Hypergeometric Groups and Dynamics on K3 Surfaces*

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Abstract

A hypergeometric group is a matrix group modeled on the monodromy group of a generalized hypergeometric differential equation. This article presents a fruitful interaction between the theory of hypergeometric groups and dynamics on K3 surfaces by showing that a certain class of hypergeometric groups and related lattices lead to a lot of K3 surface automorphisms of positive entropy, especially such automorphisms with Siegel disks.

1 Introduction

This article originates from a simple question: What happens if we put the following two topics together ? One is the theory of hypergeometric groups due to Levelt [17], Beukers and Heckman [4], and the other is dynamics on K3 surfaces due to McMullen [18, 20, 21]; see also Gross and McMullen [9]. In this article we present a fruitful interaction between them by showing that a certain class of hypergeometric groups and related lattices produce a lot of K3 surface automorphisms of positive entropy, especially such automorphisms with Siegel disks.

A hypergeometric group is a group modeled on the monodromy group of a generalized hypergeometric differential equation. It is a matrix group $H = \langle A, B \rangle \subset \mathrm{GL}(n, \mathbb{C})$ generated by two invertible matrices A and B such that $\mathrm{rank}(A - B) = 1$, which is equivalent to the condition $\mathrm{rank}(I - C) = 1$ for the third matrix $C := A^{-1}B$. In the context of an n-th order hypergeometric equation the matrices A, B, C are the local monodromy matrices around the regular singular points $z = \infty$, 0, 1, respectively. The rank condition for C is then a consequence of the property that the differential equation has $n - 1$ linearly independent holomorphic solutions around $z = 1$. Beukers and Heckman [4] established several fundamental properties of hypergeometric groups such as irreduciblity, invariant Hermitian form, signature, etc. and went on to classify finite hypergeometric groups. Along the way they also determined differential Galois groups of hypergeometric equations, that is, Zariski closures of hypergeomtric groups.

On the other hand, McMullen [18] synthesized examples of K3 surface automorphisms with Siegel disks. His constructions were based upon (i) K3 lattices and K3 structures in Salem number fields, (ii) Lefschetz and Atiyah-Bott fixed point formulas, (iii) Siegel-Sternberg theory on linearizations of nonlinear maps and small divisor problems, and (iv) Gel'fond-Baker method in transcendence theory and Diophantine approximation. In his work the characteristic polynomials of the constructed automorphisms were Salem polynomials of degree 22, so the topological entropies of them were logarithms of Salem numbers of degree 22 by the Gromov-Yomdin theorem [8, 28]. McMullen [20, 21] went on to construct K3 surface automorphisms with Salem numbers of lower degrees, especially ones with Lehmer's number λ_{L} in [16], whose logarithm was the minimum of the positive entropy spectrum for all automorphisms on compact complex surfaces [19]. He discussed the non-projective cases in [20] and the projective ones in [21], though these papers did not touch on Siegel disks. Here we should recall from [18, Theorem 7.2] that an automorphism on a projective K3 surface never admits a Siegel disk.

Our chief idea in this article is to use hypergeometric groups and associated lattices, in place of Salem number fields mentioned in item (i) above. To outline our idea we need to review the minimal basics about K3 surfaces and their automorphisms (see Barth et al. [3, Chap. VIII]). The middle cohomology group $L = H^2(X, \mathbb{Z})$ of a

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K3 surface X equipped with the intersection form is an even unimodular lattice of rank 22 and signature $(3, 19)$. The geometry of X then defines a triple, called the *K3 structure* on L, consisting of Hodge structure $L \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, positive cone $\mathcal{C}^+(X)$ and Kähler cone $\mathcal{K}(X)$. It is accompanied by the related concepts of Picard lattice $\mathrm{Pic}(X)$ (or Néron-Severi lattice $\mathrm{NS}(X)$), root system $\Delta(X)$ and Weyl group $W(X)$. The surface X is projective if and only if $\mathrm{Pic}(X)$ contains an element of positive self-intersection. Any K3 surface automorphism $f : X \to X$ induces a lattice automorphism $f^* : L \to L$ preserving the K3 structure. Conversely, thanks to the Torelli theorem and surjectivity of the period map, any automorphism of an abstract K3 lattice preserving an abstract K3 structure is realized by a unique K3 surface automorphism up to isomorphisms. An automorphism $f : X \to X$ gives rise to two basic invariants: the *entropy* $h(f) = \log \lambda(f)$ with $\lambda(f)$ being the spectral radius of $f^*|H^2(X)$, and the *special eigenvalue* $\delta(f) \in S^1$ defined by $f^*\eta = \delta(f)\,\eta$ for a nowhere vanishing holomorphic 2-form η on X, or rather the *special trace* $\tau(f) := \delta(f) + \delta(f)^{-1} = \mathrm{Tr}(f^*|H^{2,0}(X) \oplus H^{0,2}(X)) \in [-2, 2]$. Note that $\lambda(f)$ is either 1 or a Salem number $\lambda > 1$, while $\delta(f)$ is either a root of unity or a conjugate to the Salem number λ. The spectral radius $\lambda(F)$, special eigenvalue $\delta(F)$ and special trace $\tau(F)$ are conceivable for a Hodge isometry $F : L \to L$ of an abstract K3 lattice L with a Hodge structure. F is said to be *positive* if it preserves the positive cone. It falls into one of the elliptic, parabolic and hyperbolic types (see e.g. Cantat [5, §1.4]).

We turn our attention to hypergeometric groups. Let $\varphi(z)$ and $\psi(z)$ be a coprime pair of monic polynomials over $\mathbb{Z}$ of an even degree $n = 2N$ such that $\varphi(z)$ is anti-palindromic and $\psi(z)$ is palindromic, that is, $z^n\varphi(z^{-1}) = -\varphi(z)$ and $z^n\psi(z^{-1}) = \psi(z)$. Consider the hypergeometric group $H = \langle A, B\rangle \subset \mathrm{GL}(n, \mathbb{Z})$ generated by $A := Z(\varphi)$ and $B := Z(\psi)$, where $Z(f)$ is the companion matrix of a monic polynomial $f(z) = z^n + f_1 z^{n-1} + \cdots + f_n$,

$$Z(f) := \begin{pmatrix} 0 & & & & -f_n \\ 1 & 0 & & & -f_{n-1} \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -f_2 \\ & & & 1 & -f_1 \end{pmatrix}.$$

Theorem 1.1 *The matrix $C := A^{-1}B$ is a reflection, that is, it fixes a hyperplane in $\mathbb{Q}^n$ pointwise and sends a nonzero vector $\boldsymbol{r} \in \mathbb{Q}^n$ to its negative $-\boldsymbol{r}$. We have a free $\mathbb{Z}$-module of rank n,*

$$L := \langle \boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}\rangle_{\mathbb{Z}} = \langle \boldsymbol{r}, B\boldsymbol{r}, \dots, B^{n-1}\boldsymbol{r}\rangle_{\mathbb{Z}}, \tag{1}$$

stable under the action of H. There exists a unique, non-degenerate, H-invariant, $\mathbb{Z}$-valued symmetric bilinear form $(\cdot\,,\cdot) : L \times L \to \mathbb{Z}$ such that $(\boldsymbol{r}, \boldsymbol{r}) = 2$, which makes L into an even lattice. Its Gram matrix for the A-basis is given by $(A^{i-1}\boldsymbol{r}, A^{j-1}\boldsymbol{r}) = \xi_{|i-j|}$, where $\xi_0 := 2$ and $\{\xi_i\}_{i=1}^{\infty}$ is defined via the Taylor series expansion

$$\frac{\psi(z)}{\varphi(z)} = 1 + \sum_{i=1}^{\infty} \xi_i\, z^{-i} \qquad \text{around} \quad z = \infty. \tag{2}$$

The Gram matrix for the B-basis is given in a similar manner by exchanging $\varphi(z)$ and $\psi(z)$ upside down in the Taylor expansion (2). The lattice L is unimodular if and only if the resultant of $\varphi(z)$ and $\psi(z)$ is ± 1.

Since $\varphi(z)$ and $\psi(z)$ are anti-palindromic and palindromic of degree $n = 2N$, there exist monic polynomials $\Phi(w)$ and $\Psi(w)$ of degrees $N-1$ and N over $\mathbb{Z}$ such that $\varphi(z) = (z^2-1)z^{N-1}\Phi(z+z^{-1})$ and $\psi(z) = z^N\Psi(z+z^{-1})$. We refer to $\Phi(w)$ and $\Psi(w)$ as the *trace polynomials* of $\varphi(z)$ and $\psi(z)$. Being coprime, $\varphi(z)$ and $\psi(z)$ have no roots in common and the same is true for $\Phi(w)$ and $\Psi(w)$. Let $\boldsymbol{A}$ be the multi-set of all complex roots of $\Phi(w)$ counted with multiplicities. Let $\boldsymbol{A}_{\mathrm{on}}$ and $\boldsymbol{A}_{\mathrm{off}}$ be those parts of $\boldsymbol{A}$ which lie on and off the interval $[-2, 2]$ respectively. Define $\boldsymbol{B}$, $\boldsymbol{B}_{\mathrm{on}}$ and $\boldsymbol{B}_{\mathrm{off}}$ in a similar manner for $\Psi(w)$. Then $\boldsymbol{A}_{\mathrm{on}}$ and $\boldsymbol{B}_{\mathrm{on}}$ dissect each other into interlacing components, $\boldsymbol{A}_1, \dots, \boldsymbol{A}_{s+1}$ and $\boldsymbol{B}_1, \dots, \boldsymbol{B}_s$, called *trace clusters*, such that

$$-2 \le \boldsymbol{A}_{s+1} < \boldsymbol{B}_s < \boldsymbol{A}_s < \cdots < \boldsymbol{B}_1 < \boldsymbol{A}_1 \le 2, \tag{3}$$

where one or both of the end clusters $\boldsymbol{A}_1$ and $\boldsymbol{A}_{s+1}$ may be null, while any other cluster must be non-null. Let $\boldsymbol{A}_{>2}$ and $\boldsymbol{B}_{>2}$ be those parts of $\boldsymbol{A}$ and $\boldsymbol{B}$ which lie in $(2, \infty)$ respectively. We give a formula representing the index of L in terms of the trace clusters (3) as well as $\boldsymbol{A}_{>2}$ and $\boldsymbol{B}_{>2}$ (see Theorem 4.2). It in particular says that the index up to sign $\pm$ depends only on the trace clusters (3), being independent of $\boldsymbol{A}_{>2}$ and $\boldsymbol{B}_{>2}$.

We focus on the specific rank $n = 22$, i.e. $N = 11$. It is natural to ask when a hypergeometric lattice L or its negative $L(-1)$ becomes a K3 lattice with a Hodge structure such that the matrix A (or B) is a positive Hodge isometry. Here we take L or $L(-1)$ depending on whether the index of L is negative or positive, because a K3 lattice has negative index -16. This procedure is called the *renormalization* of L and the renormalized H-invariant symmetric bilinear form is referred to as the intersection form on L. To state our theorems we introduce some notation and terminology: $|\boldsymbol{A}_{\rm on}|$ stands for the cardinality of $\boldsymbol{A}_{\rm on}$ counted with multiplicities; $[\boldsymbol{A}_{\rm on}] = 0^{\nu_0} 1^{\nu_1} 2^{\nu_2} 3^{\nu_3}$ means that $\boldsymbol{A}_{\rm on}$ consists of ν_0 null clusters, ν_1 simple clusters, ν_2 double clusters, ν_3 triple clusters, where j^{ν_j} is omitted if $\nu_j = 0$. The same rule applies to $\boldsymbol{B}_{\rm on}$ and other related entities. By "doubles adjacent" we mean the situation in which there exist a unique double $\boldsymbol{A}_{\rm on}$-cluster $\boldsymbol{A}_i$ and a unique double $\boldsymbol{B}_{\rm on}$-cluster $\boldsymbol{B}_j$ and they are adjacent to each other. For such a pair if $\boldsymbol{A}_i \cup \boldsymbol{B}_j$ consists of four distinct elements $\lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4$, then λ_2 and λ_3 are referred to as the inner elements of the adjacent pair (AP).

Theorem 1.2 *Let $L = L(\varphi, \psi)$ be a unimodular hypergeometric lattice of rank 22. After renormalization, L is a K3 lattice with a Hodge structure such that the matrix A is a positive Hodge isometry, if and only if $\Phi(-2) \neq 0$ and the roots of $\Phi(w)$ and $\Psi(w)$ are all simple and have any one of the following configurations.*

- (E) *In elliptic case, any entry of Table 1 such that $\boldsymbol{A}_1$ does not contain 2, that is, $\Phi(2) \neq 0$.*
- (P) *In parabolic case, any entry of Table 1 such that $\boldsymbol{A}_1$ does contain 2, that is, $\Phi(2) = 0$.*
- (H) *In hyperbolic case, any entry of Table 2 such that $\boldsymbol{A}_1$ does not contain 2, that is, $\Phi(2) \neq 0$.*

The special trace $\tau(A)$ and the Hodge structure up to complex conjugation are uniquely determined by the pair (φ, ψ). The location of $\tau(A)$ is exhibited in the last columns of Tables 1 and 2, where we mean by "middle of TC" that $\tau(A)$ is the middle element of the unique triple $\boldsymbol{A}_{\rm on}$-cluster, and by "inner of AP" that it is the inner element in $\boldsymbol{A}_{\rm on}$ of the unique adjacent pair of double clusters in $\boldsymbol{A}_{\rm on} \cup \boldsymbol{B}_{\rm on}$.

Table 1: Conditions for A to be a positive Hodge isometry of elliptic or parabolic type.

case	s	$[\boldsymbol{A}_{\rm on}]$	$[\boldsymbol{B}_{\rm on}]$	$\boldsymbol{A}_1$	$\lvert\boldsymbol{B}_{\rm off}\rvert$	constraints	ST $\tau(A)$
1	8	$0^1 1^7 3^1$	1^8	non-null	3		middle of TC
2	8	$0^1 1^7 3^1$	$1^7 3^1$	non-null	1		middle of TC
3	9	$0^2 1^7 3^1$	$1^8 2^1$	null	1	$\lvert\boldsymbol{B}_1\rvert = 2$	middle of TC
4	8	$1^8 2^1$	1^8	non-null	3	$\lvert\boldsymbol{A}_9\rvert = 2$	min $\boldsymbol{A}_9$
5	8	$1^8 2^1$	$1^7 3^1$	non-null	1	$\lvert\boldsymbol{A}_9\rvert = 2$	min $\boldsymbol{A}_9$
6	9	$0^1 1^8 2^1$	$1^8 2^1$	non-null	1	doubles adjacent	inner of AP
7	9	$0^1 1^8 2^1$	$1^8 2^1$	null	1	$\lvert\boldsymbol{A}_{10}\rvert = 2$, $\lvert\boldsymbol{B}_1\rvert = 2$	min $\boldsymbol{A}_{10}$
8	10	$0^2 1^8 2^1$	1^{10}	null	1	$\lvert\boldsymbol{A}_2\rvert = 2$	max $\boldsymbol{A}_2$
9	9	1^{10}	$1^8 2^1$	non-null	1	$\lvert\boldsymbol{B}_9\rvert = 2$	element of $\boldsymbol{A}_{10}$

Table 2: Conditions for A to be a positive Hodge isometry of hyperbolic type.

case	s	$[\boldsymbol{A}_{\rm on}]$	$[\boldsymbol{B}_{\rm on}]$	$\lvert\boldsymbol{A}_{>2}\rvert$	$\lvert\boldsymbol{B}_{\rm off}\rvert$	constraints	ST $\tau(A)$
1	8	$0^2 1^6 3^1$	1^8	1	3		middle of TC
2	8	$0^2 1^6 3^1$	$1^7 3^1$	1	1		middle of TC
3	8	$0^1 1^7 2^1$	1^8	1	3	$\lvert\boldsymbol{A}_1\rvert = 2$	max $\boldsymbol{A}_1$
4	8	$0^1 1^7 2^1$	1^8	1	3	$\lvert\boldsymbol{A}_9\rvert = 2$	min $\boldsymbol{A}_9$
5	8	$0^1 1^7 2^1$	$1^7 3^1$	1	1	$\lvert\boldsymbol{A}_1\rvert = 2$	max $\boldsymbol{A}_1$
6	8	$0^1 1^7 2^1$	$1^7 3^1$	1	1	$\lvert\boldsymbol{A}_9\rvert = 2$	min $\boldsymbol{A}_9$
7	9	$0^2 1^7 2^1$	$1^8 2^1$	1	1	doubles adjacent	inner of AP
8	9	$0^1 1^9$	$1^8 2^1$	1	1	$\lvert\boldsymbol{A}_1\rvert = 1$, $\lvert\boldsymbol{B}_1\rvert = 2$	element of $\boldsymbol{A}_1$
9	9	$0^1 1^9$	$1^8 2^1$	1	1	$\lvert\boldsymbol{A}_{10}\rvert = 1$, $\lvert\boldsymbol{B}_9\rvert = 2$	element of $\boldsymbol{A}_{10}$

Due to the asymmetry of $\varphi(z)$ and $\psi(z)$ the corresponding result for the matrix B is somewhat different and a dominant role is played by $\boldsymbol{B}_{\mathrm{on}}$ and $\boldsymbol{A}_{\mathrm{in}} := \boldsymbol{A}_2 \cup \cdots \cup \boldsymbol{A}_s$ in place of $\boldsymbol{A}_{\mathrm{on}}$.

Theorem 1.3 *Let $L = L(\varphi, \psi)$ be a unimodular hypergeometric lattice of rank 22. After renormalization, L is a K3 lattice with a Hodge structure such that the matrix B is a positive Hodge isometry, if and only if all roots of $\Psi(w)$ are simple, $\Psi(\pm 2) \neq 0$, and the roots of $\Phi(w)$ and $\Psi(w)$ have any one of the configurations in Table 3. The special trace $\tau(B)$ and the Hodge structure up to complex conjugation are uniquely determined by the pair (φ, ψ). The location of $\tau(B)$ is exhibited in the last column of Table 3, where we mean by "middle of TC" that $\tau(B)$ is the middle element of the unique triple $\boldsymbol{B}_{\mathrm{on}}$-cluster, and by "inner of AP" that it is the inner element in $\boldsymbol{B}_{\mathrm{on}}$ of the unique adjacent pair of double clusters in $\boldsymbol{A}_{\mathrm{in}} \cup \boldsymbol{B}_{\mathrm{on}}$. The Hodge isometry B is always of hyperbolic type and any root of $\Phi(w)$ is simple except for at most one integer root of multiplicity 2 or 3.*

Table 3: Conditions for B to be a positive Hodge isometry, always of hyperbolic type.

case	s	$[\boldsymbol{A}_{\mathrm{in}}]$	$[\boldsymbol{B}_{\mathrm{on}}]$	$\lvert\boldsymbol{A}_{\mathrm{in}}\rvert$	$\lvert\boldsymbol{B}_{>2}\rvert$	constraints	ST $\tau(B)$
1	8	1^7	$1^7 3^1$	7	1		middle of TC
2	8	$1^6 3^1$	$1^7 3^1$	9	1		middle of TC
3	9	$1^7 2^1$	$1^8 2^1$	9	1	doubles adjacent	inner of AP
4	9	1^8	$1^8 2^1$	8	1	$\lvert\boldsymbol{B}_1\rvert = 2$	$\max \boldsymbol{B}_1$
5	9	1^8	$1^8 2^1$	8	1	$\lvert\boldsymbol{B}_9\rvert = 2$	$\min \boldsymbol{B}_9$
6	9	$1^7 3^1$	$1^8 2^1$	10	1	$\lvert\boldsymbol{B}_1\rvert = 2$	$\max \boldsymbol{B}_1$
7	9	$1^7 3^1$	$1^8 2^1$	10	1	$\lvert\boldsymbol{B}_9\rvert = 2$	$\min \boldsymbol{B}_9$
8	10	$1^8 2^1$	1^{10}	10	1	$\lvert\boldsymbol{A}_2\rvert = 2$	element of $\boldsymbol{B}_1$
9	10	$1^8 2^1$	1^{10}	10	1	$\lvert\boldsymbol{A}_{10}\rvert = 2$	element of $\boldsymbol{B}_{10}$

Remark 1.4 Put $\chi(z) = \varphi(z)$ and $F = A$ in Theorem 1.2, while $\chi(z) = \psi(z)$ and $F = B$ in Theorem 1.3.

(1) Any root of $\chi(z)$ is simple except for the triple root $z = 1$ in the parabolic case of Theorem 1.2.

(2) In the elliptic and parabolic cases of Theorem 1.2, any irreducible component of $\chi(z)$ is a cyclotomic polynomial and the spectral radius $\lambda(F)$ is equal to 1.

(3) In the hyperbolic case of Theorem 1.2 and in the whole case of Theorem 1.3, $\chi(z)$ has a unique Salem polynomial factor with the associated Salem number $\lambda > 1$ giving the spectral radius $\lambda(F)$, and any other irreducible component of $\chi(z)$ is a cyclotomic polynomial.

(4) In every case of Theorems 1.2 and 1.3 the special trace $\tau(F)$ lies in $(-2, 2)$, so that the special eigenvalue $\delta(F) \in S^1$ is either a root of unity different from ± 1 or a conjugate to the Salem number λ.

Let $\mathrm{Pic} := L \cap H^{1,1}$ be the "Picard lattice" arising from the constructions in Theorems 1.2 and 1.3. There are two cases: one is the *projective* case where Pic contains a vector of positive self-intersection and the other is the *non-projective* case where no such vector exists. By item (4) of Remark 1.4 the minimal polynomial $\chi_0(z)$ of $\delta(F)$ is either a cyclotomic polynomial of degree ≥ 2 or a Salem polynomial of degree ≥ 4. Note that the degree of $\chi_0(z)$ is necessarily even in either case.

Theorem 1.5 *The intersection form is non-degenerate on* Pic *and the Picard number $\rho := \operatorname{rank} \mathrm{Pic}$ is given by $\rho = 22 - \deg \chi_0(z)$, which is even and ≤ 20. The projective case occurs precisely when $\chi_0(z)$ is a cylcotomic polynomial of degree ≥ 2, while the non-projective case is exactly the case where $\chi_0(z)$ is a Salem polynomial of degree ≥ 4 and the associated Salem number yields the spectral radius $\lambda(F)$. In particular both the elliptic and parabolic cases in Theorem 1.2 are projective. If $\boldsymbol{r}$ is the vector in Theorem 1.1 and $\boldsymbol{s} := \chi_0(F)\boldsymbol{r}$, then the vectors $\boldsymbol{s}, F\boldsymbol{s}, \dots, F^{\rho-1}\boldsymbol{s}$ form a free $\mathbb{Z}$-basis of* Pic*, which we call the standard basis.*

Theorem 1.1 enables us to calculate the Gram matrix of Pic with respect to the standard basis and hence its discriminant. The root system is defined by $\varDelta := \{\boldsymbol{u} \in \mathrm{Pic} : (\boldsymbol{u}, \boldsymbol{u}) = -2\}$ and the Weyl group W is the group generated by all reflections in root vectors. Usually, F does not preserve any Weyl chamber. Fix a Weyl chamber C. It may be sent to a different Weyl chamber $F(C)$, but there exists a unique element $w_F \in W$ that brings $F(C)$ back to the original chamber C, since W acts on the set of Weyl chambers simply transitively. In this article we discuss the non-projective case only, in which the intersection form is negative definite on Pic so that the root system Δ and the Weyl group W are finite. Lexicographical order with respect to the standard basis in Theorem 1.5 leads to a set of positive roots Δ^+ and the corresponding Weyl chamber $\mathcal{K}$.

Theorem 1.6 *In the non-projective case there exists an algorithm to output the unique element $w_F \in W$ such that the modified matrix $\tilde{F} := w_F \circ F$ preserves the Weyl chamber $\mathcal{K}$ (Algorithm 7.5). This twist does not change the Hodge structure, spectral radius $\lambda(F) = \lambda(\tilde{F})$ and special trace $\tau(F) = \tau(\tilde{F})$. The modified Hodge isometry $\tilde{F} : L \to L$ lifts to a non-projective K3 surface automorphism $f : X \to X$ of positive entropy $h(f) = \log \lambda(F)$ with special trace $\tau(f) = \tau(F)$, Picard lattice $\mathrm{Pic}(X) \cong \mathrm{Pic}$ and Picard number $\rho(X) = \rho$ as in Theorem 1.5, while $\mathcal{K}$ lifting to the Kähler cone of X. The map f is uniquely determined by the pair (φ, ψ) plus the choice of $F = A$ or B up to biholomorphic conjugacy and complex conjugation.*

The Weyl chamber $\mathcal{K}$ is chosen just for the sake of algorithmic purpose. Taking any other Weyl chamber results in a modified matrix conjugate to the one in Theorem 1.6 by a unique Weyl group element. It is an interesting future problem to consider what is going on in the projective case, in which the spectral radius $\lambda(\tilde{F})$ may change from the original one $\lambda(F)$. This case should produce automorphisms of projective K3 surfaces.

Back in the non-projective case we illustrate our method by working out some examples. Let $\lambda_1, \dots, \lambda_{10}$ be the ten Salem numbers of degree 22 from McMullen [18, Table 4] reproduced in this article as Table 6, and $S_1(z), \dots, S_{10}(z)$ be the associated Salem polynomials. Let λ_{L} be Lehmer's number, the smallest Salem number ever known, and $\mathrm{L}(z)$ be Lehmer's polynomial (see (47a)). In this article we examine only three patterns: (i) $\varphi(z) = C(z)$, $\psi(z) = S_i(z)$ and matrix B; (ii) $\varphi(z) = \mathrm{L}(z) \cdot C(z)$, $\psi(z) = S_i(z)$ and matrix A; (iii) $\varphi(z) = C(z)$, $\psi(z) = \mathrm{L}(z) \cdot \tilde{C}(z)$ and matrix B, where $C(z)$ and $\tilde{C}(z)$ stand for products of cyclotomic polynomials.

Theorem 1.7 *In each of the three patterns there exists a complete enumeration of all cases that produce non-projective K3 surface automorphisms $f : X \to X$, where we have $h(f) = \log \lambda_i$ and $\rho(X) = 0$ in pattern (i), while $h(f) = \log \lambda_{\mathrm{L}}$ and $\rho(X) = 12$ in patterns (ii) and (iii); see Theorems 8.1, 8.2 and 8.5, respectively.*

Even from these restricted setups we can obtain a lot of K3 surface automorphisms of positive entropy, especially examples with Siegel disks. As for the results on Siegel disks we refer to Theorems 9.3, 9.4 and 9.5, the last of which contains the following statement.

Theorem 1.8 *There are automorphisms of non-projective K3 surfaces of the smallest positive entropy $\log \lambda_{\mathrm{L}}$, where λ_{L} is Lehmer's number, which admit a 3-periodic cycle of Siegel disks as well as some invariant curves.*

With various other Salem polynomials and in various other settings our method allows us to construct a much larger number of K3 surface automorphisms of positive entropy. If a K3 surface admits an automorphism with special eigenvalue conjugate to a Salem number, then its Picard number must be an even integer between 0 and 18. Conversely, all these Picard numbers can be realized by our hypergeometric constructions (even in the presence of Siegel disks). This result will be reported upon separately.

It is usually difficult to decide whether two different choices of the pair (φ, ψ) yield the same automorphism, or whether the construction in this article gives examples identical to those in previous works such as Oguiso [22], McMullen [20] and others, unless some of the resulting invariants, e.g. entropies, special eigenvalues, Picard numbers, Dynkin types of root systems, etc. are distinct, in which case the answer is clearly negative. This is because, given a unimodular quadratic form over $\mathbb{Z}$ with known Gram matrix with respect to some basis, there is no known theory (at least to the authors) to convert it to a canonical form. However, there exists a systematic way to compare the method of hypergeometric groups with that of Salem number fields due to McMullen [18], when $\psi(z)$ is an unramified Salem polynomial of degree 22; see Theorem 10.1 and Corollary 10.2. This comparison should be extended to Salem numbers of lower degrees.

We wonder whether our construction yields interesting examples of finite order automorphisms. By Theorem 1.5 only automorphisms of positive entropy, necessarily of infinite order, can occur in the non-projective case. So we have to examine the projective case to answer this question. This is another problem yet to be discussed.

The organization of this article is as follows. The first part (§2–§5) is devoted to enriching infrastructure in the theory of hypergeometric groups in anticipation of its application in the second part as well as other applications elsewhere. In §2, for a complex hypergeometric group $H = H(\varphi, \psi)$ we study the Gram matrix of an H-invariant Hermitian form and prove a large part of Theorem 1.1. Its index is expressed in terms of the clusters of roots of $\varphi(z)$ and $\psi(z)$ (Theorem 3.2). When H is a real hypergeometric group, the index is represented in terms of the trace clusters of roots of $\Phi(w)$ and $\Psi(w)$ (Theorem 4.2). A formula for local indices (Proposition 4.5) is also included for later use in Hodge structures. When H is defined over $\mathbb{Z}$, an H-invariant even lattice, called the hypergeometric lattice, is introduced in §5. The second part (§6–§10) is an application of hypergeometric groups to dynamics on K3 surfaces, whose main results are already stated above. A classification of all real hypergeometric groups of rank 22 and index ± 16 is given in Theorem 6.2. Theorems 1.2 and 1.3 are then proved in §6.3. Theorems 1.5, 1.6 and 1.7 are established in §7.2, §7.3 and §8 respectively.

As the final remark, Theorems 1.2 and 1.3 say that at least two eigenvalues of B must be algebraic units off the unit circle in order for a hypergeometric group to yield a K3 lattice. This offers a sharp contrast to such a situation as in the classification of finite hypergeometric groups by Beukers and Heckman [4] or in their treatment of Lorentzian hypergeometric groups by Fuchs, Meiri and Sarnak [7], in which all eigenvalues of A and B are roots of unity. Even if they are off the unit circle, hypergeometric groups can enjoy rich structures.

There appear a lot of symbols in the sections (§2–§6) concerning hypergeometric groups. For the reader's convenience a list of them with brief explanations is provided in Appendix C.

2 Hypergeometric Groups

The theory of hypergeometric groups is developed by Beukers and Heckman [4]. A lucid explanation of this concept can also be found in Heckman's lecture notes [10]. A hypergeometric group is a group $H = \langle A, B \rangle$ generated by two invertible matrices $A, B \in \mathrm{GL}(n, \mathbb{C})$ such that $\mathrm{rank}(A - B) = 1$. Let $\boldsymbol{a} = \{a_1, \dots, a_n\}$ and $\boldsymbol{b} = \{b_1, \dots, b_n\}$ be the eigenvalues of A and B respectively (they are multi-sets allowing repeated elements). Then H acts on $\mathbb{C}^n$ irreducibly if and only if

$$\boldsymbol{a} \cap \boldsymbol{b} = \emptyset \tag{4}$$

(see [10, Theorem 3.8]). Hereafter we always assume condition (4). Let $a^\dagger := \bar{a}^{-1}$ for $a \in \mathbb{C}^\times$. There then exists a non-degenerate H-invariant Hermitian form $(\,\cdot\,,\,\cdot\,) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ if and only if

$$\boldsymbol{a}^\dagger = \boldsymbol{a}, \qquad \boldsymbol{b}^\dagger = \boldsymbol{b}, \tag{5}$$

where $\boldsymbol{a}^\dagger := \{a_1^\dagger, \dots, a_n^\dagger\}$ (see [4, Theorem 4.3] and [10, Theorem 3.13]). It is unique up to a real nonzero constant multiple and hence is referred to as *the* invariant Hermitian form. We remark that the "only if" part is not mentioned there but it is an easy exercise. Hereafter we also assume condition (5).

Let $\varphi(z)$ and $\psi(z)$ be the characteristic polynomials of A and B respectively. It is interesting to describe conditions (4) and (5) in terms of $\varphi(z)$ and $\psi(z)$. Note that $\varphi(0) = (-1)^n \det A$ and $\psi(0) = (-1)^n \det B$ are non-zero since $A, B \in \mathrm{GL}(n, \mathbb{C})$. Condition (4) is equivalent to

$$\mathrm{Res}(\varphi, \psi) \neq 0, \tag{4'}$$

where $\mathrm{Res}(\varphi, \psi)$ denotes the resultant of $\varphi(z)$ and $\psi(z)$, while condition (5) is equivalent to

$$z^n \bar{\varphi}(z^{-1}) = \bar{\varphi}(0) \cdot \varphi(z), \qquad z^n \bar{\psi}(z^{-1}) = \bar{\psi}(0) \cdot \psi(z), \tag{5'}$$

where $\bar{f}(z) := \overline{f(\bar{z})}$ for $f(z) \in \mathbb{C}[z]$. Comparing the constant terms in (5′) we have

$$|\varphi(0)| = 1, \qquad |\psi(0)| = 1, \tag{6}$$

because $\varphi(z)$ and $\psi(z)$ are monic polynomials. In the other way round, start with any monic polynomials $\varphi(z)$ and $\psi(z)$ with nonzero constant terms that satisfy condition (4′). Then their companion matrices $Z(\varphi)$ and $Z(\psi)$ generate an irreducible hypergeometric group

$$H(\varphi, \psi) := \langle Z(\varphi), Z(\psi) \rangle. \tag{7}$$

Levelt's theorem [17] states that any irreducible hypergeometric group $H = \langle A, B \rangle$ is conjugate in $\mathrm{GL}(n, \mathbb{C})$ to this one (see [4, Theorem 3.5] and [10, Theorem 3.9]). In this sense H is uniquely determined by the characteristic polynomials $\varphi(z)$ and $\psi(z)$ of A and B, or equivalently by their eigenvalue sets $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus we can write a general irreducible hypergeometric group $H = \langle A, B \rangle$ as $H(\varphi, \psi)$ or $H(\boldsymbol{a}, \boldsymbol{b})$.

The structure of the H-invariant Hermitian form is discussed in [4, §4] and [10, §3.3] when $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the unit circle S^1. We can extend the discussion there to the general case without this restriction. Since $C := A^{-1}B$ is a complex reflection, that is, $\mathrm{rank}(I - C) = 1$, its determinant

$$c := \det C = \frac{\psi(0)}{\varphi(0)} = \frac{b_1 \cdots b_n}{a_1 \cdots a_n} \in S^1 \tag{8}$$

is an eigenvalue of C, called the distinguished eigenvalue. In this article we assume that

$$c \neq 1. \tag{9}$$

Let $\boldsymbol{r}$ be an eigenvector of C corresponding to the eigenvalue c. As in the proof of [10, Theorem 3.14], we have $(\boldsymbol{r},\, \boldsymbol{r}) \in \mathbb{R}^{\times}$ and

$$C\boldsymbol{v} = \boldsymbol{v} - \zeta(\boldsymbol{v}, \boldsymbol{r})\boldsymbol{r}, \qquad \boldsymbol{v} \in \mathbb{C}^n, \tag{10a}$$

$$\frac{\psi(z)}{\varphi(z)} = 1 + \zeta\,((zI - A)^{-1}A\boldsymbol{r},\, \boldsymbol{r}), \quad \text{where} \quad \zeta := \frac{1-c}{(\boldsymbol{r}, \boldsymbol{r})}. \tag{10b}$$

Recall that the H-invariant Hermitian form is uniquely determined up to scalar multiplications in $\mathbb{R}^{\times}$. To eliminate this ambiguity we take the normalization

$$(\boldsymbol{r}, \boldsymbol{r}) = |1 - c| > 0 \qquad \text{so that} \qquad \zeta = \frac{1-c}{|1-c|} \in S^1. \tag{11}$$

Let $\{\xi_i\}_{i=1}^{\infty}$ be the sequence defined by the Taylor series expansion

$$\frac{\psi(z)}{\varphi(z)} = 1 + \zeta \sum_{i=1}^{\infty} \xi_i\, z^{-i} \qquad \text{around} \quad z = \infty. \tag{12}$$

Theorem 2.1 *The vectors $\boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}$ form a basis of $\mathbb{C}^n$. The Gram matrix $G = (g_{ij})_{i,j=1}^n$ with respect to this basis is given by*

$$g_{ij} := (A^{i-1}\boldsymbol{r},\, A^{j-1}\boldsymbol{r}) = \begin{cases} \xi_{i-j} & (i \ge j \ge 1), \\ \bar{\xi}_{j-i} & (1 \le i < j), \end{cases} \tag{13}$$

where $\xi_0 := |1 - c|$ by convention. The determinant of G has absolute value

$$|\det G| = |\mathrm{Res}(\varphi, \psi)|. \tag{14}$$

Proof. First we show formula (13). Taylor expansion of (10b) around $z = \infty$ reads

$$\frac{\psi(z)}{\varphi(z)} = 1 + \zeta\,((I - z^{-1}A)^{-1}z^{-1}A\boldsymbol{r},\, \boldsymbol{r}) = 1 + \zeta \sum_{i=1}^{\infty} (A^i\boldsymbol{r},\, \boldsymbol{r}) z^{-i}.$$

Comparing this with expansion (12) together with convention $\xi_0 = |1-c|$ yields $(A^i\boldsymbol{r},\, \boldsymbol{r}) = \xi_i$ for every $i \in \mathbb{Z}_{\ge 0}$. Since the Hermitian form is A-invariant, we have

$$g_{ij} = (A^{i-1}\boldsymbol{r},\, A^{j-1}\boldsymbol{r}) = \begin{cases} (A^{i-j}\boldsymbol{r},\, \boldsymbol{r}) = \xi_{i-j} & (i \ge j \ge 1), \\ \overline{(A^{j-i}\boldsymbol{r},\, \boldsymbol{r})} = \bar{\xi}_{j-i} & (1 \le i < j). \end{cases}$$

This together with normalization (11) leads to formula (13).

Next we show formula (14). Suppose for the time being that $a_1, \dots, a_n$ are mutually distinct. Let $\boldsymbol{r} = \boldsymbol{r}_1 + \cdots + \boldsymbol{r}_n$ be the decomposition of $\boldsymbol{r}$ into eigenvectors of A, where $\boldsymbol{r}_i$ corresponds to the eigenvalue a_i. By condition $\boldsymbol{a}^{\dagger} = \boldsymbol{a}$ in (5) there exists a permutation $\sigma \in S_n$ such that $\sigma^2 = 1$ and $a_{\sigma(i)} = a_i^{\dagger}$ for $i = 1, \dots, n$. Note that $\sigma(i) = i$ if and only if $a_i \in S^1$. It follows from non-degeneracy and A-invariance of the Hermitian form that $(\boldsymbol{r}_i, \boldsymbol{r}_{\sigma(i)})$, $i = 1, \dots, n$, are non-zero while all the other Hermitian parings $(\boldsymbol{r}_i, \boldsymbol{r}_j)$ vanish. Thus equation (10b) leads to

$$\frac{\psi(z)}{\varphi(z)} = 1 + \zeta \sum_{i=1}^{n} \frac{a_i\,(\boldsymbol{r}_i,\, \boldsymbol{r}_{\sigma(i)})}{z - a_i}.$$

Taking residue at $z = a_i$ we have for $i = 1, \dots, n$,

$$\lambda_i := (\boldsymbol{r}_i,\, \boldsymbol{r}_{\sigma(i)}) = \frac{\psi(a_i)}{\zeta\, a_i\, \varphi_i(a_i)} \qquad \text{with} \quad \varphi_i(z) := \prod_{j \neq i} (z - a_j). \tag{15}$$

After rearranging $a_1, \dots, a_n$ if necessary, we may assume that σ fixes $1, \dots, l$ and exchanges $l + 2i - 1$ and $l + 2i$ for $i = 1, \dots, m$, where $n = l + 2m$. Then $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n$ form a basis of $\mathbb{C}^n$ with respect to which the Gram matrix of the invariant Hermitian form is given by

$$\varLambda = (\lambda_1) \oplus \cdots \oplus (\lambda_l) \oplus \varLambda_1 \oplus \cdots \oplus \varLambda_m \quad \text{with} \quad \varLambda_i := \begin{pmatrix} 0 & \lambda_{l+2i-1} \\ \lambda_{l+2i} & 0 \end{pmatrix}. \tag{16}$$

Note that $\lambda_1, \dots, \lambda_l \in \mathbb{R}$ and $\lambda_{l+2i-1} = \bar{\lambda}_{l+2i}$ for $i = 1, \dots, m$. From (15) and (16) we have

$$|\det \varLambda| = |(-1)^m \lambda_1 \cdots \lambda_n| = \left| \frac{(-1)^m \psi(a_1) \cdots \psi(a_n)}{\zeta^n\, a_1 \cdots a_n\, \varphi_1(a_1) \cdots \varphi_n(a_n)} \right| = \frac{|\mathrm{Res}(\varphi, \psi)|}{\prod_{i<j} |a_i - a_j|^2},$$

where $|\zeta| = 1 = |a_1 \cdots a_n|$ is used. Moreover we have $(\boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}) = (\boldsymbol{r}_1, \dots, \boldsymbol{r}_n)V$ with $V := (a_i^{j-1})_{i,j=1}^n$ being a Vandermonde matrix. This implies $G = {}^{\mathrm{t}}V \varLambda \overline{V}$ and hence

$$|\det G| = |\det \varLambda| |\det V|^2 = |\det \varLambda| \prod_{i<j} |a_i - a_j|^2 = |\mathrm{Res}(\varphi, \psi)|,$$

which proves formula (14) when $a_1, \dots, a_n$ are distinct. The formula in the general case follows by a continuity argument, which works as far as conditions (4′), (5′) and (9) are fulfilled.

Finally assumption (4′) and formula (14) imply $\det G \neq 0$ showing that $\boldsymbol{r}, A\boldsymbol{r}, \dots A^{n-1}\boldsymbol{r}$ form a basis. □

Proof of Theorem 1.1. A large part of the theorem is a special case of Theorem 2.1 where $\varphi(z)$ and $\psi(z)$ are polynomials over $\mathbb{Z}$, $c = -1$ in (9) so that $\zeta = 1$ in (11) and (12). For the rest, see the beginning of §5. □

Remark 2.2 It is obvious that if $H = \langle A, B \rangle \subset \mathrm{GL}(n, \mathbb{C})$ is a hypergeometric group then so is $H^{\mathrm{a}} := \langle -A, -B \rangle$. We refer to H^{a} as the *antipode* of H. Note that $\varphi^{\mathrm{a}}(z) = (-1)^n \varphi(-z)$ and $\psi^{\mathrm{a}}(z) = (-1)^n \psi(-z)$, hence $\boldsymbol{a}^{\mathrm{a}} = -\boldsymbol{a}$ and $\boldsymbol{b}^{\mathrm{a}} = -\boldsymbol{b}$. It follows from $C^{\mathrm{a}} = C$ and formula (13) that H and H^{a} have the same invariant Hermitian form. This simple remark will be useful in discussing Hodge structures (see Remark 6.10).

3 Index of the Invariant Hermitian Form

Let $\boldsymbol{a}_{\mathrm{on}}$ and $\boldsymbol{a}_{\mathrm{off}}$ be those components of $\boldsymbol{a}$ whose elements lie on and off S^1 respectively. We define $\boldsymbol{b}_{\mathrm{on}}$ and $\boldsymbol{b}_{\mathrm{off}}$ in a similar manner for $\boldsymbol{b}$. If both of $\boldsymbol{a}_{\mathrm{on}}$ and $\boldsymbol{b}_{\mathrm{on}}$ are nonempty then they dissect each other into an equal number of components $\boldsymbol{a}_1, \dots, \boldsymbol{a}_t$ and $\boldsymbol{b}_1, \dots, \boldsymbol{b}_t$ so that $\boldsymbol{a}_1, \boldsymbol{b}_1,\ \boldsymbol{a}_2, \boldsymbol{b}_2,\ \dots,\ \boldsymbol{a}_t, \boldsymbol{b}_t$ are located consecutively on S^1 in the positive direction (anti-clockwise) as shown in Figure 1. Each $\boldsymbol{a}_i$ is called a *cluster* of $\boldsymbol{a}_{\mathrm{on}}$ and is said to be *simple*, *double*, *triple*, etc. if $|\boldsymbol{a}_i| = 1, 2, 3$, and so on, where $|\boldsymbol{x}|$ denotes the cardinality counted with multiplicities of a multi-set $\boldsymbol{x}$. We write

$$[\boldsymbol{a}_{\mathrm{on}}] = 1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots$$

if $\boldsymbol{a}_{\mathrm{on}}$ consists of ν_1 simple clusters, ν_2 double clusters, ν_3 triple clusters, etc. Note that $|\boldsymbol{a}_{\mathrm{on}}| = \nu_1 + 2\nu_2 + 3\nu_3 + \cdots$, $|\boldsymbol{a}_{\mathrm{off}}|$ is even and $|\boldsymbol{a}_{\mathrm{on}}| + |\boldsymbol{a}_{\mathrm{off}}| = n$; the same is true for $\boldsymbol{b}$. Taking a branch-cut ℓ separating $\boldsymbol{b}_t$ and $\boldsymbol{a}_1$ as in Figure 1 we define the argument of $z \in \mathbb{C}^{\times}$ so that

$$\Theta \le \arg z < \Theta + 2\pi, \tag{17}$$

where $\Theta \in [-\pi, \pi)$ is the angle of the ray ℓ to the positive real axis.

Let $\arg a_i = 2\pi\alpha_i$ and $\arg b_i = 2\pi\beta_i$ for $i = 1, \dots, n$. Formula (8) allows us to write

$$c = \mathrm{e}^{2\pi \mathrm{i}\, \gamma} \in S^1 \qquad \text{with} \quad \gamma := \sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i \in \mathbb{R}, \tag{18}$$

where $\mathrm{i} := \sqrt{-1}$, hence condition (9) is equivalent to $\gamma \in \mathbb{R} \setminus \mathbb{Z}$, that is, $\sin \pi\gamma \in \mathbb{R}^{\times}$.

Remark 3.1 Taking another branch of arg has no effect on the contribution of $\boldsymbol{a}_{\mathrm{off}}$ and $\boldsymbol{b}_{\mathrm{off}}$ to the value of $\sin \pi\gamma$, because any pair $\lambda,\ \lambda^{\dagger} \in \boldsymbol{a}_{\mathrm{off}}$ has a common argument so the sum $\arg \lambda + \arg \lambda^{\dagger}$ alters only by an even multiple of 2π; the same is true for $\lambda,\ \lambda^{\dagger} \in \boldsymbol{b}_{\mathrm{off}}$.

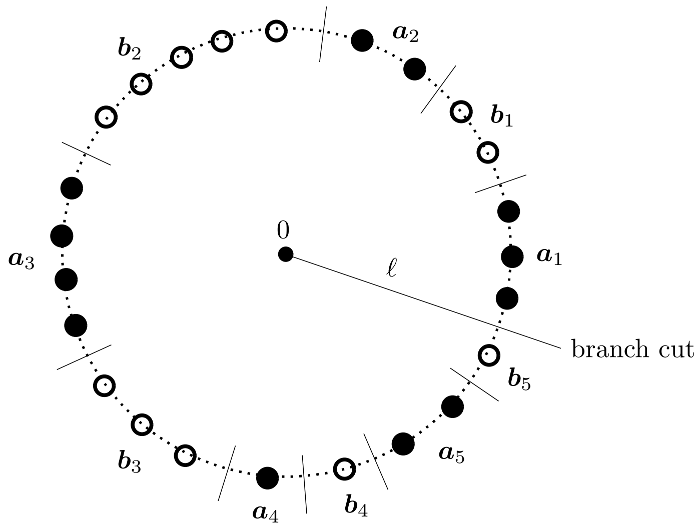

Figure 1: Clusters of $\boldsymbol{a}_{\rm on}$ and $\boldsymbol{b}_{\rm on}$ when $t = 5$.

3.1 Clusters and Index

Let (p, q) be the signature of the $H(\boldsymbol{a}, \boldsymbol{b})$-invariant Hermitian form on $\mathbb{C}^n$. Under the condition

$$\boldsymbol{a}_{\rm off} = \boldsymbol{b}_{\rm off} = \emptyset, \tag{19}$$

Beukers and Heckman [4, Theorem 4.5] gave a formula for the index $p - q$ (up to sign). We can state a refined version of it in terms of the clusters of $\boldsymbol{a}_{\rm on}$ and $\boldsymbol{b}_{\rm on}$ without assuming (19).

Theorem 3.2 *If $\boldsymbol{a}_{\rm on}$ and $\boldsymbol{b}_{\rm on}$ are nonempty then the invariant Hermitian form has index*

$$p - q = \varepsilon \sum_{k \in \mathrm{K}} (-1)^{\tau_k} \qquad \textit{with} \quad \mathrm{K} := \{\, k = 1, \dots, t \,:\, |\boldsymbol{a}_k| \equiv 1 \bmod 2 \,\}, \tag{20}$$

where $\varepsilon = \pm 1$ is the sign of $\sin \pi\gamma \in \mathbb{R}^\times$ with γ given in (18) *and τ_k is defined by*

$$\tau_1 := 0; \qquad \tau_k := |\boldsymbol{a}_1| + |\boldsymbol{b}_1| + \cdots + |\boldsymbol{a}_{k-1}| + |\boldsymbol{b}_{k-1}|, \quad k = 2, \dots, t.$$

If at least one of $\boldsymbol{a}_{\rm on}$ and $\boldsymbol{b}_{\rm on}$ is empty then the index $p - q$ is zero.

Proof. Under normalization (11) the Gram matrix G in Theorem 2.1 and hence the invariant Hermitian form depends continuously on (A, B). Its index is kept invariant by any continuous deformation of (A, B) as far as conditions (4), (5) and (9) are satisfied. While keeping these conditions, we can continuously unfold any multiple element of $\boldsymbol{a}$ into a cluster of simple elements. Thus we may assume from the beginning that $a_1, \dots, a_n$ are mutually distinct. After rearranging the indices of a_i and b_j if necessary, we may further assume that

$$\boldsymbol{a}_{\rm on} = \{a_1, \dots, a_l\}, \quad \boldsymbol{a}_{\rm off} = \{a_{l+1}, \dots, a_n\}, \quad a_{l+2i-1} = a_{l+2i}^\dagger, \quad i = 1, \dots, m,$$

$$\boldsymbol{b}_{\rm on} = \{b_1, \dots, b_d\}, \quad \boldsymbol{b}_{\rm off} = \{b_{d+1}, \dots, b_n\}, \quad b_{d+2i-1} = b_{d+2i}^\dagger, \quad i = 1, \dots, e,$$

where $n = l + 2m = d + 2e$. Suppose that both of $\boldsymbol{a}_{\rm on}$ and $\boldsymbol{b}_{\rm on}$ are nonempty, that is, $l \ge 1$ and $d \ge 1$. Since the Hermitian matrix $\varLambda_i$ in (16) has null index, we have

$$p - q = l_+ - l_-, \qquad l_\pm := \#\{\, i = 1, \dots, l \,:\, \pm\lambda_i > 0 \,\}. \tag{21}$$

For x, $y \in \mathbb{R}^\times$ we write $x \sim y$ if x and y have the same sign. We claim that

$$\lambda_i \sim \sigma_i := \varepsilon \cdot \frac{\prod_{j=1}^{d} \sin \pi(\beta_j - \alpha_i)}{\prod_{j=1}^{*\,l} \sin \pi(\alpha_j - \alpha_i)} \in \mathbb{R}^\times, \qquad i = 1, \dots, l, \tag{22}$$

where $\prod_j^*$ is the product avoiding $j = i$. Indeed, equation (15) together with (11) yields

$$\lambda_i = \frac{|1-c| \prod_{j=1}^{d}(a_i - b_j) \prod_{j=1}^{e}(a_i - b_{d+2j}^{\dagger})(a_i - b_{d+2j})}{(1-c)\, a_i \prod_{j=1}^{*\,l}(a_i - a_j) \prod_{j=1}^{m}(a_i - a_{l+2j}^{\dagger})(a_i - a_{l+2j})}$$

for $i = 1, \dots, l$. To evaluate the right-hand side we use the following identities:

$$\begin{aligned} n &= l + 2m = d + 2e, \\ 1 - c &= -c^{\frac{1}{2}}(c^{\frac{1}{2}} - c^{-\frac{1}{2}}) = -2\mathrm{i} \cdot c^{\frac{1}{2}} \cdot \sin \pi\gamma, \\ u - v &= -u^{\frac{1}{2}} v^{\frac{1}{2}}(u^{-\frac{1}{2}} v^{\frac{1}{2}} - u^{\frac{1}{2}} v^{-\frac{1}{2}}) = -2\mathrm{i} \cdot u^{\frac{1}{2}} v^{\frac{1}{2}} \cdot \sin \pi(\phi - \theta), \\ (u - w^{\dagger})(u - w) &= -uw \cdot |u - w^{\dagger}|^2 = -u(w^{\dagger} w)^{\frac{1}{2}} \cdot |w||u - w^{\dagger}|^2, \end{aligned}$$

for $u = \mathrm{e}^{2\pi \mathrm{i}\, \theta} \in S^1$, $v = \mathrm{e}^{2\pi \mathrm{i}\, \phi} \in S^1$ and $w \in \mathbb{C}^{\times}$. Some calculations yield

$$\lambda_i = \mu_i \cdot \sigma_i, \qquad \mu_i := 2^{d-l+1} \cdot \frac{\prod_{j=1}^{e} |b_{d+2j}||a_i - b_{d+2j}^{\dagger}|^2}{\prod_{j=1}^{m} |a_{l+2j}||a_i - a_{l+2j}^{\dagger}|^2} > 0$$

for $i = 1, \dots, l$ and hence claim (22) is proved.

Relation (22) readily shows that the sign $\varepsilon_i = \pm 1$ of λ_i is determined by

$$\varepsilon_i = \varepsilon \cdot (-1)^{r_i}, \quad r_i := \#\{j = 1, \dots, l \,:\, \alpha_j < \alpha_i\} + \#\{j = 1, \dots, d \,:\, \beta_j < \alpha_i\}, \tag{23}$$

where $\arg a_j = 2\pi\alpha_j$ and $\arg b_j = 2\pi\beta_j$. If a_i is the δ_i-th smallest element of $\boldsymbol{a}_k$ in the argument then $r_i = \tau_k + \delta_i - 1$ and $\varepsilon_i = \varepsilon \cdot (-1)^{\tau_k + \delta_i - 1}$ by (23). Since δ_i ranges over $1, \dots, |\boldsymbol{a}_k|$ as a_i runs through $\boldsymbol{a}_k$, one has

$$\sum_{a_i \in \boldsymbol{a}_k} (-1)^{\delta_i - 1} = \begin{cases} 1 & (|\boldsymbol{a}_k| \equiv 1 \bmod 2), \\ 0 & (|\boldsymbol{a}_k| \equiv 0 \bmod 2). \end{cases}$$

Thus it follows from formula (21) that

$$p - q = l_+ - l_- = \sum_{i=1}^{l} \varepsilon_i = \varepsilon \sum_{k=1}^{t} (-1)^{\tau_k} \sum_{a_i \in \boldsymbol{a}_k} (-1)^{\delta_i - 1} = \varepsilon \sum_{k \in \mathrm{K}} (-1)^{\tau_k},$$

which establishes formula (20). When $\boldsymbol{a}_{\mathrm{on}}$ is empty, the index is zero as we have $l = 0$ in (16). When $\boldsymbol{b}_{\mathrm{on}}$ is empty, replace $\boldsymbol{a}$ with $\boldsymbol{b}$ and proceed in a similar manner. □

Remark 3.3 The following remarks are helpful in applying Theorem 3.2.

(1) Formula (20) is invariant under any cyclic permutation of the indices k for $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$.

(2) Note that $|p-q| \le |\mathrm{K}| \le t \le n$ and $|p-q| \equiv |\mathrm{K}| \equiv n \bmod 2$; moreover $|\mathrm{K}| = t$ if and only if all $\boldsymbol{a}_{\mathrm{on}}$-clusters $\boldsymbol{a}_1, \dots, \boldsymbol{a}_t$ have odd cardinalities.

(3) We may exchange the roles of $\boldsymbol{a}$ and $\boldsymbol{b}$ in Theorem 3.2.

3.2 Lorentzian Hypergeometric Groups

It is interesting to ask when the invariant Hermitian form is definite or Lorentzian. Under condition (19) Beukers and Heckman [4, Corollary 4.7] obtained the interlacing criterion for definiteness, while Fuchs, Meiri and Sarnak [7, §2.2] derived the so-called almost interlacing criterion for the Lorentzian case. Even without assuming (19) a priori, Theorem 3.2 readily implies that the Hermitian form is definite if and only if $[\boldsymbol{a}_{\mathrm{on}}] = [\boldsymbol{b}_{\mathrm{on}}] = 1^n$ and $\boldsymbol{a}_{\mathrm{off}} = \boldsymbol{b}_{\mathrm{off}} = \emptyset$. For the Lorentzian case we have the following classification, where types 1 and 2 appear in [7].

Theorem 3.4 *The Lorentzian case is classified into five types in Table 4. In type 1 we mean by "doubles adjacent" that the unique double cluster in $\boldsymbol{a}_{\mathrm{on}}$ and the unique double cluster in $\boldsymbol{b}_{\mathrm{on}}$ must be adjacent to each other. For the other types there are no constraints on the location of multiple clusters.*

Table 4: Lorentzian case.

type	$[\boldsymbol{a}_{\mathrm{on}}]$	$[\boldsymbol{b}_{\mathrm{on}}]$	constraint	$\lvert\boldsymbol{a}_{\mathrm{off}}\rvert$	$\lvert\boldsymbol{b}_{\mathrm{off}}\rvert$
1	$1^{n-2}2^1$	$1^{n-2}2^1$	doubles adjacent	0	0
2	$1^{n-3}3^1$	$1^{n-3}3^1$		0	0
3	$1^{n-3}3^1$	1^{n-2}		0	2
4	1^{n-2}	$1^{n-3}3^1$		2	0
5	1^{n-2}	1^{n-2}		2	2

Proof. It follows from item (2) of Remark 3.3 that $|p-q| = n-2 \le |\mathrm{K}| \le t \le n$ and $|\mathrm{K}| \equiv n \bmod 2$. So we have either $|\mathrm{K}| = t = n$ or $|\mathrm{K}| = n-2$, but $t = n$ is ruled out as it would lead to the definite case. Thus $|\mathrm{K}| = n-2$ and $t = n-2,\, n-1$.

First we consider the case $t = n-1$. If follows from $n \ge |\boldsymbol{a}_{\mathrm{on}}| \ge t = n-1$; $n \equiv |\boldsymbol{a}_{\mathrm{on}}| \bmod 2$ and $|\mathrm{K}| = n-2$ that $[\boldsymbol{a}_{\mathrm{on}}] = 1^{n-2}2^1$ and $|\boldsymbol{a}_{\mathrm{off}}| = 0$. One has also $[\boldsymbol{b}_{\mathrm{on}}] = 1^{n-2}2^1$ and $|\boldsymbol{b}_{\mathrm{off}}| = 0$ by (3) of Remark 3.3. Let $k,\, l \in \{1,\dots,n-1\}$ be the indices such that $|\boldsymbol{a}_k| = 2$ and $|\boldsymbol{b}_l| = 2$. By (1) of Remark 3.3 we may assume $k = n-1$ and hence $\mathrm{K} = \{1,\dots,n-2\}$. Index formula (20) now reads $|\sum_{k=1}^{n-2}(-1)^{\tau_k}| = n-2$, which implies $\tau_k \equiv \tau_1 = 0 \bmod 2$ for $k = 2,\dots,n-2$ and hence $|\boldsymbol{a}_k| + |\boldsymbol{b}_k| = 1 + |\boldsymbol{b}_k| \equiv 0 \bmod 2$ for $k = 1,\dots,n-3$, which in turn forces $l = n-2$ or $l = n-1$, that is, $\boldsymbol{a}_k$ and $\boldsymbol{b}_l$ must be adjacent. This case falls into type 1 of Table 4.

Next we proceed to the case $t = n-2$. Since $|\mathrm{K}| = n-2 = t$, all of $|\boldsymbol{a}_1|,\dots,|\boldsymbol{a}_{n-2}|$ must be odd by item (2) of Remark 3.3. It then follows from $n \ge |\boldsymbol{a}_{\mathrm{on}}| \ge n-2$ and $n \equiv |\boldsymbol{a}_{\mathrm{on}}| \bmod 2$ that $\boldsymbol{a}$ must satisfy either (A1) $[\boldsymbol{a}_{\mathrm{on}}] = 1^{n-3}3^1$, $|\boldsymbol{a}_{\mathrm{off}}| = 0$; or (A2) $[\boldsymbol{a}_{\mathrm{on}}] = 1^{n-2}$, $|\boldsymbol{a}_{\mathrm{off}}| = 2$. By item (3) of Remark 3.3, $\boldsymbol{b}$ must also satisfy either (B1) $[\boldsymbol{b}_{\mathrm{on}}] = 1^{n-3}3^1$, $|\boldsymbol{b}_{\mathrm{off}}| = 0$; or (B2) $[\boldsymbol{b}_{\mathrm{on}}] = 1^{n-2}$, $|\boldsymbol{b}_{\mathrm{off}}| = 2$. Then the combinations (A1)-(B1), (A1)-(B3), (A3)-(B1), (A3)-(B3) lead to types 2, 3, 4, 5 in Table 4, respectively. The converse implication is easy to verify. □

3.3 Local Index

Let $E(\nu)$ be the generalized eigenspace of A corresponding to an eigenvalue $\nu \in \boldsymbol{a}$. Note that $m(\nu) := \dim E(\nu)$ is the multiplicity of ν in the multi-set $\boldsymbol{a}$. Put $E(\mu,\mu^\dagger) := E(\mu) \oplus E(\mu^\dagger)$ for $\mu \in \boldsymbol{a}_{\mathrm{off}}$. Some linear algebra shows that non-degeneracy and A-invariance of the Hermitian form lead to an orthogonal direct sum decomposition

$$\mathbb{C}^n = \bigoplus_{|\lambda|=1} E(\lambda) \,\oplus\, \bigoplus_{|\mu|>1} E(\mu,\mu^\dagger), \tag{24}$$

where λ ranges over all distinct elements in $\boldsymbol{a}_{\mathrm{on}}$ and μ ranges over all distinct elements in $\boldsymbol{a}_{\mathrm{off}}$ such that $|\mu| > 1$. The Hermitian form is non-degenerate on $E(\lambda)$ and $E(\mu,\mu^\dagger)$, though it is null on $E(\mu)$ and $E(\mu^\dagger)$ individually. Note that $m(\mu) = m(\mu^\dagger)$. It is interesting to find the index of the Hermitian form restricted to $E(\lambda)$ or $E(\mu,\mu^\dagger)$. The same problem also makes sense with B and $\boldsymbol{b}$ in place of A and $\boldsymbol{a}$, where if $\nu \in \boldsymbol{b}$ then $E(\nu)$ is understood to be the generalized ν-eigenspace of B. Thanks to (4) or (4′) the common notation $m(\nu)$ is allowed for $\nu \in \boldsymbol{a} \cup \boldsymbol{b}$, because $m(\nu)$ is the same as the multiplicity of ν in $\varphi(z) \cdot \psi(z)$.

For two distinct elements $\lambda,\, \lambda' \in S^1$ we say that λ' is *smaller* than λ if $\arg\lambda' < \arg\lambda$ with respect to the argument defined in (17). In this case we write $\lambda \prec \lambda'$. For any $\lambda \in S^1$ let

$$r(\lambda) := \#\text{ of all elements in } \boldsymbol{a}_{\mathrm{on}} \cup \boldsymbol{b}_{\mathrm{on}} \text{ that are smaller than } \lambda, \tag{25}$$

where # denotes the cardinality counted with multiplicities.

Proposition 3.5 *For each $\lambda \in \boldsymbol{a}_{\mathrm{on}} \cup \boldsymbol{b}_{\mathrm{on}}$ the Hermitian form restricted to $E(\lambda)$ has index*

$$\mathrm{idx}(\lambda) := \begin{cases} \varepsilon \cdot (-1)^{r(\lambda)} & \text{if } \lambda \in \boldsymbol{a}_{\mathrm{on}} \text{ and } m(\lambda) \text{ is odd}, \\ \varepsilon \cdot (-1)^{r(\lambda)+1} & \text{if } \lambda \in \boldsymbol{b}_{\mathrm{on}} \text{ and } m(\lambda) \text{ is odd}, \\ 0 & \text{if } m(\lambda) \text{ is even}, \end{cases} \tag{26}$$

where $\varepsilon = \pm 1$ *is the sign of* $\sin \pi\gamma$ *mentioned in Theorem* 3.2. *For each* $\mu \in \boldsymbol{a}_{\text{off}} \cup \boldsymbol{b}_{\text{off}}$ *with* $|\mu| > 1$ *the Hermitian form restricted to* $E(\mu, \mu^\dagger)$ *has null index, that is,*

$$\text{idx}(\mu) = 0. \tag{27}$$

Proof. First we show (26) and (27) for $\lambda \in \boldsymbol{a}_{\text{on}}$ and $\mu \in \boldsymbol{a}_{\text{off}}$. In the special case where $a_1, \dots, a_n$ are distinct and hence λ and μ are simple, they are direct consequences of (23) and (16) respectively. Here we remark that if $\lambda = a_i \in \boldsymbol{a}_{\text{on}}$ then $r(\lambda) = r_i$ in (23). In the general case they are obtained by a perturbation argument (see e.g. Kato [15, Chapters I-II]). If $D \subset \mathbb{C}$ is a bounded domain whose boundary contains no eigenvalue of A and $V \subset \mathbb{C}^n$ is the direct sum of generalized eigenspaces corresponding to the eigenvalues of A in D, then

$$P = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} (zI - A)^{-1} \, dz$$

yields the projection onto V. For any small continuous perturbation A_t of $A = A_0$ the associated projection P_t can be expressed as $P_t = U_t P U_t^{-1}$, where $U_0 = I$ and $U_t \in \mathrm{GL}(n, \mathbb{C})$ depends continuously on t [15, I-(4.42)], so the corresponding space V_t varies continuously in t. If h_t is a continuous family of Hermitian forms on $\mathbb{C}^n$ such that $h = h_0$ is non-degenerate on $V = V_0$, then the index of $h_t|V_t$ is locally constant in t. These general results can be applied to $V = E(\lambda)$ when $\lambda \in \boldsymbol{a}_{\text{on}}$ is a multiple element. Keeping conditions (4), (5) and (9), slightly perturb A so that $\boldsymbol{a}$ is unfolded into simple elements, with λ split into $\lambda_1 \prec \cdots \prec \lambda_m$ and $E(\lambda)$ deformed into $E(\lambda_1) \oplus \cdots \oplus E(\lambda_m)$, where $m = m(\lambda)$. It follows from the special case that $\text{idx}(\lambda_j) = \varepsilon \cdot (-1)^{r(\lambda)+j-1}$ for $j = 1, \dots, m$, and hence

$$\text{idx}(\lambda) = \sum_{j=1}^{m} \text{idx}(\lambda_j) = \varepsilon \cdot (-1)^{r(\lambda)} \sum_{j=1}^{m} (-1)^{j-1} = \begin{cases} \varepsilon \cdot (-1)^{r(\lambda)} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

which proves (26). Similarly, for a multiple $\mu \in \boldsymbol{a}_{\text{off}}$, $E(\mu, \mu^\dagger)$ is deformed into $E(\mu_1, \mu_1^\dagger) \oplus \cdots \oplus E(\mu_m, \mu_m^\dagger)$ with $\mu_1, \dots, \mu_m$ all being simple. Since $\text{idx}(\mu_j) = 0$ for $j = 1, \dots, m$, formula (27) follows.

Next we can show the results for $\lambda \in \boldsymbol{b}_{\text{on}}$ and $\mu \in \boldsymbol{b}_{\text{off}}$ in a similar manner by exchanging the roles of $\boldsymbol{a}$ and $\boldsymbol{b}$. Notice that $r_a(\lambda) := r(\lambda)$ and $\varepsilon_a := \varepsilon$ appearing in (26) are defined with respect to the branch cut $\ell_a := \ell$ lying between $\boldsymbol{b}_t$ and $\boldsymbol{a}_1$ as in Figure 1. Changing the roles of $\boldsymbol{a}$ and $\boldsymbol{b}$ we should replace ℓ_a by a new branch cut ℓ_b lying between $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$ and consider the corresponding $r_b(\lambda)$ and ε_b. For $\lambda \in \boldsymbol{b}_{\text{on}}$ one has $r_b(\lambda) = r_a(\lambda) - |\boldsymbol{a}_1|$ and $\varepsilon_b = -\varepsilon_a \cdot (-1)^{|\boldsymbol{a}_1|}$, where Remark 3.1 is used to obtain the latter relation. Thus $\varepsilon_b \cdot (-1)^{r_b(\lambda)} = \varepsilon_a \cdot (-1)^{r_a(\lambda)+1}$, which proves (26) for $\lambda \in \boldsymbol{b}_{\text{on}}$. The proof of (27) for $\mu \in \boldsymbol{b}_{\text{off}}$ is the same as that for $\mu \in \boldsymbol{a}_{\text{off}}$. □

Remark 3.6 Due to (27) the sum of $\text{idx}(\lambda)$ over all distinct elements λ of $\boldsymbol{a}_{\text{on}}$ (or of $\boldsymbol{b}_{\text{on}}$) is equal to the global index $p - q$ of the invariant Hermitian form on the whole space $\mathbb{C}^n$.

4 Real Hypergeometric Groups

A hypergeometric group $H = H(\varphi, \psi) = H(\boldsymbol{a}, \boldsymbol{b}) = \langle A, B \rangle$ is said to be *real* if

$$\varphi(z) \in \mathbb{R}[z], \quad \psi(z) \in \mathbb{R}[z], \quad \text{or equivalently} \quad \bar{\boldsymbol{a}} = \boldsymbol{a}, \quad \bar{\boldsymbol{b}} = \boldsymbol{b}.$$

In the rest of this article we always assume that H is a real hypergeometric group. We also retain the assumptions (4), (5) and (9). Let $L_{\mathbb{R}}$ be the $\mathbb{R}$-linear span of $\boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}$. These vectors form an $\mathbb{R}$-linear basis of $L_{\mathbb{R}}$ as they are a $\mathbb{C}$-linear basis of $\mathbb{C}^n$ by Theorem 2.1. Obviously A preserves $L_{\mathbb{R}}$. Since $\xi_i \in \mathbb{R}$ for every $i \in \mathbb{Z}_{\geq 0}$, formula (13) shows that the invariant Hermitian form is $\mathbb{R}$-valued on $L_{\mathbb{R}}$. The matrix $C = A^{-1}B$ acts on $L_{\mathbb{R}}$ as a real reflection because formulas (10a) and (11) read

$$C\boldsymbol{v} = \boldsymbol{v} - (\boldsymbol{v}, \boldsymbol{r})\boldsymbol{r}, \quad \boldsymbol{v} \in L_{\mathbb{R}} \quad \text{with} \quad (\boldsymbol{r}, \boldsymbol{r}) = 2. \tag{28}$$

So $L_{\mathbb{R}}$ is also preserved by $B = AC$ and hence by the whole group H, thus $H \subset O(L_{\mathbb{R}})$.

4.1 Trace Polynomials

Conditions (6) implies $\varphi(0) = \pm 1$ and $\psi(0) = \pm 1$, while assumption (9) forces

$$c = -1, \qquad \zeta = 1, \qquad (\boldsymbol{r}, \boldsymbol{r}) = 2, \tag{29}$$

in (11), hence $\varphi(0)$ and $\psi(0)$ must have opposite signs. Thus after exchanging $\varphi(z)$ and $\psi(z)$ if necessary we may assume $\varphi(0) = -1$ and $\psi(0) = 1$ so that (5′) becomes $z^n \varphi(z^{-1}) = -\varphi(z)$ and $z^n \psi(z^{-1}) = \psi(z)$, that is, $\varphi(z)$ is anti-palindromic while $\psi(z)$ is palindromic.

In general a palindromic polynomial $f(z)$ of even degree $2d$ can be expressed as $f(z) = z^d F(z + z^{-1})$ for a unique polynomial $F(w)$ of degree d, a palindromic polynomial $f(z)$ of odd degree factors as $f(z) = (z+1)g(z)$ with $g(z)$ being palindromic of even degree, and an anti-palindromic polynomial $f(z)$ factors as $f(z) = (z-1)g(z)$ with $g(z)$ being palindromic. Hence there exist unique monic real polynomials $\Phi(w)$ and $\Psi(w)$ such that

$$\varphi(z) = (z^2-1) z^{N-1} \Phi(z + z^{-1}), \quad \psi(z) = z^N \Psi(z + z^{-1}), \qquad \text{if } n = 2N; \tag{30a}$$

$$\varphi(z) = (z-1) z^{N} \Phi(z + z^{-1}), \quad \psi(z) = (z+1) z^N \Psi(z + z^{-1}), \quad \text{if } n = 2N+1. \tag{30b}$$

We refer to $\Phi(w)$ and $\Psi(w)$ as the *trace* polynomials of $\varphi(z)$ and $\psi(z)$. The real hypergeometric group $H = H(\varphi, \psi)$ can also be expressed as $H = H(\Phi, \Psi)$. The resultant of (φ, ψ) and that of (Φ, Ψ) are related by

$$\mathrm{Res}(\varphi, \psi) = (-1)^N \cdot \Psi(2) \cdot \Psi(-2) \cdot \mathrm{Res}(\Phi, \Psi)^2, \qquad \text{if } n = 2N; \tag{31a}$$

$$\mathrm{Res}(\varphi, \psi) = 2(-1)^N \cdot \Psi(2) \cdot \Phi(-2) \cdot \mathrm{Res}(\Phi, \Psi)^2, \qquad \text{if } n = 2N+1. \tag{31b}$$

To avoid digression from the main line, we give a proof of (31) in Appendix A.

It follows from (4′) and (31) that $\mathrm{Res}(\Phi, \Psi) \neq 0$ hence $\Phi(w)$ and $\Psi(w)$ have no root in common. By formulas (30) the roots $\lambda \neq \pm 1$ of $\varphi(z)$ are in two-to-one correspondence with the roots $\tau \neq \pm 2$ of $\Phi(w)$ via the relation $\tau = \lambda + \lambda^{-1}$, since $w - \tau = z^{-1}(z - \lambda)(z - \lambda^{-1})$ with $w = z + z^{-1}$. The same statement is true for $\psi(z)$ and $\Psi(w)$. Moreover we have

$$m(\lambda) = \begin{cases} M(\tau) & \text{if } \lambda \neq \pm 1, \text{ i.e. } \tau \neq \pm 2, \\ 2M(\tau) + 1 & \text{if } \lambda = \pm 1, \text{ i.e. } \tau = \pm 2, \end{cases} \tag{32}$$

under $\tau := \lambda + \lambda^{-1}$, where $M(\tau)$ is the multiplicity of $w = \tau$ in the equation $\Phi(w) \cdot \Psi(w) = 0$.

4.2 Trace Clusters and Index

In the real case the index formula (20) in Theorem 3.2 can be restated in terms of what we call trace clusters. In view of formulas (30), if n is even then $\pm 1 \in \boldsymbol{a}_{\mathrm{on}}$ and hence $\boldsymbol{a}_{\mathrm{on}}$ is nonempty (but $\boldsymbol{b}_{\mathrm{on}}$ may be empty), while if n is odd then $1 \in \boldsymbol{a}_{\mathrm{on}}$, $-1 \in \boldsymbol{b}_{\mathrm{on}}$ and hence both of $\boldsymbol{a}_{\mathrm{on}}$ and $\boldsymbol{b}_{\mathrm{on}}$ are nonempty. In any case, as far as both of them are nonempty, the clusters $\boldsymbol{a}_1, \boldsymbol{b}_1, \dots, \boldsymbol{a}_t, \boldsymbol{b}_t$ can be indexed so that $1 \in \boldsymbol{a}_1$. With this convention it is easy to see that

$$\text{if } n \text{ is even then } t = 2s \text{ is also even, } -1 \in \boldsymbol{a}_{s+1} \text{ and } |\boldsymbol{a}_1| \equiv |\boldsymbol{a}_{s+1}| \equiv 1 \bmod 2, \tag{33a}$$

$$\text{if } n \text{ is odd then } t = 2s-1 \text{ is also odd, } -1 \in \boldsymbol{b}_{s} \text{ and } |\boldsymbol{a}_1| \equiv |\boldsymbol{b}_{s}| \equiv 1 \bmod 2. \tag{33b}$$

In either case, with the convention $\boldsymbol{a}_{t+1} = \boldsymbol{a}_1$, $\boldsymbol{b}_{t+1} = \boldsymbol{b}_1$, we have

$$\bar{\boldsymbol{a}}_i = \boldsymbol{a}_{t+2-i}, \qquad \bar{\boldsymbol{b}}_i = \boldsymbol{b}_{t+1-i}, \qquad i = 1, \dots, s. \tag{34}$$

Note that $\boldsymbol{a}_2, \dots, \boldsymbol{a}_s$ are exactly those $\boldsymbol{a}_{\mathrm{on}}$-clusters which lie in the upper half-plane $\mathrm{Im}\, z > 0$.

Let $\boldsymbol{A}$ be the multi-set of all complex roots of $\Phi(w)$ and $\boldsymbol{B}$ be its $\Psi(w)$-counterpart. Let $\boldsymbol{A}_{\mathrm{on}}$ resp. $\boldsymbol{A}_{\mathrm{off}}$ be the component of $\boldsymbol{A}$ whose elements lie on resp. off $[-2, 2]$. Let $\boldsymbol{B}_{\mathrm{on}}$ and $\boldsymbol{B}_{\mathrm{off}}$ be defined in a similar manner for $\boldsymbol{B}$. Notice that both of $\boldsymbol{a}_{\mathrm{on}}$ and $\boldsymbol{b}_{\mathrm{on}}$ are nonempty if and only if

$$\text{either } n \text{ is even and } \boldsymbol{B}_{\mathrm{on}} \text{ is nonempty, or } n \text{ is odd}, \tag{35}$$

in which case $\boldsymbol{A}_{\mathrm{on}}$ and $\boldsymbol{B}_{\mathrm{on}}$ dissect each other into interlacing components called *trace clusters*,

$$\boldsymbol{A}_{s+1}, \boldsymbol{B}_s, \boldsymbol{A}_s, \dots, \boldsymbol{B}_1, \boldsymbol{A}_1 \quad \text{if } n \text{ is even}; \quad \boldsymbol{B}_s, \boldsymbol{A}_s, \dots, \boldsymbol{B}_1, \boldsymbol{A}_1 \quad \text{if } n \text{ is odd},$$

where one or both of the end clusters may be empty but all the other clusters must be nonempty. Put

$$\boldsymbol{A}_{\mathrm{in}} := \boldsymbol{A}_2 \cup \cdots \cup \boldsymbol{A}_s. \tag{36}$$

Finally, let $\boldsymbol{A}_{>2}$ be the components of $\boldsymbol{A}_{\mathrm{off}}$ whose elements are real numbers greater than 2 and $\boldsymbol{B}_{>2}$ be defined in a similar manner for $\boldsymbol{B}$.

Lemma 4.1 *Let $\gamma \in \mathbb{R} \setminus \mathbb{Z}$ be the number defined in* (18). *Under assumption* (35) *we have*

$$\varepsilon = \sin \pi\gamma = (-1)^{|\boldsymbol{A}_1|+|\boldsymbol{A}_{>2}|+|\boldsymbol{B}_{>2}|}. \tag{37}$$

Proof. We consider how each component of $\boldsymbol{a}$ contributes to the sum $2\pi\alpha := 2\pi\alpha_1 + \cdots + 2\pi\alpha_n$, where $2\pi\alpha_i := \arg a_i$. The 1's in $\boldsymbol{a}_{\mathrm{on}}$ has no contribution. The -1's in $\boldsymbol{a}_{\mathrm{on}}$ has contribution πm_a^- where m_a^- is the multiplicity of -1 in $\boldsymbol{a}$. For each non-real pair $\lambda, \bar{\lambda} \in \boldsymbol{a}_{\mathrm{on}}$ with $\operatorname{Im}\lambda > 0$, the sum $\arg\lambda + \arg\bar{\lambda}$ is 0 if $\lambda \in \boldsymbol{a}_1$ and 2π if $\lambda \notin \boldsymbol{a}_1$. So the total contribution of $\boldsymbol{a}_{\mathrm{on}}$ is given by

$$\pi m_a^- + 2\pi \cdot \frac{|\boldsymbol{a}_{\mathrm{on}}| - |\boldsymbol{a}_1| - m_a^-}{2} = \pi(|\boldsymbol{a}_{\mathrm{on}}| - |\boldsymbol{a}_1|).$$

Let $\boldsymbol{a}_{<-1}$ resp. $\boldsymbol{a}_{>1}$ be the component of $\boldsymbol{a}_{\mathrm{off}}$ whose elements are real numbers < -1 resp. > 1. For each real pair λ, $\lambda^\dagger \in \boldsymbol{a}_{\mathrm{off}}$, the sum $\arg\lambda + \arg\lambda^\dagger$ is 0 if $\lambda \in \boldsymbol{a}_{>1}$ and 2π if $\lambda \in \boldsymbol{a}_{<-1}$. For each non-real quartet λ, $\bar{\lambda}$, $\lambda^\dagger$, $\bar{\lambda}^\dagger \in \boldsymbol{a}_{\mathrm{off}}$ with $|\lambda| > 1$ and $\operatorname{Im}\lambda > 0$ we have

$$\arg\lambda + \arg\bar{\lambda} + \arg\lambda^\dagger + \arg\bar{\lambda}^\dagger = \begin{cases} 0 & (0 < \arg\lambda \le |\Theta|), \\ 4\pi & (|\Theta| < \arg\lambda < \pi), \end{cases}$$

where Θ is the number appearing in (17), which belongs to the interval $(-\pi, 0)$ due to assumption (35). Therefore the contribution of $\boldsymbol{a}_{\mathrm{off}}$ to the sum $2\pi\alpha$ is $2\pi|\boldsymbol{a}_{<-1}| \bmod 4\pi\mathbb{Z}$. In total we have $2\pi\alpha \equiv \pi(|\boldsymbol{a}_{\mathrm{on}}| - |\boldsymbol{a}_1|) + 2\pi|\boldsymbol{a}_{<-1}| \bmod 4\pi\mathbb{Z}$, which yields a modulo 2 congruence

$$\alpha \equiv \frac{|\boldsymbol{a}_{\mathrm{on}}| - |\boldsymbol{a}_1|}{2} + |\boldsymbol{a}_{<-1}| \mod 2.$$

In a similar manner we consider how each component of $\boldsymbol{b}$ contributes to the sum $2\pi\beta := 2\pi\beta_1 + \cdots + 2\pi\beta_n$, where $2\pi\beta_i := \arg b_i$. Taking $1 \notin \boldsymbol{b}$ into account we find that

$$\beta \equiv \frac{|\boldsymbol{b}_{\mathrm{on}}|}{2} + |\boldsymbol{b}_{<-1}| \mod 2.$$

Since $\gamma = \beta - \alpha$, we use relations $|\boldsymbol{a}_{\mathrm{on}}| + |\boldsymbol{a}_{\mathrm{off}}| = n$, $|\boldsymbol{a}_{\mathrm{off}}| \equiv 2|\boldsymbol{a}_{<-1}| + 2|\boldsymbol{a}_{>1}| \bmod 4$ and their $\boldsymbol{b}$-counterparts together with $|\boldsymbol{a}_1| = 2|\boldsymbol{A}_1| + 1$ to obtain a modulo 2 congruence

$$\begin{aligned}\gamma &\equiv \frac{|\boldsymbol{b}_{\mathrm{on}}| - |\boldsymbol{a}_{\mathrm{on}}| + |\boldsymbol{a}_1|}{2} + |\boldsymbol{b}_{<-1}| - |\boldsymbol{a}_{<-1}| \equiv \frac{|\boldsymbol{a}_1|}{2} + |\boldsymbol{a}_{>1}| - |\boldsymbol{b}_{>1}| \\ &= \frac{1}{2}(2|\boldsymbol{A}_1| + 1) + |\boldsymbol{A}_{>2}| - |\boldsymbol{B}_{>2}| \equiv \frac{1}{2} + |\boldsymbol{A}_1| + |\boldsymbol{A}_{>2}| + |\boldsymbol{B}_{>2}| \bmod 2.\end{aligned}$$

This establishes formula (37). □

In the real case the index formula (20) in Theorem 3.2 can be restated as follows.

Theorem 4.2 *Let $H = H(\Phi, \Psi) = \langle A, B \rangle$ be a real hypergeometric group of rank n. If condition* (35) *is satisfied then the index of the H-invariant Hermitian form is given by*

$$p - q = \varepsilon(1 + \delta - 2S), \tag{38}$$

where $\varepsilon = \pm 1$ is given in formula (37) *while δ and S are defined by*

$$\delta := \begin{cases} (-1)^{|\boldsymbol{A}_{\mathrm{in}}|+|\boldsymbol{B}_{\mathrm{on}}|+1} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \tag{39a}$$

$$S := \sum_{i \in \mathrm{I}} (-1)^{\sigma_i} \qquad \text{with} \quad \mathrm{I} := \{\, i = 2, \dots, s : |\boldsymbol{A}_i| \equiv 1 \bmod 2 \,\}, \tag{39b}$$

with $\boldsymbol{A}_{\rm in}$ *being as in* (36) *and* σ_i *defined by* $\sigma_2 := |\boldsymbol{B}_1|$ *and*

$$\sigma_i := |\boldsymbol{B}_1| + |\boldsymbol{A}_2| + |\boldsymbol{B}_2| + \cdots + |\boldsymbol{A}_{i-1}| + |\boldsymbol{B}_{i-1}|, \qquad i = 3, \dots, s.$$

If n *is even and* $\boldsymbol{B}_{\rm on}$ *is empty then the index* $p - q$ *is zero.*

Proof. If n is even and $\boldsymbol{B}_{\rm on}$ is nonempty, then (33a) and (34) imply $|\boldsymbol{a}_1| = 2|\boldsymbol{A}_1| + 1$, $|\boldsymbol{a}_{s+1}| = 2|\boldsymbol{A}_{s+1}| + 1$, $|\boldsymbol{a}_i| = |\boldsymbol{a}_{2s+2-i}| = |\boldsymbol{A}_i|$ for $i = 2, \dots, s$, and $|\boldsymbol{b}_i| = |\boldsymbol{b}_{2s+1-i}| = |\boldsymbol{B}_i|$ for $i = 1, \dots, s$, hence $\mathrm{I} = \mathrm{K} \cap \{2, \dots, s\}$ and $\mathrm{K} = \{1, s+1\} \sqcup \mathrm{I} \sqcup \{2s + 2 - i \,:\, i \in \mathrm{I}\}$, where K is defined in (20). For each $i \in \mathrm{I}$ we have $\tau_i = |\boldsymbol{a}_1| + \sigma_i \equiv 1 + \sigma_i \bmod 2$ and

$$\begin{aligned} \tau_{2s+2-i} &= |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_{2s+1-i}| + |\boldsymbol{b}_1| + \cdots + |\boldsymbol{b}_{2s+1-i}| \\ &\equiv |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_i| + |\boldsymbol{a}_{s+1}| + |\boldsymbol{b}_1| + \cdots + |\boldsymbol{b}_{i-1}| \bmod 2 \\ &= \tau_i + |\boldsymbol{a}_i| + |\boldsymbol{a}_{s+1}| \equiv \tau_i \bmod 2. \end{aligned}$$

Moreover, $\tau_{s+1} = |\boldsymbol{a}_1| + |\boldsymbol{A}_{\rm in}| + |\boldsymbol{B}_{\rm on}| \equiv 1 + |\boldsymbol{A}_{\rm in}| + |\boldsymbol{B}_{\rm on}| \bmod 2$. Formula (20) then yields

$$\begin{aligned} \varepsilon(p-q) &= \sum_{i \in \mathrm{K}} (-1)^{\tau_i} = (-1)^{\tau_1} + \sum_{i \in \mathrm{I}} (-1)^{\tau_i} + (-1)^{\tau_{s+1}} + \sum_{i \in \mathrm{I}} (-1)^{\tau_{2s+2-i}} \\ &= 1 + (-1)^{\tau_{s+1}} + 2 \sum_{i \in \mathrm{I}} (-1)^{\tau_i} = 1 + (-1)^{\tau_{s+1}} - 2 \sum_{i \in \mathrm{I}} (-1)^{\sigma_i} = 1 + \delta - 2S. \end{aligned}$$

Note that 1, δ and $-2S$ are the contributions of $\boldsymbol{a}_1$, $\boldsymbol{a}_{s+1}$ and $\boldsymbol{a}_{\rm in} := \boldsymbol{a}_2 \cup \overline{\boldsymbol{a}}_2 \cup \cdots \cup \boldsymbol{a}_s \cup \overline{\boldsymbol{a}}_s$ respectively.

If n is odd then (33b) and (34) imply $|\boldsymbol{a}_1| = 2|\boldsymbol{A}_1| + 1$, $|\boldsymbol{a}_i| = |\boldsymbol{a}_{2s+2-i}| = |\boldsymbol{A}_i|$ for $i = 2, \dots, s$, $|\boldsymbol{b}_s| = 2|\boldsymbol{B}_s| + 1$, and $|\boldsymbol{b}_i| = |\boldsymbol{b}_{2s-i}| = |\boldsymbol{B}_i|$ for $i = 1, \dots, s-1$, hence $\mathrm{I} = \mathrm{K} \cap \{2, \dots, s\}$ and $\mathrm{K} = \{1\} \sqcup \mathrm{I} \sqcup \{2s + 1 - i \,:\, i \in \mathrm{I}\}$. For each $i \in \mathrm{I}$ we have $\tau_i = |\boldsymbol{a}_1| + \sigma_i \equiv 1 + \sigma_i \bmod 2$ and

$$\begin{aligned} \tau_{2s+1-i} &= |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_{2s-i}| + |\boldsymbol{b}_1| + \cdots + |\boldsymbol{b}_{2s-i}| \\ &\equiv |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_i| + |\boldsymbol{b}_1| + \cdots + |\boldsymbol{b}_{i-1}| + |\boldsymbol{b}_s| \bmod 2 \\ &= \tau_i + |\boldsymbol{a}_i| + |\boldsymbol{b}_s| \equiv \tau_i \bmod 2. \end{aligned}$$

Thus formula (20) in Theorem 3.2 then yields

$$\begin{aligned} \varepsilon(p-q) &= \sum_{i \in \mathrm{K}} (-1)^{\tau_i} = (-1)^{\tau_1} + \sum_{i \in \mathrm{I}} (-1)^{\tau_i} + \sum_{i \in \mathrm{I}} (-1)^{\tau_{2s+1-i}} \\ &= 1 + 2 \sum_{i \in \mathrm{I}} (-1)^{\tau_i} = 1 + 0 - 2 \sum_{i \in \mathrm{I}} (-1)^{\sigma_i} = 1 + \delta - 2S. \end{aligned}$$

Here 1 and $-2S$ are the contributions of $\boldsymbol{a}_1$ and $\boldsymbol{a}_{\rm in}$ respectively, while $\delta = 0$ is a dummy term.

In either case we have obtained formula (38). If n is even and $\boldsymbol{B}_{\rm on}$ is empty, then $\boldsymbol{b}_{\rm on}$ is empty and hence the index $p - q$ is zero by the last part of Theorem 3.2. □

Lemma 4.3 *There are the following numerical constraints*

$$S \equiv |\mathrm{I}| \equiv |\boldsymbol{A}_{\rm in}| \bmod 2, \quad |S| \le |\mathrm{I}| \le s - 1 \le \frac{|\mathrm{I}| + |\boldsymbol{A}_{\rm in}|}{2} \le |\boldsymbol{A}_{\rm in}|, \quad s \le |\boldsymbol{B}_{\rm on}|. \tag{40}$$

Proof. Congruence $S \equiv |\mathrm{I}| \bmod 2$ follows from the definition of S in (39b) and $(-1)^{\sigma_i} \equiv 1 \bmod 2$. Since $|\boldsymbol{A}_i| \ge 1$ for $i = 2, \dots, s$, the definition of S and the inclusion $\mathrm{I} \subset \{2, \dots, s\}$ imply $|S| \le |\mathrm{I}| \le s - 1 \le |\boldsymbol{A}_2| + \cdots + |\boldsymbol{A}_s| = |\boldsymbol{A}_{\rm in}|$. As $|\boldsymbol{A}_i|$ is odd for $i \in \mathrm{I}$ and even for $i \in \{2, \dots, s\} \setminus \mathrm{I}$, we have $|\mathrm{I}| \equiv |\boldsymbol{A}_{\rm in}| \bmod 2$ and $|\mathrm{I}| + 2(s - 1 - |\mathrm{I}|) \le |\boldsymbol{A}_{\rm in}|$, that is, $s - 1 \le \frac{1}{2}(|\mathrm{I}| + |\boldsymbol{A}_{\rm in}|)$. Moreover we have $s \le |\boldsymbol{B}_1| + \cdots + |\boldsymbol{B}_s| = |\boldsymbol{B}_{\rm on}|$ because $|\boldsymbol{B}_i| \ge 1$ for $i = 1, \dots, s$. Putting all these together lead to the constraints (40). □

Remark 4.4 If n is even then the reflection of the trace clusters in $\boldsymbol{A}_{\rm in} \cup \boldsymbol{B}_{\rm on}$,

$$\boldsymbol{B}_s, \boldsymbol{A}_s, \boldsymbol{B}_{s-1}, \dots, \boldsymbol{B}_2, \boldsymbol{A}_2, \boldsymbol{B}_1 \quad \xrightarrow{\text{reflection}} \quad \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2, \dots, \boldsymbol{B}_{s-1}, \boldsymbol{A}_s, \boldsymbol{B}_s \tag{41}$$

results in the change of signs $S \to \delta S$ and $p - q \to \delta(p - q)$, where δ is defined in (39a).

4.3 Local Index in the Real Case

Proposition 3.5 gives formula (26) for the local index $\mathrm{idx}(\lambda)$ at $\lambda \in \boldsymbol{a}_{\mathrm{on}} \cup \boldsymbol{b}_{\mathrm{on}}$. It can be restated in terms of $\tau := \lambda + \lambda^{-1} \in \boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}}$, where we are allowed to write $\mathrm{idx}(\lambda) = \mathrm{idx}(\lambda^{-1}) = \mathrm{Idx}(\tau)$ if $\lambda \neq \pm 1$, i.e. $\tau \neq \pm 2$. To state the result let $\rho : [-2, 2] \to \mathbb{Z}_{\geq 0}$ be a function defined by

$$\rho(\tau) := \#\text{ of all real roots of } \Phi(w) \cdot \Psi(w) \text{ that are } \begin{cases} > \tau & \text{if } \tau \in [-2, 2), \\ \geq 2 & \text{if } \tau = 2, \end{cases} \tag{42}$$

where # denotes the cardinality counted with multiplicities.

Proposition 4.5 *For any $\tau \in \boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}}$ with $\tau \neq \pm 2$ we have*

$$\mathrm{Idx}(\tau) := \begin{cases} (-1)^{\rho(\tau)+1} & \text{if } \tau \in \boldsymbol{A}_{\mathrm{on}} \text{ and } M(\tau) \text{ is odd}, \\ (-1)^{\rho(\tau)} & \text{if } \tau \in \boldsymbol{B}_{\mathrm{on}} \text{ and } M(\tau) \text{ is odd}, \\ \ 0 & \text{if } M(\tau) \text{ is even}, \end{cases} \tag{43}$$

where $M(\tau)$ is the multiplicity of τ in $\boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}}$. Moreover we have

$$\mathrm{idx}(1) = (-1)^{\rho(2)}, \qquad \mathrm{idx}(-1) = (-1)^{\rho(-2)+n+1}. \tag{44}$$

Proof. First we prove (43). Take an element $\lambda \in \boldsymbol{a}_{\mathrm{on}} \cup \boldsymbol{b}_{\mathrm{on}}$ such that $\lambda + \lambda^{-1} = \tau$. We may assume $\mathrm{Im}\,\lambda > 0$ since $\mathrm{idx}(\lambda) = \mathrm{idx}(\lambda^{-1})$. If $\tau \in \boldsymbol{A}_{\mathrm{on}}$, that is, $\lambda \in \boldsymbol{a}_{\mathrm{on}}$ then definition (25) and relation $|\boldsymbol{a}_1| = 2|\boldsymbol{A}_1| + 1$ give $r(\lambda) = |(\boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}})_{>\tau}| + |\boldsymbol{A}_1| + 1$, which together with (37) and (42) yields $\varepsilon \cdot (-1)^{r(\lambda)} = (-1)^{|\boldsymbol{A}_1| + |\boldsymbol{A}_{>2}| + |\boldsymbol{B}_{>2}|} \cdot (-1)^{|(\boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}})_{>\tau}| + |\boldsymbol{A}_1| + 1} = (-1)^{\rho(\tau)+1}$. In a similar manner, if $\tau \in \boldsymbol{B}_{\mathrm{on}}$ then $\varepsilon \cdot (-1)^{r(\lambda)+1} = (-1)^{\rho(\tau)}$. Thus (43) follows from formula (26).

Next we prove (44). By (25) and (42) together with (32) we have

$$r(1) = |\boldsymbol{A}_1| - M(2), \qquad r(-1) = |\boldsymbol{A}_{\mathrm{on}}| + |\boldsymbol{B}_{\mathrm{on}}| - M(-2) + |\boldsymbol{A}_1| + 1,$$

$$\rho(2) = M(2) + |\boldsymbol{A}_{>2}| + |\boldsymbol{B}_{>2}|, \qquad \rho(-2) = |\boldsymbol{A}_{\mathrm{on}}| + |\boldsymbol{B}_{\mathrm{on}}| - M(-2) + |\boldsymbol{A}_{>2}| + |\boldsymbol{B}_{>2}|.$$

Putting these equations into (26) and using (32) and (37) we establish (44), where we take into account that $1 \in \boldsymbol{a}_{\mathrm{on}}$ while $-1 \in \boldsymbol{a}_{\mathrm{on}}$ if n is even and $-1 \in \boldsymbol{b}_{\mathrm{on}}$ if n is odd. □

Suppose that the rank n is even; the odd case is omitted as it is not necessary in this article. Let $\boldsymbol{A}_1^{\circ} := (\boldsymbol{A}_1)_{<2}$ and $\boldsymbol{A}_{s+1}^{\circ} := (\boldsymbol{A}_{s+1})_{>-2}$. For a multi-set $\boldsymbol{X}$ given in boldface, its calligraphic style $\mathcal{X}$ denotes the ordinary set of all *distinct* elements of *odd* multiplicity in $\boldsymbol{X}$. We apply this rule to $\boldsymbol{X} = \boldsymbol{A}_1^{\circ}, \boldsymbol{A}_{s+1}^{\circ}, \boldsymbol{A}_{\mathrm{in}}, \boldsymbol{B}_{\mathrm{on}}$ to define $\mathcal{X} = \mathcal{A}_1^{\circ}, \mathcal{A}_{s+1}^{\circ}, \mathcal{A}_{\mathrm{in}}, \mathcal{B}_{\mathrm{on}}$ respectively. For these sets we put $\mathrm{Idx}(\mathcal{X}) := \sum_{\tau \in \mathcal{X}} \mathrm{Idx}(\tau)$. Note that $0 \leq |\boldsymbol{X}| - |\mathcal{X}| \equiv 0 \bmod 2$. Moreover we denote by $\mathrm{Par}(m)$ the parity of $m \in \mathbb{Z}$, that is, $\mathrm{Par}(m) = 0$ for m even and $\mathrm{Par}(m) = 1$ for m odd.

Theorem 4.6 *In the situations of Theorem 4.2 with the rank n being even we have*

$$\mathrm{idx}(1) = \varepsilon \cdot (-1)^{|\boldsymbol{A}_1^{\circ}|}, \qquad \mathrm{idx}(-1) = \varepsilon\delta \cdot (-1)^{|\boldsymbol{A}_{s+1}^{\circ}|}, \tag{45a}$$

$$\mathrm{Idx}(\mathcal{A}_1^{\circ}) = \varepsilon \cdot \mathrm{Par}(|\boldsymbol{A}_1^{\circ}|), \qquad \mathrm{Idx}(\mathcal{A}_{s+1}^{\circ}) = \varepsilon\delta \cdot \mathrm{Par}(|\boldsymbol{A}_{s+1}^{\circ}|), \tag{45b}$$

$$\mathrm{Idx}(\mathcal{A}_{\mathrm{in}}) = -\varepsilon S, \qquad \mathrm{Idx}(\mathcal{B}_{\mathrm{on}}) = (p-q)/2. \tag{45c}$$

Proof. Formulas in (45a) are easy consequences of (44). The right-hand side of index formula (38) consists of three parts ε, $\varepsilon\delta$ and $-2\varepsilon S$. As mentioned in the proof of Theorem 4.2, they are the contributions of $\boldsymbol{a}_1$, $\boldsymbol{a}_{s+1}$ and $\boldsymbol{a}_{\mathrm{in}} := \boldsymbol{a}_2 \cup \overline{\boldsymbol{a}}_2 \cup \cdots \cup \boldsymbol{a}_s \cup \overline{\boldsymbol{a}}_s$ respectively, and hence equal to the sums of local indices of all distinct elements in the three respective sets. Since $\lambda \mapsto \tau = \lambda + \lambda^{-1}$ induces two-to-one maps $(\boldsymbol{a}_1)_{\neq 1} \to \boldsymbol{A}_1^{\circ}$, $(\boldsymbol{a}_{s+1})_{\neq -1} \to \boldsymbol{A}_{s+1}^{\circ}$, $\boldsymbol{a}_{\mathrm{in}} \to \boldsymbol{A}_{\mathrm{in}}$ and $\mathrm{idx}(\lambda) = \mathrm{idx}(\lambda^{-1}) = \mathrm{Idx}(\tau)$, we have

$$\varepsilon = \mathrm{idx}(1) + 2\mathrm{Idx}(\mathcal{A}_1^{\circ}), \quad \varepsilon\delta = \mathrm{idx}(-1) + 2\mathrm{Idx}(\mathcal{A}_{s+1}^{\circ}), \quad -2\varepsilon S = 2\mathrm{Idx}(\mathcal{A}_{\mathrm{in}}),$$

which in turn lead to the formulas in (45b) and the first formula in (45c). Finally, the total index $p - q$ is the sum of local indices over all distinct elements in $\boldsymbol{b}_{\mathrm{on}}$. In view of the two-to-one correspondence $\boldsymbol{b}_{\mathrm{on}} \to \boldsymbol{B}_{\mathrm{on}}$ we have the second formula in (45c). □

5 Hypergeometric Lattices

A hypergeometric group $H = H(\varphi, \psi) = H(\Phi, \Psi) = \langle A, B \rangle$ is said to be *integral* if

$$\varphi(z) \in \mathbb{Z}[z], \quad \psi(z) \in \mathbb{Z}[z], \qquad \text{or equivalently} \quad \Phi(w) \in \mathbb{Z}[w], \quad \Psi(w) \in \mathbb{Z}[w].$$

In this case let $L = L(\varphi, \psi) = L(\Phi, \Psi)$ be the $\mathbb{Z}$-linear span of $\boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}$. Then L is a free $\mathbb{Z}$-module with free basis $\boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r}$, the H-invariant Hermitian form is $\mathbb{Z}$-valued on L and $H \subset O(L)$. It follows from (28) that $B\boldsymbol{r} = -A\boldsymbol{r}$ and $B^k\boldsymbol{r} = -A^k\boldsymbol{r} + \mathbb{Z}$-linear combination of $A\boldsymbol{r}, \dots, A^{k-1}\boldsymbol{r}$ for every $k \ge 2$, hence $\boldsymbol{r}, B\boldsymbol{r}, \dots, B^{n-1}\boldsymbol{r}$ also form a $\mathbb{Z}$-linear basis of L. Thus we have the equation in (1), that is,

$$L = \langle \boldsymbol{r}, A\boldsymbol{r}, \dots, A^{n-1}\boldsymbol{r} \rangle_{\mathbb{Z}} = \langle \boldsymbol{r}, B\boldsymbol{r}, \dots, B^{n-1}\boldsymbol{r} \rangle_{\mathbb{Z}}$$

By the A-invariance of the Hermitian form and the normalization $(\boldsymbol{r}, \boldsymbol{r}) = 2$ we have $(\boldsymbol{v}, \boldsymbol{v}) \in 2\mathbb{Z}$ for all $\boldsymbol{v} \in L$. Thus L equipped with the H-invariant form becomes an even lattice, called a *hypergeometric lattice*. We say that the group H is *unimodular* if so is the lattice L.

5.1 Unimodularity

Formula (14) in Theorem 2.1 implies that the hypergeometric lattice L is unimodular if and only if $\mathrm{Res}(\varphi, \psi) = \pm 1$. Then (31a) shows that when n is even this condition is equivalent to

$$\Psi(2) = \pm 1, \quad \Psi(-2) = \pm 1; \quad \mathrm{Res}(\Phi, \Psi) = \pm 1, \tag{46}$$

while (31b) shows that when n is odd L cannot be unimodular. We say that Ψ is *unramified* if it satisfies the first and second conditions in (46). Note that there is no unramified polynomial of degree one. With irreducible decompositions $\Phi(w) = \prod_i \Phi_i(w)$ and $\Psi(w) = \prod_j \Psi_j(w)$ in $\mathbb{Z}[w]$, the conditions (46) can be restated as

$$\Psi_j(2) = \pm 1, \quad \Psi_j(-2) = \pm 1; \quad \mathrm{Res}(\Phi_i, \Psi_j) = \pm 1 \qquad \text{for all} \quad i, j, \tag{46$'$}$$

because there is a factorization $\mathrm{Res}(\Phi, \Psi) = \prod_i \prod_j \mathrm{Res}(\Phi_i, \Psi_j)$ over $\mathbb{Z}$. This observation provides us with a recipe to construct a unimodular hypergeometric lattice.

Recipe 5.1 Given an integer $N \in \mathbb{Z}_{\ge 2}$, find a finite set of monic irreducible polynomials $\Phi_i(w)$, $\Psi_j(w) \in \mathbb{Z}[w]$ that satisfies the unimodularity condition (46′) and the degree condition

$$\sum_i \deg \Phi_i = N - 1, \qquad \sum_j \deg \Psi_j = N.$$

Take the products $\Phi(w) := \prod_i \Phi_i(w)$ and $\Psi(w) := \prod_j \Psi_j(w)$ and consider the integral hypergeometric group $H = H(\Phi, \Psi)$. Then the associated lattice L is an even unimodular lattice of rank $n = 2N$.

We are especially interested in the settings where the irreducible factors $\Phi_i(w)$ and $\Psi_j(w)$ are cyclotomic trace polynomials or Salem trace polynomials.

5.2 Cyclotomic Trace Polynomials

For $k \in \mathbb{Z}_{\ge 3}$ let $\mathrm{C}_k(z) \in \mathbb{Z}[z]$ denote the k-th cyclotomic polynomial. It is a monic irreducible polynomial of degree $\phi(k)$, where $\phi(k)$ is Euler's totient function. For $k = 1, 2$, in view of our purpose, it is convenient to employ an unconventional definition

$$\mathrm{C}_1(z) := (z-1)^2, \quad \mathrm{C}_2(z) := (z+1)^2, \qquad \phi(1) = \phi(2) = 2.$$

Note that $\mathrm{C}_k(x) \ge 0$ for every $x \in \mathbb{R}$. We say that $\mathrm{C}_k(z)$ is *unramified* if $\mathrm{C}_k(\pm 1) = 1$.

For any $k \in \mathbb{Z}_{\ge 1}$ Euler's totient $\phi(k)$ is an even integer and $\mathrm{C}_k(z)$ is a monic palindromic polynomial of degree $\phi(k)$, so there exists a unique monic polynomial $\mathrm{CT}_k(w) \in \mathbb{Z}[w]$ of degree $\phi(k)/2$, called the k-th *cyclotomic trace polynomial*, such that

$$\mathrm{C}_k(z) = z^{\phi(k)/2}\, \mathrm{CT}_k(z + z^{-1}).$$

Note that $\mathrm{CT}_1(w) = w - 2$, $\mathrm{CT}_2(w) = w + 2$ and $\mathrm{CT}_k(w)$ is irreducible for every $k \ge 1$. We say that $\mathrm{CT}_k(w)$ is *unramified* if so is $\mathrm{C}_k(z)$, in which case $\mathrm{CT}_k(2) = 1$ and $\mathrm{CT}_k(-2) = (-1)^{\phi(k)/2}$.

The following lemma is helpful in checking the unimodularity condition (46′) for cyclotomic trace factors of $\Phi(w)$ and $\Psi(w)$.

Lemma 5.2 *Let k and m be positive integers such that $k > m$.*

(1) $\mathrm{Res}(\mathrm{CT}_k, \mathrm{CT}_m) = \pm 1$ *if and only if the ratio k/m is not a prime power,*

(2) $\mathrm{CT}_m(w)$ *is unramified if and only if neither m nor $m/2$ is a prime power,*

where a prime power is an integer of the form p^l with a prime number p and a positive integer l.

Proof. Apostol [1, Theorems 1, 3, 4] evaluates the resultants of cyclotomic polynomials:

$$\mathrm{Res}(\mathrm{C}_k, \mathrm{C}_m) = \begin{cases} p^{\phi(m)} & \text{if } k/m \text{ is a power of a prime } p, \\ 1 & \text{otherwise,} \end{cases}$$

for $k > m$. In particular $\mathrm{C}_k(z)$ is unramified if and only if neither k nor $k/2$ is a prime power. Lemma 5.2 then readily follows from the relation $\mathrm{Res}(\mathrm{C}_k, \mathrm{C}_m) = \mathrm{Res}(\mathrm{CT}_k, \mathrm{CT}_m)^2$, which in turn follows from Lemma A.1. □

Table 5: Cyclotomic trace polynomials $\mathrm{CT}_k(z)$ of degree ≤ 10.

deg	k	deg	k
1	1, 2, 3, 4, 6	6	13, $\underline{21}$, 26, $\underline{28}$, $\underline{36}$, $\underline{42}$
2	5, 8, 10, $\underline{12}$	8	17, 32, 34, $\underline{40}$, $\underline{48}$, $\underline{60}$
3	7, 9, 14, 18	9	19, 27, 38, 54
4	$\underline{15}$, 16, $\underline{20}$, $\underline{24}$, $\underline{30}$	10	25, $\underline{33}$, $\underline{44}$, 50, $\underline{66}$
5	11, 22		

Lemma 5.3 *There are exactly* 41 *cyclotomic trace polynomials* $\mathrm{CT}_k(w)$ *of degree* ≤ 10*, among which* 15 *are unramified. They are given in Table* 5 *with unramified ones being underlined.*

Proof. Indeed, the cases $k = 1$ and $k = 2$ are trivial. For $k \geq 3$ Euler's totient $\phi(k)$ admits a lower bound

$$\phi(k) > \phi_0(k) := \frac{k}{\mathrm{e}^{\gamma} \log\log k + \frac{3}{\log\log k}},$$

where γ is Euler's gamma constant (see [2, Theorem 8.8.7] and [23, Theorem 15]). Thus the condition $20 \geq \phi(k)$ implies $20 \geq \phi_0(k)$ and a careful analysis of the function $\phi_0(x)$ in real variable $x \geq 3$ shows that the latter condition holds exactly for $k = 3, \dots, 93$. A case-by-case check in this range gives all solutions k to the bound $\deg \mathrm{CT}_k \leq 10$ as in Table 5. □

5.3 Salem Trace Polynomials

A Salem number is an algebraic unit $\lambda > 1$ whose conjugates other than $\lambda^{\pm 1}$ lie on the unit circle S^1 (see Salem [25]). The monic minimal polynomial of a Salem number is called a Salem polynomial. A Salem polynomial is a palindromic polynomial of even degree. A Salem trace is an algebraic integer $\tau > 2$ whose other conjugates lie in the interval $[-2, 2]$. The monic minimal polynomial of a Salem trace is called a Salem trace polynomial. Salem numbers λ and Salem traces τ are in one-to-one correspondence via the relation $\tau = \lambda + \lambda^{-1}$. A Salem polynomial $S(z)$ and the associated Salem trace polynomial $R(w)$ are related by

$$S(z) = z^d R(z + z^{-1}) \qquad \text{with} \quad d := \deg R(w).$$

We say that $S(z)$ and $R(w)$ are *unramified* if $|S(\pm 1)| = 1$ and $|R(\pm 2)| = 1$ respectively.

Example 5.4 McMullen [18, Table 4] gives a list of ten unramified Salem numbers λ_i of degree 22 and the associated Salem polynomials $S_i(z)$ and Salem trace polynomials $R_i(w)$. Approximate values of λ_i and exact formulas for $R_i(w)$ are given in Table 6.

Table 6: Salem trace polynomials of degree 11 from McMullen [18, Table 4].

i	λ_i	Salem trace polynomial $R_i(w)$
1	1.37289	$w(w-1)(w+1)^2(w^2-4)(w^5-6w^3+8w-2)-1$
2	1.45099	$(w+1)(w^2-4)(w^8-8w^6-w^5+19w^4+3w^3-12w^2+1)-1$
3	1.48115	$w(w-1)(w+1)^2(w^2-4)(w^5-6w^3-w^2+8w+1)-1$
4	1.52612	$w^2(w+1)(w^2-4)(w^2+w-1)(w^4-w^3-5w^2+3w+5)-1$
5	1.55377	$(w-1)(w+1)^2(w^2-4)(w^6-7w^4-w^3+12w^2+2w-1)-1$
6	1.60709	$w(w+1)^2(w^2-4)(w^2+w-1)(w^4-2w^3-3w^2+6w-1)-1$
7	1.6298	$(w+1)(w^2-4)(w^8-8w^6-w^5+19w^4+2w^3-14w^2+2)-1$
8	1.6458	$w(w+1)^2(w^2-4)(w^6-w^5-7w^4+5w^3+13w^2-6w-2)-1$
9	1.66566	$w(w+1)(w^2-4)(w^7-9w^5-2w^4+25w^3+9w^2-20w-8)-1$
10	1.69496	$w(w+1)^2(w^2-3)(w^2-4)(w^4-w^3-4w^2+2w+1)-1$

Example 5.5 Lehmer's number $\lambda_{\mathrm{L}} \approx 1.17628$ discovered in [16] is the smallest Salem number ever known (see e.g. Hironaka [11]). The associated Salem polynomial and Salem trace polynomial, that is, Lehmer's polynomial and Lehmer's trace polynomial are given by

$$\mathrm{L}(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1, \tag{47a}$$

$$\mathrm{LT}(w) = (w+1)(w^2-1)(w^2-4)-1, \tag{47b}$$

respectively. Note that they are unramified. Lehmer's trace $\tau_{\mathrm{L}} := \lambda_{\mathrm{L}} + \lambda_{\mathrm{L}}^{-1}$ is approximately 2.02642.

6 Hypergeometric K3 Lattices

An even unimodular lattice of rank 22 and signature $(3, 19)$, that is, index -16 is called a *K3 lattice*. It is well known that the second cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface X equipped with the intersection form is a K3 lattice. We wonder whether a K3 lattice can be realized as a hypergeometric lattice.

Definition 6.1 A *hypergeometric K3 lattice* is a unimodular hypergeometric lattice of rank 22 and index ± 16, where the index is calculated with respect to the invariant Hermitian form normalized by $(\boldsymbol{r}, \boldsymbol{r}) = 2$ as in (29), which we call *hypergeometric normalization*. In the context of K3 lattice we should employ another normalization that makes the index always equal to -16, which we call *K3 normalization*. When the index is $+16$ in the former normalization we can switch to the latter one by negating the invariant Hermitian form; this amounts to taking the reversed normalization $(\boldsymbol{r}, \boldsymbol{r}) = -2$. The invariant Hermitian form with K3 normalization is often referred to as the *intersection form*.

6.1 Real of Rank 22 and Index ±16

Putting aside the integral structure and unimodularity condition for the moment we shall classify all real hypergeometric group of rank 22 and index ± 16. We employ the hypergeometric normalization in Definition 6.1 and use the notations in §4. We now have $\deg \Phi(w) = |\boldsymbol{A}| = 10$ and $\deg \Psi(w) = |\boldsymbol{B}| = 11$.

Theorem 6.2 *A real hypergeometric group of rank $n = 22$ has index $p - q = \pm 16$, if and only if $\boldsymbol{A}_{\mathrm{in}}$ and $\boldsymbol{B}_{\mathrm{on}}$ have any one of the configurations in Table 7, where in case 5 we mean by "doubles adjacent" that the unique double cluster in $\boldsymbol{A}_{\mathrm{in}}$ and the unique double cluster in $\boldsymbol{B}_{\mathrm{on}}$ must be adjacent to each other. Each of cases 6, 7, 8 divides into two subcases as indicated in the constraints column.*

Proof. We use Theorem 4.2 and Lemma 4.3 with $n = 22$. Equation (38) yields $|1 + \delta - 2S| = |p - q| = 16$, which is the case if and only if

$$(\delta, S) = (1, -7), (1, 9), (-1, \mp 8). \tag{48}$$

Table 7: Real of rank 22 and index ±16, where $\varepsilon = \pm$ is defined in (37).

case	s	$\|\boldsymbol{A}_{\rm in}\|$	$\|\boldsymbol{B}_{\rm on}\|$	$[\boldsymbol{A}_{\rm in}]$	$[\boldsymbol{B}_{\rm on}]$	constraints	$\varepsilon(p-q)$	δ	S
1	8	7	8	1^7	1^8		16	1	-7
2	8	9	8	$1^6 3^1$	1^8		16	1	-7
3	8	7	10	1^7	$1^7 3^1$		16	1	-7
4	8	9	10	$1^6 3^1$	$1^7 3^1$		16	1	-7
5	9	9	10	$1^7 2^1$	$1^8 2^1$	doubles adjacent	16	1	-7
6	9	8	10	1^8	$1^8 2^1$	$\|\boldsymbol{B}_{5\pm4}\| = 2$	±16	-1	∓8
7	9	10	10	$1^7 3^1$	$1^8 2^1$	$\|\boldsymbol{B}_{5\pm4}\| = 2$	±16	-1	∓8
8	10	10	10	$1^8 2^1$	1^{10}	$\|\boldsymbol{A}_{6\pm4}\| = 2$	±16	-1	∓8

If $|\boldsymbol{B}_{\rm on}|$ is odd then $\delta = (-1)^{|\boldsymbol{A}_{\rm in}|}$ by (39a) and hence $|\boldsymbol{A}_{\rm in}| \not\equiv S \bmod 2$ by (48), which contradicts the congruence $S \equiv |\boldsymbol{A}_{\rm in}| \bmod 2$ in (40). Hence $|\boldsymbol{B}_{\rm on}|$ must be even and $\delta = -(-1)^{|\boldsymbol{A}_{\rm in}|}$. It follows from (40) that $8 \le |S|+1 \le s \le |\boldsymbol{B}_{\rm on}| \le |\boldsymbol{B}| = 11$, so that we have either

$$|\boldsymbol{B}_{\rm on}| = 8, \quad s = 8; \qquad \text{or} \qquad |\boldsymbol{B}_{\rm on}| = 10, \quad s = 8, 9, 10. \tag{49}$$

A careful inspection shows that the only ten cases in Table 8 can meet the constraints (40), (48), (49) and $|\boldsymbol{A}_{\rm in}| \le |\boldsymbol{A}| = 10$. Indeed, when $s = 8$ these constraints force $(\delta, S) = (1, -7)$, $|\mathrm{I}| = 7$, $|\boldsymbol{A}_{\rm in}| = 7$, 9 and $|\boldsymbol{B}_{\rm on}| = 8$, 10, leading to cases 1–4 in Table 8. Here, for example, if $|\boldsymbol{A}_{\rm in}| = 7$ and $|\boldsymbol{B}_{\rm on}| = 8$ then $[\boldsymbol{A}_{\rm in}] = 1^7$ and $[\boldsymbol{B}_{\rm on}] = 1^8$, while if $|\boldsymbol{A}_{\rm in}| = 9$ and $|\boldsymbol{B}_{\rm on}| = 10$ then $[\boldsymbol{A}_{\rm in}] = 1^6 3^1$ and $[\boldsymbol{B}_{\rm on}] = 1^7 3^1$, $1^6 2^2$. When $s = 9$ those constraints yield either (i) $(\delta, S) = (1, -7)$, $|\mathrm{I}| = 7$, $|\boldsymbol{A}_{\rm in}| = 9$ and $|\boldsymbol{B}_{\rm on}| = 10$, which is case 5; or (ii) $(\delta, S) = (-1, \mp8)$, $|\mathrm{I}| = 8$, $|\boldsymbol{A}_{\rm in}| = 8$, 10 and $|\boldsymbol{B}_{\rm on}| = 10$, leading to cases 6 and 7. When $s = 10$ one has (iii) $(\delta, S) = (-1, \mp8)$, $|\mathrm{I}| = 8$ and $|\boldsymbol{A}_{\rm in}| = |\boldsymbol{B}_{\rm on}| = 10$; or (iv) $(\delta, S) = (1, -7)$, $|\mathrm{I}| = |\boldsymbol{A}_{\rm in}| = 9$ and $|\boldsymbol{B}_{\rm on}| = 10$; or (v) $(\delta, S) = (1, 9)$, $|\mathrm{I}| = |\boldsymbol{A}_{\rm in}| = 9$ and $|\boldsymbol{B}_{\rm on}| = 10$, which lead to cases 8, 9, 10 respectively. Let us make a case-by-case treatment of these ten cases. In what follows "cases" refer to those in Table 8.

Table 8: Ten cases.

case	s	δ	S	$\|\mathrm{I}\|$	$\|\boldsymbol{A}_{\rm in}\|$	$\|\boldsymbol{B}_{\rm on}\|$	$[\boldsymbol{A}_{\rm in}]$	$[\boldsymbol{B}_{\rm on}]$
1	8	1	-7	7	7	8	1^7	1^8
2	8	1	-7	7	9	8	$1^6 3^1$	1^8
3	8	1	-7	7	7	10	1^7	$1^7 3^1$, $1^6 2^2$
4	8	1	-7	7	9	10	$1^6 3^1$	$1^7 3^1$, $1^6 2^2$
5	9	1	-7	7	9	10	$1^7 2^1$	$1^8 2^1$
6	9	-1	∓8	8	8	10	1^8	$1^8 2^1$
7	9	-1	∓8	8	10	10	$1^7 3^1$	$1^8 2^1$
8	10	-1	∓8	8	10	10	$1^8 2^1$	1^{10}
9	10	1	-7	9	9	10	1^9	1^{10}
10	10	1	9	9	9	10	1^9	1^{10}

In cases 1–4 we have $\mathrm{I} = \{2, \dots, 8\}$. In cases 1 and 2, since σ_i is odd for every $i = 2, \dots, 8$, $S = -7$ is actually realized. The same is true in cases 3 and 4 with $[\boldsymbol{B}_{\rm on}] = 1^7 3^1$. In cases 3 and 4 with $[\boldsymbol{B}_{\rm on}] = 1^6 2^2$, let $|\boldsymbol{B}_k| = |\boldsymbol{B}_l| = 2$ with $1 \le k < l \le 8$. Up to reflection (41) we may assume (i) $k = 1$ and $l = 8$; or (ii) $2 \le k < l \le 8$. In case (i) σ_i is even for every $i = 2, \dots, 8$, so that $S = 7$ contradicting $S = -7$. In case (ii) σ_i is odd for $2 \le i \le k$ or $l+1 \le i \le 8$ and even for $k+1 \le i \le l$, so $S = -(k-1)-(8-l)+(l-k) = 2(l-k)-7 \ge -5$, which again contradicts $S = -7$. Thus cases 3 and 4 with $[\boldsymbol{B}_{\rm on}] = 1^6 2^2$ cannot occur.

In case 5 let $|\boldsymbol{A}_k| = |\boldsymbol{B}_l| = 2$ with $2 \le k \le 9$ and $1 \le l \le 9$, in which case $\mathrm{I} = \{2, \dots, \hat{k}, \dots, 9\}$, Up to reflection (41) we may assume $l < k$. Then σ_i is odd for $2 \le i \le l$ or $k+1 \le i \le 9$ and even for $l+1 \le i \le k-1$, so that $S = -(l-1) - (9-k) + (k-l-1) = 2(k-l) - 9 = -7$ implies $l = k-1$. Taking reflection (41) we have also $l = k$. Thus $S = -7$ is realized if and only if $|\boldsymbol{A}_k| = |\boldsymbol{B}_{k-1}| = 2$ or $|\boldsymbol{A}_k| = |\boldsymbol{B}_k| = 2$ holds for some $2 \le k \le 9$, that is, the double cluster in $\boldsymbol{A}_{\mathrm{in}}$ and the one in $\boldsymbol{B}_{\mathrm{on}}$ must be adjacent to each other.

In cases 6 and 7 we have $\mathrm{I} = \{2, \dots, 9\}$. Let $|\boldsymbol{B}_k| = 2$ with $1 \le k \le 9$. Up to reflection (41) we may assume $5 \le k \le 9$. Then σ_i is odd for $2 \le i \le k$ and even for $k+1 \le i \le 9$, so that $S = -(k-1)+(9-k) = 2(5-k) = \mp 8$ implies $(k, S) = (9, -8)$. Taking reflection (41) we have also $(k, S) = (1, 8)$. In summary $S = \mp 8$ forces $|\boldsymbol{B}_{5\pm 4}| = 2$.

In case 8 let $|\boldsymbol{A}_k| = 2$ and $\mathrm{I} = \{2, \dots, \hat{k}, \dots, 10\}$ with $2 \le k \le 10$. Then σ_i is odd for $2 \le i \le k-1$ and even for $k+1 \le i \le 10$, so $S = -(k-2) + (10-k) = 2(6-k) = \mp 8$, which implies $k = 6 \pm 4$, that is, $|\boldsymbol{A}_{6\pm 4}| = 2$.

In cases 9 and 10 we have $[\boldsymbol{B}_{\mathrm{on}}] = 1^{10}$ and $\mathrm{I} = \{2, \dots, 10\}$, so σ_i is odd for $i = 2, \dots, 10$, and hence $S = -9$, i.e. neither $S = -7$ nor $S = 9$ occurs. Thus these cases cannot happen. All these observations lead to the classification in Table 7. □

Some important information about $\boldsymbol{A}$ and $\boldsymbol{B}$ can be extracted from Table 7. Recall that an end cluster in $\boldsymbol{A}_{\mathrm{on}}$ may be empty. If this is the case then it is called a *null* cluster.

Lemma 6.3 *The following are valid for $\boldsymbol{A}$ and the same is true with $\boldsymbol{B}$.*

(1) *Any non-null $\boldsymbol{A}_{\mathrm{on}}$-cluster is simple except for at most one cluster of multiplicity 2 or 3.*

(2) *We have $|\boldsymbol{A}_{\mathrm{off}}| \le 3$ and if $|\boldsymbol{A}_{\mathrm{off}}| \ge 2$ then any non-null $\boldsymbol{A}_{\mathrm{on}}$-cluster is simple.*

(3) *Any element of $\boldsymbol{A}$ is simple except for at most one element of multiplicity 2 or 3.*

Proof. Table 7 tells us $|\boldsymbol{A}_{\mathrm{in}}| \ge 7$ and so $|\boldsymbol{A}_1| + |\boldsymbol{A}_{s+1}| + |\boldsymbol{A}_{\mathrm{off}}| = 10 - |\boldsymbol{A}_{\mathrm{in}}| \le 3$, in particular $|\boldsymbol{A}_{\mathrm{off}}| \le 3$, which gives the first part of assertion (2). We have also $|\boldsymbol{A}_1| + |\boldsymbol{A}_{s+1}| \le 10 - |\boldsymbol{A}_{\mathrm{in}}|$. Using this inequality we consider how $\boldsymbol{A}_{\mathrm{in}}$ can be extended to $\boldsymbol{A}_{\mathrm{on}}$ by adding $\boldsymbol{A}_1$ and $\boldsymbol{A}_{s+1}$. A careful inspection of all cases in Table 7 leads to assertion (1). Suppose $|\boldsymbol{A}_{\mathrm{off}}| \ge 2$ and hence $|\boldsymbol{A}_{\mathrm{in}}| \le 8$. Then we must be in one of the cases 1, 3, 6 in Table 7, where $\boldsymbol{A}_{\mathrm{in}}$ consists of simple clusters only. Then $\boldsymbol{A}_{\mathrm{on}}$ can also contain simple or null clusters only, since $|\boldsymbol{A}_1| + |\boldsymbol{A}_{s+1}| \le 1$. This proves the second part of assertion (2). By assertion (1) any element of $\boldsymbol{A}_{\mathrm{on}}$ is simple except for at most one element which is of multiplicity 2 or 3. Moreover it follows from $|\boldsymbol{A}_{\mathrm{off}}| \le 3$ that $\boldsymbol{A}_{\mathrm{off}}$ can contain at most one multiple element, which is of multiplicity 2 or 3. If there is such an element then the second part of assertion (2) implies that there is no multiple element in $\boldsymbol{A}_{\mathrm{on}}$. This proves assertion (3). As for $\boldsymbol{B}$ assertions (1) and (2) can be seen directly from Table 7 and then assertion (3) is proved just in the same manner as in the case of $\boldsymbol{A}$. □

We now take the integral structure and unimodularity condition into account.

Lemma 6.4 *If $L(\Phi, \Psi)$ is a hypergeometric K3 lattice then any root of $\Phi(w)$ is simple except for at most one integer root of multiplicity 2 or 3, whereas any root of $\Psi(w)$ is simple.*

Proof. It follows from (3) of Lemma 6.3 that $\Phi(w)$ admits at most one multiple root. If $\Phi(w)$ actually contains such a root τ, then any conjugate τ' to τ is also a multiple root, so uniqueness forces $\tau' = \tau$, which means that τ must be an integer. For the same reason any root of $\Psi(w)$ is simple except for at most one integer root, but unramifiedness of $\Psi(w)$ rules out this exception because there is no unramified polynomial of degree one. □

Remark 6.5 If $\Phi(w)$ admits a multiple root $\tau \in \mathbb{Z}$ then the last condition in (46) yields $\Psi(\tau) = \pm 1$. This equation help us find the multiple root of $\Phi(w)$ if it exists.

6.2 Hodge Isometry and Special Eigenvalue

Let L be a K3 lattice and $L_{\mathbb{C}} := L \otimes \mathbb{C}$ be its complexification equipped with the induced Hermitian form. A *Hodge structure* on L is an orthogonal decomposition

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \tag{50}$$

of signatures $(1,0) \oplus (1,19) \oplus (1,0)$ such that $\overline{H^{i,j}} = H^{j,i}$. We remark that $H^{1,1}_{\mathbb{R}} := H^{1,1} \cap L_{\mathbb{R}}$ with $L_{\mathbb{R}} := L \otimes \mathbb{R}$ is a real Lorentzian space of signature $(1, 19)$ and the set of time-like vectors $\mathcal{C} := \{\boldsymbol{v} \in H^{1,1}_{\mathbb{R}} \,:\, (\boldsymbol{v}, \boldsymbol{v}) > 0\}$ consists of two disjoint connected cones, one of which is referred to as the *positive cone* $\mathcal{C}^+$ and the other as the negative cone $\mathcal{C}^- = -\mathcal{C}^+$.

A *Hodge isometry* is a lattice automorphism $F : L \to L$ preserving the Hodge structure (50). We then have either $F(\mathcal{C}^{\pm}) = \mathcal{C}^{\pm}$ or $F(\mathcal{C}^{\pm}) = \mathcal{C}^{\mp}$, according to which F is said to be *positive* or *negative*, respectively. For any positive Hodge isometry F there is a trichotomy (see e.g. Cantat [5, §1.4]):

(E) There exists a line in $\mathcal{C} \cup \{0\}$ preserved by F. In this case the line is fixed pointwise by F and all eigenvalues of F lie on S^1.

(P) There exists a unique line in $\overline{\mathcal{C}}$ preserved by F and this line is on the light-cone $\partial\mathcal{C}$. In this case the line is fixed pointwise by F and all eigenvalues of F lie on S^1.

(H) There exists a real number $\lambda > 1$ such that $\lambda^{\pm 1}$ are the only eigenvalues of F outside S^1. In this case the eigenvalues $\lambda^{\pm 1}$ are simple and their eigen-lines are on the light-cone $\partial\mathcal{C}$.

In the respective cases F is said to be of *elliptic*, *parabolic* or *hyperbolic type*. Since F preserves the free $\mathbb{Z}$-module L, any eigenvalue of F must be a root of unity in the elliptic and parabolic cases, whereas it is either a root of unity or a conjugate to a unique Salem number $\lambda > 1$ in the hyperbolic case. In any case F restricted to $H^{1,1}$ has a real eigenvalue $\lambda \geq 1$.

Remark 6.6 A Hodge isometry F is positive and of hyperbolic type, if $F|H^{1,1}$ has a real eigenvalue $\lambda > 1$.

Any Hodge isometry $F : L \to L$ admits a number $\delta \in S^1$ such that $F|H^{2,0} = \delta I$ and $F|H^{0,2} = \delta^{-1} I$, where I is an identity map. Note that if $\delta = \pm 1$ then $F|H^{2,0} \oplus H^{0,2} = \pm I$. We refer to $\delta^{\pm 1}$ and $\tau := \delta + \delta^{-1} \in [-2, 2]$ as the *special eigenvalues* and the *special trace* of F respectively. This observation leads us to the following.

Definition 6.7 Let $F : L \to L$ be an automorphism of a K3 lattice L. An eigenvalue $\delta \in S^1$ of F is said to be *special*, if $\delta \neq \pm 1$ and there exists a line $\ell \subset L_{\mathbb{C}}$ such that $F|\ell = \delta I$ and the induced Hermitian form is positive definite on ℓ; or if $\delta = \pm 1$ and there exists a plane $P \subset L_{\mathbb{C}}$ such that $F|P = \pm I$, $\overline{P} = P$ and the induced Hermitian form is positive definite on P. We refer to $\tau := \delta + \delta^{-1} \in [-2, 2]$ as the *special trace* of F.

Remark 6.8 Since the K3 lattice L has signature $(3, 19)$, the special trace τ of F is unique if it exists. Moreover, if $\tau \neq \pm 2$ then the pair (δ, ℓ) in Definition 6.7 is uniquely determined by F up to exchange of (δ, ℓ) for $(\delta^{-1}, \overline{\ell})$.

If a lattice automorphism $F : L \to L$ admits a special eigenvalue $\delta \neq \pm 1$ with associated line $\ell \subset L_{\mathbb{C}}$, then

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} := \ell \oplus (\ell \oplus \overline{\ell})^{\perp} \oplus \overline{\ell} \tag{51}$$

gives a unique Hodge structure up to complex conjugation such that F is a Hodge isometry. Here the description of the case $\delta = \pm 1$ is omitted as it is not used in this article.

Let $L = L(A, B)$ be a hypergeometric K3 lattice in K3 normalization (see Definition 6.1). It is natural to ask if A (or B) admits a special eigenvalue or equivalently a special trace. First we focus on the matrix A. Recall from §3.3 that $E(\lambda)$ stands for the generalized eigenspace of A corresponding to an eigenvalue $\lambda \in \boldsymbol{a}_{\mathrm{on}}$ and $m(\lambda) = \dim E(\lambda)$ is the multiplicity of λ. Denote by $V(\lambda)$ the λ-eigenspace of A in the narrow sense.

Lemma 6.9 *For any eigenvalue $\lambda \in \boldsymbol{a}_{\mathrm{on}}$ one has $\dim V(\lambda) = 1$ and if $m(\lambda) \geq 2$ then $V(\lambda)$ must be isotropic with respect to the invariant Hermitian form. In particular if A admits a special eigenvalue δ then it is simple i.e. $m(\delta) = 1$ and different from ± 1, so that the special trace $\tau := \delta + \delta^{-1}$ is different from ± 2.*

Proof. Theorem 2.1 says that $\boldsymbol{r}$ is a cyclic vector for the matrix A. Let $\boldsymbol{v}$ be its projection down to $E(\lambda)$ relative to the direct sum decomposition (24). Then $\boldsymbol{v}$ is a cyclic vector of $A|E(\lambda)$, so if we put $\boldsymbol{v}_j := (A - \lambda I)^{j-1} \boldsymbol{v}$ for $j \in \mathbb{Z}_{\geq 1}$ then $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ with $m := m(\lambda)$ form a basis of $E(\lambda)$. Since $\boldsymbol{v}_{m+1} = \boldsymbol{0}$, we have $A\boldsymbol{v}_m = \lambda \boldsymbol{v}_m$ and hence $V(\lambda) = \mathbb{C}\boldsymbol{v}_m$ is 1-dimensional. If $m \geq 2$, using $\boldsymbol{v}_m = A\boldsymbol{v}_{m-1} - \lambda \boldsymbol{v}_{m-1}$ we have

$$\begin{aligned} \lambda(\boldsymbol{v}_m, \boldsymbol{v}_m) &= (A\boldsymbol{v}_m, A\boldsymbol{v}_{m-1} - \lambda \boldsymbol{v}_{m-1}) = (A\boldsymbol{v}_m, A\boldsymbol{v}_{m-1}) - (\lambda \boldsymbol{v}_m, \lambda \boldsymbol{v}_{m-1}) \\ &= (\boldsymbol{v}_m, \boldsymbol{v}_{m-1}) - |\lambda|^2 (\boldsymbol{v}_m, \boldsymbol{v}_{m-1}) = 0, \end{aligned}$$

by A-invariance of the Hermitian form and $\lambda \in S^1$, so the Hermitian form vanishes on $V(\lambda)$. Let δ be a special eigenvalue of A. If $\delta = \pm 1$ then the associated plane P in Definition 6.7 must be contained in the line $V(\delta)$, but this is impossible. So we have $\delta \neq \pm 1$ and $\ell = V(\delta)$. Since the Hermitian form is positive-definite on $V(\delta)$, we must have $m(\delta) = 1$. It is evident that the special trace τ is different from ± 2. □

Remark 6.10 If A admits a special trace then the recipe (51) yields a unique Hodge structure on L up to complex conjugation such that A is a Hodge isometry. Since $\pm A$ induce the same Hodge structure, it makes sense to speak of A being a positive or negative Hodge isometry. The hypergeometric group $H = \langle A, B \rangle$ can then be normalized so that A is positive, otherwise by replacing H with its antipode H^{a} mentioned in Remark 2.2. Lemma 6.9 and these remarks are also valid for the matrix B.

6.3 Determination of Special Trace

To consider when a special trace exists and to determine it explicitly, we begin with the following.

Lemma 6.11 *Suppose that A admits a special trace and becomes a positive Hodge isometry with respect to the Hodge structure* (51)*. Then the pair $(M(2), \mathrm{idx}(1))$ is $(0, 1)$ in the elliptic case; $(1, -1)$ in the parabolic case; and $(0, -1)$ in the hyperbolic case respectively, while one has $M(-2) = 0$ and $\mathrm{idx}(-1) = -1$ in every case, where the index is taken in K3 normalization in Definition* 6.1*. Moreover any root of $\Phi(w)$ is simple.*

Proof. Let $\delta^{\pm 1}$ be the special eigenvalues of A. Consider the associated Hodge structure (51) and use Lemma 6.9 repeatedly. Since $\delta \neq \pm 1$ we have $V(\pm 1) \subset E(\pm 1) \subset H^{1,1}$. Let $V_{\mathbb{R}}(\pm 1) := V(\pm 1) \cap L_{\mathbb{R}} \subset H^{1,1}_{\mathbb{R}}$. If the line $V_{\mathbb{R}}(-1)$ lies in $\overline{\mathcal{C}}$ then A sends $\mathcal{C}^{\pm}$ to $\mathcal{C}^{\mp}$ as $A = -I$ on $V_{\mathbb{R}}(-1)$. This contradicts the positivity of A, so $V_{\mathbb{R}}(-1)$ must be in the space-like region. Thus the Hermitian form h is negative-definite on $V(-1)$ and we must have $m(-1) = 2M(-2) + 1 = 1$, i.e. $M(-2) = 0$ and $\mathrm{idx}(-1) = -1$. In the elliptic and parabolic cases $V_{\mathbb{R}}(1)$ is the unique line in $\overline{\mathcal{C}}$ preserved by A. Accordingly, in the elliptic case h is positive definite on $V(1)$, so we have $m(1) = 2M(2) + 1 = 1$, i.e. $M(2) = 0$ and $\mathrm{idx}(1) = 1$. Similarly, in the parabolic case h vanishes on $V(1)$ and has signature $(u, v) = (M(2), M(2) + 1)$ or $(M(2) + 1, M(2))$ on $E(1)$, which forces $M(2) = 1$ and $(u, v) = (1, 2)$, i.e. $\mathrm{idx}(1) = -1$, since $E(1) \subset H^{1,1}$ and $H^{1,1}$ has signature $(1, 19)$. In the hyperbolic case h must be negative-definite on $E(1)$ because h has signature $(1, 1)$ on $E(\lambda) \oplus E(\lambda^{-1}) \subset H^{1,1}$ where λ is the real eigenvalue of A strictly greater than 1. This makes $M(2) = 0$ and $\mathrm{idx}(1) = -1$. Let U be $E(1)$ in the elliptic or parabolic case and $E(\lambda) \oplus E(\lambda^{-1})$ in the hyperbolic case. Then h is negative-definite on $U^{\perp} \cap H^{1,1}$. Hence by Propositions 3.5 and 4.5 any root of $\Phi(w)$ other than $\tau = \delta + \delta^{-1}$ and 2 is simple. Simpleness of τ also follows from the same propositions, while if 2 is a root then it is simple because $M(2) = 1$ as just shown. □

We are now in a position to prove Theorems 1.2 and 1.3 stated in the Introduction, §1.

Proof of Theorem 1.2. Suppose that A is a positive Hodge isometry admitting a special trace $\tau = \tau(A)$. By Lemma 6.11 we have $\mathcal{A}_1^{\circ} = \boldsymbol{A}_1^{\circ}$, $\mathcal{A}_{s+1}^{\circ} = \boldsymbol{A}_{s+1}^{\circ} = \boldsymbol{A}_{s+1}$ and $\mathcal{A}_{\mathrm{in}} = \boldsymbol{A}_{\mathrm{in}}$ in the notation of Theorem 4.6. An inspection of Table 7 allows us to rewrite (45) in K3 normalization (see Definition 6.1). Indeed, since $p - q = -16$ in this normalization, we have $\varepsilon = -1$ from $\varepsilon(p - q) = 16$ in cases 1–5; and $\varepsilon = \mp 1$ from $\varepsilon(p - q) = \pm 16$ in cases 6–8 of Table 7. Moreover we have $\delta = 1$ and $\delta = -1$ in respective cases. Formulas (45a) now read

$$\mathrm{idx}(1) = -(-1)^{|\boldsymbol{A}_1^{\circ}|}, \qquad \mathrm{idx}(-1) = -(-1)^{|\boldsymbol{A}_{s+1}|} \qquad \text{in cases 1–5,}$$
$$\mathrm{idx}(1) = \mp(-1)^{|\boldsymbol{A}_1^{\circ}|}, \qquad \mathrm{idx}(-1) = \pm(-1)^{|\boldsymbol{A}_{s+1}|} \qquad \text{in cases 6–8 of Table 7.}$$

These equations are combined with Lemma 6.11 to determine the parities of $|\boldsymbol{A}_1^{\circ}|$ and $|\boldsymbol{A}_{s+1}|$. Then $\mathrm{Idx}(\boldsymbol{A}_1^{\circ})$ and $\mathrm{Idx}(\boldsymbol{A}_{s+1})$ are evaluated by the formulas (45b), which now look like

$$\mathrm{Idx}(\boldsymbol{A}_1^{\circ}) = -\mathrm{Par}(|\boldsymbol{A}_1^{\circ}|), \qquad \mathrm{Idx}(\boldsymbol{A}_{s+1}^{\circ}) = -\mathrm{Par}(|\boldsymbol{A}_{s+1}|) \qquad \text{in cases 1–5,}$$
$$\mathrm{Idx}(\boldsymbol{A}_1^{\circ}) = \mp\mathrm{Par}(|\boldsymbol{A}_1^{\circ}|), \qquad \mathrm{Idx}(\boldsymbol{A}_{s+1}^{\circ}) = \pm\mathrm{Par}(|\boldsymbol{A}_{s+1}|) \qquad \text{in cases 6–8.}$$

Recall also from Lemma 6.11 that $\boldsymbol{A}_1 = \boldsymbol{A}_1^{\circ}$ in the elliptic and hyperbolic cases, while $\boldsymbol{A}_1 = \boldsymbol{A}_1^{\circ} \cup \{2\}$ in the parabolic case. The information so obtained and a further use of Lemma 6.11 confine the possibilities of $\boldsymbol{A}_1$ and $\boldsymbol{A}_{s+1}$. In particular, subcase $|\boldsymbol{B}_9| = 2$ of case 7 and subcase $|\boldsymbol{A}_{10}| = 10$ of case 8 are excluded in the elliptic case, whereas cases 7 and 8 are altogether ruled out in the parabolic and hyperbolic cases.

The special trace is the (unique) element τ such that $\mathrm{idx}(\tau) = 1$. In cases 1–5 of Table 7 the first formula in (45c) reads $\mathrm{Idx}(\boldsymbol{A}_{\mathrm{in}}) = -7$. In cases 1 and 3 where $|\boldsymbol{A}_{\mathrm{in}}| = 7$, this implies that all elements of $\boldsymbol{A}_{\mathrm{in}}$ have local index -1, so τ must lie either in $\boldsymbol{A}_1$ or in $\boldsymbol{A}_{s+1} = \boldsymbol{A}_9$. Which of them contains τ can be seen as follows. In elliptic case we have $|\boldsymbol{A}_1| + |\boldsymbol{A}_9| = 3$, $|\boldsymbol{A}_1|$ odd, $|\boldsymbol{A}_9|$ even, $\mathrm{Idx}(\boldsymbol{A}_1) = -1$, $\mathrm{Idx}(\boldsymbol{A}_9) = 0$, hence (i) $|\boldsymbol{A}_1| = 3$, $|\boldsymbol{A}_9| = 0$ and $\tau \in \boldsymbol{A}_1$; or (ii) $|\boldsymbol{A}_1| = 1$, $|\boldsymbol{A}_9| = 2$ and $\tau \in \boldsymbol{A}_9$. In parabolic case we have $|\boldsymbol{A}_1^{\circ}| + |\boldsymbol{A}_9| = 2$, $|\boldsymbol{A}_1^{\circ}|$ and $|\boldsymbol{A}_9|$ both even, $\mathrm{Idx}(\boldsymbol{A}_1^{\circ}) = \mathrm{Idx}(\boldsymbol{A}_9) = 0$, hence (i) $|\boldsymbol{A}_1^{\circ}| = 2$, $|\boldsymbol{A}_9| = 0$ and $\tau \in \boldsymbol{A}_1^{\circ}$; or (ii) $|\boldsymbol{A}_1^{\circ}| = 0$, $|\boldsymbol{A}_9| = 2$

and $\tau \in \boldsymbol{A}_9$. In hyperbolic case we have $|\boldsymbol{A}_1| + |\boldsymbol{A}_9| = 2$, $|\boldsymbol{A}_1|$ and $|\boldsymbol{A}_9|$ both even, $\mathrm{Idx}(\boldsymbol{A}_1) = \mathrm{Idx}(\boldsymbol{A}_9) = 0$, hence (i) $|\boldsymbol{A}_1| = 2$, $|\boldsymbol{A}_9| = 0$ and $\tau \in \boldsymbol{A}_1$; or (ii) $|\boldsymbol{A}_1| = 0$, $|\boldsymbol{A}_9| = 2$ and $\tau \in \boldsymbol{A}_9$. In cases 2, 4, 5 where $|\boldsymbol{A}_{\mathrm{in}}| = 9$, there is a unique element of $\boldsymbol{A}_{\mathrm{in}}$ with local index 1 (which is exactly τ) and all the other elements have local index -1. Cases 6–8 can be treated in a similar manner, where the first formula in (45c) reads $\mathrm{Idx}(\boldsymbol{A}_{\mathrm{in}}) = -8$ with $|\boldsymbol{A}_{\mathrm{in}}|$ being either 8 or 10. In any case, once the configuration of the clusters is fixed, formula (43) tells us exactly where τ is located. In particular τ lies in the unique multiple cluster, if it exists, and τ is its middle element if it is triple. Exhausting all possibilities we obtain all conditions in Theorem 1.2 as well as in Tables 1 and 2. By a case-by-case check it can be seen that these necessary conditions are also sufficient. □

Proof of Theorem 1.3. Suppose that B is a positive Hodge isometry admitting a special trace $\tau = \tau(B)$. If B is of elliptic or parabolic type then $|\boldsymbol{B}_{\mathrm{off}}| = 0$, while if B is of hyperbolic type then $|\boldsymbol{B}_{\mathrm{off}}| = |\boldsymbol{B}_{>2}| = 1$. On the other hand, since $|\boldsymbol{B}| = 11$, we observe in Table 7 that $|\boldsymbol{B}_{\mathrm{off}}| = 3$ in cases 1–2, and $|\boldsymbol{B}_{\mathrm{off}}| = 1$ in cases 3–8 respectively. Thus cases 1–2 cannot occur and we must be in cases 3–8 with $|\boldsymbol{B}_{\mathrm{off}}| = |\boldsymbol{B}_{>2}| = 1$. In particular B must always be of hyperbolic type. Note that $\Psi(\pm 2) \neq 0$ is an immediate consequence of $\boldsymbol{B}_{\mathrm{on}} \subset (-2, 2)$. Assertion that all roots of $\Psi(w)$ are simple and those of $\Phi(w)$ are simple except for at most one exception follows from Lemma 6.4. This in particulat implies $\mathcal{B}_{\mathrm{on}} = \boldsymbol{B}_{\mathrm{on}}$ in the notation of Theorem 4.6. The second formula in (45c) adapted in K3 normalization reads $\mathrm{Idx}(\boldsymbol{B}_{\mathrm{on}}) = -8$. Since $|\boldsymbol{B}_{\mathrm{on}}| = 10$ in cases 3–8, this shows that there exists a unique element of $\boldsymbol{B}_{\mathrm{on}}$ with local index 1 (which is exactly τ) and all the other nine elements have index -1. Indeed, only in this case we have $\mathrm{Idx}(\boldsymbol{B}_{\mathrm{on}}) = 1 \cdot 1 + 9 \cdot (-1) = -8$. The location of τ can be determined by formula (43). Exhausting all possibilities we obtain all conditions in Theorem 1.3 as well as in Table 3. By a case-by-case check it can be seen that these necessary conditions are also sufficient. □

7 K3 Structure

Given a K3 lattice L with a Hodge structure (50), the Picard lattice and the root system are defined by $\mathrm{Pic} := L \cap H^{1,1}$ and $\Delta := \{\boldsymbol{u} \in \mathrm{Pic} : (\boldsymbol{u}, \boldsymbol{u}) = -2\}$ respectively. Given a positive cone $\mathcal{C}^+ \subset \mathcal{C}$, a *set of positive roots* is by definition a subset $\Delta^+ \subset \Delta$ such that $\Delta = \Delta^+ \amalg (-\Delta^+)$ and

$$\mathcal{K} := \{\boldsymbol{v} \in \mathcal{C}^+ : (\boldsymbol{v}, \boldsymbol{u}) > 0 \;\text{ for any }\; \boldsymbol{u} \in \Delta^+\}$$

is nonempty, in which case $\mathcal{K}$ is called the *Kähler cone* associated with Δ^+. Note that Δ^+ determines a basis Δ_{b} of the root system Δ and vice versa. A *Picard-Lefschetz reflection* is a lattice automorphism $\sigma_{\boldsymbol{u}} : L \to L$ defined by $\sigma_{\boldsymbol{u}}(\boldsymbol{v}) := \boldsymbol{v} + (\boldsymbol{v}, \boldsymbol{u})\boldsymbol{u}$ for each $\boldsymbol{u} \in \Delta^+$. The group $W := \langle \sigma_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in \Delta^+ \rangle$ generated by those reflections is referred to as the *Weyl group*. It acts on $\mathcal{C}^+$ properly discontinuously and the closure in $\mathcal{C}^+$ of the Kähler cone $\mathcal{K}$ is a fundamental domain of this action. Each fundamental domain is called a *Weyl chamber*.

Definition 7.1 A *K3 structure* on a K3 lattice L is a specification of (i) a Hodge structure, (ii) a positive cone $\mathcal{C}^+$ and (iii) a set of positive roots Δ^+. These data determines the associated Kähler cone $\mathcal{K}$. Alternatively we can specify (i), (ii) and (iii′) a Weyl chamber $\mathcal{K} \subset \mathcal{C}^+$ which we call the Kähler cone, in place of (iii). In this case the set of positive roots Δ^+ is associated afterwards. See McMullen [20, §6].

7.1 Synthesis of Automorphisms

Any K3 surface X induces a K3 structure on $H^2(X, \mathbb{Z})$; its Hodge structure is given by the Hodge-Kodaira decomposition of $H^2(X, \mathbb{C})$; its Kähler cone $\mathcal{K}(X)$ is the set of all Kähler classes on X; its positive cone $\mathcal{C}^+(X)$ is the connected component containing $\mathcal{K}(X)$; the set of positive roots $\Delta^+(X)$ is that of all effective (-2)-classes. Any automorphism $f : X \to X$ induces a lattice automorphism $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ that preserves the K3 structure. Conversely, the Torelli theorem and surjectivity of the period mapping tells us the following.

Theorem 7.2 *Let L be a K3 lattice. Any K3 structure on L is realized by a unique marked K3 surface (X, ι) up to isomorphism and any lattice automorphism $F : L \to L$ preserving the K3 structure is realized by a unique K3 surface automorphism $f : X \to X$ up to biholomorphic conjugacy, that is, there exists a commutative diagram:*

$$\begin{array}{ccc} H^2(X, \mathbb{Z}) & \xrightarrow{\;\iota\;} & L \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ H^2(X, \mathbb{Z}) & \xrightarrow{\;\iota\;} & L. \end{array} \tag{52}$$

The K3 surface automorphism $f : X \to X$ so obtained is called the *lift* of F. Thus constructing a K3 surface automorphism amounts to constructing an automorphism of a K3 lattice preserving a K3 structure. There may be curves in the surface X and it will later be necessary to see how irreducible curves are permuted by f.

Lemma 7.3 *In Theorem 7.2 suppose that the intersection form is negative-definite on* Pic*. Then any irreducible curve $C \subset X$ is a (-2)-curve and hence the unique effective divisor representing the nodal class $[C] \in \mathrm{Pic}(X)$. Through the marking isomorphism ι in (52) the (-2)-curves in X are in one-to-one correspondence with the simple roots in Δ^+, that is, the elements of $\Delta_\mathrm{b} \subset L$ and how these curves are permuted by f is faithfully represented by the action of F on Δ_b or equivalently on its Dynkin diagram.*

Proof. By assumption $\mathrm{Pic} \subset L$ is negative-definite and even, so is $\mathrm{Pic}(X) \subset H^2(X, \mathbb{Z})$. Since any irreducible curve C represents a non-trivial effective class $[C] \in \mathrm{Pic}(X)$, its self-intersection number (C, C) is an even negative integer. But the irreducibility of C forces $(C, C) \geq -2$ and so $(C, C) = -2$, which implies that C is a (-2)-curve. The remaining assertions follow from Barth et al. [3, Chap. VIII, (3.7) Proposition]. □

In particular the irreducible f-invariant curves in X are in one-to-one correspondence with those elements of Δ_b which are fixed by F. The union $\mathcal{E} = \mathcal{E}(X)$ of all (-2)-curves in X is called the *exceptional set*. The dual graph of its intersection relations is represented by the Dynkin diagram mentioned in Lemma 7.3.

If $F : L \to L$ is an automorphism of a K3 lattice L admitting a special trace, then F is a Hodge isometry with respect to the Hodge structure (51). We may assume that F is a positive Hodge isometry, otherwise by replacing F with $-F$. For any K3 structure having (51) as its Hodge structure, F obviously preserves the Hodge structure and the positive cone $\mathcal{C}^+$, but it is not always true that F preserves the Kähler cone $\mathcal{K}$; it may be sent to a different Weyl chamber $F(\mathcal{K})$. However, there exists a unique element $w_F \in W$ that brings $F(\mathcal{K})$ back to $\mathcal{K}$, since the Weyl group W acts simply transitively on the set of Weyl chambers. The modified map

$$\tilde{F} := w_F \circ F \tag{53}$$

preserves the K3 structure and hence lifts to a K3 surface automorphism via Theorem 7.2. In §7.3 we shall apply this scenario to a hypergeometric K3 lattice with the map F being the matrix A or B.

7.2 Picard Lattice

Let L be a K3 lattice with Hodge structure (50) and $F : L \to L$ be a positive Hodge isometry with special eigenvalues $\delta(F)^{\pm 1}$. If $\chi_0(z)$ is their minimal polynomial then the characteristic polynomial $\chi(z)$ of F factors as

$$\chi(z) = \chi_0(z) \cdot \chi_1(z) \tag{54}$$

for some monic polynomial $\chi_1(z) \in \mathbb{Z}[z]$. Consider the Picard lattice $\mathrm{Pic} := L \cap H^{1,1}$. It is projective or not depending on whether it contains a vector of positive self-intersection or not.

Theorem 7.4 *Suppose that $\delta(F)$ is different from ± 1 and that $\chi_0(z)$ and $\chi_1(z)$ are coprime. Then*

$$\mathrm{Pic} = \{\boldsymbol{v} \in L : \chi_1(F)\boldsymbol{v} = \boldsymbol{0}\}, \tag{55}$$

the intersection form is non-degenerate on Pic *and the Picard number $\rho := \mathrm{rank}\,\mathrm{Pic}$ is given by $\rho = 22 - \deg \chi_0(z)$, which is necessarily even and ≤ 20. In the projective case $\chi_0(z)$ is a cyclotomic polynomial of degree ≥ 2 and* Pic *has signature $(1, \rho - 1)$. In the non-projective case $\chi_0(z)$ is a Salem polynomial of degree ≥ 4, the associated Salem number gives the spectral radius $\lambda(F)$ and* Pic *has signature $(0, \rho)$.*

Proof. There exist orthogonal direct sum decompositions

$$L_K = L_0(K) \oplus L_1(K) \qquad \text{for} \quad K = \mathbb{Q},\, \overline{\mathbb{Q}},\, \mathbb{C}, \tag{56}$$

where $L_K := L \otimes K$ and $L_i(K) := \{\boldsymbol{v} \in L_K : \chi_i(F)\boldsymbol{v} = \boldsymbol{0}\}$ for $i = 0, 1$. Indeed, since $\chi_0(z)$ and $\chi_1(z)$ are coprime, we have $\phi_0(z) \cdot \chi_0(z) + \phi_1(z) \cdot \chi_1(z) = d$ for some $d \in \mathbb{Z}_{\geq 1}$ and $\phi_0(z)$, $\phi_1(z) \in \mathbb{Z}[z]$, so the orthogonal projection $P_i : L_\mathbb{Q} \to L_i(\mathbb{Q})$ is given by $P_i := d^{-1}\phi_{1-i}(F) \cdot \chi_{1-i}(F)$ for $i = 0, 1$. This yields (56) for $K = \mathbb{Q}$, which in turn gives (56) for $K = \overline{\mathbb{Q}}, \mathbb{C}$. Let $\boldsymbol{u} \in L_0(\overline{\mathbb{Q}})$ be a $\delta(F)$-eigenvector of F. Since $\boldsymbol{u} \in H^{2,0}$ and $\mathrm{Pic} \subset H^{1,1}$, we have $\boldsymbol{u} \perp \mathrm{Pic}$ and so $\sigma(\boldsymbol{u}) \perp \mathrm{Pic}$ for any $\sigma \in G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This implies $L_0(\overline{\mathbb{Q}}) \perp \mathrm{Pic}$

and hence $\mathrm{Pic} \subset L_1(\overline{\mathbb{Q}}) \subset H^{1,1}$ by (56) for $K = \overline{\mathbb{Q}}$, because the vectors $\{\sigma(\boldsymbol{u})\}_{\sigma \in G}$ span $L_0(\overline{\mathbb{Q}})$. Therefore $\mathrm{Pic} = L \cap L_1(\overline{\mathbb{Q}}) = \{\boldsymbol{v} \in L : \chi_1(F)\boldsymbol{v} = \boldsymbol{0}\}$, which yields (55). Moreover we have

$$H^{2,0} \oplus H^{0,2} \subset L_0(\mathbb{C}), \qquad \mathrm{Pic}_{\mathbb{Q}} = L_1(\mathbb{Q}), \tag{57}$$

where $\mathrm{Pic}_{\mathbb{Q}} := \mathrm{Pic} \otimes \mathbb{Q}$. Non-degeneracy of Pic and $\rho = 22 - \deg \chi_0(z)$ follow from (56), the latter part of (57), non-degeneracy of L and $\mathrm{rank}\, L = 22$. Recall that $L_{\mathbb{C}}$ and $H^{2,0} \oplus H^{0,2}$ have signatures $(3, 19)$ and $(2, 0)$ respectively. Thus the former part of (57) implies that the total signature $(3, 19)$ decomposes into either (i) $(2, 20-\rho) \oplus (1, \rho - 1)$; or (ii) $(3, 19-\rho) \oplus (0, \rho)$, along the orthogonal decomposition (56) for $K = \mathbb{C}$.

On the other hand, $\chi_0(z)$ is either (C) a cyclotomic polynomial; or (S) a Salem polynomial. In case (C), since $\delta(F) \neq \pm 1$, the polynomial $\chi_0(z)$ is of degree even and has distinct roots $\lambda_1, \dots, \lambda_m, \bar{\lambda}_1, \dots, \bar{\lambda}_m \in S^1$ with $\deg \chi_0(z) = 2m \ge 2$. Then $L_0(\mathbb{C})$ admits an orthogonal decomposition $L_0(\mathbb{C}) = \bigoplus_{j=1}^m E(\lambda_j) \oplus E(\bar{\lambda}_j)$ by eigenspaces, where $E(\lambda_j)$ and $E(\bar{\lambda}_j)$ have the same signature $(1, 0)$ or $(0, 1)$. Thus $L_0(\mathbb{C})$ must have a signature of the form (even, even), so we are in case (i) and $L_1(\mathbb{C})$ must have signature $(1, \rho - 1)$. In case (S) the polynomial $\chi_0(z)$ has distinct roots $\lambda^{\pm 1}$ and $\lambda_1, \dots, \lambda_m, \bar{\lambda}_1, \dots, \bar{\lambda}_m \in S^1$ with $\lambda > 1$ being a Salem number and $\deg \chi_0(z) = 2(m+1) \ge 4$. Then $L_0(\mathbb{C})$ admits an orthogonal decomposition $L_0(\mathbb{C}) = E(\lambda, \lambda^{-1}) \oplus \bigoplus_{j=1}^m E(\lambda_j) \oplus E(\bar{\lambda}_j)$ as in (24), where $E(\lambda, \lambda^{-1})$ has signature $(1, 1)$ while $E(\lambda_j)$ and $E(\bar{\lambda}_j)$ have the same signature $(1, 0)$ or $(0, 1)$. Thus $L_0(\mathbb{C})$ must have a signature of the form (odd, odd), so we are in case (ii) and $L_1(\mathbb{C})$ must have signature $(0, \rho)$. These observations and the latter part of (57) then lead to the dichotomy in the theorem. Clearly cases (C) and (S) correspond to the projective and non-projective cases respectively. □

Let $L = L(\varphi, \psi)$ be a hypergeometric K3 lattice and let $F = A$ or B. Suppose that F admits a special eigenvalue $\delta(F)$ such that F is a positive Hodge isometry with respect to the Hodge structure (51). Let $\chi_0(z)$ be the minimal polynomial of $\delta(F)^{\pm 1}$. There then exists a factorization of polynomials (54).

Proof of Theorem 1.5. Recall from item (4) of Remark 1.4 that $\delta(F)$ is different from ± 1. By item (1) of the same remark any root of $\chi(z)$ is simple except for the triple root $z = 1$ in the parabolic case of Theorem 1.2. In any case $\chi_0(z)$ and $\chi_1(z)$ are coprime, since $z = 1$ cannot be a root of $\chi_0(z)$. Thus Theorem 7.4 establishes all assertions in Theorem 1.5 other than the one on standard basis. Formula (1) shows that $\boldsymbol{r}$ is a cyclic vector for both $F = A$ and B. So any vector $\boldsymbol{v} \in L$ can be expressed as $\boldsymbol{v} = \chi_2(F)\boldsymbol{r}$ for a unique polynomial $\chi_2(z) \in \mathbb{Z}[z]$ such that $\deg \chi_2(z) \le 21$. In view of (55) we have $\boldsymbol{v} \in \mathrm{Pic}$ if and only if $\chi_1(F) \cdot \chi_2(F)\boldsymbol{r} = \boldsymbol{0}$, which is the case exactly when $\chi(z)$ divides $\chi_1(z) \cdot \chi_2(z)$, that is, $\chi_0(z)$ divides $\chi_2(z)$. Upon writing $\chi_2(z) = \chi_3(z) \cdot \chi_0(z)$, any vector $\boldsymbol{v} \in \mathrm{Pic}$ can be represented as $\boldsymbol{v} = \chi_3(F)\boldsymbol{s}$ with $\boldsymbol{s} := \chi_0(F)\boldsymbol{r}$ for a unique polynomial $\chi_3(z) \in \mathbb{Z}[z]$ such that $\deg \chi_3(z) \le \rho - 1$. Hence $\boldsymbol{s}_1, \dots, \boldsymbol{s}_\rho$ form a free $\mathbb{Z}$-basis of Pic. □

7.3 Bringing-back Algorithm

In the non-projective case there exists an algorithm to output the Weyl group element $w_F \in W$ in (53) for $F = A$ or B. Before stating it we make a remark about Kähler cone. There exists an orthogonal decomposition $H^{1,1}_{\mathbb{R}} = V_{\mathbb{R}} \oplus \mathrm{Pic}_{\mathbb{R}}$ of signatures $(1, 19 - \rho) \oplus (0, \rho)$, where $\mathrm{Pic}_{\mathbb{R}} := \mathrm{Pic} \otimes \mathbb{R}$ and the Weyl group W acts on $V_{\mathbb{R}}$ trivially. Thus the intersection form $(\boldsymbol{u}, \boldsymbol{v})$ restricted to $\mathrm{Pic}_{\mathbb{R}}$ is negative-definite. For the sake of convenience in Algorithm 7.5 below we turn it positive-definite by putting $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := -(\boldsymbol{u}, \boldsymbol{v})$. Let $\mathcal{C}_0^+ := \mathcal{C}^+ \cap V_{\mathbb{R}}$ and

$$\mathcal{K}_1 := \{\boldsymbol{v}_1 \in \mathrm{Pic}_{\mathbb{R}} : (\boldsymbol{v}_1, \boldsymbol{u}) = -\langle \boldsymbol{v}_1, \boldsymbol{u} \rangle > 0 \ \text{ for any } \ \boldsymbol{u} \in \Delta^+\}. \tag{58}$$

We refer to $\mathcal{K}_1$ as the essential part of the Kähler cone $\mathcal{K}$, since the latter is represented as

$$\mathcal{K} = \{\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 : \boldsymbol{v}_0 \in \mathcal{C}_0^+, \, \boldsymbol{v}_1 \in \mathcal{K}_1, \, (\boldsymbol{v}_0, \boldsymbol{v}_0) + (\boldsymbol{v}_1, \boldsymbol{v}_1) > 0\}.$$

Algorithm 7.5 Take $A = Z(\varphi)$ and $B = Z(\psi)$ as in (7) and specify $F = A$ or B.

1. Gram matrix. The Gram matrix $\langle \boldsymbol{s}_i, \boldsymbol{s}_j \rangle$ for the standard basis $\boldsymbol{s}_1, \dots, \boldsymbol{s}_\rho$ of Pic (see Theorem 1.5) can be evaluated explicitly by using formula (2) in Theorem 1.1 (or its B-basis version). Note that $\langle \boldsymbol{s}_i, \boldsymbol{s}_j \rangle$ depends only on $|i - j|$. We give an algorithm to find all roots of the root system Δ.

2. Finding all roots. We have an even, positive-definite quadratic form

$$Q(t_1, \dots, t_\rho) := \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \sum_{i,j=1}^{\rho} \langle \boldsymbol{s}_i, \boldsymbol{s}_j \rangle \, t_i t_j \qquad \text{in} \qquad (t_1, \dots, t_\rho) \in \mathbb{Z}^\rho,$$

by expressing each element $\boldsymbol{u} \in \mathrm{Pic}$ as a $\mathbb{Z}$-linear combination $\boldsymbol{u} = t_1 \boldsymbol{s}_1 + \cdots + t_\rho \boldsymbol{s}_\rho$. Finding all roots in Δ amounts to finding all integer solutions to the inequality $Q(t_1, \dots, t_\rho) \le 2$. Indeed, all but the trivial solution $\mathbf{0} = (0, \dots, 0)$ lead to the solutions of the equation $Q(t_1, \dots, t_\rho) = 2$, because $Q(t_1, \dots, t_\rho)$ cannot take value 1. As a positive-definite quadratic form, $Q(t_1, \dots, t_\rho)$ can be expressed as

$$Q(t_1, \dots, t_\rho) = \sum_{j=1}^{\rho} c_j \{t_j - p_j(t_1, \dots, t_{j-1})\}^2, \qquad c_j \in \mathbb{Q}_{>0}, \quad j = 1, \dots, \rho, \tag{59}$$

where $p_1 = 0$ and for $j = 2, \dots, \rho$ the expression $p_j(t_1, \dots, t_{j-1})$ is a linear form over $\mathbb{Q}$ in $t_1, \dots, t_{j-1}$. Note that c_j and p_j can be calculated explicitly in terms of the coefficients of $Q(t_1, \dots, t_\rho)$.

Let $Q_k(t_1, \dots t_{k-1}; t_k)$ be the partial sum of (59) summed over $j = 1, \dots, k$; it is a quadratic function of t_k, once $t_1, \dots, t_{k-1}$ are given. We define a *rooted forest* in the following manner. First, let all integer solutions t_1 of the inequality $Q_1(t_1) \le 2$ be the roots (in graph theory) of the forest. Next, for each root t_1, let all integer solutions t_2 of $Q_2(t_1; t_2) \le 2$ be the children of t_1. Inductively, given a parent t_k with its ancestors $t_1, \dots, t_{k-1}$, let all integer solutions t_{k+1} of $Q_{k+1}(t_1, \dots, t_k; t_{k+1}) \le 2$ be the children of t_k. Consider all paths from roots to leaves, say, $(t_1, \dots, t_k)$. Some of them may continue to the ρ-th generation, that is, $k = \rho$, while others may not. All paths $(t_1, \dots, t_\rho)$ that continue to the ρ-th generation yield all integer solutions to the inequality $Q(t_1, \dots, t_\rho) \le 2$ and hence all elements $\boldsymbol{u} \in \Delta$ along with the origin $\mathbf{0}$.

3. Positive roots, simple roots and the Kähler cone. Provide Pic with a lexicographic order in the following manner: for $\boldsymbol{u} = t_1 \boldsymbol{s}_1 + \cdots + t_\rho \boldsymbol{s}_\rho$, $\boldsymbol{u}' = t_1' \boldsymbol{s}_1 + \cdots + t_\rho' \boldsymbol{s}_\rho \in \mathrm{Pic}$,

$$\boldsymbol{u} \succ \boldsymbol{u}' \overset{\text{def}}{\Longleftrightarrow} \text{there exists an } i \in \{1, \dots, \rho\} \text{ such that } t_i > t_i' \text{ and } t_j = t_j' \text{ for all } j < i. \tag{60}$$

Then in the previous step all solutions $\boldsymbol{u} = t_1 \boldsymbol{s}_1 + \cdots + t_\rho \boldsymbol{s}_\rho \succ \mathbf{0}$ give the set of positive roots Δ^+. An element $\boldsymbol{u} \in \Delta^+$ is a simple root if and only if $\boldsymbol{u} - \boldsymbol{u}' \notin \Delta^+$ for any $\boldsymbol{u}' \in \Delta^+$. This test can easily be carried out by computer and we obtain the set of simple roots, say Δ_{b}, that is, the basis of Δ relative to Δ^+. Looking at Δ_{b} we can draw the Dynkin diagram of Δ, which indicates the irreducible decomposition of Δ as well as the Dynkin type of each irreducible component. The essential part $\mathcal{K}_1$ of the Kähler cone $\mathcal{K}$ is defined by (58). It is the *negative* of the Weyl chamber $C(\boldsymbol{\delta})$ containing the regular vector $\boldsymbol{\delta} := \frac{1}{2} \sum_{\boldsymbol{u} \in \Delta^+} \boldsymbol{u} \in \mathrm{Pic}_{\mathbb{R}}$.

4. Bringing back. The matrix F sends $-\mathcal{K}_1 = C(\boldsymbol{\delta})$ to the Weyl chamber $C(\boldsymbol{d})$ containing the regular vector $\boldsymbol{d} := F\boldsymbol{\delta} \in \mathrm{Pic}_{\mathbb{R}}$. Let $w_F \in W$ be an element maximizing $\langle w(\boldsymbol{d}), \boldsymbol{\delta} \rangle$ for $w \in W$. We claim that w_F brings $C(\boldsymbol{d})$ back to $C(\boldsymbol{\delta})$. Indeed, for any $\boldsymbol{u} \in \Delta_{\mathrm{b}}$ one has $\sigma_{\boldsymbol{u}}(\boldsymbol{\delta}) = \boldsymbol{\delta} - \boldsymbol{u}$ by Humphreys [12, §10.2, Lemma B] and hence the defining property of w_F and $\sigma_{\boldsymbol{u}}$-invariance of the inner product yield

$$\langle w_F(\boldsymbol{d}), \boldsymbol{\delta} \rangle \ge \langle (\rho_{\boldsymbol{u}} w_F)(\boldsymbol{d}), \boldsymbol{\delta} \rangle = \langle w_F(\boldsymbol{d}), \rho_{\boldsymbol{u}}(\boldsymbol{\delta}) \rangle = \langle w_F(\boldsymbol{d}), \boldsymbol{\delta} - \boldsymbol{u} \rangle = \langle w_F(\boldsymbol{d}), \boldsymbol{\delta} \rangle - \langle w_F(\boldsymbol{d}), \boldsymbol{u} \rangle,$$

that is, $\langle w_F(\boldsymbol{d}), \boldsymbol{u} \rangle \ge 0$. This implies more strictly that $\langle w_F(\boldsymbol{d}), \boldsymbol{u} \rangle > 0$ for any $\boldsymbol{u} \in \Delta_{\mathrm{b}}$, since $w_F(\boldsymbol{d})$ is a regular vector. Thus we have $w_F(\boldsymbol{d}) \in C(\boldsymbol{\delta})$ and so $w_F(C(\boldsymbol{d})) = C(\boldsymbol{\delta})$. Note that the element $w_F \in W$ is unique because W acts on the set of Weyl chambers simply transitively.

5. Product of Picard-Lefschetz reflections. To determine w_F explicitly, let PL be the set of all Picard-Lefschetz reflections together with the identity transformation 1. Start with $\boldsymbol{d}_0 := \boldsymbol{d}$ and find an element $\sigma_1 \in \mathrm{PL}$ maximizing $\langle \sigma(\boldsymbol{d}_0), \boldsymbol{\delta} \rangle$ for $\sigma \in \mathrm{PL}$. If $\sigma_1 = 1$, stop here; otherwise, put $\boldsymbol{d}_1 := \sigma_1(\boldsymbol{d}_0)$ and find $\sigma_2 \in \mathrm{PL}$ maximizing $\langle \sigma(\boldsymbol{d}_1), \boldsymbol{\delta} \rangle$ for $\sigma \in \mathrm{PL}$. Inductively, given $\sigma_k \in \mathrm{PL}$ and $\boldsymbol{d}_{k-1} \in \mathrm{Pic}_{\mathbb{R}}$, if $\sigma_k = 1$, stop at this stage; otherwise, put $\boldsymbol{d}_k := \sigma_k(\boldsymbol{d}_{k-1})$ and find yet another $\sigma_{k+1} \in \mathrm{PL}$ maximizing $\langle \sigma(\boldsymbol{d}_k), \boldsymbol{\delta} \rangle$ for $\sigma \in \mathrm{PL}$. We claim that

$$\text{if } \sigma_{k+1} = 1 \text{ then } \boldsymbol{d}_k \in C(\boldsymbol{\delta}); \text{ otherwise, } \langle \sigma_{k+1}(\boldsymbol{d}_k), \boldsymbol{\delta} \rangle > \langle \boldsymbol{d}_k, \boldsymbol{\delta} \rangle. \tag{61}$$

Indeed, if $\sigma_{k+1} = 1$ then for any $\boldsymbol{u} \in \Delta_{\mathrm{b}}$ we have from Humphreys [12, §10.2, Lemma B],

$$\langle \boldsymbol{d}_k, \boldsymbol{\delta} \rangle \ge \langle \sigma_{\boldsymbol{u}}(\boldsymbol{d}_k), \boldsymbol{\delta} \rangle = \langle \boldsymbol{d}_k, \sigma_{\boldsymbol{u}}(\boldsymbol{\delta}) \rangle = \langle \boldsymbol{d}_k, \boldsymbol{\delta} - \boldsymbol{u} \rangle = \langle \boldsymbol{d}_k, \boldsymbol{\delta} \rangle - \langle \boldsymbol{d}_k, \boldsymbol{u} \rangle,$$

that is, $\langle \boldsymbol{d}_k, \boldsymbol{u} \rangle \ge 0$, which in turn implies $\langle \boldsymbol{d}_k, \boldsymbol{u} \rangle > 0$ for any $\boldsymbol{u} \in \Delta_{\mathrm{b}}$, since $\boldsymbol{d}_k$ is a regular vector. Thus $\boldsymbol{d}_k \in C(\boldsymbol{\delta})$ and the first part of (61) is proved. Next, suppose that $\sigma_{k+1} \ne 1$ is the Picard-Lefschetz reflection associated with a positive root $\boldsymbol{u}_{k+1} \in \Delta^+$. Then

$$\langle \sigma_{k+1}(\boldsymbol{d}_k), \boldsymbol{\delta} \rangle = \langle \boldsymbol{d}_k - \langle \boldsymbol{d}_k, \boldsymbol{u}_{k+1} \rangle \boldsymbol{u}_{k+1}, \boldsymbol{\delta} \rangle = \langle \boldsymbol{d}_k, \boldsymbol{\delta} \rangle - \langle \boldsymbol{d}_k, \boldsymbol{u}_{k+1} \rangle \langle \boldsymbol{u}_{k+1}, \boldsymbol{\delta} \rangle,$$

where $\langle \boldsymbol{d}_k, \boldsymbol{u}_{k+1}\rangle\langle \boldsymbol{u}_{k+1}, \boldsymbol{\delta}\rangle$ is nonzero, since $\boldsymbol{d}_k$ and $\boldsymbol{\delta}$ are regular vectors. On the other hand, by the defining property of σ_{k+1} we have $\langle \sigma_{k+1}(\boldsymbol{d}_k), \boldsymbol{\delta}\rangle \geq \langle \boldsymbol{d}_k, \boldsymbol{\delta}\rangle$ and thus $\langle \sigma_{k+1}(\boldsymbol{d}_k), \boldsymbol{\delta}\rangle > \langle \boldsymbol{d}_k, \boldsymbol{\delta}\rangle$.

Since $\{\langle w(\boldsymbol{d}), \boldsymbol{\delta}\rangle : w \in W\}$ is a finite set, it follows from (61) that the step-by-step procedure mentioned above eventually terminates with $\sigma_{k+1} = 1$ and leads to the desired representation $w_F = \sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1$ as a product of Picard-Lefschetz reflections.

6. Modified matrix. Let w_F act on the whole lattice L by extending it as identity to the orthogonal complement of Pic. The modified matrix $\tilde{F} := w_F \circ F$ then preserves the Kähler cone $\mathcal{K}$ and hence preserves the K3 structure constructed from F. Thanks to factorization (54) and the non-projective assumption the characteristic polynomial $\tilde{\chi}(z)$ of $\tilde{F}$ factors as

$$\tilde{\chi}(z) = \chi_0(z) \cdot \tilde{\chi}_1(z), \qquad \deg \chi_0(z) = 22 - \rho, \quad \deg \tilde{\chi}_1(z) = \rho, \tag{62}$$

where the Salem polynomial component $\chi_0(z)$ is the same as that in (54), while $\tilde{\chi}_1(z)$ is the characteristic polynomial of $\tilde{F}|\mathrm{Pic}$. Thus $\tilde{F}$ has the same spectral radius as F.

7. Action on the Dynkin diagram. Since $\tilde{F}$ preserves Δ^+, it also preserves Δ_{b}. Thus $\tilde{F}$ induces an automorphism of the corresponding Dynkin diagram. Describe this action explicitly.

Now we are able to obtain a K3 surface automorphism $f : X \to X$ as the lift of the modified Hodge isometry $\tilde{F} : L \to L$. This establishes Theorem 1.6. Step 7 is used to apply Lemma 7.3 to the constructed map f.

Remark 7.6 The Picard number ρ is always positive for $F = A$, while it can be zero for $F = B$. If $\rho = 0$ then Pic is trivial, the Kähler cone $\mathcal{K}$ coincides with the positive cone $\mathcal{C}^+$, so there is no need to modify B by a Weyl group element. In this case $\psi(z)$ is an unramified Salem polynomial of degree 22.

8 K3 Surface Automorphisms

We illustrate the method of hypergeometric groups by constructing many examples of non-projective K3 surface automorphisms of positive entropy. In this article we present only a couple of cases involving ten Salem numbers of degree 22 and Lehmer's number. Much wider applications of our method will be reported upon elsewhere. Tables in this section are created by implementing the Mathematica programs in our webpage [13].

8.1 Salem Numbers of Degree 22

Things are simpler with Salem numbers of degree 22 by Remark 7.6. McMullen [18, Table 4] gives a list of ten unramified Salem polynomials $S_i(z)$ of degree 22, $i = 1, \dots, 10$, for each of which he constructs a K3 surface automorphism $f : X \to X$ with a Siegel disk such that the induced map $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ has $S_i(z)$ as its characteristic polynomial (see [18, Theorem 10.1]). The Salem trace polynomials $R_i(w)$ associated with $S_i(z)$ are given in Table 6. Applying our method to them we are able to construct a much greater number of K3 surface automorphisms with a Siegel disk. We refer to §9 for discussions about Siegel disks.

Setup and Tests. In order to deal with Salem numbers of degree 22 it is necessary to use the matrix B. Let $R(w)$ be an unramified Salem trace polynomial of degree 11. We look for all hypergeomtric K3 lattices L such that $\Psi(w) = R(w)$ and $\Phi(w)$ is a product of cyclotomic trace polynomials of the form

$$\Phi(w) = \mathrm{CT}_{\boldsymbol{k}}(w) := \prod_{k \in \boldsymbol{k}} \mathrm{CT}_k(w) \quad \text{with} \quad \sum_{k \in \boldsymbol{k}} \deg \mathrm{CT}_k(w) = 10, \tag{63}$$

where $\boldsymbol{k}$ is a multi-set of positive integers whose elements k come from Table 5. By the last part of Theorem 1.3 any element of $\boldsymbol{k}$ is simple except for at most one element of multiplicity 2 or 3, where the multiple element must be one of 1, 2, 3, 4, 6. An inspection of Table 5 shows that $\boldsymbol{k}$ has at most seven distinct elements and this number reduces to six if $\boldsymbol{k}$ has a triple element. With this setup our tests proceed as follows.

(1) Find all $\boldsymbol{k}$'s satisfying the unimodularity condition (46′), which reads $\mathrm{Res}(\mathrm{CT}_k, R) = \pm 1$ for every $k \in \boldsymbol{k}$.

(2) Judge which of the $\boldsymbol{k}$'s above are K3 lattices according to the criterion in Theorem 1.3.

(3) Identify the special trace $\tau \in (-2, 2)$ of the matrix B by using Theorem 1.3 again.

For each of the data in (63) passing two tests in steps (1) and (2), the information from step (3) allow us to provide L with the Hodge structure (51), with respect to which B is a positive Hodge isometry, since $\Psi(w) = R(w)$ is a Salem trace polynomial (see Remark 6.6). Specify one component of $\mathcal{C}$ as the positive cone $\mathcal{C}^+$ as well as the Kähler cone $\mathcal{K}$. Thanks to Theorem 7.2 the Hodge isometry $B : L \to L$ then lifts to a K3 surface automorphism $f : X \to X$.

We carry out the above-mentioned tests for the ten Salem trace polynomials $R_i(w)$ in Table 6. For each $i = 1, \dots, 10$, let $y_{10} < \cdots < y_2 < y_1$ be the roots of $R_i(w)$ in the interval $(-2, 2)$. Numerical values of them are given in Table 9, where the roots greater than $\tau_0 := 1 - 2\sqrt{2}$ are separated from those smaller than τ_0 by a line; this information will be used to discuss Siegel disks in §9. Note that y_{10} is always smaller than τ_0.

Table 9: Roots smaller than 2 of the Salem trace polynomials in Table 6.

	R_1	R_2	R_3	R_4	R_5
y_1	1.993294	1.988033	1.995401	1.995839	1.994419
y_2	1.205977	1.76766	1.339130	1.495199	1.594614
y_3	0.9297159	0.7304257	0.9728311	0.5691842	0.9860031
y_4	0.3600253	0.3763138	0.1183310	0.2303372	0.2512725
y_5	−0.1005899	−0.3628799	−0.3009327	−0.2778793	−0.5016395
y_6	−0.8098076	−0.8420136	−0.736344	−0.6300590	−0.7481516
y_7	−1.19931	−1.280397	−1.297751	−1.18174	−1.209765
y_8	−1.667161	−1.677966	−1.492648	−1.515876	−1.746739
y_9	−1.842436	−1.891176	−1.770639	−1.897604	−1.866712
y_{10}	−1.970982	−1.948177	−1.983677	−1.96877	−1.950669
	R_6	R_7	R_8	R_9	R_{10}
y_1	1.997218	1.996192	1.992823	1.988808	1.995293
y_2	1.368950	1.521855	1.816402	1.855143	1.738103
y_3	0.526038	0.8486504	0.6683560	1.144637	0.773170
y_4	0.3786577	0.4924207	0.08744302	0.02840390	0.0659823
y_5	−0.1559579	−0.4803898	−0.3825477	−0.4829143	−0.4868621
y_6	−0.7702764	−0.8293021	−0.7409521	−0.8693841	−0.6959829
y_7	−1.274529	−1.297520	−1.234218	−1.419774	−1.265297
y_8	−1.464669	−1.694755	−1.698505	−1.704027	−1.548546
y_9	−1.85815	−1.838895	−1.790300	−1.868712	−1.904399
y_{10}	−1.97660	−1.961625	−1.971905	−1.938202	−1.956408

Theorem 8.1 *In the above setting we have a hypergeometric K3 lattice such that the matrix B admits a special eigenvalue δ conjugate to the Salem number λ_i, if and only if $\Psi(w) = R_i(w)$ and $\boldsymbol{k}$ are as in Table* 10*, where the special trace $\tau := \delta + \delta^{-1}$ is given in the "ST" column, while the "case" column refers to the case in Table* 3*. For each entry of the table the positive Hodge isometry $B : L \to L$ lifts to a K3 surface automorphism $f : X \to X$ of entropy $h(f) = \log \lambda_i$ with special trace $\tau(f) = \tau$ and Picard number $\rho(X) = 0$. The "S/H" column is explained in Theorem* 9.3.

Proof. Carrying out the tests for each polynomial $R_i(w)$ in Table 6 by computer, we have all possible values of $\boldsymbol{k}$ as in Table 10 along with the information about cases and special traces. Each of these data leads to a K3 surface automorphism $f : X \to X$ via Theorem 7.2. □

Table 10: K3 surface automorphisms from Salem numbers of degree 22.

Ψ	Case	$\boldsymbol{k}$	ST	S/H	Ψ	Case	$\boldsymbol{k}$	ST	S/H
R_1	1	$1, 1, 1, 3, 4, 6, 16$	y_8	S	R_1	1	$1, 1, 2, 3, 4, 6, 16$	y_8	S
R_1	1	$2, 2, 1, 3, 4, 6, 16$	y_8	S	R_1	1	$2, 2, 2, 3, 4, 6, 16$	y_8	S
R_1	2	$3, 3, 1, 3, 4, 6, 16$	y_8	S	R_1	2	$3, 3, 2, 3, 4, 6, 16$	y_8	S
R_1	2	$4, 4, 1, 3, 4, 6, 16$	y_8	S	R_1	2	$4, 4, 2, 3, 4, 6, 16$	y_8	S

continued

Ψ	Case	$\boldsymbol{k}$	ST	S/H	Ψ	Case	$\boldsymbol{k}$	ST	S/H
R_1	2	$6,6,1,3,4,6,16$	y_8	S	R_1	2	$6,6,2,3,4,6,16$	y_8	S
R_1	3	$1,3,16,30$	y_2	S	R_1	3	$2,3,16,30$	y_2	S
R_1	3	$1,3,5,7,9$	y_2	S	R_1	3	$2,3,5,7,9$	y_2	S
R_1	3	$1,3,17$	y_2	S	R_1	3	$2,3,17$	y_2	S
R_1	5	$1,2,3,4,5,6,7$	y_{10}	H	R_1	5	$1,1,3,4,5,6,7$	y_{10}	H
R_1	5	$2,2,3,4,5,6,7$	y_{10}	H	R_1	7	$3,3,3,4,5,6,7$	y_{10}	H
R_1	7	$4,4,3,4,5,6,7$	y_{10}	H	R_1	7	$6,6,3,4,5,6,7$	y_{10}	H
R_1	8	$3,6,16,30$	y_1	S	R_1	8	$4,5,7,20$	y_1	S
R_1	8	$3,5,6,7,9$	y_1	S	R_1	8	$3,6,17$	y_1	S
R_2	1	$1,1,1,9,24$	y_7	S	R_2	1	$1,1,2,9,24$	y_7	S
R_2	1	$2,2,1,9,24$	y_7	S	R_2	1	$2,2,2,9,24$	y_7	S
R_2	3	$1,3,5,13$	y_8	S	R_2	3	$1,3,5,42$	y_8	S
R_2	3	$2,3,5,13$	y_8	S	R_2	3	$2,3,5,42$	y_8	S
R_2	3	$3,3,1,9,24$	y_7	S	R_2	3	$3,3,2,9,24$	y_7	S
R_2	3	$1,3,5,12,24$	y_4	S	R_2	3	$2,3,5,12,24$	y_4	S
R_2	3	$1,3,12,13$	y_3	S	R_2	3	$1,3,12,42$	y_3	S
R_2	3	$2,3,12,13$	y_3	S	R_2	3	$2,3,12,42$	y_3	S
R_2	3	$1,3,24,30$	y_2	S	R_2	3	$2,3,24,30$	y_2	S
R_2	5	66	y_{10}	H	R_2	5	$1,2,3,5,22$	y_{10}	H
R_2	5	$1,1,3,5,22$	y_{10}	H	R_2	5	$2,2,3,5,22$	y_{10}	H
R_2	7	$14,16,18$	y_{10}	H	R_2	7	$3,3,3,5,22$	y_{10}	H
R_2	8	$3,12,18,24$	y_1	S	R_3	1	$1,1,1,4,36$	y_8	S
R_3	1	$1,1,2,4,36$	y_8	S	R_3	1	$2,2,1,4,36$	y_8	S
R_3	1	$2,2,2,4,36$	y_8	S	R_3	1	$1,1,1,4,13$	y_7	S
R_3	1	$1,1,2,4,13$	y_7	S	R_3	1	$2,2,1,4,13$	y_7	S
R_3	1	$2,2,2,4,13$	y_7	S	R_3	2	$3,3,1,4,36$	y_8	S
R_3	2	$3,3,2,4,36$	y_8	S	R_3	2	$4,4,1,4,36$	y_8	S
R_3	2	$4,4,2,4,36$	y_8	S	R_3	2	$6,6,1,4,36$	y_8	S
R_3	2	$6,6,2,4,36$	y_8	S	R_3	2	$4,4,1,4,13$	y_7	S
R_3	2	$4,4,2,4,13$	y_7	S	R_3	2	$6,6,1,4,13$	y_7	S
R_3	2	$6,6,2,4,13$	y_7	S	R_3	2	$1,3,8,42$	y_3	S
R_3	2	$2,3,8,42$	y_3	S	R_3	3	$1,3,7,11$	y_9	S
R_3	3	$2,3,7,11$	y_9	S	R_3	3	$1,3,4,7,30$	y_9	S
R_3	3	$2,3,4,7,30$	y_9	S	R_3	3	$3,3,1,4,13$	y_7	S
R_3	3	$3,3,2,4,13$	y_7	S	R_3	3	$1,3,11,18$	y_2	S
R_3	3	$2,3,11,18$	y_2	S	R_3	3	$1,3,4,18,30$	y_2	S
R_3	3	$2,3,4,18,30$	y_2	S	R_3	8	$3,4,8,13$	y_1	S
R_3	8	$3,6,11,18$	y_1	S	R_3	8	$3,4,6,18,30$	y_1	S
R_4	1	$1,1,1,4,5,24$	y_7	S	R_4	1	$1,1,2,4,5,24$	y_7	S
R_4	1	$2,2,1,4,5,24$	y_7	S	R_4	1	$2,2,2,4,5,24$	y_7	S
R_4	2	$4,4,1,4,5,24$	y_7	S	R_4	2	$4,4,2,4,5,24$	y_7	S
R_4	3	$3,3,1,4,5,24$	y_7	S	R_4	3	$3,3,2,4,5,24$	y_7	S
R_4	3	$1,3,4,5,11$	y_3	S	R_4	3	$2,3,4,5,11$	y_3	S
R_4	3	$1,3,17$	y_3	S	R_4	3	$2,3,17$	y_3	S
R_4	3	$1,3,24,30$	y_2	S	R_4	3	$2,3,24,30$	y_2	S
R_4	8	$4,13,18$	y_1	S	R_4	8	$3,4,9,11$	y_1	S
R_5	1	$1,1,17$	y_9	H	R_5	1	$1,2,17$	y_9	H
R_5	1	$2,2,17$	y_9	H	R_5	1	$1,1,1,7,24$	y_7	S
R_5	1	$1,1,2,7,24$	y_7	S	R_5	1	$2,2,1,7,24$	y_7	S
R_5	1	$2,2,2,7,24$	y_7	S	R_5	1	$1,1,1,9,30$	y_6	S
R_5	1	$1,1,2,9,30$	y_6	S	R_5	1	$2,2,1,9,30$	y_6	S
R_5	1	$2,2,2,9,30$	y_6	S	R_5	2	$3,3,17$	y_9	H
R_5	2	$6,6,17$	y_9	H	R_5	2	$1,6,7,9,10$	y_7	S

continued

Ψ	Case	$\boldsymbol{k}$	ST	S/H	Ψ	Case	$\boldsymbol{k}$	ST	S/H
R_5	2	$2,6,7,9,10$	y_7	S	R_5	2	$6,6,1,7,24$	y_7	S
R_5	2	$6,6,2,7,24$	y_7	S	R_5	2	$1,6,7,9,12$	y_6	S
R_5	2	$2,6,7,9,12$	y_6	S	R_5	2	$6,6,1,9,30$	y_6	S
R_5	2	$6,6,2,9,30$	y_6	S	R_5	3	$3,3,1,7,24$	y_7	S
R_5	3	$3,3,2,7,24$	y_7	S	R_5	3	$3,3,1,9,30$	y_6	S
R_5	3	$3,3,2,9,30$	y_6	S	R_5	3	$1,3,24,30$	y_2	S
R_5	3	$2,3,24,30$	y_2	S	R_5	8	$13,16$	y_1	S
R_5	8	$16,42$	y_1	S	R_5	8	$3,6,24,30$	y_1	S
R_5	8	$3,7,12,24$	y_1	S	R_5	8	$3,9,10,30$	y_1	S
R_6	1	$1,1,1,4,5,24$	y_7	S	R_6	1	$1,1,2,4,5,24$	y_7	S
R_6	1	$2,2,1,4,5,24$	y_7	S	R_6	1	$2,2,2,4,5,24$	y_7	S
R_6	2	$21,24$	y_7	S	R_6	2	$4,4,1,4,5,24$	y_7	S
R_6	2	$4,4,2,4,5,24$	y_7	S	R_6	2	$1,3,5,36$	y_4	S
R_6	2	$2,3,5,36$	y_4	S	R_6	2	$1,3,8,42$	y_3	S
R_6	2	$2,3,8,42$	y_3	S	R_6	3	$3,3,1,4,5,24$	y_7	S
R_6	3	$3,3,2,4,5,24$	y_7	S	R_6	3	$1,3,24,30$	y_2	S
R_6	3	$2,3,24,30$	y_2	S	R_6	7	$4,8,14,16$	y_{10}	H
R_6	8	$3,4,5,8,24$	y_1	S	R_6	8	$4,19$	y_1	S
R_7	1	$1,1,1,7,16$	y_7	S	R_7	1	$1,1,1,7,24$	y_7	S
R_7	1	$1,1,2,7,16$	y_7	S	R_7	1	$1,1,2,7,24$	y_7	S
R_7	1	$2,2,1,7,16$	y_7	S	R_7	1	$2,2,1,7,24$	y_7	S
R_7	1	$2,2,2,7,16$	y_7	S	R_7	1	$2,2,2,7,24$	y_7	S
R_7	3	$3,3,1,7,16$	y_7	S	R_7	3	$3,3,1,7,24$	y_7	S
R_7	3	$3,3,2,7,16$	y_7	S	R_7	3	$3,3,2,7,24$	y_7	S
R_7	3	$1,3,7,11$	y_5	S	R_7	3	$2,3,7,11$	y_5	S
R_7	3	$1,3,5,7,9$	y_5	S	R_7	3	$2,3,5,7,9$	y_5	S
R_7	3	$1,3,16,30$	y_2	S	R_7	3	$1,3,24,30$	y_2	S
R_7	3	$2,3,16,30$	y_2	S	R_7	3	$2,3,24,30$	y_2	S
R_8	1	$1,1,1,4,12,30$	y_6	S	R_8	1	$1,1,2,4,12,30$	y_6	S
R_8	1	$2,2,1,4,12,30$	y_6	S	R_8	1	$2,2,2,4,12,30$	y_6	S
R_8	1	$1,1,1,3,12,30$	y_5	S	R_8	1	$1,1,2,3,12,30$	y_5	S
R_8	1	$2,2,1,3,12,30$	y_5	S	R_8	1	$2,2,2,3,12,30$	y_5	S
R_8	2	$4,4,1,4,12,30$	y_6	S	R_8	2	$4,4,2,4,12,30$	y_6	S
R_8	2	$1,12,14,16$	y_5	S	R_8	2	$2,12,14,16$	y_5	S
R_8	2	$3,3,1,3,12,30$	y_5	S	R_8	2	$3,3,2,3,12,30$	y_5	S
R_8	2	$1,3,12,36$	y_4	S	R_8	2	$2,3,12,36$	y_4	S
R_8	3	$1,3,4,7,30$	y_9	S	R_8	3	$2,3,4,7,30$	y_9	S
R_8	3	$1,3,5,42$	y_8	S	R_8	3	$2,3,5,42$	y_8	S
R_8	3	$1,3,5,7,18$	y_8	S	R_8	3	$2,3,5,7,18$	y_8	S
R_8	3	$3,3,1,4,12,30$	y_6	S	R_8	3	$3,3,2,4,12,30$	y_6	S
R_8	3	$4,4,1,3,12,30$	y_5	S	R_8	3	$4,4,2,3,12,30$	y_5	S
R_8	3	$1,3,12,42$	y_3	S	R_8	3	$2,3,12,42$	y_3	S
R_8	3	$1,3,7,12,18$	y_3	S	R_8	3	$2,3,7,12,18$	y_3	S
R_8	3	$1,3,4,5,7,12$	y_2	S	R_8	3	$2,3,4,5,7,12$	y_2	S
R_8	6	$3,4,7,12,14$	y_1	S	R_9	1	$1,2,3,4,28$	y_9	H
R_9	1	$1,1,3,4,28$	y_9	H	R_9	1	$2,2,3,4,28$	y_9	H
R_9	1	$1,1,1,4,8,24$	y_8	S	R_9	1	$1,1,2,4,8,24$	y_8	S
R_9	1	$2,2,1,4,8,24$	y_8	S	R_9	1	$2,2,2,4,8,24$	y_8	S
R_9	1	$1,1,1,7,24$	y_7	S	R_9	1	$1,1,2,7,24$	y_7	S
R_9	1	$1,1,1,4,12,24$	y_7	S	R_9	1	$1,1,2,4,12,24$	y_7	S
R_9	1	$2,2,1,7,24$	y_7	S	R_9	1	$2,2,2,7,24$	y_7	S
R_9	1	$2,2,1,4,12,24$	y_7	S	R_9	1	$2,2,2,4,12,24$	y_7	S
R_9	2	$3,3,3,4,28$	y_9	H	R_9	2	$4,4,3,4,28$	y_9	H

continued

Ψ	Case	$\boldsymbol{k}$	ST	S/H	Ψ	Case	$\boldsymbol{k}$	ST	S/H
R_9	2	$3,3,1,4,8,24$	y_8	S	R_9	2	$3,3,2,4,8,24$	y_8	S
R_9	2	$4,4,1,4,8,24$	y_8	S	R_9	2	$4,4,2,4,8,24$	y_8	S
R_9	2	$4,4,1,7,24$	y_7	S	R_9	2	$4,4,2,7,24$	y_7	S
R_9	2	$4,4,1,4,12,24$	y_7	S	R_9	2	$4,4,2,4,12,24$	y_7	S
R_9	3	$3,3,1,7,24$	y_7	S	R_9	3	$3,3,2,7,24$	y_7	S
R_9	3	$3,3,1,4,12,24$	y_7	S	R_9	3	$3,3,2,4,12,24$	y_7	S
R_9	3	$1,3,12,42$	y_3	S	R_9	3	$2,3,12,42$	y_3	S
R_9	5	$1,2,3,16,18$	y_{10}	H	R_9	5	$1,1,3,16,18$	y_{10}	H
R_9	5	$2,2,3,16,18$	y_{10}	H	R_9	7	$3,3,3,16,18$	y_{10}	H
R_9	7	$4,4,3,16,18$	y_{10}	H	R_9	8	$30,42$	y_1	S
R_9	8	$3,12,18,24$	y_1	S	R_{10}	1	$1,2,3,4,36$	y_9	H
R_{10}	1	$1,1,3,4,36$	y_9	H	R_{10}	1	$2,2,3,4,36$	y_9	H
R_{10}	1	$1,1,1,4,12,24$	y_7	S	R_{10}	1	$1,1,2,4,12,24$	y_7	S
R_{10}	1	$2,2,1,4,12,24$	y_7	S	R_{10}	1	$2,2,2,4,12,24$	y_7	S
R_{10}	2	$3,3,3,4,36$	y_9	H	R_{10}	2	$4,4,3,4,36$	y_9	H
R_{10}	2	$4,4,1,4,12,24$	y_7	S	R_{10}	2	$4,4,2,4,12,24$	y_7	S
R_{10}	3	$3,3,1,4,12,24$	y_7	S	R_{10}	3	$3,3,2,4,12,24$	y_7	S
R_{10}	8	$3,12,18,24$	y_1	S					

8.2 Minimum Entropy

McMullen [19, Theorem (A.1)] shows that any automorphism f of a compact complex surface with a positive entropy $h(f) > 0$ has a lower bound $h(f) \ge \log \lambda_{\mathrm{L}}$, where λ_{L} is Lehmer's number in Example 5.5. In [20] he obtains some non-projective K3 examples that actually attain this lower bound and he goes on to construct projective ones in [21]. Our method enables us to synthesize a goodly number of non-projective K3 surface automorphisms with minimum entropy $\log \lambda_{\mathrm{L}}$, which necessarily have Picard number 12.

8.2.1 Use of Matrix $\boldsymbol{A}$

Suppose that $\Phi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{\boldsymbol{k}}(w)$, where $\mathrm{LT}(w)$ is Lehmer's trace polynomial in (47b) and $\mathrm{CT}_{\boldsymbol{k}}(w)$ is a product of cyclotomic trace polynomials of the form

$$\mathrm{CT}_{\boldsymbol{k}}(w) := \prod_{k \in \boldsymbol{k}} \mathrm{CT}_k(w) \qquad \text{with} \qquad \sum_{k \in \boldsymbol{k}} \deg \mathrm{CT}_k(w) = 5, \tag{64}$$

where $\boldsymbol{k}$ is a set of positive integers whose elements k come from Table 5 with deg ≤ 5. Suppose that $\Psi(w) = R_i(w)$ is one of the Salem trace polynomials in Table 6. We remark that $\boldsymbol{k}$ is not a multi-set but an ordinary set because an assertion of Theorem 1.2 rules out the occurrence of multiple elements. To construct K3 surface automorphisms we utilize the matrix A and its modification $\tilde{A}$. Let $x_4 < x_3 < x_2 < x_1$ denote the roots in $[-2, 2]$ of Lehmer's trace polynomial $\mathrm{LT}(w)$ in (47b), whose numerical values are given by

$$x_4 \approx -1.88660, \quad x_3 \approx -1.46887, \quad x_2 \approx -0.584663, \quad x_1 \approx 0.913731. \tag{65}$$

Theorem 8.2 *In the above setting we have a hypergeometric K3 lattice such that the matrix A admits a special eigenvalue δ conjugate to Lehmer's number λ_{L}, if and only if $\Psi(w) = R_i(w)$ and $\boldsymbol{k}$ are as in Table* 11*, where the special trace $\tau := \delta + \delta^{-1}$ is given in the "ST" column. The modified positive Hodge isometry $\tilde{A} := w_A \circ A : L \to L$ lifts to a K3 surface automorphism $f : X \to X$ of entropy $h(f) = \log \lambda_{\mathrm{L}}$ with special trace $\tau(f) = \tau$ and Picard number $\rho(X) = 12$, having exceptional set $\mathcal{E}(X)$ of Dynkin type indicated in Table* 11*. The matrix $\tilde{A}$ or the map $f^*|H^2(X,\mathbb{Z})$ has characteristic polynomial $\tilde{\varphi}(z) = \mathrm{L}(z) \cdot \tilde{\varphi}_1(z)$, where $\mathrm{L}(z)$ is Lehmer's polynomial in* (47a) *and $\tilde{\varphi}_1(z)$ is given in Table* 11*. The value of $\mathrm{Tr}\,\tilde{A} = \mathrm{Tr}\, f^*|H^2(X,\mathbb{Z})$ is also included there. The "S/H" column is explained in Theorems* 9.4 *and* 9.5.

Table 11: K3 surface automorphisms of minimum entropy from matrix $\tilde{A}$.

Ψ	$\boldsymbol{k}$	ST	Dynkin type	$\tilde{\varphi}_1(z)$	$\mathrm{Tr}\,\tilde{A}$	S/H
R_1	$4,20$	x_4	$\mathrm{E}_6 \oplus \mathrm{E}_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
R_1	$4,6,7$	x_2	D_{10}	$(z-1)^9(z+1)(z^2+1)$	7	S
R_3	$3,15$	x_4	A_2	$(z-1)^2(z^2+z+1)\,\mathrm{C}_{15}(z)$	1	S
R_3	$3,4,6,8$	x_3	$\mathrm{E}_6 \oplus \mathrm{E}_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
R_3	$4,6,18$	x_2	D_{10}	$(z-1)^9(z+1)(z^2+1)$	7	S
R_4	$3,4,9$	x_4	D_{10}	$(z-1)^9(z+1)(z^2+1)$	7	H
R_4	$4,24$	x_2	$\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$(z-1)^9(z+1)(z^2+1)$	7	S
R_4	$4,20$	x_1	$\mathrm{E}_6 \oplus \mathrm{E}_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
R_5	$7,12$	x_3	$\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
R_9	$4,20$	x_4	$\mathrm{E}_6 \oplus \mathrm{E}_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
R_9	$12,18$	x_3	$\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
R_9	$4,30$	x_1	$\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
R_{10}	$4,16$	x_4	$\mathrm{E}_6 \oplus \mathrm{E}_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
R_{10}	$4,24$	x_2	$\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$(z-1)^9(z+1)(z^2+1)$	7	S
R_{10}	$3,15$	x_1	A_2	$(z-1)^2(z^2+z+1)\,\mathrm{C}_{15}(z)$	1	S

Proof. First, pick out all pairs $(i, \boldsymbol{k})$ such that $\Phi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{\boldsymbol{k}}(w)$ and $\Psi(w) = R_i(w)$ satisfy unimodularity condition (46), where $\boldsymbol{k}$ must be subject to the degree constraint in (64). Secondly, determine all the correct solutions as well as their special traces by using the hyperbolic case of Theorem 1.2. Since the special trace comes from Lehmer's polynomial factor, the matrix $F = A$ is a positive Hodge isometry by Remark 6.6. Thirdly, for each solution run Algorithm 7.5 to find the set of positive roots Δ^+, its basis Δ_{b}, its Dynkin type and the modified matrix $\tilde{A}$. The characteristic polynomial $\tilde{\varphi}(z) = \varphi_0(z) \cdot \tilde{\varphi}_1(z)$ of $\tilde{A}$ is determined by the formulas (54) and (62) together with the non-projectivity assumption $\varphi_0(z) = \mathrm{L}(z)$, where $\chi(z) = \varphi(z)$ and $\chi_i(z) = \varphi_i(z)$. $\square$

We illustrate how Algorithm 7.5 works for an entry in Table 11.

Example 8.3 Consider the case where $\Phi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{\boldsymbol{k}}(w)$ with $\boldsymbol{k} = \{3, 4, 6, 8\}$ and $\Psi(w) = R_3(w)$ in Table 11. Steps 1–3 of Algorithm 7.5 return us 72 elements $\mathbf{0} \prec \boldsymbol{u}_1 \prec \boldsymbol{u}_2 \prec \cdots \prec \boldsymbol{u}_{72}$ in the lexicographic order (60) for the set of positive roots Δ^+ and then 12 elements for the basis Δ_{b}, implying that the root system is of type $\mathrm{E}_6 \oplus \mathrm{E}_6$. If the basis $\Delta_{\mathrm{b}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_6, \boldsymbol{e}'_1, \dots, \boldsymbol{e}'_6\}$ is labeled as in Figure 2 then the simple roots are given by

$$\begin{aligned} &\boldsymbol{e}_1 = \boldsymbol{u}_{23}, \quad \boldsymbol{e}_2 = \boldsymbol{u}_8, \quad \boldsymbol{e}_3 = \boldsymbol{u}_1, \quad \boldsymbol{e}_4 = \boldsymbol{u}_3, \quad \boldsymbol{e}_5 = \boldsymbol{u}_{16}, \quad \boldsymbol{e}_6 = \boldsymbol{u}_{25}, \\ &\boldsymbol{e}'_1 = \boldsymbol{u}_7, \quad \boldsymbol{e}'_2 = \boldsymbol{u}_{24}, \quad \boldsymbol{e}'_3 = \boldsymbol{u}_2, \quad \boldsymbol{e}'_4 = \boldsymbol{u}_5, \quad \boldsymbol{e}'_5 = \boldsymbol{u}_9, \quad \boldsymbol{e}'_6 = \boldsymbol{u}_{35}. \end{aligned}$$

Steps 4–6 tell us that the Weyl group element $w_A \in W$ bringing $A(\mathcal{K})$ back to $\mathcal{K}$ is

$$w_A = \sigma_5 \circ \sigma_{23} \circ \sigma_{35} \circ \sigma_{41} \circ \sigma_{62} \circ \sigma_{57} \circ \sigma_{72},$$

where σ_j denotes the Picard-Lefschetz reflection corresponding to the j-th positive root $\boldsymbol{u}_j$. Step 7, that is, how the modified matrix $\tilde{A}$ acts on Δ_{b} will be mentioned in §8.2.3.

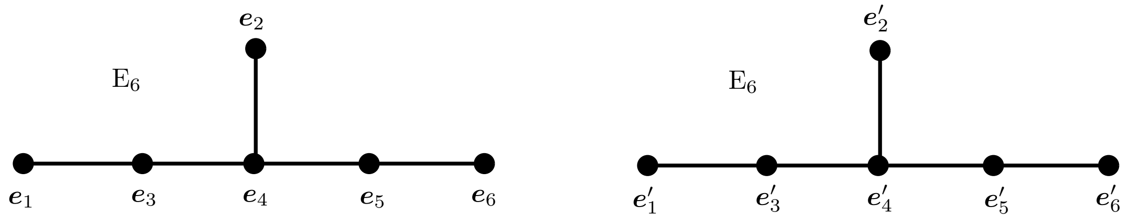

Figure 2: Dynkin diagram of type $\mathrm{E}_6 \oplus \mathrm{E}_6$.

Remark 8.4 A careful inspection shows that all entries in Table 11 fall into case 7 of Table 2. However, other cases can occur in other settings, although empirically they are rather rare. The following two examples are in cases 1 and 9 of Table 2, respectively:

$$\Phi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{18}(w) \cdot \mathrm{CT}_5(w), \quad \Psi(w) = \{(w+1)(w^2-4)-1\} \cdot \mathrm{CT}_{60}(w),$$

$$\Phi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{15}(w) \cdot \mathrm{CT}_4(w), \quad \Psi(w) = \{w(w^2-1)(w^2-3)(w^2-4)-1\} \cdot \mathrm{CT}_{24}(w).$$

8.2.2 Use of Matrix B

Suppose that $\Phi(w)$ is a product of cyclotomic trace polynomials of the form (63), while $\Psi(w)$ is a product of Lehmer's trace polynomial and cyclotomic trace polynomials

$$\Psi(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{\boldsymbol{l}}(w) \qquad \text{such that} \qquad \sum_{l \in \boldsymbol{l}} \deg \mathrm{CT}_l(w) = 6.$$

To construct K3 surface automorphisms we utilize the matrix B and its modification $\tilde{B}$. By Theorem 1.3 any element of $\boldsymbol{l}$ must be simple and the unramifiedness condition in (46) reduces the possibilities of $\boldsymbol{l}$ considerably, confining $\Psi(w)$ into one of the following possibilities.

$$L_1(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{21}(w), \qquad L_2(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{28}(w),$$
$$L_3(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{36}(w), \qquad L_4(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{42}(w),$$
$$L_5(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{12}(w) \cdot \mathrm{CT}_{15}(w), \qquad L_6(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{12}(w) \cdot \mathrm{CT}_{20}(w),$$
$$L_7(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{12}(w) \cdot \mathrm{CT}_{24}(w), \qquad L_8(w) = \mathrm{LT}(w) \cdot \mathrm{CT}_{12}(w) \cdot \mathrm{CT}_{30}(w).$$

Table 12: K3 surface automorphisms of minimum entropy from matrix $\tilde{B}$.

Ψ	case	$\boldsymbol{k}$	ST	Dynkin type	$\tilde{\psi}_1(z)$	$\mathrm{Tr}\,\tilde{B}$	S/H
L_3	2	$3,6,10,21$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_3	2	$1,6,8,28$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	2	$2,6,8,28$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	2	$8,10,42$	x_1	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	3	$1,3,5,6,11$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	3	$1,3,7,11$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	3	$2,3,5,6,11$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	3	$2,3,7,11$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_3	3	50	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_6	1	$1,1,8,13$	x_4	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	S
L_6	1	$1,2,8,13$	x_4	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	S
L_6	1	$2,2,8,13$	x_4	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	S
L_7	1	$1,1,17$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_7	1	$1,2,17$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_7	1	$2,2,17$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_7	1	$1,27$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_7	1	$2,27$	x_4	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	H
L_7	3	$16,42$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_7	3	$5,20,30$	x_2	$E_6 \oplus E_6$	$(z-1)^4(z+1)^4(z^2+1)^2$	-1	S
L_8	1	$1,1,1,7,16$	x_3	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
L_8	1	$1,1,2,7,16$	x_3	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
L_8	1	$2,2,1,7,16$	x_3	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
L_8	1	$2,2,2,7,16$	x_3	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	H
L_8	2	$7,9,20$	x_3	$E_8 \oplus A_2 \oplus A_2$	$(z-1)^9(z+1)(z^2+1)$	7	H

Theorem 8.5 *In the above setting we have a hypergeometric K3 lattice such that the matrix B admits a special eigenvalue δ conjugate to Lehmer's number λ_{L}, if and only if $\Psi(w)$ and $\boldsymbol{k}$ are as in Table 12, where the special trace $\tau := \delta + \delta^{-1}$ is given in the "ST" column while the "case" column refers to the case in Table 3. The modified positive Hodge isometry $\tilde{B} := w_B \circ B : L \to L$ lifts to a K3 surface automorphism $f : X \to X$ of entropy $h(f) = \log \lambda_{\mathrm{L}}$ with special trace $\tau(f) = \tau$ and Picard number $\rho(X) = 12$, having exceptional set $\mathcal{E}(X)$ of Dynkin type indacated in Table 12. The matrix $\tilde{B}$ or the map $f^*|H^2(X, \mathbb{Z})$ has characteristic polynomial $\tilde{\psi}(z) = \mathrm{L}(z) \cdot \tilde{\psi}_1(z)$ with $\tilde{\psi}_1(z)$ given in Table 12. The value of $\mathrm{Tr}\, \tilde{B} = \mathrm{Tr}\, f^*|H^2(X, \mathbb{Z})$ is also included there. The "S/H" column is explained in Theorems 9.4 and 9.5.*

Proof. First, pick out all pairs $(\boldsymbol{k}, i)$ such that $\Phi(w) = \mathrm{CT}_{\boldsymbol{k}}(w)$ and $\Psi(w) = L_i(w)$ satisfy unimodularity condition (46), where $\boldsymbol{k}$ must be subject to the degree constraint in (63). Secondly, determine all the correct solutions as well as their special traces by using Theorem 1.3. Since the special trace comes from Lehmer's polynomial factor, the matrix $F = B$ is a positive Hodge isometry by Remark 6.6. Thirdly, for each solution run Algorithm 7.5 to find the set of positive roots Δ^+, its basis Δ_{b}, its Dynkin type and the modified matrix $\tilde{B}$. The characteristic polynomial $\tilde{\psi}(z) = \psi_0(z) \cdot \tilde{\psi}_1(z)$ of $\tilde{B}$ is determined by the formulas (54) and (62) together with the non-projectivity assumption $\psi_0(z) = \mathrm{L}(z)$, where $\chi(z) = \psi(z)$ and $\chi_i(z) = \psi_i(z)$. □

In Table 12 only four polynomials $L_3(w)$, $L_6(w)$, $L_7(w)$, $L_8(w)$ appear as $\Psi(w)$.

8.2.3 Action on Dynkin Diagram

Denote by $\tilde{F}$ the modified matrix $\tilde{A}$ or $\tilde{B}$. We are interested in how $\tilde{F}$ acts on the simple roots Δ_{b}. Let $\tilde{\chi}_1(z)$ denote the polynomial $\tilde{\varphi}_1(z)$ in Table 11 or $\tilde{\psi}_1(z)$ in Table 12. Then $\tilde{\chi}_1(z)$ is divisible by the characteristic polynomial of $\tilde{F}|\mathrm{Span}\, \Delta_{\mathrm{b}}$. Observe that Tables 11 and 12 contain only root systems of types

$$\mathrm{E}_6 \oplus \mathrm{E}_6, \qquad \mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2, \qquad \mathrm{D}_{10}, \qquad \mathrm{A}_2. \tag{66}$$

With the polynomials $\tilde{\chi}_1(z)$ in Tables 11 and 12 we can identify the action of $\tilde{F}$ on Δ_{b} in a unique way by enumerating all Dynkin automorphisms of types (66) and their characteristic polynomials. This may not be true if the root system is different from (66) or if $\tilde{\chi}_1(z)$ is a different polynomial. Moreover, it is a consequence of direct calculations that all entries of the same Dynkin type have the same polynomial $\tilde{\chi}_1(z)$. We do not know if this is just by accident or with any reason. In any case, within Tables 11 and 12, how $\tilde{F}$ acts on Δ_{b} depends only on the Dynkin type of Δ_{b}.

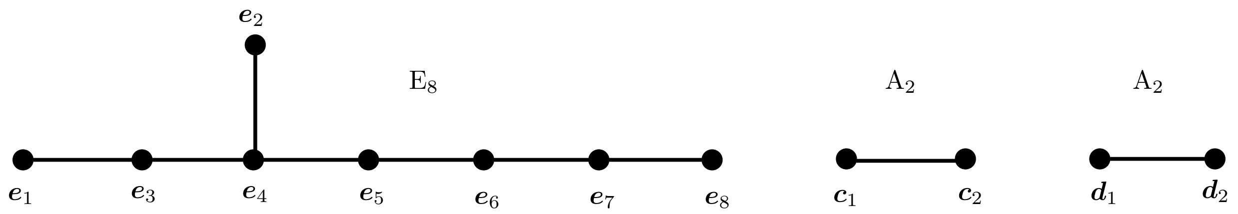

Figure 3: Dynkin diagram of type $\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$.

Observation 8.6 According to Dynkin types we have the following observations, where $(c_1, c_2, \dots, c_k)$ stands for the cyclic permutation $c_1 \to c_2 \to \cdots \to c_k \to c_1$.

(1) In case of type $\mathrm{E}_6 \oplus \mathrm{E}_6$ the matrix $\tilde{F}$ acts on the simple roots in Figure 2 by

$$(\boldsymbol{e}_1, \boldsymbol{e}_1', \boldsymbol{e}_6, \boldsymbol{e}_6')(\boldsymbol{e}_3, \boldsymbol{e}_3', \boldsymbol{e}_5, \boldsymbol{e}_5')(\boldsymbol{e}_2, \boldsymbol{e}_2')(\boldsymbol{e}_4, \boldsymbol{e}_4').$$

In particular $\tilde{F}$ exchanges the two connected E_6-components.

(2) In case of type $\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$ the matrix $\tilde{F}$ fixes $\boldsymbol{e}_1, \dots, \boldsymbol{e}_8$ in the E_8-component while it acts on the simple roots in the $\mathrm{A}_2 \oplus \mathrm{A}_2$-component by $(\boldsymbol{c}_1, \boldsymbol{d}_1, \boldsymbol{c}_2, \boldsymbol{d}_2)$ in Figure 3.

(3) In case of type D_{10} the matrix $\tilde{F}$ fixes $\boldsymbol{e}_1, \dots, \boldsymbol{e}_8$ and exchanges $\boldsymbol{e}_9$ and $\boldsymbol{e}_{10}$ in Figure 4.

(4) In case of type A_2 the matrix $\tilde{F}$ fixes all simple roots.

Observation 8.6 shows how the (-2)-curves in X are permuted by the K3 surface automorphism $f : X \to X$ arising from the Hodge isometry $\tilde{F} : L \to L$ (see Lemma 7.3).

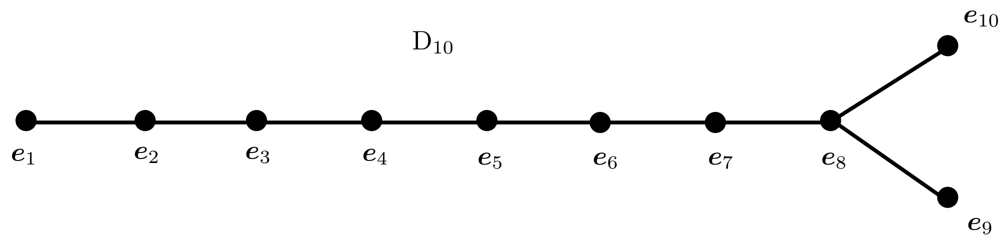

Figure 4: Dynkin diagram of type D_{10}.

9 Siegel Disks

Let $\mathbb{D}$ and S^1 be the unit open disk and the unit circle in $\mathbb{C}$ respectively. Given $(\alpha_1, \alpha_2) \in T := S^1 \times S^1$, the map $g : \mathbb{D}^2 \to \mathbb{D}^2$, $(z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$ is said to be an *irrational rotation* if $g : T \to T$ has dense orbits; this condition is equivalent to saying that α_1 and α_2 are multiplicatively independent, meaning that $\alpha_1^{m_1} \alpha_2^{m_2} = 1$, $m_1, m_2 \in \mathbb{Z}$, implies $m_1 = m_2 = 0$. Let $f : X \to X$ be an automorphism of a complex surface X, and U be an open neighborhood of a point $p \in X$. We say that f has a *Siegel disk* U centered at p if $f(U, p) = (U, p)$ and $f|(U, p)$ is biholomorphically conjugate to an irrational rotation $g|(\mathbb{D}^2, 0)$. Namely, a Siegel disk is an invariant subset modeled on an irrational rotation around the origin.

9.1 Existence of a Siegel Disk

McMullen [18] synthesizes examples of K3 surface automorphisms with a Siegel disk from Salem numbers of degree 22. To this end he establishes a sufficient condition for the existence of a Siegel disk (see [18, Theorems 7.1 and 9.2]). We shall relax his criterion so that it may be applied to Salem numbers of lower degrees in the presence of exceptional sets. Let $F : L \to L$ be an automorphism of a K3 lattice L admitting a special eigenvalue $\delta \in S^1$ and preserving a K3 structure. Let $f : X \to X$ be the K3 surface automorphism obtained as the lift of F. In [18] the number δ is called the *determinant* of f because δ is equal to the determinant of the holomorphic tangent map $(df)_p : T_pX \to T_pX$ at any fixed point p of f. Thus the eigenvalues of $(df)_p$ can be expressed as $\alpha_1 = \delta^{1/2}\, \alpha$ and $\alpha_2 = \delta^{1/2}\, \alpha^{-1}$ for some $\alpha \in \mathbb{C}^{\times}$.

Proposition 9.1 *Suppose that the special eigenvalue δ is conjugate to a Salem number and there exists a rational function $q(w) \in \mathbb{Q}(w)$ such that $(\alpha + \alpha^{-1})^2 = q(\tau)$ for the special trace $\tau := \delta + \delta^{-1} \in (-2, 2)$. If τ satisfies $0 \le q(\tau) \le 4$ and admits a conjugate $\tau' \in (-2, 2)$ such that $q(\tau') > 4$, then the fixed point $p \in X$ is the center of a Siegel disk for the map f. If τ satisfies $q(\tau) > 4$ then p is a hyperbolic fixed point of f.*

Proof. Since $\tau \in (-2, 2)$, we have $\delta \in S^1$ and $\delta^{1/2} \in S^1$. It follows from $0 \le q(\tau) \le 4$ that $\alpha \in S^1$ and hence α_1, $\alpha_2 \in S^1$. Note that α is an algebraic number. To show α_1, $\alpha_2 \in \overline{\mathbb{Q}}$ are multiplicatively independent we mimic the proof of [18, Lemma 7.5]. Let δ' and α' be the conjugates of δ and α corresponding to τ'. We have $\delta' \in S^1$ and $\alpha' \not\in S^1$ by $\tau' \in (-2, 2)$ and $\{\alpha' + (\alpha')^{-1}\}^2 = q(\tau') > 4$. If $\alpha_1^m \alpha_2^n = \delta^{(m+n)/2} \alpha^{m-n} = 1$ with m, $n \in \mathbb{Z}$ then $(\delta')^{(m+n)/2} (\alpha')^{m-n} = 1$ and hence $|\alpha'|^{m-n} = 1$, which forces $m = n$ and $\delta^m = 1$, but the latter equation implies $m = 0$ because δ is not a root of unity. The Gel'fond-Baker method then shows that they are jointly Diophantine and the Siegel-Sternberg theory tells us that there exists a Siegel disk centered at p (see [18, Theorems 5.2 and 5.3]). On the other hand, if $q(\tau) > 4$ then $\delta^{1/2} \in S^1$ and $\alpha \not\in S^1$, so we have $|\alpha_i| < 1 < |\alpha_j|$ for $(i, j) = (1, 2)$ or $(2, 1)$. Thus p is a hyperbolic fixed point of f. □

When f has no fixed curves in X, McMullen [18] uses Proposition 9.1 with the following setting

$$\text{(a)} \quad \mathrm{Tr}\,[F : L \to L] = -1; \qquad \text{(b)} \quad q(w) = \frac{(w+1)^2}{w+2}, \tag{67}$$

by showing that under condition (a) the Lefschetz fixed point formula implies the existence of a unique transverse fixed point $p \in X$ of f and the Atiyah-Bott formula determines $q(w)$ in the form (b). To guarantee the non-existence of fixed curves, he takes Salem numbers of degree 22 so that the Picard lattice and root system in it are empty. His conclusion is that the unique fixed point p is the center of a Siegel disk, provided

$$\tau > \tau_0 := 1 - 2\sqrt{2} \approx -1.8284271 \text{ and } \tau \text{ has a conjugate } \tau' < \tau_0. \tag{68}$$

To any simple root $\boldsymbol{u} \in \Delta_{\mathrm{b}}$ the corresponding (-2)-curve in X is denoted by the same symbol $\boldsymbol{u}$. If F fixes a simple root $\boldsymbol{u}$ then f preserves the curve $\boldsymbol{u} \cong \mathbb{P}^1$, inducing a Möbius transformation on it, so that $\boldsymbol{u}$ is a fixed curve of f precisely when the induced map is identity. In the possible occurrence of fixed curves we have to use fixed point formulas stronger than the classical Lefschetz and Atiyah-Bott formulas (see §9.3). Any version of Lefschetz-type formula involves the value of $\mathrm{Tr}\, f^*|H^2(X)$ and the following remark is helpful in this respect.

Remark 9.2 The *trace* of a monic polynomial $P(z) = z^d + c_1 z^{d-1} + \cdots + c_d$ is defined by $\mathrm{Tr}\, P := -c_1$, the sum of its roots. A palindromic polynomial of even degree and its trace polynomial have the same trace. Let $\chi(z)$ be the characteristic polynomial of $F : L \to L$. It is a palindromic polynomial of degree 22. We can calculate $\mathrm{Tr}\, f^*|H^2(X) = \mathrm{Tr}\, F$ as the trace of $\chi(z)$ or equivalently as the trace of its trace polynomial.

9.2 Salem Numbers of Degree 22

Our result on Siegel disks for Salem numbers of degree 22 is stated as follows.

Theorem 9.3 *In Theorem* 8.1 *each K3 surface automorphism $f : X \to X$ has a unique fixed point $p \in X$, which is either the center of a Siegel disk or a hyperbolic fixed point, where the former case is indicated by "S" while the latter case by "H" in the last columns of Table* 10.

Proof. Observe from Table 6 that $\mathrm{Tr}\, R_i = -1$ for $i = 1, \dots, 10$. Condition (a) in (67) is checked by Remark 9.2. Following the framework (67)-(68) the theorem can be established by Proposition 9.1. □

We remark that Table 10 contains a total of 255 entries, among which 222 are "S" and 33 are "H". A majority of the entries have $\boldsymbol{k}$'s with a multiple element.

9.3 Minimum Entropy

We discuss the existence of Siegel disks for K3 surface automorphisms constructed in Theorems 8.2 and 8.5. Let $x_4 < x_3 < x_2 < x_1$ be the roots in $[-2, 2]$ of Lehmer's trace polynomial $\mathrm{LT}(w)$ as in (65). We denote by $\mathcal{E}$ the exceptional set in X (see Lemma 7.3). Even when $\mathcal{E}$ is nonempty, the framework (67)-(68) remains valid if f has no irreducible fixed curve in $\mathcal{E}$. In particular this is the case if $\tilde{F}$ acts on Δ_{b} freely.

Theorem 9.4 *For each entry of Dynkin type* $\mathrm{E}_6 \oplus \mathrm{E}_6$ *in Tables* 11 *and* 12*, the K3 surface automorphism $f : X \to X$ has no fixed point on $\mathcal{E}$ and a unique fixed point $p \in X \setminus \mathcal{E}$. If the special trace τ is any one of x_1, x_2, x_3 then p is the center of a Siegel disk, while if $\tau = x_4$ then p is a hyperbolic fixed point; this information is indicated in the "S/H" columns of the tables.*

Proof. It follows from item (1) of Observation 8.6 that f exchanges the two E_6-components of $\mathcal{E}$, having no fixed point there. Thus f has no fixed curve on X and so (67)-(68) can be applied to show the existence of a unique transverse fixed point $p \in X \setminus \mathcal{E}$. In view of $x_4 < \tau_0 < x_3$ Proposition 9.1 leads to the theorem. □

To discuss the cases where the nontrivial automorphism $f : X \to X$ may have fixed curves, we use S. Saito's fixed point formula [24, (0.2)]. In the notation of Iwasaki and Uehara [14, Theorem 1.2] the formula reads

$$L(f) := \sum_{i=0}^{4} (-1)^i \mathrm{Tr} f^*|H^i(X) = \sum_{p \in X_0(f)} \nu_p(f) + \sum_{C \in X_{\mathrm{I}}(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{\mathrm{II}}(f)} \tau_C \cdot \nu_C(f), \tag{69}$$

where $X_0(f)$ is the set of fixed points of f while $X_{\mathrm{I}}(f)$ and $X_{\mathrm{II}}(f)$ are the sets of irreducible fixed curves of types I and II respectively, χ_C is the Euler number of the normalization of C and τ_C is the self-intersection number of C. The definitions of $\nu_p(f)$, $\nu_C(f)$, $X_{\mathrm{I}}(f)$ and $X_{\mathrm{II}}(f)$ are given in Appendix B. Originally this formula was stated for projective surfaces, but it remains valid for compact Kähler surfaces (Dinh et al. [6, Theorem 4.3]).

We also use the Toledo-Tong fixed point formula [27, Theorem (4.10)] in the special case where X is a compact complex surface, $f : X \to X$ is a holomorphic map, E is a holomorphic line bundle on X and $\phi : f^*E \to E$ is a holomorphic bundle map. Suppose that any isolated fixed point p is *transverse*, that is, the tangent map $(df)_p : T_pX \to T_pX$ does not have eigenvalue 1. Suppose moreover that any connected component

C of the 1-dimensional fixed point set is *transverse*, that is, C is a smooth curve and the induced differential map $d^N f$ on the normal line bundle $N = N_C$ to C does not have eigenvalue 1. The formula is then given by

$$L(f, \phi) := \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(f, \phi)^* | H^i(X, \mathcal{O}(E)) = \sum_p \nu_p(f, \phi) + \sum_C \nu_C(f, \phi), \tag{70}$$

where the sums are taken over all isolated fixed points p and all connected fixed curves C. Let λ_C be the eigenvalue of $d^N f : N_C \to N_C$; μ_p the eigenvalue of $\phi_p : E_p \to E_p$; and μ_C the eigenvalue of $\phi|_C : E|_C \to E|_C$, where $E|_C$ is the restriction of the line bundle E to C and $\phi|_C$ is the restriction of the bundle map $\phi : f^*E \to E$ to $E|_C$. Note that λ_C and μ_C are constants since they are holomorphic functions on the compact curve C. Moreover $\lambda_C \neq 1$ by transversality of C. The indices $\nu_p(f, \phi)$ and $\nu_C(f, \phi)$ in (70) are expressed as

$$\nu_p(f, \phi) = \frac{\mu_p}{1 - \operatorname{Tr}(df)_p + \det(df)_p}, \tag{71a}$$

$$\nu_C(f, \phi) = \int_C \operatorname{td}(C) \cdot \{1 - \lambda_C \operatorname{ch}(\check{N})\}^{-1} \cdot \mu_C \operatorname{ch}(E), \tag{71b}$$

where $\operatorname{td}(C)$ is the Todd class of C, $\operatorname{ch}(E)$ is the Chern character of E, $\check{N}$ is the dual line bundle to N and the integral sign stands for evaluation on the fundamental cycle of C.

A 3-*cycle of Siegel disks* for f is a sequence of open subsets U, $f(U)$, $f^2(U)$ in X such that U is a Siegel disk for f^3 centered at a point $p \in U$ which is a periodic point of period 3, that is, p, $f(p)$, $f^2(p)$ are distinct but $f^3(p) = p$. Note that U, $f(U)$, $f^2(U)$ are Siegel disks for f^3 centered at p, $f(p)$, $f^2(p)$, respectively.

Theorem 9.5 *For each entry of type* $\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$ *in Tables* 11 *and* 12*, the K3 surface automorphism* $f : X \to X$ *has a unique periodic orbit* p*,* $f(p)$*,* $f^2(p) \in X$ *of period* 3*, which lie in* $X \setminus \mathcal{E}$*. If the special trace* τ *is either* x_2 *or* x_4 *then the three points are the centers of a* 3*-cycle of Siegel disks, while if* τ *is either* x_1 *or* x_3 *then they are hyperbolic periodic points; this information is indicated in the "S/H" columns of the tables.*

Proof. It follows from (2) of Observation 8.6 that f has no fixed point on the $\mathrm{A}_2 \oplus \mathrm{A}_2$-component of $\mathcal{E}$. On the other hand f preserves each (-2)-curve on the E_8-component, inducing a Möbius transformation on it. A Möbius transformation falls into one of the three categories according to the number n of its fixed points; parabolic for $n = 1$, non-parabolic for $n = 2$, and the identity for $n \geq 3$. In the parabolic case the derivative at the unique fixed point is 1. In the non-parabolic case, if the derivatives at one fixed points is c then the derivative at the other fixed point is c^{-1}, where $c \neq 1$. Notice that $\boldsymbol{e}_4$ is a fixed curve of f since $\boldsymbol{e}_4$ is fixed at the three intersections $\boldsymbol{e}_4 \cap \boldsymbol{e}_2$, $\boldsymbol{e}_4 \cap \boldsymbol{e}_3$, $\boldsymbol{e}_4 \cap \boldsymbol{e}_5$ (see Figure 5).

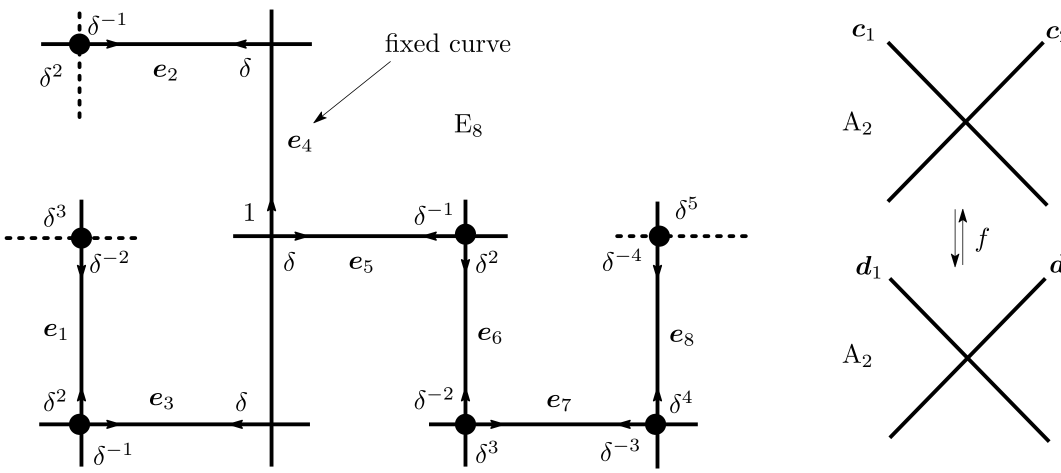

Figure 5: Eigenvalues at the isolated fixed points on the exceptional set $\mathcal{E}$.

Consider the arm $\boldsymbol{a} := \boldsymbol{e}_5 \cup \boldsymbol{e}_6 \cup \boldsymbol{e}_7 \cup \boldsymbol{e}_8$ emanating from $\boldsymbol{e}_4$. Let q_0, q_1, q_2, q_3 be the points at $\boldsymbol{e}_4 \cap \boldsymbol{e}_5$, $\boldsymbol{e}_5 \cap \boldsymbol{e}_6$, $\boldsymbol{e}_6 \cap \boldsymbol{e}_7$, $\boldsymbol{e}_7 \cap \boldsymbol{e}_8$. Note that they are fixed points of f. We use the fact that $\det(df)_q = \delta$ at any fixed point

q of f. Since $(df)_{q_0} = 1$ along $\boldsymbol{e}_4$ we have $(df)_{q_0} = \delta \neq 1$ along $\boldsymbol{e}_5$. Thus f induces a non-parabolic Möbius transformation on $\boldsymbol{e}_5$ having derivative δ^{-1} at q_1. Then f induces a non-parabolic Möbius transformation on $\boldsymbol{e}_6$ having derivatives δ^2 at q_1 and δ^{-2} at q_2. Repeating this argument shows that f has four fixed points on $\boldsymbol{a}$, three of which are q_1, q_2, q_3 and the final one q_4 is on $\boldsymbol{e}_8$, and that $(df)_{q_j}$ has eigenvalues δ^{-j} and δ^{j+1} for $j = 1, 2, 3, 4$. Similar statements can be made for the shorter arms $\boldsymbol{b} := \boldsymbol{e}_1 \cup \boldsymbol{e}_3$ and $\boldsymbol{e}_2$. In total there are seven isolated fixed points indicated by $\bullet$ and one fixed curve $\boldsymbol{e}_4$ on $\mathcal{E}$.

We apply Saito's formula (69) to f. For each entry of type $\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$ in Tables 11 and 12 we have $\mathrm{Tr}\, \tilde{F} = 7$ where $\tilde{F} = \tilde{A}$ or $\tilde{B}$, and so $L(f) = 2 + \mathrm{Tr}\, \tilde{F} = 9$. The seven isolated fixed point on $\mathcal{E}$ are transverse and hence of index 1. The fixed curve $\boldsymbol{e}_4$ is of type I and has index 1, since df has eigenvalue $\delta \neq 1$ in its normal direction. Any point on $\boldsymbol{e}_4$ has index 0. The Euler number of $\boldsymbol{e}_4 \cong \mathbb{P}^1$ is 2. There is no fixed curve of type II. Thus formula (69) reads

$$9 = 7 + \sum_{p \in X_0(f) \setminus \mathcal{E}} \nu_p(f) + 2, \qquad \text{i.e.} \qquad \sum_{p \in X_0(f) \setminus \mathcal{E}} \nu_p(f) = 0, \tag{72}$$

where 7 and 2 on the RHS are the contributions of the seven isolated fixed point on $\mathcal{E}$ and the fixed curve $\boldsymbol{e}_4$ respectively. This implies that f has no fixed point on $X \setminus \mathcal{E}$.

Next we apply the Toledo-Tong formula (70) to f with $E = K_X := \wedge^2 T^* X$ and $\phi = f^* : f^* K_X \to K_X$, the pull-back mapping. Notice that $\mu_p = \lambda_C = \mu_C = \delta$ with $C = \boldsymbol{e}_4$ in (71). If η is a nowhere vanishing holomorphic 2-form on X, then $T_p X \to T_p^* C$, $v \mapsto \eta_p(v, \cdot)$ induces an isomorphism $N_p := T_p X / T_p C \cong T_p^* C$ at each $p \in C$, hence $\check{N} \cong TC$. A little calculation yields $\nu_C(f, f^*) = \delta(\delta + 1)/(\delta - 1)^2$ with $C = \boldsymbol{e}_4$ in (71b). It is easy to see that $L(f, f^*) = 1 + \delta$. Formula (70) then asserts that $z = \delta$ is a solution to the equation

$$\begin{aligned} 1 + z = {} & \sum_{j=1}^{4} \frac{z}{1 - (z^{-j} + z^{j+1}) + z} + \sum_{j=1}^{2} \frac{z}{1 - (z^{-j} + z^{j+1}) + z} \\ & + \frac{z}{1 - (z^{-1} + z^2) + z} + \frac{z(z+1)}{(z-1)^2}, \end{aligned}$$

where the first three terms in the RHS are the contributions of the fixed points on the arms $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{e}_2$ and the last term is that of the fixed curve $\boldsymbol{e}_4$. A careful inspection shows that the difference $D(z)$ of the LHS from the RHS above admits a clean factorization

$$D(z) = \frac{\mathrm{L}(z)}{(z+1) \cdot \mathrm{C}_1(z) \cdot \mathrm{C}_3(z) \cdot \mathrm{C}_5(z)} = \frac{z \cdot \mathrm{LT}(w)}{(z+1) \cdot \mathrm{CT}_1(w) \cdot \mathrm{CT}_3(w) \cdot \mathrm{CT}_5(w)}, \tag{73}$$

where $w := z + z^{-1}$. So the Toledo-Tong formula for f is nothing other than the tautological fact that δ is a root of Lehmer's polynomial, but the formula (73) itself will be useful later.

We again apply Saito's formula (69) this time to f^3. If $\tilde{F}$ has characteristic polynomial $\tilde{\chi}(z) = z^{22} - e_1 z^{21} + e_2 z^{20} - e_3 z^{19} + \cdots$ then $\mathrm{Tr}(\tilde{F}^3) = p_3 = 3(e_3 - e_1 e_2) + e_1^3$ by the relation between power sums and elementary symmetric polynomials. We have $e_1 = 7$, $e_2 = 20$, $e_3 = 29$ and $p_3 = 10$ in the present cases. This implies $L(f) = 2 + \mathrm{Tr}(\tilde{F}^3) = 12$, hence the equation (72) turns into

$$12 = 7 + \sum_{p \in X_0(f^3) \setminus \mathcal{E}} \nu_p(f^3) + 2, \qquad \text{i.e.} \qquad \sum_{p \in X_0(f^3) \setminus \mathcal{E}} \nu_p(f^3) = 3.$$

Thus f^3 has three fixed points on $X \setminus \mathcal{E}$ counted with multiplicity. Let p be one of them. Since f has no fixed point on $X \setminus \mathcal{E}$, we have $p \neq f(p)$ and hence $f(p) \neq f^2(p)$ and $f^2(p) \neq f^3(p) = p$. Accordingly, p, $f(p)$, $f^2(p)$ must be distinct and $\nu_{f^j(p)}(f^3) = 1$ for $j = 0, 1, 2$.

The tangent maps $(df^3)_{f^j(p)} : T_{f^j(p)} X \to T_{f^j(p)} X$, $j = 0, 1, 2$, have common determinant δ^3 and common eigenvalues of the form $\delta^{3/2} \alpha^{\pm 1}$ with $\alpha \in \mathbb{C}^{\times}$. The Toledo-Tong formula (70) for f^3 is then expressed as

$$D(\delta^3) = \frac{3\delta^3}{1 - \delta^{3/2}(\alpha + \alpha^{-1}) + \delta^3},$$

in terms of the rational function $D(z)$ in (73). Solving this equation we have

$$q(\tau) := (\alpha+\alpha^{-1})^2 = \delta^{-3}\left\{1+\delta^3-\frac{3\delta^3}{D(\delta^3)}\right\}^2 = \frac{(1+\delta^3)^2\mathrm{M}(\delta^3)^2}{\delta^3\,\mathrm{L}(\delta^3)^2}$$
$$= \frac{(\delta^3+\delta^{-3}+2)\,\mathrm{MT}(\delta^3+\delta^{-3})^2}{\mathrm{LT}(\delta^3+\delta^{-3})^2} = \frac{(\tau+2)(\tau-1)^2\,\mathrm{MT}(\tau^3-3\tau)^2}{\mathrm{LT}(\tau^3-3\tau)^2},$$

where $\mathrm{M}(z)$ and $\mathrm{MT}(z)$ are Salem polynomial and its trace polynomial defined by

$$\mathrm{M}(z) := z^{10}-2z^9-z^7+2z^6-z^5+2z^4-z^3-2z+1,$$
$$\mathrm{MT}(w) := (w+1)(w-2)(w^3-w^2-4w+1)-1.$$

Here we observe that $0 < q(x_j) < 4$ for $j = 2, 4$ and $q(x_j) > 4$ for $j = 1, 3$. Thus Proposition 9.1 leads to the conclusion of the theorem. □

Remark 9.6 In case of type D_{10} it follows from (3) of Observation 8.6 that f has no fixed curve and nine transverse fixed points on $\mathcal{E}$. Thus the Lefschetz formula implies that f has no fixed point in $X \setminus \mathcal{E}$. The map f^2 has a unique fixed curve corresponding to the simple root e_8 in Figure 4 and nine transverse fixed points on $\mathcal{E}$. Saito's formula (69) then shows that f^2 has no fixed point in $X \setminus \mathcal{E}$. Moreover, f^3 has no fixed curve and nine transverse fixed points on $\mathcal{E}$, so the Lefschetz formula and an argument as in the proof of Theorem 9.5 imply that f has a unique transverse periodic cycle p, $f(p)$, $f^2(p) \in X \setminus \mathcal{E}$ of period 3. The Atiyah-Bott formula tells us that for the eigenvalues $\delta^{3/2}\alpha^{\pm1}$ of $(df^3)_{f^j(p)}$, $j = 0, 1, 2$, we have $(\alpha+\alpha^{-1})^2 = q(\tau)$, where

$$q(w) := \frac{(w+2)(w-1)^2\mathrm{NT}(w^3-3w)^2}{\mathrm{LT}(w^3-3w)^2},$$

with $\mathrm{NT}(w) := (w^2-2w-2)(w^3-3w+1)-1$ being a Salem trace polynomial. Since $0 < q(x_j) < 4$ for $j = 1$, 2, 3 and $q(x_4) > 4$, Proposition 9.1 shows that if $\tau = x_1$, x_2, x_3 then p, $f(p)$, $f^2(p)$ are the centers of a 3-cycle of Siegel disks for f, and if $\tau = x_4$ they are hyperbolic periodic points of period 3. Actually we have $\tau = x_2$, x_4 for the entries of type D_{10} in Table 11. Notice that there is no entry of type D_{10} in Table 12.

Remark 9.7 In case of type A_2, let $\boldsymbol{e}_\pm$ be the two (-2)-curves in $\mathcal{E}$ and p_0 be their intersection. Saito's formula (69) and the Toledo-Tong formula (70) rule out the possibility that either $\boldsymbol{e}_+$ or $\boldsymbol{e}_-$ is fixed by f. Notice that p_0 is a fixed point of f, which is either (i) transverse or (ii) of multiplicity 2. But case (ii) can be ruled out by using an Atiyah-Bott formula allowing isolated multiple fixed points, whose local indices can be expressed in terms of Grothendieck residues (Toledo [26, formula (6.3)]). In case (i), thinking of the Möbius transformations on $\boldsymbol{e}_\pm$ induced by f and a use of the Lefschetz formula show that f has two transverse fixed points $p_\pm \in \boldsymbol{e}_\pm \setminus \{p_0\}$ and no fixed point in $X \setminus \mathcal{E}$. If the eigenvalues of $(df)_{p_0}$ are $\delta^{1/2}\alpha^{\pm1}$ then those of $(df)_{p_\varepsilon}$ are $\delta^{1/2}(\delta^\varepsilon\alpha)^{\pm1}$ for $\varepsilon = \pm1$. The Atiyah-Bott formula yields $q(w) = (w^2-3)^2/(w+2)$ at the point p_0. We have $0 < q(x_j) < 4$ for $j = 1$, 3, 4 and $q(x_2) > 4$, hence Proposition 9.1 is applicable. The multiplicative independence of the eigenvalues of $(df)_{p_\varepsilon}$, $\varepsilon = \pm1$, can be verified in the same manner as that of the eigenvalues of $(df)_{p_0}$ (see the proof of Proposition 9.1). We can now conclude that the three points p_0 and $p_\pm$ are the centers of Siegel disks, since we have $\tau = x_1$, x_4 for the entries of type A_2 in Table 11; there is no entry of type A_2 in Table 12.

For proofs of the results in these remarks we refer to an archive version of this article: arXiv:`2003.13943v4`.

10 Lattices in Number Fields

It is interesting to discuss the relationship between the method of hypergeometric groups and that of Salem number fields by McMullen [18] and Gross and McMullen [9]. We briefly review the construction in [18]. Let $S(z)$ be an unramified Salem polynomial of degree 22 with the associated Salem trace polynomial $R(w)$ where $w = z + z^{-1}$. Consider the $\mathbb{Z}$-algebra $L_S := \mathbb{Z}[z]/(S(z))$ together with its field of fractions $K := \mathbb{Q}[z]/(S(z))$. Let $U(w)$ be a unit in $\mathbb{Z}[w]/(R(w))$ such that the polynomial $R(w)$ admits a unique root

$$\tau \in (-2,\,2) \quad \text{that satisfies} \quad U(\tau)\,R'(\tau) > 0, \qquad \text{see [18, Theorem 8.3].} \tag{74}$$

This root τ corresponds to the "special trace" in our method. By defining the inner product

$$(g_1, g_2)_S := \mathrm{Tr}_{\mathbb{Q}}^{K}\left(\frac{U(w)\, g_1(z)\, g_2(z^{-1})}{R'(w)}\right), \qquad g_1(z),\, g_2(z) \in \mathbb{Z}[z] \bmod S(z), \tag{75}$$

one can make L_S into a K3 lattice equipped with a Hodge structure such that multiplication by z, that is, $M_z : L_S \to L_S$, $g(z) \mapsto zg(z)$ induces a Hodge isometry. The map M_z lifts to a K3 surface automorphism $f : X \to X$ such that $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ has characteristic polynomial $S(z)$.

Table 13: Table 4 in McMullen [18] vs. Table 10 in this article.

case	R	ST in [18]	$(1,1)_S$	ST in this article
1	R_1	$\boldsymbol{y_8}$	0	$y_1,\, y_2,\, \boldsymbol{y_8},\, y_{10}$
2	R_2	$\boldsymbol{y_4}$	$\mathbf{-2}$	$y_1,\, y_2,\, y_3,\, \boldsymbol{y_4},\, y_7,\, y_8,\, y_{10}$
3	R_3	$\boldsymbol{y_7}$	0	$y_1,\, y_2,\, y_3,\, \boldsymbol{y_7},\, y_8,\, y_9$
4	R_4	y_5	$\mathbf{-2}$	$y_1,\, y_2,\, y_3,\, y_7$
5	R_5	y_4	$\mathbf{-2}$	$y_1,\, y_2,\, y_6,\, y_7$
6	R_6	y_8	0	$y_1,\, y_2,\, y_3,\, y_4,\, y_7,\, y_{10}$
7	R_7	$\boldsymbol{y_7}$	0	$y_2,\, y_5,\, \boldsymbol{y_7}$
8	R_8	$\boldsymbol{y_6}$	0	$y_1,\, y_2,\, y_3,\, y_4,\, y_5,\, \boldsymbol{y_6},\, y_8,\, y_9$
9	R_9	y_6	$\mathbf{-2}$	$y_1,\, y_3,\, y_7,\, y_8,\, y_9,\, y_{10}$
10	R_{10}	y_5	$\mathbf{-2}$	$y_1,\, y_7,\, y_9$

McMullen [18, Table 4] gives a list of ten triples $(S_i(z), R_i(w), U_i(w))$, $i = 1, \ldots, 10$, to which his method is applied. Table 13 gives a comparison between his Table 4 and our Table 10 (in Theorem 8.1), showing to what extent the special traces are common or not, where the common ones are shown in boldface.

An interesting thought occurs to us when we notice a similarity between the lattice in number field L_S and the hypergeometric lattice $L_H = L(A, B) = L(\Phi, \Psi)$, that is,

$$L_S = \langle 1, z, z^2, \ldots, z^{21}\rangle_{\mathbb{Z}} \circlearrowleft M_z \quad \text{and} \quad L_H = \langle \boldsymbol{r}, B\boldsymbol{r}, \ldots, B^{21}\boldsymbol{r}\rangle_{\mathbb{Z}} \circlearrowleft B.$$

Recall from (29) and Definition 6.1 that the vector $\boldsymbol{r}$ is normalized as $(\boldsymbol{r},\, \boldsymbol{r})_H = \pm 2$. So we wonder whether when $(1,\, 1)_S = \pm 2$ we can go back and forth between L_S and L_H via the correspondences $1 \leftrightarrow \boldsymbol{r}$ and $M_z \leftrightarrow B$. The value of $(1,\, 1)_S$ is twice the coefficient of w^{10} in $U(w)$, so cases 2, 4, 5, 9, 10 in Table 13 are relevant to this question. In these cases we can recover the matrix $A = BC$ and hence the polynomail $\Phi(w)$ from McMullen's data, since the reflection in the vector $1 \in L_S$ corresponds to the matrix C, that is, the reflection in $\boldsymbol{r} \in L_H$. The results are given in Table 14, where $\Phi(w)$ contains a non-cyclotomic trace factor, which is Salem in cases 4 and 10, but not Salem in cases 2, 5 and 9. These are not covered by Table 10, because there $\Phi(w)$ is restricted to a product of cyclotomic trace polynomials (see (63)).

Table 14: Examples in McMullen [18, Table 4] recovered by hypergeometric method.

case	R	$\Phi(w)$
2	R_2	$\mathrm{CT}_3(w) \cdot (w^9 - 9w^7 + 25w^5 - 2w^4 - 21w^3 + 7w^2 + 2w - 2)$
4	R_4	$\mathrm{CT}_4(w) \cdot \mathrm{CT}_{42}(w) \cdot (w^3 - w^2 - 3w + 1)$
5	R_5	$w^{10} - 10w^8 - 2w^7 + 33w^6 + 12w^5 - 37w^4 - 16w^3 + 6w^2 - 3w - 3$
9	R_9	$w^{10} - 11w^8 - 3w^7 + 42w^6 + 22w^5 - 62w^4 - 49w^3 + 23w^2 + 33w + 8$
10	R_{10}	$\mathrm{CT}_4(w) \cdot (w^9 - w^8 - 10w^7 + 7w^6 + 35w^5 - 14w^4 - 48w^3 + 7w^2 + 18w - 1)$

In the other way round some hypergeometric lattices can be realized as lattices in number fields. Thinking a little bit more generally, let $S(z) \in \mathbb{Z}[z]$ be a monic, irreducible, palindromic polynomial of degree $2N$ with the associated trace polynomial $R(w) \in \mathbb{Z}[w]$. Let L_H be an even unimodular lattice of rank $2N$ equipped with an isometry $F : L_H \to L_H$ whose characteristic polynomial is $S(z)$. Suppose that F admits a *cyclic vector* $\boldsymbol{r}$ over $\mathbb{Z}$, that is,

$$L_H = \langle \boldsymbol{r}, F\boldsymbol{r}, \dots, F^{2N-1}\boldsymbol{r} \rangle_{\mathbb{Z}}. \tag{76}$$

Let $P_j(w) := z^j + z^{-j}$ for $j \in \mathbb{Z}_{\geq 0}$ and think of them as polynomials in $w = z + z^{-1}$. Then they satisfy three-term recurrence relation

$$P_0(w) = 2, \qquad P_1(w) = w, \qquad P_{j+1}(w) - w\, P_j(w) + P_{j-1}(w) = 0, \quad j \geq 1,$$

which shows that if $j \geq 1$ then $P_j(w)$ is a monic polynomial of degree j in $\mathbb{Z}[w]$. Given a polynomial $g(w) \in \mathbb{Z}[w]$, we denote by $[g(w)]_R \in \mathbb{Z}$ the coefficient of w^{N-1} in the remainder of $g(w)$ divided by $R(w)$. We define the integers $u_1, \dots, u_N \in \mathbb{Z}$ inductively by

$$u_1 = \frac{1}{2}(\boldsymbol{r},\, \boldsymbol{r})_H, \qquad u_j = (F^{j-1}\boldsymbol{r},\, \boldsymbol{r})_H - \sum_{k=1}^{j-1} c_{jk}\, u_k, \quad j = 2, \dots, N, \tag{77}$$

where $c_{jk} := [P_{j-1}(w) \cdot w^{N-k}]_R \in \mathbb{Z}$. Note that $u_1 \in \mathbb{Z}$ since L_H is an even lattice.

Theorem 10.1 *Under condition* (76) *the pair* $(L_H,\, F)$ *is isomorphic to* $(L_S,\, M_z)$ *with* $L_S := \mathbb{Z}[z]/(S(z))$ *and* M_z *being multiplication by* z*, where* L_S *carries the inner product* (75) *with*

$$U(w) = u_1\, w^{N-1} + u_2\, w^{N-2} + \cdots + u_N \in \mathbb{Z}[w], \tag{78}$$

whose coefficients are determined by the recurrence (77)*. It is a unit in* $\mathbb{Z}[w]/(R(w))$*.*

Proof. Identify $(L_H,\, F)$ and $(L_S,\, M_z)$ via $\boldsymbol{r} \leftrightarrow 1$ and $F \leftrightarrow M_z$. Export the inner product on L_H to L_S isometrically. The existence in $\mathbb{Q}[w]/(R(w))$ of $U(w)$ that makes (75) valid is mentioned in Gross and McMullen [9, §4, Remark], where they suppose $\mathcal{O}_K = \mathbb{Z}[z]/(S(z))$ but the existence can be proved without this assumption. To determine the coefficients of $U(w)$, substitute $g_1(z) = z^{j-1}$, $j = 1, \dots, N$, and $g_2(z) = 1$ into (75) and use (78) to have

$$(F^{j-1}\boldsymbol{r},\, \boldsymbol{r})_H = (z^{j-1},\, 1)_S = \mathrm{Tr}_{\mathbb{Q}}^{K}\left(\frac{U(w) \cdot z^{j-1}}{R'(w)}\right) = \mathrm{Tr}_{\mathbb{Q}}^{J} \circ \mathrm{Tr}_{J}^{K}\left(\frac{U(w) \cdot z^{j-1}}{R'(w)}\right)$$

$$= \mathrm{Tr}_{\mathbb{Q}}^{J}\left(\frac{U(w)\, P_{j-1}(w)}{R'(w)}\right) = [U(w)\, P_{j-1}(w)]_R = \sum_{k=1}^{N} c_{jk}\, u_k = \sum_{k=1}^{j} c_{jk}\, u_k,$$

where $J := \mathbb{Q}[w]/(R(w))$, the fifth equality is by residue calculus as in [18, page 222] and the final equality follows from $c_{jk} = 0$ for $j < k \leq N$. For $j = 1$ we have $(\boldsymbol{r},\, \boldsymbol{r})_H = c_{11}\, u_1 = 2u_1$, which yields the first equality in (77). For $j \geq 2$ we have the second equality in (77), since $c_{jj} = 1$. Thus $U(z) \in \mathbb{Q}[w]$ belongs to $\mathbb{Z}[w]$. The discriminant of L_S is $\mathrm{disc}(L_S) = \pm \det^2 M_{U(w)}$, where $M_{U(w)} : \mathbb{Z}[w]/(R(w)) \to \mathbb{Z}[w]/(R(w))$ is multiplication by $U(w)$. Since $L_S \cong L_H$ is unimodular, i.e. $\mathrm{disc}(L_S) = \pm 1$, we have $\det M_{U(w)} = \pm 1$ and hence $U(w)$ is a unit in $\mathbb{Z}[w]/(R(w))$. □

Theorem 10.1 can be applied to the unimodular hypergeometric lattice $L_H = L(A, B)$ that arises from an irreducible hypergeometric group $H = H(A, B)$, since it satisfies condition (76) for $F = B$. In particular for hypergeometric K3 lattices we have the following.

Corollary 10.2 *In the situation of Theorem* 1.3 *suppose that the polynomial* $\psi(z)$ *is an unramified Salem polynomial* $S(z)$ *of degree* 22*. Then the pair* $(L_H,\, B)$ *admits McMullen's construction with unit* $U(w) \in \mathbb{Z}[w]/(R(w))$ *determined by* (77) *and* (78)*, and the special trace* τ *satisfies the compatibility condition* (74)*.*

As an illustration we give an example from the top entry of Table 10. The B-basis version of formula (2) gives the values of $(B^{j-1}\boldsymbol{r},\, \boldsymbol{r})_H$ for $j = 1, \dots, 11$, then formulas (77) and (78) yield

$$U(w) = -w^{10} + 6w^9 - 7w^8 - 22w^7 + 54w^6 - 4w^5 - 70w^4 + 36w^3 + 24w^2 - 16w.$$

As for non-projective K3 surface automorphisms with minimum entropy, McMullen [20] gives only one such example. It has characteristic polynomial $\mathrm{L}(z) \cdot (z-1)^9(z+1)(z^2+1)$, special trace x_4 in (65), and root system of Dynkin type $\mathrm{E}_8 \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$. Our Table 12 contains three entries with the same data. It is interesting to compare them in a deeper level, but it needs to discuss gluing of lattices.

A Calculations of Resultants

We give a proof of the formulas in (31), which relies on the following lemma.

Lemma A.1 *Let $f(z)$ and $g(z)$ be monic palindromic polynomials of even degrees. If $F(w)$ and $G(w)$ are the trace polynomials of $f(z)$ and $g(z)$ respectively, then* $\mathrm{Res}(f,g) = \mathrm{Res}(F,G)^2$.

Proof. We have $f(z) = z^m F(z+z^{-1})$ and $g(z) = z^n G(z+z^{-1})$, where $\deg F(w) = m$ and $\deg G(w) = n$. Let $F(w) = \prod_{i=1}^m (w - A_i)$ and $G(w) = \prod_{j=1}^n (w - B_j)$. Set $\alpha_i + \alpha_i^{-1} = A_i$ and $\beta_j + \beta_j^{-1} = B_j$. We then have

$$f(z) = z^m \prod_{i=1}^m (z + z^{-1} - \alpha_i - \alpha_i^{-1}) = \prod_{i=1}^m \{z^2 - (\alpha_i + \alpha_i^{-1})z + 1\} = \prod_{i=1}^m (z - \alpha_i)(z - \alpha_i^{-1}),$$

and similarly $g(z) = \prod_{j=1}^n (z - \beta_j)(z - \beta_j^{-1})$. Setting $\varDelta_{ij} := (\alpha_i - \beta_j)(\alpha_i - \beta_j^{-1})(\alpha_i^{-1} - \beta_j)(\alpha_i^{-1} - \beta_j^{-1})$ we have

$$\varDelta_{ij} = (\alpha_i - \beta_j) \cdot \frac{\alpha_i\beta_j - 1}{\beta_j} \cdot \frac{1 - \alpha_i\beta_j}{\alpha_i} \cdot \frac{\beta_j - \alpha_i}{\alpha_i\beta_j} = \left\{ \frac{(\alpha_i - \beta_j)(\alpha_i\beta_j - 1)}{\alpha_i\beta_j} \right\}^2,$$

$$A_i - B_j = \alpha_i + \frac{1}{\alpha_i} - \beta_j - \frac{1}{\beta_j} = \alpha_i - \beta_j + \frac{\beta_j - \alpha_i}{\alpha_i\beta_j} = \frac{(\alpha_i - \beta_j)(\alpha_i\beta_j - 1)}{\alpha_i\beta_j},$$

and hence $\varDelta_{ij} = (A_i - B_j)^2$. Thus $\mathrm{Res}(f,g) = \prod_{i=1}^m \prod_{j=1}^n \varDelta_{ij} = \prod_{i=1}^m \prod_{j=1}^n (A_i - B_j)^2 = \mathrm{Res}(F,G)^2$. □

Proof of Formulas (31a) *and* (31b). We use the following properties of the resultant: if $\deg f = m$ and $\deg g = n$ then $\mathrm{Res}(g,f) = (-1)^{mn}\mathrm{Res}(f,g)$; $\mathrm{Res}(f_1 f_2,\, g) = \mathrm{Res}(f_1, g) \cdot \mathrm{Res}(f_2, g)$; $\mathrm{Res}(z - \alpha, g) = g(\alpha)$ for $\alpha \in \mathbb{C}$. Put $\varphi_1(z) := z^{N-1}\Phi(z + z^{-1})$ in (30a). Then it follows from these properties and Lemma A.1 that

$$\mathrm{Res}(\varphi, \psi) = \psi(1) \cdot \psi(-1) \cdot \mathrm{Res}(\varphi_1, \psi) = \Psi(2) \cdot (-1)^N \Psi(-2) \cdot \mathrm{Res}(\Phi, \Psi)^2,$$

which verifies (31a). Similarly, putting $\varphi_1(z) := z^N \Phi(z + z^{-1})$ and $\psi_1(z) := z^N \Psi(z + z^{-1})$ in (30b) we have

$$\mathrm{Res}(\varphi, \psi) = (z+1)|_{z=1} \cdot \psi_1(1) \cdot \varphi_1(-1) \cdot \mathrm{Res}(\varphi_1, \psi_1) = 2 \cdot \Psi(2) \cdot (-1)^N \Phi(-2) \cdot \mathrm{Res}(\Phi, \Psi)^2,$$

which proves (31b) as desired. □

B Saito's Indices

We define the indices $\nu_p(f)$ for $p \in X_0(f)$ and $\nu_C(f)$ for $C \in X_1(f)$ in Saito's formula (69), where $X_0(f)$ is the set of all fixed points and $X_1(f)$ is the set of all irreducible fixed curves of f. We also introduce the decomposition $X_1(f) = X_{\mathrm{I}}(f) \amalg X_{\mathrm{II}}(f)$. These tasks are mostly ring-theoretic. For details see [24] and [14, §3].

Let $A := \mathbb{C}[\![z_1, z_2]\!]$ and $\mathfrak{m}$ be its maximal ideal. Let $\sigma : A \to A$ be a nontrivial ring-endomorphism continuous in $\mathfrak{m}$-adic topology. It can be expressed as $\sigma(z_i) = z_i + g \cdot h_i$, $i = 1, 2$, for some g, h_1, $h_2 \in A$, where g is nonzero and h_1, h_2 are coprime. Consider the ideals $\mathfrak{a} = (g)$ and $\mathfrak{b} := (h_1, h_2)$ in A. Since h_1 and h_2 are coprime,

$$\delta(\sigma) := \dim_{\mathbb{C}}(A/\mathfrak{b}) < \infty.$$

Let $\varLambda(\sigma)$ be the set of all prime ideals $\mathfrak{p}$ of height 1 in A that divides $\mathfrak{a}$. we define

$$\nu_{\mathfrak{p}}(\sigma) := \max\{m \in \mathbb{Z}_{\ge 1} : \mathfrak{a} \subset \mathfrak{p}^m\} \qquad \text{for each} \quad \mathfrak{p} \in \varLambda(\sigma). \tag{79}$$

Let $\kappa[\mathfrak{p}]$ be the normalization of the ring $A/\mathfrak{p}$. It is isomorphic to $\mathbb{C}[\![t]\!]$ for some prime element t. Let

$$\hat{\Omega}^1_{A/\mathbb{C}} := \varprojlim_n \Omega^1_{A_n/\mathbb{C}}, \qquad \hat{\Omega}^1_{\kappa[\mathfrak{p}]/\mathbb{C}} := \varprojlim_n \Omega^1_{\kappa[\mathfrak{p}]_n/\mathbb{C}},$$

where $A_n := A/\mathfrak{m}^n$ and $\kappa[\mathfrak{p}]_n := \mathbb{C}[\![t]\!]/(t)^n$. Then $\hat{\Omega}^1_{A/\mathbb{C}} = A\,dz_1 \oplus A\,dz_2$ and $\hat{\Omega}^1_{\kappa[\mathfrak{p}]/\mathbb{C}} = \kappa[\mathfrak{p}]\,dt$. Let

$$\tau_{\mathfrak{p}} : \hat{\Omega}^1_{A/\mathbb{C}} \to \hat{\Omega}^1_{\kappa(\mathfrak{p})/\mathbb{C}} := \hat{\Omega}^1_{\kappa[\mathfrak{p}]/\mathbb{C}} \otimes_{\kappa[\mathfrak{p}]} \kappa(\mathfrak{p})$$

be the homomorphism induced from the natural map $A \to \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p})$ is the quotient field of $\kappa[\mathfrak{p}]$. Put

$$\Lambda_{\mathrm{I}}(\sigma) := \{\mathfrak{p} \in \Lambda(\sigma) : \tau_{\mathfrak{p}}(\varpi_\sigma) \neq 0\}, \qquad \Lambda_{\mathrm{II}}(\sigma) := \{\mathfrak{p} \in \Lambda(\sigma) : \tau_{\mathfrak{p}}(\varpi_\sigma) = 0\},$$

where $\varpi_\sigma := h_2 \cdot dz_1 - h_1 \cdot dz_2 \in \hat{\Omega}^1_{A/\mathbb{C}}$. There exists a nonzero element $a \in \kappa[\mathfrak{p}]$ such that

$$\tau_{\mathfrak{p}}(\varpi_\sigma) = a \cdot dt \quad \text{if } \mathfrak{p} \in \Lambda_{\mathrm{I}}(\sigma); \qquad \varpi_\sigma = a \cdot du \quad \text{if } \mathfrak{p} \in \Lambda_{\mathrm{II}}(\sigma),$$

where $u \in A$ is a prime element such that $\mathfrak{p} = (u)$. Under the identification $\kappa[\mathfrak{p}] = \mathbb{C}[\![t]\!]$ we define

$$\mu_{\mathfrak{p}}(\sigma) := \max\{m \in \mathbb{Z}_{\ge 0} : (a) \subset (t)^m\}, \qquad \text{for each} \quad \mathfrak{p} \in \Lambda(\sigma),$$
$$\nu_A(\sigma) := \delta(\sigma) + \sum_{\mathfrak{p} \in \Lambda(\sigma)} \nu_{\mathfrak{p}}(\sigma) \cdot \mu_{\mathfrak{p}}(\sigma). \tag{80}$$

Back in the geometric situation we first define $\nu_p(f)$ at $p \in X_0(f)$. The completion $\hat{\mathcal{O}}_{X,p}$ of the local ring $\mathcal{O}_{X,p}$ is identified with $A = \mathbb{C}[\![z_1, z_2]\!]$ and the map f induces a nontrivial ring-endomorphism $\sigma := f_p^* : A \to A$ continuous in $\mathfrak{m}$-adic topology. Then $\nu_p(f)$ is defined to be $\nu_A(\sigma)$ in (80). Next we define $\nu_C(f)$ for $C \in X_1(f)$ as follows. Take a point $p \in C$ and consider $\sigma := f_p^* : A \to A$ again upon identifying $\hat{\mathcal{O}}_{X,p}$ with A. Let C_p be the germ of C at p and let $\Lambda(C_p)$ be the set of all prime ideals in A determined by the irreducible components of C_p. Then one has $\Lambda(C_p) \subset \Lambda(\sigma)$, so $\nu_{\mathfrak{p}}(\sigma)$ makes sense for any $\mathfrak{p} \in \Lambda(C_p)$ via (79). Moreover the value of $\nu_{\mathfrak{p}}(\sigma)$ is independent of $p \in C$ and $\mathfrak{p} \in \Lambda(C_p)$. This common value is just $\nu_C(f)$ by definition. We can also show that either $\Lambda(C_p) \subset \Lambda_{\mathrm{I}}(\sigma)$ or $\Lambda(C_p) \subset \Lambda_{\mathrm{II}}(\sigma)$ holds with this dichotomy independent of $p \in C$. We have $C \in X_{\mathrm{I}}(f)$ in the former case while $C \in X_{\mathrm{II}}(f)$ in the latter case. If C is smooth and transverse then $C \in X_{\mathrm{I}}(f)$.

C Symbols for Hypergeometric Groups

There are a lot of symbols in the sections (§2–§6) concerning hypergeometric groups. Here is a list of them with brief explanations including the places where they appear for the first time.

- H, hypergeometric group in $\mathrm{GL}(n, \mathbb{C})$, §2,
- A, B, matrices generating H, §2,
- $\boldsymbol{a} = \{a_1, \dots, a_n\}$, eigenvalues of A, §2,
- $\boldsymbol{b} = \{b_1, \dots, b_n\}$, eigenvalues of B, §2,
- $a^\dagger := \bar{a}^{-1}$, reciprocal of conjugate of $a \in \mathbb{C}^\times$, §2,
- $\varphi(z)$, characteristic polynomials of A, §2,
- $\psi(z)$, characteristic polynomials of B, §2,
- $\mathrm{Res}(\varphi, \psi)$, resultant of $\varphi(z)$ and $\psi(z)$, §2,
- $C := A^{-1}B$, complex reflection, $c := \det C$, §2,
- $\boldsymbol{r}$, eigenvector of C corresponding to c, §2,
- $(\,\cdot\,,\,\cdot\,)$, H-invariant Hermitian form, §2,
- $\boldsymbol{a}_{\mathrm{on}}/\boldsymbol{a}_{\mathrm{off}}$, parts of $\boldsymbol{a}$ lying on/off S^1, §3,
- $\boldsymbol{b}_{\mathrm{on}}/\boldsymbol{b}_{\mathrm{off}}$, parts of $\boldsymbol{b}$ lying on/off S^1, §3,
- $\boldsymbol{a}_1, \dots, \boldsymbol{a}_t$, clusters in $\boldsymbol{a}_{\mathrm{on}}$, §3,
- $\boldsymbol{b}_1, \dots, \boldsymbol{b}_t$, clusters in $\boldsymbol{b}_{\mathrm{on}}$, §3,
- $[\boldsymbol{a}_{\mathrm{on}}]$, $[\boldsymbol{b}_{\mathrm{on}}]$, configurations of $\boldsymbol{a}_{\mathrm{on}}$, $\boldsymbol{b}_{\mathrm{on}}$, §3,
- $2\pi\alpha_i$, $2\pi\beta_i$, arguments of a_i, b_i, §3,
- $\gamma := \beta_1 + \cdots + \beta_n - (\alpha_1 + \cdots + \alpha_n)$, §3,
- $\varepsilon = \pm 1$, signature of $\sin \pi\gamma$, §3.1,
- $p - q$, index of invariant Hermitian form, §3.1,
- $E(\lambda)$, generalized eigenspace for $\lambda \in \boldsymbol{a} \cup \boldsymbol{b}$, §3.3,
- $m(\lambda) := \dim E(\lambda)$, multiplicity of λ, §3.3,
- $\mathrm{idx}(\lambda)$, local index at $\lambda \in \boldsymbol{a}_{\mathrm{on}} \cup \boldsymbol{b}_{\mathrm{on}}$, §3.3,
- $E(\mu, \mu^\dagger) := E(\mu) \oplus E(\mu^\dagger)$ for $\mu \in \boldsymbol{a}_{\mathrm{off}} \cup \boldsymbol{b}_{\mathrm{off}}$, §3.3,
- $\Phi(w)$, trace polynomials of $\varphi(z)$, §4.1,
- $\Psi(w)$, trace polynomials of $\psi(z)$, §4.1,
- $M(\lambda)$, multiplicity of $w - \lambda$ in $\Phi(w) \cdot \Psi(w)$, §4.1,
- $\boldsymbol{A}$, $\boldsymbol{B}$, multi-sets of roots of $\Phi(w)$, $\Psi(w)$, §4.2,
- $\boldsymbol{A}_{\mathrm{on}}/\boldsymbol{A}_{\mathrm{off}}$, parts of $\boldsymbol{A}$ lying on/off $[-2, 2]$, §4.2,
- $\boldsymbol{B}_{\mathrm{on}}/\boldsymbol{B}_{\mathrm{off}}$, parts of $\boldsymbol{B}$ lying on/off $[-2, 2]$, §4.2,
- $\boldsymbol{A}_1, \dots, \boldsymbol{A}_{s+1}$, trace clusters in $\boldsymbol{A}_{\mathrm{on}}$, §4.2,
- $\boldsymbol{B}_1, \dots, \boldsymbol{B}_s$, trace clusters in $\boldsymbol{B}_{\mathrm{on}}$, §4.2,
- $\boldsymbol{A}_{\mathrm{in}} := \boldsymbol{A}_2 \cup \cdots \cup \boldsymbol{A}_s$, §4.2,
- $\delta := (-1)^{|\boldsymbol{A}_{\mathrm{in}}| + |\boldsymbol{B}_{\mathrm{on}}| + 1}$ when n is even, §4.2,
- $\mathrm{Idx}(\tau)$, local index at $\tau \in \boldsymbol{A}_{\mathrm{on}} \cup \boldsymbol{B}_{\mathrm{on}}$, §4.3,
- $\boldsymbol{A}_1^\circ := (\boldsymbol{A}_1)_{<2}$, $\boldsymbol{A}_{s+1}^\circ := (\boldsymbol{A}_{s+1})_{>-2}$, §4.3,
- $\mathrm{Idx}(\mathcal{X}) := \sum_{\tau \in \mathcal{X}} \mathrm{Idx}(\tau)$, local index on $\mathcal{X}$, §4.3,
- L, hypergeometric lattice, §5,
- $\mathrm{C}_k(z)$, k-th cyclotomic polynomial, §5.2,
- $\mathrm{CT}_k(w)$, k-th cyclotomic trace polynomial, §5.2,
- λ_i, Salem numbers from McMullen [18], §5.3,
- $S_i(z)$, minimal polynomial of λ_i, §5.3,

- $R_i(w)$, minimal trace polynomial of λ_i, §5.3,
- $\mathrm{L}(z)$, Lehmer's polynomial, §5.3,
- $\mathrm{LT}(w)$, Lehmer's trace polynomial, §5.3,
- $V(\lambda)$, λ-eigenspace of A in narrow sense, §6.2,
- $\tau = \tau(A)$, $\tau(B)$, special traces of A, B, §6.3.

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References

[1] T.M. Apostol, *Resultants of cyclotomic polynomials*, Proc. Amer. Math. Soc. **24** (1970), no. 3, 457–462.

[2] E. Bach and J. Shallit, Algorithmic Number Theory, Vol I: Efficient Algorithms, MIT Press Series in the Foundations of Computing, MIT Press, Cambridge, MA, 1996.

[3] W.P. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven, Compact Complex Surfaces. Second Enlarged Edition, Springer-Verlag, Heidelberg, Berlin, 2004.

[4] F. Beukers and G.J. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95** (1989), 325–354.

[5] S. Cantat, *Dynamique des automorphismes des surface K3*, Acta Math. **187** (2001), no. 1, 1–57.

[6] T.-C. Dinh, V.-A. Nguyên and T.T. Truong, *Growth of the number of periodic points for meromorphic maps*, Bull. London Math. Soc. **49** (2017), 947–964.

[7] E. Fuchs, C. Meiri and P. Sarnak, *Hyperbolic monodoromy groups for the hypergeometric equation and Cartan involutions*, J. Eur. Math. Soc. **16** (2014), no. 8, 1617–1671.

[8] M. Gromov, *On the entropy of holomorphic maps*, L'Enseignement Math. **49** (2003), 217–235.

[9] B. Gross and C.T. McMullen, *Automorphisms of even unimodular lattices and unramified Salem numbers*, J. Algebra **257** (2002), 265–290.

[10] G.J. Heckman, Tsinghua Lectures on Hypergeometric Functions, Radboud University of Nijmegen, December 8, 2015.

[11] E. Hironaka, *What is ... Lehmer's number?*, Notices Amer. Math. Soc. **56** (2009), no. 3, 374–375.

[12] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, GTM **9**, Springer-Verlag, New York, Heidelberg, 1972.

[13] K. Iwasaki and Y. Takada, Mathematica Programs for Hypergeometric Groups and Dynamics on K3 Surfaces, Hokkaido University, 2021. `https://www.math.sci.hokudai.ac.jp/~iwasaki/`

[14] K. Iwasaki and T. Uehara, *Periodic points for area-preserving birational maps of surfaces*, Math. Z. **266** (2010), no. 2, 289–318.

[15] T. Kato, Perturbation Theorey for Linear Operators, Springer-Verlag, Berlin, 1980.

[16] D.H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. Math. (2) **34** (1933), 461–479.

[17] A.H.M. Levelt, Hypergeometric Functions, Ph.D. thesis, University of Amsterdam, 1961.

[18] C.T. McMullen, *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002), 201–233.

[19] C.T. McMullen, *Dynamics on blowups of the projective plane*, Publ. Math. IHES **105** (2007), 49–89.

[20] C.T. McMullen, *K3 surfaces, entropy and glue*, J. Reine Angew. Math. **658** (2011), 1–25.

[21] C.T. McMullen, *Automorphisms of projective K3 surfaces with minimum entropy*, Invent. Math. **203** (2016), 179–215.

[22] K. Oguiso, *The third smallest Salem number in automorphisms of K3 surfaces*, Adv. Stud. Pure Math. **60** (2010), 331–360.

[23] J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962) no. 1, 64–94.

[24] S. Saito, *General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings*, Amer. J. Math. **109** (1987), no. 6, 1009–1042.

[25] R. Salem, Algebraic Numbers and Fourier Analysis, D.C. Heath and Company, Boston, 1963.

[26] D. Toledo, *On the Atiyah-Bott formula for isolated fixed points*, J. Differential Geometry **8** (1973), 401–436.

[27] D. Toledo and Y.L. Tong, *Duality and intersection theory in complex manifolds. II. The holomorphic Lefschetz formula*, Ann. Math. (2) **108** (1978), no. 3, 519–538.

[28] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.