Identifying All Combinations of Boundary Conditions for In-plane Vibration of Isotropic and Anisotropic Rectangular Plates

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Keywords:
In-plane vibration
Rectangular plate
Combination
Boundary condition
Ritz method

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ABSTRACT
This work presents a combined study of an effective vibration analysis and a counting theory to identify all combinations of boundary conditions for in-plane vibration of isotropic, specially orthotropic and symmetrically laminated rectangular plates. First, a method of analysis is proposed to obtain the in-plane natural frequencies of plates under any in-plane boundary conditions of free, clamp and two types of simple supports, and this method makes it possible to calculate the frequencies of rectangular plates subject to 256(=4 powered by 4) sets of boundary conditions. Secondly, Polya counting theory is introduced to determine theoretically the total number of distinct combinations of plate boundary conditions, and to reduce the 256 sets into the essentially identical subsets of frequencies. In numerical experiments, all sets of natural frequencies are calculated for the rectangular plates with different aspect ratio, material property and lamination, and are sorted into the classes with identical sets of frequencies. The distinct combinations are thus obtained for in-plane vibration of the plates, and it is shown that the number of combinations from the experiment exactly agrees with prediction by Polya counting theory.

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1. Introduction

Plate components play significantly important role for engineering structures in civil, architectural, automobile, aerospace and other industries, and their vibrational characteristics are of essential design concern. The plate vibration may be subdivided into two categories, out-of-plane (bending) vibration and in-plane vibration, except for some coupling cases where plates have asymmetric lamination or slight curvature. It is widely known that the bending vibration has generally lower values of natural frequencies than those obtained for in-plane vibration, and the amount of research devoted to bending vibration is much voluminous, as compiled in a detailed monograph by Leissa [1]. There are a number of good textbooks as well available on this topic, for example, in [2,3]. In contrast, the research on in-plane vibration has received sparse treatment. Nevertheless, excitation of in-plane vibration takes place, for example, when plates are under tangential flow, or plates are transmitting vibration noises in the plane.

For in-plane vibration of isotropic rectangular plates, Bardell and others [4] obtained in 1996 the natural frequencies and mode shapes of totally free, simply supported and clamped plates by Rayleigh-Ritz method. In 2004, Gorman introduced a method of superposition to solve problem of free plates [5] and applied to simply supported and clamped plates [6]. He derived exact solutions for cases when two opposite edges are simply supported [7]. Xing and Liu [8] also derived exact solutions for the problem. Houmat [9] developed the finite element to study in-plane vibration of plates with curvilinear planform. Liu and Banerjee [10] used dynamic stiffness method to solve the problem for fifty-five possible combinations of boundary conditions. Those studies were extended to include elastic springs along edges in the plane [11,12]. When the plate is reinforced by unidirectional fibers parallel to the edge, such specially orthotropic rectangular plates are analyzed by Gorman [13] and Liu and Xing [14], and the same problem with elastic springs along the edges was considered [15] in 2014. For laminated or single-layer plates with the principal material axes being not parallel to the edges, Woodcock and others [16] studied the effect of ply-orientation on the natural frequencies. Dozio [17] presented thorough list of frequencies for plates with wide range of edge conditions, and he claimed that there are no previous results available for comparison of laminated rectangular plates.

For editing monograph or design data book on vibration of plates, it is necessary to clarify the maximum number of combinations for the distinct sets of frequencies with respect to boundary condition, because the free vibration behaviors are strongly affected by the edge constraints. For bending vibration, Leissa [18] already listed the bending frequencies of isotropic rectangular plates for all twenty-one combinations of free, simply supported and clamped edges under variable aspect ratio. Bert and Malik [19] clarified that there are fifty-five possible combinations, when four boundary conditions are considered along four edges, including guided edge condition. Eisenberger and Deutsch [20] recently derived frequency solutions for all possible combinations of boundary conditions of rectangular plates. Thus, all the combinations of the classical boundary conditions for bending vibration of isotropic rectangular plate were already clarified.

For in-plane vibration of rectangular plates, there has been virtually no established method to seek for possible combinations of four boundary conditions, i.e., free, clamp and two types of simply support in the plane. In 2000, the first present author proposed use of Polya counting theory for such counting problem involving symmetry of plate shape and material anisotropy, and successfully found the total number of combinations with boundary conditions for rectangular plates [21], and recently he presented application of the theory to isotropic plates of general shapes [22]. Polya, mathematician in combinatorics and other related fields, proposed Polya counting theory (also known as Polya enumeration theorem and Redfield–Polya theorem), and developed the theory [23] as a powerful tool in combinatorics and graph theory (for example, refer to textbook [24]). Particularly, it has a wide range of applications to enumeration of chemical compounds. For example, Haigh and Baker [25] applied it to enumeration of the isomers in chemistry.

The present paper is composed of two stages: first a simple but very effective method is proposed to calculate natural frequencies under any sets of in-plane boundary conditions, and secondly application of Polya theory abovementioned is made. For four in-plane boundary conditions (free, clamped and two types of simple supports) along four edges, there are 256 (i.e., 4^4=256) sets of frequencies but some sets are exactly identical each other because physically they are the same (namely, one plate is obtained from another after rotating or flipping). The accuracy of the analysis is established by convergence test and comparison with existing results. Numerical examples include isotropic, orthotropic and laminated square and rectangular plates. In numerical experiments, all the natural frequencies for 256 sets of boundary conditions are calculated and are sorted out into the classes with the same sets of frequencies. When a plate is subjected to rigid-body motions, types of rigid-body motions (translational or rotational around axes) are clarified. Such numerical sorting proves that the number of combinations exactly matches the prediction of Polya counting theory.

2. Problem statement and solution procedure

2.1 Rectangular plate under arbitrary boundary conditions

Figure 1 shows a rectangular plate in the coordinate system with the origin located in the center. The dimension of the whole plate is given by $a \times b \times h$ (thickness). The $u(x,y,t)$ and $v(x,y,t)$ are in-plane displacement components in x and y directions, respectively. The plate can be composed of isotropic, specially orthotropic and symmetrically laminated materials. For each layer in laminates, the major and minor principal axes are denoted by the L and T axes, and an angle of L axis is given by θ with respect to the x-axis. Four edges are labelled as Edge(1), Edge(2), Edge(3) and Edge(4) in counter-clockwise, starting from Edge(1) along x=-a/2, as shown in the figure.

In bending (out-of-plane) vibration, three classical boundary conditions exist along each edge as free, simply supported and clamped condition. Analogous set of boundary condition is used here also for in-plane vibration of a rectangular plate, although formulation of the edges is a little different. Particularly, two distinct sets of simple support are physically possible and in the present paper these are designated by the symbol S1 and S2. In most of relevant literature, for example in Refs.[5-7], SS1 and SS2 have been used (instead of S1 and S2), but due to large amount of results presented here for all combinations of boundary conditions, short expressions S1 and S2 are used throughout without loss of clarity in this paper.

A boundary labelled as S1 is characterized by the fact that the in-plane displacement parallel to the edge is constrained (zero-displacement) but normal in-plane displacement to the edge is not constrained (i.e., normal stress is free). The set S2 of simple support implies that the in-plane displacement parallel to the edge is free (i.e., shear stress along the edge is free) but the displacement normal to the edge is constrained (zero displacement). The symbol C means that two in-plane displacements u and v are both rigidly constrained, and the symbol F does both displacements are totally free (i.e., both normal and shear stresses are free along the edge). Therefore, there can exist 4^4=256 combinations seemingly for a rectangular plate with each of four edges under any of four boundary conditions, F, S1, S2 and C.

2.2 Ritz method for any sets of in-plane boundary conditions

For solving for such in-plane problem, the in-plane strains at an arbitrary point (x,y) are assumed in the linear elastic theory as

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{1}$$

The stress-strain equations for an element of material in the k th layer can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2}$$

where the constants $\bar{Q}_{ij}^{(k)}$ are the elastic constants of the k th layer. The $\bar{Q}_{ij}^{(k)}$ are determined from the transformation relationships using the fiber orientation angle θ and the elastic constants

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\nu_{TL}},\ Q_{12} = \nu_{TL} Q_{11},\ Q_{22} = \frac{E_T}{1-\nu_{LT}\nu_{TL}},\ Q_{66} = G_{LT} \tag{3}$$

where E_L and E_T are the moduli of elasticity in the L and T directions, respectively, G_{LT} is the shear modulus and ν_{LT} and ν_{TL} are the major and minor Poisson's ratios in the layer. Superscript (k) is omitted for simplicity in Eq.(3). For isotropic plate, these are reduced to $Q_{11}=Q_{22}=E/(1-\nu^2)$, $Q_{12}=\nu\ Q_{11}$ and $Q_{66}=G=E/2(1+\nu)$.

The force resultants are obtained by integrating the stresses over the thickness h, and are written in matrix form as

$$\{N\} = [A]\{\varepsilon\} \tag{4}$$

where $\{N\}$ and $\{\varepsilon\}$ are the vectors of force resultants and in-plane strains, respectively, as

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix},\quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{5a,b}$$

and $[A]$ is the matrix of stiffness coefficients defined by

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \tag{6}$$

The stiffness coefficients in Eq.(6) are determined by

$$A_{ij} = \sum_{k=1}^{K} \bar{Q}_{ij}^{(k)} \left(z_k - z_{k-1} \right) \tag{7}$$

(i,j=1,2,6) where z_k is the distance from the mid-surface to the upper surface of the k th layer and K is the total number of layers.

The free vibration problem can be solved by means of the Ritz method. This requires the evaluation of energy functional. The strain energy due to stretching is given by

$$V = \frac{1}{2}\iint \{\varepsilon\}^T [A]\{\varepsilon\}\, dArea \tag{8}$$

The kinetic energy of the plate due to in-plane motion only is given by

$$T = \frac{1}{2}\rho h \iint \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 \right] dArea \tag{9}$$

where ρ [kg/m^3] is the average mass density.

In this formulation, the following dimensionless coordinates are introduced,

$$\xi = \frac{2x}{a},\ \eta = \frac{2y}{b} \tag{10}$$

and the displacements may be assumed in the form

$$u(\xi,\eta,t)=u^{*}(\xi,\eta)\sin\omega t=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}P_{ij}X_i(\xi)Y_j(\eta)\sin\omega t \tag{11a}$$

$$v(\xi,\eta,t)=v^{*}(\xi,\eta)\sin\omega t=\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}Q_{kl}X_k(\xi)Y_l(\eta)\sin\omega t \tag{11b}$$

where P_{ij} and Q_{kl} are unknown coefficients and $X_i(\xi)$, $Y_j(\eta)$, $X_k(\xi)$ and $Y_l(\eta)$ are the functions that satisfy at least the kinematical boundary conditions at the edges. The upper limit *M, N* in each of the summations (11) is arbitrary but is taken equal in the convergence test.

After substituting Eqs.(11) into the functional L

$$L=T_{max}-V_{max} \tag{12}$$

composed of the maximum strain and kinetic energies obtained from Eqs.(8) and (9), the stationary value is obtained by

$$\frac{\partial L}{\partial P_{ij}}=0,\ \frac{\partial L}{\partial Q_{kl}}=0 \qquad (i,k=0,1,2,..,(M-1);\ j,l=0,1,2,...,(N-1)) \tag{13}$$

The result of the minimization process (13) yields a set of homogeneous, linear simultaneous equations in the unknowns $\{P_{ij},Q_{kl}\}$. For non-trivial solutions, the determinant of the coefficient matrix is set to zero. The $(M\times N)\times 2$ eigenvalues may be extracted and the lower eigenvalues (natural frequencies) are important from a practical viewpoint.

A frequency parameter is defined by

$$\Omega=\omega a\left(\frac{\rho(1-\nu_{LT}\nu_{TL})}{E_T}\right)^{1/2} \text{ and } \Omega=\omega a\left(\frac{\rho(1-\nu^2)}{E}\right)^{1/2} \tag{14}$$

for laminated and isotropic plates, respectively.

The above procedure is a standard routine of the Ritz method, and is modified here to incorporate arbitrary edge conditions [26,27]. This approach introduces the following polynomials

$$X_i(\xi)=\xi^i(1+\xi)^{Bu1}(1-\xi)^{Bu3},\ Y_j(\eta)=\eta^j(1+\eta)^{Bu2}(1-\eta)^{Bu4} \tag{15a,b}$$

$$X_k(\xi)=\xi^k(1+\xi)^{Bv1}(1-\xi)^{Bv3},\ Y_l(\eta)=\eta^l(1+\eta)^{Bv2}(1-\eta)^{Bv4} \tag{15c,d}$$

where B_{rs} $(r=u,v;\ s=1,2,3,4)$ is the boundary index which is used to satisfy the kinematic boundary conditions. The capital letter B stands for Boundary. The first subscript letter "r" in B_{rs} indicates which displacement (u or v) is dealt with and the second subscript number "s" indicates which edge, Edge(1),.. or Edge(4) in Fig.1, is under consideration.

For in-plane displacements u and v, $B_{rs}=0$ ($r=u$ or v, s=1,2,3,4) denotes that the specified displacement along the specified edge is free and $B_{rs}=1$ does that the displacement is fixed. With such boundary index, one can accommodate arbitrary sets of in-plane boundary conditions in the vibration analysis.

The introduction of the boundary index makes it possible to deal with wide variations of edge conditions in the analysis. The number of combinations in the boundary condition is 4^4=256, when one of the two conditions (free or fixed) in u and v along each of four edges. This is larger than the case of bending analysis with typical three edge conditions. The present vibration analysis can calculate natural frequencies for any sets of the in-plane boundary combinations.

The boundary index is written at the Edge (1) along $x=-a/2$ as

$B_{u1}= B_{v1}=0$ for the free edge (no constraint), denoted by F (16a)

$B_{u1}=0$, $B_{v1}=1$ for the first-type simply supported edge, by S1 (16b)

$B_{u1}=1$, $B_{v1}=0$ for the second-type simply supported edge, by S2 (16c)

$B_{u1}= B_{v1}=1$ for the clamped edge (fully constrained edge), by C (16d)

in the plane. Along the entire four edges, for an example of plate with C-S1-S2-F, the boundary index can be expressed by

$B_{u1}= B_{v1}=1$ for the clamped edge along Edge(1), (17a)

$B_{u2}=1$, $B_{v2}=0$ for the simply supported edge S1, along Edge(2), (17b)

$B_{u3}=1$, $B_{v3}=0$ for the simply supported edge S2, along Edge(3), (17c)

$B_{u4}= B_{v4}=0$ for the free edge along Edge(4) (17d)

Note that the index values (17b) and (17c) may appear identical for different simple supports but Edge(2) and Edge(3) are perpendicular each other.

When a rectangular plate under consideration is symmetric in terms of boundary condition with respect to x and y axis in Fig.1, namely opposite edges have the same boundary condition, such as C-F-C-F, the mode shapes can be separated into four types of modes and the series solutions (11) take the even or odd integers:

(i-mode) i=odd and j=even for u; k=even and l=odd for v (18a)

(ii-mode) i=odd and j=odd for u; k=even and l=even for v (18b)

(iii-mode) i=even and j=even for u; k=odd and l=odd for v (18c)

(iv-mode) i=even and j=odd for u; k=odd and l=even for v (18d)

The i-mode indicates that u-displacement is antisymmetric about y axis and symmetric about x axis, and that v-displacement is symmetric about y axis and anti-symmetric about x axis. Likewise, other modes take symmetric or anti-symmetric shape in u and v. The present mode separation is found similarly in shallow shell vibration [28, 29]. For shallow shells, the i-mode corresponds to doubly-symmetric mode in w (out-of-plane displacement), and other three modes, ii-mode, iii-mode and iv-mode in Eqs.(18), do to symmetric-antisymmetric, antisymmetric-symmetric and doubly antisymmetric modes with respect to w, respectively. If you intentionally assume other sets of integers, such as all i, j, k, l being odd, there appear physically non-existing fictious modes that are not included in the computational results using full terms.

An advantage of using this mode separation is the reduction of matrix size in the frequency equation. For example, use of full ten terms ($M\times N$=10×10) results in the matrix size (200×200), but use of five terms (odd or even terms only) causes to reduce to smaller matrix size (50×50). For sets of boundary conditions with only one symmetry about the x axis, for example C-F-F-F, there are only two separable modes, and the series solution (11) take all positive integers in x direction. The matrix size in this case ($M\times N$=10×10) is (100×100).

3. Polya counting theory and application to the present problem

The combinatorics approach is outlined here. In Group theory, a permutation is defined as a one-to-one mapping from a set D onto D. For example, when there is a set D composed of four elements {1,2,3,4}, i.e., D={1,2,3,4}, a permutation of transposing 1→2, 2→4, 3→3, 4→1 can be written in cyclic notation as

$$P=\{(124)(3)\} \tag{19}$$

For such two permutations, a product P_1P_2 of P_1 and P_2 also becomes a permutation. In Group theory, a set G of all permutations acting on a set D can be considered as a finite group. Figure 2(Ex.1) presents an isotropic square plate, and four edges are numbered as Edge(1), Edge(2), Edge(3) and Edge(4) as shown in Fig.1. One can assume that D is a set of such edge numbers. When the plate has four edges fixed in the space under four different boundary conditions (F, S1, S2, C), the number of sets in boundary condition is seemingly given by 4^4=256. However, two cases of C-S1-F-F and F-C-S1-F are identical because the former C-S1-F-F is obtained by rotating the latter F-C-S1-F with 90 degree in clockwise. These two cases are said to belong to the same "class". In contrast, other cases of C-S1-F-F and C-F-S1-F cannot be in the same class, because one case cannot be realized neither by rotating nor flipping plates. Thus, "class" is important notation is this counting problem, and in this context, the first character is capitalized as Class in this paper.

In other words, this paper considers counting the number of Classes when a cyclic permutation group $\boldsymbol{G}$ (action of rotation or flipping) acts on a set $\boldsymbol{D}$ (four edges). Polya counting theory [23] derives a kind of polynomial "cyclic polynomial" in order to calculate the number of different Classes (i.e., number of essentially different combinations). When a cyclic group, acting on a finite group, is denoted by $\boldsymbol{G}$ and $C_k(\boldsymbol{G})$ is a number of elements in $\boldsymbol{G}$ with the cyclic number k, the cyclic polynomial for a group $\boldsymbol{G}$ acting on $\boldsymbol{D}$ is given by

$$Z_G(x)=\frac{1}{|G|}\sum_{k=1}^{|G|}C_k(G)x^k \tag{20}$$

Figure 2 illustrates six examples Ex.1-Ex.6 used to figure out all combinations of boundary conditions under four in-plane boundary conditions F, S1, S2 and C. Ex.1 and Ex.2 are isotropic, square (a/b=1) and rectangular plates (a/b=1.5), respectively. Ex.3 is a specially orthotropic square plate (i.e., the fiber orientation angle is zero). Ex.4 is an angle-ply symmetric four-layer plate with laminate construction of [30°/-30°]s. Subscript "s" means symmetric lamination. Due to in-plane behavior, the plates with [30°/-30°]s and [-30°/30°]s present the same in-plane characteristics, because the in-plane stiffness A_{ij} are identical for these two, unlike bending stiffness D_{ij} being dependent on the laminate sequence (distance from the middle plate surface). As explained next, Ex.1 has 55 Classes, and three examples Ex.2, Ex.3 and Ex.4 present the same number of Classes of 100, because they commonly share the in-plane stiffness of $A_{16}=A_{26}=0$. Ex.5 is an unidirectionally stiffened square plate [30°]. Ex.6 is a laminated square plate [0/30°]s. Both Ex.5 and Ex.6 have non-zero stiffness A_{16}, A_{26}, and this results in more number of Classes of 136 than Ex.2-4.

In Ex.1, an isotropic square plate has four edges of equal length and four symmetrical axes. Two axes are x and y axis parallel to the edges, and the remaining two axes are diagonal axes. For rotation around the central point, it has four possible rotational motions in counter-clockwise as

$$G_r=\{(1)(2)(3)(4),(1234),(13)(24),(1432)\} \quad (21)$$

with rotating angles 0°, 90°, 180° and 270°, respectively. For flipping, it has four possible motions as

$$G_f=\{(1)(3)(24),(12)(34),(2)(4)(13),(14)(23)\} \quad (22)$$

by flipping with respect to the four symmetric axes. By adding two sets of motions, one gets

$$G=G_r+G_f,\ |G|=8 \quad (23)$$

and number of elements in Eqs.(21) and (22) are determined for each cyclic number k as

$$C_4=1,\ C_3=2,\ C_2=3,\ C_1=2 \quad (24)$$

From Eqs.(23) and (24), cyclic polynomial in this example is

$$Z_G(x)=\frac{1}{8}\left(x^4+2x^3+3x^2+2x\right) \quad (25)$$

For Ex.2, Ex.3 and Ex.4, there are no diagonal symmetric axes, leaving only two symmetric axes parallel to the edges. A cyclic polynomial in this example is

$$Z_G(x)=\frac{1}{4}\left(x^4+2x^3+x^2\right) \quad (26)$$

Due to skewed principal material axes in Ex.5 and Ex.6, a cyclic polynomial becomes

$$Z_G(x)=\frac{1}{2}\left(x^4+x^2\right) \quad (27)$$

When all the combinations of F, S1, S2 and C are taken into account in the examples in Fig.2, the cyclic polynomials result in the number of Class, i.e., distinct combinations of boundary conditions, as

$$Z_G(4)=55,\ Z_G(4)=100,\ Z_G(4)=136 \quad (28)$$

for Ex.1, Ex.2-4 and Ex-5&6, respectively. These predicted numbers in Eqs.(28) will be computationally collaborated in numerical experiment.

4. Results and discussions

4.1 Convergence and Comparison

Table 1 presents convergence study for lowest five frequency parameters of Ex.1 and Ex.6 with respect to the number of series terms in Eq.(11). The boundary condition C-S1-S2-F in Ex.1 and Ex.6 is chosen to include all of F, S1, S2 and C, and there are no mode separation. Also considered are uniformly constrained as S1-S1-S1-S1 and C-C-C-C. Due to the symmetry, four separated modes exist as explained in Eqs.(18) and are written in the table. There are two iv-modes among the lowest five frequencies in both Ex.1 and Ex.6. The series-terms are increased from 6×6 to 12×12, and the matrix size of frequency equation also changes from 72×72 to 288×288. In Ex.1, even the solution of smallest terms 6×6 yields the converged values within four significant figures. For a little more complicated case in Ex.6, although the convergence speed is almost unchanged, the solution of 10×10 gives well converged values. In numerical calculation hereafter, the 10×10 solution is used throughout.

When the plate material has directional property in Ex.3-Ex.6 in Fig.2, the following material constants are used:

$$E_L=150 \text{ GPa},\ E_T=10 \text{ GPa},\ G_{LT}=5 \text{ GPa},\ \nu_{LT}=0.3 \qquad (29)$$

These are the averaged and discretized values of the material constants in a number of recent studies on carbon fiber reinforced composites, and detailed explanation on validity of using values (29) is given in Ref. [30].

Comparison of the present results is conducted in Table 2 with other existing results to further confirm validity of the present method. The first three comparisons are made in Ex.1 (isotropic square plate) for uniformly constrained edges of F-F-F-F, and S1-S1-S1-S1 and C-C-C-C with those of Refs.[6][10] and [11]. Next two comparisons are in Ex.3 (Specially orthotropic square plate) for S2-S2-S2-S2 and S2-C-S2-C with Ref.[14]. Because these five cases have mode separation both in x and y directions, mode types defined in Eq.(18) are written in the table.

Last two comparisons are in Ex.4 (Angle-ply laminated square plate, [30°/-30°]s) for C-F-F-F (cantilever) with Ref.[17]. The first case in Ex.4 assumes relatively weak material orthotropy of $E_L/E_T=2$, and the second case does strong orthotropy of $E_L/E_T=20$. The solution for strong orthotropy exhibits slight differences. In all comparisons in the table, the present results are in very good agreement with the existing results, and the accuracy of the present Ritz method is well established.

4.2 Results under all combinations of in-plane boundary conditions

Table 3 presents a full list of the lowest five frequency parameters of an isotropic square plate (Ex.1) for all fifty five Classes (combinations) of F, S1, S2 and C, starting from a totally free plate F-F-F-F (Class 1) and ending with a totally clamped plate C-C-C-C (Class 55). The set of boundary index is increased from 0 to 1 one by one in the sequence of v at Edge(1), u at Edge(1), v at Edge(2), u at Edge(2),

v at Edge(3), u at Edge(3), v at Edge(4) and u at Edge(4). Roughly speaking, the edge constraints are gradually increased. Except for it, however, value of Class is just a sequence number and has practically no physical meanings. As predicted by Polya counting theory, all 256 cases are reduced to 55 Classes in Eq.(28), with each Class containing physically identical sets of plate frequencies. In numerical experiments, any two sets of frequencies among 256 sets are compared and when the sum of absolute differences for lowest six frequencies is less than 0.001, they are regarded as the same Class.

In each Class of the table, sets of boundary conditions are listed together that give the same sets of frequency parameters. For example, at Class 2, it is shown that the same sets of frequencies are obtained for isotropic square plates with S1-F-F-F, F-F-S1-F, F-S1-F-F and F-F-F-S1. Thus, it is numerically demonstrated that Polya counting theory gives the exact number of Classes, i.e., essentially distinct combinations in boundary conditions.

Also given in the table is the number of modes for rigid-body motions (RBM). At Class 1, there are three such modes: two translational motions in x and y directions and one in-plane rotational motion. At Class 2, there exist one translational and one rotation motion, and basically number of rigid-body motions decreases as more constraints are added along four edges, as observed in the table. The present Ritz method yields zero natural frequencies in the calculation for modes of rigid-body motion, and these are easily identified and excluded in the frequency table.

Such rigid-body motions are detailed in Table 4. For all boundary conditions in each Class with such motions, translational and/or rotational rigid-body motions are identified with circle ○(existing) or cross ×(non-existing). TRX stands for TRanslational motion in x direction, and TRY does for that in y direction. ROT does ROTational motion in the x-y plane. For more details describing rotational motions, footnotes 1-9 for ROT give information on which corner point the plate exhibits rotation. The four corners are denoted as A, B, C and D in the figure. For example, the footnote *2 indicates that the plate can rotate around the axis at A or B perpendicular to the points in the plane.

Tables 5, 6 and 7 present 100 Classes of frequencies for Ex.2, Ex.3 and Ex.4, respectively. The number of Class 100 was predicted as in Eq.(28). Due to space limitation, sets of frequencies are presented one Class every ten Classes, namely, Class 1, 10, 20, .. ,100. It is seen in the tables that the maximum number of boundary conditions is four per Class while in Table 3 there are Classes with eight boundary conditions. Also observed is that in the three tables the same sets of boundary conditions appear every ten Classes, although values of natural frequencies are all different because of each physical setting (i.e., Ex.2 isotropic rectangular plate, Ex.3 specially orthotropic square plate, Ex.4 angle-ply square plate). It is noted that the three plates have common property of A_{16}=A_{26}=0 to allow flipping around y axis to give the same result. For rigid-body motion in Ex.2-4, basic characteristics are the same as Ex.1, and the behaviors of translation and rotation in Table 4 are applicable.

Tables 8 and 9 present 136 Classes, sets of frequencies for Ex.5 and Ex.6, respectively. These two tables are in the same format as in Tables 5-7 except for the number 136 of Class. The number of Class 136 agrees with the prediction in Eq.(28). In this case also, the same sets of boundary conditions are found every ten Classes, and likewise the natural frequencies vary depending upon physical settings (i.e., Ex.5 unidirectional square plate [30°], Ex.6 generally laminated square plate, [0°/30°]s), and the difference of number of Classes from Ex.2-4 is originated by anisotropic property of $A_{16}\neq0$, $A_{26}\neq0$.

For illustration of transition, Figure 3 shows examples of how one Class for identical sets of frequencies splits into two smaller Classes in the process of loosing symmetric axes of the problem. Only examples stemming from first five Classes in Ex.1 are presented due to space limitation. There are four symmetric axes (two axes on x and y coordinate axes, and two diagonal axes) in Ex1 (isotropic square plate), and in Ex.2-4 two axes exist after loosing diagonally symmetric axes. In the figure, Class 2, 3, 4 and 5 in Ex.1 are subdivided into two Classes with smaller number of sets of boundary conditions (BCs) in Ex.2-4.

When one looks at the transition of all Classes, the 55 Classes in Ex.1 are composed of 15 Classes with 8 BCs, 30 Classes with 4 BCs, 6 Classes with 2 BCs and 4 Classes with 1 BC. In the first transition of Ex.1 to Ex.2-4, they now become 36 Classes with 4 BCs, 48 Classes with 2 BCs and 16 Classes with 1 BC but no Classes with 8 BCs. As a result, total number of Class increases from 55 to 100. Likewise, as partly shown in the figure, Classes in Ex.2-4 next become, in Ex.5&6, 120 Classes with 2 BCs and 16 Classes with 1 BC, but no Classes any more with 4 BCs. So the total number of Classes is 136.

5. Conclusions

The objective of this study was first to propose an effective semi-analytical method to obtain natural frequencies for in-plane vibration of rectangular plates under any sets of boundary conditions and to actually determine the frequency values of plates composed of isotropic and anisotropic materials. Next, Polya counting theory was introduced to determine theoretically the essential number of combination of in-plane boundary conditions that give identical frequency results. In numerical study, the frequencies were computed in good accuracy for all combinations of boundary conditions of free, clamp and two types of simple support, and such in-plane vibration frequencies are sorted into 55, 100 and 136 subsets (Classes) with identical results from 256 original sets for various material construction and aspect ratio. Those number of sorted subsets exactly agreed with theoretical predictions by Polya counting theory.

It was well demonstrated that a simple but numerically efficient semi-analytical method can provide with a complete set of in-plane natural frequencies for rectangular plates with complicating effects and the frequencies obtained can be grouped into the subsets with identical results, and the number of such subsets can be predicted by the counting theory. It is hoped that this proposal of joint work on vibration analysis and counting theory will be a good technical approach enough to support design data handling in vibration engineering.

References

[1] A.W. Leissa, Vibration of Plates. Acoustical Society of America, 1993 (previously, NASA SP-160, U.S. Government Printing Office, Washington D.C., 1969).

[2] D.J. Gorman, Vibration Analysis of Plates by the Method of Superposition. 1999, World Scientific Publishing Company.

[3] A.W. Leissa, M.S. Qatu, Vibration of Continuous Systems. 2011, McGraw-Hill Professional Pub.

[4] N.S. Bardell, R.S. Langley, J.M. Dunson, On the free in-plane vibration of isotropic rectangular plates, J Sound Vib. 191 (1996) 459-467.

[5] D.J. Gorman, Free in-plane vibration analysis of rectangular plates by the method of superposition, J Sound Vib. 272 (2004) 831-851.

[6] D.J. Gorman, Accurate analytical type solutions for the free in-plane vibration of clamped and simply supported rectangular plates, J Sound Vib. 276 (2004) 311-333.

[7] D.J. Gorman, Exact solutions for the free in-plane vibration of rectangular plates with two opposite edges simply supported, J Sound Vib. 294 (2006) 131-161.

[8] Y.F. Xing, B. Liu, Exact solutions for the free in-plane vibrations of rectangular plates, Int J Mech Sci. 51 (2009) 246-255.

[9] A. Houmat, In-plane vibration of plates with curvilinear plan-forms by a trigonometrically enriched curved triangular p-element, Thin-Walled Struct. 46 (2008) 103-111.

[10] X. Liu, J.R. Banerjee, A spectral dynamic stiffness method for free vibration analysis of plane elastodynamic problems, Mech Syst Signal Pr. 87 (2017) 136-160.

[11] D.J. Gorman, Free in-plane vibration analysis of rectangular plates with elastic support normal to the boundaries, J Sound Vib. 285 (2005) 941-966.

[12] J. Du, W.L. Li, G. Jin, T. Yang, Z. Liu, An analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges, J Sound Vib. 306 (2007) 908-927.

[13] D.J. Gorman, Accurate in-plane free vibration analysis of rectangular orthotropic plates, J Sound Vib. 323 (2009) 426-443.

[14] B. Liu, Y. Xing, Comprehensive exact solutions for free in-plane vibrations of orthotropic rectangular plates, Eur J Mech A/Solid. 30 (2011) 383-395.

[15] Y. Zhang, J. Du, T. Yang, Z. Liu, A series solution for the in-plane vibration analysis of orthotropic rectangular plates with elastically restrained edges, Int J Mech Sci. 79 (2014) 15-24.

[16] R.L. Woodcock, R.B. Bhat, I.G. Stiharu, Effect of ply orientation on the in-plane vibration of single-layer composite plates, J Sound Vib. 312 (2008) 94-108

[17] L. Dozio, In-plane free vibrations of single-layer and symmetrically laminated rectangular composite plates, Compos Struct. 93 (2011) 1787-1800.

[18] A.W. Leissa, The free vibration of rectangular plates, J Sound Vib. 31 (1973) 257-293.

[19] C.W. Bert, M. Malik, Frequency equations and modes of free vibrations of rectangular plates with various edge conditions, P I Mech Eng C-J Mec. 208 (1994) 307-319.

[20] M. Eisenberger, A. Deutsch, Solution of thin rectangular plate vibrations for all combinations of boundary conditions, J Sound Vib. 452 (2019) 1-12.

[21] Y. Narita, Combinations for the free-vibration behaviors of anisotropic rectangular plates under general edge conditions, ASME J Appl Mech. 67 (2000) 568-573.

[22] Y. Narita, Polya counting theory applied to combination of edge conditions for generally shaped isotropic plates, EPI Int J Eng. 2 (2019) 194-202; DOI: 10.25042/epi-ije.082019.16.

[23] G. Pólya, R.C. Read, Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds, 1987, Springer-Velag, ISBN-13: 978-1461246640.

[24] J.M. Harris, J.L. Hirst, M.J. Mossinghoff, 2000, Combinatorics and Graph Theory, 2008, Springer, ISBN-13: 978-0387797113.

[25] C.W. Haigh, P.K. Baker, Use of Polya's theorem to enumerate the isomers of seven-coordinate complexes with capped octahedral geometry, Polyhedron. 13 (1994) 417-433.

[26] Y. Narita, Series and Ritz-type buckling analysis, in: G.J. Turvey, I.H. Marshall (Eds.), Buckling and Postbuckling of Composite Plates, 1995, Chapman & Hall, London, 33–57 (Chapter 2).

[27] Y. Narita, P. Robinson, Maximizing the fundamental frequency of laminated cylindrical panels using layerwise optimization, Int J Mech Sci. 48 (2006) 1516-1524.

[28] Y. Narita and A.W. Leissa, Vibrations of completely free shallow shells of curvilinear planform, Journal of Applied Mechanics, Trans. ASME, 53 (1986) 647-651.

[29] A.W. Leissa, Y. Narita, Vibrations of completely free shallow shells of rectangular planform, J Sound Vib. 96 (1984) 207-218.

[30] Y. Narita, M. Innami, D. Narita, The effect of using different elastic moduli on vibration of laminated CFRP rectangular plates, EPI Int J Eng. 2 (2019) 19-27; DOI: 10.25042/epi-ije.022019.05.

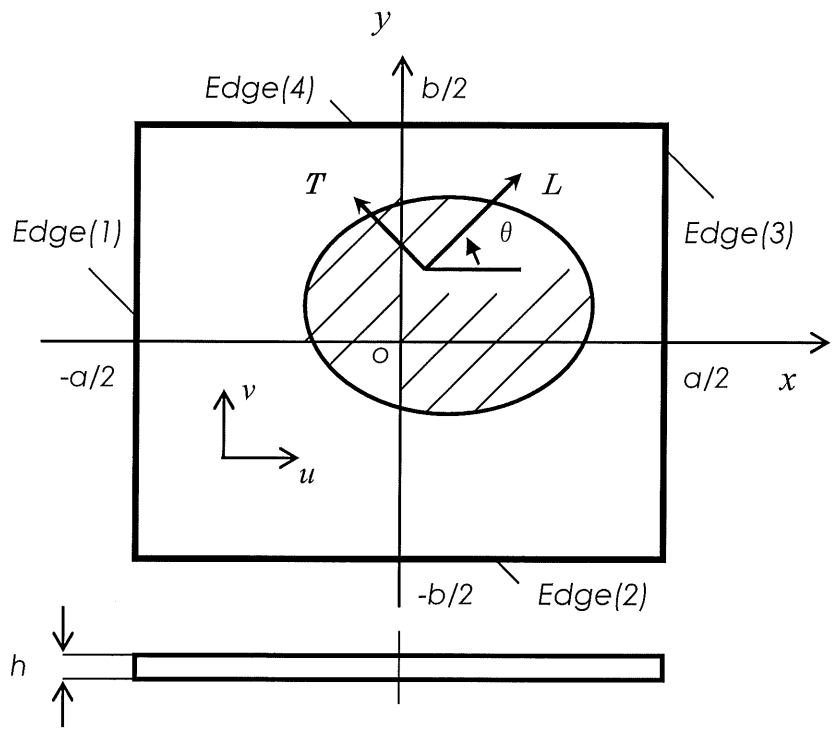


Fig. 1 Laminated composite rectangular plate and coordinate system

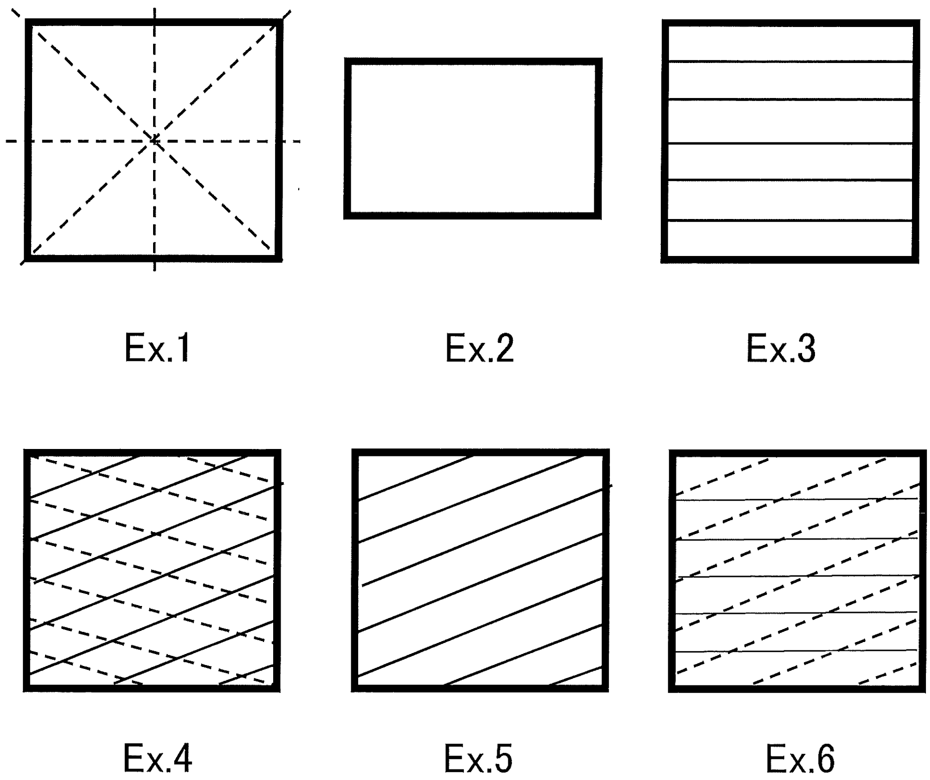


Fig.2 Numerical examples (Ex.1: Isotropic square plate, Ex.2: Isotropic rectangular plate (*a/b*=1.5), Ex.3: Specially orthotropic square plate [0°], Ex.4 Alternating angle-ply square plate [30°/-30°]s, Ex.5 Unidirectional square plate [30°], Ex.6: Generally laminated square plate [0°/30°]s)

Ex.1 Class	Ex.1 BC		Ex.2-4 Class	Ex.2-4 BC		Ex.5&6 Class	Ex.5&6 BC
1	F -F -F -F	⇒	1	F -F -F -F	⇒	1	F -F -F -F
2	S1-F -F -F	⇒	2	S1-F -F -F	⇒	2	S1-F -F -F
	F -F -S1-F			F -F -S1-F			F -F -S1-F
	F -S1-F -F	⇒	9	F -S1-F -F	⇒	9	F -S1-F -F
	F -F -F -S1			F -F -F -S1			F -F -F -S1
3	S2-F -F -F	⇒	3	S2-F -F -F	⇒	3	S2-F -F -F
	F -F -S2-F			F -F -S2-F			F -F -S2-F
	F -S2-F -F	⇒	5	F -S2-F -F	⇒	5	F -S2-F -F
	F -F -F -S2			F -F -F -S2			F -F -F -S2
4	C -F -F -F	⇒	4	C -F -F -F	⇒	4	C -F -F -F
	F -F -C -F			F -F -C -F			F -F -C -F
	F -C -F -F	⇒	13	F -C -F -F	⇒	13	F -C -F -F
	F -F -F -C			F -F -F -C			F -F -F -C
5	S1-S2-F -F	⇒	6	S1-S2-F -F	⇒	6	S1-S2-F -F
	F -F -S1-S2			F -F -S1-S2			F -F -S1-S2
	F -S2-S1-F			F -S2-S1-F	⇒	20	F -S2-S1-F
	S1-F -F -S2			S1-F -F -S2			S1-F -F -S2
	S2-S1-F -F	⇒	11	S2-S1-F -F	⇒	11	S2-S1-F -F
	F -F -S2-S1			F -F -S2-S1			F -F -S2-S1
	F -S1-S2-F			F -S1-S2-F	⇒	38	F -S1-S2-F
	S2-F -F -S1			S2-F -F -S1			S2-F -F -S1
total 55 Classes			total 100 Classes			total 136 Classes	

Fig.3 Transition of Classes from first five Classes in Ex.1 to Ex.2-4, and to Ex.5&6 with decreasing symmetric axes.

Table 1
Convergence of frequency parameters Ω of square plates (Ex.1 and Ex.6).

Number of term $M \times N$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Ex.1 Isotropic plate (C-S1-S2-F)					
6x6	0.847	2.548	2.797	3.325	3.596
8x8	0.847	2.547	2.797	3.324	3.596
10x10	0.847	2.547	2.797	3.324	3.596
12x12	0.847	2.547	2.797	3.324	3.596
Ex.1 Isotropic plate (S1-S1-S1-S1)					
	(ii-mode)	(iii-mode)	(i-mode)	(iv-mode)	(iv-mode)
6x6	1.859	1.859	2.628	3.717	3.717
8x8	1.859	1.859	2.628	3.717	3.717
10x10	1.859	1.859	2.628	3.717	3.717
12x12	1.859	1.859	2.628	3.717	3.717
Ex.6 Four-layer plate [0°/30°]s (C-S1-S2-F)					
6x6	1.778	3.730	5.252	5.752	6.997
8x8	1.778	3.729	5.246	5.749	6.975
10x10	1.778	3.729	5.245	5.748	6.974
12x12	1.777	3.729	5.245	5.748	6.974
Ex.6 Four-layer plate [0°/30°]s (C-C-C-C)					
	(ii-mode)	(iii-mode)	(iv-mode)	(iv-mode)	(i-mode)
6x6	4.839	7.326	7.635	9.003	10.009
8x8	4.839	7.326	7.635	9.003	9.993
10x10	4.839	7.326	7.635	9.003	9.993
12x12	4.839	7.326	7.635	9.003	9.993

Table 2
Comparison of frequency parameters Ω of square plates.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Ex.1 Isotropic square plate (F-F-F-F)					
	(iv-mode)	(ii-mode)	(iii-mode)	(i-mode)	(i-mode)
Present	2.321	2.472	2.472	2.628	2.987
Ref.[10]	2.321	2.472	2.472	2.628	2.987
Ref.[11]	2.320	2.472	2.472	2.628	2.988
Ex.1 Isotropic square plate (S1-S1-S1-S1)					
	(ii-mode)	(iii-mode)	(i-mode)	(iv-mode)	(iv-mode)
Present	1.859	1.859	2.628	3.717	3.717
Ref.[6]	1.859	1.859	2.628	3.717	3.717
Ref.[10]	1.859	1.859	2.628	3.717	3.717
Ex.1 Isotropic square plate (C-C-C-C)					
	(ii-mode)	(iii-mode)	(iv-mode)	(iv-mode)	(i-mode)
Present	3.555	3.555	4.235	5.186	5.859
Ref.[6]	3.555	3.555	4.235	5.186	5.859
Ref.[10]	3.555	3.555	4.235	5.186	5.859
Ex.3 Specially orthotropic plate (S2-S2-S2-S2)					
	(ii-mode)	(i-mode)	(iii-mode)	(ii-mode)	(iii-mode)
Present	3.142	3.229	4.443	5.035	5.095
Ref.[14]	3.141	3.229	4.442	5.034	5.095
Ex.3 Specially orthotropic plate (S2-C-S2-C)					
	(ii-mode)	(iv-mode)	(iii-mode)	(ii-mode)	(iv-mode)
Present	3.142	3.593	4.714	5.162	6.187
Ref.[14]	3.141	3.592	4.714	5.162	6.187
Ex.4 Angle-ply square plate (C-F-F-F)[30°/-30°]s, (E_L/E_T=2)					
Present	0.836	2.023	2.253	3.208	3.612
Ref.[17]	0.836	2.023	2.253	3.208	3.612
Ex.4 Angle-ply square plate (C-F-F-F)[30°/-30°]s, (E_L/E_T=20)					
Present	1.900	3.381	4.775	5.132	6.085
Ref.[17]	1.897	3.379	4.772	5.131	6.085

Table 3
All Classes for frequency parameters Ω of (Ex.1) isotropic square plates with corresponding boundary conditions (RBM: Rigid-Body Motion).

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	boundary conditions
1	3	2.321	2.472	2.472	2.628	2.987	FFFF
2	2	1.634	2.363	2.628	2.683	3.375	S1-F-F-F, F-F-S1-F, F-S1-F-F, F-F-F-S1
3	1	0.977	1.480	2.392	2.602	3.073	S2-F-F-F, F-F-S2-F, F-S2-F-F, F-F-F-S2
4	0	0.628	1.507	1.691	2.686	2.897	C-F-F-F, F-F-C-F, F-C-F-F, F-F-F-C
5	1	1.236	1.862	2.484	3.050	3.257	S1-S2-F-F, F-F-S1-S2, F-S2-S1-F, S1-F-F-S2 S2-S1-F-F, F-F-S2-S1, F-S1-S2-F, S2-F-F-S1,
6	0	1.314	1.494	1.726	2.523	3.022	S2-S2-F-F, F-S2-S2-F, S2-F-F-S2, F-F-S2-S2
7	0	1.450	1.611	1.949	2.847	3.541	C-S2-F-F, S2-C-F-F, F-C-S2-F, F-S2-C-F, C-F-F-S2, F-F-C-S2, S2-F-F-C, F-F-S-2C
8	1	1.160	2.152	2.628	2.643	3.421	S1-S1-F-F, F-S1-S1-F, S1-F-F-S1, F-F-S1-S1
9	0	0.780	1.606	2.222	2.919	3.070	C-S1-F-F, S1-C-F-F, F-C-S1-F, F-S1-C-F, C-F-F-S1, F-F-C-S1, S1-F-F-C, F-F-S1-C
10	0	1.470	1.884	2.255	3.403	3.843	C-C-F-F, F-C-C-F, C-F-F-C, F-F-C-C
11	1	1.408	1.859	2.628	3.249	3.364	S1-F-S1-F, F-S1-F-S1
12	0	0.514	1.480	2.334	2.444	2.987	S2-F-S1-F, S1-F-S2-F, F-S1-F-S2, F-S2-F-S1
13	0	1.486	1.507	2.446	3.058	3.303	C-F-S1-F, S1-F-C-F, F-C-F-S1, F-S1-F-C
14	1	1.859	1.866	3.059	3.423	3.425	S1-S2-S1-F, S1-F-S1-S2, S2-S1-F-S1, F-S1-S2-S1
15	0	1.314	1.682	2.517	2.607	3.801	S2-S2-S1-F, S1-S2-S2-F, S2-S1-F-S2, S2-F-S1-S2, S1-F-S2-S2, F-S1-S2-S2, S2-S2-F-S1, F-S2-S2-S1
16	0	1.509	1.951	2.545	3.481	3.839	C-S2-S1-F, S1-S2-C-F, C-F-S1-S2, S1-F-C-S2, S2-C-F-S1, F-C-S2-S1, S2-S1-F-C, F-S1-S2-C
17	0	0.929	1.624	2.628	2.788	3.391	S1-S1-S1-F, S1-S1-F-S1, S1-F-S1-S1, F-S1-S1-S1
18	0	0.704	1.749	2.518	2.776	3.285	S2-S1-S1-F, S1-S1-S2-F, S1-S1-F-S2, F-S1-S1-S2, S1-S2-F-S1, S2-F-S1-S1, F-S2-S1-S1, S1-F-S2-S1
19	0	1.560	1.872	2.899	3.351	3.543	C-S1-S1-F, S1-S1-C-F, S1-C-F-S1, C-F-S1-S1, F-C-S1-S1, S1-F-C-S1, S1-S1-F-C, F-S1-S1-C
20	0	0.929	1.978	2.788	3.062	3.439	S1-C-S1-F, C-S1-F-S1, F-S1-C-S1, S1-F-S1-C
21	0	1.598	1.760	2.650	3.238	4.125	S2-C-S1-F, S1-C-S2-F, C-S1-F-S2, F-S1-C-S2, C-S2-F-S1, F-S2-C-S1, S2-F-S1-C, S1-F-S2-C
22	0		2.085	3.184	3.548	4.216	C-C-S1-F, S1-C-C-F, C-C-F-S1, F-C-C-S1, C-S1-F-C, C-F-S1-C, S1-F-C-C, F-S1-C-C
23	1	1.408	2.628	3.142	3.249	3.364	S2-F-S2-F, F-S2-F-S2
24	0	0.756	2.376	2.782	3.259	3.490	C-F-S2-F, S2-F-C-F, F-C-F-S2, F-S2-F-C
25	0	1.571	1.866	3.059	3.423	3.425	S2-S2-S2-F, S2-S2-F-S2, S2-F-S2-S2, F-S2-S2-S2
26	0	1.675	2.641	3.100	3.487	4.201	C-S2-S2-F, S2-S2-C-F, S2-C-F-S2, C-F-S2-S2, F-C-S2-S2, S2-F-C-S2, S2-S2-F-C, F-S2-S2-C
27	1	1.624	2.628	3.142	3.391	3.449	S2-S1-S2-F, S1-S2-F-S2, F-S2-S1-S2, S2-F-S2-S1
28	0	0.847	2.547	2.797	3.324	3.596	C-S1-S2-F, S2-S1-C-F, S1-C-F-S2, F-C-S1-S2, C-F-S2-S1, S2-F-C-S1, S1-S2-F-C, F-S2-S1-C

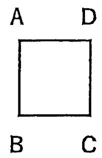
(continued on next page)

Table 3 (*continued*)

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	boundary conditions
29	0	1.571	1.978	3.062	3.439	4.162	S2-C-S2-F, C-S2-F-S2, F-S2-C-S2, S2-F-S2-C
30	0	1.708	2.726	3.112	4.078	4.405	C-C-S2-F, S2-C-C-F, C-C-F-S2, F-C-C-S2, C-S2-F-C, C-F-S2-C, S2-F-C-C, F-S2-C-C
31	0	1.694	3.019	3.121	3.351	3.745	C-F-C-F, F-C-F-C
32	0	2.262	3.047	3.232	3.829	4.536	C-S2-C-F, C-F-C-S2, S2-C-F-C, F-C-S2-C
33	0	1.774	3.078	3.399	3.425	3.890	C-S1-C-F, C-F-C-S1, S1-C-F-C, F-C-S1-C
34	0	2.270	3.163	3.409	4.306	4.718	C-C-C-F, C-C-F-C, C-F-C-C, F-C-C-C
35	1	1.859	2.628	3.142	3.717	4.156	S1-S2-S1-S2, S2-S1-S2-S1
36	0	1.571	2.078	3.351	3.512	3.832	S2-S2-S1-S2, S1-S2-S2-S2, S2-S1-S2-S2, S2-S2-S2-S1
37	0	1.571	2.205	3.422	3.865	4.241	C-S2-S1-S2, S1-S2-C-S2, S2-C-S2-S1, S2-S1-S2-C
38	0	0.929	2.078	2.788	3.351	3.512	S1-S1-S1-S2, S1-S2-S1-S1, S2-S1-S1-S1, S1-S1-S2-S1
39	0	1.314	2.221	2.939	2.939	3.943	S2-S1-S1-S2, S1-S1-S2-S2, S2-S2-S1-S1, S1-S2-S2-S1
40	0	1.638	2.479	2.999	3.809	4.153	C-S1-S1-S2, S1-S1-C-S2, C-S2-S1-S1, S2-C-S1-S1, S1-C-S2-S1, S1-S2-C-S1, S2-S1-S1-C, S1-S1-S2-C
41	0	0.929	2.788	2.811	3.450	4.265	S1-C-S1-S2, C-S1-S2-S1, S2-S1-C-S1, S1-S2-S1-C
42	0	1.747	2.814	3.513	3.570	4.682	S2-C-S1-S2, C-S1-S2-S2, S1-C-S2-S2, S2-S1-C-S2, C-S2-S2-S1, S2-S2-C-S1, S2-S2-S1-C, S1-S2-S2-C
43	0	1.778	2.947	3.557	4.292	4.727	C-C-S1-S2, S1-C-C-S2, C-C-S2-S1, S2-C-C-S1, C-S2-S1-C, C-S1-S2-C, S1-S2-C-C, S2-S1-C-C
44	0	2.628	3.142	3.142	4.156	4.156	S2-S2-S2-S2
45	0	2.811	3.142	3.450	4.265	4.327	C-S2-S2-S2, S2-C-S2-S2, S2-S2-C-S2, S2-S2-S2-C
46	0	2.929	3.354	3.641	4.359	5.010	C-C-S2-S2, S2-C-C-S2, C-S2-S2-C, S2-S2-C-C
47	0	3.142	3.275	3.495	4.411	4.957	C-S2-C-S2, S2-C-S2-C
48	0	2.386	3.091	3.750	4.133	4.683	C-S1-C-S2, C-S2-C-S1, S2-C-S1-C, S1-C-S2-C
49	0	3.189	3.524	3.804	4.858	5.300	C-C-C-S2, C-C-S2-C, C-S2-C-C, S2-C-C-C
50	0	1.859	1.859	2.628	3.717	3.717	S1-S1-S1-S1
51	0	1.859	2.205	3.422	3.717	3.865	C-S1-S1-S1,S1-C-S1-S1,S1-S1-C-S1,S1-S1-S1-C
52	0	2.118	2.593	3.798	3.900	4.360	C-C-S1-S1, S1-C-C-S1, C-S1-S1-C, S1-S1-C-C
53	0	1.859	3.275	3.495	3.717	4.411	C-S1-C-S1, S1-C-S1-C
54	0	2.395	3.356	4.070	4.499	4.758	C-C-C-S1, C-C-S1-C, C-S1-C-C, S1-C-C-C
55	0	3.555	3.555	4.235	5.186	5.859	C-C-C-C

Table 4
Type of rigid-body motion (RBM) of isotropic square plate for possible combinations.

Class	1	2			
BC	F-F-F-F	S1-F-F-F	F-F-S1-F	F-S1-F-F	F-F-F-S1
TRX	O	O	O	×	×
TRY	O	×	×	O	O
ROT	O*1	O*2	O*3	O*4	O*5



Class	3				5	
BC	S2-F-F-F	F-F-S2-F	F-S2-F-F	F-F-F-S2	S1-S2-F-F	F-F-S1-S2
TRX	×	×	O	O	O	O
TRY	O	O	×	×	×	×
ROT	×	×	×	×	×	×

Class	5					
BC	F-S2-S1-F	S1-F-F-S2	S2-S1-F-F	F-F-S2-S1	F-S1-S2-F	S2-F-F-S1
TRX	O	O	×	×	×	×
TRY	×	×	O	O	O	O
ROT	×	×	×	×	×	×

Class	8				11	
BC	S1-S1-F-F	F-S1-S1-F	S1-F-F-S1	F-F-S1-S1	S1-F-S1-F	F-S1-F-S1
TRX	×	×	×	×	O	×
TRY	×	×	×	×	×	O
ROT	O*6	O*7	O*8	O*9	×	×

Class	14				23	
B.C.	S1-S2-S1-F	S1-F-S1-S2	S2-S1-F-S1	F-S1-S2-S1	S2-F-S2-F	F-S2-F-S2
TRX	O	O	×	×	×	O
TRY	×	×	O	O	O	×
ROT	×	×	×	×	×	×

Class	27				35	
BC	S2-S1-S2-F	S1-S2-F-S2	F-S2-S1-S2	S2-F-S2-S1	S1-S2-S1-S2	S2-S1-S2-S1
TRX	×	O	O	×	O	×
TRY	O	×	×	O	×	O
ROT	×	×	×	×	×	×

(Rotation with respect to) *1: any point, *2: A or B, *3: C or D, *4: B or C, *5: D or A, *6: B, *7: C, *8: D, *9: A.

Table 5
Every ten Classes with frequency parameters Ω and boundary conditions of (Ex.2) isotropic rectangular plates ($a/b = 1.5$).

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		boundary conditions				
1	3	2.196	2.881	2.915	3.938	3.971
		FFFF				
10	1	1.440	2.297	3.123	3.543	4.133
		S1-S1-F-F, F-S1-S1-F, S1-F-F-S1, F-F-S1-S1				
20	1	2.228	2.788	3.169	3.520	4.233
		S1-S2-S1-F, S1-F-S1-S2				
30	0	0.660	2.215	2.963	3.910	4.270
		C-F-S2-F, S2-F-C-F				
40	0	2.791	3.325	3.738	5.244	5.292
		C-C-C-F, C-F-C-C				
50	0	1.311	3.181	3.891	4.313	4.725
		S1-C-F-S2, F-C-S1-S2, S1-S2-F-C, F-S2-S1-C				
60	0	2.042	4.160	4.216	5.041	5.176
		S2-C-S1-S2, S1-C-S2-S2, S2-S2-S1-C, S1-S2-S2-C				
70	0	3.342	4.755	4.800	5.996	6.118
		C-C-C-S2, C-S2-C-C				
80	0	0.929	2.788	2.939	3.943	4.647
		S2-S1-S1-S1, S1-S1-S2-S1				
90	0	2.933	3.745	4.581	5.658	5.999
		C-C-C-S1, C-S1-C-C				
100	0	4.113	4.925	5.403	6.564	6.603
		C-C-C-C				

Table 6
Every ten Classes with frequency parameters Ω and boundary conditions of (Ex.3) specially orthotropic square plates ($a/b = 1$).

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		boundary conditions				
1	3	2.653	2.775	3.132	4.165	4.658
		FFFF				
10	1	1.388	2.684	3.250	3.542	4.398
		S1-S1-F-F, F-S1-S1-F, S1-F-F-S1, F-F-S1-S1				
20	1	2.215	2.585	4.430	4.553	5.061
		S1-S2-S1-F, S1-F-S1-S2				
30	0	1.053	3.204	3.284	4.369	5.408
		C-F-S2-F, S2-F-C-F				
40	0	2.668	4.582	5.168	6.325	6.702
		C-C-C-F, C-F-C-C				
50	0	0.991	2.930	3.469	4.545	5.207
		S1-C-F-S2, F-C-S1-S2, S1-S2-F-C, F-S2-S1-C				
60	0	3.273	4.528	6.112	6.227	6.334
		S2-C-S1-S2, S1-C-S2-S2, S2-S2-S1-C, S1-S2-S2-C				
70	0	3.834	5.391	6.638	7.323	7.564
		C-C-C-S2, C-S2-C-C				
80	0	1.107	3.256	3.322	4.524	5.537
		S2-S1-S1-S1, S1-S1-S2-S1				
90	0	2.712	4.689	5.190	6.389	6.820
		C-C-C-S1, C-S1-C-C				
100	0	3.835	5.395	6.640	7.325	7.578
		C-C-C-C				

Table 7

Every ten Classes with frequency parameters Ω and boundary conditions of (Ex.4) angle-ply laminated square plates ($a/b = 1$), [30°/-30°]s.

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		boundary conditions				
1	3	2.959	3.145	5.181	5.264	5.652
		FFFF				
10	1	2.826	3.685	4.378	5.463	6.399
		S1-S1-F-F, F-S1-S1-F, S1-F-F-S1, F-F-S1-S1				
20	1	3.804	4.388	5.452	7.230	7.374
		S1-S2-S1-F, S1-F-S1-S2				
30	0	2.152	4.053	6.127	6.241	6.685
		C-F-S2-F, S2-F-C-F				
40	0	5.175	7.413	7.922	8.842	9.997
		C-C-C-F, C-F-C-C				
50	0	2.006	4.009	5.580	6.532	7.368
		S1-C-F-S2, F-C-S1-S2, S1-S2-F-C, F-S2-S1-C				
60	0	3.416	5.785	6.498	7.376	7.462
		S2-C-S1-S2, S1-C-S2-S2, S2-S2-S1-C, S1-S2-S2-C				
70	0	6.512	8.796	8.822	9.255	10.485
		C-C-C-S2, C-S2-C-C				
80	0	2.726	2.852	6.230	6.775	7.045
		S2-S1-S1-S1, S1-S1-S2-S1				
90	0	5.738	7.594	8.740	9.633	10.073
		C-C-C-S1, C-S1-C-C				
100	0	6.597	9.094	9.301	9.846	11.703
		C-C-C-C				

Table 8
Every ten Classes with frequency parameters Ω and boundary conditions of (Ex.5) unidirectional square plates ($a/b = 1$), [30°].

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		boundary conditions				
1	3	2.765	2.918	3.140	3.634	3.841
		FFFF				
10	1	1.998	2.995	3.307	3.692	4.468
		S1-S1-F-F, F-F-S1-S1				
20	1	1.448	2.554	2.998	4.155	4.625
		F-S2-S1-F, S1-F-F-S2				
30	0	1.746	3.086	4.144	4.922	5.567
		S2-C-S1-F, S1-F-S2-C				
40	1	2.670	3.342	3.713	4.908	5.499
		S2-S1-S2-F, S2-F-S2-S1				
50	0	2.798	4.721	5.204	6.194	7.256
		C-S2-C-F, C-F-C-S2				
60	1	2.195	3.258	3.763	4.752	5.190
		S1-S2-F-S2, F-S2-S1-S2				
70	0	3.093	4.479	5.487	6.133	6.404
		C-C-F-S2, F-S2-C-C				
80	0	2.914	3.740	5.269	5.983	6.288
		S2-C-S1-S2, S1-S2-S2-C				
90	0	3.531	3.993	5.087	6.037	6.363
		S2-C-S2-S2, S2-S2-S2-C				
100	0	3.802	5.502	6.357	7.195	7.962
		C-C-C-S2, C-S2-C-C				
110	0	1.649	2.766	4.025	4.920	5.688
		S2-S1-S1-S1, S1-S1-S2-S1				
120	0	1.771	3.793	4.512	5.084	6.204
		S2-C-S2-S1, S2-S1-S2-C				
136	0	3.808	5.559	6.384	7.272	8.017
		C-C-C-C				

Table 9
Every ten Classes with frequency parameters Ω and boundary conditions of (Ex.6) generally laminated square plates ($a/b = 1$), [0°/30°]s.

Class	RBM	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		boundary conditions				
1	3	2.891	3.098	4.539	5.522	5.782
		FFFF				
10	1	2.303	3.022	4.577	5.216	5.607
		S1-S1-F-F, F-F-S1-S1				
20	1	1.525	3.234	4.210	5.275	6.084
		F-S2-S1-F, S1-F-F-S2				
30	0	1.912	4.578	4.701	5.998	6.943
		S2-C-S1-F, S1-F-S2-C				
40	1	2.844	4.036	4.468	6.212	6.647
		S2-S1-S2-F, S2-F-S2-S1				
50	0	3.765	5.933	6.821	7.854	8.620
		C-S2-C-F, C-F-C-S2				
60	1	2.560	4.065	5.591	6.140	6.616
		S1-S2-F-S2, F-S2-S1-S2				
70	0	3.660	5.364	6.080	6.411	7.236
		C-C-F-S2, F-S2-C-C				
80	0	3.409	5.217	6.226	7.016	7.268
		S2-C-S1-S2, S1-S2-S2-C				
90	0	3.580	4.772	6.844	7.058	7.264
		S2-C-S2-S2, S2-S2-S2-C				
100	0	4.820	7.305	7.546	8.878	9.984
		C-C-C-S2, C-S2-C-C				
110	0	1.804	3.342	5.119	5.758	6.356
		S2-S1-S1-S1, S1-S1-S2-S1				
120	0	1.746	3.932	5.509	5.766	7.028
		S2-C-S2-S1, S2-S1-S2-C				
136	0	4.839	7.326	7.635	9.003	9.993
		C-C-C-C				