}_0), t > 0, 0 < x < K, \\ \partial_t \boldsymbol{U}_j = D\partial_{xx}\boldsymbol{U}_j + \boldsymbol{F}(\boldsymbol{U}_j), t > 0,\ x > 0\, (j = 1, \ldots, m_L), \\ \partial_t \widetilde{\boldsymbol{U}}_i = D\partial_{xx}\widetilde{\boldsymbol{U}}_i + \boldsymbol{F}(\widetilde{\boldsymbol{U}}_i), t > 0,\ x > 0\, (i = 1, \ldots, m_R), \\ \partial_x \boldsymbol{U}_0(t, +0) + \displaystyle\sum_{j=1}^{m_L} \partial_x \boldsymbol{U}_j(t, +0) = \boldsymbol{0}, \boldsymbol{U}_0(t, +0) = \boldsymbol{U}_j(t, +0)\, (j = 1, \ldots, m_L), \\ \partial_x \boldsymbol{U}_0(t, K) = \displaystyle\sum_{i=1}^{m_R} \partial_x \widetilde{\boldsymbol{U}}_i(t, +0), \boldsymbol{U}_0(t, K) = \widetilde{\boldsymbol{U}}_i(t, +0)\, (i = 1, \ldots, m_R) \end{cases} \tag{5.1.1}$$

for $K \gg 1$, where $\boldsymbol{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function. Here $\boldsymbol{U}(t, \boldsymbol{x}) \in \mathbb{R}^N$ is a vector valued function of t and $\boldsymbol{x} \in \Omega$, and $\boldsymbol{U}_0(t, x) := \boldsymbol{U}(t, x\boldsymbol{e}_0)$ and $\boldsymbol{U}_j(t, x) := \boldsymbol{U}(t, x\boldsymbol{e}_j)$ and $\widetilde{\boldsymbol{U}}_i(t, x) := \boldsymbol{U}(t, x\tilde{\boldsymbol{e}}_i + K\boldsymbol{e}_0)$. We call boundary conditions of (5.1.1) at junction points $\boldsymbol{O} := (0, 0)$ and $\boldsymbol{K} := (K, 0)$ "Kirchhoff's boundary condition" in this thesis.

From now on, we will also express (5.1.1) as

$$\partial_t \boldsymbol{U} = D\Delta_\Omega \boldsymbol{U} + \boldsymbol{F}(\boldsymbol{U}) \tag{5.1.2}$$

on Ω with the Kirchhoff's boundary condition.

We call $\Omega \cup \{\boldsymbol{O}\} \cup \{\boldsymbol{K}\}$ "H-shaped metric graph"(Fig 6) in this thesis. An H-shaped metric graph connects each vertex of the two star-shaped metric graphs by a line segment and is an extended domain of the star-shaped metric graph. If $K = 0$, the domain is the same as that of the star-shaped metric graph. Such a graph connected the origin of multiple star-shaped metric graphs by line segments is more important from the application viewpoint than star-shaped metric graphs. In nature, channel geometries and nerve fibers are branching geometries formed by connecting the origin points of multiple star-shaped metric graphs with line segments. Such geometries are often seen in nature. However, analyzing reaction-diffusion systems in such a region is difficult because the shape is more complicated than star-shaped metric graphs. As a first step to considering the problem in such a domain, the issue of reaction-diffusion systems on H-shaped metric graphs, which are relatively easy to handle, has been studied in recent years. In particular, [21] obtains a pioneering result for a scalar reaction-diffusion equation on H-shaped metric graphs. [21] has been reported on the behavior of front-progressive and front-type stationary solutions. Moreover, this result tells us that the length K of the line segment connecting the origin points is closely related to the existence of front-type stationary solutions and front-type traveling wave solutions. On the other hand, as mentioned in chapter 4, there is no result for the pulse dynamics for reaction-diffusion systems on H-shaped metric graphs. However, If $K \gg 1$ holds, we can directly apply the theory of [10] to (5.1.2). As a result, I have obtained some results. I report some results in this chapter.

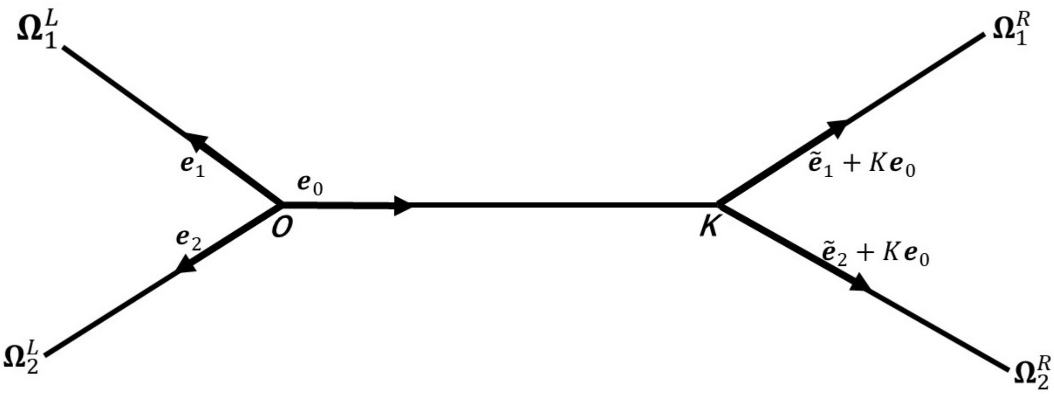

Figure 6: The H-shaped metric graph in the case of $m_L = 2, m_R = 2$.

5.2 Main results of the dynamics on an H-shaped metric graph

In this section, The assumptions and notation are the same as in chapter 4. Moreover we denote $L^2_K := \{L^2(0,K)\}^N$ for $K>0$. The inner product L^2_K by $\langle U, V\rangle_{L^2_K} := \int_0^K \langle \boldsymbol{U}(x), \boldsymbol{V}(x)\rangle dx$ for $\boldsymbol{U}, \boldsymbol{V} \in L^2_K$. H^2_K, L^∞_K are similarly defined by $H^2_K := \{H^2(0,K)\}^N$ and $L^\infty_K := \{L^\infty(0,K)\}^N$.

We use notion $h(l;K) := \{l, K-l\}$ and $\underline{h}(l;K) := \min h(l;K)$ and $\overline{h}(l;K) := \max h(l;K)$ in this section.

5.2.1 The motion of a single pulse solution on an H-shaped metric graph.

We define

$$\boldsymbol{S}(\boldsymbol{x}, l_0) := \begin{cases} \boldsymbol{S}(x-l_0), \boldsymbol{x} = x\boldsymbol{e}_0 \in \Omega_0, & \\ \quad \boldsymbol{0}, \qquad \boldsymbol{x} = x\boldsymbol{e}_j \in \Omega^L_j & (j=1,\dots,m_L), \\ \quad \boldsymbol{0}, \qquad \boldsymbol{x} = x\tilde{\boldsymbol{e}}_i + K\boldsymbol{e}_0 \in \Omega^R_i & (i=1,\dots,m_R) \end{cases}$$

for $m_L, m_R \in \mathbb{N}$ satisfying $m_L \geq 2$ and $m_R \geq 2$. Fix arbitrarily one positive integer $m_L \geq 2$ and $m_R \geq 2$. Then the following holds.

Theorem 5.1. Suppose there exist sufficiently large positive constants K_0, l^* such that $l^* < \underline{h}(l^*;K_0) \leq \overline{h}(l^*;K_0) < 2\underline{h}(l^*;K_0)$. Moreover suppose for any $K > K_0$, there exists a positive constant l^0 such that $l^* < \underline{h}(l^0;K) \leq \overline{h}(l^0;K) < 2\underline{h}(l^0;K)$ and the initial value $\boldsymbol{U}(0,\boldsymbol{x})$ is sufficiently close to $\boldsymbol{S}(\boldsymbol{x};l^0)$. Then there exists $0 < l_0(t) < K$ such that the solution $\boldsymbol{U}(t,\boldsymbol{x})$ of (5.1.2) satisfies

$$\begin{cases} ||\boldsymbol{U}_0(t,\cdot) - \boldsymbol{S}(x-l_0(t))||_{L^\infty_K} \leq O(e^{-\alpha\underline{h}(l_0(t);K)}), \\ ||\boldsymbol{U}_j(t,\cdot)||_{L^\infty_+} \leq O(e^{-\alpha\underline{h}(l_0(t);K)}) \ (j=1,\dots,m_L), \\ ||\widetilde{\boldsymbol{U}}_i(t,\cdot)||_{L^\infty_+} \leq O(e^{-\alpha\underline{h}(l_0(t);K)}) \ (i=1,\dots,m_R) \end{cases}$$

as long as $l^* < \underline{h}(l_0(t);K) \leq \overline{h}(l_0(t);K) < 2\underline{h}(l_0(t);K)$. Moreover $l_0(t)$ satisfies

$$\frac{dl_0}{dt} = -M_0\left(\frac{m_L-1}{m_L+1}e^{-2\alpha l_0} - \frac{m_R-1}{m_R+1}e^{-2\alpha(K-l_0)}\right) + (e^{-\alpha l_0} + e^{-\alpha(K-l_0)})O(e^{-\gamma'\underline{h}(l_0;K)})$$

as long as $l^* < \underline{h}(l_0(t);K) \leq \overline{h}(l_0(t);K) < 2\underline{h}(l_0(t);K)$ for a positive constant $\gamma' > \alpha$, where $M_0 := 2\alpha\langle D\boldsymbol{a}, \boldsymbol{a}^*\rangle$.

Remark 5.1. If $m_L + 1 = R$ and $m_R = 1$, the leading term in Theorem 5.1 is $\dot{l}_0 = -M_0 \dfrac{R-2}{R} e^{-2\alpha l_0}$. This is consistent with Theorem 4.1 ([10], Theorem 2.2) for $R \geq 3$ and $r = 1$.

We define $H_K(l_0) := -M_0 \left(\dfrac{m_L - 1}{m_L + 1} e^{-2\alpha l_0} - \dfrac{m_R - 1}{m_R + 1} e^{-2\alpha(K - l_0)} \right)$, let $\bar{l} > 0$ be an equilibrium satisfying $H_K(\bar{l}) = 0$ and $l^* < \min h(\bar{l}; K) \leq \max h(\bar{l}; K) < 2 \min h(\bar{l}; K)$. Then the following corollary 5.1 for Theorem 5.1 holds.

Corollary 5.1. There exists $K > K_0$ such that an equilibrium of $H_K(l)$ satisfying the above, say $\bar{l}$, $\bar{l} = \dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{m^*}$. Then if $M_0 < 0\,(> 0)$, there exists the stable(unstable) stationary solution $\boldsymbol{U}^*(\boldsymbol{x})$ of (5.1.2) such that

$$\begin{cases} ||\boldsymbol{U}_0^* - \boldsymbol{S}(\cdot - \bar{l})||_{L_K^\infty} \leq O(e^{-\alpha \underline{h}(\bar{l};K)}), \\ ||\boldsymbol{U}_j^*||_{L_+^\infty} \leq O(e^{-\alpha \underline{h}(\bar{l};K)}) \ \ (j = 1, \dots, m_L), \\ ||\widetilde{\boldsymbol{U}}_i^*||_{L_+^\infty} \leq O(e^{-\alpha \underline{h}(\bar{l};K)}) \ \ (i = 1, \dots, m_R), \end{cases}$$

where $m^* := \dfrac{(m_R + 1)(m_L - 1)}{(m_R - 1)(m_L + 1)}$.

When $m_L, m_R \in \mathbb{N}$ satisfy $m_L \geq 2$ and $m_R \geq 2$, the following holds for m^*.

Proposition 5.1.
(1) If $m_L = m_R$ holds, $m^* = 1$ holds.
(2) If $m_L > m_R$ holds, $m^* > 1$ holds.
(3) If $m_L < m_R$ holds, $0 < m^* < 1$ holds.

Proof. I only prove (2). Since $2m_L > 2m_R$ holds,

$$m_L m_R + m_L - m_R - 1 > m_L m_R + m_R - m_L - 1$$

holds. Thus, we see

$$(m_R + 1)(m_L - 1) > (m_R - 1)(m_L + 1). \tag{5.2.3}$$

(5.2.3) means $m^* > 1$ holds. □

5.2.2 The motion of a single front solution on an H-shaped metric graph

We define

$$\boldsymbol{S}(\boldsymbol{x},l_0):=\begin{cases}\boldsymbol{S}(x-l_0), & \boldsymbol{x}=x\boldsymbol{e}_0\in\Omega_0,\\ \quad \boldsymbol{S}_-, & \boldsymbol{x}=x\boldsymbol{e}_j\in\Omega_j^L \qquad\quad (j=1,\dots,m_L),\\ \quad \boldsymbol{S}_+, & \boldsymbol{x}=x\tilde{\boldsymbol{e}}_i+K\boldsymbol{e}_0\in\Omega_i^R\ \ (i=1,\dots,m_R)\end{cases}$$

for $m_L,m_R\in\mathbb{N}$ satisfying $m_L\geq 2$ and $m_R\geq 2$. Fix arbitrarily one positive integer $m_L\geq 2$ and $m_R\geq 2$. Then the following holds from [8] and [10].

Theorem 5.2. Suppose that there exist sufficiently large positive constants K_0,l^* such that $l^*<\underline{h}(l^*;K_0)\leq\overline{h}(l^*;K_0)<2\underline{h}(l^*;K_0)$. Moreover suppose for any $K>K_0$, there exists a positive constant l^0 such that if $l^*<\underline{h}(l^0;K)\leq\overline{h}(l^0;K)<2\underline{h}(l^0;K)$ and the initial value $\boldsymbol{U}(0,\boldsymbol{x})$ is sufficiently close to $\boldsymbol{S}(\boldsymbol{x};l^0)$. Then there exists $0<l_0(t)<K$ such that the solution $\boldsymbol{U}(t,\boldsymbol{x})$ of (5.1.2) satisfies

$$\begin{cases}||\boldsymbol{U}_0(t,\cdot)-\boldsymbol{S}(x-l_0(t))||_{L^\infty_K}\leq O(e^{-\gamma\underline{h}(l_0(t);K)}),\\ ||\boldsymbol{U}_j(t,\cdot)-\boldsymbol{S}_-||_{L^\infty_+}\leq O(e^{-\gamma\underline{h}(l_0(t);K)})\ (j=1,\dots,m_L),\\ ||\widetilde{\boldsymbol{U}}_i(t,\cdot)-\boldsymbol{S}_+||_{L^\infty_+}\leq O(e^{-\gamma\underline{h}(l_0(t);K)})\ (i=1,\dots,m_R)\end{cases}$$

as long as $l^*<\underline{h}(l_0(t);K)\leq\overline{h}(l_0(t);K)<2\underline{h}(l_0(t);K)$ for $\gamma:=\min\{\alpha_-,\alpha_+\}$. Moreover $l_0(t)$ satisfies

$$\frac{dl_0}{dt}=\left(-M_-\frac{m_L-1}{m_L+1}e^{-2\alpha_-l_0}+M_+\frac{m_R-1}{m_R+1}e^{-2\alpha_+(K-l_0)}\right)+(e^{-\alpha_-l_0}+e^{-\alpha_+(K-l_0)})O(e^{-\gamma'\underline{h}(l_0;K)})$$

as long as $l^*<\underline{h}(l_0(t);K)\leq\overline{h}(l_0(t);K)<2\underline{h}(l_0(t);K)$ for $\gamma'>\max\{\alpha_-,\alpha_+\}$, where $M_\pm:=\pm2\alpha_\pm\langle D\boldsymbol{a}_\pm,\boldsymbol{a}_\pm^*\rangle$.

Remark 5.2. If $m_L+1=R$ and $m_R=1$, the leading term in Theorem 5.1 is $\dot{l}_0=-M_-\dfrac{R-2}{R}e^{-2\alpha l_0}$. This is consistent with Theorem 4.2 ([10], Theorem 2.3) for $R\geq 3$ and $r=1$.

We define $H_K(l_0):=-M_-\dfrac{m_L-1}{m_L+1}e^{-2\alpha_-l_0}+M_+\dfrac{m_R-1}{m_R+1}e^{-2\alpha_+(K-l_0)}$. Let $\bar{l}>0$ be an equilibrium satisfying $H(\bar{l})=0$ and $l^*<\min h(\bar{l};K)\leq\max h(\bar{l};K)<2\min h(\bar{l};K)$. Moreover we assume $\alpha^+=\alpha^-,\boldsymbol{a}_+=-\boldsymbol{a}_-$ and $\boldsymbol{a}_+^*=\boldsymbol{a}_-^*$.

Corollary 5.2. There exists $K > K_0$ such that an equilibrium of $H_K(l)$ satisfying the above, say $\bar{l}$, $\bar{l} = \dfrac{K}{2} + \dfrac{1}{2\alpha}\log\sqrt{m^*}$. If $M_- < 0\,(> 0)$, there exists the stable(unstable) stationary solution $\boldsymbol{U}^*(\boldsymbol{x})$ of (5.1.2) such that

$$\begin{cases} ||\boldsymbol{U}_0^*(t,\cdot) - \boldsymbol{S}(x - l^*)||_{L^\infty_K} \leq O(e^{-\alpha\underline{h}(\bar{l};K)}), \\ ||\boldsymbol{U}_j^*(t,\cdot) - \boldsymbol{S}_-||_{L^\infty_+} \leq O(e^{-\alpha\underline{h}(\bar{l};K)})\,(j = 1,\ldots,m_L), \\ ||\widetilde{\boldsymbol{U}}_i^*(t,\cdot) - \boldsymbol{S}_+||_{L^\infty_+} \leq O(e^{-\alpha\underline{h}(\bar{l};K)})\,(i = 1,\ldots,m_R), \end{cases}$$

where $m^* := \dfrac{(m_R+1)(m_L-1)}{(m_R-1)(m_L+1)}$ and $\alpha := \alpha_\pm$.

Proposition 5.1 also holds for m^*.

5.3 Applications to reaction-diffusion systems on an H-shaped metric graph

5.3.1 The dynamics of a single front solution for the Allen-Cahn equation

We consider a single front solution of the Allen-Cahn equation on the following H-shaped metric graph

$$\partial_t u = \Delta_\Omega u + \frac{1}{2}u(1-u^2),\ t > 0,\ x \in \Omega. \tag{5.3.4}$$

We note the above Allen-Cahn equation on $\mathbb{R}$ has a stable standing front solution $S(x) = \tanh\left(\dfrac{x}{2}\right)$([13]). Since $S(x)$ is an odd function for x and $M_- < 0$, (5.3.4) has a stable front-type stationary solution stated in Collorary 5.2. Here $S_- = -1, S_+ = 1$, $\alpha_\pm = 1$, and $M_\pm = -12$.

(a) The case of $m_L = 2, m_R = 2$: The dynamics of a front solution is essentially governed by

$$\frac{dl_0}{dt} = 4(e^{-2l_0} - e^{-2(K-l_0)}).$$

Thus, we see that $\dot{l}_0 > 0$ for $0 < l_0(0) < \dfrac{K}{2}$ or $\dot{l}_0 < 0$ for $\dfrac{K}{2} < l_0(0) < K$(Fig 7).

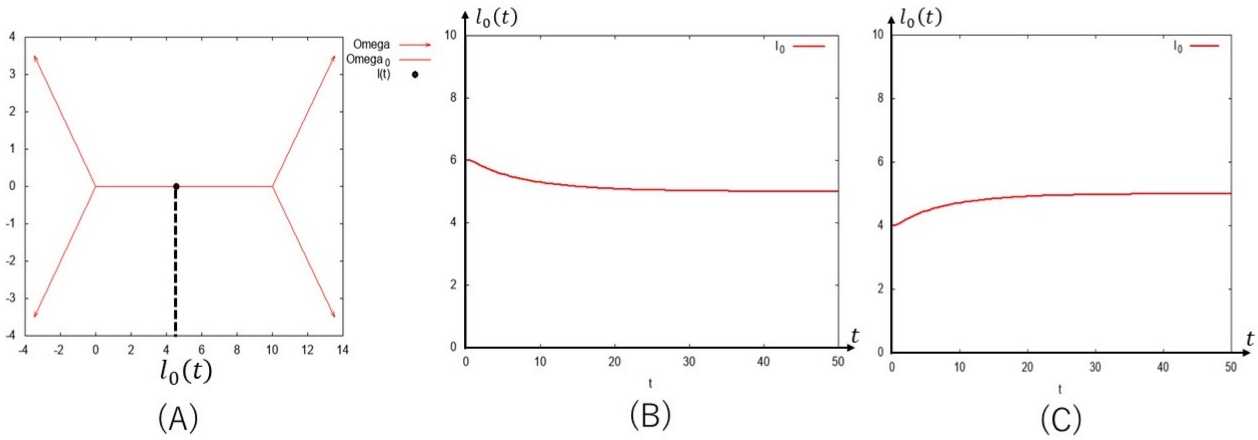

Figure 7(provided by Mr.Ken.Mitsuzono): Behavior of the front solution on the H-shaped graph when $K = 10$. (A) represents the zero point $l_0(t)$ of the front solution on Ω_0. (B) represents the time evolution of $l_0(t)$ when $l_0(0) > \dfrac{K}{2}$. (C) represents the time evolution of $l_0(t)$ when $l_0(0) < \dfrac{K}{2}$. From (B) and (C), we see that $l_0(\infty) = \dfrac{K}{2}$.

(b) The case of $m_L = 2, m_R = 3$: The dynamics of a front solution is essentially governed by

$$\frac{dl_0}{dt} = 2(2e^{-2l_0} - 3e^{-2(K-l_0)}).$$

Thus, we see that $\dot{l}_0 > 0$ for $0 < l_0(0) < \dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{2}{3}}$ or $\dot{l}_0 < 0$ for $\dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{2}{3}} < l_0(0) < K$. We note $\dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{2}{3}} < \dfrac{K}{2}$.

(c) The case of $m_L = 3, m_R = 2$: The dynamics of a front solution is essentially governed by

$$\frac{dl_0}{dt} = 2(3e^{-2l_0} - 2e^{-2(K-l_0)}).$$

Thus, we see that $\dot{l}_0 > 0$ for $0 < l_0(0) < \dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{3}{2}}$ or $\dot{l}_0 < 0$ for $\dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{3}{2}} < l_0(0) < K$. We note $\dfrac{K}{2} + \dfrac{1}{2}\log\sqrt{\dfrac{3}{2}} > \dfrac{K}{2}$.

5.3.2 The dynamics of a single pulse solution for the Gray-Scott-model

As taking the same parameter as [8], we consider a single pulse solution of the Gray-Scott-model on the following H-shaped metric graph

$$\begin{cases} \partial_t u = \Delta_\Omega u - uv^2 + \epsilon^2(1-u), t \in (0,\infty), \boldsymbol{x} \in \Omega, \\ \partial_t v = \epsilon^2 \Delta_\Omega v - \epsilon^{1/2} v + uv^2, \end{cases} \tag{5.3.5}$$

where $0 < \epsilon \ll 1$. As in chapter 3, the Gray-Scott model on $\mathbb{R}$ has a stable symmetric pulse-type stationary solution $\boldsymbol{S}(x)$([5]) and $M_0 > 0$ holds([8]). (5.3.5) has an unstable pulse-type stationary solution stated in Collorary 5.1.

(a) The case of $m_L = 2, m_R = 2$: The dynamics of a pulse solution is essentially governed by

$$\frac{dl_0}{dt} = -\frac{M_0}{3}(e^{-2\alpha l_0} - e^{-2\alpha(K-l_0)}).$$

Thus, the behavior is opposite to that of the Allen-Cahn equation.That is, $\dot{l}_0 < 0$ for $0 < l_0(0) < \dfrac{K}{2}$ or $\dot{l}_0 > 0$ for $\dfrac{K}{2} < l_0(0) < K$.

(b) The case of $m_L = 2, m_R = 3$: The dynamics of a front solution is essentially governed by

$$\frac{dl_0}{dt} = -M_0 \left(\frac{1}{3} e^{-2\alpha l_0} - \frac{1}{2} e^{-2\alpha(K-l_0)} \right).$$

Thus, we see that $\dot{l}_0 < 0$ for $0 < l_0(0) < \dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{2}{3}}$ or $\dot{l}_0 > 0$ for $\dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{2}{3}} < l_0(0) < K$. We note $\dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{2}{3}} < \dfrac{K}{2}$.

(c) The case of $m_L = 3, m_R = 2$: The dynamics of a front solution is essentially governed by

$$\frac{dl_0}{dt} = -M_0 \left(\frac{1}{2} e^{-2\alpha l_0} - \frac{1}{3} e^{-2\alpha(K-l_0)} \right).$$

Thus, we see that $\dot{l}_0 < 0$ for $0 < l_0(0) < \dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{3}{2}}$ or $\dot{l}_0 > 0$ for $\dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{3}{2}} < l_0(0) < K$. We note $\dfrac{K}{2} + \dfrac{1}{2\alpha} \log \sqrt{\dfrac{3}{2}} > \dfrac{K}{2}$.

5.4 Proof of Theorem 5.1

Proof. Hereafter we denote $l := l_0$. Fix one each of l and K satisfying $1 \ll \min h(l;K) \le \max h(l;K) < 2\min h(l;K)$. We define the function $\boldsymbol{G}(\boldsymbol{x};l;K)$ on Ω satisfying

$$\begin{cases} \boldsymbol{0} = D\partial_{xx}\boldsymbol{G}_p^L + \boldsymbol{F}'(\boldsymbol{0})\boldsymbol{G}_p^L \ \ (p = 0,\dots,m_L), \\ \partial_x\boldsymbol{S}(-l) + \partial_x\boldsymbol{G}_0^L(0) + \sum_{j=1}^{m_L}\partial_x\boldsymbol{G}_j^L(0) = \boldsymbol{0}, \\ \boldsymbol{S}(-l) + \boldsymbol{G}_0^L(0) = \boldsymbol{G}_j^L(0) \ (1 \le j \le m_L), \\ \boldsymbol{G}_0^L(K) = \boldsymbol{0}, \\ \boldsymbol{G}_j^L(+\infty) = \boldsymbol{0} \ (1 \le j \le m_L), \end{cases} \quad \begin{cases} \boldsymbol{0} = D\partial_{xx}\boldsymbol{G}_q^R + \boldsymbol{F}'(\boldsymbol{0})\boldsymbol{G}_q^R \ \ (q = 0,\dots,m_R), \\ \partial_x\boldsymbol{S}(K-l) + \partial_x\boldsymbol{G}_0^R(K) = \sum_{i=1}^{m_R}\partial_x\boldsymbol{G}_i^R(0), \\ \boldsymbol{S}(K-l) + \boldsymbol{G}_0^R(K) = \boldsymbol{G}_i^R(0) \ (1 \le i \le m_R), \\ \boldsymbol{G}_0^R(+0) = \boldsymbol{0}, \\ \boldsymbol{G}_i^R(+\infty) = \boldsymbol{0} \ (1 \le i \le m_R), \end{cases} \tag{5.4.6}$$

where $\boldsymbol{G}_0(x) = \boldsymbol{G}_0^L(x;l;K) + \boldsymbol{G}_0^R(x;l;K) := \boldsymbol{G}(x\boldsymbol{e}_0;l;K), \boldsymbol{G}_j^L(x) = \boldsymbol{G}_j^L(x;l;K) := \boldsymbol{G}(x\boldsymbol{e}_j;l;K)\,(j = 1,\dots,m_L), \boldsymbol{G}_i^R(x) = \boldsymbol{G}_i^R(x;l;K) := \boldsymbol{G}(x\boldsymbol{e}_i;l;K)\,(i = 1,\dots,m_R)$. Since $\boldsymbol{m}_k(x)\,(k = 1,\dots,2N)$ are the fundamental functions of the ODE

$$\boldsymbol{0} = D\partial_{xx}\boldsymbol{m}_k(x) + \boldsymbol{F}'(\boldsymbol{0})\boldsymbol{m}_k(x),$$

$\boldsymbol{G}_0^L(x;l;K), \boldsymbol{G}_j^L(x;l;K), \boldsymbol{G}_0^R(x;l;K), \boldsymbol{G}_i^R(x;l;K)$ is expresses that

$$\begin{aligned} \boldsymbol{G}_0^L(x;l;K) &= c_{01}^L(l;K)(e^{-\alpha x} - e^{\alpha(x-2K)})\boldsymbol{a} + \boldsymbol{b}_0^L(x), && 0 < x < K, \\ \boldsymbol{G}_j^L(x;l;K) &= c_{j1}^L(l;K)e^{-\alpha x}\boldsymbol{a} + \boldsymbol{b}_j^L(x), && x > 0, \\ \boldsymbol{G}_0^R(x;l;K) &= c_{01}^R(l;K)(e^{-\alpha x} - e^{\alpha x})\boldsymbol{a} + \boldsymbol{b}_0^R(x), && 0 < x < K, \\ \boldsymbol{G}_i^R(x;l;K) &= c_{i1}^R(l;K)e^{-\alpha x}\boldsymbol{a} + \boldsymbol{b}_i^R(x), && x > 0 \end{aligned}$$

by $\boldsymbol{G}_0^L(K) = \boldsymbol{0}, \boldsymbol{G}_j^L(+\infty) = \boldsymbol{0}\,(j = 1,\dots,m_L), \boldsymbol{G}_0^R(0) = \boldsymbol{0}, \boldsymbol{G}_i^R(+\infty) = \boldsymbol{0}\,(i = 1,\dots,m_R)$ for $\gamma > \alpha$. Here $\boldsymbol{b}_0^L(x) := \sum_{\substack{k=2 \\ k\neq N+1}}^{2N} c_{0k}^L\boldsymbol{m}_k(x)$ with $\sum_{\substack{k=2 \\ k\neq N+1}}^{2N} c_{0k}^L\boldsymbol{m}_k(K) = \boldsymbol{0}$. $\boldsymbol{b}_0^R(x) := \sum_{\substack{k=2 \\ k\neq N+1}}^{2N} c_{0k}^R\boldsymbol{m}_k(x)$ with $\sum_{\substack{k=2 \\ k\neq N+1}}^{2N} c_{0k}^R\boldsymbol{m}_k(0) = \boldsymbol{0}$. $\boldsymbol{b}_j^L(x) := \sum_{k=2}^{N} c_{jk}^L\boldsymbol{m}_k(x)$, and $\boldsymbol{b}_i^R(x) := \sum_{k=2}^{N} c_{ik}^R\boldsymbol{m}_k(x)$. Then we have

Lemma 5.1. The coefficients $\{c_{pk}^L(l)\}$ of $\boldsymbol{G}_p^L(x)\,(p = 0,1,\dots,m_L)$, and $\{c_{qk}^R(l)\}$ of $\boldsymbol{G}_q^L(x)\,(q =$

$0, 1, \ldots, m_R)$ are given by

$$
\begin{cases}
c_{01}^L(l;K) = C_0^L(l;K) := -\dfrac{(m_L-1)e^{-\alpha l}}{(m_L+1)-(m_L-1)e^{-2\alpha K}}, \\
c_{j1}^L(l;K) = C_j^L(l;K) := \dfrac{2e^{-\alpha l}}{(m_L+1)-(m_L-1)e^{-2\alpha K}} & (1 \le j \le m_L), \\
c_{01}^R(l;K) = C_0^R(l;K) := \dfrac{(m_R-1)e^{-\alpha(2K-l)}}{(m_R+1)-(m_R-1)e^{-2\alpha K}}, \\
c_{i1}^R(l;K) = C_i^L(l;K) := \dfrac{2e^{-\alpha(K-l)}}{(m_R+1)-(m_R-1)e^{-2\alpha K}} & (1 \le i \le m_R), \\
c_{0k}^L(l;K) = O(e^{-\gamma_1 l}) & (2 \le k \le N), \\
c_{0k}^L(l;K) = O(e^{-\gamma_1(2K+l)}) & (N+2 \le k \le 2N), \\
c_{jk}^L(l;K) = O(e^{-\gamma_1 l}) & (1 \le j \le m_L, 2 \le k \le N), \\
c_{0k}^R(l;K) = O(e^{-\gamma_2(2K-l)}) & (k \ne 1, k \ne N+1) \\
c_{ik}^R(l;K) = O(e^{-\gamma_2(K-l)}) & (1 \le i \le m_R, 2 \le k \le N),
\end{cases}
\tag{5.4.7}
$$

as $\min h(l;K) \to +\infty$ for $\gamma_1 > \alpha, \gamma_2 > \alpha$.

Proof. I prove Lemma 5.1 by the quite similar way to Lemma 4.1 ([10], Lemma 4.1). Since $S(x-l)$ has the asymptotic profile as $\min h(l;K) \to +\infty$, $S(x-l) \to e^{\alpha(x-l)}\boldsymbol{a}$ together with $\partial_x S(x-l) \to \alpha e^{\alpha(x-l)}\boldsymbol{a}$ in a neighborhood of $x=0$ and $S(x-l) \to e^{-\alpha(x-l)}\boldsymbol{a}$ together with $\partial_x S(x-l) \to -\alpha e^{-\alpha(x-l)}\boldsymbol{a}$ in a neighborhood of $x = K$. Thus $S(-l) \to e^{-\alpha l}\boldsymbol{a} + O(e^{-\gamma_l})$, $\partial_x S(-l) \to \alpha e^{-\alpha l}\boldsymbol{a} + O(e^{-\gamma_l})$, $S(K-l) \to e^{-\alpha(K-l)}\boldsymbol{a} + O(e^{-\gamma(K-l)})$, $\partial_x S(K-l) \to -\alpha e^{-\alpha(K-l)}\boldsymbol{a} + O(e^{-\gamma(K-l)})$ for some $\gamma > \alpha$. Then substituting these profiles and $\boldsymbol{G}_0^L(x;l;K), \partial_x \boldsymbol{G}_0^L(x;l;K), \boldsymbol{G}_j^L(x;l;K), \partial_x \boldsymbol{G}_j^L(x;l;K), \boldsymbol{G}_0^R(x;l;K), \partial_x \boldsymbol{G}_0^R(x;l;K)$, $\boldsymbol{G}_i^R(x;l;K), \partial_x \boldsymbol{G}_i^R(x;l)$ into the Kirchhoff boundary condition of (5.4.6), we have

$$
\begin{cases}
\{\alpha e^{-\alpha l} - \alpha c_{01}^L(l;K)(1+e^{-2\alpha K})\}\boldsymbol{a} + O(e^{-\gamma l}) + \partial_x \boldsymbol{b}_0^L(0) - \left(\displaystyle\sum_{j=1}^{m_L} \{\alpha c_{j1}^L(l;K)\boldsymbol{a} - \partial_x \boldsymbol{b}_j^L(0)\}\right) = \boldsymbol{0}, \\
\{e^{-\alpha l} + c_{01}^L(l;K)(1-e^{-2\alpha K})\}\boldsymbol{a} + O(e^{-\gamma l}) + \boldsymbol{b}_0^L(0) = c_{j1}^L(l;K)\boldsymbol{a} + \boldsymbol{b}_j^L(0), \\
-\alpha\{e^{-\alpha(K-l)} + c_{01}^R(l;K)(e^{-\alpha K} + e^{\alpha K})\}\boldsymbol{a} + O(e^{-\gamma(K-l)}) + \partial_x \boldsymbol{b}_0^R(K) \\
= -\left(\displaystyle\sum_{i=1}^{m_R} \{\alpha c_{i1}^R(l;K)\boldsymbol{a} - \partial_x \boldsymbol{b}_i^R(0)\}\right), \\
\{e^{-\alpha(K-l)} + c_{01}^R(l;K)(e^{-\alpha K} - e^{\alpha K})\}\boldsymbol{a} + O(e^{-\gamma(K-l)}) + \boldsymbol{b}_0^R(K) = c_{i1}^R(l;K)\boldsymbol{a} + \boldsymbol{b}_i^R(0).
\end{cases}
\tag{5.4.8}
$$

Since $C_p^L(l), C_q^R(l)$ defined (5.4.7) satisfy

$$\begin{cases} e^{-\alpha l} - C_0^L(l;K)(1+e^{-2\alpha K}) = \sum_{j=1}^{m_L} C_j^L(l;K), \\ e^{-\alpha l} + C_0^L(l;K)(1-e^{-2\alpha K}) = C_j^L(l;K), \\ e^{-\alpha(K-l)} + C_0^R(l;K)(e^{-\alpha K}+e^{\alpha K}) = \sum_{i=1}^{m_R} C_i^R(l;K), \\ e^{-\alpha(K-l)} + C_0^R(l;K)(e^{-\alpha K}-e^{\alpha K}) = C_i^R(l;K), \end{cases}$$

we have

$$\begin{cases} O(e^{-\gamma l}) + \partial_x \boldsymbol{b}_0^L(0) + \sum_{j=1}^{m_L} \partial_x \boldsymbol{b}_j^L(0) = \boldsymbol{0}, \\ O(e^{-\gamma l}) + \boldsymbol{b}_0^L(0) = \boldsymbol{b}_j^L(0), \\ O(e^{-\gamma(K-l)}) + \partial_x \boldsymbol{b}_0^R(K) = \sum_{i=1}^{m_R} \partial_x \boldsymbol{b}_i^R(0), \\ O(e^{-\gamma(K-l)}) + \boldsymbol{b}_0^R(K) = \boldsymbol{b}_i^R(0). \end{cases} \tag{5.4.9}$$

(5.4.9) means that $\boldsymbol{b}_0^L(0)$ and $\partial_x \boldsymbol{b}_0^L(0)$ can be taken in $O(e^{-\gamma l})$, $\boldsymbol{b}_j^L(0)$ and $\partial_x \boldsymbol{b}_j^L(0)$ can be taken in $O(e^{-\gamma l})\,(2 \le k \le N)$ for $\gamma > \alpha$. That is, we can also take $c_{pk}^L(l) = O(e^{-\gamma_1 l})\,(0 \le p \le m_L, 2 \le k \le N)$ and $c_{0k}^L(l) = O(e^{-\gamma_1(2K+l)})\,(N+2 \le k \le 2N)$ for $\gamma_1 > \alpha$ by $\boldsymbol{b}_j^L(K) = \boldsymbol{0}$. Furthermore, (5.4.9) also means that $\boldsymbol{b}_0^R(K)$ and $\partial_x \boldsymbol{b}_0^R(K)$ can be taken in $O(e^{-\gamma'(K-l)})$, $\boldsymbol{b}_i^R(0)$ and $\partial_x \boldsymbol{b}_i^R(0)$ can be taken in $O(e^{-\gamma'(K-l)})\,(2 \le k \le N)$ for $\gamma' > \alpha$. That is, we can also take $c_{ik}^R(l) = O(e^{-\gamma_2(K-l)})\,(1 \le i \le m_R, 2 \le k \le N)$ and $c_{0k}^R(l) = O(e^{-\gamma_2(2K-l)})\,(k \ne 1, k \ne N+1)$ for $\gamma_2 > \alpha$ by $\boldsymbol{b}_0^R(0) = \boldsymbol{0}$. $\square$

We define $\boldsymbol{G}_0(x;l;K) := \boldsymbol{G}_0^L(x;l;K) + \boldsymbol{G}_0^R(x;l;K)$. We express the solution $\boldsymbol{U}_0(t,x), \boldsymbol{U}_j(t,x)$ $(j = 1,\dots,m_L), \boldsymbol{U}_i(t,x)\,(i = 1,\dots,m_R)$ of (5.1.1) by $\boldsymbol{U}_0(t,x) = \boldsymbol{S}(x-l) + \boldsymbol{G}_0(x;l;K) + \boldsymbol{V}_0(t,x), \boldsymbol{U}_j(t,x) = \boldsymbol{G}_j^L(x;l;K) + \boldsymbol{V}_j(t,x), \widetilde{\boldsymbol{U}}_i(t,x) = \boldsymbol{G}_i^R(x;l;K) + \widetilde{\boldsymbol{V}}_i(t,x)$. Then the equation (5.1.1) become the equation of $\boldsymbol{V}_0, \boldsymbol{V}_j, \widetilde{\boldsymbol{V}}_i$

$$\begin{cases} \partial_t\{\boldsymbol{S}(\cdot - l) + \boldsymbol{G}_0 + V_0\} = L(l)\boldsymbol{V}_0 + L(l)\boldsymbol{G}_0 + \boldsymbol{Z}_0(l,\boldsymbol{V}_0), \\ \partial_t\{\boldsymbol{G}_j^L + \boldsymbol{V}_j\} = L_0\boldsymbol{V}_j + L_0\boldsymbol{G}_j^L + \boldsymbol{Z}_j(l,\boldsymbol{V}_j), \\ \partial_t\{\boldsymbol{G}_i^R + \widetilde{\boldsymbol{V}}_i\} = L_0\tilde{\boldsymbol{V}}_i + L_0\boldsymbol{G}_i^R + \widetilde{\boldsymbol{Z}}_i(l,\tilde{\boldsymbol{V}}_i), \\ \partial_x\boldsymbol{V}_0(t,+0) + \sum_{j=1}^{m_L} \partial_x\boldsymbol{V}_j(t,+0) = \boldsymbol{0}, \boldsymbol{V}_0(t,+0) = \boldsymbol{V}_j(t,+0)\,(j=1,\dots,m_L), \\ \partial_x\boldsymbol{V}_0(t,K) = \sum_{i=1}^{m_R} \partial_x\widetilde{\boldsymbol{V}}_i(t,+0), \boldsymbol{V}_0(t,K) = \widetilde{\boldsymbol{V}}_i(t,+0)\,(i=1,\dots,m_R), \end{cases}$$

where $L(l) := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{S}(x-l)), L_0 := D\partial_{xx} + \boldsymbol{F}'(\boldsymbol{0})$. $\boldsymbol{Z}_0(l, \boldsymbol{V}_0), \boldsymbol{Z}_j(l, \boldsymbol{V}_j)\,(j = 1, \ldots, m_L), \widetilde{\boldsymbol{Z}}_i(l, \widetilde{\boldsymbol{V}}_i)\,(i = 1, \ldots, m_R)$ are functions satisfying $|\boldsymbol{Z}_0(l, \boldsymbol{V}_0)| \leq O(|\boldsymbol{G}_0(x, l; K)|^2 + |\boldsymbol{V}_0(t, x)|^2), |\boldsymbol{Z}_j(l, \boldsymbol{V}_j)| \leq O(|\boldsymbol{G}_j^L(x; l; K)|^2 + |\boldsymbol{V}_j(t, x)|^2), |\widetilde{\boldsymbol{Z}}_i(l, V_i)| \leq O(|\boldsymbol{G}_i^R(x; l; K)|^2 + |\widetilde{\boldsymbol{V}}_i(t, x)|^2)$. The rest can be proved in quite a similar way as Theorem 4.1([10], Theorem 2.2). □

5.5 Proof of Corollary 5.1

Proof. I prove Corollary 5.1 by quite a similar way to "Proof of corollary 1" in [9]. We define $\boldsymbol{h}_K(l) := \boldsymbol{h}(l; K)$. Hereafter we denote $l := l_0$.

Assume $M_0 < 0$. Moreover, we define

$$H_0(l; K) := -M_0 \left(\frac{m_L - 1}{m_L + 1} e^{-2\alpha l} - \frac{m_R - 1}{m_R + 1} e^{-2\alpha(K-l)} \right).$$

Then, we see

$$\frac{dH_0}{dl}(l; K) = 2\alpha M_0 \left(\frac{m_L - 1}{m_L + 1} e^{-2\alpha l} + \frac{m_R - 1}{m_R + 1} e^{-2\alpha(K-l)} \right) < 0. \tag{5.5.10}$$

for m_L, $m_R \in \boldsymbol{N}$ with $m_L, m_R \geq 2$.

By quite a similar way to "Proof of corollary 1" in [9], we have

$$\frac{dl}{dt} = H_0(l; K) + H_1(l; K),$$

where $H_1(l; K)$ is a function satisfying

$$H_1(l; K) = O(e^{-(\alpha+\gamma) \min h_K(l)}), \tag{5.5.11}$$

$$|H_1(l; K) - H_1(l'; K)| \leq O\{[e^{-(\alpha+\gamma') \min h_K(l)} + e^{-(\alpha+\gamma') \min h_K(l')}]|l - l'|\}, \tag{5.5.12}$$

for $\gamma > \alpha, \gamma' > \alpha$. Let $\bar{l}$ be the sufficiently large positive constant satisfying $H_0(\bar{l}; K) = 0$ and $\frac{dH_0}{dl}(\bar{l}; K) := -\beta < 0$ with $l^* < \min h(\bar{l}; K) \leq \max h(\bar{l}; K) < 2 \min h(\bar{l}; K)$ for a positive constant β. Then,

$$\beta = 2\alpha |M_0| \left(\frac{m_L - 1}{m_L + 1} e^{-2\alpha \bar{l}} + \frac{m_R - 1}{m_R + 1} e^{-2\alpha(K-\bar{l})} \right) > 0 \tag{5.5.13}$$

by (5.5.10).

We define $H_K(l) := H_0(l; K) + H_1(l; K)$. In exactly the same way as in "Proof of corollary 1" in [9], we suffices to show the existence of an equilibrium $\tilde{l}$ of $H_K(l)$ satisfying $\tilde{l} := \bar{l}(1 + O(e^{-\gamma_1 \min h_K(\bar{l})}))$ for $\gamma_1 > 0$ and $H_K(l)$ is monotone decreasing in the neighborhood of $\tilde{l}$.

By substituting $\tilde{l} = \bar{l} + p$ into $H_K(\bar{l} + p) = 0$ for $|p| \ll 1$, we have $0 = -\beta p + O(p^2) + H_1(\bar{l} + p; K)$ and

$$p = \frac{1}{\beta} H_1(\bar{l} + p; K) + O(p^2). \tag{5.5.14}$$

Since $H_1(\bar{l} + p; K) = O(e^{-(\alpha+\gamma)\min h_K(\bar{l})})$ and $\beta = O(e^{-2\alpha \min h_K(\bar{l})})$ hold by (5.5.11) and (5.5.13), $\frac{1}{\beta} H_1(\bar{l} + p; K) = O(e^{-\gamma_2 \min h_K(\bar{l})})$ holds for $\gamma_2 > 0$, which means to become sufficiently small by taking $\min h_K(\bar{l}) > l^*$ sufficiently large. Then it is easy to see the right-hand side of (5.5.14) is the contraction in the set $W := \{|p| \leq \rho e^{-\gamma_2 \min h_K(\bar{l})}\}$ for an appropriate $\rho > 0$. Thus the fixed value, say $p^* = O(e^{-\gamma_2 \min h_K(\bar{l})})$, gives the equilibrium $\tilde{l} = \bar{l} + p^*$ of $H_K(l)$.

In the neighborhood of $\tilde{l}$, $H_K(l)$ is represented by

$$H_K(\tilde{l} + p) = -\beta p + O(p^2 + (p^*)^2) + H_1(\bar{l} + p^* + p; K) - \beta p^*.$$

Thus we see

$$H_K(\tilde{l} + p) - H_K(\tilde{l} + p') = \{-\beta + O(|p| + |p'| + e^{-(\alpha+\gamma')\min h_K(l)})\}(p - p')$$

by (5.5.12). $\gamma' > \alpha$ means $H_K(\tilde{l}+p)$ is monotone decreasing for sufficiently small p and p'. It is easy to see that $\bar{l} = \frac{K}{2} + \frac{\log\sqrt{m^*}}{2\alpha}$ satisfies the above $\bar{l}$ for $K > \max\{K_0, \frac{3|\log\sqrt{m^*}|}{\alpha}, \frac{|\log\sqrt{m^*}|}{\alpha} + 2l^*\}$.

Other cases are shown in precisely the same way. $\square$

5.6 Concluding remarks

This chapter dealt with the following special situation:
1) Just one pulse (or front) lies only Ω_0.
2) Diffusion matrices on Ω_0, Ω_j^L $(j = 1, \ldots, m_L)$, Ω_i^R $(i = 1, \ldots, m_R)$ are all same.

As for 1), the results in this chapter can be extended to the pulse dynamics under the situation where at most one pulse is placed on each Ω_0, Ω_j^L $(j = 1, \ldots, m_L), \Omega_i^R$ $(i = 1, \ldots, m_R)$ by [10].

As For 2), this can be extended to the following problem with different diffusion matrices

D on each Ω

$$\begin{cases} \partial_t \boldsymbol{U}_0 = D_0 \partial_{xx} \boldsymbol{U}_0 + \boldsymbol{F}(\boldsymbol{U}_0), t > 0, x > 0, \, \boldsymbol{U}_0 \in \mathbb{R}^N, \\ \partial_t \boldsymbol{U}_j = D_j \partial_{xx} \boldsymbol{U}_j + \boldsymbol{F}(\boldsymbol{U}_j), t > 0, x > 0, \, \boldsymbol{U}_j \in \mathbb{R}^N \, (j = 1, \dots, m_L), \\ \partial_t \widetilde{\boldsymbol{U}}_i = \widetilde{D}_i \partial_{xx} \widetilde{\boldsymbol{U}}_i + \boldsymbol{F}(\widetilde{\boldsymbol{U}}_i), t > 0, x > 0, \, \widetilde{\boldsymbol{U}}_i \in \mathbb{R}^N \, (i = 1, \dots, m_R), \\ D_0 \partial_x \boldsymbol{U}_0(t, +0) + \displaystyle\sum_{j=1}^{m_L} D_j \partial_{x_j} \boldsymbol{U}_j(t, +0) = \boldsymbol{0}, \\ \boldsymbol{U}_0(t, +0) = \boldsymbol{U}_j(t, +0) \, (j = 1, \dots, m_L), \\ D_0 \partial_x \boldsymbol{U}_0(t, K) = \displaystyle\sum_{i=1}^{m_R} \widetilde{D}_i \partial_x \widetilde{\boldsymbol{U}}_i(t, +0), \\ \boldsymbol{U}_0(t, K) = \widetilde{\boldsymbol{U}}_i(t, +0) \, (i = 1, \dots, m_R). \end{cases}$$

We plan to consider it based on [10] in the future.

6 The front dynamics for reaction-duffusion equations on a loop-edge-metric graph

6.1 Introduction

Various problems for reaction-diffusion equations on $\mathbb{R}^N$ has been studied for a long time and is still being studied intensively. However, they are more difficult to analyze than the problems on $\mathbb{R}$. In such a situation, Methods of attributing

$$\begin{cases} \partial_t u = \Delta u + F(u), t > 0, \boldsymbol{x} \in \Omega^\epsilon, \\ \dfrac{\partial}{\partial \boldsymbol{\nu}} u = 0, \qquad\qquad\qquad \boldsymbol{x} \in \partial\Omega^\epsilon \end{cases}$$

for $0 < \epsilon \ll 1$, where $\Omega^\epsilon := \{(x, y) \in \mathbb{R}^2; 0 < x < K, 0 < y < \epsilon w(x)\}$ to

$$\begin{cases} \partial_t u = \dfrac{1}{w(x)} \partial_x \{w(x) \partial_x u\} + F(u), t > 0, 0 < x < K, \\ \partial_x u(t, +0) = \partial_x u(t, K) = 0 \end{cases}$$

as $\epsilon \to 0$ is still one of the most powerful methods (see e.g. [40]). Here $\Delta := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $\boldsymbol{\nu}$ is the outward unit normal vector for $\partial\Omega^\epsilon$. Moreover, both $F : \mathbb{R} \to \mathbb{R}$ and $w : \mathbb{R} \to \mathbb{R}$ are a smooth function.

Remark 6.1. When $w(x)$ is constant, by the above limit equation(e.g. [41]), we see

$$\begin{cases} \partial_t u = \partial_{xx} u + F(u), t > 0, 0 < x < K, \\ \partial_x u(t, +0) = \partial_x u(t, K) = 0. \end{cases}$$

Later, [40] was extended by Prof. Yanagida, and [41] reported the following results on the stability of stationary solutions of the reaction-diffusion equation on graphs containing metric graphs with circles. Moreover, [41] are famous results on the stability of stationary solutions for reaction-diffusion equations on graphs containing metric graphs with circles.

On the other hand, because analysis for pulse dynamics and front dynamics on metric graphs with circles is complicated, the result is almost nonexistent. Therefore, as a first step, in this chapter, I report a result obtained for front dynamics in reaction-diffusion equations on the following metric graphs Ω(Fig 8) resembling Type 3 in [41].

$\Omega := \Omega_1 \cup \Omega_2 \cup \{\boldsymbol{O}\}$, where $\Omega_1 := \{(x,0) \in \mathbb{R}^2\,; x > 0\}$ and

$$\Omega_2 := \left\{ r_2 \left(\cos\frac{x}{r_2} - 1, \sin\frac{x}{r_2} \right) \in \mathbb{R}^2\,;\, 0 < x < K \right\},$$

where $K := 2\pi r_2$ with $r_2 > 0$ is the radius of the circle. Furthermore, We denote the restriction of v to Ω_1 as $v_1(x) := v((x,0))$, and v to Ω_2 as $v_2(x) := v\left(r_2\left(\cos\frac{x}{r_2} - 1, \sin\frac{x}{r_2}\right)\right)$ for a function v on Ω. Then we consider

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + F(u_1), t > 0, x > 0, \\ \partial_t u_2 = \partial_{xx} u_2 + F(u_2), t > 0, 0 < x < K, \\ w_1 \dfrac{\partial u_1}{\partial x}(t,+0) + w_2 \dfrac{\partial u_2}{\partial x}(t,+0) = w_2 \dfrac{\partial u_2}{\partial x}(t,K), \\ u_1(t,+0) = u_2(t,+0) = u_2(t,K) \end{cases} \tag{6.1.1}$$

for $K > 0, w_j > 0\,(j = 1,2)$, where $F : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function. Moreover, $u(t,\boldsymbol{x}) \in \mathbb{R}$ is a scalar valued function of t and $\boldsymbol{x} \in \Omega$, and $u_1(t,x) := u(t,(x,0))$ and $u_2(t,x) := u\left(t, r_2\left(\cos\frac{x}{r_2} - 1, \sin\frac{x}{r_2}\right)\right)$. Note that boundary conditions in (6.1.1) are the same as part of those in [41], and as $\epsilon \to 0$, w_j gives a width of a thin tubular domain by ϵw_j (see e.g. [40], [41]). We call Ω a loop-edge-metric graph in this chapter. I report front dynamics for (6.1.1) in this chapter.

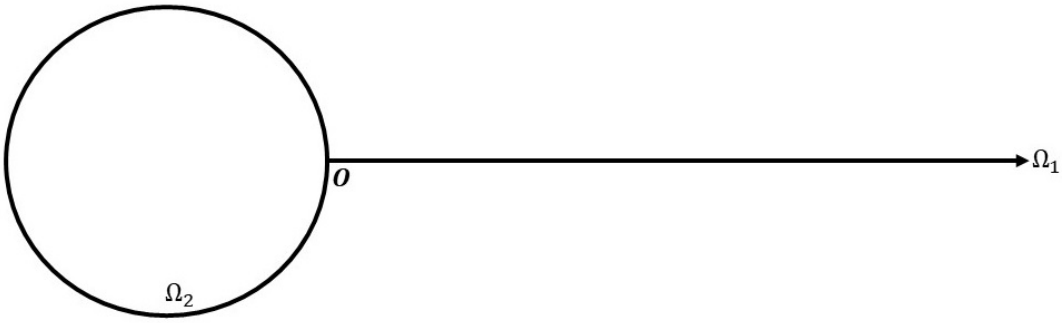

Figure 8: A loop-edge-metric graph.

6.2 The result of the front dynamics on a loop-edge-metric graph

This section assumes (H1)'-(H4)', (A2) and the assumptions and notation are precisely the same as in chapter 4 and chapter 5. Furthermore, the following proposition holds immediately from [8].

Proposition 6.1([8]). If $L = L^*$ holds, $M_- < 0$ holds, where $M_- := -2\alpha_- \langle D\boldsymbol{a}_-, \boldsymbol{a}_-^* \rangle$.

We consider (6.1.1) based on the above. In this section, we discuss the front dynamics results for the scalar reaction-diffusion equation in (6.1.1).

Theorem 6.1. Fix arbitrarily one $w_1 > 0, w_2 > 0, K > 0$. Suppose there exists $l^* \gg 1$ such that the initial date $u_1(0, x)$ sufficiently close to $S(x - l_0)$ and $u_2(0, x)$ to S_- for $l_0 > l^*$. Then the solution $u_j(t, x)$ $(j = 1, 2)$ of (6.1.1) satisfies

$$||u_1(t, \cdot) - S(\cdot - l(t))||_{L^\infty_+} \leq O(e^{-\alpha_- l}),$$
$$||u_2(t, \cdot) - S_-||_{L^\infty_K} \leq O(e^{-\alpha_- l})$$

as long as $l(t) > l^*$. Moreover $l(t)$ satisfies

$$\frac{dl}{dt} = M_- \frac{e^{-2\alpha_- l}[(w_1 - 2w_2) + (w_1 + 2w_2)e^{-\alpha_- K}]}{(w_1 + 2w_2) + (w_1 - 2w_2)e^{-\alpha_- K}} + e^{-\alpha_- l} O(e^{-\gamma l})$$

as long as $l(t) > l^*$ for $\gamma > \alpha_-$.

We define $H^-(l) := M_- \dfrac{e^{-2\alpha_- l}[(w_1 - 2w_2) + (w_1 + 2w_2)e^{-\alpha_- K}]}{(w_1 + 2w_2) + (w_1 - 2w_2)e^{-\alpha_- K}}$ and

$J := \left\{(w_1, w_2, K) \in \mathbb{R}^3\,;\,(w_1 - 2w_2) + (w_1 + 2w_2)e^{-\alpha_- K} \neq 0, 0 < w_1 < 2w_2, K > 0\right\}.$

Then by Proposition 6.1([8]), we can immediately see that the following.

Proposition 6.2. If $(w_1, w_2, K) \in J$, The following (1)-(3) hold.
(1) If $w_1 \geq 2w_2$ holds, $H^-(l) < 0$ holds for $K > 0$.
(2) If $w_1 < 2w_2$ and $0 < K < \dfrac{1}{\alpha_-} \log\left(\dfrac{2w_2 + w_1}{2w_2 - w_1}\right)$ hold, $H^-(l) < 0$ holds.
(3) If $w_1 < 2w_2$ and $K > \dfrac{1}{\alpha_-} \log\left(\dfrac{2w_2 + w_1}{2w_2 - w_1}\right)$ hold, $H^-(l) > 0$ holds.

Remark 6.2. Proposition 6.2 means if (1) or (2) hold, $\dfrac{dl}{dt} < 0$ holds, while if (3) holds, $\dfrac{dl}{dt} > 0$ holds.

6.3 Applications to the Allen-Cahn equation

We consider

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + \dfrac{1}{2} u(1 - u^2), t > 0, x > 0, \\ \partial_t u_2 = \partial_{xx} u_2 + \dfrac{1}{2} u(1 - u^2), t > 0, 0 < x < K, \\ w_1 \dfrac{\partial u_1}{\partial x}(t, +0) + w_2 \dfrac{\partial u_2}{\partial x}(t, +0) = w_2 \dfrac{\partial u_2}{\partial x}(t, K), \\ u_1(t, +0) = u_2(t, +0) = u_2(t, K) \end{cases}$$

for $K > 0$. Exactly as in chapters 5, we treat $S(x) = \tanh\left(\dfrac{x}{2}\right)$, where $S_\pm = \pm 1, \alpha_\pm = 1, M_\pm = -12$([13]). Then the dynamics is essentially given by

$$\frac{dl}{dt} = H^-(l) = \frac{-12e^{-2l}[(w_1 - 2w_2) + (w_1 + 2w_2)e^{-K}]}{(w_1 + 2w_2) + (w_1 - 2w_2)e^{-K}}.$$

(I) The case of $w_1 \geq 2w_2$.
(I-a) When $w_1 : w_2 = 2 : 1$, we see that $l(t)$ comes close to O by (1) of Proposition 6.2.
(I-b) When $w_1 : w_2 = 3 : 1$, we see that $l(t)$ comes close to O by (1) of Proposition 6.2.

(II) The case of $w_1 < 2w_2$.

(II-a) When $w_1 : w_2 = 1 : 2$ and $K < \log \dfrac{5}{3}$, we see that $l(t)$ comes close to O by (2) of Proposition 6.2.
(II-b) When $w_1 : w_2 = 1 : 2$ and $K > \log \dfrac{5}{3}$, we see that $l(t)$ goes apart from O by (3) of Proposition 6.2.

The above results show that the dynamics for the front solution are affected by w_1 and w_2 and K.

6.4 Proof of Theorem 6.1

Proof. Fix one each of $l \gg 1$ and $D = 1$. We define the function $G^-(\boldsymbol{x}; l)$ on Ω satisfying

$$\begin{cases} 0 = D\partial_{xx}G_1^- + F'(S_-)G_1^-, x > 0, \\ 0 = D\partial_{xx}G_2^- + F'(S_-)G_2^-, 0 < x < K, \\ w_1(\partial_x S(-l) + \partial_x G_1^-(0)) = w_2(\partial_x G_2^-(K) - \partial_x G_2^-(0)), \\ S(-l) + G_1^-(0) = S_- + G_2^-(0) = S_- + G_2^-(K), \\ G_1^-(+\infty) = 0, \end{cases} \tag{6.4.2}$$

where $G_j^-(\boldsymbol{x}; l) := G^-(\boldsymbol{x}; l), \boldsymbol{x} \in \Omega_j\ (j = 1, 2)$. Let $m_{j,1}^-(x)$ and $m_{j,2}^-(x)\ (j = 1, 2)$ be the fundamental functions of the ODE

$$0 = D\partial_{xx}m_{j,1}^-(x) + F'(S_-)m_{j,2}^-(x).$$

Then $G_1^-(x; l)$ is expresses that

$$G_1^-(x; l) = c_1^-(l)e^{-\alpha_- x}a_-$$

by (H4)' and $G_1^-(\infty) = 0$. Moreover, since $S_- + G_2^-(0) = S_- + G_2^-(K)$ holds, $G_2^-(x; l)$ is expresses that

$$G_2^-(x; l) = c_2^-(l)m_{2,1}^-(x) + \tilde{c}_2^-(l)m_{2,2}^-(x) = c_2^-(l)e^{-\alpha_- x}a_- + c_2^-(l)e^{\alpha_-(x-K)}a_-$$

by (H4)'.

Lemma 6.1. The coefficients $c_j^-\ (j = 1, 2)$ of $G_j^-(x; l)\ (j = 1, 2)$ are given by

$$c_1^-(l) = C_1^-(l) := \frac{[(w_1 - 2w_2) + (w_1 + 2w_2)e^{-\alpha_- K}]e^{-\alpha_- l}}{(w_1 + 2w_2) + (w_1 - 2w_2)e^{-\alpha_- K}}, \tag{6.4.3}$$

$$c_2^- = C_2^-(l) := \frac{2w_1 e^{-\alpha_- l}}{(w_1 + 2w_2) + (w_1 - 2w_2)e^{-\alpha_- K}} \tag{6.4.4}$$

as $l \to +\infty$.

Proof. I prove Lemma 6.1 by quite a similar way to the proof of Lemma 4.1([10], Lemma 4.1) in Chapter 4. Since $S(x-l)$ has the asymptotic profile as $l \to \infty$, $S(x-l) \to e^{\alpha_-(x-l)}a_-$ together with $\partial_x S(x-l) \to \alpha_- e^{\alpha_-(x-l)}a_-$ in a neighborhood of $x=0$. Then substituting these profiles and $G_j^-(0;l), \partial_x G_j^-(0;l)\,(j=1,2)$ into the Kirchhoff boundary condition of (6.4.2), we have

$$\begin{cases} w_1\alpha_-(e^{-\alpha_- l} - c_1^-(l))a_- = 2w_2\alpha_- c_2^-(l)(1-e^{-\alpha_- K})a_-, \\ S_- + e^{-\alpha_- l}a_- + c_1^-(l)a_- = S_- + c_2^-(l)(1+e^{-\alpha_- K})a_-. \end{cases} \tag{6.4.5}$$

Moreover $C_1^-(l), C_2^-(l)$ defined (6.4.3), (6.4.4) satisfy

$$\begin{cases} w_1 e^{-\alpha_- l} - w_1 C_1^-(l) = 2w_2 C_2^-(l)(1-e^{-\alpha_- K}), \\ e^{-\alpha_- l} + C_1^-(l) = C_2^-(l)(1+e^{-\alpha_- K}). \end{cases}$$

Thus (6.4.5) holds by taking $c_1^-(l) = C_1^-(l), c_2^-(l) = C_2^-(l)$. □

We put $G_1(x;l) := G_1^-(x;l), G_2(x;l) := G_2^-(x;l)$. We express the solution $u_j(t,x)\,(j=1,2)$ of (6.1.1) by $u_1(t,x) = S(x-l)+G_1(x;l)+V_1(t,x), u_2(t,x) = S_- + G_2(x;l)+V_2(t,x)$. Then the equation (6.1.1) become the equation of V_j

$$\begin{cases} \partial_t\{S(\cdot - l) + G_1 + V_1\} = L(l)V_1 + L(l)G_1 + Z_1(l,V_1), \\ \partial_t\{G_2 + V_2\} = L_- V_2 + L_- G_2 + Z_2(l,V_2), \\ w_1\partial_x V_1(t,+0) + w_2\partial_x V_2(t,+0) = w_2\partial_x V_2(t,K), \\ V_1(t,+0) = V_2(t,+0) = V_2(t,K), \end{cases}$$

where $L(l) := D\partial_{xx} + F'(S(x-l)), L_- = D\partial_{xx} + F'(S_-)$, and $Z_j(l,V_j)\,(j=1,2)$ are functions satisfying $|Z_j(l,V_j)| \le O(|G_j(x;l)|^2 + |V_j(t,x)|^2)$.

The rest can be proved in quite a similar way to Theorem 4.1([10], Theorem 2.2) in Chapter 4. □

6.5 Concluding remarks

In this chapter, I discussed the front dynamics for the scalar reaction-diffusion equation(6.1.1). In recently, we have studied front dynamics for the following two types of problems:

(1)(Type 3 in [41])

$\Omega := \Omega_1 \cup \Omega_2 \cup \{\boldsymbol{O}\} \cup \{\boldsymbol{K}_1\}$ and $\Omega_1 := \{(x, 0) \in \mathbb{R}^2; 0 < x < K_1\}$, where $\boldsymbol{K}_1 := (K_1, 0) \in \mathbb{R}^2$ and

$$\Omega_2(r_2) := \left\{ r_2 \left(\cos \frac{x}{r_2} - 1, \sin \frac{x}{r_2} \right) \in \mathbb{R}^2;\ 0 < x < K_2 \right\}$$

and $K_2 := 2\pi r_2$, r_2 is the radius of the circle. Then we consider

$$\begin{cases} \partial_t u_j = \partial_{xx} u_j + F(u_j), t > 0, 0 < x < K_j \, (j = 1, 2), \\ w_1 \dfrac{\partial u_1}{\partial x}(t, +0) + w_2 \dfrac{\partial u_2}{\partial x} = w_2 \dfrac{\partial u_2}{\partial x}(t, K_2), \\ u_1(t, +0) = u_2(t, +0) = u_2(t, K_2), \\ \dfrac{\partial u_1}{\partial x}(t, K_1) = 0 \end{cases}$$

for $K_1 \gg 1, K_2 > 0, w_j > 0 \, (j = 1, 2)$, and $F : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function.

(2)(Type 5 in [41])

Let $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \{\boldsymbol{O}\} \cup \{\boldsymbol{K}_1\}$ be the finite metric graph, where Ω_1 and Ω_2 are exactly the same as (1). Moreover

$$\Omega_3(r_3) := \left\{ \left(-r_3 \left(\cos \frac{x}{r_3} - 1 \right) + K_1, r_3 \sin \frac{x}{r_3} \right) \in \mathbb{R}^2;\ 0 < x < K_3 \right\},$$

where $K_3 := 2\pi r_3$, r_3 is the radius of the circle. Furthermore, exactly as in (1), we denote $u_3(t, x) := u\left(t, \left(-r_3 \left(\cos \dfrac{x}{r_3} - 1\right) + K_1, r_3 \sin \dfrac{x}{r_3}\right)\right)$, $t > 0, \boldsymbol{x} \in \Omega_3$. Then we consider

$$\begin{cases} \partial_t u_j = \partial_{xx} u_j + F(u_j), t > 0, 0 < x < K_j \, (j = 1, 2, 3), \\ w_1 \dfrac{\partial u_1}{\partial x}(t, +0) + w_2 \dfrac{\partial u_2}{\partial x}(t, +0) = w_2 \dfrac{\partial u_2}{\partial x}(t, K_2), u_1(t, +0) = u_2(t, +0) = u_2(t, K_2), \\ w_3 \dfrac{\partial u_3}{\partial x}(t, +0) = w_1 \dfrac{\partial u_1}{\partial x}(t, K_1) + w_3 \dfrac{\partial u_3}{\partial x}(t, K_3), u_3(t, +0) = u_1(t, K_1) = u_3(t, K_3) \end{cases}$$

for $K_1 \gg 1, K_2 > 0, K_3 > 0, w_j > 0 \, (j = 1, 2, 3)$, and $F : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function.

We note that both (1)(Type 3 in [41]) and (2)(Type 5 in [41]) are domains dealt with in [41]. To develop front dynamics on (1)(Type 2 in [41]) and (2)(Type 5 in [41]), we have analyzed the above two problems using the method of [10] currently. We will continue to study the above two issues. In addtion, we will also analyze pulse/front dynamics on Type 3, Type 4 and Type 5 in [41] for reaction diffusion systems in the future.

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