Proceedings of 48th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
Y. Liu, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya

Series #185, 2023

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

#166 S. -I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 41st Sapporo Symposium on Partial Differential Equations, 110 pages. 2016.

#167 Kubo, Hideo and Ozawa, Tohru and Takamura, Hiroyuki, Mathematical Analysis for Stability in Nonlinear Dynamics - in honor of Professor Vladimir Georgiev on his 60th birthday -, 79 pages. 2016.

#168 F. Nakamura, Y. Aikawa, K. Asahara, M. Abe, D. Komori, H. Saito, S. Handa, K. Fujisawa, S. Honda, A. Rodrguez Mulet, 第 13 回数学総合若手研究集会, 501 pages. 2017.

#169 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations I, 177 pages. 2017.

#170 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations II, 519 pages. 2017.

#171 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, Y. Tonegawa, and K. Tsutaya, Proceedings of the 42nd Sapporo Symposium on Partial Differential Equations - In memory of Professor Taira Shirota - , 102 pages. 2017.

#172 Shigeru Kuroda, Takao Namiki, Akihiro Yamaguchi, Yutaka Yamaguchi, 複雑系数理の新展開：津田一郎教授退職記念研究集会報告集, 69 pages. 2018.

#173 S. Handa, D. Komori, K. Fujisawa, A. Rodriguez Mulet, M. Aoki, Y. kamijima, I. Fukuda, T. Yabu, 第 14 回数学総合若手研究集会, 580 pages. 2018.

#174 Yoshikazu Giga, Hideo Kubo, Tohru Ozawa, The 11th Mathematical Society of Japan - Seasonal Institute - The Role of Metrics in the Theory of Partial Differential Equations- , 160 pages. 2018.

#175 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Proceedings of 43rd Sapporo Symposium on Partial Differential Equations, 99 pages. 2018.

#176 I. Fukuda, M. Aoki, Y. Ueda, Y. Kamijima, T. Yabu, T. Niimura, N. Satoh, H. Toyokawa, K. Matsuzaka, S. Yamagata, K. Yoshida, 第 15 回数学総合若手研究集会, 657 pages. 2019.

#177 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 44th Sapporo Symposium on Partial Differential Equations, 115 pages. 2019.

#178 H. Toyokawa, Y. Otsuto, N. Satoh, K. Matsuzaka, S. Yamagata, K. Yoshida, R. Ashikaga, S. Okida, M. Oyama, K. Kuwata K. Misu, H. Ishii, T. Sukekawa, 第 16 回数学総合若手研究集会, 692 pages. 2020.

#179 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, K. Tsutaya, Proceedings of 45th Sapporo Symposium on Partial Differential Equations, 106 pages. 2020.

#180 K. Misu, S. Okida, R. Ashikaga, M. Oyama, K. Kuwata, H. Ishii, I. Kudo, T. Sukekawa, M. Seki, A. Hasegawa, K. Hatano, 第 17 回数学総合若手研究集会, 595 pages. 2021.

#181 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 46th Sapporo Symposium on Partial Differential Equations, 116 pages. 2021.

#182 Tsubasa Sukekawa(Representative organizer), Hiroshi Ishii, Isamu Kudo, Motoki Seki, Aoi Hasegawa, Kohei Hatano, Keisuke Abiko, Yu Tajima, Katsunori Fujie, Shinpei Makida, 第 18 回数学総合若手研究集会, 862 pages. 2022.

#183 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 47th Sapporo Symposium on Partial Differential Equations, 120 pages. 2022.

#184 Keisuke Abiko(Representative organizer), Katsunori Fujie, Yuta Takada, Yu Tajima, Sakumi Sugawara, Yu Ohno, Takuya Saito, Taichi Togashi, Kotaro Hata, Kazuya Hirose, 第 19 回数学総合若手研究集会, 820 pages. 2023.

Proceedings of 48th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
Y. Liu, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya

Sapporo, 2023

Partially supported by Grant-in-Aid for Scientific Research, the Japan Society for the Promotion of Science.

日本学術振興会科学研究費補助金（基 盤 研 究 B 課 題 番 号 22H01129）
日本学術振興会科学研究費補助金（基 盤 研 究 B 課 題 番 号 19H01795）
日本学術振興会科学研究費基金　（挑戦的研究開拓 課題番号 20K20342）

Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 16 through August 18 in 2023 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium more than 40 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Hokkaido University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
H. Kuroda (Hokkaido University)
S. Masaki (Hokkaido University)
Y. Liu (Hokkaido University)
T. Ozawa (Waseda University)
T. Sakajo (Kyoto University)
K. Tsutaya (Hirosaki University)

Contents

The 48th Sapporo Symposium on Partial Differential Equations

第 48 回偏微分方程式論札幌シンポジウム

Period	August 16 , 2023 – August 18, 2023
Organizers	Shin-Ichiro Ei, Satoshi Masaki, Hirotoshi Kuroda
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Yikan Liu, Satoshi Masaki, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	https://www.math.sci.hokudai.ac.jp/sympo/sapporo/program230816.html

August 16, 2023 (Wednesday)

13:10-13:20 Opening

13:20-14:20 北直泰（熊本大学）Naoyasu Kita (Kumamoto University)
Large time behavior of solutions to a system of dissipative nonlinear Schrödinger equations

14:35-15:15 高棹圭介（京都大学） Keisuke Takasao (Kyoto University)
Convergence of Allen-Cahn equation with non-local term to volume preserving mean curvature flow

15:15-15:40 ＊

15:40-16:10 Chueh-Hsin Chang（National Chung Cheng University）
Front-back-pulse solutions of a three-species Lotka-Volterra competition diffusion system through heteroclinic bifurcation approach

16:20-16:50 片山翔（東京大学） Sho Katayama (The University of Tokyo)
A supercritical Hénon equation with a forcing term

17:00-17:30 勝呂剛志（大阪公立大学）Takeshi Suguro (Osaka Metropolitan University)
Stability of the logarithmic Sobolev inequality for the Tsallis entropy and its application

August 17, 2023 (Thursday)

10:00-11:00	Junping Shi (College of William＆Mary) Uniqueness of positive solution to some coupled cooperative variational elliptic systems
11:15-11:45	岡本潤（京都大学）Jun Okamoto (Kyoto University) A singular limit of the Kobayashi--Warren--Carter system
11:45-13:30	＊
13:30-14:30	小池茂昭（早稲田大学）Shigeaki Koike (Waseda University) ABP maximum principle with upper contact sets for fully nonlinear elliptic PDEs
14:45-15:15	廣瀬和也（北海道大学）Kazuya Hirose (Hokkaido University) A dynamical approach to lower gradient estimates for viscosity solutions of Hamilton-Jacobi equations
15:15-15:30	＊
15:30-16:10	Michał Łasica (The Polish Academy of Sciences) Bounds on the gradient of minimizers in variational denoising
16:20-16:50	齋藤平和（電気通信大学）Hirokazu Saito (The University of Electro-Communications) Time decay estimates of L_p-L_q type for the Stokes semigroup arising from two-phase incompressible viscous flows
17:00-17:30	Chih-Chiang Huang (National Chung Cheng University) Boundary Conditions for Cylindrical Traveling Waves of Reaction-Diffusion Equations
17:30-18:00	Free discussion with speakers

August 18, 2023 (Friday)

10:00-11:00	肥田野久二男（三重大学）Kunio Hidano (Mie University) D'Alembert formula approach to semi-linear systems of wave equations with or without the null condition
11:15-11:55	高津飛鳥（東京都立大学）Asuka Takatsu (Tokyo Metropolitan University) Characterization of concavity preserved by the Dirichlet heat flow
11:55-12:00	Closing

(＊: breaks/discussions)

Large time behavior of solutions to a system of dissipative nonlinear Schrödinger equations

Naoyasu Kita
Faculty of Advanced Science and Technology, Kumamoto University

1. Introduction

This is a joint work with Y. Nakamura (Kumamoto University) and Y. Sagawa (Kumamoto University). We consider a Cauchy problem for a system of nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t \mathbf{u} + \frac{1}{2}\partial_x^2 \mathbf{u} = \mathbf{f}(\mathbf{u}) \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \end{cases} \tag{1.1}$$

where $(t,x) \in [0,\infty)\times\mathbb{R}$ and $\mathbf{u} = \mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^t \in \mathbb{C}^n$ with $n \geq 1$. The nonlinearity is assumed to be of $\mathbb{C}^n$-valued gauge-invariant cubic power type, i.e.,

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \cdots, f_n(\mathbf{u}))^t \text{ with } f_j(\mathbf{u}) \in \mathbb{C}\ (j = 1, 2, \cdots, n), \tag{1.2}$$

$$\mathbf{f}(e^{i\theta}\mathbf{u}) = e^{i\theta}\mathbf{f}(\mathbf{u}) \text{ for any } \theta \in \mathbb{R} \text{ and } \mathbf{u} \in \mathbb{C}^n, \tag{1.3}$$

$$\mathbf{f}(\rho\mathbf{u}) = \rho^3\mathbf{f}(\mathbf{u}) \text{ for any } \rho > 0 \text{ and } \mathbf{u} \in \mathbb{C}^n. \tag{1.4}$$

The first aim of this manuscript is to obtain the L^∞-decay estimate of the solution without size restriction on the initial data $\mathbf{u}_0$. To this end, a structural assumption indicating strong nonlinear dissipation is required, i.e., we assume that there exists some $\rho_1 > 0$ such that, for any $\mathbf{p} = (p_1, p_2, \cdots, p_n)^t \in \mathbb{C}^n$,

$$\mathrm{Im}\left\{\overline{\mathbf{p}}^t \cdot \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})\mathbf{p} \pm \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{\mathbf{p}}\right)\right\} \leq -\rho_1 \sum_{j=1}^{n} |u_j|^2|p_j|^2, \tag{1.5}$$

where $\overline{\mathbf{p}} = (\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n})^t$, $\partial\mathbf{f}/\partial\mathbf{u}$ (resp. $\partial\mathbf{f}/\partial\overline{\mathbf{u}}$) is a matrix, the jk-entry of which is given by $\partial f_j/\partial u_k$ (resp. $\partial f_j/\overline{u_k}$). We call (1.5) *the strong dissipative condition.* There are some examples of $\mathbf{f}(\mathbf{u})$ satisfying (1.2)–(1.5). In the case of $n = 1$, i.e., single equation, we have $\mathbf{f}(u) = \lambda|u|^2u$ with $\lambda \in \mathbb{C}, \mathrm{Im}\lambda < 0$ and $|\mathrm{Re}\lambda| < \sqrt{3}|\mathrm{Im}\lambda|$ (or equivalently $2|\mathrm{Im}\lambda| - |\lambda| > 0$). In fact, we see that, for $p \in \mathbb{C}$,

$$\begin{aligned} \mathrm{Im}\left\{\overline{p} \cdot \left(\frac{\partial \mathbf{f}}{\partial u}(u)p \pm \frac{\partial \mathbf{f}}{\partial \overline{u}}(u)\overline{p}\right)\right\} &= 2\mathrm{Im}\lambda|u|^2|p|^2 \pm \mathrm{Im}(\lambda u^2\overline{p}^2) \\ &\leq -(2|\mathrm{Im}\lambda| - |\lambda|)|u|^2|p|^2, \end{aligned}$$

and we may take $\rho_1 = 2|\mathrm{Im}\lambda| - |\lambda| > 0$. In the case of $n = 2$, we have

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \lambda_{11}|u_1|^2u_1 + \lambda_{12}|u_2|^2u_1 \\ \lambda_{21}|u_1|^2u_2 + \lambda_{22}|u_2|^2u_2 \end{pmatrix}$$

2010 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35B40.
Key Words and Phrases. system of nonlinear Schrödinger equations, critical dissipative nonlinearity, decay estimate, asymptotic profile.

with $\mathbf{u} = (u_1, u_2)^t$, $\lambda_{jk} \in \mathbb{C}$ $(j, k = 1, 2)$ and

$$\begin{aligned} &\mathrm{Im}\lambda_{jk} < 0, \\ &2|\mathrm{Im}\lambda_{11}| > |\lambda_{11}| + 2|\lambda_{12}| + |\lambda_{21}|, \\ &2|\mathrm{Im}\lambda_{22}| > |\lambda_{22}| + |\lambda_{12}| + 2|\lambda_{21}|. \end{aligned} \tag{1.6}$$

In fact, we see that

$$\begin{aligned} \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}} &= \left(\frac{\partial \mathbf{f}(\mathbf{u})}{\partial u_1} \ \frac{\partial \mathbf{f}(\mathbf{u})}{\partial u_2} \right) = \begin{pmatrix} 2\lambda_{11}|u_1|^2 + \lambda_{12}|u_2|^2 & \lambda_{12}\overline{u_2}u_1 \\ \lambda_{21}\overline{u_1}u_2 & \lambda_{21}|u_1|^2 + 2\lambda_{22}|u_2|^2 \end{pmatrix}, \\ \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{\mathbf{u}}} &= \left(\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{u_1}} \ \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{u_2}} \right) = \begin{pmatrix} \lambda_{11}u_1^2 & \lambda_{12}u_2u_1 \\ \lambda_{21}u_1u_2 & \lambda_{22}u_2^2 \end{pmatrix}, \end{aligned}$$

and so, by Young's inequality $2|u_1||u_2||p_1||p_2| \le |u_1|^2|p_1|^2 + |u_2|^2|p_2|^2$ and (1.6), we have

$$\begin{aligned} &\mathrm{Im}\left\{ \overline{\mathbf{p}}^t \cdot \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})\mathbf{p} \pm \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{\mathbf{p}} \right) \right\} \\ &\le -(2|\mathrm{Im}\lambda_{11}| - |\lambda_{11}| - 2|\lambda_{12}| - |\lambda_{21}|)|u_1|^2|p_1|^2 \\ &\quad - (2|\mathrm{Im}\lambda_{22}| - |\lambda_{22}| - |\lambda_{12}| - 2|\lambda_{21}|)|u_2|^2|p_2|^2. \end{aligned}$$

Therefore, by taking $\rho_1 = \min\{2|\mathrm{Im}\lambda_{11}| - |\lambda_{11}| - 2|\lambda_{12}| - |\lambda_{21}|, 2|\mathrm{Im}\lambda_{22}| - |\lambda_{22}| - |\lambda_{12}| - 2|\lambda_{21}|\} > 0$, we see that the nonlinearity satisfies (1.5).

We here introduce the known results on the dissipative NLS. For the single case, i.e.,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda|u|^2 u \\ u(0, x) = u_0(x), \end{cases} \tag{1.7}$$

there are some works on the asymptotic dynamics of the solutions. If $\mathrm{Im}\lambda = 0$, Hayashi-Naumkin [6] proved that, under the smallness assumption on the initial data, the solution to (1.7) decays like $O(t^{-1/2})$ as $t \to \infty$ in L^∞. However the asymptotic leading term of $u(t)$ is given by the phase correction added to the free profile – it suggests that the nonlinear effect is visible in phase. If $\mathrm{Im}\lambda < 0$, Shimomura [23] proved that the small-data-solutions decays like $O(t^{-1/2}(\log t)^{-1/2})$ in L^∞ – the nonlinear effect is now visible in the decay-rate as well as in the asymptotic leading term. Under some smallness assumptions on the data, the effect caused by nonlinear dissipation has been studied not only for (1.7) but also for NLS of super-critical and sub-critical nonlinearity (see [4, 10, 17]). For higher space-dimensional case, see [1, 2, 3]. For derivative type of nonlinearities, see [7].

There are also some works in which the size-restriction on the initial data is removed. Kita-Shimomura [16] assumed the strong dissipative condition on the nonlinearity, i.e.,

$$\mathrm{Im}\lambda < 0, \quad |\mathrm{Re}\lambda| \le \sqrt{3}|\mathrm{Im}\lambda|,$$

and obtained $\|u(t)\|_{L^\infty} = O(t^{-1/2}(\log t)^{-1/2})$ as well as an asymptotic leading term of $u(t)$. The key estimate to remove the smallness assumption is $\|Ju(t)\|_{L^2} \le \|xu_0\|_{L^2}$, where J is an operator described by $J = x + it\partial_x$. The upper bound of $\|Ju(t)\|_{L^2}$ is derived by the strong dissipative condition ($n = 1$ version of (1.5)). Jin-Jin-Li [11] and Hayashi-Li-Naumkin [5] applied another kind of strong dissipative condition to the sub-critical nonlinearity, and proved the decay estimate and modified asymptotic leading term of the solutions without size-restriction. For the generalization of the initial data, see also [8, 9].

Let us next focus on works concerning systems of NLS. In early researches, the dissipative structure was not considered. Therefore the decay estimate and asymptotic behavior of the solutions have been considered only for small data solution. This is because the estimate of $\|J\mathbf{u}(t)\|_{L^2}$, which is the key to control the error terms, indicated that $\|J\mathbf{u}(t)\|_{L^2}$ grows and its growth-rate depended on the size of initial data. Therefore, to guarantee the rapid decay of the error terms, one can not help restricting the size of data. Anyway, under the smallness assumption of the data, Katayama-Sakoda [15] considered the case where the nonlinearity contains derivatives of unknown variables, and proved the asymptotic behavior of solutions by imposing a certain structural condition on the nonlinearity – it has something to do with the method of Lyapunov function. In most cases of the systems, it becomes more difficult than in the case of single equation, to detect a concrete form of the asymptotic leading term of the solution, since the well-known technique of gauge-transform does not work so well. However, in [15], the asymptotic leading term is described by a function satisfying the ordinary differential equation associated with the system of NLS.

Masaki [21] completely classified types of the 2-component systems of the ordinary differential equations associated with NLS systems of gauge-invariant cubic nonlinearities. This work is instructive in the point that we know which nonlinearities are essential for the asymptotic dynamics of nonlinear dispersive equations. For some nonlinearities included in the classification by [21], Masaki-Segata-Uriya [22] and Kita-Masaki-Segata-Uriya [14] considered coupled nonlinear Schrödinger equations, for which the solutions decay less rapidly than those of free Schrödinger equations.

On the works where some dissipative structures were taken into account for systems, Kim [13] considered the case of distinct masses, and proved that the L^∞-norm of the solutions decays like $O(t^{-1/2}(\log t)^{-1/2})$. Li-Nishii-Sagawa-Sunagawa [18] treated another system, and proved the existence of solutions indicating that one component rapidly decays but the other is asymptotically free. The works introduced above suggest that solutions to the systems of NLS exhibit complicated asymptotic profiles which are not simply expected by the single NLS. For the case of derivative nonlinearities, see [19, 20, 24].

As we have seen above, several results on the systems of NLS were obtained. Nevertheless, all of these works impose the size restriction on the data. When it comes to removing the size-restriction on the initial data, there seems to be no result on the asymptotic analysis in the case of system.

The aim of this research is to find sufficient conditions under which the L^∞-decay estimate and asymptotic profile of the solutions are obtained even for the system of NLS.

Theorem 1.1 (existence of global solutions & L^∞-decay). *Let* $\mathbf{f}(\mathbf{u})$ *satisfy* (1.2)–(1.4) *and* (1.5). *Then, if* $\mathbf{u}_0 \in H^1$ *and* $x\mathbf{u}_0 \in L^2$, *there exists a unique solution to* (1.1) *such that*

$$\mathbf{u} \in C([0,\infty); H^1) \cap C^1([0,\infty); H^{-1}), \quad x\mathbf{u} \in C([0,\infty); L^2). \tag{1.8}$$

Furthermore there exists some constant $C > 0$ *such that, for* $t > 2$,

$$\|\mathbf{u}(t,\cdot)\|_{L^\infty} \le Ct^{-1/2}(\log t)^{-1/2}. \tag{1.9}$$

The result on the asymptotic behavior is the following.

Theorem 1.2 (asymptotic behavior). *Under the same assumption as in Theorem* 1.1, *there exists some* $\mathbb{C}^n$*-valued function* $\mathbf{w}_0 \in L^\infty$ *(depending on* $\mathbf{u}_0$ *) such that the solution to* (1.1) *satisfies*

$$\mathbf{u}(t,x) = (it)^{-1/2} e^{ix^2/2t}\mathbf{w}(t, x/t) + o(t^{-1/2}(\log t)^{-1/2}) \tag{1.10}$$

as $t \to \infty$ *in* L^∞, *where* $\mathbf{w} = \mathbf{w}(t,x) \in C^1([T,\infty); L^\infty)$ *is a solution to an initial value problem of the ordinary differential equation:*

$$\begin{cases} i\partial_t \mathbf{w} = t^{-1}\mathbf{f}(\mathbf{w}) \\ \mathbf{w}(T,x) = \mathbf{w}_0(x) \end{cases} \tag{1.11}$$

for some $T > 0$.

If the equation is single, it is easy to solve (1.11) by applying the gauge transform, and the asymptotic leading term of the solution is explicitly described. On the other hand, if the equation is a system, (1.11) is usually hard to be solved. This is why the asymptotic leading term is implicit for the system of NLS.

Before closing this section, let us introduce some notation. The function space L^q $(1 \le q \le \infty)$ for $\mathbb{C}^n$-valued functions is defined by

$$L^q = \{\mathbf{u}(x) = (u_1(x), u_2(x), \cdots, u_n(x))^t \ ; \ \|\mathbf{u}\|_{L^q} < \infty\},$$

where $\|\mathbf{u}\|_{L^q} = \left(\sum_{j=1}^n \int_{\mathbb{R}} |u_j(x)|^q \, dx\right)^{1/q}$ if $1 \le q < \infty$, and $\|\mathbf{u}\|_{L^\infty} = \max_{1\le j\le n}\{\text{ess.sup}_{x\in\mathbb{R}}|u_j(x)|\}$. The Sobolev space H^1 is endowed with

$$\|\mathbf{u}\|_{H^1} = \left(\|\mathbf{u}\|_{L^2}^2 + \|\partial_x \mathbf{u}\|_{L^2}^2\right)^{1/2}.$$

The negatively indexed Sobolev space H^{-1} is the dual of H^1. We define the dilation operator by

$$(D\phi)(x) = (it)^{-1/2}\phi(x/t) \tag{1.12}$$

and define $M = e^{ix^2/2t}$ for $t \ne 0$. Then the Schrödinger group $U(t) = \exp(it\partial_x^2/2)$ possesses a factorization formula such as

$$U(t) = MD\mathcal{F}M,$$

where $\mathcal{F}$ is the Fourier transform. Since $U(-t) = U(t)^{-1}$, we also have

$$U(-t) = M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}.$$

The infinitesimal generator of the Galilei transformation is given by

$$J\left(t\right) = U\left(t\right) xU\left(-t\right) = x + it\partial_x.$$

It commutes with the linear differential operator $i\partial_t + \frac{1}{2}\partial_x^2$.

2. Proof of Theorem 1.1

We proceed in the proof step by step.

• **Global Existence**. Let $J = x - it\partial_x$. Then the global existence of the solution follows from the boundedness of $\|\mathbf{u}(t)\|_{H^1}$ and $\|J\mathbf{u}(t)\|_{L^2}$. It is stated in next lemma.

Lemma 2.1. *Let* $\mathbf{u}_0 \in H^1$ *and* $x\mathbf{u}_0 \in L^2$. *We assume* (1.2)–(1.5) *for the nonlinearity. Then the solution to (1.1) satisfies*

$$\|\mathbf{u}(t)\|_{L^2} \leq \|\mathbf{u}_0\|_{L^2}, \tag{2.1}$$

$$\|J\mathbf{u}(t)\|_{L^2} \leq \|x\mathbf{u}_0\|_{L^2}, \tag{2.2}$$

$$\|\partial_x\mathbf{u}(t)\|_{L^2} \leq \|\partial_x\mathbf{u}_0\|_{L^2}. \tag{2.3}$$

Proof of Lemma 2.1. By (1.3), we see that

$$e^{i\theta}\mathbf{f}(\mathbf{u}) = \mathbf{f}(e^{i\theta}\mathbf{u}).$$

Taking the derivative with respect to θ and substituting $\theta = 0$, we have

$$\mathbf{f}(\mathbf{u}) = \frac{\partial\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}}\mathbf{u} + \frac{\partial\mathbf{f}(\mathbf{u})}{\partial\overline{\mathbf{u}}}\overline{\mathbf{u}}.$$

Then (1.5) leads to

$$\frac{d\|\mathbf{u}(t)\|_{L^2}^2}{dt} = \mathrm{Im}\left\{\mathbf{u}^t\left(\frac{\partial\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}}\mathbf{u} + \frac{\partial\mathbf{f}(\mathbf{u})}{\partial\overline{\mathbf{u}}}\overline{\mathbf{u}}\right)\right\} \leq 0.$$

Hence (2.1) follows. We next prove (2.2). Note that

$$\begin{aligned} J\mathbf{f}(\mathbf{u}) &= \sum_{k=1}^{n}\frac{\partial\mathbf{f}}{\partial u_k}(\mathbf{u})Ju_k - \sum_{k=1}^{n}\frac{\partial\mathbf{f}}{\partial\bar{u}_k}(\mathbf{u})\overline{Ju_k} \\ &= \frac{\partial\mathbf{f}}{\partial\mathbf{u}}(\mathbf{u})J\mathbf{u} - \frac{\partial\mathbf{f}}{\partial\overline{\mathbf{u}}}(\mathbf{u})\overline{J\mathbf{u}}. \end{aligned}$$

The strong dissipative condition (1.5) plays an important role to estimate $\mathrm{Im}\left\{(\overline{J\mathbf{u}})^t \cdot J\mathbf{f}(\mathbf{u})\right\}$. In fact, by (1.5) with $\mathbf{p} = J\mathbf{u}$, the solution $\mathbf{u}$ satisfies

$$\begin{aligned} \frac{d\|J\mathbf{u}(t)\|_{L^2}^2}{dt} &= \int \mathrm{Im}\left\{\overline{J\mathbf{u}}\cdot\left(\frac{\partial\mathbf{f}}{\partial\mathbf{u}}(\mathbf{u})J\mathbf{u} - \frac{\partial\mathbf{f}}{\partial\overline{\mathbf{u}}}(\mathbf{u})\overline{J\mathbf{u}}\right)\right\}\, dx \\ &\leq -\rho_1\int\sum_{j=1}^{n}|u_j|^2|Ju_j|^2\, dx \\ &\leq 0. \end{aligned}$$

Therefore, for $t \geq 0$, we have (2.2). By the analogy in deriving (2.2), we have (2.3). □

By (2.1)–(2.3) in Lemma 2.1, we have the global existence of the solution such that $u \in C([0,\infty); H^1)$ and $xu \in C([0,\infty); L^2)$.

• $\mathbf{L^\infty}$**-Decay**. We will next derive (1.9). Let $\mathbf{v} = FU(-t)\mathbf{u}$, where $\mathbf{v} = \mathbf{v}(t,\xi) = (v_1, v_2, \cdots, v_n)^t$ with $v_j = FU(-t)u_j(t)$. Then we see that

$$\frac{\partial \mathbf{v}}{\partial t} = -it^{-1}\mathbf{f}(\mathbf{v}) + \mathbf{R}(t), \tag{2.4}$$

where the remainder term is described by

$$\begin{aligned}\mathbf{R}(t) &= -it^{-1}FM^{-1}F^{-1}\mathbf{f}(FMF^{-1}\mathbf{v})\\ &= -it^{-1}F(M^{-1}-1)F^{-1}\mathbf{f}(FMF^{-1}\mathbf{v}) - it^{-1}\left\{\mathbf{f}(FMF^{-1}\mathbf{v}) - \mathbf{f}(\mathbf{v})\right\}\\ &\equiv \mathbf{R}_1(t) + \mathbf{R}_2(t).\end{aligned} \tag{2.5}$$

The remainder term $\mathbf{R}(t)$ decays rapidly so that it is integrable around $t = \infty$. It is stated in the next lemma.

Lemma 2.2. *Let $\mathbf{u}(t)$ be the time-global solution to (1.1). Then we have*

$$\|\mathbf{R}(t)\|_{L^\infty} \le Ct^{-5/4}\|\mathbf{u}(t)\|_{L^2}\|J\mathbf{u}(t)\|_{L^2}^2. \tag{2.6}$$

Furthermore, by applying Lemma 2.1, we see that $\|\mathbf{R}(t)\|_{L^\infty} \le Ct^{-5/4}$ for $t > 1$.

Proof of Lemma 2.2. As for $\mathbf{R}_1(t)$ in (2.5), the Gagliardo-Nirenberg inequality, Plancherel's identity and $|M^{-1}-1| \le |x|/\sqrt{t}$ lead to

$$\begin{aligned}\|\mathbf{R}_1(t)\|_{L^\infty} &\le C\|\mathbf{R}_1(t)\|_{L^2}^{1/2} \cdot \|\partial_\xi \mathbf{R}_1(t)\|_{L^2}^{1/2}\\ &\le Ct^{-5/4}\|\partial_\xi \mathbf{f}(FMF^{-1}\mathbf{v})\|_{L^2}\\ &\le Ct^{-5/4}\|FMF^{-1}\mathbf{v}\|_{L^\infty}^2 \cdot \|xU(-t)\mathbf{u}\|_{L^2}.\end{aligned}$$

By $\|xU(-t)\mathbf{u}\|_{L^2} = \|J\mathbf{u}\|_{L^2}$ and repeated use of the Gagliardo-Norenberg inequality for the L^∞-norm, we see that

$$\begin{aligned}\|\mathbf{R}_1(t)\|_{L^\infty} &\le Ct^{-5/4}\|FMF^{-1}\mathbf{v}\|_{L^2}^2 \cdot \|J\mathbf{u}\|_{L^2}^2\\ &= Ct^{-5/4}\|\mathbf{u}\|_{L^2} \cdot \|J\mathbf{u}\|_{L^2}^2.\end{aligned} \tag{2.7}$$

As for $\mathbf{R}_2(t)$ in (2.5), it is easy to see

$$\|\mathbf{R}_2(t)\|_{L^\infty} \le Ct^{-1}(\|FMF^{-1}\mathbf{v}\|_{L^\infty}^2 + \|\mathbf{v}\|_{L^\infty}^2) \cdot \|F(M-1)F^{-1}\mathbf{v}\|_{L^\infty}.$$

Then the Gagliardo-Nirenberg inequality and analogous estimate for $\mathbf{R}_1(t)$ yields

$$\|\mathbf{R}_2(t)\|_{L^\infty} \le Ct^{-5/4}\|\mathbf{u}\|_{L^2} \cdot \|J\mathbf{u}\|_{L^2}^2. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain (2.6). □

We now turn back to the proof of L^∞-decay. From (2.4), it follows that

$$\frac{\partial |\mathbf{v}|^2}{\partial t} = 2t^{-1}\mathrm{Im}\{\overline{\mathbf{v}}^t \cdot \mathbf{f}(\mathbf{v})\} + 2\mathrm{Re}\{\overline{\mathbf{v}}^t \cdot \mathbf{R}(t)\}. \tag{2.9}$$

We are going to focus on the estimate of $\mathrm{Im}\{\overline{\mathbf{v}}^t \cdot \mathbf{f}(\mathbf{v})\}$ in (2.14) for a while. Since we have the gauge-invariance of the nonlinearity, i.e., $e^{i\theta}\mathbf{f}(\mathbf{v}) = \mathbf{f}(e^{i\theta}\mathbf{v})$, the derivative with respective to θ leads to

$$ie^{i\theta}\mathbf{f}(\mathbf{v}) = \sum_{j=1}^{n} ie^{i\theta} v_j \frac{\partial \mathbf{f}}{\partial u_j}(e^{i\theta}\mathbf{v}) - \sum_{j=1}^{n} ie^{-i\theta} \bar{v}_j \frac{\partial \mathbf{f}}{\partial \bar{u}_j}(e^{i\theta}\mathbf{v}).$$

Taking $\theta = 0$ and removing i in the above identity, we have

$$\mathbf{f}(\mathbf{v}) = \sum_{j=1}^{n} v_j \frac{\partial \mathbf{f}}{\partial u_j}(\mathbf{v}) - \sum_{j=1}^{n} \bar{v}_j \frac{\partial \mathbf{f}}{\partial \bar{u}_j}(\mathbf{v}). \tag{2.10}$$

By (1.5) and Cauchy-Schwarz' inequality, $\mathrm{Im}\{\overline{\mathbf{v}}^t \cdot \mathbf{f}(\mathbf{v})\}$ is estimated in such a way that

$$\mathrm{Im}\{\overline{\mathbf{v}}^t \cdot \mathbf{f}(\mathbf{v})\} \le -\rho_1 \sum_{j=1}^{n} |v_j|^4 \le -\rho_1 n^{-1} |\mathbf{v}|^4. \tag{2.11}$$

By (2.11), we see that

$$\frac{\partial |\mathbf{v}|^2}{\partial t} \le -2\rho_1 n^{-1} t^{-1} |\mathbf{v}|^4 + 2\mathrm{Re}\{\overline{\mathbf{v}}^t \cdot \mathbf{R}(t)\}. \tag{2.12}$$

Hereafter we follow the idea of Katayama-Matsumura-Sunagawa [12] to obtain an L^∞-decay estimate of $\mathbf{v} = \mathbf{v}(t, \xi)$. From (2.12), it follows that

$$\begin{aligned} \frac{\partial (\log t)^2 |\mathbf{v}|^2}{\partial t} &= 2t^{-1} \log t |\mathbf{v}|^2 - 2\rho_1 n^{-1} t^{-1} (\log t)^2 |\mathbf{v}|^4 \\ &\quad + 2(\log t)^2 \mathrm{Re}\{\overline{\mathbf{v}}^t \cdot \mathbf{R}(t)\}. \end{aligned} \tag{2.13}$$

By Young's inequality, we have $(\log t)|\mathbf{v}|^2 \le (n/4\rho_1) + \rho_1 n^{-1} (\log t)^2 |\mathbf{v}|^4$. Then, from (2.13), it follows that

$$\frac{\partial (\log t)^2 |\mathbf{v}|^2}{\partial t} \le \frac{n}{2\rho_1 t} + 2(\log t)^2 \mathrm{Re}\{\overline{\mathbf{v}}^t \cdot \mathbf{R}(t)\}.$$

Integrating the above from 1 to t, we have

$$\begin{aligned} |\mathbf{v}|^2 &\le \frac{n}{2\rho_1} (\log t)^{-1} + 2(\log t)^{-2} \int_1^t (\log \tau)^2 \mathrm{Re}\{\overline{\mathbf{v}}^t \cdot \mathbf{R}(\tau)\}\, d\tau \\ &\le \frac{n}{2\rho_1} (\log t)^{-1} + C(\log t)^{-2}, \end{aligned} \tag{2.14}$$

where we applied $\|\mathbf{R}(\tau)\|_{L^\infty} \le C\tau^{-5/4}$ (Lemma 2.2) and rough estimate $\|\mathbf{v}\|_{L^\infty} \le C\|\mathbf{v}\|_{H^1} \le C(\|\mathbf{u}_0\|_{L^2} + \|x\mathbf{u}_0\|_{L^2})$ (Lemma 2.1) to the integral. By (2.14), we have

$$\|\mathbf{v}(t)\|_{L^\infty} \le C(\log t)^{-1/2} \tag{2.15}$$

for $t > 2$, and obtain $\|\mathbf{u}(t)\|_{L^\infty} \le Ct^{-1/2}(\log t)^{-1/2}$. The proof for Theorem 1.1 is complete. $\square$

3. Proof of Theorem 1.2

Recall that the function $\mathbf{v}(t,\xi) = FU(-t)\mathbf{u}(t)$ satisfies

$$\frac{\partial \mathbf{v}}{\partial t} = -it^{-1}\mathbf{f}(\mathbf{v}) + \mathbf{R}(t). \tag{3.1}$$

When we take $t \to \infty$, the remainder $\mathbf{R}(t)$ expectedly decays more rapidly than $-it^{-1}\mathbf{f}(\mathbf{v})$ of (3.1). Hence the large-time behavior of $\mathbf{v}$ expectedly coincides with a solution to the ordinary differential equation such as

$$\frac{\partial \mathbf{w}}{\partial t} = -it^{-1}\mathbf{f}(\mathbf{w}). \tag{3.2}$$

To justify it, we let $\mathbf{w}(t,\xi) = \mathbf{v}(t,\xi) + \mathbf{p}(t,\xi)$ where $\mathbf{p}$ is a perturbation of $\mathbf{v}$, and we will find the equation of $\mathbf{p}$. In addition, we will consider which initial data is the best for (3.2) so that $\mathbf{w}$ well-approximates $\mathbf{v}$ as $t \to \infty$.

We first generates a differential equation of $\mathbf{p}$. Substitute $\mathbf{w} = \mathbf{v} + \mathbf{p}$ into (3.2), and note that $\mathbf{v}$ satisfies (3.1). Then we see that

$$\frac{\partial \mathbf{p}}{\partial t} = -it^{-1}\left\{\mathbf{f}(\mathbf{v}+\mathbf{p}) - \mathbf{f}(\mathbf{v})\right\} + \mathbf{R}(t). \tag{3.3}$$

Applying Taylor's expansion, we have

$$\mathbf{f}(\mathbf{v}+\mathbf{p}) - \mathbf{f}(\mathbf{v}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})\overline{\mathbf{p}} + \mathbf{S}(\mathbf{p}), \tag{3.4}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})\mathbf{p} = \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial u_j}(\mathbf{v})p_j, \quad \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})\overline{\mathbf{p}} = \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial \overline{u_j}}(\mathbf{v})\overline{p_j} \tag{3.5}$$

and

$$\mathbf{S}(\mathbf{p}) = \mathbf{f}(\mathbf{v}+\mathbf{p}) - \mathbf{f}(\mathbf{v}) - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})\mathbf{p} - \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})\overline{\mathbf{p}}. \tag{3.6}$$

Note here that $\mathbf{S}(\mathbf{p})$ consists of the quadratic and cubic terms of $\mathbf{p}$. Substitute (3.4) into (3.3), and we have

$$\frac{\partial \mathbf{p}}{\partial t} = -it^{-1}\left\{\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})\overline{\mathbf{p}}\right\} - it^{-1}\mathbf{S}(\mathbf{p}) + \mathbf{R}(t). \tag{3.7}$$

We want to regard (3.7) as a couple of equations of $\mathbf{p}$ and $\overline{\mathbf{p}}$. Let

$$\mathbf{q} = \begin{pmatrix} \mathbf{p} \\ \overline{\mathbf{p}} \end{pmatrix}, \quad \mathbf{M}(\mathbf{v}) = \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v}) & \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v}) \\ -\overline{\frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})} & -\overline{\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})} \end{pmatrix} \tag{3.8}$$

and

$$\mathbf{N}_S = \begin{pmatrix} it^{-1}\mathbf{S}(\mathbf{p}) \\ -it^{-1}\overline{\mathbf{S}(\mathbf{p})} \end{pmatrix}, \quad \mathbf{I}_R = \begin{pmatrix} \mathbf{R}(t) \\ \overline{\mathbf{R}(t)} \end{pmatrix} \tag{3.9}$$

Then (3.7) is transformed into

$$\frac{\partial \mathbf{q}}{\partial t} = -it^{-1}\mathbf{M}(\mathbf{v})\mathbf{q} + \mathbf{N}_S + \mathbf{I}_R. \tag{3.10}$$

If the equation (3.10) possesses a solution $\mathbf{q} = \mathbf{q}(t)$ decaying more rapidly than $\mathbf{v}(t)$ in L^∞, it implies that there exists a solution $\mathbf{w} = \mathbf{w}(t)$ satisfying the ODE (3.2) so that $\mathbf{w}(t)$ provides the asymptotic leading term of $\mathbf{u}(t)$ (the solution to (1.1)).

• **Linear Problem**. To deduce the asymptotic behavior of $\mathbf{q}$ satisfying (3.10), we first consider the Cauchy problem of the linearized equation, i.e.,

$$\left\{ \begin{array}{l} \partial_t \mathbf{q} = -it^{-1}\mathbf{M}(\mathbf{v})\mathbf{q} \\ \mathbf{q}(s) = \mathbf{b}, \end{array} \right. \tag{3.11}$$

where the initial data $\mathbf{b}$ belongs to

$$\mathbb{D} = \left\{ \left(\begin{array}{c} \mathbf{a} \\ \overline{\mathbf{a}} \end{array} \right) \in \mathbb{C}^{2n} \ ; \ \mathbf{a} \in \mathbb{C}^n \right\}.$$

Since the nonlinearity $\mathbf{f}(\mathbf{u})$ is cubic, there exists some $\rho_2 > 0$ such that, for any $\mathbf{p} \in \mathbb{C}^n$,

$$\mathrm{Im}\left\{ \overline{\mathbf{p}^t} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v})\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{v})\overline{\mathbf{p}} \right) \right\} \geq -\rho_2 |\mathbf{v}|^2 |\mathbf{p}|^2. \tag{3.12}$$

Then we have an estimate of the solution to (3.11).

Lemma 3.1. *Let $\mathbf{b} = \mathbf{b}(\xi)$ belong to L^∞. Then, for some $T > 0$, there exists a unique solution to* (3.11) *such that*

$$\mathbf{q} \in C^1([T, \infty); L^\infty). \tag{3.13}$$

Furthermore, if $T < t_1 \leq t_2$, the solution $\mathbf{q}$ satisfies

$$\|\mathbf{q}(t_1, \cdot)\|_{L^\infty} \leq e^{C(\log T)^{-1}} \left(\frac{\log t_2}{\log t_1} \right)^{\frac{n\rho_2}{2\rho_1}} \|\mathbf{q}(t_2, \cdot)\|_{L^\infty}. \tag{3.14}$$

Remark. Let $\mathbf{q}(t) = \mathbf{U}(t, s)\mathbf{b}$. Then (3.14) implies that, if $T < t \leq s$, we have

$$\|\mathbf{U}(t, s)\mathbf{b}\|_{L^\infty} \leq e^{C(\log T)^{-1}} \left(\frac{\log s}{\log t} \right)^{\frac{n\rho_2}{2\rho_1}} \|\mathbf{b}\|_{L^\infty}. \tag{3.15}$$

Proof of Lemma 3.1. By (3.11), we see that

$$\begin{aligned} \frac{\partial |\mathbf{q}(t)|^2}{\partial t} &= 2t^{-1}\mathrm{Im}(\overline{\mathbf{q}}^t \mathbf{M}(\mathbf{v})\mathbf{q}) \\ &= 2t^{-1}\mathrm{Im}\left\{ \overline{\mathbf{p}}^t \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}\overline{\mathbf{p}} \right) + \mathbf{p}^t \left(-\overline{\frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}}\mathbf{p} - \overline{\frac{\partial \mathbf{f}}{\partial \mathbf{u}}}\overline{\mathbf{p}} \right) \right\} \\ &= 4t^{-1}\mathrm{Im}\left\{ \overline{\mathbf{p}}^t \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}\overline{\mathbf{p}} \right) \right\}. \end{aligned} \tag{3.16}$$

By (3.12) and $|\mathbf{p}|^2 = |\mathbf{q}|^2/2$, we have

$$\frac{\partial |\mathbf{q}(t)|^2}{\partial t} \geq -4\rho_2 t^{-1}|\mathbf{v}|^2|\mathbf{p}|^2 = -2\rho_2 t^{-1}|\mathbf{v}|^2|\mathbf{q}|^2. \tag{3.17}$$

Applying (2.14) to $\mathbf{v}$ of (3.17), we have

$$\frac{\partial |\mathbf{q}(t)|^2}{\partial t} \geq -\left(\frac{n\rho_2}{\rho_1}t^{-1}(\log t)^{-1} + Ct^{-1}(\log t)^{-2} \right) |\mathbf{q}|^2.$$

By Gronwall's inequality, we see that, if $0 < t_1 < t_2$,

$$|\mathbf{q}(t_2)|^2 \geq |\mathbf{q}(t_1)|^2 \cdot \left(\frac{\log t_1}{\log t_2}\right)^{\frac{n\rho_2}{\rho_1}} \exp\left\{C((\log t_2)^{-1} - (\log t_1)^{-1})\right\}. \square$$

• **Nonlinear Problem**. We next solve the nonlinear equation (3.10). To this end, we rewrite (3.10) as an integral equation. We note that

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathbf{U}(t,\tau)\mathbf{q}(\tau) &= \mathbf{U}(t,\tau)(i\tau^{-1}\mathbf{M}(\mathbf{v})) + \mathbf{U}(t,\tau)\frac{\partial \mathbf{q}(\tau)}{\partial\tau}\\ &= \mathbf{U}(t,\tau)(i\tau^{-1}\mathbf{M}(\mathbf{v})) + \mathbf{U}(t,\tau)(-i\tau^{-1}\mathbf{M}(\mathbf{v}) + \mathbf{N}_S + \mathrm{I}_R)\\ &= \mathbf{U}(t,\tau)(\mathbf{N}_S + \mathbf{I}_R),\end{aligned}$$

where $\mathbf{U}(t,\tau)$ is the solution operator for the linearized equation (3.11). Taking the integral over $[t,s]$, we have

$$\mathbf{U}(t,s)\mathbf{q}(s) - \mathbf{q}(t) = \int_t^s \mathbf{U}(t,\tau)\mathbf{I}_R\, d\tau + \int_t^s \mathbf{U}(t,\tau)\mathbf{N}_S\, d\tau.$$

Since $\mathbf{q}(s)$ is expected to decay rapidly enough as $s \to \infty$, we assume that

$$\lim_{s\to\infty} \|\mathbf{U}(t,s)\mathbf{q}(s)\|_{L^\infty} = 0.$$

Then the integral equation we are going to solve is

$$\begin{aligned}\mathbf{q}(t) &= -\int_t^\infty \mathbf{U}(t,\tau)\mathbf{I}_R\, d\tau - \int_t^\infty \mathbf{U}(t,\tau)\mathbf{N}_S\, d\tau\\ &\equiv \Phi(\mathbf{q}(t)).\end{aligned} \tag{3.18}$$

We will solve (3.18) by the contraction mapping principle. To see which function space is appropriate, we consider the estimate of $\int_t^\infty \mathbf{U}(t,\tau)\mathbf{I}_R\, d\tau$. Applying (3.15) to $\mathbf{U}(t,\tau)$ and Lemma 2.2 to $\mathbf{I}_R$, we see that, if $T < t$, there exists some constant $C_0 > 0$ independent of T such that

$$\begin{aligned}\left\|\int_t^\infty \mathbf{U}(t,\tau)\mathbf{I}_R\, d\tau\right\|_{L^\infty} &\leq C(\log t)^{-n\rho_2/2\rho_1}\int_t^\infty (\log\tau)^{n\rho_2/2\rho_1}\tau^{-5/4}\, d\tau\\ &\leq C_0 t^{-1/4}.\end{aligned} \tag{3.19}$$

Then the function space applied to the contraction mapping principle is

$$X_T = \{\mathbf{q}(t,\xi)\ ;\ \|\mathbf{q}\|_{X_T} < \infty\}, \tag{3.20}$$

where

$$\|\mathbf{q}\|_{X_T} = \sup_{T\leq t<\infty} t^{1/4}\|\mathbf{q}(t,\cdot)\|_{L^\infty}. \tag{3.21}$$

Let $\overline{B}_{2C_0}(X_T)$ be a closed ball in X_T with radius of $2C_0$, where C_0 is the constant in (3.19).

We here consider the estimate of $\mathbf{N}_S$ in (3.10).

Lemma 3.2. *Let $\mathbf{q}(t) \in \overline{B}_{2C_0}(X_T)$. Then there exists some $C > 0$ such that, for $t \in [T,\infty)$,*

$$\|\mathbf{N}_S\|_{L^\infty} \leq CC_0^2(\log t)^{-1/2}t^{-3/2} + CC_0^3 t^{-7/4}. \tag{3.22}$$

Proof of Lemma 3.2. By (3.6), we find that $\mathbf{N}_S$ consists of quadratic terms of $\mathbf{q}(t)$ (whose variable coefficients are given by the entries of $\mathbf{v}(t)$) and cubic terms. Then it follows that

$$\|\mathbf{N}_S\|_{L^\infty} \le Ct^{-1}(\|\mathbf{v}(t)\|_{L^\infty}\|\mathbf{q}(t)\|_{L^\infty}^2 + \|\mathbf{q}(t)\|_{L^\infty}^3).$$

Applying (2.15) to $\|\mathbf{v}(t)\|_{L^\infty}$ and $\mathbf{q}(t) \in \overline{B}_{2C_0}(X_T)$ to $\|\mathbf{q}(t)\|_{L^\infty}$, we have

$$\begin{aligned}\|\mathbf{N}_S\|_{L^\infty} &\le Ct^{-1}(C_0^2(\log t)^{-1/2}t^{-1/2} + CC_0^3 t^{-3/4})\\ &\le C(C_0^2(\log t)^{-3/2}t^{-1/2} + CC_0^3 t^{-7/4}). \quad \square\end{aligned}$$

By Lemma 3.1 and 3.2, we see that, for $\mathbf{q} \in \overline{B}_{2C_0}(X_T)$,

$$\begin{aligned}&\|\Phi(\mathbf{q})\|_{X_T}\\ &\le C_0 + C \sup_{T\le t<\infty} t^{1/4}(\log t)^{-n\rho_2/2\rho_1}\int_t^\infty (\log\tau)^{n\rho_2/2\rho_1}\|\mathbf{N}_S(\tau,\cdot)\|_{L^\infty}\, d\tau\\ &\le C_0 + C \sup_{T\le t<\infty} t^{1/4}(\log t)^{-n\rho_2/2\rho_1}\int_t^\infty (\log\tau)^{n\rho_2/2\rho_1}\\ &\qquad\qquad \times (C_0^2(\log\tau)^{-1/2}\tau^{-3/2} + C_0^3\tau^{-7/4})\, d\tau\\ &\le C_0 + C(C_0^2 + C_0^3)(\log T)^{-1/2}T^{-1/4}. \end{aligned} \tag{3.23}$$

Thus, if $T > 0$ is taken so large in (3.23), $\Phi(\mathbf{q}) \in \overline{B}_{2C_0}(X_T)$. We next consider the estimate of $\|\Phi(\mathbf{q}_1) - \Phi(\mathbf{q}_2)\|_{X_T}$ for $\mathbf{q}_1, \mathbf{q}_2 \in \overline{B}_{2C_0}(X_T)$. We see that

$$\|\Phi(\mathbf{q}_1) - \Phi(\mathbf{q}_2)\|_{X_T} \le C(C_0 + C_0^2)(\log T)^{-1/2}T^{-1/4}\|\mathbf{q}_1 - \mathbf{q}_2\|_{X_T}. \tag{3.24}$$

Thus, if $T > 0$ is taken so large in (3.24), Φ is a contraction map. We have now solved (3.18), and hence Theorem 1.2 is proved. $\square$

References

[1] T.Cazenave, Z.Han, *Asymptotic behavior for a Schrödinger equation with nonlinear subcritical dissipation*, Discrete Contin. Dyn. Syst. **40** (2020), 4801–4819.

[2] T.Cazenave, Z.Han, I.Naumkin, *Asymptotic behavior for a dissipative nonlinear Schrödinger equation*, Nonlinear Anal. **205** (2021), 112243, 37 pp.

[3] T.Cazenave, I.Naumkin, *Modified scattering for the critical nonlinear Schrödinger equation*, J. Funct. Anal. **274** (2018), 402–432.

[4] N.Hayashi, E.I.Kaikina, P.I.Naumkin, *Large time behavior of solutions to the generalized derivative nonlinear Schrödinger equation*, Discrete Contin. Dynam. Systems **5** (1999), 93–106.

[5] N.Hayashi, C.Li, P.I.Naumkin, *Time decay for nonlinear dissipative Schrödinger equations in optical fields*, Adv. Math. Phys. Art. ID 3702738, 7 (2016).

[6] N.Hayashi, P.I.Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. **120** (1998), 369-389.

[7] N.Hayashi, P.I.Naumkin, H.Sunagawa, *On the Schrödinger equation with dissipative nonlinearities of derivative type*, SIAM J. Math. Anal. 40 (2008), no.1, 278-291.

[8] G.Hoshino, *Dissipative nonlinear Schrödinger equations for large data in one space dimension*, Communications on Pure and Applied Analysis **19**(2020), 967-981.

[9] G.Hoshino, *Asymptotic behavior for solutions to the dissipative nonlinear Scrödinger equations with the fractional Sobolev space*, Journal of Mathematical Physics **60** (2019), 111504.

[10] G.Hoshino, *Scattering for solutions of a dissipative nonlinear Schrödinger equation*, J. Differential Equations **266** (2019), 4997–5011.

[11] G.Jin, Y.Jin, C.Li, *The initial value problem for nonlinear Schrödinger equations with a dissipative nonlinearity in one space dimension*, Journal of Evolution Equations **16** (2016) 983–995.

[12] S.Katayama, A.Matsumura, H.Sunagawa, *Energy decay for systems of semilinear wave equations with dissipative structure in two space dimensions*, Nonlinear Differential Equations Appl. **22** (2015), 601–628.

[13] D.Kim, *A note on decay rates of solutions to a system of cubic nonlinear Schrödinger equations in one space dimension*,Asymptotic Analysis **98**(2016), 79–90.

[14] N.Kita, S.Masaki, J.Segata, K.Uriya, *Polynomial deceleration for a system of cubic nonlinear Schrödinger equations in one space dimension*, Nonlinear Analysis, TMA 230 (2023), 113216, 22 pp.

[15] S.Katayama, D.Sakoda, *Asymptotic behavior for a class of derivative nonlinear Schrödinger systems*, Partial Differ. Equ. Appl. 1 (2020), no. 3, Paper No.12.

[16] N.Kita, A.Shimomura, *Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data*, J. Math. Soc. Japan **61** (2009), 39–64.

[17] N.Kita, A.Shimomura, *Asymptotic behavior of solutions to Schrödinger equations with a subcritical dissipative nonlinearity*, J. Differential Equations **242** (2007), 192–210.

[18] C.Li, Y.Nishii, Y.Sagawa, H.Sunagawa, *Large time asymptotics for a cubic nonlinear Schrödinger system in one space dimension*, Funkcialaj Ekvacioj **64** (2021), 361–377.

[19] C.Li, H.Sunagawa, *Remarks on derivative nonlinear Schrödinger systems with multiple masses*, Adv. Stud. Pure Math. **81**, Mathematical Society of Japan, Tokyo, 2019, 173–195.

[20] C.Li, H.Sunagawa, *On Schrödinger systems with cubic dissipative nonlinearities of derivative type*. Nonlinearity **29** (2016), 1537–1563.

[21] S.Masaki, *Classification of a class of systems of cubic ordinary differential equations*, J. Differential Equations **344** (2023), 471–508.

[22] S.Masaki, J.Segata, K.Uriya, *Asymptotic behavior in time of solution to system of cubic nonlinear Schrödinger equations in one space dimension*, To appear in Proceedings of the workshop "Mathematical Physics and its Interactions" Proceedings in Mathematics & Statistics series by Springer.

[23] A.Shimomura, *Asymptotic behavior of solutions to Schrödinger equations with dissipative nonlinearities*, Comm. Partial Differential Equations **31** (2006), 1407–1423.

[24] D.Sakoda, H.Sunagawa, *Small data global existence for a class of quadratic derivative nonlinear Schrödinger systems in two space dimensions*, J. Differential Equations **268** (2020), 1722–1749.

FACULTY OF ADVANCED SCIENCE AND TECHNOLOGY
KUMAMOTO UNIVERSITY
KUROKAMI 2-39-1, KUMAMOTO, 860-8555 JAPAN
E-mail address: `nkita@kumamoto-u.ac.jp`

CONVERGENCE OF ALLEN-CAHN EQUATION WITH NON-LOCAL TERM TO VOLUME PRESERVING MEAN CURVATURE FLOW

KEISUKE TAKASAO
DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY,
KITASHIRAKAWA-OIWAKECHO SAKYO KYOTO 606-8502, JAPAN

1. Introduction

In this note we consider the existence of the weak solution to the volume preserving mean curvature flow via the phase field method. This note is mainly about the existence theorem obtained in [17].

Let T be a positive constant and $d \geq 2$ be a positive integer. Suppose that $U_t \subset \mathbb{R}^d$ is a bounded open set with smooth boundary $M_t = \partial U_t$ for any $t \in (0,T)$. We say that the family of hypersurfaces $\{M_t\}_{t\in[0,T)}$ is a volume preserving mean curvature flow if the normal velocity vector $\vec{v}$ satisfies

$$\vec{v} = \vec{h} - \left(\frac{1}{\mathscr{H}^{d-1}(M_t)} \int_{M_t} \vec{h} \cdot \vec{\nu} \, d\mathscr{H}^{d-1} \right) \vec{\nu}, \quad \text{on } M_t, \text{ for any } t \in (0,T), \tag{1.1}$$

where $\vec{h}$ is the mean curvature vector and $\vec{\nu}$ is the inner unit normal vector of M_t, respectively. Note that the volume preserving property is immediately obtained from the following formula:

$$\frac{d}{dt}\mathscr{L}^d(U_t) = -\int_{M_t} \vec{v} \cdot \vec{\nu} \, d\mathscr{H}^{d-1} = 0, \quad t \in (0,T). \tag{1.2}$$

In addition, the solution to (1.1) satisfies

$$\mathscr{H}^{d-1}(M_{t_2}) + \int_{t_1}^{t_2} \int_{M_t} |\vec{v}|^2 \, d\mathscr{H}^{d-1} dt = \mathscr{H}^{d-1}(M_{t_1}) \qquad \text{for any } 0 \leq t_1 < t_2 < T. \tag{1.3}$$

Throughout this note, we denote $W(s) = \dfrac{(1-s^2)^2}{2}$ and $\Omega = (\mathbb{R}/\mathbb{T})^d$ (that is, we consider the periodic boundary condition). First we discuss the following standard Allen-Cahn equation:

$$\begin{cases} \varepsilon \varphi^\varepsilon_t = \varepsilon \Delta \varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega \times (0,T), \\ \varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where ε is a positive constant. Let $\{M_t\}_{t\in[0,T)}$ be a classical solution to the mean curvature flow (without the volume constraint). It is well known that the solution to (1.4) converges to $\{M_t\}_{t\in[0,T)}$ as $\varepsilon \to 0$ if φ^ε_0 approximates M_0 well (see [2, 4, 6]). Moreover, Ilmanen [8] proved the existence of the weak solution to the mean curvature flow in the sense of varifolds (Brakke flow [1]) via (1.4).

2020 *Mathematics Subject Classification.* Primary 35K93, Secondary 53E10.
Key words and phrases. volume preserving mean curvature flow, Allen–Cahn equation, phase field method, varifolds.

Remark 1.1. Let $\{\varepsilon_i\}_{i=1}^\infty$ be a positive sequence with $\varepsilon_i \to 0$ and φ^{ε_i} be a solution to (1.4) with ε_i instead of ε. In [8], one of the key estimate for the existence theorem is the vanishing of the discrepancy measure, that is,

$$\lim_{j\to\infty} \int_0^T \int_\Omega \left| \frac{\varepsilon_{i_j} |\nabla \varphi^{\varepsilon_{i_j}}(x,t)|^2}{2} - \frac{W(\varphi^{\varepsilon_{i_j}}(x,t))}{\varepsilon_{i_j}} \right| dxdt = 0 \tag{1.5}$$

for some subsequence $\{\varepsilon_{i_j}\}_{j=1}^\infty$. Ilmanen proved (1.5) by the monotonicity formula for (1.4) and the maximum principle for the function $\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}$. In fact, this strategy works for our equation (1.13) below.

Based on the results, it is natural to add a suitable term to (1.4) in order to solve (1.1). The most simple and well-known phase field model for (1.1) was studied by Rubinstein and Sternberg [14]. They considered the following Allen-Cahn equation:

$$\begin{cases} \varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda_{RS}^\varepsilon, & (x,t) \in \Omega \times (0,T), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where

$$\lambda_{RS}^\varepsilon(t) = \frac{1}{\mathscr{L}^d(\Omega)} \int_\Omega \frac{W'(\varphi^\varepsilon(x,t))}{\varepsilon} \, dx.$$

Set

$$E^\varepsilon(t) = \int_\Omega \left(\frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) dx. \tag{1.7}$$

Under appropriate conditions, we have $\frac{1}{\sigma}E^\varepsilon(t) \approx \mathscr{H}^{d-1}(M_t^\varepsilon)$, where $\sigma = \int_{-1}^1 \sqrt{2W(s)}\, ds$ and $M_t^\varepsilon = \{x \in \Omega \mid \varphi^\varepsilon(x,t) = 0\}$. Note that the solution to (1.6) has properties corresponding to (1.2) and (1.3), that is,

$$\frac{d}{dt} \int_\Omega \varphi^\varepsilon \, dx = 0, \qquad t \in (0,T)$$

and

$$E^\varepsilon(t_2) + \int_{t_1}^{t_2} \int_\Omega \varepsilon(\varphi_t^\varepsilon)^2 \, dxdt = E^\varepsilon(t_1) \qquad \text{for any } 0 \le t_1 < t_2 < T. \tag{1.8}$$

Chen, Hilhorst, and Logak [5] showed that the solution to (1.6) converges to (1.1) if the classical solution to (1.1) exists and M_0^ε approximates M_0 well. Related to this, Laux and Simon [10] proved that the solution to (1.6) converges to the measure theoretic weak solution to the volume preserving mean curvature flow under an assumption

$$\lim_{\varepsilon\to 0} \int_0^T \frac{1}{\sigma} E^\varepsilon(t)\, dt = \int_0^T \mathscr{H}^{d-1}(\partial^* U_t)\, dt, \tag{1.9}$$

where $U_t = \{x \in \Omega \mid \lim_{\varepsilon\to 0} \varphi^\varepsilon(x,t) = 1\}$ and $\partial^* U_t$ is the reduced boundary of U_t (this condition is related to the convergence assumption in [11]). Moreover, they also showed the convergence of the vector-valued version of (1.6) to the multi-phase volume preserving mean curvature flow under a assumption similar to (1.9).

Remark 1.2. Under suitable conditions, the vanishing of the discrepancy measure (1.5) can be obtained from the convergence assumption (1.9) (see [10, Lemma 2.11]).

Remark 1.3. To the best of our knowledge, it is not known whether the convergence theorem of [10] can be shown if the assumption (1.9) is removed. If one try to prove the convergence by following [8], then the estimate

$$\sup_{\varepsilon>0} \int_0^T \int_\Omega \varepsilon \left(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \right)^2 dxdt < \infty \tag{1.10}$$

is needed. This requirement is natural because (1.10) corresponds to the L^2 estimate of the mean curvature of (1.1). From (1.8), the estimate (1.10) holds if

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^T |\lambda_{RS}^\varepsilon|^2 \, dt < \infty.$$

However, the best estimate known for this is $\sup_{\varepsilon>0} \int_0^T |\lambda_{RS}^\varepsilon|^2 \, dt < \infty$ by Bronsard and Stoth [3].

For (1.1), Golovaty [7] studied the following Allen-Cahn equation:

$$\begin{cases} \varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda_G^\varepsilon \sqrt{2W(\varphi^\varepsilon)}, & (x,t) \in \Omega \times (0,T), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{1.11}$$

where

$$\lambda_G^\varepsilon(t) = \frac{-\int_\Omega \sqrt{2W(\varphi^\varepsilon)} \, (\varepsilon\Delta\varphi^\varepsilon - \varepsilon^{-1}W'(\varphi^\varepsilon)) \, dx}{2\int_\Omega W(\varphi^\varepsilon) \, dx}.$$

Note that (1.11) also has the properties such as (1.2) and (1.3), namely, the solution satisfies (1.8) and

$$\frac{d}{dt} \int_\Omega G(\varphi^\varepsilon) \, dx = 0, \qquad t \in (0,T),$$

where $G(s) = \int_0^s \sqrt{2W(a)} \, da = s - \frac{1}{3}s^3$. In [16], for $d = 2,3$, the author proved the existence of the weak solution to (1.1) via (1.11), without any assumption for the convergence such as (1.9).

Remark 1.4. The function $\sqrt{2W(\varphi^\varepsilon)} = 1 - (\varphi^\varepsilon)^2$ is behaves like a cut-off function for the zero level set of φ^ε. It is natural that the non-local term has $\sqrt{2W(\varphi^\varepsilon)}$, since the support of the non-local term of (1.1) is on M_t, not $\mathbb{R}^d$. In fact, this function removes the problem mentioned in Remark 1.3 (see [16, 17]).

Next we explain the phase field model studied in [17]. For this, first we consider the mean curvature flow with penalty for the volume below. Assume that $\delta > 0$ and U_t^δ is an open set with smooth boundary M_t^δ for any $t \in [0,T)$. The approximate solutions for (1.1) studied in [13] and [9] correspond to the following flow $\{M_t^\delta\}_{t\in[0,T)}$ with penalty:

$$\vec{v} = \vec{h} - \lambda^\delta \vec{\nu}, \quad \text{on } M_t^\delta, \ t \in (0,T), \tag{1.12}$$

where

$$\lambda^\delta(t) = \frac{1}{\delta}(\mathscr{L}^d(U_0^\delta) - \mathscr{L}^d(U_t^\delta)).$$

We define $F^\delta(t)$ by

$$F^\delta(t) = \mathscr{H}^{d-1}(M_t^\delta) + \frac{1}{2\delta}(\mathscr{L}^d(U_0^\delta) - \mathscr{L}^d(U_t^\delta))^2.$$

Then we have

$$\frac{d}{dt}F^\delta(t) = -\int_{M_t^\delta} |\vec{v}|^2 \, d\mathscr{H}^{d-1} \le 0 \qquad \text{for any } t \in (0,T).$$

Therefore (1.12) is a L^2-gradient flow of $F^\delta(t)$. The penalty of the volume imply

$$(\mathscr{L}^d(U_0^\delta) - \mathscr{L}^d(U_t^\delta))^2 \le 2\delta F^\delta(t) \le 2\delta F^\delta(0) = 2\delta \mathscr{H}^{d-1}(M_0^\delta).$$

Hence, formally, we can expect that $\{M_t^\delta\}_{t\in[0,T)}$ converges to the volume preserving mean curvature flow as $\delta \to 0$. Mugnai, Seis, and Spadaro [13] studied a minimizing movement scheme for (1.12) and proved the existence theorem under an assumption corresponding to (1.9). Kim and Kwon [9] proved the existence of the viscosity solution to (1.1) via (1.12).

Let $\alpha \in (0,1)$. In this note, we consider the following phase field model for (1.12):

$$\begin{cases} \varepsilon \varphi_t^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda^\varepsilon \sqrt{2W(\varphi^\varepsilon)}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{1.13}$$

where

$$\lambda^\varepsilon(t) = \frac{1}{\varepsilon^\alpha}\left(\int_\Omega G(\varphi_0^\varepsilon(x))\, dx - \int_\Omega G(\varphi^\varepsilon(x,t))\, dx \right). \tag{1.14}$$

We remark that the solution to (1.13) exists under suitable assumptions for φ_0^ε (see [17]). Set

$$E_P^\varepsilon(t) = \frac{1}{2\varepsilon^\alpha}\left(\int_\Omega G(\varphi_0^\varepsilon(x))\, dx - \int_\Omega G(\varphi^\varepsilon(x,t))\, dx \right)^2$$

and $\tilde{E}^\varepsilon(t) = E^\varepsilon(t) + E_P^\varepsilon(t)$, where $E^\varepsilon(t)$ is given by (1.7) with the solution φ^ε to (1.13). By integration by parts, the solution φ^ε to (1.13) satisfies

$$\frac{d}{dt}\tilde{E}^\varepsilon(t) = -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\, dx \le 0 \qquad \text{for any } t \in (0,\infty),$$

$$\tilde{E}^\varepsilon(T) + \int_0^T \int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\, dxdt = \tilde{E}^\varepsilon(0) = E^\varepsilon(0) \qquad \text{for any } T \ge 0,$$

and

$$\left(\int_\Omega G(\varphi_0^\varepsilon(x))\, dx - \int_\Omega G(\varphi^\varepsilon(x,t))\, dx \right)^2 = 2\varepsilon^\alpha E_P^\varepsilon(t) \le 2\varepsilon^\alpha E^\varepsilon(0) \quad \text{for any } t \in [0,\infty).$$

Note that these properties correspond to those of (1.12).

In [17], the equation (1.13) was used to construct the measure theoretic weak solution (L^2-flow) to the volume preserving mean curvature flow. The detail is provided in the next section.

Remark 1.5. We denote $\tilde{\varphi}^\varepsilon(\tilde{x},\tilde{t}) := \varphi^\varepsilon(\varepsilon\tilde{x}, \varepsilon^2\tilde{t})$. Then we obtain

$$\tilde{\varphi}_{\tilde{t}}^\varepsilon = \Delta_{\tilde{x}}\tilde{\varphi}^\varepsilon - W'(\tilde{\varphi}^\varepsilon) + \varepsilon\lambda^\varepsilon(\varepsilon^2\tilde{t})\sqrt{2W(\tilde{\varphi}^\varepsilon)}, \tag{1.15}$$

where $\Delta_{\tilde{x}}$ is a Laplacian with respect to $\tilde{x}$. If $\sup_x |\varphi_0^\varepsilon(x)| < 1$ then the maximum principle implies $\sup_{x,t} |\varphi^\varepsilon(x,t)| < 1$. Therefore we have

$$\sup_{\tilde{t}\ge 0} |\varepsilon\lambda^\varepsilon(\varepsilon^2\tilde{t})\sqrt{2W(\tilde{\varphi}^\varepsilon)}| \le \sup_{\tilde{t}\ge 0} |\varepsilon\lambda^\varepsilon(\varepsilon^2\tilde{t})| \le \frac{4}{3}\mathscr{L}^d(\Omega)\varepsilon^{1-\alpha}.$$

Here we used $\max_{s\in[-1,1]}|G(s)| = \frac{2}{3}$. Since $\alpha \in (0,1)$, the rescaled equation (1.15) can be treated like the standard Allen-Cahn equation and we can prove the rectifiability and the integrality of the varifolds below.

2. Main results

In this section we describe the existence theorem obtained in [17].

First we recall the varifolds and refer to [15, 18] for more details. For $d \times d$ matrix $A = (a_{ij})$ and $B = (b_{ij})$, we define $A \cdot B := \sum_{i,j} a_{ij}b_{ij}$. For $d, k \in \mathbb{N}$ with $k < d$, we denote the space of k-dimensional subspace of $\mathbb{R}^d$ by $\mathbb{G}(d,k)$. For an open set $U \subset \mathbb{R}^d$, we define $G_k(U) := U \times \mathbb{G}(d,k)$. The measure V is called a general k-varifold on U if V is a Radon measure on $G_k(U)$. We define the set of all general k-varifolds on U by $\mathbb{V}_k(U)$. For $V \in \mathbb{V}_k(U)$, we denote the weight measure $\|V\|$ by

$$\|V\|(\phi) := \int_{G_k(U)} \phi(x)\, dV(x,S) \qquad \text{for any } \phi \in C_c(U).$$

The varifold $V \in \mathbb{V}_k(U)$ is called rectifiable if there exist a $\mathscr{H}^k$-measurable k-countably rectifiable set $M \subset U$ and positive function $\theta \in L^1_{loc}(\mathscr{H}^k\lfloor_M)$ such that

$$V(\phi) = \int_M \phi(x, T_xM)\theta(x)\, d\mathscr{H}^k \qquad \text{for any } \phi \in C_c(G_k(U)),$$

where T_xM is the approximate tangent space of M at x, with respect to θ (see [15]). In addition, V is called integral if $\theta \in \mathbb{N}$ $\mathscr{H}^k$-a.e. on M. For $V \in \mathbb{V}_k(U)$, the first variation δV is defined by

$$\delta V(\vec{\phi}) := \int_{G_k(U)} \nabla\vec{\phi}(x) \cdot S\, dV(x,S) \qquad \text{for any } \vec{\phi} \in C^1_c(U;\mathbb{R}^d).$$

Here, $S \in \mathbb{G}(d,k)$ is regarded as the orthogonal projection matrix of $\mathbb{R}^d$ onto S.

Assume that δV satisfies

$$\sup\{|\delta V(\vec{\phi})| \mid \vec{\phi} \in C^1_c(U;\mathbb{R}^d),\ |\vec{\phi}| \le 1,\ \operatorname{spt}\vec{\phi} \subset K\} < \infty$$

for any compact set $K \subset U$. Then the domain of δV can be extended to $C_c(U;\mathbb{R}^d)$ uniquely, and the Riesz representation theorem implies that there exist a Radon measure $\|\delta V\|$ and a $\|\delta V\|$-measurable function $\vec{\sigma} : U \to \mathbb{R}^d$ such that

$$\delta V(\vec{\phi}) = \int_U \vec{\phi} \cdot \vec{\sigma}\, d\|\delta V\|, \qquad \text{for any } \vec{\phi} \in C_c(U).$$

In addition, if $\|\delta V\| \ll \|V\|$, then the Radon-Nikodym theorem tells us that there exists a measurable vector field $\vec{h} = -\frac{d\|\delta V\|}{d\|V\|}\vec{\sigma}$ such that

$$\delta V(\vec{\phi}) = -\int_U \vec{\phi}(x) \cdot \vec{h}(x)\, d\|V\|(x) \qquad \text{for any } \vec{\phi} \in C_c(U;\mathbb{R}^d).$$

We call $\vec{h}$ the generalized mean curvature vector of V.

Next we give the definition of the weak solution considered in this note. The definition is similar to that of the Brakke flow (see [1, 18]).

Definition 2.1 (L^2-flow [12])**.** Let $T > 0$, $U \subset \mathbb{R}^d$ be an open set, and $\{\mu_t\}_{t\in[0,T)}$ be a family of Radon measures on U. Let $d\mu := d\mu_t dt$. The family of Radon measures $\{\mu_t\}_{t\in[0,T)}$ is called an L^2-flow with a generalized velocity vector $\vec{v}$ if the following hold:

(1) For a.e. $t \in (0,T)$, μ_t is $(d-1)$-integral, that is, there exists a $(d-1)$-integral varifold $V_t \in \mathbb{V}_{d-1}(U)$ such that $\mu_t = \|V_t\|$. In addition, V_t has a generalized mean curvature vector $\vec{h} \in L^2(\mu_t; \mathbb{R}^d)$.

(2) $\vec{v} \in L^2(0,T; (L^2(\mu_t))^d)$ and

$$\vec{v}(x,t) \perp T_x\mu_t \quad \text{for } \mu\text{-a.e. } (x,t) \in U \times (0,T),$$

where $T_x\mu_t \in \mathbb{G}(d, d-1)$ is the approximate tangent space of V_t at x.

(3) There exists $C_T > 0$ such that

$$\left| \int_0^T \int_U (\eta_t + \nabla \eta \cdot \vec{v})\, d\mu_t dt \right| \le C_T \|\eta\|_{C^0(U\times(0,T))} \tag{2.1}$$

for any $\eta \in C_c^1(U \times (0,T))$.

Remark 2.2. Assume that $M_t \subset U$ is a smooth hypersurface with the normal velocity $\vec{v}$ for any $t \in (0,T)$ and it holds that $\int_0^T \int_{M_t} |\vec{v} \cdot \vec{h}|\, d\mathscr{H}^{d-1} dt < \infty$. Then (2.1) holds with $d\mu_t = d\mathscr{H}^{d-1}\lfloor_{M_t}$ and $C_T = \int_0^T \int_{M_t} |\vec{v}\cdot\vec{h}|\, d\mathscr{H}^{d-1}dt$, since we have

$$\frac{d}{dt}\int_{M_t} \eta\, d\mathscr{H}^{d-1} = \int_{M_t} (\nabla^\perp \eta - \eta\vec{h})\cdot\vec{v} + \eta_t\, d\mathscr{H}^{d-1} \qquad \text{for any } \eta \in C_c^1(U\times(0,T)).$$

Here $\nabla^\perp\eta = \nabla\eta - (\nabla\eta\cdot\vec{\nu})\vec{\nu}$. Moreover, if there exists a constant $C > 0$ such that

$$\left| \int_0^T \int_{M_t} (\eta_t + \nabla\eta\cdot\vec{w})\, d\mathscr{H}^{d-1} dt \right| \le C\|\eta\|_{C^0(U\times(0,T))} \qquad \text{for any } \eta \in C_c^1(U\times(0,T)),$$

then we have $\vec{v} \equiv \vec{w}$. This property can be proved in the same way as in [18, Proposition 2.1].

We define a Radon measure μ_t^ε on Ω by

$$\mu_t^\varepsilon(\phi) := \frac{1}{\sigma}\int_\Omega \phi\left(\frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}\right) dx, \qquad \phi \in C_c(\Omega), \tag{2.2}$$

where φ^ε is a solution to (1.13). The approximate velocity vector $\vec{v}^\varepsilon$ is given by

$$\vec{v}^\varepsilon = \begin{cases} \frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|}\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}, & \text{if } |\nabla\varphi^\varepsilon| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following is the existence theorem for (1.1) in the sense of L^2-flow.

Theorem 2.3 (see [17])**.** Let $d \ge 2$ and $U_0 \subset \Omega$ be an open set with C^1 boundary M_0. Then there exists a family of solutions $\{\varphi^{\varepsilon_i}\}_{i=1}^\infty$ to (1.13) with ε_i instead of ε such that the following hold.

(a) There exist a countable subset $B \subset [0,\infty)$ and a family of $(d-1)$-integral Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on Ω such that

$$\mu_0 = \mathscr{H}^{d-1}\lfloor_{M_0}, \qquad \mu_t^{\varepsilon_i} \rightharpoonup \mu_t \quad \text{as Radon measures for any } t \ge 0,$$

and

$$\mu_s(\Omega) \le \mu_t(\Omega) \qquad \text{for any } s,t \in [0,\infty)\setminus B \text{ with } 0 \le t < s < \infty.$$

(b) There exists a function $\psi \in BV_{loc}(\Omega\times[0,\infty)) \cap C_{loc}^{\frac{1}{2}}([0,\infty); L^1(\Omega))$ such that the following hold.

(b1) Set $\psi^{\varepsilon_i} = \frac{1}{2}(\varphi^{\varepsilon_i} + 1)$. Then we have $\psi^{\varepsilon_i} \to \psi$ in $L^1_{loc}(\Omega\times[0,\infty))$ and a.e. pointwise.

(b2) $\psi|_{t=0} = \chi_{U_0}$ a.e. on Ω.

(b3) For any $t \in [0, \infty)$, it holds that $\psi(\cdot, t) = 1$ or 0 a.e. on Ω. In addition, we have

$$\int_{\Omega} \psi(x, t)\, dx = \mathscr{L}^d(U_0) \qquad \text{for any } t \in [0, \infty).$$

(b4) For any non-negative function $\phi \in C_c(\Omega)$, we have

$$\|\nabla \psi(\cdot, t)\|(\phi) \le \mu_t(\phi) \qquad \text{for any } t \in [0, \infty).$$

(c) We have

$$\sup_{i \in \mathbb{N}} \int_0^T |\lambda^{\varepsilon_i}|^2\, dt < \infty \qquad \text{for any } T > 0,$$

where λ^{ε_i} is given by (1.14). Moreover, there exists $\lambda \in L^2_{loc}(0, \infty)$ such that $\lambda^{\varepsilon_i} \rightharpoonup \lambda$ weakly in $L^2(0, T)$ for any $T > 0$.

(d) The family of Radon measures $\{\mu_t\}_{t \in [0, \infty)}$ is an L^2-flow with a generalized velocity vector $\vec{v}$, where $\vec{v}$ satisfies

$$\vec{v} = \vec{h} - \lambda \frac{d\|\nabla \psi(\cdot, t)\|}{d\mu_t} \vec{\nu} \qquad \mu\text{-a.e. on } \Omega \times (0, \infty).$$

Here $\vec{h} \in L^2_{loc}([0, \infty); (L^2(\mu_t))^d)$ is the generalized mean curvature vector of μ_t and $\vec{\nu}$ is the inner unit normal vector of $\{\psi(\cdot, t) = 1\}$ on $\operatorname{spt} \|\nabla \psi(\cdot, t)\|$. In addition, for any $\vec{\phi} \in C_c(\Omega \times [0, \infty); \mathbb{R}^d)$, we have

$$\lim_{i \to \infty} \int_0^\infty \int_\Omega \vec{v}^{\varepsilon_i} \cdot \vec{\phi}\, d\mu_t^{\varepsilon_i} dt = \int_0^\infty \int_\Omega \vec{v} \cdot \vec{\phi}\, d\mu_t dt. \tag{2.3}$$

Note that if there exists $T > 0$ such that $\mu_t = \|\nabla \psi(\cdot, t)\|$ for a.e. $t \in (0, T)$, then we obtain $\vec{v} = \vec{h} - \lambda \vec{\nu}$ in the sense of L^2-flow. In fact, it is true if one adds the assumption that the initial data M_0 is close to a sphere. We explain this in detail below. We denote $B_r(0) = \{x \in \mathbb{R}^d \mid |x| < r\}$ for $r > 0$ and $U_t = \{x \in \Omega \mid \psi(x, t) = 1\}$ for any $t > 0$.

Theorem 2.4 (see [17]). For any $r \in (0, \frac{1}{4})$, there exists $\delta_1 > 0$ depending only on d and r with the following property: Suppose that an open set $U_0 \subset (\frac{1}{4}, \frac{3}{4})^d$ satisfies $\mathscr{L}^d(U_0) = \mathscr{L}^d(B_r(0))$ and has a C^1 boundary M_0 with $\mathscr{H}^{d-1}(M_0) \le 2\mathscr{H}^{d-1}(\partial B_r(0))$. In addition, assume that U_0 satisfies

$$\mathscr{H}^{d-1}(M_0) - d\omega_d^{\frac{1}{d}} (\mathscr{L}^d(U_0))^{\frac{d-1}{d}} \le \delta_1. \tag{2.4}$$

Let $\{\mu_t\}_{t \in [0, \infty)}$ be the L^2-flow with initial data $\mu_0 = \mathscr{H}^{d-1}\lfloor_{M_0}$, obtained by Theorem 2.3. Then there exists $T_1 = T_1(d, r, M_0) > 0$ such that

$$\mu_t = \|\nabla \psi(\cdot, t)\| = \mathscr{H}^{d-1}\lfloor_{\partial^* U_t} \qquad \text{for a.e. } t \in [0, T_1),$$

where $\partial^* U_t$ is the reduced boundary of U_t and ψ is the function given by Theorem 2.3.

Acknowledgements

This work was supported by JSPS KAKENHI Grant Numbers JP20K14343, JP18H03670, JP23K03180, JP23H00085, and JSPS Leading Initiative for Excellent Young Researchers (LEADER) operated by Funds for the Development of Human Resources in Science and Technology.

References

[1] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

[2] Bronsard, L. and Kohn, R. V., *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations, **90** (1991), 211–237.

[3] Bronsard, L. and Stoth, B., *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation*, SIAM J. Math. Anal., **28** (1997), 769–807.

[4] Chen, X., *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations, **96** (1992), 116–141.

[5] Chen, X., Hilhorst, D., and Logak, E., *Mass conserving Allen-Cahn equation and volume preserving mean curvature flow*, Interfaces Free Bound., **12** (2010), 527–549.

[6] Evans, L. C., Soner, H. M., and Souganidis, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., **45** (1992), 1097–1123.

[7] Golovaty, D., *The volume-preserving motion by mean curvature as an asymptotic limit of reaction-diffusion equations*, Quart. Appl. Math., **55** (1997), 243–298.

[8] Ilmanen, T., *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., **38** (1993), 417–461.

[9] Kim, I. and Kwon, D., *Volume preserving mean curvature flow for star-shaped sets*, Calc. Var. Partial Differential Equations, **59** (2020), Paper No. 81, 40.

[10] Laux, T. and Simon, T. M., *Convergence of the Allen-Cahn equation to multiphase mean curvature flow*, Comm. Pure Appl. Math., **71** (2018), 1597–1647.

[11] Luckhaus, S. and Sturzenhecker, T., *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations, **3** (1995), 253–271.

[12] Mugnai, L. and Röger, M., *The Allen-Cahn action functional in higher dimensions*, Interfaces Free Bound., **10** (2008), 45–78.

[13] Mugnai, L., Seis, C. and Spadaro, E., *Global solutions to the volume-preserving mean-curvature flow*, Calc. Var. Partial Differential Equations, **55** (2016), Art. 18, 23.

[14] Rubinstein, J. and Sternberg, P., *Nonlocal reaction-diffusion equations and nucleation*, IMA Journal of Applied Mathematics, **48** (1992), 249–264.

[15] Simon, L., *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[16] Takasao, K., *Existence of weak solution for volume preserving mean curvature flow via phase field method*, Indiana Univ. Math. J., **66** (2017), 2015–2035.

[17] Takasao, K., *The Existence of a Weak Solution to Volume Preserving Mean Curvature Flow in Higher Dimensions*, Arch. Ration. Mech. Anal., **247** (2023), 52 pages.

[18] Tonegawa, Y., *Brakke's mean curvature flow*, SpringerBriefs in Mathematics, Springer, Singapore, (2019).

E-mail address: `k.takasao@math.kyoto-u.ac.jp`

Front-back-pulse solutions of a three-species Lotka-Volterra competition diffusion system through heteroclinic bifurcation approach

Chueh-Hsin Chang

Department of Mathematics, National Chung Cheng University, Chia-Yi, Taiwan

1 Introduction

We consider the existence of traveling wave solutions of the three-species Lotka-Volterra competition-diffusion system

$$\begin{cases} u_{1t} = u_{1xx} + u_1(1 - u_1 - c_{12}u_2 - c_{13}u_3), \\ u_{2t} = d_v u_{2xx} + u_2(r_2 - c_{21}u_1 - u_2 - c_{23}u_3), \\ u_{3t} = d_w u_{3xx} + u_3(r_3 - c_{31}u_1 - c_{32}u_2 - u_3). \end{cases} \tag{1.1}$$

Here $(u_1, u_2, u_3)(x,t) \in \mathbb{R}^3$ denotes the population densities of the three species at spatial position x and time t. For $i,\ j = 1, 2, 3$, the parameters d_i, c_{ii} and c_{ij} $(i \neq j)$ are all positive constants denoting the diffusion rates, intrinsic growth rate, intra-specific and inter-specific competition rates, respectively.

$$\begin{cases} u_{1t} = u_{1xx} + u_1(1 - u_1 - c_{12}u_2 - c_{13}u_3), \\ u_{2t} = d_v u_{2xx} + u_2(r_2 - c_{21}u_1 - u_2 - c_{23}u_3), \\ u_{3t} = d_w u_{3xx} + u_3(r_3 - c_{31}u_1 - c_{32}u_2 - u_3). \end{cases} \tag{1.2}$$

For simplicity, we consider the scaled system (1.2) instead of (1.1) in the following study. Traveling waves solutions of (1.2) having the form $(u_1, u_2, u_3)(x,t) = (u, v, w)(z)$, where $z = x - st$ and s is the wave propagating speed. Substituting this ansatz into (1.2), it becomes

$$\begin{cases} u'' + su' + u(1 - u - c_{12}v - c_{13}w) = 0, \\ d_v v'' + sv' + v(r_2 - c_{21}u - v - c_{23}w) = 0, \qquad z \in \mathbb{R}, \\ d_w w'' + sw' + w(r_3 - c_{31}u - c_{32}v - w) = 0, \end{cases} \tag{1.3}$$

where $(\cdot)' = \frac{d}{dz}$. There are many possibilities for the boundary conditions of $(u, v, w)(z)$ as $z \to \pm\infty$. We consider the following condition with u, v and w components to be of front, back and pulse type profiles,respectively :

$$(u, v, w)(-\infty) = (0, r_2, 0), \ (u, v, w)(+\infty) = (1, 0, 0). \tag{1.4}$$

Let us call such wave a front-back-pulse (FBP for short) solution for convenience. One motivation to consider such solution is, assume that the two species (u, v) are natively living in the environment, and w invades into the (u, v) systems. If u and v compete with each other with a strong competition, i.e.,

$$\frac{1}{c_{12}} < r_2 < c_{21}. \tag{1.5}$$

Then from the results by Kan-on [24], for fixed d_v, the two-species system

$$\begin{cases} u'' + su' + u(1 - u - c_{12}v) = 0, \\ d_v v'' + sv' + v(r_2 - c_{21}u - v) = 0 \end{cases}$$

with

$$(u, v)(-\infty) = (0, r_2), \; (u, v)(+\infty) = (1, 0)$$

has a unique monotone solution with a unique speed s. Then the invade species w can only survive in the middle region where u and v are not so dominant. Hence w has a pulse profile. The two-species parabolic competition system is a monotone system. One can use the comparison principle to analyze the structure of its solutions. See Volpert's book [34]. However, the three-species system (1.2) is not a monotone system. Due to the lack of maximum principles, it is not easy to construct solutions of (1.3) and (1.4). There are some know results for the three-species traveling wave solutions, e.g., in [19], [20], [21], [22], [27], [30] and [31], the studies of three-species problems mainly rely on singular perturbation methods which assume smallness of some diffusion coefficients of (1.3). Chen, Hung, Mimura and Ueyama [5] found FBP waves numerically by the software package AUTO. In [4], together with M. Tohma, they further constructed semi-exact solutions of two-hump waves and analyzed related bifurcation behaviors of these solutions numerically. Contento, Mimura, Tohma [9] and Mimura and Tohma [32] used FBP waves to construct spiral waves, wedge waves, and other very interesting new dynamical patterns on a two-dimensional domain via numerical simulations. Recently Chang et al. [3] consider the existence and asymptotic stability of FBP solutions by considering c_{13} and c_{23} sufficiently small enough. Chang and Chen [2] study the existence FBP solutions by the gluing bifurcation theory developed from [29]. When $c_{13} = 0$ and $c_{31} = 0$, (1.3) becomes monotone system and Guo et al. [16] apply maximum principle to obtain monotone traveling wave solutions. The stability of monotone traveling wave solutions when was established by Chang [1]. On the other hand, for the case $c_{13} < 0$ and $c_{31} < 0$, (1.3) becomes a competitive-cooperative system and and the maximum principle can be applied also [18]. More recently, Ei, Ikeda and Ogawa [11], [12] they give the bifurcation diagram with numerical simulation for the FBP solutions. In this talk we consider the authors' recent work [3] and [2] for the construction of FBP solutions.

2 Weak invasion of the alien species

In the work [3] we consider the following type of (1.3):

$$
(\varepsilon\text{-}\mathbf{P})\begin{cases} u''+su'+u(1-u-c_{12}v-\varepsilon_1 w)=0,\\ d_v v''+sv'+v(r_2-c_{21}u-v-\varepsilon_2 w)=0,\\ d_w w''+sw'+w(r_3-c_{31}u-c_{32}v-w)=0,\end{cases} \tag{2.1}
$$

we come to the following assumption.

(A1) (**Bistability**) $c_{31}>r_3$, $c_{32}r_2>r_3$, $c_{21}>r_2>\dfrac{1}{c_{12}}$.

A direct consequence from (A1) is that the points $(1,0,0)$ and $(0,r_2,0)$ are stable equilibria of the diffusion-less system of (1.1). Our main theorem is as follows:

Theorem 2.1. *([3]) Assume that $c_{12}, d_2, r_2,$ and c_{21} satisfy*

(*a*) $d_2>0$ *and* $c_{21}>r_2>\dfrac{1}{c_{12}}>0$*; and*

(*b*) *either* $c_{21}\geq b_0(r_2,d_2)$ *or* $c_{12}\geq c_0(r_2,d_2)$*, where*

$$
\begin{aligned}
&b_0(r_2,d_2)\\
&=r_2+d_2+\begin{cases} 2r_2d_2^2\left(1-\dfrac{1}{d_2}\right)\left(1+\sqrt{1+\dfrac{1}{r_2d_2}}\right), & \text{if } d_2\geq 1,\\ 2d_2\left(\dfrac{1}{d_2}-1\right)\left(\sqrt{2}-1\right), & \text{if } 0<d_2<1;\end{cases}\\
&c_0(r_2,d_2)\\
&=\frac{1}{r_2}+\frac{1}{d_2}+\begin{cases} 2\left(1-\dfrac{1}{d_2}\right)\left(\sqrt{2}-1\right), & \text{if } d_2\geq 1,\\ \dfrac{2}{r_2d_2}\left(\dfrac{1}{d_2}-1\right)\left(\sqrt{1+r_2d_2}+1\right) & \text{if } 0<d_2<1.\end{cases}
\end{aligned} \tag{2.2}
$$

Then there exist positive d_3, r_3, c_{31}, c_{32}, c_{33} and δ such that (ε-P) has a stable positive solution for $0\leq\varepsilon_1<\delta$ and $0\leq\varepsilon_2<\delta$, where $\delta=\delta(c_{12},d_2,r_2,c_{21},d_3,r_3,c_{31},c_{32},c_{33})$.

The main idea in the proofs of Theorem 2.1 is asfollows. Since we consider the case when the effects of w on both u and v are small. Along this thinking, it is natural to study the extreme case where both $\varepsilon_1=0$ and $\varepsilon_2=0$ first. That is,

$$
(\mathbf{0\text{-}P})\begin{cases} u''+su'+u(1-u-c_{12}v)=0,\\ d_v v''+sv'+v(r_2-c_{21}u-v)=0,\\ d_w w''+sw'+w(r_3-c_{31}u-c_{32}v-w)=0,\\ (u,v,w)(-\infty)=(0,r_2,0),\ (u,v,w)(\infty)=(1,0,0)\end{cases} \tag{2.3}
$$

and then the first two equations of (0-P) are decoupled from the third equation. By the results of Kan-on [24] (see also [14]), the first two equations have a solution (u_0,v_0) for some s_0. Then we apply the method of sub-sup solutions to the third equation and obtain a solution (u_0, v_0, w_0) for (0-P). Once (0-P) is solved, we solve (ε-P) by perturbing $(\varepsilon_1, \varepsilon_2)$ from $(0, 0)$ to a pair of small numbers. To state the perturbation result which Theorem 2.1 rely on, we consider the following existence condition of (0-P) as a starting point:

(A2) (**A priori existence**) Assume the system (0-P) has a positive solution $U_0(z) = (u_0, v_0, w_0)(z)$ with wave speed $s = s_0$ and $u_0'(z) > 0$, $v_0'(z) < 0$ for $z \in \mathbb{R}$.

As mentioned above, by the standard heteroclinic bifurcation theory [29], we have the existence of the solution $U^\varepsilon(z) := (u_\varepsilon, v_\varepsilon, w_\varepsilon)(z)$ of Problem (ε-P) with $s = s_\varepsilon$ for some speed $s_\varepsilon \sim s_0$ when $(\varepsilon_1, \varepsilon_2) \sim (0, 0)$. For the stability of U^ε, we consider the spectrum of the linearized operator L_0 of (2.3) around U_0 first. We prove that U_0 is stable first by using the fact that (u_0, v_0) is stable [26] and the comparison principle for the third equation of L_0. Then by the perturbation theory from, e.g., [15] or [28], we have that the number of eigenvalues of the operator L_0 and that of operator L_ε, the linearized operator of (2.1) around U^ε are the same for $(\varepsilon_1, \varepsilon_2) \sim (0, 0)$. Since U_0 is stable, this number equals one (it is the eigenvalue at the origin due to translation invariance of traveling waves) and hence U^ε is stable.

Theorem 2.2. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Suppose that hypotheses (A1) and (A2) hold. Then there exists $\delta > 0$ such that Problem (ε-P) has a positive solution $U^\varepsilon(z) = (u_\varepsilon, v_\varepsilon, w_\varepsilon)^T(z)$ with $s = s_\varepsilon$ depending on ε if $0 \le \varepsilon_1 \le \delta$ and $0 \le \varepsilon_2 \le \delta$. Moreover the solution $U^\varepsilon(z)$ is stable.*

3 Gluing bifurcation

In our second approach, we use the gluing bifurcation for the construction of FBP solutions. we introduce the gluing orbits first. If we let $v \equiv 0$, then (1.3) reduces to the two-species system

$$\begin{cases} u'' + su' + u(1 - u - c_{13}w) = 0, \\ d_w w'' + sw' + w(r_3 - c_{31}u - w) = 0, \end{cases} \quad z \in \mathbb{R} \tag{3.1}$$

In particular, for the strong competition case, i.e., the parameters in (3.1) satisfying

$$\frac{1}{c_{13}} < r_3 < c_{31}, \tag{3.2}$$

the existence and stability of a traveling wave solution $(u_0, w_R)(z)$ of (3.1) with wave speed $s = s_2$ satisfying

$$(u_0, w_R)(-\infty) = (0, r_3), \quad (u_0, w_R)(+\infty) = (1, 0)$$

were proved in Kan-on [24], Kan-on and Fang [26]. With (u_0, w_R), we obtain a solution $(u_0, 0, w_R)(z)$ of (1.3) with wave speed $s = s_2$ satisfying the boundary condition

$$(u_0, 0, w_R)(-\infty) = (0, , 0, r_3), \quad (u_0, 0, w_R)(+\infty) = (1, 0, 0),$$

which we call a trivial three-species traveling wave solution. Similarly, if $u \equiv 0$, (1.3) reduces to the system

$$\begin{cases} d_v v'' + sv' + v(r_2 - v - c_{23}w) = 0, \\ d_w w'' + sw' + w(r_3 - c_{32}v - w) = 0, \end{cases} \quad z \in \mathbb{R}. \tag{3.3}$$

Under the strong competition case

$$\frac{1}{c_{32}} < \frac{r_2}{r_3} < c_{23}, \tag{3.4}$$

by Kan-on's result again, there also exist a traveling wave solution $(w_L, v_0)(z)$ of (3.3) with wave speed $s = s_1$ connecting the stable steady states $(0, r_2)$ and $(r_3, 0)$:

$$(w_L, v_0)(-\infty) = (0, r_2), \quad (w_L, v_0)(+\infty) = (r_3, 0).$$

Therefore there also exist a unique trivial three-species traveling wave $(0, v_0, w_L)(z)$ of (1.3) with wave speed $s = s_1$ whose u-component equals to 0, and

$$(0, v_0, w_L)(-\infty) = (0, r_2, 0), \quad (0, v_0, w_L)(+\infty) = (0, 0, r_3).$$

We explain the idea to study the existence of FFP solutions of (1.3) by the heteroclinic bifurcation theory developed by Kokubu [29] and Chow et al. [6], [7], [8] and [17]. in the following steps:

1. Let d_v and d_w be given constants. Rewrite (1.3) as an equivalent first order system with (r_i, c_{ij}) as parameters satisfying (3.2), (3.4) and the strong competition between u and v species (1.5). That is, the 3 species u, v and w are strongly competitive with one another. Then from Kan-on's results [24], we have the existence of trivial three-species waves $(0, v_0, w_L)$ and $(u_0, 0, w_R)$ which are equivalent to the heteroclinic orbits of the first order system. Generically their wave speed s_1 and s_2 are not equal.

2. We further assume that there exists some $r_{i,0}$, and $c_{ij,0}$ such that for $(r_i, c_{ij}) = (r_{i,0}, c_{ij,0})$, $(0, v_0, w_L)$ and $(u_0, 0, w_R)$ have the same traveling speed s_0. In other words, when $(s, r_i, c_{ij}) = (s_0, r_{i,0}, c_{ij,0})$, there exists two heteroclinic orbits $(0, v_0, w_L)$ and $(u_0, 0, w_R)$ connecting a common equilibrium $(0, 0, r_3)$.

3. Under the assumption of the existence of $(s_0, r_{i,0}, c_{ij,0})$, we intend to find a connecting orbit from $(0, r_2, 0)$ to $(1, 0, 0)$ in a neighborhood of the union of $(0, v_0, w_L)$, $(u_0, 0, w_R)$ and the limiting three equilibria if some hypotheses of the preliminary orbits $(0, v_0, w_L)$, $(u_0, 0, w_R)$ are satisfied. We will find conditions of $(s_0, r_{i,0}, c_{ij,0})$ to achieve these hypotheses of the preliminary orbits. Then such a connection from $(0, r_2, 0)$ to $(1, 0, 0)$ can be seen as a FFP solution of (1.3).

4. We find examples of exact solutions of $(0, v_0, w_L)$ and $(u_0, 0, w_R)$ from [33] such that the existence of $r_{i,0}$, $c_{ij,0}$ and s_0 is non-empty.

Our assumption of parameters (r_i, c_{ij}) are the same as that in Ei et al. [13], but different from that in Chen et al. [5] and [32]. In general, it may be difficult to find the existence of bifurcation points $(s_0, r_{i,0}, c_{ij,0})$. In the earlier work to find N-pulse and N-front solutions, bifurcation points were constructed by using the symmetry of the equations [25], or the geometrical singular perturbation theory [10]. In our situation, since all the three equilibria are distinct and the orbits $(0, v_0, w_L)$ and $(u_0, 0, w_R)$ are rather different from each other, it causes more difficulty. We also give explicit examples to examine that there exists $r_{i,0}$, $c_{ij,0}$ and s_0 such that we have the existence of the trivial three-species waves $(0, v_0, w_L)$ connecting $(0, r_2, 0)$ to $(0, 0, r_3)$ and $(u_0, 0, w_R)$ connecting $(0, r_3, 0)$ to $(1, 0, 0)$ when $(s, r_i, c_{ij}) = (s_0, r_{i,0}, c_{ij,0})$.

References

[1] C.-H. Chang, The stability of traveling wave solutions for a diffusive competition system of three species, J. Math. Anal. Appl., 459 (2018), pp. 564-576.

[2] C.-H. Chang and C.-C. Chen, Existence of front-back-pulse solutions of a three-species lotka-volterra competition-diffusion system, Journal of Dynamics and Differential Equations, 35 (2023), pp. 1273–1308.

[3] C.-H. Chang, C.-C. Chen, L.-C. Hung, M. Mimura, T. Ogawa, Existence and stability of non-monotone travelling wave solutions for the diffusive Lotka-Volterra system of three competing species, Nonlinearity, 33 (2020), pp. 5080-5110.

[4] C. C. Chen, L, C, Hung, M. Mimura, M. Tohma, and D. Ueyama, Semi-exact equilibrium solutions for three-species competition-diffusion systems, Hiroshima Mathematical Journal,43 (2013) no.2 pp.179-206..

[5] C. C. Chen, L, C, Hung, M. Mimura, and D. Ueyama, Exact travelling wave solutions of three-species competition-diffusion systems, Discrete Contin. Dyn. Syst. Ser. B, 17 (2012), pp. 2653–2669.

[6] S.-N. Chow, B. Deng, and B. Fiedler, Homoclinic bifurcation at resonant eigenvalues, J. Dynamical Systems and Differential Equations, 2 (1990), pp. 177-244.

[7] S.-N. Chow, B. Deng, and D. Terman, The bifurcation of homoclinic and periodic orbits from two heteroclinic orbits, SIAM J. Math Anal., 21 (1990), pp. 179–204.

[8] S.-N. Chow, B. Deng, and D. Terman, The bifurcation of a homoclinic orbit from two heteroclinic orbits-a topological approach. Appi. Anal, 42 (1991), pp. 275-296.

[9] L. Contento, M. Mimura and M. Tohma, Two-dimensional traveling waves arising from planar front interaction in a three-species competition–diffusion system. Jpn J Ind Appl Math 32(3) (2015), pp. 707–747.

[10] B. Deng, The existence of infinitely many travelling front and back waves in the FitzHugh-Nagumo equations. SIAM J. Math. Anal. 22 (1991), pp. 1631–1650.

[11] Shin-Ichiro Ei, Hideo Ikeda, Toshiyuki Ogawa, Bifurcation of co-existing traveling wave solutions in a three-component competition–diffusion system, Physica D: Nonlinear Phenomena, Volume 448, 2023, 133703

[12] Shin-Ichiro Ei, Hideo Ikeda, Toshiyuki Ogawa. Alien invasion into the buffer zone between two competing species. Discrete and Continuous Dynamical Systems - B. doi: 10.3934/dcdsb.2023081

[13] S. Ei, R. Ikota, and M. Mimura, Segregating partition problem in competition-diffusion systems, Interfaces Free Bound., 1 (1999), pp. 57–80.

[14] R. A. Gardner, Existence and stability of travelling wave solutions of competition models: A degree theoretic approach, J. Differential Equations, 44 (1982), pp. 343–364.

[15] I. T. Gokhberg and M. G. Krein, Introductory to the Theory of Linear Non-selfadjoint Operators, Volume 18 of Translations of Mathematical Monographs (1969), American Mathematical Society.

[16] J. S. Guo, Y. Wang, C. H. Wu, C. C. Wu, The minimal speed of traveling wave solutions for a diffusive three species competition system, Taiwanese J. Math. 19 (2015), No. 6, pp. 1805–1829.

[17] A. J. Homburg and B. Sandstede, Homoclinic and Heteroclinic Bifurcations in Vector Fields, Hand-book of Dynamical Systems III: H. Broer, F. Takens and B. Hasselblatt, ed., Elsevier, 2010, pp. 379–524.

[18] L.-C. Hung, Traveling wave solutions of competitive-cooperative Lotka-Volterra systems of three species, Nonlinear Analysis: Real World Applications, 12 (2011), pp. 3691–3700.

[19] H. Ikeda, Travelling wave solutions of three-component systems with competition and diffusion. Math. J. Toyama Univ. 24 (2001), pp. 37–66.

[20] H. Ikeda, Multiple travelling wave solutions of three-component systems with competition and diffusion. Methods Appl. Anal. 8 (2001), no. 3, pp. 479–496.

[21] H. Ikeda, Global bifurcation phenomena of standing pulse solutions for three-component systems with competition and diffusion, Hiroshima Math. J., 32 (2001), pp. 87–124.

[22] H. Ikeda, Dynamics of weakly interacting front and back waves in three-component systems. Toyama Math. J. 30 (2007), pp. 1–34.

[23] H. Ikeda, Private communication.

[24] Y. Kan-on, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, SIAM J. Math. Anal. 26 (1995), pp. 340–363.

[25] Y. Kan-on, Existence of standing waves for competition-diffusion equations, Japan J. Indust. Appl. Math. 13 (1996), pp. 117–133.

[26] Y. Kan-on and Q. Fang, Stability of monotone travelling waves for competition-diffusion equations, Japan J. Indust. Appl. Math., 13 (1996), pp. 343–349.

[27] Y. Kan-on and M. Mimura, Singular perturbation approach to a 3-component reaction-diffusion system arising in population dynamics, SIAM J. Math. Anal. 29 (1998), pp. 1519–1536.

[28] T. Kato, Perturbation theory for linear operators, vol. 132, Springer Verlag, 1995.

[29] H. Kokubu, Homoclinic and heteroclinic bifurcations of vector fields, Japan J. Appl. Math., 5 (1988), pp. 455–501.

[30] P. D. Miller, Nonmonotone waves in a three species reaction-diffusion model, Methods and Applications of Analysis 4 (1997), pp. 261–282.

[31] P. D. Miller, Stability of non-monotone waves in a three-species reaction-diffusion model, Proceedings of the Royal Society of Edinburgh 129A(1999), pp. 125–152.

[32] M. Mimura, M. Tohma, Dynamic coexistence in a three-species competitiondiffusion system, Ecological Complexity, 21 (2015), pp. 215-232

[33] M. Rodrigo and M. Mimura, Exact solutions of reaction-diffusion systems and nonlinear wave equations. Japan J. Indust. Appl. Math. 18 (2001), pp. 657–696.

[34] A. I. Volpert, V. A. Volpert and V. A. Volpert. Traveling Wave Solutions of Parabolic Systems. Translations of Mathematical Monographs 140, Amer. Math. Soc., Providence (1994).

A supercritical Hénon equation with a forcing term

Sho Katayama

Graduate School of Mathematical Sciences, The University of Tokyo

1 Introduction

This talk is based on the paper [11]. We consider a Hénon equation with a forcing term

$$\begin{cases} -\Delta u = |x|^a u^p + \kappa\mu & \text{in} \quad \mathbb{R}^N, \\ \quad u > 0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{P}$$

Here $N \geq 3$, $p > 1$, $a > -2$, μ is a nontrivial Radon measure on $\mathbb{R}^N$, and $\kappa > 0$ is a parameter. Our aim is to give a complete classification of existence and nonexistence of solutions to problem (P) with respect to the parameter κ, under a suitable assumptions on the exponent p and the nonhomogeneous term μ. In particular, we prove a threshold constant $\kappa^* \in (0, \infty)$ with the following properties.

(A1) If $0 < \kappa < \kappa^*$, then problem (P) possesses a solution.

(A2) If $\kappa = \kappa^*$, then problem (P) possesses a unique solution.

(A3) If $\kappa > \kappa^*$, then problem (P) possesses no solutions.

Nonlinear elliptic equations with forcing terms in $\mathbb{R}^N$ arise naturally in the study of stochastic processes. In particular, problem (P) with $a = 0$ appeared in establishing some limit theorems for super-Brownian motion. See e.g. [3, 5, 7, 15] for a brief history and background of problem (P). One of the main difficulties of Hénon equations on $\mathbb{R}^N$ is that techniques of calculus of variations are not applicable since the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, |x|^a dx)$ does not hold, except for the case $a \in (-2, 0]$, $p = (N + 2 + 2a)/(N - 2)$ (in this sense, problem (P) is "supercritical"). See e.g. [10] for the case $a = 0$, $p = (N + 2)/(N - 2)$.

1.1 Some known results and remarks

There are some results concerning properties (A1) and (A3). We first remark that the condition

$$a > -2, \quad p > \frac{N + a}{N - 2}$$

is a necessary condition for the existence of solution to problem (P). Indeed, there are no positive measurable functions u satisfying

$$-\Delta u \geq |x|^a u^p \quad \text{in} \quad \mathbb{R}^N$$

if $a \le -2$ or $1 < p \le (N+a)/(N-2)$. See e.g. [4, Theorem 3.3] (see also [9], [18, Section 8.1] for the case of $a = 0$).

On the other hand, Bae [1, Theorem 3.2] proved that the existence of a threshold with properties (A1) and (A3) hold if

- $a > -2$, $p > \dfrac{N+a}{N-2}$, and $|\cdot|^{\frac{2p+a}{p-1}}\mu \in L^\infty(\mathbb{R}^N)$.

See also e.g. [2, Proposition 3.3] for the case $a = 0$.

For a previous result concerning the property (A2), we refer the paper [12], which this research is motivated by. It treats the scalar field equation with a forcing term

$$\begin{cases} -\Delta u + u = u^p + \kappa\mu & \text{in} \quad \mathbb{R}^N, \\ u > 0 & \text{in} \quad \mathbb{R}^N, \\ u(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{S}$$

where $N \ge 2$, $p > 1$, and μ is a nontrivial Radon measure in $\mathbb{R}^N$ with compact support. In [12], under the following condition on μ:

- $G * \mu \in L^q(\mathbb{R}^N)$ for some $q \in (p, \infty]$ with $q > \dfrac{N}{2}(p-1)$, where G is the fundamental solution to the elliptic operator $-\Delta + 1$ in $\mathbb{R}^N$,

the existence of a threshold with properties (A1)–(A3) was proved for problem (S) in the case of $1 < p < p_{JL}$, where p_{JL} is the Joseph–Lundgren exponent, that is,

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - \sqrt{4N-4}} & \text{if} \quad N > 10, \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Unfortunately, the arguments in [12] depend heavily on the exponential decay of the fundamental solution G at the space infinity, and they are not applicable to our problem (P). This difference makes it difficult to control decay of solutions to problem (P), which is another main difficulty There are no previous results concerning the property (A2) of problem (P). See e.g. [13,14,16,17] for related results on problem (S).

1.2 Main results

To state our result, we first define solutions to problem (P). We let $N \ge 3$ and denote by Γ the fundamental solution to the elliptic operator $-\Delta$ in $\mathbb{R}^N$, that is

$$\Gamma(z) := \frac{1}{N(N-2)\omega_N}|z|^{-N+2} \quad \text{for} \quad z \in \mathbb{R}^N \setminus \{0\},$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

Definition 1.1. Let μ be a nontrivial Radon measure on $\mathbb{R}^N$, $a > 0$, and $\kappa > 0$.

(1) Let u be a nonnegative, measurable, finite and positive almost everywhere function in $\mathbb{R}^N$. We say that u is a solution to problem (P) if u satisfies

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x-y)|y|^a u(y)^p dy + \kappa \int_{\mathbb{R}^N} \Gamma(x-y) d\mu(y) \tag{1.2}$$

for almost all (a.a.) $x \in \mathbb{R}^N$. We also say that u is a supersolution to problem (P) if u satisfies

$$u(x) \geq \int_{\mathbb{R}^N} \Gamma(x-y)|y|^a u(y)^p dy + \kappa \int_{\mathbb{R}^N} \Gamma(x-y) d\mu(y)$$

for a.a. $x \in \mathbb{R}^N$.

(2) Let u be a solution to problem (P). We say that u is a minimal solution to problem (P) if, for any solution v to problem (P), the inequality $u(x) \leq v(x)$ holds for a.a. $x \in \mathbb{R}^N$. (Obviously, a minimal solution is uniquely determined.)

We also define a function space $L^\infty_{c,d}$ for any $c, d \in \mathbb{R}$ by

$$L^\infty_{c,d} := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) : \|f\|_{L^\infty_{c,d}} := \sup_{x \in \mathbb{R}^N \setminus \{0\}} \frac{|f(x)|}{\omega_{c,d}(x)} < \infty \right\},$$

$$\omega_{c,d}(x) := \begin{cases} |x|^c & \text{if} \quad |x| \leq 1, \\ |x|^d & \text{if} \quad |x| > 1. \end{cases}$$

Now we are ready to state our results. Our first theorem concerns properties (A1) and (A3).

Theorem 1.1. *Let $N \geq 3$, $a > -2$ and $p > (N+a)/(N-2)$. Let μ be a nontrivial Radon measure on $\mathbb{R}^N$ and assume that*

$$\Gamma * \mu \in L^\infty_{c,d} \quad \textit{for some} \quad -\frac{2+a}{p-1} < c \leq 0, \quad 2 - N \leq d < -\frac{2+a}{p-1}.$$

Then there is a constant $\kappa^* \in (0, \infty)$ with the following properties.

(i) *If $0 < \kappa < \kappa^*$, then problem* (P) *possesses a minimal solution u^κ.*

(ii) *If $\kappa > \kappa^*$, then problem* (P) *possesses no solutions.*

Furthermore, there is a constant $\kappa_* \in (0, \kappa^*]$ such that if $0 < \kappa < \kappa_*$, then u^κ belongs to $L^\infty_{c,d}$.

Existence results with two thresholds, the lower one of which divides the existence of a solution in a fixed function space, are typical (see e.g. [1,5]). It is generally open whether those constants coincide, although it is a natural question.

Our second and main theorem concerns property (A2) and completes a classification of existence and nonexistence of solutions to problem (P) under a suitable condition on the exponent p. This result also includes the coincidence of κ^* and κ_*. Set $a_- := \min\{a, 0\}$ and

$$p^*(a) := \begin{cases} 1 + \dfrac{2(2+a)}{N - 4 - a - \sqrt{(2+a)(2N-2+a)}} & \text{if} \quad N > 10 + 4a, \\ \infty & \text{otherwise}, \end{cases}$$

$$p_*(a) := 1 + \frac{2(2+a)}{N - 4 - a + \sqrt{(2+a)(2N-2+a)}}.$$

Theorem 1.2. *Assume the same condition as in Theorem* 1.1 *and $p_*(a) < p < p^*(a_-)$. Let κ^* and κ_* be as in Theorem 1.1. Then $\kappa^* = \kappa_*$ and*

(iii) *if $\kappa = \kappa^*$, then problem* (P) *possesses a unique solution u^{κ^*}. Furthermore, u^{κ^*} belongs to $L^\infty_{c,d}$.*

We remark that it is open whether problem (P) with $\kappa = \kappa^*$ possesses a solution if $p \geq p^*(a_-)$ or $1 < p \leq p_*(a)$. We do not know whether the equality $\kappa^* = \kappa_*$ holds in such cases either.

The critical exponent $p^*(a)$ appears naturally in studies of structures of radial solutions and stability of solutions to the Hénon equation $-\Delta u = |x|^a u^p$ in $\mathbb{R}^N$, and acts analogously to the Joseph–Lundgren exponent (see e.g. [6, 19]). We also remark that $p^*(0) = p_{JL}$.

2 The Kelvin transform

Our strategy of overcoming the difficulty of a bad decay property is the use of the Kelvin transform

$$u^\sharp(x) := |x|^{-N+2} u(|x|^{-2}x) \quad \text{for} \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Indeed, it suffices to make a local estimate of $u^\sharp$ around 0 for a decay estimate of u. Furthermore, a formal calculation shows that $u^\sharp$ is a solution to another Hénon equation,

$$\begin{cases} -\Delta u^\sharp = |x|^{a^\sharp}(u^\sharp)^p + \kappa\nu & \text{in} \quad \mathbb{R}^N, \\ \quad u^\sharp > 0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{P$^\sharp$}$$

Here $a^\sharp := (N-2)(p-1) - 4 - a$ and $\nu(x) = |x|^{-N-2}\mu(|x|^{-2}x)$. Rigorously, we obtain the following proposition by using a change of variables $y = |x|^{-2}x$ on the integral equation (1.2).

Proposition 2.1. *Assume the same condition as in Theorem* 1.1*. Let u be a solution to problem* (P)*. Then the following properties hold.*

(i) *$u^\sharp$ is a solution to problem* (P$^\sharp$) *in the sense of Definition* 1.1 (1)*, with $\Gamma * \nu = (\Gamma * \mu)^\sharp$. Furthermore, if u is a minimal solution to problem* (P)*, then $u^\sharp$ is a minimal solution to problem* (P$^\sharp$)*.*

(ii) *$a^\sharp$ satisfies the inequality $p > (N + a^\sharp)/(N-2)$. Furthermore, $\Gamma * \nu \in L^\infty_{c^\sharp, d^\sharp}$ with*

$$c^\sharp := -N - 2 - d > -\frac{2 + a^\sharp}{p-1}, \quad d^\sharp := -N - 2 - c < -\frac{2 + a^\sharp}{p-1}.$$

More explicitly, ν is a Radon measure in $\mathbb{R}^N$ given by

$$\nu(S) = \int_{\{y : |y|^{-2}y \in S\}} |y|^{N-2} d\mu(y)$$

for a Borel set $S \subset \mathbb{R}^N \setminus \{0\}$. Assertion (ii) implies that the same condition as in Theorem 1.1 also holds for problem (P$^\sharp$), and thus we may directly refer estimates obtained of u to obtain estimates of $u^\sharp$.

3 Proof of Theorem 1.2

Since the proof of Theorem 1.1 involves only a standard supersolution method, we omit it here. We also omit the uniqueness of a solution to problem (P) with $\kappa = \kappa^*$, since the proof is almost same as in [12].

3.1 The existence of a solution to problem (P) with $\kappa = \kappa_*$

Assume the same conditions as in Theorem 1.1. In order to prove the existence of a solution to problem (P) with $\kappa = \kappa_*$, we obtain a uniform estimate of the minimal solutions $\{u^\kappa\}_{\kappa\in(0,\kappa_*)}$ to problem (P). Our method of a uniform estimate of u^κ involves a test function method and elliptic regularity theorems. We have to reduce problem (P) into an elliptic problem on the Dirichlet space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ first. We define approximate solutions U_j^κ to problem (P) by

$$U_{-1}^\kappa \equiv 0, \quad U_j^\kappa := \Gamma * (U_{j-1}^\kappa)^p + \kappa\Gamma * \mu \quad \text{for} \quad j = 0, 1, \dots.$$

Lemma 3.1. *There is a large number j_* such that for any $\kappa \in (0, \kappa_*)$, function $w^\kappa := u^\kappa - U_{j_*}^\kappa$ belongs to $L^\infty_{0,-N+2} \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ and satisfies*

$$-\Delta w^\kappa = (w^\kappa + U_{j_*}^\kappa)^p - (U_{j_*-1}^\kappa)^p \quad \textit{in} \quad \mathbb{R}^N, \quad w^\kappa > 0 \quad \textit{in} \quad \mathbb{R}^N$$

in weak sense.

Futhermore, by considering an eigenvalue problem

$$-\Delta\phi = \lambda p(u^\kappa)^{p-1}\phi \quad \text{in} \quad \mathbb{R}^N, \quad \phi \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{E}$$

we obtain the following stability assertion.

Lemma 3.2. *For any $\kappa \in (0, \kappa_*)$, we have*

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 dx \le \int_{\mathbb{R}^N} p(u^\kappa)^{p-1}\psi^2 dx \quad \textit{for any} \quad \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{3.1}$$

The proof of Lemma 3.2 is similar to [12, Lemma 4.6]. By Lemma 3.2, we derive the following uniform integral estimate of w^κ.

Lemma 3.3. *Let $\nu \ge 1$ and assume that $\nu^2/(2\nu - 1) < p$. Then*

$$\sup_{\kappa\in(0,\kappa^*)} \int_{B(0,2)} (w^\kappa)^{\frac{2N\nu}{N-2}} dx < \infty.$$

The proof of Lemma 3.3 involves a uniform energy estimate

$$\int_{\mathbb{R}^N} |\nabla(\zeta(\overline{w}^\kappa)^\nu)|^2 dx < \infty, \quad \overline{w}^\kappa := w^\kappa + M$$

with a large constant $M > 0$ and an appropriate cutoff function ζ, using a test function method and the stability assertion Lemma 3.2 (see e.g. [8, Proposition 6] and [12, Lemmas 5.2 and 5.3] for related estimates).

In particular, if

$$\text{there is a constant } \nu \ge 1 \text{ with} \quad \frac{\nu^2}{2\nu - 1} < p \quad \text{and} \quad \frac{(N-2)(p-1)}{2N\nu} + \frac{-a_-}{N} < \frac{2}{N}, \tag{3.2}$$

then Lemma 3.3 implies that

$$\sup_{\kappa\in(0,\kappa^*)} \big\| |\cdot|^a (w^\kappa)^{p-1} \big\|_{L^q(B(0,2))} < \infty \quad \text{with} \quad \frac{1}{q} := \frac{(N-2)(p-1)}{2N\nu} + \frac{-a_-}{N} < \frac{2}{N}.$$

This together with the elliptic regularity theorem implies that

$$\sup_{\kappa\in(0,\kappa^*)} \|w^\kappa\|_{L^\infty(B(0,1))} < \infty.$$

To obtain a uniform decay estimate of w^κ, we use this estimate on $(w^\kappa)^\sharp$. As a result, we obtain

$$\sup_{\kappa\in(0,\kappa^*)} \||\cdot|^{N-2} w^\kappa\|_{L^\infty(\mathbb{R}^N\setminus B(0,1))} = \sup_{\kappa\in(0,\kappa^*)} \|(w^\kappa)^\sharp\|_{L^\infty(B(0,1))} < \infty,$$

if

$$\text{there is a constant } \nu' \geq 1 \text{ with } \quad \frac{\nu'^2}{2\nu'-1} < p \quad \text{and} \quad \frac{(N-2)(p-1)}{2N\nu'} + \frac{-(a_\sharp)_-}{N} < \frac{2}{N}. \tag{3.3}$$

Combining these estimates, we have a uniform $L^\infty_{0,-N+2}$-estimate of $\{w^\kappa\}_{\kappa\in(0,\kappa_*)}$ provided (3.2) and (3.3). Furthermore, by elementary (but very hard!) calculations, we have the following equivalences.

Lemma 3.4. (i) *The condition* (3.2) *is equivalent to* $1 < p < p^*(a_-)$.

(ii) *The condition* (3.3) *is equivalent to* $p_*(b) < p < p^*(0)$.

Thus if $p_*(b) < p < p^*(a_-)$, then the limit $u^* := U^{\kappa_*}_{j_*} + \lim_{\kappa\to\kappa_*} w^\kappa$ belongs to $L^\infty_{c,d}$ and it is a solution to problem (P) with $\kappa = \kappa_*$.

3.2 $\kappa^* = \kappa_*$

We finally assume the same conditions as in Theorem 1.2 and prove that $\kappa^* = \kappa_*$. We use the following lemmas.

Lemma 3.5. *The first eigenvalue of the eigenvalue problem* (E) *with* $\kappa = \kappa_*$ *is* 1.

Lemma 3.6. *Assertion* (3.1) *holds for any* $\kappa \in (0,\kappa^*)$, *even if* $\kappa > \kappa_*$.

The proof of Lemma 3.5 is similar to [12, Lemma 6.2]. To prove Lemma 3.6, we approximate $p(u^\kappa)^{p-1}$ by $\eta \in L^\infty_c(\mathbb{R}^N\setminus\{0\})$ from below and consider the eigenvalue problem

$$-\Delta\phi = \lambda\eta\phi \quad \text{in} \quad \mathbb{R}^N, \quad \phi \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{E_η}$$

Then a similar argument as in the proof of Lemma 3.2 derives

$$\int_{\mathbb{R}^N} \eta\psi^2 dx \leq \int_{\mathbb{R}^N} |\nabla\psi|^2 dx \quad \text{for any} \quad \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Letting $\eta \nearrow p(u^\kappa)^{p-1}$, we obtain (3.1).

Assume on the contrary that $\kappa_* < \kappa^*$ and let $\kappa \in (\kappa_*,\kappa^*)$. Then by the supersolution method, we observe that $u^\kappa \geq (\kappa/\kappa_*)u^{\kappa_*}$. Let ϕ^* be a first eigenfunction to eigenvalue problem (E) with $\kappa = \kappa_*$. We have

$$\int_{\mathbb{R}^N} |\nabla\phi^*|^2 dx = \int_{\mathbb{R}^N} p(u^{\kappa_*})^{p-1}(\phi^*)^2 dx \leq \left(\frac{\kappa_*}{\kappa}\right)^{p-1} \int_{\mathbb{R}^N} p(u^\kappa)^{p-1}(\phi^*)^2 dx < \int_{\mathbb{R}^N} |\nabla\phi^*|^2 dx,$$

which is a contradiction.

References

[1] S. Bae, *Existence of positive entire solutions of semilinear elliptic equations*, Nonlinear Anal. **71** (2009), e607–e615.

[2] S. Bae and W.-M. Ni, *Existence and infinite multiplicity for an inhomogeneous semilinear elliptic equation on* $\mathbb{R}^n$, Math. Ann. **320** (2001), 191–210.

[3] G. Bernard, *An inhomogeneous semilinear equation in entire space*, J. Differential Equations **125** (1996), 184–214.

[4] M.-F. Bidaut-Véron and S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84** (2001), 1–49.

[5] H. Chen, P. Felmer, and J. Yang, *Weak solutions of semilinear elliptic equation involving Dirac mass*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), 729–750.

[6] E. N. Dancer, Y. Du, and Z. Guo, *Finite Morse index solutions of an elliptic equation with supercritical exponent*, J. Differential Equations **250** (2011), no. 8, 3281–3310.

[7] Y. Deng, Y. Guo, and Y. Li, *Existence and decay properties of positive solutions for an inhomogeneous semilinear elliptic equation*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 301–322.

[8] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. (9) **87** (2007), 537–561 (English, with English and French summaries).

[9] B. Gidas, *Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic equations*, Lecture Notes in Pure and Appl. Math., vol. 54, Dekker, New York, 1980, pp. 255–273.

[10] N. Hirano, A. M. Micheletti, and A. Pistoia, *Multiple existence of solutions for a nonhomogeneous elliptic problem with critical exponent on* $\mathbb{R}^N$, Nonlinear Anal. **65** (2006), 501–513.

[11] K. Ishige and S. Katayama, *Supercritical Hénon type equation with a forcing term*, preprint.

[12] K. Ishige, S. Okabe, and T. Sato, *A supercritical scalar field equation with a forcing term*, J. Math. Pures Appl. **128** (2019), 183–212.

[13] ———, *Thresholds for the existence of solutions to inhomogeneous elliptic equations with general exponential nonlinearity*, Advances in Nonlinear Analysis **11** (2022), 968–992.

[14] ———, *Existence of non-minimal solutions to an inhomogeneous elliptic equation with supercritical nonlinearity*, Adv. Nonlinear Stud. **23** (2023), 2022–0073.

[15] T.-Y. Lee, *Some limit theorems for super-Brownian motion and semilinear differential equations*, Ann. Probab. **21** (1993), 979–995.

[16] Y. Naito and T. Sato, *Positive solutions for semilinear elliptic equations with singular forcing terms*, J. Differential Equations **235** (2007), 439–483.

[17] ———, *Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent*, Ann. Mat. Pura Appl. **191** (2012), 25–51.

[18] P. Quittner and P. Souplet, *Superlinear parabolic problems*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, Cham, 2019.

[19] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc. **337** (1993), 549–590.

STABILITY OF THE LOGARITHMIC SOBOLEV INEQUALITY FOR THE TSALLIS ENTROPY AND ITS APPLICATION

Takeshi Suguro[1]

Osaka Central Advanced Mathematical Institute
Osaka Metropolitan University
Osaka 558-8585, Japan

1. Introduction

We consider the Boltzmann–Shannon entropy

$$H[f] \equiv -\int_{\mathbb{R}^n} f(x)\log f(x)\,dx \tag{1.1}$$

for a nonnegative and integrable function $f:\mathbb{R}^n \to \mathbb{R}$ with $\|f\|_{L^1}=1$. We call such a function a probability density function. The entropy (1.1) was introduced by Ludwig Boltzmann in statistical mechanics and Shannon [17] in information theory. In the paper [17], Shannon considered the maximum problem of entropy (1.1) under the condition that the second moment of probability density functions is finite. This problem leads to the following inequality: For any nonnegative function $f \in L^1_2(\mathbb{R}^n)$ with $\|f\|_{L^1}=1$,

$$-\int_{\mathbb{R}^n} f(x)\log f(x)\,dx \le \frac{n}{2}\log\left(\frac{2\pi e}{n}\int_{\mathbb{R}^n}|x|^2 f(x)\,dx\right). \tag{1.2}$$

The constant $2\pi e/n$ is the best possible and is attained by the Gauss function

$$G_t(x) \equiv (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \tag{1.3}$$

for $t>0$ and $x\in\mathbb{R}^n$. Here, $\|\cdot\|_{L^p}$ denotes the $L^p(\mathbb{R}^n)$-norm, and we define the weighted Lebesgue space as $L^1_b(\mathbb{R}^n) \equiv \{f \in L^1(\mathbb{R}^n);\ |x|^b f \in L^1(\mathbb{R}^n)\}$ for $b>0$. In what follows, we call the inequality (1.2) the Shannon inequality. More generally, the Shannon inequality (1.2) is extended to any b-th moment and the logarithmic weight by Ogawa–Wakui [14] and Kubo–Ogawa–Suguro [12], respectively (see also Ogawa–Seraku [15]).

It is well-known that the Gauss function (1.3) is the fundamental solution to the heat equation:

$$\partial_t u = \Delta u, \quad t>0,\ x\in\mathbb{R}^n. \tag{1.4}$$

The Boltzmann H-theorem implies that for a nonnegative solution u to the equation (1.4),

$$\frac{d}{dt}H[u(t)] = I[u(t)] \equiv \int_{\mathbb{R}^n}\frac{1}{u(t)}|\nabla u(t)|^2 dx \ge 0, \tag{1.5}$$

where the right-hand side in (1.5) is called the Fisher information of u. Concerning the relation between the heat equation (1.4) and the Boltzmann–Shannon entropy (1.1), the dissipation estimate of a solution to the heat equation (1.4) is equivalent to the logarithmic Sobolev inequality, which implies a lower bound of the Boltzmann–Shannon entropy by the Fisher information: For any nonnegative smooth function $f\in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1}=1$,

$$\int_{\mathbb{R}^n} f(x)\log f(x)\,dx \le \frac{n}{2}\log\left(\frac{1}{2n\pi e}\int_{\mathbb{R}^n}\frac{1}{f(x)}|\nabla f(x)|^2\,dx\right). \tag{1.6}$$

[1]E-mail adress: suguro@omu.ac.jp

Similarly to the Shannon inequality (1.2), the Gauss function $G_t(x-a)$ also attains the equality for any $t>0$ and $a \in \mathbb{R}^n$. The inequality (1.6) is equivalent to the following inequality:

$$\int_{\mathbb{R}^n} f(x)\log f(x)\,dx \le t\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)|^2\,dx - \frac{n}{2}\log(4\pi e^2 t) \tag{1.7}$$

for any $t>0$. The equality is also attained by the Gauss function G_t.

In this paper, we consider the one-parameter extension of the Boltzmann–Shannon entropy (1.1) motivated by an extension of the heat equation (1.4): For $\alpha>0$ with $\alpha \neq 1$,

$$\partial_t u - \Delta u^\alpha = 0, \quad t>0,\ x\in\mathbb{R}^n. \tag{1.8}$$

In the case of $\alpha>1$, the equation (1.8) is called by the porous-medium equation, and in the case of $\alpha<1$, the equation (1.8) is called by the fast-diffusion equation. While the Boltzmann–Shannon entropy (1.1) corresponds to the heat equation (1.4), the entropy corresponding to this equation (1.8) is called the Tsallis entropy, which was first introduced by Tsallis [21] in statistical mechanics. We denote the Tsallis entropy by

$$H_\alpha[f] \equiv \frac{1}{1-\alpha}\left(\int_{\mathbb{R}^n} f(x)^\alpha\,dx - 1\right) \tag{1.9}$$

for a nonnegative function $f\in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1}=1$. This entropy appears in the nonextensive statistical mechanics, the so-called Tsallis statistical mechanics (see Suyari [20], Tsallis [21]). The Tsallis entropy is expressed using the q-logarithmic function as follows:

$$H_\alpha[f] = -\int_{\mathbb{R}^n} f(x)\ln_{2-\alpha} f(x)\,dx,$$

where $\ln_q(x)$ is the q-logarithmic function defined by $\ln_q(x)\equiv (x^{1-q}-1)/(1-q)$ for $q\neq 1$. The Tsallis entropy is a one-parameter extension of the Boltzmann–Shannon entropy (1.1). In fact, the Tsallis entropy (1.9) converges to the Boltzmann–Shannon entropy (1.1) as $\alpha\to 1$.

The logarithmic Sobolev inequality for the Tsallis entropy is also known and is expressed as follows:

Proposition 1.1. *Let* $\max\{1-2/(n+2), 1-1/n\} < \alpha < 1$ *or* $\alpha > 1$. *For any nonnegative smooth function* $f\in L^1_2(\mathbb{R}^n)$ *with* $\|f\|_{L^1}=1$,

$$-H_\alpha[f] \le \frac{1}{2(n(\alpha-1)+1)}\int_{\mathbb{R}^n}\frac{1}{f(x)}|\nabla f(x)^\alpha|^2\,dx + D_\alpha, \tag{1.10}$$

where the best possible constant

$$D_\alpha \equiv \frac{E_\alpha[U_\alpha]}{n(\alpha-1)+1} - \frac{1}{\alpha-1}.$$

Moreover, the constant D_α *is attained by* $U_\alpha(\cdot - a)$ *for any* $a\in\mathbb{R}^n$, *where* U_α *is the Zel'dovich–Kompaneets–Barenblatt (ZKB) function defined by*

$$U_\alpha(x) \equiv \left(\gamma_\alpha - \frac{\alpha-1}{2\alpha}|x|^2\right)_+^{\frac{1}{\alpha-1}}. \tag{1.11}$$

Here, $\kappa \equiv n(\alpha-1)+2$, $f_+(x)\equiv \max\{f(x),0\}$, *and*

$$\gamma_\alpha = \begin{cases} \left(\dfrac{1-\alpha}{2\alpha\pi}\right)^{\frac{n(1-\alpha)}{\kappa}} \left(\dfrac{\Gamma\left(\frac{1}{1-\alpha}-\frac{n}{2}\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}\right)^{\frac{2(1-\alpha)}{\kappa}} & \text{if } \alpha<1,\\[2ex] \left(\dfrac{\alpha-1}{2\alpha\pi}\right)^{\frac{n(\alpha-1)}{\kappa}} \left(\dfrac{\Gamma\left(\frac{\alpha}{\alpha-1}+\frac{n}{2}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}\right)^{\frac{2(\alpha-1)}{\kappa}} & \text{if } \alpha>1.\end{cases}$$

When $\alpha \to 1$, the inequality (1.10) coincides with the inequality (1.7) with $t = 1/2$ and the ZKB function (1.11) converges to the Gauss function (1.3) as $\alpha \to 1$. We note that the logarithmic Sobolev inequality (1.10) follows from the upper bound of the Lyapunov functional for an equation. For a probability density function f, we set

$$E_\alpha[f] \equiv \frac{1}{\alpha - 1}\int_{\mathbb{R}^n} f(x)^\alpha\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |x|^2 f(x)\,dx,$$

which is the Lyapunov functional for a nonlinear Fokker–Planck equation

$$\partial_t u = \Delta u^\alpha + \nabla\cdot(xu), \quad t > 0,\ x \in \mathbb{R}^n.$$

Then, the logarithmic Sobolev inequality (1.10) is equivalent to the following inequality: For any nonnegative smooth function f with $\|f\|_{L^1} = 1$,

$$E_\alpha[f] - E_\alpha[U_\alpha] \le \frac{1}{2}\int_{\mathbb{R}^n} f(x)\left|\frac{\alpha}{\alpha-1}\nabla f(x)^{\alpha-1} + x\right|^2 dx,$$

where U_α is defined by (1.11).

For $t > 0$, we define the profile

$$\mathcal{B}_\alpha(t,x) = t^{-\frac{n}{\kappa}} U_\alpha(t^{-\frac{1}{\kappa}}x).$$

We state the stability of the profile $\mathcal{B}_\alpha$ for the logarithmic Sobolev inequality (1.10) using a parameter:

Theorem 1.2 ([19])**.** *Let* $\max\{1 - 2/(n+2), 1 - 1/n\} < \alpha < 1$ *or* $\alpha > 1$. *For* $t > 0$, *suppose that a nonnegative smooth function* $f \in L^1_2(\mathbb{R}^n)$ *satisfies*

$$\|f\|_{L^1} = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} |x|^2 f(x)\,dx \le \int_{\mathbb{R}^n} |x|^2 \mathcal{B}_\alpha(t,x)\,dx = \frac{2n\alpha\gamma_\alpha}{n(\alpha-1)+2\alpha} t^{\frac{2}{\kappa}}. \tag{1.12}$$

Then it holds that

$$\begin{aligned} &\frac{t}{2(n(\alpha-1)+1)}\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)^\alpha|^2\,dx + H_\alpha[f] + D_\alpha(t) \\ &\quad \ge \phi_\alpha\big(H_\alpha[f] - H_\alpha[\mathcal{B}_\alpha(t)]; t\big), \end{aligned} \tag{1.13}$$

where

$$\phi_\alpha(s;t) \equiv \begin{cases} C_\alpha(t)^{\frac{\kappa}{n(1-\alpha)}}\left[(B_*(t) - s)^{-\frac{2(\kappa-1)}{n(1-\alpha)}} - B_*(t)^{-\frac{2(\kappa-1)}{n(1-\alpha)}}\right] - s & \text{if } \alpha < 1, \\ C_\alpha(t)^{-\frac{\kappa}{n(\alpha-1)}}\left[(B_*(t) + s)^{\frac{2(\kappa-1)}{n(\alpha-1)}} - B_*(t)^{\frac{2(\kappa-1)}{n(\alpha-1)}}\right] - s & \text{if } \alpha > 1 \end{cases}$$

and $B_*(t) \equiv \|\mathcal{B}_\alpha(t)\|^\alpha_{L^\alpha}/|\alpha - 1|$,

$$C_\alpha(t) = \frac{\|\mathcal{B}_\alpha(t)\|^\alpha_{L^\alpha}}{|\alpha-1|}\left(\frac{2(\kappa-1)}{n|\alpha-1|}\right)^{\frac{n(\alpha-1)}{\kappa}}, \text{ and } D_\alpha(t) = \frac{1}{\alpha-1}\left(\frac{\kappa\|\mathcal{B}_\alpha(t)\|^\alpha_{L^\alpha}}{2(\kappa-1)} - 1\right).$$

Furthermore, if assume that $\alpha < 2$ *and* $t = 1$ *in addition, then it holds that*

$$\frac{1}{2(n(\alpha-1)+1)} I_\alpha[f] + H_\alpha[f] + D_\alpha \ge \phi_\alpha\big(c_\alpha^{-2}\|f - U_\alpha\|^2_{L^1}; 1\big), \tag{1.14}$$

where

$$c_\alpha = \left(\frac{2}{\alpha}\int_{\mathbb{R}^n} U_\alpha(x)^{2-\alpha}\,dx\right)^{\frac{1}{2}}.$$

The inequality (1.13) is the estimate of the deficit term of the one-parameter extension of the logarithmic Sobolev inequality (1.10), and the inequality (1.14) implies the L^1-stability of the optimizer U_α for the logarithmic Sobolev inequality (1.10). For the Sobolev inequality, Brezis–Lieb [4] estimated the deficit term, and Bianchi–Egnell [2] showed the stability result (see also

Dolbeault–Esteban–Figalli–Frank–Loss [7]). As a corollary of Theorem 1.2, we obtain the deficit estimate of the logarithmic Sobolev inequality (1.7):

Corollary 1.3. *For $t > 0$, suppose that a nonnegative function $f \in L^1_2(\mathbb{R}^n)$ satisfies*

$$\|f\|_{L^1} = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} |x|^2 f(x)\, dx \le 2nt.$$

Then it holds that

$$t\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)|^2\, dx - \int_{\mathbb{R}^n} f(x)\log f(x)\, dx - \frac{n}{2}\log(4\pi e^2 t) \ge \phi\left(\int_{\mathbb{R}^n} f(x)\log\frac{f(x)}{G_t(x)}\, dx\right), \tag{1.15}$$

where for $s \ge 0$,

$$\phi(s) \equiv \frac{n}{2}\left(e^{\frac{2}{n}s} - 1 - \frac{2}{n}s\right). \tag{1.16}$$

Remark. The functional inside of ϕ in the inequality (1.15) is called the Kullback–Leibler divergence, which is nonnegative according to the Shannon inequality (1.2). We note that the function ϕ defined in (1.16) is a nonnegative function such that $\phi(0) = 0$. Furthermore, it holds that $\phi(s) \ge s^2/2n$ for any $s \in \mathbb{R}$. This fact and the inequality (1.15) imply that

$$t\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)|^2\, dx - \int_{\mathbb{R}^n} f(x)\log f(x)\, dx - \frac{n}{2}\log(4\pi e^2 t) \ge \frac{1}{2n}\left(\int_{\mathbb{R}^n} f(x)\log\frac{f(x)}{G_t(x)}\, dx\right)^2.$$

Between the Kullback–Leibler divergence and L^1-norm, the Csiszár–Kullback–Pinsker inequality holds as follows:

$$\|f-g\|_{L^1}^2 \le 2\int_{\mathbb{R}^n} f(x)\log\frac{f(x)}{g(x)}\, dx \tag{1.17}$$

holds for any nonnegative functions $f, g \in L^1(\mathbb{R}^n)$. By this inequality (1.17), we obtain

$$t\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)|^2\, dx - \int_{\mathbb{R}^n} f(x)\log f(x)\, dx - \frac{n}{2}\log(4\pi e^2 t) \ge \frac{1}{8n}\|f - G_t\|_{L^1}^4. \tag{1.18}$$

When we consider the case $t = 1/2$, the inequality (1.18) implies that

$$\frac{1}{2}I[f] + H[f] - \frac{n}{2}\log(2\pi e^2) \ge \frac{1}{8n}\|f - G_{1/2}\|_{L^1}^4,$$

which coincides the result given by Dolbeault–Toscani [9]. Recently, Indrei–Kim [11] showed the estimate of the deficit term by the L^1-norm under the condition that the second moment of the probability density function is bounded. Bobkov–Gozlan–Roberto–Samson [3] has studied the estimate of the deficit term by the Wasserstein metric (see also Bez–Nakamura–Tsuji [1]).

2. Strategy of the proof of Theorem 1.2

By the scaling argument, the logarithmic Sobolev inequality (1.10) can be expressed by the following inequality:

$$-\frac{1}{1-\alpha}\log\int_{\mathbb{R}^n} f(x)^\alpha dx \le \frac{n}{2+n(\alpha-1)}\log\left(\frac{B_\alpha}{\|f\|_{L^\alpha}^\alpha}\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)^\alpha|^2\, dx\right), \tag{2.1}$$

where the best possible constant

$$B_\alpha = \begin{cases} \dfrac{1-\alpha}{2n\alpha\pi}\left[\dfrac{2\alpha - n(1-\alpha)}{2\alpha}\right]^{\frac{\kappa}{n(1-\alpha)}} \left(\dfrac{\Gamma\left(\frac{1}{1-\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha} - \frac{n}{2}\right)}\right)^{\frac{2}{n}} & \text{for } 1 - \dfrac{2}{n+2} < \alpha < 1, \\ \dfrac{\alpha-1}{2n\alpha\pi}\left[\dfrac{2\alpha}{2\alpha + n(\alpha-1)}\right]^{\frac{\kappa}{n(\alpha-1)}} \left(\dfrac{\Gamma\left(\frac{\alpha}{\alpha-1} + \frac{n}{2}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}\right)^{\frac{2}{n}} & \text{for } \alpha > 1 \end{cases} \tag{2.2}$$

We note that the left-hand side in the inequality (2.1) is called the Rényi entropy, which is also a one-parameter extension of the Boltzmann–Shannon entropy (1.1). Furthermore, the inequality (2.1) is equivalent to some of the Gagliardo–Nirenberg inequalities. In the case $\max\{1 - 2/(n+2), 1 - 1/n\} < \alpha < 1$, let $\psi \in L^{p+1}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ for $p = 1/(2\alpha - 1)$. If we put $f \equiv \psi^{2p}/\|\psi\|_{2p}^{2p}$ to the inequality (1.10), then we obtain the Gagliardo–Nirenberg inequality

$$\|\psi\|_{L^{2p}} \leq \tilde{B}_\alpha^{\frac{\theta}{2}} \|\psi\|_{L^{p+1}}^{1-\theta} \|\nabla\psi\|_{L^2}^{\theta} \quad \text{with } \theta = \frac{n(p-1)}{p(n+2-p(n-2))} = \frac{n(2\alpha-1)(1-\alpha)}{n(\alpha-1)+2\alpha}, \tag{2.3}$$

where we set

$$\tilde{B}_\alpha = \left(\frac{2\alpha}{2\alpha-1}\right)^2 B_\alpha$$

and B_α is defined by (2.2). When $\alpha > 1$, we take $\psi \in L^{2p}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ for $p = 1/(2\alpha - 1)$. Putting $f \equiv \psi^{2p}/\|\psi\|_{2p}^{2p}$ to the inequality (1.10), we obtain the Gagliardo–Nirenberg inequality

$$\|\psi\|_{L^{p+1}} \leq \tilde{B}_\alpha^{\frac{\theta}{2}} \|\psi\|_{L^{2p}}^{1-\theta} \|\nabla\psi\|_{L^2}^{\theta} \quad \text{with } \theta = \frac{n(1-p)}{(p+1)(n-p(n-2))} = \frac{n(2\alpha-1)(\alpha-1)}{2\alpha(n(\alpha-1)+1)}. \tag{2.4}$$

The best constants of these Gagliardo–Nirenberg inequalities were derived by Del Pino–Dolbeault [6] (see also Dolbeault–Toscani [8]).

We apply the following self-improvements to the Gagliardo–Nirenberg inequalities (2.3) and (2.4):

- Let $0 < \sigma < 1$. If $A, B, C \geq 0$ satisfies $A^\sigma B^{1-\sigma} \geq C$,
$$A + B - \delta C \geq \phi(B_* - B),$$
where $\delta \equiv \sigma^{-\sigma}(1-\sigma)^{-(1-\sigma)}$,
$$\phi(s) \equiv C^{\frac{1}{\sigma}}[(B_* - s)^{1-\frac{1}{\sigma}} - B_*^{1-\frac{1}{\sigma}}] - s, \text{ and } B_* \equiv C\left(\frac{1-\sigma}{\sigma}\right)^{\sigma}.$$
- Let $0 < \tau < 1$. If $A, B, C \geq 0$ satisfies $A^{-\tau}B^{1+\tau} \leq C$,
$$A - B + \delta C \geq \psi(B - B_*),$$
where $\delta \equiv \tau^\tau(1+\tau)^{-(1+\tau)}$,
$$\psi(s) \equiv C^{-\frac{1}{\tau}}[(B_* + s)^{1+\frac{1}{\tau}} - B_*^{1+\frac{1}{\tau}}] - s, \text{ and } B_* = C\left(\frac{\tau}{1+\tau}\right)^{\tau}.$$

By choosing suitable coupling (A, B, σ) or (A, B, τ), we obtain the inequality (1.13).

In order to prove the stability result of Theorem 1.2, we consider the Csiszár–Kullback–Pinsker inequality for the Tsallis entropy. We define the Bregman divergence by

$$E_\alpha[f|g] \equiv \frac{1}{\alpha-1}\int_{\mathbb{R}^n} f(x)^\alpha\,dx - \frac{1}{\alpha-1}\int_{\mathbb{R}^n} g(x)^\alpha\,dx - \frac{\alpha}{\alpha-1}\int_{\mathbb{R}^n} g(x)^{\alpha-1}(f(x)-g(x))\,dx. \tag{2.5}$$

for $\alpha > 0$ with $\alpha \neq 1$. Then the following inequality holds:

Proposition 2.1 ([5])**.** *Let $0 < \alpha < 2$ with $\alpha \neq 1$. Suppose that a nonnegative function f be in $L^1(\mathbb{R}^n)$ with $\|f\|_{L^1} = 1$. Assume that a nonnegative function $g \in L^1(\mathbb{R}^n)$ satisfies $\|g\|_{L^1} = 1$ and*

$$C_g \equiv \left(\frac{2}{\alpha}\int_{\mathbb{R}^n} g(x)^{2-\alpha}\,dx\right)^{\frac{1}{2}} < +\infty.$$

Then it holds that

$$\|f - g\|_{L^1} \le C_g\sqrt{E_\alpha[f|g]}. \tag{2.6}$$

If we set $g = U_\alpha$, then the Bregman divergence (2.5) is written by

$$E_\alpha[f|U_\alpha] = \frac{1}{\alpha-1}\int_{\mathbb{R}^n} f(x)^\alpha\,dx - \frac{1}{\alpha-1}\int_{\mathbb{R}^n} U_\alpha(x)^\alpha\,dx - \frac{\alpha}{\alpha-1}\int_{\mathbb{R}^n} U_\alpha(x)^{\alpha-1}(f(x) - U_\alpha(x))\,dx.$$

Since we assume the condition (1.12), we see that

$$E_\alpha[f|U_\alpha] \le \frac{1}{\alpha-1}\int_{\mathbb{R}^n} f(x)^\alpha\,dx - \frac{1}{\alpha-1}\int_{\mathbb{R}^n} U_\alpha(x)^\alpha\,dx = H_\alpha[f] - H_\alpha[U_\alpha]. \tag{2.7}$$

Combining the Csiszár–Kullback–Pinsker inequality (2.6) with $g = U_\alpha$ and (2.7), we obtain

$$H_\alpha[f] - H_\alpha[U_\alpha] \ge c_\alpha^{-2}\|f - U_\alpha\|_{L^1}^2, \tag{2.8}$$

where

$$c_\alpha = \left(\frac{2}{\alpha}\int_{\mathbb{R}^n} U_\alpha(x)^{2-\alpha}\,dx\right)^{\frac{1}{2}}.$$

Thus, the inequality (1.14) follows from the inequalities (1.13) and (2.8).

3. Application to the uncertainty relation inequality

Combining the Shannon inequality (1.2) and the logarithmic Sobolev inequality (1.6), we obtain the following Cramér–Rao inequality: For any smooth probability density function $f \in L^1_2(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |x|^2 f(x)\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)|^2\,dx\right)^{\frac{1}{2}} \ge n. \tag{3.1}$$

The constant n is the best possible, and the equality is attained by the Gauss function (1.3). For $\psi \in H^1(\mathbb{R}^n)$, if we put $f \equiv \psi^2$ to the Cramér–Rao inequality (3.1), then we obtain the Heisenberg uncertainty relation inequality:

$$\left(\int_{\mathbb{R}^n} |x|^2|\psi(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n} |\nabla\psi(x)|^2\,dx\right)^{\frac{1}{2}} \ge \frac{n}{2}\|\psi\|_{L^2}^2.$$

In the paper [18], we showed the Shannon inequality for the Rényi entropy as follows:

$$\frac{1}{1-\alpha}\log\int_{\mathbb{R}^n} f(x)^\alpha\,dx \le \frac{n}{2}\log\left(C_\alpha\int_{\mathbb{R}^n} |x|^2 f(x)\,dx\right), \tag{3.2}$$

where the optimal constant C_2 is given by

$$C_\alpha = \frac{1}{n}\|U_\alpha\|_{L^\alpha}^{\frac{\kappa\alpha}{n(1-\alpha)}}.$$

By combining the inequalities (2.1) and (3.2), we obtain a one-parameter extension of the Cramér–Rao inequality:

$$\left(\int_{\mathbb{R}^n} |x|^2 f(x)\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n} \frac{1}{f(x)}|\nabla f(x)^\alpha|^2\,dx\right)^{\frac{1}{2}} \ge n\int_{\mathbb{R}^n} f(x)^\alpha\,dx. \tag{3.3}$$

We note that $B_\alpha = 1/(n^2C_\alpha)$. Ozawa–Yuasa [16] considered the estimate of the deficit term of Cramér–Rao inequality (3.1). Recently, Fathi [10] and McCurdy–Venkatraman [13] showed the stability of the Cramér–Rao inequality (3.1). As an application of the inequality (1.13), we obtain the stability of the inequality (3.3):

Theorem 3.1 ([19]). *Let $\alpha \geq \max\{1-2/(n+2), 1-1/n\}$. For $t > 0$, suppose that a nonnegative smooth function $f \in L^1_2(\mathbb{R}^n)$ satisfies the condition* (1.12). *Then it holds that*

$$\left(\int_{\mathbb{R}^n} |x|^2 f(x)\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{f(x)} |\nabla f(x)^\alpha|^2\,dx\right)^{\frac{1}{2}}$$
$$\geq n\|f\|^\alpha_{L^\alpha} \left(1 + \frac{2(n(\alpha-1)+1)}{n\|f\|^\alpha_{L^\alpha}} \phi_\alpha\big(H_\alpha[f] - H_\alpha[\mathcal{B}_\alpha]; t\big)\right)^{\min\{\frac{1}{2}, \frac{1}{\kappa}\}},$$

where ϕ_α is defined in Theorem 1.2.

Appendix A. Shannon–Khinchin axiom

Shannon [17] defined the entropy for the discrete probability distribution as follows:

$$S_1^{(N)}(p_1, \ldots, p_N) \equiv -\sum_{j=1}^N p_j \log p_j \quad \text{for } p_j \geq 0 \text{ with } \sum_{j=1}^N p_j = 1. \tag{A.1}$$

For $N \in \mathbb{N}$, let Δ_N be the set of the all discrete probability distributions:

$$\Delta_N \equiv \left\{ (p_1, p_2, \ldots, p_N);\ p_i \geq 0, \quad \sum_{j=1}^N p_j = 1 \right\}.$$

The Shannon entropy S_1 is uniquely determined by the following Shannon–Khinchin axiom:

(1) Continuity: For any $N \in \mathbb{N}$, the function $S_1^{(N)}(p_1, p_2, \ldots, p_N)$ is continuous with respect to $(p_1, p_2, \ldots, p_N) \in \Delta_N$.
(2) Maximality: For any $N \in \mathbb{N}$ and $(p_1, p_2, \ldots, p_N) \in \Delta_N$,
$$S_1^{(N)}(p_1, p_2, \ldots, p_N) \leq S_1^{(N)}\left(\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}\right).$$
(3) Shannon additivity: For any $N \in \mathbb{N}$ and $(p_1, p_2, \ldots, p_N) \in \Delta_N$, and $0 < \theta < 1$,
$$S_1^{(N+1)}(\theta p_1, (1-\theta)p_1, p_2, \ldots, p_N) = S_1^{(N)}(p_1, p_2, \ldots, p_N) + p_1 S_1^{(2)}(\theta, 1-\theta).$$
(4) Expandability: For any $N \in \mathbb{N}$ and $(p_1, p_2, \ldots, p_N) \in \Delta_N$,
$$S_1^{(N+1)}(p_1, p_2, \ldots, p_N, 0) = S_1^{(N)}(p_1, p_2, \ldots, p_N).$$

The Tsallis entropy for discrete probability distributions is defined by

$$S_\alpha^{(N)}(p_1, \ldots, p_N) \equiv \frac{1}{1-\alpha}\left(\sum_{j=1}^N p_j^\alpha - 1\right) \tag{A.2}$$

for $(p_1, \ldots, p_N) \in \Delta_N$. Then the Tsallis entropy $S_\alpha^{(N)}$ is also uniquely determined by the Shannon–Khinchin axiom replacing (3) with the following:

(3') Generalized Shannon additivity: For any $N \in \mathbb{N}$ and $(p_1, p_2, \ldots, p_N) \in \Delta_N$, and $0 < \theta < 1$,
$$S_\alpha^{(N+1)}(\theta p_1, (1-\theta)p_1, p_2, \ldots, p_N) = S_\alpha^{(N)}(p_1, p_2, \ldots, p_N) + p_1^\alpha S_\alpha^{(2)}(\theta, 1-\theta).$$

In this sense, the Tsallis entropy (A.2) are generalizations of the Boltzmann–Shannon entropy (A.1), respectively.

References

[1] Bez, N., Nakamura, S., and Tsuji, H., *Stability of hypercontractivity, the logarithmic Sobolev inequality, and Talagrand's cost inequality*, arXiv:2201.12478.

[2] Bianchi, G. and Egnell, H., *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), 18–24.

[3] Bobkov, S. G., Gozlan, N., Roberto, C., and Samson, P.-M., *Bounds on the deficit in the logarithmic Sobolev inequality*, J. Funct. Anal. **267** (2014), 4110–4138.

[4] Brezis, H. and Lieb, E. H., *Sobolev inequalities with remainder terms*, J. Funct. Anal. **62** (1985), 73–86.

[5] Carrillo, J. A., Jüngel, A., Markowich, P. A., Toscani, G., and Unterreiter, A., *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math. **133** (2001), 1–82.

[6] Del Pino, M. and Dolbeault, J., *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. **81** (2002), 847-875.

[7] Dolbeault, J., Esteban, M. J., Figalli, A., Frank, R. L., and Loss, M., *Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence*, arXiv:2209.08651.

[8] Dolbeault, J. and Toscani, G., *Improved interpolation inequalities, relative entropy and fast diffusion equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 917-934.

[9] Dolbeault, J. and Toscani, G., *Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities*, Int. Math. Res. Not. IMRN (2016), 473–498.

[10] Fathi, M., *A short proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality*, Nonlinear Anal. **210** (2021), Paper No. 112403, 3pp.

[11] Indrei, E. and Kim, D., *Deficit estimates for the logarithmic Sobolev inequality*, Differential Integral Equations **34** (2021), 437–466.

[12] Kubo, H., Ogawa, T., and Suguro, T., *Beckner type of the logarithmic Sobolev and a new type of Shannon's inequalities and an application to the uncertainty principle*, Proc. Amer. Math. Soc. **147** (2019), 1511–1518.

[13] McCurdy, S. and Venkatraman, R., *Quantitative stability for the Heisenberg-Pauli-Weyl inequality*, Nonlinear Anal. **202** (2021), Paper No. 112147, 13pp.

[14] Ogawa, T. and Wakui, H., *Non-uniform bound and finite time blow up for solutions to a drift-diffusion equation in higher dimensions*, Anal. Appl. **14** (2016), 145-183.

[15] Ogawa, T. and Seraku, K., *Logarithmic Sobolev and Shannon's inequalities and an application to the uncertainty principle*, Comm. Pure Appl. Anal. **17** (2018), 1651-1669.

[16] Ozawa, T. and Yuasa, K., *Uncertainty relations in the framework of equalities*, J. Math. Anal. Appl. **445** (2017), 998–1012.

[17] Shannon, C. E., *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379-423, 623-656.

[18] Suguro, T., *Shannon's inequality for the Rényi entropy and an application to the uncertainty principle*, J. Funct. Anal. **283** (2022), 26pp.

[19] Suguro, T., *Stability of the logarithmic Sobolev inequality and uncertainty principle for the Tsallis entropy*, preprint.

[20] Suyari, H., *Generalization of Shannon-Khinchin axioms to nonextensive systems and the uniqueness theorem for the nonextensive entropy*, IEEE Trans. Inform. Theory **50** (2004), 1783–1787.

[21] Tsallis, C., *Possible generalization of Boltzmann-Gibbs statistics*, J. Statist. Phys. **52** (1988), 479–487.

Uniqueness of positive solution to some coupled cooperative variational elliptic systems

Junping Shi*

Department of Mathematics, College of William & Mary, Williamsburg, VA 23187-8795, USA

Email: jxshix@wm.edu

Abstract

The uniqueness of positive solutions to some semilinear elliptic systems with variational structure arising from mathematical physics is proved. The key ingredient of the proof is the oscillatory behavior of solutions to linearized equations for cooperative semilinear elliptic systems of two equations on one-dimensional domains, and it is shown that the stability of the positive solutions for such semilinear system is closely related to the oscillatory behavior.

Keywords: Semilinear elliptic systems, uniqueness, cooperative, variational

1 Introduction

Systems of nonlinear elliptic type partial differential equations arise from many models in mathematical physics, such as the nonlinear static Chern-Simons-Higgs equations of classical field theory [5, 7, 8, 9, 25], and standing wave solutions of coupled nonlinear Schödinger equations from Bose-Einstein condensation [1, 6, 19, 24]. In the case of two interacting particles or waves, the static equation is in form

$$\Delta u_1 + f(u_1, u_2) = 0, \quad \Delta u_2 + g(u_1, u_2) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$. While the existence of positive solutions to (1.1) have been obtained through various variational or other methods, the uniqueness or exact multiplicity of solutions have been mostly open. Here we provide a rather general approach of proving the uniqueness of positive solution to the system in one dimensional space [3].

Here we provide a rather general approach of proving the uniqueness of positive solution to the system in one dimensional space. To achieve that, we prove some general properties of associated linearized system which resembles the classic Sturm comparison principle, and with these properties, for some important systems with a variational structure, we prove the uniqueness of the solution of

$$\begin{cases} u_1'' + f(u_1, u_2) = 0, & x \in \mathbb{R}, \\ u_2'' + g(u_1, u_2) = 0, & x \in \mathbb{R}, \\ u_1(x) > 0, \;\; u_2(x) > 0, & x \in \mathbb{R}, \\ u_1(x) \to 0, \;\; u_2(x) \to 0, & |x| \to \infty, \end{cases} \tag{1.2}$$

*Partially supported by NSF grants DMS-1022648 and DMS-1313243.

or the solution of the related Dirichlet boundary value problem:

$$
\begin{cases} u_1'' + f(u_1, u_2) = 0, & -R < x < R, \\ u_2'' + g(u_1, u_2) = 0, & -R < x < R, \\ u_1(x) > 0, \;\; u_2(x) > 0, & -R < x < R, \\ u_1(\pm R) = 0, \;\; u_2(\pm R) = 0. \end{cases} \tag{1.3}
$$

2 One-dimensional Cooperative Systems

We assume that the nonlinear functions f, g in (1.2) and (1.3) satisfy

(f1) $f, g \in C^1(\mathbb{R}^2_+)$;

(f2) (Cooperativeness) Define the Jacobian of the vector field (f, g) to be

$$
J(u_1, u_2) = \begin{pmatrix} \dfrac{\partial f}{\partial u_1}(u_1, u_2) & \dfrac{\partial f}{\partial u_2}(u_1, u_2) \\ \dfrac{\partial g}{\partial u_1}(u_1, u_2) & \dfrac{\partial g}{\partial u_2}(u_1, u_2) \end{pmatrix} \equiv \begin{pmatrix} f_1(u_1, u_2) & f_2(u_1, u_2) \\ g_1(u_1, u_2) & g_2(u_1, u_2) \end{pmatrix}. \tag{2.1}
$$

Then (f, g) is said to be *cooperative* if $f_2(u_1, u_2) \geq 0$ and $g_1(u_1, u_2) \geq 0$ for $(u_1, u_2) \in \mathbb{R}^2_+$, and $f_2(u_1, u_2) > 0$ and $g_1(u_1, u_2) > 0$ for $(u_1, u_2) \in int(\mathbb{R}^2_+)$.

Under the conditions (f1) and (f2), it is well-known that a positive solution $(u_1(x), u_2(x))$ of (1.3) must be an even function in the sense that $u_i(-x) = u_i(x)$ and $u_i'(x) < 0$ for $x \in (0, R)$ (see [27]), and the symmetry properties for positive solutions to (1.2) have also been established in [4, 14] under some additional assumptions on f and g at $(u_1, u_2) = (0, 0)$. These work are natural extensions of the classical results in [15, 16] for the scalar equation since the maximum principle also holds for elliptic systems with cooperative nonlinearities [26].

Our first result is a Sturm comparison type result for positive solutions to system (1.2) or (1.3). We note that the Sturm comparison lemma can be regarded as another aspect of maximum principle in one-dimensional space. A simplified version of the classical Sturm comparison lemma is: suppose that $w_1(x)$ and $w_2(x)$ are two linear independent solutions of $w'' + q(x)w = 0$ where q is continuous on $[a, b]$, and $w_1(a) = w_1(b) = 0$, then w_2 has a zero in (a, b). A straightforward application of this lemma is for a solution of

$$
u'' + g(u) = 0, \;\; x \in (0, R), \;\; u'(0) = 0, \; u(0) = \alpha, \tag{2.2}
$$

such that $u(x) > 0$, $u'(x) < 0$ in $(0, R)$, then any solution ϕ of the linearized equation

$$
\phi'' + g'(u(x))\phi = 0, \;\; x \in (0, R) \tag{2.3}
$$

changes sign at most once in $(0, R)$ since $u'(x)$ is also a solution of (2.3), $u'(0) = 0$ and $u'(x) < 0$ in $(0, R)$. Our result for solutions of linearized equation around a positive solution to (1.2) or (1.3) resembles the one above for the scalar equation. More precisely, we have

Lemma 2.1. *Let (u_1, u_2) be a solution of the initial value problem:*

$$
\begin{cases} u_1'' + f(u_1, u_2) = 0, & x > 0, \\ u_2'' + g(u_1, u_2) = 0, & x > 0, \\ u_1'(0) = u_2'(0) = 0, \\ u_1(0) = \alpha, \;\; u_2(0) = \beta, \end{cases} \tag{2.4}
$$

where $\alpha > 0$ and $\beta > 0$, such that $u_1(x) > 0$, $u_2(x) > 0$ for $x \in (0, R)$, and $u_1'(x) < 0$ and $u_2'(x) < 0$ for $x \in (0, R]$ (when R is ∞, then $u_i'(x) < 0$ for $x \in (0, R)$), and let (ϕ_1, ψ_1) (resp.(ϕ_2, ψ_2)) be a solution of

$$
\begin{cases}
\phi'' + f_1\phi + f_2\psi = 0, & 0 < x < R,\\
\psi'' + g_1\phi + g_2\psi = 0, & 0 < x < R,\\
\phi'(0) = \psi'(0) = 0,\\
(\phi(0), \psi(0)) = (1, 0), \quad (\textit{ resp.} \ \ (0, 1)).
\end{cases}
\tag{2.5}
$$

Assume that (f, g) is cooperative as defined in (f2). Then

1. *$\phi_1(x)$ changes sign at most once, and $\psi_1(x) < 0$ for $x \in (0, R)$;*
2. *$\psi_2(x)$ changes sign at most once, and $\phi_2(x) < 0$ for $x \in (0, R)$.*

It is well-known that oscillatory properties of solutions to the linearized equation is critical for the stability and uniqueness of positive solution of semilinear elliptic equations [2, 18, 22, 23], hence the non-oscillatory property for solutions of (2.5) is very useful for the stability and uniqueness of positive solutions to (1.2) or (1.3). We remark that such property usually does not hold for higher dimensional radial Laplacian $Lu = r^{1-n}(r^{n-1}u')'$ even for the scalar case, hence the spatial dimension $n = 1$ is a critical assumption here.

Our second result is related to the stability of a positive solution to (1.2) or (1.3). It is known that for (2.2), the number of sign-changes of the solution of linearized equation is related to stability of a positive solution. If the solution of linearized equation does not change sign, then the positive solution is linearly stable; while the solution of linearized equation changes sign once, then the positive solution is linearly unstable. Here we also establish such a connection between the number of sign-changes of the solution to the linearized equation (2.5) and the stability of a positive solution of the system (1.2). More precisely we show that,

Proposition 2.2. *Let (u_1, u_2) be a solution of* (1.3) *such that $u_1'(R) < 0$ and $u_2'(R) < 0$ and let $(A_c, B_c) = (\phi_1, \psi_1) + c(\phi_2, \psi_2)$ where (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are defined in* (2.5)*. Assume that (f, g) is cooperative as defined in (f2).*

1. *(u_1, u_2) is stable if and only if for some $c > 0$, $A_c(x) > 0$, $B_c(x) > 0$ in $(0, R]$;*
2. *(u_1, u_2) is neutrally stable if and only if for some $c > 0$, $A_c(x) > 0$ and $B_c(x) > 0$ in $(0, R)$ and $A_c(R) = B_c(R) = 0$;*
3. *(u_1, u_2) is unstable if and only if for all $c \geq 0$, at least one of $A_c(x)$ or $B_c(x)$ is not positive in $(0, R)$.*

The non-oscillatory results above are proved under rather general conditions (f1) and (f2) on (f, g), and these results pave the way for the stability, non-degeneracy and uniqueness of the positive solution to (1.2) or (1.3) from a wide range of applications. Two additional structures on (f, g) would be needed for these further results: (i) the growth rate of functions f and g; and (ii) a variational structure for the vector field (f, g).

The growth rate of the functions f and g plays an important role in the qualitative behavior of the solutions to (1.2) and (1.3). Here we define several conditions on the growth rate of f and g:

(f3) (Superlinear) The vector field (f, g) is said to be *superlinear* if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 + f_2u_2 - f \geq 0, \quad g_1u_1 + g_2u_2 - g \geq 0; \tag{2.6}$$

(f3') (Strongly superlinear) The vector field (f, g) is said to be *strongly superlinear*, if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 - f \geq 0, \quad g_2u_2 - g \geq 0; \tag{2.7}$$

(f4) (Sublinear) The vector field (f, g) is said to be *sublinear* if for all $(u_1, u_2) \in \mathbb{R}_+^2$,

$$f_1 u_1 + f_2 u_2 - f \leq 0, \quad g_1 u_1 + g_2 u_2 - g \leq 0; \tag{2.8}$$

(f4') (Weakly sublinear) The vector field (f, g) is said to be *weakly sublinear*, if for all $(u_1, u_2) \in \mathbb{R}_+^2$,

$$f_1 u_1 - f \leq 0, \quad g_2 u_2 - g \leq 0. \tag{2.9}$$

We remark that all definitions above are actually for a single function $f : \mathbb{R}_+^2 \to \mathbb{R}$, but we assume that f and g have the same type of growth rate in this article. Note that under the cooperativeness assumption (f2), condition (f3') implies (f3), while condition (f4) implies (f4'), which is the reason for the "strongly" and "weakly" in the definition. On the other hand, we notice that a function f can be both weakly sublinear and superlinear. The notion of superlinear and sublinear growth rate for a uni-variable function $g : \mathbb{R}_+ \to \mathbb{R}$ was considered in [23], and definition here can be considered as a generalization of the definition in [23] to multi-variable functions. It is known that sublinear/superlinear properties are related to the stability of positive solutions of (1.3). It can be shown that when (f, g) is sublinear, then any positive solution of (1.3) is stable, while when (f, g) is superlinear, then any positive solution of (1.3) is unstable (see [10]).

The final assumption for non-degeneracy and uniqueness of positive solution is the variational structure on the vector field (f, g). Here two possible variational structure can be defined as in [11, 12]. The system (1.2) or (1.3) is a *Hamiltonian system* if there exists a differentiable function $H(u_1, u_2)$ such that

$$f(u_1, u_2) = \frac{\partial H(u_1, u_2)}{\partial u_2}, \quad \text{and} \quad g(u_1, u_2) = \frac{\partial H(u_1, u_2)}{\partial u_1}; \tag{2.10}$$

Clearly a Hamiltonian system satisfies $f_1 = g_2$. For a Hamiltonian system, if $(u_1(x), u_2(x))$ is a solution of (2.4), we define

$$H_0(x) = u_1'(x) u_2'(x) + H(u_1(x), u_2(x)), \tag{2.11}$$

then $H_0'(x) = 0$ hence $H_0(x) \equiv H_0(0)$ for $x > 0$. On the other hand, the system (1.2) or (1.3) is a *gradient system* if there exists a differentiable function $F(u_1, u_2)$ such that

$$f(u_1, u_2) = \frac{\partial F(u_1, u_2)}{\partial u_1}, \quad \text{and} \quad g(u_1, u_2) = \frac{\partial F(u_1, u_2)}{\partial u_2}. \tag{2.12}$$

Clearly a gradient system satisfies $f_2 = g_1$ hence the Jacobian matrix is symmetric and the corresponding linearized equation is self-adjoint. For a gradient system, if $(u_1(x), u_2(x))$ is a solution of (2.4), we define

$$F_0(x) = \frac{1}{2}[u_1'(x)]^2 + \frac{1}{2}[u_2'(x)]^2 + F(u_1(x), u_2(x)), \tag{2.13}$$

then $F_0'(x) = 0$ hence $F_0(x) \equiv F_0(0)$ for $x > 0$. Both the energy functions H_0 and F_0 are generalizations of the energy function $G_0(x) = \frac{1}{2}[u'(x)]^2 + G(u(x))$ for (2.2) where $G(u) = \int_0^u g(s)ds$. For the scalar equation

$$u'' + g(u) = 0, \ x \in \mathbb{R}, \ u'(0) = 0, \ \lim_{|x| \to \infty} u(x) = 0, \tag{2.14}$$

the energy function G_0 alone can guarantee the uniqueness of positive solution to (2.14), which in general is not the case for the system (1.3). But with the Hamiltonian of Gradient structure, the uniqueness of positive solution to (1.3) can be proved by combining with oscillatory property.

Our third key result is that assume (u_1, u_2) is a positive solution of (1.3), (f, g) satisfies (f1) and (f2), (f, g) is superlinear (thus (u_1, u_2) is unstable), and in addition, if (f, g) is weakly sublinear, and it is a Hamiltonian or gradient system, then (u_1, u_2) must be non-degenerate, which often suggests uniqueness. More precisely, we have

Lemma 2.3. *Let* (u_1, u_2) *be a solution of* (1.3) *such that* $u_1'(R) < 0$ *and* $u_2'(R) < 0$. *Define*

$$(2.15) \qquad c_1 = c_1(R) = \begin{cases} -\dfrac{\phi_1(R)}{\phi_2(R)}, & \text{if } \phi_1(R) > 0, \\ 0, & \text{if } \phi_1(R) \le 0; \end{cases} \quad c_2 = c_2(R) = \begin{cases} -\dfrac{\psi_1(R)}{\psi_2(R)}, & \text{if } \psi_2(R) > 0, \\ \infty, & \text{if } \psi_2(R) \le 0. \end{cases}$$

Assume that (2.4) *is a Hamiltonian system or is a gradient system, and* (f, g) *is cooperative as defined in (f2). Then*

1. *If* (f, g) *is sublinear, then* $c_1 > c_2$*, and for any* $c_2 < c < c_1$*, each of* $A_c(x)$ *and* $B_c(x)$ *is positive in* $(0, R]$*;*
2. *If* (f, g) *is superlinear and weakly sublinear, then* $c_1 < c_2$*, and for any* $c_1 < c < c_2$*, each of* $A_c(x)$ *and* $B_c(x)$ *changes sign exactly once in* $(0, R)$ *and* $A_c(R) < 0$*,* $B_c(R) < 0$*. Moreover for any* $c \ge c_2$ *or* $c \le c_1$*,* $A_c(R)B_c(R) \le 0$*.*

3 Uniqueness

Combining the cooperative and variational structure, the weakly sublinear and superlinear properties, we can prove that the positive solution to (1.3) for certain (f, g) is unique for any given $R > 0$, and it also implies the uniqueness of positive solution to (1.2) when it exists. Here we consider the following example of a Hamiltonian Schrödinger system:

$$(3.1) \qquad \begin{cases} u_1'' - u_1 + h_2(u_2) = 0, & x \in (0, R), \\ u_2'' - u_2 + h_1(u_1) = 0, & x \in (0, R), \\ u_1(x) > 0, \;\; u_2(x) > 0, & x \in (0, R), \\ u_1'(0) = u_2'(0) = 0, \;\; u_1(R) = u_2(R) = 0, \end{cases}$$

and the ground state solutions satisfy

$$(3.2) \qquad \begin{cases} u_1'' - u_1 + h_2(u_2) = 0, & x \in (0, \infty), \\ u_2'' - u_2 + h_1(u_1) = 0, & x \in (0, \infty), \\ u_1(x) > 0, \; u_2(x) > 0, \; u_1'(x) < 0, \; u_2'(x) < 0, & x \in (0, \infty), \\ u_1'(0) = u_2'(0) = 0. \end{cases}$$

Here we assume that for $i = 1, 2$,

$$(3.3) \qquad h_i(0) = 0, \quad h_i'(u_i) > 0, \quad \text{and} \quad h_i'(u_i)u_i - h_i(u_i) > 0 \text{ for } \; u_i > 0;$$

and

$$(3.4) \qquad h_i'(0) = 0, \quad \lim_{u_i \to \infty} h_i'(u_i) = \infty.$$

Notice that (3.3) implies that (f, g) is superlinear but not strongly superlinear. Also (f, g) is weakly sublinear.

According to the signs of f and g, we define the following regions in $\mathbb{R}^2_+$:

$$(3.5) \qquad \begin{aligned} I &= \{(u_1, u_2) \in \mathbb{R}^2_+ : f(u_1, u_2) > 0, g(u_1, u_2) > 0\}, \\ II &= \{(u_1, u_2) \in \mathbb{R}^2_+ : f(u_1, u_2) < 0, g(u_1, u_2) < 0\}, \\ III &= \{(u_1, u_2) \in \mathbb{R}^2_+ : f(u_1, u_2) < 0, g(u_1, u_2) > 0\}, \\ IV &= \{(u_1, u_2) \in \mathbb{R}^2_+ : f(u_1, u_2) > 0, g(u_1, u_2) < 0\}. \end{aligned}$$

Since we assume that h_i satisfies (3.3) and (3.4), then the curves $f(u_1, u_2) = 0$ and $g(u_1, u_2) = 0$ are monotone ones, and they have a unique intersection point (u_1^*, u_2^*). For $(\alpha, \beta) \in II \cup III \cup IV$, $u_1' > 0$ or $u_2' > 0$ in $(0, \delta)$, hence it cannot be a solution of (3.1). For $(\alpha, \beta) \in I$, $u_1' < 0$ and $u_2' < 0$ in $(0, \delta)$. We recall $R = R(\alpha, \beta)$ to be the right endpoint of the maximal interval $(0, R(\alpha, \beta))$ so that $u_i(x) > 0$ and $u_i'(x) < 0$ in $(0, R(\alpha, \beta))$, $i = 1, 2$. We partition I into the following classes:

$$
\begin{aligned}
\mathcal{B} &= \{(\alpha,\beta) \in I : R < \infty, u_1(R) = 0, u_1'(R) < 0, u_2(R) > 0, u_2'(R) < 0\},\\
\mathcal{G} &= \{(\alpha,\beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) = 0, u_2(R) > 0, u_2'(R) < 0\},\\
\mathcal{R} &= \{(\alpha,\beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) < 0, u_2(R) = 0, u_2'(R) < 0\},\\
\mathcal{Y} &= \{(\alpha,\beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) < 0, u_2(R) > 0, u_2'(R) = 0\},\\
\mathcal{S} &= \{(\alpha,\beta) \in I : R < \infty, u_1(R) = 0, u_1'(R) < 0, u_2(R) = 0, u_2'(R) < 0\},\\
\mathcal{Q} &= \{(\alpha,\beta) \in I : R = \infty, \lim_{x\to\infty} u_1(x) = \lim_{x\to\infty} u_2(x) = 0\},\\
\mathcal{P} &= I \setminus (\mathcal{B} \cup \mathcal{G} \cup \mathcal{R} \cup \mathcal{Y} \cup \mathcal{S} \cup \mathcal{Q}).
\end{aligned}
\tag{3.6}
$$

It is clear that if $(\alpha, \beta) \in \mathcal{S}$, then the corresponding solution (u_1, u_2) is a solution of (3.1), while each element in $\mathcal{Q}$ defines a ground state solution in the whole space.

We also define

$$\hat{R} = \hat{R}(\alpha, \beta) = \sup\{r > 0 : u_1(x) > 0, u_2(x) > 0, x \in (0, r)\} \geq R(\alpha, \beta).$$

For $\hat{R}$, we define

$$
\begin{aligned}
\mathcal{U} &= \{(\alpha,\beta) \in \mathbb{R}_+^2 : \hat{R} < \infty, u_1 > 0, u_2 > 0, x \in (0, \hat{R}),\ u_1(\hat{R}) > 0, u_2(\hat{R}) = 0\},\\
\mathcal{V} &= \{(\alpha,\beta) \in \mathbb{R}_+^2 : \hat{R} < \infty, u_1 > 0, u_2 > 0, x \in (0, \hat{R}),\ u_1(\hat{R}) = 0, u_2(\hat{R}) > 0\}.
\end{aligned}
\tag{3.7}
$$

Then we can prove the uniqueness of the solution to (3.1) and (3.2) by the following steps:

1. $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}_+^2$ such that $\mathcal{U} \supset \overline{III}$ and a portion of I and II adjacent to III, and $\mathcal{V} \supset \overline{IV}$ and a portion of I and II adjacent to IV.
2. Suppose that $(\alpha_0, \beta_0) \in \mathcal{S}$, then $(\alpha, \beta_0) \in \mathcal{U}$ for $0 < \alpha < \alpha_0$, and $(\alpha_0, \beta) \in \mathcal{U}$ for $\beta > \beta_0$; $(\alpha, \beta_0) \in \mathcal{V}$ for $\alpha > \alpha_0$, and $(\alpha_0, \beta) \in \mathcal{V}$ for any $0 < \beta < \beta_0$.
3. For $\alpha > 0$, define

   $$\Phi_1(\alpha) = \inf\{\beta > 0 : (\alpha, \beta) \in \mathcal{V}\}, \quad \Phi_2(\alpha) = \sup\{\beta > 0 : (\alpha, \beta) \in \mathcal{U}\}. \tag{3.8}$$

   Then Φ_i $(i = 1, 2)$ are well-defined. Moreover there exists $\alpha_* > 0$ such that for $\alpha \geq \alpha_*$, $\Phi_1(\alpha) = \Phi_2(\alpha) \equiv \Phi(\alpha)$, where $\Phi : (\alpha_*, \infty) \to \mathbb{R}_+$ is a continuously differentiable, strictly increasing function. Moreover $\mathcal{S} = \{(\alpha, \Phi(\alpha)) : \alpha > \alpha_*\}$ are the initial value with crossing solutions, and $\mathcal{Q} = \{(\alpha_*, \Phi(\alpha_*))\}$ is the initial value for ground state solution.
4. Now all crossing solutions are on the curve defined by $R(\alpha) = R(\alpha, \Phi(\alpha))$. We prove that $R(\alpha)$ is strictly decreasing. We differentiate $u_i(R(\alpha); \alpha, \Phi(\alpha)) = 0$ with respect to α for $i = 1, 2$, then

   $$
   \begin{aligned}
   u_1'(R(\alpha))R'(\alpha) + \phi_1(R(\alpha)) + \Phi'(\alpha)\phi_2(R(\alpha)) = 0,\\
   u_2'(R(\alpha))R'(\alpha) + \psi_1(R(\alpha)) + \Phi'(\alpha)\psi_2(R(\alpha)) = 0,
   \end{aligned}
   \tag{3.9}
   $$

   where (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are fundamental solutions of linearized equations defined in (2.5). Let $c = \Phi'(\alpha) > 0$, then (3.9) is equivalent to

   $$u_1'(R(\alpha))R'(\alpha) = -A_c(R(\alpha)), \quad u_2'(R(\alpha))R'(\alpha) = -B_c(R(\alpha)), \tag{3.10}$$

where $(A_c, B_c) = (\phi_1, \psi_1) + c(\phi_2, \psi_2)$. Since $u_1'(R(\alpha)) < 0$ and $u_2'(R(\alpha)) < 0$, then (3.10) implies that $A_c(R(\alpha))B_c(R(\alpha)) > 0$. Then from Lemma 2.3, $c_1 < c < c_2$, $A_c(R(\alpha)) < 0$ and $B_c(R(\alpha)) < 0$, which implies $R'(\alpha) < 0$. The monotonicity of $R(\alpha)$ implies the uniqueness of positive solution to (1.3) for a given $R > 0$.

5. From the existence theory, for any $R > 0$, the equation has a positive solution, so the range of $R(\alpha)$ is $(0, \infty)$, and $\lim_{\alpha \to \alpha_*^+} R(\alpha) = \infty$, thus $(\alpha_*, \Phi(\alpha_*)) \in \mathcal{Q}$ (ground state). We can prove the ground state is unique.

This proves the following uniqueness result:

Theorem 3.1. *Suppose that h_i, $(i = 1, 2)$, satisfy* (3.3) *and* (3.4)*, then for any $R > 0$,* (3.1) *has a unique positive solution $(u_1(x; R), u_2(x; R))$. If $R_1 > R_2$, then $u_1(0; R_1) < u_1(0; R_2)$ and $u_2(0; R_1) < u_2(0; R_2)$. Moreover* (3.2) *has a unique solution (U_1, U_2).*

An example for Theorem 3.1 is $f(u_1, u_2) = -u_1 + u_2^q$, $g(u_1, u_2) = -u_2 + u_1^p$, where $p, q > 1$ (see [13, 14, 17]), and another example that we can prove the uniqueness is a gradient system $f(u_1, u_2) = -bu_1 + u_1u_2$, $g(u_1, u_2) = -cu_2 + u_1^2/2$, where $b, c > 0$ (see [20, 21, 28]). The general approach described above can be applied to prove the uniqueness of positive solution of (1.3) as long as (f, g) is (i) cooperative, (ii) weakly sublinear and superlinear, and (iii) Hamiltonian or gradient.

References

[1] A. Ambrosetti and E. Colorado. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc. (2)*, 75(1):67–82, 2007.

[2] W. O. Amrein, A. M. Hinz, and D. B. Pearson, editors. *Sturm-Liouville theory: Past and present*. Birkhäuser Verlag, Basel, 2005.

[3] Y.-L. An, J.-L. Chern, and J.-P. Shi. Uniqueness of positive solutions to some coupled cooperative variational elliptic systems. *Trans. Amer. Math. Soc.*, 370(7):5209–5243, 2018.

[4] J. Busca and B. Sirakov. Symmetry results for semilinear elliptic systems in the whole space. *J. Differential Equations*, 163(1):41–56, 2000.

[5] H. Chan, C.-C. Fu, and C.-S. Lin. Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation. *Comm. Math. Phys.*, 231(2):189–221, 2002.

[6] Z.-J. Chen and W.-M. Zou. Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent. *Arch. Ration. Mech. Anal.*, 205(2):515–551, 2012.

[7] Z.-Y. Chen and J.-L. Chern. Uniqueness of topological multivortex solutions in the Maxwell-Chern-Simons model. *J. Funct. Anal.*, 270(6):2073–2125, 2016.

[8] J.-L. Chern, Z.-Y. Chen, and C.-S. Lin. Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles. *Comm. Math. Phys.*, 296(2):323–351, 2010.

[9] J.-L. Chern and S.-G. Yang. Evaluating solutions on an elliptic problem in a gravitational gauge field theory. *J. Funct. Anal.*, 265(7):1240–1263, 2013.

[10] R.-H. Cui, P. Li, J.-P. Shi, and Y.-W. Wang. Existence, uniqueness and stability of positive solutions for a class of semilinear elliptic systems. *Topol. Methods Nonlinear Anal.*, 42(1):91–104, 2013.

[11] D. G. de Figueiredo. Semilinear elliptic systems. In *Nonlinear functional analysis and applications to differential equations (Trieste, 1997)*, pages 122–152. World Sci. Publ., River Edge, NJ, 1998.

[12] D. G. de Figueiredo. Semilinear elliptic systems: existence, multiplicity, symmetry of solutions. In *Handbook of differential equations: stationary partial differential equations. Vol. V*, Handb. Differ. Equ., pages 1–48. Elsevier/North-Holland, Amsterdam, 2008.

[13] D. G. de Figueiredo and P. L. Felmer. On superquadratic elliptic systems. *Trans. Amer. Math. Soc.*, 343(1):99–116, 1994.

[14] D. G. De Figueiredo and J.-F. Yang. Decay, symmetry and existence of solutions of semilinear elliptic systems. *Nonlinear Anal.*, 33(3):211–234, 1998.

[15] B. Gidas, W.-M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[16] B. Gidas, W.-M. Ni, and L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. In *Mathematical analysis and applications, Part A*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 369–402. Academic Press, New York, 1981.

[17] J. Hulshof and R. van der Vorst. Differential systems with strongly indefinite variational structure. *J. Funct. Anal.*, 114(1):32–58, 1993.

[18] M.-K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.

[19] T.-C. Lin and J.-C. Wei. Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^n$, $n \leq 3$. *Comm. Math. Phys.*, 255(3):629–653, 2005.

[20] O. Lopes. Stability of solitary waves of some coupled systems. *Nonlinearity*, 19(1):95–113, 2006.

[21] O. Lopes. Uniqueness of a symmetric positive solution to an ODE system. *Electron. J. Differential Equations*, pages No. 162, 8, 2009.

[22] T.-C. Ouyang and J.-P. Shi. Exact multiplicity of positive solutions for a class of semilinear problems. *J. Differential Equations*, 146(1):121–156, 1998.

[23] T.-C. Ouyang and J.-P. Shi. Exact multiplicity of positive solutions for a class of semilinear problem. II. *J. Differential Equations*, 158(1):94–151, 1999.

[24] B. Sirakov. Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. *Comm. Math. Phys.*, 271(1):199–221, 2007.

[25] J. Spruck and Y.-S. Yang. The existence of nontopological solitons in the self-dual Chern-Simons theory. *Comm. Math. Phys.*, 149(2):361–376, 1992.

[26] G. Sweers. Strong positivity in $C(\overline{\Omega})$ for elliptic systems. *Math. Z.*, 209(2):251–271, 1992.

[27] W. C. Troy. Symmetry properties in systems of semilinear elliptic equations. *J. Differential Equations*, 42(3):400–413, 1981.

[28] A. C. Yew, A. R. Champneys, and P. J. McKenna. Multiple solitary waves due to second-harmonic generation in quadratic media. *J. Nonlinear Sci.*, 9(1):33–52, 1999.

A singular limit of the Kobayashi–Warren–Carter system

Jun Okamoto*

Abstract

We consider the singular limit problem of a single-well Modica–Mortola energy and the Kobayashi–Warren–Carter energy. In this study, we introduce a finer topology of sliced graph convergence of functions into the function space and derive the singular limit of a single-well Modica–Mortola energy and the Kobayashi–Warren–Carter energy energies in the sense of Gamma-convergence. The energy functional obtained as this singular limit is also shown to have the remarkable property of a minimizing function that is concave concerning the strength of jumps of a function.

1 Introduction

We consider the Kobayashi–Warren–Carter energy, which is a sum of a weighted total variation and a single-well Modica–Mortola energy. Their explicit forms are

$$E^{\varepsilon}_{\mathrm{KWC}}(u,v) := \int_{\Omega} \alpha(v)|Du| + E^{\varepsilon}_{\mathrm{sMM}}(v), \tag{1.1}$$
$$E^{\varepsilon}_{\mathrm{sMM}}(v) := \frac{\varepsilon}{2}\int_{\Omega} |\nabla v|^2 \mathrm{d}\mathcal{L}^N + \frac{1}{2\varepsilon}\int_{\Omega} F(v)\mathrm{d}\mathcal{L}^N,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with the Lebesgue measure $\mathcal{L}^N$, $\alpha \geq 0$, $\varepsilon > 0$ is a small parameter, and F is a single-well potential which takes its minimum at $v = 1$. Typical examples of α and F are $\alpha(v) = v^2$ and $F(v) = (v-1)^2$, respectively. These are the original choices in [KWC1, KWC3]. The first term in (1.1) is a weighted total variation with weight $\alpha(v)$. This energy was first introduced by [KWC1, KWC3] to model motion of grain boundaries of polycrystal which have some structures like the averaged angle of each grain. This energy is quite popular in materials science.

The gradient flow of the Kobayashi–Warren–Carter energy $E^{\varepsilon}_{\mathrm{KWC}}$ is proposed in [KWC1] (see also [KWC2, KWC3]) to model grain boundary motion when each grain has some structure. Its

*Kyoto University, okamoto.jun.8n@kyoto-u.ac.jp

explicit form is

$$\tau_1 v_t = s\Delta v + (1-v) - 2sv|\nabla v|,$$
$$\tau_0 v^2 u_t = s \operatorname{div}\left(v^2 \frac{\nabla u}{|\nabla u|} \right),$$

where τ_0, τ_1, and s are positive parameters. This system is regarded as the gradient flow of $E^\varepsilon_{\mathrm{KWC}}$ with $F(v) = (v-1)^2$, $\varepsilon = 1$, and $\alpha(v) = v^2$. Because of the presence of the singular term $\nabla u/|\nabla u|$, the meaning of the solution itself is non-trivial since, even if $v \equiv 1$, the flow is the total variation flow, and a non-local quantity determines the speed [KG]. At this moment, the well-posedness of its initial-value problem is an open question. If the second equation is replaced by

$$\tau_0 (v^2 + \delta) u_t = s \operatorname{div}\left((v^2 + \delta')\nabla u/|\nabla u| + \mu \nabla u \right)$$

with $\delta > 0$, $\delta' \geq 0$ and $\mu \geq 0$ satisfying $\delta' + \mu > 0$, the existence and large-time behavior of solutions are established in [IKY, MoSh, MoShW1, SWat, SWY, WSh] under several homogeneous boundary conditions. However, its uniqueness is only proved in a one-dimensional setting under $\mu > 0$ [IKY, Theorem 2.2]. These results can be extended to the cases of non-homogeneous boundary conditions. Under non-homogeneous Dirichlet boundary conditions, we are able to find various structural patterns of steady states; see [MoShW2].

We are interested in a singular limit of the Kobayashi–Warren–Carter energy $E^\varepsilon_{\mathrm{KWC}}$ as ε tends to zero. If we assume boundedness of $E^\varepsilon_{\mathrm{KWC}}$ for a sequence (u, v_ε) for fixed u, then v_ε tends to a unique minimum of F as $\varepsilon \to 0$ in the L^2 sense. However, if u has a jump discontinuity, its convergence is not uniform near such places, suggesting that we have to introduce a finer topology than L^2 or L^1 topology.

2 The definition of the sliced graph convergence

We next recall the notation often used in the slicing argument [FL]. Let S be a set in $\mathbb{R}^N$. Let S^{N-1} denote the unit sphere in $\mathbb{R}^N$ centered at the origin, i.e.,

$$S^{N-1} = \left\{ \nu \in \mathbb{R}^N \;\middle|\; |\nu| = 1 \right\}.$$

For a given ν, let Π_ν denote the hyperplane whose normal equals ν. In other words,

$$\Pi_\nu := \left\{ x \in \mathbb{R}^N \;\middle|\; \langle x, \nu \rangle = 0 \right\},$$

where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^N$. For $x \in \Pi_\nu$, let $S_{x,\nu}$ denote the intersection of S and the whole line with direction ν, which contains x; that is,

$$S_{x,\nu} := \left\{ x + t\nu \;\middle|\; t \in S^1_{x,\nu} \right\},$$

where

$$S^1_{x,\nu} := \left\{ t \in \mathbb{R} \;\middle|\; x + t\nu \in S \right\} \subset \mathbb{R}.$$

We also set

$$S_\nu := \left\{ x \in \Pi_\nu \;\middle|\; S_{x,\nu} \neq \emptyset \right\}.$$

For a given function f on S, we associate it with a function $f_{x,\nu}$ on $S^1_{x,\nu}$ defined by

$$f_{x,\nu}(t) := f(x + t\nu).$$

Let Ω be a bounded domain in $\mathbb{R}^N$, and $\mathcal{T}$ denote the set of all Lebesgue measurable (closed) set-valued function $\Gamma : \Omega \to 2^{\mathbb{R}}$. For $\nu \in S^{N-1}$, we consider $\Omega^1_{x,\nu} \subset \mathbb{R}$ and the (sliced) set-valued function $\Gamma_{x,\nu}$ on $\Omega^1_{x,\nu}$ defined by $\Gamma_{x,\nu}(t) = \Gamma(x + t\nu)$. Let $\overline{\Gamma_{x,\nu}}$ denote its closure defined on the closure of $\overline{\Omega^1_{x,\nu}}$. Namely, it is uniquely determined so that the graph of $\overline{\Gamma_{x,\nu}}$ equals the closure of $\operatorname{graph}\Gamma_{x,\nu}$ in $\mathbb{R} \times \mathbb{R}$. As with usual measurable functions, $\Gamma^{(1)}$ and $\Gamma^{(2)}$ belonging to $\mathcal{T}$ are identified if $\Gamma^{(1)}(z) = \Gamma^{(2)}(z)$ for $\mathcal{L}^N$-a.e. $z \in \Omega$. By Fubini's theorem, $\Gamma^{(1)}_{x,\nu}(t) = \Gamma^{(2)}_{x,\nu}(t)$ for $\mathcal{L}^1$-a.e. t for $\mathcal{L}^{N-1}$-a.e. $x \in \Omega_\nu$. With this identification, we consider its equivalence class, and we call each $\Gamma^{(1)}$, $\Gamma^{(2)}$ a representative of this equivalence class. For $\nu \in S^{N-1}$, we define the subset $\mathcal{B}_\nu \subset \mathcal{T}$ as follows: $\Gamma \in \mathcal{B}_\nu$ if, for a.e. $x \in \Omega_\nu$,

- There is a representative of $\Gamma_{x,\nu}$ such that $\overline{\Gamma_{x,\nu}} = \Gamma_{x,\nu}$ on $\Omega^1_{x,\nu}$;
- $\operatorname{graph}\overline{\Gamma_{x,\nu}}$ is compact in $\overline{\Omega^1_{x,\nu}} \times \mathbb{R}$.

We note that if $\Gamma^{(1)}, \Gamma^{(2)} \in \mathcal{B}_\nu$, then $\overline{\Gamma^{(1)}_{x,\nu}}, \overline{\Gamma^{(2)}_{x,\nu}} \in \mathcal{C}$ with $M = \overline{\Omega^1_{x,\nu}}$ by a suitable choice of representative of $\Gamma^{(1)}_{x,\nu}, \Gamma^{(2)}_{x,\nu}$, which follows from the definition.

We now introduce a metric on $\mathcal{B}_\nu$ of form

$$d_\nu\left(\Gamma^{(1)}, \Gamma^{(2)}\right) := \int_{\Omega_\nu} \frac{d_g\left(\overline{\Gamma^{(1)}_{x,\nu}}, \overline{\Gamma^{(2)}_{x,\nu}}\right)}{1 + d_g\left(\overline{\Gamma^{(1)}_{x,\nu}}, \overline{\Gamma^{(2)}_{x,\nu}}\right)} \, \mathrm{d}\mathcal{L}^{N-1}(x)$$

for $\Gamma^1, \Gamma^2 \in \mathcal{B}_\nu$, where $\mathcal{L}^{N-1}$ denotes the Lebesgue measure on Π_ν. We identify $\Gamma^{(1)}, \Gamma^{(2)} \in \mathcal{B}_\nu$ if $\Gamma^{(1)}_{x,\nu} = \Gamma^{(2)}_{x,\nu}$ for a.e. x. With this identification, $(\mathcal{B}_\nu, d_\nu)$ is indeed a metric space. By a standard argument, we see that $(\mathcal{B}_\nu, d_\nu)$ is a complete metric space; we do not give proof since we do not use this fact.

Let D be a countable dense set in S^{N-1}. We set

$$\mathcal{B}_D := \bigcap_{\nu \in D} \mathcal{B}_\nu.$$

It is a metric space with metric

$$d_D\left(\Gamma^{(1)}, \Gamma^{(2)}\right) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_{\nu_j}\left(\Gamma^{(1)}, \Gamma^{(2)}\right)}{1 + d_{\nu_j}\left(\Gamma^{(1)}, \Gamma^{(2)}\right)},$$

where $D = \{\nu_j\}_{j=1}^{\infty}$. (This is also a complete metric space.)

We shall fix D. The convergence with respect to d_D is called the *sliced graph convergence.* If $\{\Gamma_k\} \subset \mathcal{B}_D$ converges to $\Gamma \in \mathcal{B}_D$ with respect to d_D, we write $\Gamma_k \xrightarrow{sg} \Gamma$ (as $k \to \infty$). Roughly speaking, $\Gamma_k \xrightarrow{sg} \Gamma$ if the graph of the slice Γ_k converges to that of Γ for a.e. $x \in \Omega_\nu$ for any $\nu \in D$. For a function v on Ω, we associate a set-valued function Γ_v by $\Gamma_v(x) = \{v(x)\}$. If $\Gamma_k = \Gamma_{v_k}$ for some v_k, we shortly write $v_k \xrightarrow{sg} \Gamma$ instead of $\Gamma_{v_k} \xrightarrow{sg} \Gamma$. We note that if $v \in H^1(\Omega)$, the L^2-Sobolev space of order 1, then $\Gamma_v \in \mathcal{B}_D$ for any D.

3 Singular limit of the Kobayashi–Warren–Carter energy

We first recall the Kobayashi–Warren–Carter energy. For a given $\alpha \in C(\mathbb{R})$ with $\alpha \geq 0$, we consider the Kobayashi–Warren–Carter energy of the form

$$E^{\varepsilon}_{\mathrm{KWC}}(u, v) = \int_\Omega \alpha(v)|Du| + E^{\varepsilon}_{\mathrm{sMM}}(v)$$

for $u \in BV(\Omega)$ and $v \in H^1(\Omega)$. The first term is the weighted total variation of u with weight $w = \alpha(v)$, defined by

$$\int_\Omega w|Du| := \sup\left\{ -\int_\Omega u \operatorname{div} \varphi \ \mathrm{d}\mathcal{L}^N \;\middle|\; |\varphi(z)| \leq w(z) \text{ a.e. } x,\ \varphi \in C^1_c(\Omega) \right\}$$

for any non-negative Lebesgue measurable function w on Ω.

We shall assume that

(F1) $F \in C^1(\mathbb{R})$ is non-negative, and $F(v) = 0$ if and only if $v = 1$,

(F2) $\liminf_{|v| \to \infty} F(v) > 0$. We occasionally impose a stronger growth assumption than (F2):

(F2') (monotonicity condition) $F'(v)(v-1) \geq 0$ for all $v \in \mathbb{R}$.

We are interested in the Gamma limit of $E^\varepsilon_{\mathrm{KWC}}$ as $\varepsilon \to 0$ under the sliced graph convergence. We define the subset $\mathcal{A}_0 := \mathcal{A}_0(\Omega) \subset \mathcal{B}_D$ as follows: $\Xi \in \mathcal{A}_0(\Omega)$ if there is a countably $N-1$ rectifiable set $\Sigma \subset \Omega$ such that

$$\Xi(z) = \left\{ \begin{array}{l} 1,\ z \in \Omega\backslash\Sigma \\ \left[\xi^-, \xi^+\right],\ z \in \Sigma \end{array} \right. \tag{3.1}$$

with $\mathcal{H}^{N-1}$-measurable function $\xi_\pm$ on Σ and $\xi^-(z) \leq 1 \leq \xi^+(z)$ for $\mathcal{H}^{N-1}$-a.e. $z \in \Sigma$. For the definition of countably $N-1$ rectifiability. Here $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.

We briefly remark on the compactness of the graph of $\Xi \in \mathcal{A}_0$. By definition, if Ξ is of form (3.1), then $\Xi(z)$ is compact. However, there may be a chance that graph $\overline{\Gamma_{x,\nu}}$ is not compact, even for the one-dimensional case ($N=1$). Indeed, if a set-valued function on $(0,1)$ is of form

$$\Xi(z) = \left\{ \begin{array}{ll} [1, m] & \text{for } z = 1/m \\ \{1\} & \text{otherwise,} \end{array} \right.$$

then $\overline{\Xi}$ is not compact in $[0,1] \times \mathbb{R}$. It is also possible to construct an example that $\overline{\Xi} \neq \Xi$ in $(0,1)$, which is why we impose $\Xi \in \mathcal{B}_D$ in the definition of $\mathcal{A}_0$.

For $\Xi \in \mathcal{A}_0$, we define a functional

$$E^0_{\mathrm{sMM}}(\Xi, \Omega) := 2\int_\Sigma \left\{ G(\xi^-) + G(\xi^+) \right\} \mathrm{d}\mathcal{H}^{N-1}, \quad \text{where} \quad G(\sigma) := \left| \int_1^\sigma \sqrt{F(\tau)}\, \mathrm{d}\tau \right|.$$

For later applications, it is convenient to consider a more general functional. Let J be a countably $N-1$ rectifiable set, and $\alpha : \mathbb{R} \to [0,\infty)$ be continuous. Let j be a non-negative $\mathcal{H}^{N-1}$-measurable function on J. We denote the triplet (J, j, α) by $\mathcal{J}$. We set

$$E^{0,\mathcal{J}}_{\mathrm{sMM}}(\Xi, \Omega) = E^0_{\mathrm{sMM}}(\Xi, \Omega) + \int_{J\cap\Sigma} \left(\min_{\xi^- \leq \xi \leq \xi^+} \alpha(\xi) \right) \mathrm{d}\mathcal{H}^{N-1}.$$

For S, we also set

$$E^{\varepsilon,\mathcal{J}}_{\mathrm{sMM}}(v) := E^\varepsilon_{\mathrm{sMM}}(v) + \int_J \alpha(v) j \, \mathrm{d}\mathcal{H}^{N-1},$$

which is important to study the Kobayashi–Warren–Carter energy.

We next define the functional, which turns out to be a singular limit of the Kobayashi–Warren–Carter energy. For $\Xi \in \mathcal{A}_0(\Omega)$, let Σ be its singular set in the sense that

$$\Sigma = \left\{ z \in \Omega \;\middle|\; \Xi(z) \neq \{1\} \right\}.$$

For $u \in BV(\Omega)$, let J_u denote the set of its jump discontinuities. In other words,

$$J_u = \left\{ z \in \Omega \backslash \Sigma_0 \;\middle|\; j(z) := |u(z + 0\nu) - u(z - 0\nu)| > 0 \right\}.$$

Here ν denotes the approximate normal of J_u, and $u(z \pm 0\nu)$ denotes the trace of u in the direction of $\pm\nu$. We consider a triplet $\mathcal{J}(u) = (J_u, j, \alpha)$ and consider $E^{0,\mathcal{J}}_{\mathrm{sMM}}(\Xi, \Omega)$, whose explicit form is

$$E^{0,\mathcal{J}}_{\mathrm{sMM}}(\Xi, \Omega) = E^{0}_{\mathrm{sMM}}(\Xi, \Omega) + \int_{J \cap \Sigma} j \min_{\xi^- \leq \xi \leq \xi^+} \alpha(\xi) \; \mathrm{d}\mathcal{H}^{N-1},$$

where $\Xi(z) = [\xi^-(z), \xi^+(z)]$ for $z \in \Sigma$. We then define the limit Kobayashi–Warren–Carter energy:

$$E^0_{\mathrm{KWC}}(u, \Xi, \Omega) = \int_{\Omega \backslash J_u} \alpha(1)|Du| + E^{0,\mathcal{J}(u)}_{\mathrm{sMM}}(\Xi, \Omega),$$

in which the explicit representation of the second term is

$$E^{0,\mathcal{J}(u)}_{\mathrm{sMM}}(\Xi, \Omega) = E^{0}_{\mathrm{sMM}}(\Xi, \Omega) + \int_{J_u \cap \Sigma} |u^+ - u^-| \alpha_0(z) \; \mathrm{d}\mathcal{H}^{N-1}(z)$$

with $u^\pm = u(z \pm 0\nu)$ and

$$\alpha_0(z) := \min \left\{ \alpha(\xi) \;\middle|\; \xi^-(z) \leq \xi \leq \xi^+(z) \right\}.$$

Here $u^\pm$ are defined by

$$u^+(x) := \inf \left\{ t \in \mathbb{R} \;\middle|\; \lim_{r \to 0} \frac{\mathcal{L}^N \left(B_r(x) \cap \{u > t\} \right)}{r^N} = 0 \right\},$$
$$u^-(x) := \sup \left\{ t \in \mathbb{R} \;\middle|\; \lim_{r \to 0} \frac{\mathcal{L}^N \left(B_r(x) \cap \{u < t\} \right)}{r^N} = 0 \right\},$$

where $B_r(x)$ is the closed ball of radius r centered at x in $\mathbb{R}^N$. This is a measure-theoretic upper and lower limit of u at x. If $u^+(x) = u^-(x)$, we say that u is approximately continuous. For more detail, see [Fe]. We are now in a position to state our main results rigorously.

Theorem 1 (Γ-convergence)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that F satisfies (F1) and (F2) and that $\alpha \in C(\mathbb{R})$ is non-negative.*

(i) (liminf inequality) Assume that $\{u_\varepsilon\}_{0<\varepsilon<1} \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in L^1, i.e., $\|u_\varepsilon - u\|_{L^1} \to 0$. Assume that $\{u_\varepsilon\}_{0<\varepsilon<1} \subset H^1(\Omega)$. If $v_\varepsilon \xrightarrow{sg} \Xi$ and $\Xi \in \mathcal{A}_0$, then

$$E^0_{\mathrm{KMC}}(u, \Xi \; \Omega) \leq \liminf_{\varepsilon \to 0} E^\varepsilon_{\mathrm{KMC}}(u_\varepsilon, v_\varepsilon).$$

(ii) (limsup inequality) We further assume that F satisfies (F2') . For any $\Xi \in \mathcal{A}_0$ and

$u \in BV(\Omega)$, there exists a family of Lipschitz functions $\{w_\varepsilon\}_{0<\varepsilon<1}$ such that

$$E^0_{\mathrm{KMC}}(u, \Xi, \Omega) = \lim_{\varepsilon \to 0} E^\varepsilon_{\mathrm{KMC}}(u, w_\varepsilon).$$

If one minimizes E^0_{KWC} in the Ξ variable, i.e.,

$$TV_{\mathrm{KWC}}(u) := \inf_{\Xi \in \mathcal{A}_0} E^0_{\mathrm{KWC}}(u, \Xi),$$

this can be calculated as

$$TV_{\mathrm{KWC}}(u) = \int_\Sigma \sigma\left(|u^+ - u^-|\right) \mathrm{d}\mathcal{H}^{N-1} + \int_{\Omega \setminus J_u} |Du|$$

with

$$\begin{aligned}\sigma(r) &:= \min_{\xi^-, \xi^+} \left\{ r \min_{\xi^- \le \xi \le \xi^+} \alpha(\xi) + 2\left(G(\xi^-) + G(\xi^+)\right) \right\} \\ &= \min_{\xi^-} \left\{ r \min_{\xi^- \le \xi \le 1} \alpha(\xi) + 2G(\xi^-) \right\}, \ r \ge 0\end{aligned}$$

if $\alpha(v) \ge \alpha(1)$ for $v \ge 1$. This σ is always concave. If $F(v) = (v-1)^2$, then

$$\sigma(r) = \min_{\xi^-} \left\{ r(\xi^-_+)^2 + (\xi^- - 1)^2 \right\} = \frac{r}{r+1}.$$

In other words,

$$TV_{\mathrm{KWC}}(u) = \int_\Sigma \frac{|u^+ - u^-|}{1 + |u^+ - u^-|} \mathrm{d}\mathcal{H}^{N-1} + \int_{\Omega \setminus J_u} |Du|.$$

This functional is a kind of total variation but has different aspects. For example, if u is a piecewise constant monotone increasing function in a one-dimensional setting, the total variation $TV(u) = \int_\Omega |Du|$ equals $\sup u - \inf u$. This case is often called a staircase problem since TV does not care about the number and size of jumps for monotone functions. In contrast to TV, the TV_{KWC} costs less if the number of jumps is smaller, provided that each jump is the same size and $\sup u - \inf u$ is the same. The energy like TV_{KWC} for a piecewise constant function is derived as the surface tension of grain boundaries in polycrystals [LL], which is an active area, as studied by [GaSp].

Acknowledgments

The talk and abstract are based on the joint work with Yoshikazu Giga (the University of Tokyo), Koya Sakakibara (Kanazawa University) and Masaaki Uesaka (the University of Tokyo) .

References

[Fe] H. Federer, Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153 *Springer-Verlag New York Inc., New York,* 1969.

[FL] I. Fonseca and P. Liu, The weighted Ambrosio–Tortorelli approximation scheme. *SIAM J. Math. Anal.* 49 (2017), 4491–4520.

[GaSp] A. Garroni and E. Spadaro, Derivation of surface tension of grain boundaries in polycystals. Lecture at the workshop “Free boundary problems and related evolution equations” organized by G. Bellettini et al., Erwin Schrödinger Institute, Feb. 21–25, 2022, Vienna.

[GOU] Y. Giga, J. Okamoto and M. Uesaka, A finer singular limit of a single-well Modica–Mortola functional and its applications to the Kobayashi–Warren–Carter energy. *Adv. Calc. Var.*, published online (2021).

[IKY] A. Ito, N. Kenmochi and N. Yamazaki, A phase-field model of grain boundary motion. *Appl. Math.* 53 (2008), no. 5, 433–454.

[KG] R. Kobayashi and Y. Giga, Equations with singular diffusivity. *J. Stat. Phys.* 95 (1999), no. 5–6, 1187–1220.

[KWC1] R. Kobayashi, J. A. Warren and W. C. Carter, A continuum model of grain boundaries. *Phys. D* 140 (2000), no. 1–2, 141–150.

[KWC2] R. Kobayashi, J. A. Warren and W. C. Carter, Grain boundary model and singular diffusivity. in: *Free Boundary Problems: Theory and Applications,* GAKUTO Internat. Ser. Math. Sci. Appl. 14, Gakkōtosho, Tokyo (2000), 283–294.

[KWC3] R. Kobayashi, J. A. Warren and W. C. Carter, Modeling grain boundaries using a phase field technique. *J. Crystal Growth* 211 (2000), no. 1–4, 18–20.

[LL] G. Lauteri and S. Luckhaus, An energy estimate for dislocation configurations and the emergence of Cosserat-type structures in metal plasticity. arXiv: 1608.06155 (2016).

[MoSh] S. Moll and K. Shirakawa, Existence of solutions to the Kobayashi–Warren–Carter system. *Calc. Var. Partial Differential Equations* 51 (2014), no. 3–4, 621–656.

[MoShW1] S. Moll, K. Shirakawa and H. Watanabe, Energy dissipative solutions to the Kobayashi–Warren–Carter system. *Nonlinearity* 30 (2017), no. 7, 2752–2784.

[MoShW2] S. Moll, K. Shirakawa and H. Watanabe, Kobayashi–Warren–Carter type systems with nonhomogeneous Dirichlet boundary data for crystalline orientation, in preparation.

[SWat] K. Shirakawa and H. Watanabe, Energy-dissipative solution to a one-dimensional phase field model of grain boundary motion. *Discrete Contin. Dyn. Syst. Ser. S* 7 (2014), no. 1, 139–159.

[SWY] K. Shirakawa, H. Watanabe and N. Yamazaki, Solvability of one-dimensional phase field systems associated with grain boundary motion. *Math. Ann.* 356 (2013), no. 1, 301–330.

[WSh] H. Watanabe and K. Shirakawa, Qualitative properties of a one-dimensional phase-field system associated with grain boundary, in: *Nonlinear Analysis in Interdisciplinary Sciences – Modellings, Theory and Simulations,* GAKUTO Internat. Ser. Math. Sci. Appl. 36, Gakkōtosho, Tokyo (2013), 301–328.

ABP maximum principle with upper contact sets for fully nonlinear elliptic PDEs

Shigeaki Koike (Waseda University)

1 Introduction

In a celebrated work [1] by L. A. Caffarelli in 1989, it has turned out that the Aleksandrov-Bakelman-Pucci's maximum principle (ABP for short) is the key tool for the regularity theory of fully nonlinear PDEs. See also [2] for the details of [1]. Afterwards, the notion of weak solutions named L^p-viscosity solutions was introduced by Caffarelli-Crandall-Kocan-Święch [3] in 1996 to enable us to deal with viscosity solutions even when inhomogeneous terms in PDE of nondivergence type may be unbounded.

In our series of works with A. Święch, we have studied ABP when coefficients to the drift term are L^q functions provided $q \geq n$, where n is the dimension. In our results for ABP, the maximum of L^p-viscosity solutions is estimated by the L^p norm of the inhomogeneous term, where the L^p norm is taken over the whole domain. However, it is known that the ABP with upper contact sets holds true for strong solutions even when the coefficient to the drift term belongs to L^n. Also, in [3], it is known that ABP with upper contact sets holds for L^p-viscosity solutions when the coefficient to the drift term is bounded. In this talk, we point out the importance of ABP with upper contact sets, and then present recent results in this direction.

This talk is based on a joint work [9] with A. Święch (Georgia Institute of Technology).

2 Known results

Fix a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. We denote by d_Ω the diameter of Ω.

For fixed $0 < \lambda \leq \Lambda$, we denote the Pucci operators $\mathcal{P}^\pm : \mathbb{S}(n) \to \mathbb{R}$ by

$$\mathcal{P}^+(X) := \max\{-\mathrm{trace}(AX) \mid \lambda I \leq A \leq \Lambda I, A \in \mathbb{S}(n)\}, \quad \mathcal{P}^-(X) = -\mathcal{P}^+(-X),$$

where $\mathbb{S}(n)$ is the set of symmetric matrices of order n. The following inequalities allow us to deal with $\mathcal{P}^{\pm}$ as if these were Laplacian: for $X, Y \in \mathbb{S}(n)$,

$$\mathcal{P}^-(X)+\mathcal{P}^-(Y) \leq \mathcal{P}^-(X+Y) \leq \mathcal{P}^-(X)+\mathcal{P}^+(Y) \leq \mathcal{P}^+(X+Y) \leq \mathcal{P}^+(X)+\mathcal{P}^+(Y).$$

Given $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ and $f \in L^p(\Omega)$, we consider general PDE:

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{2.1}$$

We recall the definition of L^p-viscosity solutions of (2.1) for $p > n/2$.

Definition 2.1. *We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of* (2.1) *if whenever $u - \psi$ attains its local maximum (resp., minimum) at $x \in \Omega$ for $\psi \in W^{2,p}_{loc}(\Omega)$, it follows that*

$$\mathit{ess}\liminf_{y \to x}\{F(y, u(y), D\psi(y), D^2\psi(y)) - f(y)\} \leq 0$$

$$\left(\mathit{resp.,}\ \mathit{ess}\limsup_{y \to x}\{F(y, u(y), D\psi(y), D^2\psi(y)) - f(y)\} \geq 0\right).$$

We also call $u \in C(\Omega)$ an L^p-viscosity solution of (2.1) *if it is a C-viscosity subsolution and supersolution of* (2.1).

We also recall the notion of standard viscosity solutions, named C-viscosity solutions here, of (2.1) when F, f are continuous.

Definition 2.2. *We call $u \in C(\Omega)$ a C-viscosity subsolution (resp., supersolution) of* (2.1) *if whenever $u - \psi$ attains its local maximum (resp., minimum) at $x \in \Omega$ for $\psi \in C^2(\Omega)$, it follows that*

$$F(x, u(x), D\psi(x), D^2\psi(x)) - f(x) \leq 0 \quad (\mathit{resp.,}\ \geq 0)\,.$$

We also call $u \in C(\Omega)$ a C-viscosity solution of (2.1) *if it is a C-viscosity subsolution and supersolution of* (2.1).

For the sake of simplicity of presentations, we will only consider the case when F does not depend on the second variable.

Since we suppose that there is nonnegative $\mu \in L^q(\Omega)$ for $q \geq n$ such that

$$\mathcal{P}^-(X) - \mu(x)|\xi| \leq F(x, \xi, X) - F(x, \xi, Y) \leq \mathcal{P}^+(X - Y) + \mu(x)|\xi| \tag{2.2}$$

for $(x, \xi, X, Y) \in \Omega \times \mathbb{R}^n \times \mathbb{S}(n) \times \mathbb{S}(n)$, and that

$$F(x, 0, O) = 0 \quad \text{for } x \in \Omega, \tag{2.3}$$

we will only study on ABP for L^p-viscosity subsolutions of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega. \tag{2.4}$$

We will suppose one of two hypotheses: for $q > n$ and $n > p > p_0$, where $p_0 \in (\frac{n}{2}, n)$ is the constant in [10],

$$\begin{cases} (H1) & \mu \in L^q(\Omega), f \in L^n(\Omega) \text{ for } L^n\text{-viscosity solutions} \\ (H2) & \mu \in L^n(\Omega), f \in L^n(\Omega) \text{ for } L^p\text{-viscosity solutions.} \end{cases} \tag{2.5}$$

For $u \in C(\overline{\Omega})$, we shall introduce the so-called upper contact set of u over Ω;

$$\Gamma[u] := \{x \in \Omega \mid \exists p \in \mathbb{R}^n \text{ such that } u(y) \le u(x) + \langle p, y-x\rangle \text{ for } y \in \Omega\}$$

We will also use subsets of $\Gamma[u]$ when the size of p is smaller than $r > 0$:

$$\Gamma_r[u] := \{x \in \Omega \mid \exists p \in \overline{B}_r \text{ such that } u(y) \le u(x) + \langle p, y-x\rangle \text{ for } y \in \Omega\},$$

where $B_r := \{p \in \mathbb{R}^n \mid |p| < r\}$. Notice $\Gamma[u] = \bigcup_{r>0} \Gamma_r[u]$.

We recall ABP with upper contact sets. The first is a classical one but we refer the proof of Proposition 2.12 in [3] for the readers.

Proposition 2.1. *There exists $C = C(n, \lambda, \Lambda) > 0$ such that if $u \in C(\overline{\Omega}) \cap W^{2,n}(\Omega)$ satisfies*

$$\mathcal{P}^-(D^2u(x)) - \mu(x)|Du(x)| \le f(x) \quad \textit{a.e. in } \Omega$$

for $\mu, f \in L^n(\Omega)$, then it follows that

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + Cd_\Omega \|f^+\|_{L^n(\Gamma[u])}.$$

The next is a version of ABP with upper contact sets for L^n-viscosity solutions.

Proposition 2.2. (*Proposition 2.12 in [3]*)
There exists $C = C(n, \lambda, \Lambda) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of (2.4) *for $\mu \in L^\infty(\Omega)$ and $f \in L^n(\Omega)$, then it follows that*

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + Cd_\Omega \|f^+\|_{L^n(\Gamma[u])}.$$

Giving up having upper contact sets, we have ABP for L^n-viscosity solutions of (2.4) with the whole domain Ω when μ may be unbounded, i.e. $\mu \in L^q(\Omega)$ for $q > n$.

Proposition 2.3. (*cf. Proposition 2.8 in [7]*)
There exists $C = C(n, \lambda, \Lambda) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of (2.4) *for $\mu \in L^q(\Omega)$ for $q > n$ and $f \in L^n(\Omega)$, then it follows that*

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + Cd_\Omega \|f^+\|_{L^n(\Omega)}.$$

3 Why ABP with upper contact sets ?

In this section, to recall an advantage of the ABP maximum principle with upper contact sets in [3], we introduce some terminologies.

Definition 3.1. *For $u \in C(\Omega)$, we denote by $J^{2,\pm}u(x)$ the set of sub- and super-jets of order 2 for u at $x \in \Omega$:*

$$\left\{ (\xi, X) \in \mathbb{R}^n \times \mathbb{S}(n) \;\middle|\; \begin{array}{c} \pm\left(u(y) - u(x) - \langle \xi, y - x\rangle - \frac{1}{2}\langle X(y-x), y-x\rangle\right) \\ \leq o(|y-x|^2) \text{ as } y \in \Omega \to x \end{array} \right\}$$

We call $x \in \Omega$ a twice subdifferentiable (resp., superdifferentiable) point if $J^{2,-}u(x)$ (resp., $J^{2,+}u(x)$) $\neq \emptyset$. Also, we call $x \in \Omega$ a twice differentiable point if it is a twice sub- and superdifferentiable point.

For $u \in C(\Omega)$, we denote by $E_+[u]$ (resp., $E_-[u]$, $E[u]$) the set of all twice superdifferentiable (resp., subdifferentiable, differentiable) points of u.

Proposition 3.1. (*Proposition 3.5 in [3]*) *Assume* (2.2) *for $\mu \in L^\infty(\Omega)$,* (2.3), *and $f \in L^n(\Omega)$. If $u \in C(\Omega)$ is an L^n-viscosity subsolution (resp., supersolution, solution) of* (2.1), *then $m(\Omega \setminus E^+[u]) = 0$ (resp., $m(\Omega \setminus E^-[u])$, $m(\Omega \setminus E^+[u] \cap E^-[u])$ $= 0$, where m is the Lebesgue measure for $\mathbb{R}^n$.*

Proposition 3.2. (*Proposition 3.4 in [3]*) *Under the hypotheses in Proposition 3.1, it follows that an L^n-viscosity subsolution (resp., supersolution, solution) of* (2.1) *satisfies*

$$F(x, Du(x), D^2u(x)) - f(x) \leq \ (resp., \ \geq, \ =)\ 0 \quad a.e. \ in \ \Omega.$$

4 Main results

Our main results are as follows.

Theorem 4.1. (*Theorem 1.1 and 1.2 in K-Święch [9]*)
(i) Assume that (H1) holds. There exists $C_0 = C_0(n, \lambda, \Lambda, q) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of (2.4), *then it follows that*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C_0 d_\Omega \|f\|_{L^n(\Gamma[u])}. \tag{4.1}$$

(ii) Assume that (H2) holds. Fix $p \in (p_0, n)$. There exists $C_0 = C_0(n, \lambda, \Lambda, p) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (2.4) *for $n > p > p_0$, then it follows that*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C_0 d_\Omega \|f\|_{L^n(\Gamma[u])}. \tag{4.2}$$

We shall show a difficulty to prove (i) of the assertions of Theorem 4.1. In [3], in order to reduce to the case when $f \in C(\Omega)$ while $\mu \in L^\infty(\Omega)$, we utilize a strong solution $v_k \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ of

$$\begin{cases} \mathcal{P}^+(D^2 v_k) + \mu |Dv_k| &= f_k - f \qquad \text{a.e. in } \Omega, \\ v_k &= 0, \end{cases}$$

where $f_k \in C(\Omega)$ satisfies $\lim_{k\to\infty} \|f_k - f\|_{L^n(\Omega)} = 0$. Notice that

$$\|v_k\|_{L^\infty(\Omega)} \le C\|f_k - f\|_{L^n(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

We observe that $u_k := u + v_k$ is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2 u_k) - \mu|Du_k| = f_k \quad \text{in } \Omega.$$

When $\mu \in L^\infty(\Omega)$, this implies

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega}} u_k + C\|f_k - f\|_{L^n(\Omega)} \le \max_{\partial\Omega} u + C\|f_k - f\|_{L^n(\Omega)} + C\|f_k\|_{L^n(\Gamma[u_k])}.$$

Hence, recalling $\Gamma[u_k]$ converges $\Gamma[u]$ in a suitable sense (see Lemma A.1 in [3]), we conclude ABP with upper contact sets when $\mu \in L^\infty(\Omega)$.

Thus, we shall suppose $f \in C(\Omega) \cap L^n(\Omega)$ in what follows.

On the other hand, when we assume merely $\mu \in L^q(\Omega)$ for $q > n$, we have to approximate μ by continuous $\mu_k \in C(\Omega)$ such that

$$\lim_{k\to\infty} \|\mu_k - \mu\|_{L^q(\Omega)} = 0.$$

We observe that u is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2) - \mu_k|Du| = f + (\mu - \mu_k)|Du|.$$

To avoid the second term of the right hand side of the above, for any $\varepsilon > 0$, we shall consider the following PDE, where u is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2) - \mu_k|Du| - \varepsilon|Du|^{\frac{q}{q-n}} = f + C_\varepsilon|\mu_k - \mu|^{\frac{q}{n}}.$$

Here, notice that a superlinear growth in Du in the left hand side of the above arises.

To avoid the second term of the right hand side of the above, $C_\varepsilon|\mu_k - \mu|^{\frac{q}{n}}$, we need the strong solvability of extremal PDEs with superlinear terms in Du in [8].

Proposition 4.2. (*Theorem 3.1 (iii) in [8]*) *There exists $w_k \in W^{2,n}(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \mathcal{P}^+(D^2 w_k) + \mu_k|Dw_k| + \varepsilon C_*|Dw_k|^{\frac{q}{q-n}} &= -C_\varepsilon|\mu_k - \mu|^{\frac{q}{n}} \qquad \text{a.e. in } \Omega, \\ w_k &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

and $\|w_k\|_{L^\infty(\Omega)} \le C\|\mu_k - \mu\|^{\frac{q}{n}}_{L^q(\Omega)}$, *where* $C_* := 2^{\frac{n}{q-n}}$.

In order to prove (ii), we need the existence of strong solutions of (4.3) when $\mu_k \in L^n(\Omega) \cap C(\Omega)$. To this end, applying Theorem 1.2 in [10] by N. V. Krylov, we need a fixed point theorem as in [8]. See [9] for the details.

Remark 4.1. *Note that* $(a+b)^{\frac{q}{q-n}} \le C_*(a^{\frac{q}{q-n}} + b^{\frac{q}{q-n}})$ *for* $a, b \ge 0$. *Thus, we easily verify that* $v_k := u_k + w_k$ *is an* L^n*-viscosity subsolution of*

$$\mathcal{P}^-(D^2 v_k) - \mu_k |Dv_k| - \varepsilon C_* |Dv_k|^{\frac{q}{q-n}} = f \quad \text{in } \Omega. \tag{4.4}$$

Notice that all ingredients, μ_k, f, *are continuous. Thus,* v_k *is also a* C*-viscosity subsolution of* (4.4). *Moreover, intuitively,*

We still need more tools to establish the ABP maximum principle with upper contact sets for (4.4).

5 Some tools from viscosity solution theory

Furthermore, we will need some technical tools in the viscosity solution theory, which were first mentioned in [11], and have been developed in [5], [4].

We barrow some notations from [5].

Definition 5.1. *For bounded continuous functions* $v : \mathbb{R}^n \to \mathbb{R}$ *and* $\alpha > 0$, *we denote the sup-convolution* (*resp., inf-convolution*) *of* v *by*

$$A_\alpha^+[v](x) := \sup_{y \in \mathbb{R}^n} \left(v(y) - \frac{|x-y|^2}{2\alpha} \right), \quad A_\alpha^-[v](x) := \inf_{y \in \mathbb{R}^n} \left(v(y) + \frac{|x-y|^2}{2\alpha} \right).$$

We notice $A_\alpha^-[A_{\alpha+\beta}^+[v]] \ge A_\beta^+[v]$ from the difinition.

Proposition 5.1. (*Proposition 4.4 in [4]*) *Assume* $v \in C(\mathbb{R}^n)$ *is bounded. For* $\alpha, \beta > 0$, *setting* $v^{\alpha,\beta} := A_\alpha^-[A_{\alpha+\beta}^+[v]]$, *we have*

$$v^{\alpha,\beta} \in C^{1,1}(\mathbb{R}^n), \quad -\frac{1}{\beta} \le D^2 v^{\alpha,\beta}(x) \le \frac{1}{\alpha} I \quad \text{a.e. in } \Omega.$$

Moreover, if $v^{\alpha,\beta}(\hat{x}) > A_\beta^+[v](\hat{x})$, *and* $v^{\alpha,\beta}$ *is twice differentiable at* $\hat{x} \in \Omega$, *then* $\frac{1}{\alpha}$ *is one of eigenvalues of* $D^2 v^{\alpha,\beta}(\hat{x})$.

For the details, we refer to [6] written in Japanese.

We will choose $\alpha(\beta) > 0$ for each $\beta > 0$, and $\Omega_\beta := \{x \in \Omega \mid d(x, \partial\Omega) > C_0\sqrt{\beta}\}$ for some $C_0 > 0$. We first notice that $u^\beta := A_{\alpha(\beta)}^-[A_{\alpha(\beta),\beta}^+[u]] \ge A_\beta^+[u]$ in Ω_β, which implies when the equality holds at $x \in \Omega_\beta$, we have

$$J^{2,+} u^\beta(x) \subset J^{2,+} A_\beta^+[u](x). \tag{5.1}$$

Using Proposition 5.1, for continuous μ_k, f, we claim that if u is a C-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu_k|Du| - \varepsilon|Du|^{\frac{q}{q-n}} = f \quad \text{in } \Omega,$$

then u^β is a C-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu_k|Du| - \varepsilon|Du|^{\frac{q}{q-n}} = f + O(\beta) \quad \text{in } \Omega_\beta.$$

Indeed, for each $\beta > 0$, there exists a small $\alpha(\beta) > 0$ such that for $(\xi, X) \in J^{2,+}u^\beta(x)$, we have

$$\mathcal{P}^-(X) \leq -\frac{\Lambda}{\alpha(\beta)} + \frac{\lambda(n-1)}{\beta} \leq -\frac{\lambda}{\beta},$$

which, for small $\beta > 0$, yields

$$\mathcal{P}^-(X) - \mu_k(x)|\xi| - \varepsilon|\xi|^{\frac{q}{q-n}} \leq f(x) \quad \text{in } \Omega_\beta.$$

Hence, with this choice of $\alpha(\beta) > 0$, it is easy to verify that for small $\beta > 0$, $(\xi, X) \in J^{2,+}u^\beta(x)$ for $x \in \Omega_\beta$ satisfies

$$\mathcal{P}^-(X) - \mu_k(x)|\xi| - \varepsilon|\xi|^{\frac{q}{q-n}} \leq f(x) + O(\beta)$$

provided $u^\beta(x) > A^+_\beta[u](x)$ while the same inequality holds in the viscosity sense provided $u^\beta(x) = A^+_\beta[u](x)$ by (5.1).

References

[1] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130 (1989), no. 1, 189–213.

[2] L.A. Caffarelli and X. Cabré, Fully nonlinear elliptic equations. American Mathematical Society Colloquium Publications, 43, American Mathematical Society, Providence, RI, 1995.

[3] L.A. Caffarelli, M.G. Crandall, M. Kocan and A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math. 49 (4), (1996), 365–398.

[4] M.G. Crandall, M. Kocan, P. Soravia and A. Święch, On the equivalence of various weak notions of solutions of elliptic PDEs with measurable ingredients, Progress in elliptic and parabolic partial differential equations (Capri, 1994), 136–162, Pitman Res. Notes Math. Ser., 350, Longman, Harlow, 1996.

[5] R. R. Jensen, Uniformly elliptic PDEs with bounded, measurable coefficients, J. Fourier Anal. Appl., 2 (1996), no. 3, 237-259.

[6] S. Koike, Nenseikai, Kyoritsu Shuppan, 2016.

[7] S. Koike and A. Święch, Maximum principle for fully nonlinear equations via the iterated comparison function method, Math. Ann. 339 (2007), no. 2, 461–484.

[8] S. Koike and A. Święch, Existence of strong solutions of Pucci extremal equations with superlinear growth in Du, J. Fixed Point Theory Appl. 5 (2009), no. 2, 291–304.

[9] S. Koike and A. Święch, Aleksandrov-Bakelman-Pucci maximum principle for L^p-viscosity solutions of equations with unbounded terms, J. Math. Pures Appl. (9) **168** (2022), 192–212.

[10] N. V. Krylov, Linear and fully nonlinear elliptic equations with L_d-drift. Comm. Partial Differential Equations 45 (2020), no. 12, 1778–1798.

[11] J. M. Lasry and P. L. Lions, A remark on regularization in Hilbert spaces, Israel J. Math., 55 (1986), no. 3, 257–266.

A dynamical approach to lower gradient estimates for viscosity solutions of Hamilton-Jacobi equations *

Kazuya Hirose †
Department of Mathematics, Graduate School of Science, Hokkaido University.

1 Introduction

1.1 Equation and goals

In this study, we consider the first-order Hamilton-Jacobi equation of the form

$$u_t(x,t) + H(x,t,D_x u(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Here $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is the unknown function, and $u_t = \partial_t u$, $D_x u = (\partial_{x_i} u)_{i=1}^n$ denote its derivatives. Moreover, the Hamiltonian $H : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous in $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n$, and the initial datum $u_0 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}^n$.

The goal of this paper is to present a new approach to derive lower bounds for weak spatial gradients of a viscosity solution to a Hamilton-Jacobi equation when the Hamiltonian is convex. More precisely, for a fixed point $(x,t) \in \mathbb{R}^n \times (0,T)$, we obtain the following inequality in the viscosity solution sense:

$$|D_x u(x,t)| \geqq C, \tag{1.3}$$

where C depends on the initial subdifferentials over some region in $\mathbb{R}^n$. We also study the case where (1.1) is a level-set equation appearing in surface evolution problems. In this case, we will derive sharper gradient estimates.

A lower bound for gradients of solutions to (1.1)–(1.2) has already been studied in [L.01] with a different approach. Comparison with the results in [L.01] is also discussed and we show that our results give better estimates in several senses.

It is known that the lower gradient estimate is useful to prove the uniqueness of solutions to some nonlocal equations ([BLM.12]). Despite its importance, however, there is little work on lower bounds for gradients. Unlike the upper gradient estimate, one cannot apply a weak Bernstein method for viscosity solutions ([B.91]) to derive lower bounds.

It would also be worth mentioning here that the lower gradient estimate of the type (1.3) does not necessarily hold unless H is convex. Thus it is a crucial and difficult step to find how we use the structure of the Hamiltonian H. In the previous work [L.01], the author employs a notion of Barron-Jensen solutions ([BJ.90]) and derives the gradient estimates by carefully studying the inf-convolution of the solution. One of the key facts in the proof is that the inf-convolution is a

*This is a joint work with Professor Nao Hamamuki (Hokkaido University).
†E-mail: hirose.kazuya.w2@elms.hokudai.ac.jp

subsolution of (1.1) with an appropriate error. In this paper, we derive lower bounds for gradients with the aid of Hamiltonian systems. We utilize recent results in [ACS.20] for convex Hamiltonians and study how the initial gradients propagate along the solutions of approximate Hamiltonian systems. To do this, a suitable approximation of the equation and careful error estimates are needed.

1.2 Assumptions, methods, and main result 1.

Our assumptions on H are as follows:

(H1) There exist $C_1 \geqq 0$ and $\beta \in \{0,1\}$ such that

$$|H(x,t,p) - H(y,t,p)| \leqq C_1(\beta + |p|)|x-y|$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

(H2) There exist $A_2, B_2 \geqq 0$ such that

$$|H(x,t,p) - H(x,t,q)| \leqq (A_2|x| + B_2)|p-q|$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $q \in \mathbb{R}^n$;

(H3) $p \mapsto H(x,t,p)$ is convex in $\mathbb{R}^n$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$;

(H4) For every $R > 0$, H is bounded and uniformly continuous in $\mathbb{R}^n \times [0,T] \times \overline{B_R(0)}$.

Here $|\cdot|$ stands for the standard Euclidean norm, $B_r(x)$ denotes the open ball with radius r centered at x, and $\overline{B_r(x)}$ is its closure. The assumptions (H1)–(H3) are the same as the ones in [L.01], while we impose (H4) for uniqueness and existence of viscosity solutions to (1.1)–(1.2). Moreover, the unique solution u is Lipschitz continuous in $\mathbb{R}^n \times [0,T)$. For these results, see [L.01, Theorem 4.1], [II.08, Appendix A] and [ABIL.11, Chapter 2, Sections 5 and 8] for instance. One can relax (H4) (see, e.g., [I.86]), but we use it just for simplification.

To estimate the gradients of solutions, we consider the Hamiltonian system:

$$\begin{cases} \xi'(s) = D_pH(\xi(s), s, \eta(s)), \\ \eta'(s) = -D_xH(\xi(s), s, \eta(s)). \end{cases} \tag{1.4}$$

Here we temporarily assume that H is smooth. When the solution u of (1.1)–(1.2) is smooth enough, the relation between u and the solution (ξ, η) of (1.4) is well-known in the theory of classical dynamics ([CS.04, Chapter 1], [E.10, Chapter 3]). The curve ξ is often called a classical characteristic. We also recall that $\eta(s)$ represents the spatial derivative of u at $(\xi(s), s)$, i.e.,

$$\eta(s) = D_xu(\xi(s), s).$$

The theory above is generalized for a possibly nonsmooth viscosity solution u ([CS.04, Chapters 5 and 6], [I.07]).

One of the important ingredients for our gradient estimates is a recent result obtained in [ACS.20]. It is shown in [ACS.20, Theorem 3.2] that, for any $y \in \mathbb{R}^n$, there is a one-to-one correspondence between $p \in D^-_{pr}u_0(y)$ and a classical characteristic ξ with $\xi(0) = y$. Here $D^-_{pr}u_0(y)$ denotes the proximal subdifferential defined by

$$D^-_{pr}u_0(x) := \left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \text{there exist some } K, r > 0 \text{ such that} \\ u_0(y) \geqq u_0(x) + \langle p, y-x \rangle - \frac{K}{2}|y-x|^2 \\ \text{for all } y \in B_r(x) \subset \Omega \end{array} \right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product.

Before stating the main result, let us illustrate the idea for our gradient estimates at a differentiable point (x,t) of u. We take the solution (ξ,η) of (1.4) with the terminal condition $(\xi(t),\eta(t)) = (x, D_x u(x,t))$. Then, u is differentiable at every point $(\xi(s),s)$ $(s \in (0,t))$ along ξ. By [ACS.20, Theorem 3.2], we see that

$$\eta(0) \in D^-_{pr} u_0(\xi(0)). \tag{1.5}$$

We then apply Gronwall's lemma to discover bounds for $|\eta(t)-\eta(0)|$ and $|\xi(t)-\xi(0)|$. On the one hand, the bound for $|\eta(t)-\eta(0)|$ yields the estimate for $|\eta(t)| = |D_x u(x,t)|$ by $|\eta(0)|$. The estimate will be of the form

$$|D_x u(x,t)| \geqq |\eta(0)|e^{-C_1 t} - \beta(1-e^{-C_1 t}),$$

where C_1 and β are the constants in (H1). On the other hand, we find possible positions of $\xi(0)$ from the bound for $|\xi(t)-\xi(0)| = |x-\xi(0)|$. We will derive

$$|x-\xi(0)| \leqq R(x,t),$$

where $R(x,t)$ is defined as

$$R(x,t) := \begin{cases} \left(\dfrac{B_2}{A_2} + |x|\right)(e^{A_2 t}-1) & \text{if } A_2 > 0, \\ B_2 t & \text{if } A_2 = 0. \end{cases} \tag{1.6}$$

Here A_2 and B_2 are the constants in (H2). From the above estimates and (1.5), we deduce the gradient estimate of the form (1.3). For a nonsmooth viscosity solution, the above argument is justified via approximation.

To apply the results of [ACS.20], we need more conditions on H than (H1)–(H4), which are smoothness and strict convexity as follows:

(H5) $H \in C^2(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$;

(H3)$_{\mathrm{st}}$ $D_{pp}H(x,t,p)$ is positive definite for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

See [ACS.20, page 1413, **(H)**]. We first derive gradient estimates when H satisfies (H5) and (H3)$_{\mathrm{st}}$, and remove these additional conditions by approximating H by H_ε satisfying (H5) and (H3)$_{\mathrm{st}}$. In this case, we study the following equation

$$(u_\varepsilon)_t(x,t) + H_\varepsilon(x,t,D_x u_\varepsilon(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \tag{1.7}$$

Applying the results in the case H satisfies (H5) and (H3)$_{\mathrm{st}}$, we derive the gradient estimates for (1.1). This is our first main result.

To state the result, we prepare

Definition 1.1. Let $(x,t) \in \mathbb{R}^n \times (0,T)$. We define

$$\overline{S}(x,t;u_0) := \lim_{\delta\to+0} \sup\left\{|p| \;\middle|\; p \in D^-_{pr}u_0(y),\ y \in \overline{B_{R(x,t)+\delta}(x)}\right\},$$
$$\underline{I}(x,t;u_0) := \lim_{\delta\to+0} \inf\left\{|p| \;\middle|\; p \in D^-_{pr}u_0(y),\ y \in \overline{B_{R(x,t)+\delta}(x)}\right\}.$$

Our first main result is the following:

Theorem 1.2 (Gradient estimates)**.** *Assume that H satisfies* (H1)–(H4)*. Let u be the viscosity solution of* (1.1)–(1.2)*. Let $(x,t) \in \mathbb{R}^n \times (0,T)$.*

(1) *If $p \in D_x^- u(x,t)$, then*

$$\underline{I}(x,t;u_0)e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}(x,t;u_0)e^{C_1 t} + \beta(e^{C_1 t} - 1).$$

(2) *If $p \in D_x^+ u(x,t)$, then $|p| \leqq \overline{S}(x,t;u_0)e^{C_1 t} + \beta(e^{C_1 t} - 1)$.*

Though our main interest lies in lower bounds for gradients of solutions, we will simultaneously present upper bounds as above because the upper bounds are obtained in an almost parallel way.

Our method provides a rather straightforward way to derive lower bounds although careful approximation is needed. Thanks to this, we can find the lower bounds in an explicit way as in Theorem 1.2, and they are optimal as shown in [HH.p, Section 7]. These facts are advantages of our method as well as contributions to this paper.

1.3 Surface evolution problem and main result 2.

We also study a Hamiltonian H which is positively homogeneous of degree 1 with respect to p. More precisely, we assume

(H6) $H(x,t,\lambda p) = \lambda H(x,t,p)$ for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $\lambda \geqq 0$.

Such a Hamiltonian appears in the application of a level-set method to surface evolution problems. When H satisfies (H6), improved gradient estimates are available.

To explain the results, let us review the level-set approach to surface evolution problems. The reader is referred to [G.06] for the details. For a given initial hypersurface $\Gamma(0)$ in $\mathbb{R}^n$, we consider its evolution $\{\Gamma(t)\}_{t \in [0,T)}$. To track the motion, we represent $\Gamma(t)$ as the zero level-set of an auxiliary function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$, that is,

$$\Gamma(t) = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}. \tag{1.8}$$

We now assume that the evolution law of $\Gamma(t)$ is given by

$$V = g(x,t,\mathbf{n}) \quad \text{on } \Gamma(t). \tag{1.9}$$

Here g is a given function, $\mathbf{n} = \mathbf{n}(x,t) \in \mathbb{R}^n$ is the unit normal vector to $\Gamma(t)$ at x from $\{x \in \mathbb{R}^n \mid u(x,t) > 0\}$ to $\{x \in \mathbb{R}^n \mid u(x,t) < 0\}$, and $V = V(x,t) \in \mathbb{R}$ is the normal velocity of $\Gamma(t)$ at x in the direction of $\mathbf{n}$. If u is smooth near (x,t) and $D_x u(x,t) \neq 0$, we have

$$\mathbf{n} = -\frac{D_x u(x,t)}{|D_x u(x,t)|}, \quad V = \frac{u_t(x,t)}{|D_x u(x,t)|}.$$

Substituting these formulas for (1.9), we are led to (1.1) with the Hamiltonian

$$H(x,t,p) = -|p| g\left(x,t,-\frac{p}{|p|}\right). \tag{1.10}$$

The corresponding equation (1.1) is often called a level-set equation. Clearly, (1.10) satisfies (H6) for $p \neq 0$. When H has continuous extension to $p = 0$ and satisfies (H1)–(H4), our results can be applied to H.

Carrying out the level-set method, we first choose the initial datum u_0 so that $\Gamma(0) = \{x \in \mathbb{R}^n \mid u_0(x) = 0\}$ and next solve (1.1)–(1.2). Then the desired surfaces $\Gamma(t)$ are given by (1.8) with the solution u. The family of $\Gamma(t)$ obtained in this manner is called a level-set solution. An important feature of this method is that level-set solutions are unique. Namely, the set $\{x \in \mathbb{R}^n \mid u(x,t) = 0\}$ depends only on the initial level-set $\{x \in \mathbb{R}^n \mid u_0(x) = 0\}$ and independent of the other levels of u_0. This fact naturally raises the question whether the same holds for the gradients, i.e., whether the gradients of u on $\{x \in \mathbb{R}^n \mid u(x,t) = 0\}$ depend only on the initial gradients of u_0 on $\{x \in \mathbb{R}^n \mid u_0(x) = 0\}$.

We give a positive answer to this question. We demonstrate that the sub- and super-differentials $D_x^{\pm}u(x,t)$ with $u(x,t) = 0$ depend only on $D_{pr}^{-}u_0(y)$ for y with $u_0(y) \approx 0$. (We prove this fact not only for the zero level-set but also for any $\gamma \in \mathbb{R}$ level-set.) This is our second main theorem, Theorem 1.4 below. A key observation for this theorem is that the solution u is constant along $(\xi(s), s)$ for the solution ξ of (1.4); the rigorous proof needs a suitable approximation of H and some error estimates.

Let us state our second main result.

Definition 1.3. Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\gamma \in \mathbb{R}$. We define

$$\overline{S}(x,t;u_0,\gamma) := \lim_{\delta \to +0} \sup \left\{ |p| \;\middle|\; p \in D_{pr}^{-}u_0(y),\ y \in \overline{B_{R(x,t)+\delta}(x)},\ |u_0(y) - \gamma| \leqq \delta \right\},$$
$$\underline{I}(x,t;u_0,\gamma) := \lim_{\delta \to +0} \inf \left\{ |p| \;\middle|\; p \in D_{pr}^{-}u_0(y),\ y \in \overline{B_{R(x,t)+\delta}(x)},\ |u_0(y) - \gamma| \leqq \delta \right\}.$$

Theorem 1.4 (Gradient estimates for a homogeneous Hamiltonian)**.** *Assume that H satisfies* (H1)–(H4), (H6) *and that*

$$H \text{ is locally Lipschitz continuous in } \mathbb{R}^n \times [0,T] \times \mathbb{R}^n. \tag{1.11}$$

Let u be the viscosity solution of (1.1)–(1.2). *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and set $\gamma := u(x,t)$.*

(1) *If $p \in D_x^{-}u(x,t)$, then*

$$\underline{I}(x,t;u_0,\gamma)e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}(x,t;u_0,\gamma)e^{C_1 t} + \beta(e^{C_1 t} - 1).$$

(2) *If $p \in D_x^{+}u(x,t)$, then $|p| \leqq \overline{S}(x,t;u_0,\gamma)e^{C_1 t} + \beta(e^{C_1 t} - 1)$.*

As with Theorem 1.2, we derive an upper bound for gradients together with the lower bound. Both bounds seem to be new for solutions to level-set equations.

We also show that the technique to prove Theorem 1.4 applies to m-homogeneous Hamiltonian with $m > 1$. For the details, see [HH.p, Theorems 5.4 and 5.7].

1.4 Literature overview

As we have already mentioned, lower bound gradient estimates for solutions to (1.1)–(1.2) are derived in [L.01] with a different approach. We will describe the results and compare them with our results in Section 6. In [L.01] the author employs a notion of Barron-Jensen solutions ([BJ.90]) and derives the gradient estimates by carefully studying the inf-convolution of the solution. One of the key facts in the proof is that the inf-convolution is a subsolution of (1.1) with an appropriate error. In [BLM.12] lower gradient bounds are obtained for solutions to second-order geometric equations. For the proof, continuous dependence property for solutions is used in a crucial way. This gradient estimate is applied to prove short time uniqueness of solutions to nonlocal geometric equations. See also [BL.06, BCLM.08, BCLM.09] for related results for nonlocal first-order equations.

We mention other types of lower bound estimates for gradients of solutions. A lower bound for the spatially Lipschitz constant $\|D_x u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ is investigated in [F.18, HK.p]. In [F.18] lower bounds with the optimal order of t are derived for linear parabolic equations and Hamilton-Jacobi equations. More general fully nonlinear parabolic equations are studied in [HK.p]. For surface evolution problems, improved level-set equations are proposed in [HN.16, H.19] so that initial gradients are preserved on the zero level-sets of solutions.

2 Proof of the main results

In this section, we present the outline of the proof of Theorem 1.2 and 1.4. For the sake of simplicity, we only prove the lower bounds.

Outline of the proof of Theorem 1.2. We recall the basic idea stated in the introduction. When H satisfies the additional conditions (H5) and $(\mathrm{H3})_{\mathrm{st}}$, for a differentiable point (x,t) of u, we obtain the following estimates

$$|D_x u(x,t)| \geqq |\eta(0)|e^{-C_1 t} - \beta(1-e^{-C_1 t}), \quad |x-\xi(0)| \leqq R(x,t),$$

where (ξ,η) is the solution of (1.4). Let $\varepsilon \in (0,1]$ and H_ε be the approximate Hamiltonian satisfying (H5) and $(\mathrm{H3})_{\mathrm{st}}$ (for the approximation methods, see [HH.p, section 2.3]). Let u_ε be the viscosity solution of (1.7)–(1.2). By the stability result, u_ε converges to u locally uniformly in $\mathbb{R}^n \times [0,T)$ as $\varepsilon \to +0$.

Suppose that $p \in D_x^- u(x,t)$. By Lemma [HH.p, Lemma A.1], there exist sequences $\{(x_\varepsilon,t_\varepsilon)\}_{\varepsilon\in(0,1]} \subset \mathbb{R}^n \times (0,T)$ and $\{p_\varepsilon\}_{\varepsilon\in(0,1]} \subset \mathbb{R}^n$ such that

$$p_\varepsilon \in D_x^- u_\varepsilon(x_\varepsilon,t_\varepsilon), \quad (x_\varepsilon,t_\varepsilon) \to (x,t), \quad p_\varepsilon \to p \quad \text{as } \varepsilon \to +0. \tag{2.1}$$

Then we obtain

$$|p_\varepsilon| \geqq I_\varepsilon(x_\varepsilon,t_\varepsilon;u_0)e^{-C_1 t_\varepsilon} - \beta(1-e^{-C_1 t_\varepsilon}),$$

where

$$I_\varepsilon(x,t;u_0) := \inf\left\{ |p| \;\middle|\; p \in D^-_{pr} u_0(y),\, y \in \overline{B_{R_\varepsilon(x,t)}(x)} \right\},$$

$$R_\varepsilon(x,t) := \begin{cases} \left(\dfrac{B_2+\varepsilon}{A_2} + |x|\right)(e^{A_2 t}-1) & \text{if } A_2 > 0, \\ (B_2+\varepsilon)t & \text{if } A_2 = 0. \end{cases}$$

Take an arbitrary $\delta > 0$. Then, since $R_\varepsilon(y,s) \to R(x,t)$ as $(y,s,\varepsilon) \to (x,t,+0)$, there exists some $\theta \in (0,1]$ such that

$$R_\varepsilon(y,s) < R(x,t) + \delta \quad \text{for all } (y,s) \in B_\theta(x,t) \text{ and } \varepsilon \in (0,\theta). \tag{2.2}$$

For $\varepsilon > 0$ small enough, we have $(x_\varepsilon,t_\varepsilon) \in B_\theta(x,t)$ and $\varepsilon \in (0,\theta)$, and so $R_\varepsilon(x_\varepsilon,t_\varepsilon) < R(x,t)+\delta$ by (2.2). This implies that

$$|p_\varepsilon| \geqq \inf\left\{ |q| \;\middle|\; q \in D^-_{pr} u_0(y),\, y \in \overline{B_{R(x,t)+\delta}(x)} \right\} \cdot e^{-C_1 t_\varepsilon} - \beta(1-e^{-C_1 t_\varepsilon})$$

Sending $\varepsilon \to +0$ and $\delta \to +0$, we obtain the desired inequalities. □

Outline of the proof of Theorem 1.4. We first consider the case where H is a smooth Hamiltonian satisfying (H6). Then we have

$$\langle D_pH(x,t,p),p\rangle = H(x,t,p) \tag{2.3}$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$. In fact, differentiating both the sides of $H(x,t,\lambda p) = \lambda H(x,t,p)$ with respect to $\lambda > 0$, we find that $\langle D_pH(x,t,\lambda p),p\rangle = H(x,t,p)$. Letting $\lambda = 1$, we obtain (2.3).

Making a use of (2.3), one can observe that the viscosity solution u of (1.1)–(1.2) keeps its value along ξ for each $(x,t) \in \mathbb{R}^n \times (0,T)$, that is,

$$u(x,t) = u_0(\xi(0)). \tag{2.4}$$

Since $\eta(0) \in D^-_{pr}u_0(\xi(0))$, this observation suggests that the generalized gradients of u at (x,t) depend on $D^-_{pr}u_0(y)$ only for y such that $u(x,t) = u_0(y)$. In other words, the gradients depend on the initial gradients on the same level-set.

In the case where H is not smooth, we approximate H by smooth H_ε. Then it is naturally expected that

$$H_\varepsilon(x,t,p) \approx \langle D_pH_\varepsilon(x,t,p),p\rangle.$$

Let us define an error function E_ε as

$$E_\varepsilon(x,t,p) := H_\varepsilon(x,t,p) - \langle D_pH_\varepsilon(x,t,p),p\rangle. \tag{2.5}$$

Then, by [HH.p, Lemma 5.2], we have

$$|E_\varepsilon(x,t,p)| \leqq (A_2|x| + A_2\varepsilon + B_2 + 1)\varepsilon \tag{2.6}$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

Let u_ε be the viscosity solution of (1.7)–(1.2). Suppose that $p \in D^-_x u(x,t)$. Then (2.1) holds for some $\{(x_\varepsilon,t_\varepsilon)\}_{\varepsilon\in(0,1]} \subset \mathbb{R}^n \times (0,T)$ and $\{p_\varepsilon\}_{\varepsilon\in(0,1]} \subset \mathbb{R}^n$. We let $(\xi_\varepsilon,\eta_\varepsilon) \in C^1([0,1])^2$ be the solution of the approximate Hamiltonian system:

$$\begin{cases} \xi'_\varepsilon(s) = D_pH_\varepsilon(\xi_\varepsilon(s),s,\eta_\varepsilon(s)), \\ \eta'_\varepsilon(s) = -D_xH_\varepsilon(\xi_\varepsilon(s),s,\eta_\varepsilon(s)), \end{cases}$$

with the terminal condition

$$\xi_\varepsilon(t_\varepsilon) = x_\varepsilon, \quad \eta_\varepsilon(t_\varepsilon) = p_\varepsilon = D_xu_\varepsilon(x_\varepsilon,t_\varepsilon).$$

By simple calculation, we find that

$$u_\varepsilon(x_\varepsilon,t_\varepsilon) = u_0(\xi_\varepsilon(0)) - \int_0^{t_\varepsilon} E_\varepsilon(\xi_\varepsilon(s),s,\eta_\varepsilon(s))\,ds.$$

Now, by [HH.p, Lemma 3.5], we have

$$\xi_\varepsilon(s) \in \overline{B_{R_\varepsilon(x_\varepsilon,t_\varepsilon)}(x_\varepsilon)} \quad (s \in [0,t_\varepsilon]), \tag{2.7}$$

which implies that there exists some $K > 0$ independent of ε and s such that $|\xi_\varepsilon(s)| \leqq K$ for all $\varepsilon \in (0,1]$ and $s \in [0,t_\varepsilon]$. Then, by (2.6)

$$|E_\varepsilon(\xi_\varepsilon(s),s,\eta_\varepsilon(s))| \leqq (A_2|\xi_\varepsilon(s)| + A_2\varepsilon + B_2 + 1)\varepsilon \leqq (A_2K + A_2\varepsilon + B_2 + 1)\varepsilon,$$

and therefore

$$|u_\varepsilon(x_\varepsilon,t_\varepsilon)-u_0(\xi_\varepsilon(0))| \leqq \int_0^{t_\varepsilon} |E_\varepsilon(\xi_\varepsilon(s),s,\eta_\varepsilon(s))|\,ds \leqq (A_2K+A_2\varepsilon+B_2+1)\varepsilon t_\varepsilon. \tag{2.8}$$

Take any $\delta>0$ and choose ε small enough that

$$|x-x_\varepsilon|<\frac{\delta}{2}, \quad R_\varepsilon(x_\varepsilon,t_\varepsilon)<R(x,t)+\frac{\delta}{2} \tag{2.9}$$

(see (2.2)) and

$$|u_\varepsilon(x_\varepsilon,t_\varepsilon)-u(x,t)|<\frac{\delta}{2}, \quad (A_2K+A_2\varepsilon+B_2+1)\varepsilon t_\varepsilon<\frac{\delta}{2}. \tag{2.10}$$

Using (2.7) and (2.9), we have

$$|x-\xi_\varepsilon(0)| \leqq |x-x_\varepsilon|+|x_\varepsilon-\xi_\varepsilon(0)| < \frac{\delta}{2}+R_\varepsilon(x_\varepsilon,t_\varepsilon)<R(x,t)+\delta. \tag{2.11}$$

In addition, (2.8) and (2.10) imply that

$$|u_0(\xi_\varepsilon(0))-\gamma| \leqq |u_0(\xi_\varepsilon(0))-u_\varepsilon(x_\varepsilon,t_\varepsilon)|+|u_\varepsilon(x_\varepsilon,t_\varepsilon)-u(x,t)|<\delta. \tag{2.12}$$

Since u is differentiable at $(\xi_\varepsilon,\eta_\varepsilon)$, we have

$$|p_\varepsilon| \geqq |\eta_\varepsilon(0)|e^{-C_1t_\varepsilon}-\beta(1-e^{-C_1t_\varepsilon}).$$

Recalling that $\eta_\varepsilon(0)\in D^-_{pr}u_0(\xi_\varepsilon(0))$ by [ACS.20, Theorem 3.2], we deduce from (2.11) and (2.12) that

$$\begin{aligned}|p_\varepsilon| \geqq \inf\left\{|q| \;\middle|\; q\in D^-_{pr}u_0(y),\, y\in\overline{B_{R(x,t)+\delta}(x)},\, |u_0(y)-\gamma|\leqq\delta\right\}\cdot e^{-C_1t_\varepsilon}&\\ -\beta(1-e^{-C_1t_\varepsilon}).&\end{aligned}$$

Finally, we send $\varepsilon\to+0$ and $\delta\to+0$ to conclude the proof. □

3 Comparison with [L.01]

In this section, we compare our results with the ones in [L.01]. Let us start by stating the results obtained in [L.01]. Moreover, we rephrase our gradient estimates under the same assumptions as [L.01].

For a given $x_0\in\mathbb{R}^n$ and $r>0$, the following assumption (U1) is imposed on u_0 in [L.01]:

(U1) There exists a constant $\theta>0$ such that

$$|p|\geqq\theta \quad \text{for all } x\in B_r(x_0) \text{ and } p\in D^-u_0(x).$$

When H satisfies (H6), the next assumption (U2) is considered:

(U2) There exists a constant $\theta>0$ such that

$$|u_0(x)|+|p|\geqq\theta \quad \text{for all } x\in\mathbb{R}^n \text{ and } p\in D^-u_0(x).$$

Let us further define

$$r(x,t) := e^{(A_2+B_2+A_2|x|)t} - 1$$

and

$$\mathcal{D}(x_0,r) := \{(x,t) \in \mathbb{R}^n \times (0,T) \mid (r(x_0,t)+1)(|x-x_0|+1) - 1 < r\}.$$

Following [L.01], we call the set $\mathcal{D}(x_0,r)$ the *domain of dependence* with the base $B_r(x_0)$. The lower bound gradient estimates in [L.01] are stated as follows:

Theorem 3.1 ([L.01, Theorem 4.2 and its proof])**.** *Assume that H satisfies* (H1)–(H3)*. Let u be a viscosity solution of* (1.1)–(1.2)*.*

(1) *Let $x_0 \in \mathbb{R}^n$, $r > 0$ and assume* (U1)*. Then, for any $t_0 \in (0,T]$ satisfying*

$$\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 T} t_0 > 0,$$

we have

$$|p| \geqq \tilde{\theta} e^{-\frac{5}{4}C_1 t} \quad \textit{for all } (x,t) \in \mathcal{D}(x_0,r) \cap (\mathbb{R}^n \times (0,t_0)) \textit{ and } p \in D_x^- u(x,t), \tag{3.1}$$

where $\tilde{\theta} := \sqrt{\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 T} t_0}$, and β and C_1 are the constants in (H1)*.*

(2) *Assume furthermore that H satisfies* (H6) *and that $\beta = 0$ in* (H1)*. Assume* (U2)*. Then there exists a constant $C = C(\theta) > 0$ such that*

$$|u(x,t)| + \frac{1}{4} e^{\frac{5}{2}C_1 t} |p|^2 \geqq C \quad \textit{for all } (x,t) \in \mathbb{R}^n \times (0,T) \textit{ and } p \in D_x^- u(x,t).$$

Next, we restate our results under the assumptions (U1) and (U2). For a given $x_0 \in \mathbb{R}^n$ and $r > 0$, let us define

$$\mathcal{E}(x_0,r) := \{(x,t) \in \mathbb{R}^n \times (0,T) \mid R(x,t) + |x-x_0| < r\},$$

where $R(x,t)$ is the constant defined in (1.6). The set $\mathcal{E}(x_0,r)$ is the domain of dependence with the base $B_r(x_0)$ obtained in this paper. The lower bound gradient estimates in [HH.p] are stated as follows:

Theorem 3.2 (Gradient estimates under (U1), (U2))**.** *Assume that H satisfies* (H1)–(H4)*. Let u be the viscosity solution of* (1.1)–(1.2)*.*

(1) *Let $x_0 \in \mathbb{R}^n$, $r > 0$ and assume* (U1)*. Then,*

$$|p| \geqq \theta e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \quad \textit{for all } (x,t) \in \mathcal{E}(x_0,r) \textit{ and } p \in D_x^- u(x,t), \tag{3.2}$$

where β and C_1 are the constants in (H1)*.*

(2) *Assume furthermore that H satisfies* (H6) *and* (1.11)*. Assume* (U2)*. Then,*

$$|u(x,t)| + e^{C_1 t}|p| \geqq \theta - \beta(e^{C_1 t} - 1) \quad \textit{for all } (x,t) \in \mathbb{R}^n \times (0,T) \textit{ and } p \in D_x^- u(x,t).$$

In particular, if $\beta = 0$ in (H1)*, then*

$$|u(x,t)| + e^{C_1 t}|p| \geqq \theta \quad \textit{for all } (x,t) \in \mathbb{R}^n \times (0,T) \textit{ and } p \in D_x^- u(x,t).$$

We consider the lower bound in (3.1). Since $t < t_0 \leqq T$, the lower bound satisfies

$$\tilde{\theta} e^{-\frac{5}{4}C_1 t} = e^{-\frac{5}{4}C_1 t}\sqrt{\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 T} t_0} \leqq e^{-\frac{5}{4}C_1 t}\sqrt{\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 t} t}.$$

We compare the right-hand side with our lower bound obtained in (3.2).

Let us prepare notations.

Definition 3.3. For $\theta > 0$, we define

$$l(t) = e^{-\frac{5}{4}C_1 t}\sqrt{\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 t} t}, \quad L(t) = \theta e^{-C_1 t} - \beta(1 - e^{-C_1 t}),$$

where β and C_1 are the constants in (H1). When $\beta = 1$, we define $t_l, t_L > 0$ as the unique numbers such that $l(t_l) = 0$ and $L(t_L) = 0$.

Remark 3.4. We have $l(0) = L(0) = \theta$, and both l and L are decreasing with respect to t. Moreover, for the constant t_0 in Theorem 3.1 (1), we have $l(t_0) > 0$. Therefore, $t_0 < t_l$ when $\beta = 1$.

The next theorems show that our result gives a sharper lower bound, holds for a longer time, and is obtained in a larger set.

Theorem 3.5 (Lower bound)**.** *Let $\theta > 0$. Assume that $C_1 > 0$.*

(1) If $\beta = 0$, then $l(t) < L(t)$ for all $t \in (0, \infty)$.

(2) If $\beta = 1$, then $t_l < t_L$ and $l(t) < L(t)$ for all $t \in (0, t_l]$.

Theorem 3.6 (Domain of dependence)**.** *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Then*

$$\mathcal{D}(x_0, r) = \mathcal{E}(x_0, r) \quad \text{if } (A_2, B_2) = (0, 0), \quad \mathcal{D}(x_0, r) \subsetneq \mathcal{E}(x_0, r) \quad \text{if } (A_2, B_2) \neq (0, 0).$$

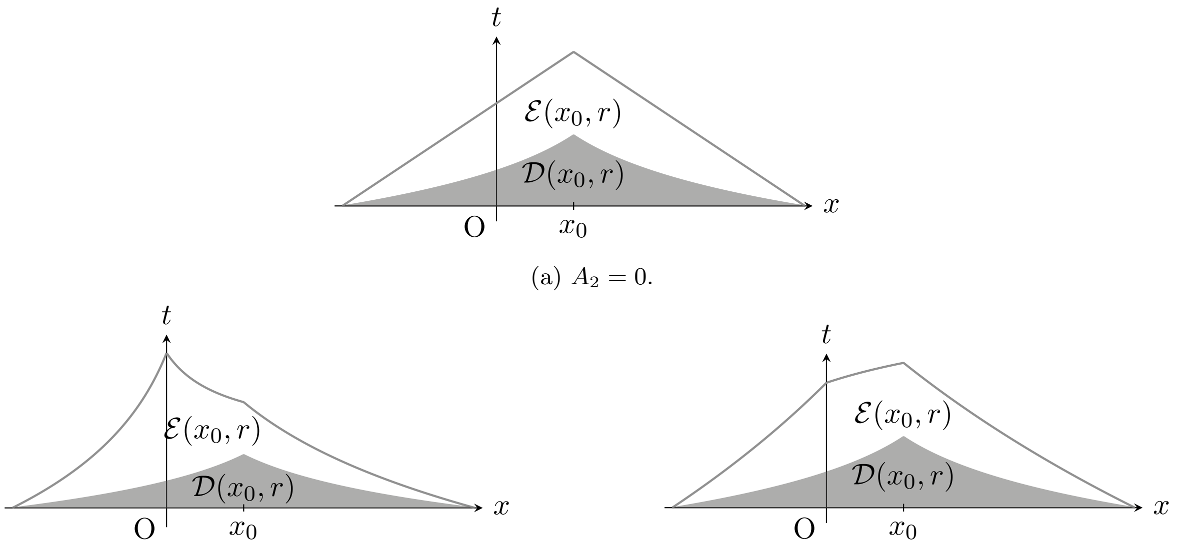
Figures 1 shows $\mathcal{D}(x_0, r)$ and $\mathcal{E}(x_0, r)$ for several A_2 and B_2 in the case where $|x_0| < r$. For more details, see [HH.p, Remark 6.7].

4 Future work

- Does the lower bound estimate hold in some sense if the Hamiltonian $H = H(x, t, p)$ is not convex with respect to p?
- How about the case where Hamiltonian H depends on u?
- How about the initial boundary value problems?
- How about a second-order Hamilton-Jacobi equation such as the mean curvature equation?
- How about on a metric space?

Acknowledgments

The author was partially supported by the Ushio Foundation, and JST SPRING, Grant Number JPMJSP2119.



(a) $A_2 = 0$.

(b) $A_2(r - |x_0|) > B_2$.

(c) $A_2(r - |x_0|) < B_2$.

Figure 1: $\mathcal{D}(x_0, r)$ and $\mathcal{E}(x_0, r)$.

References

[ABIL.11] Y. Achdou, G. Barles, H. Ishii, G. L. Litvinov, *Hamilton–Jacobi equations: approximations, numerical analysis and applications*, Lecture Notes in Mathematics **2074**, Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2011.

[ACS.20] P. Albano, P. Cannarsa, C. Sinestrari, *Generation of singularities from the initial datum for Hamilton-Jacobi equations*, Journal of Differential Equations **268** (2020), 1412–1426.

[B.91] G. Barles, *A weak Bernstein method for fully nonlinear elliptic equations*, Differential Integral Equations **4** (1991), 241–262.

[BCLM.08] G. Barles, P. Cardaliaguet, O. Ley, R. Monneau, *Global existence results and uniqueness for dislocation equations*, SIAM J. Math. Anal. **40** (2008), 44–69.

[BCLM.09] G. Barles, P. Cardaliaguet, O. Ley, A. Monteillet, *Uniqueness results for nonlocal Hamilton-Jacobi equations*, J. Funct. Anal. **257** (2009), 1261–1287.

[BL.06] G. Barles, O. Ley, *Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics*, Comm. Partial Differential Equations **31** (2006), 1191–1208.

[BLM.12] G. Barles, O. Ley, H. Mitake, *Short time uniqueness results for solutions of nonlocal and non-monotone geometric equations*, Math. Ann. **352** (2012), 409–451.

[BJ.90] E. N. Barron, R. Jensen, *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations **15** (1990), 1713–1742.

[CS.04] P. Cannarsa, C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications **58**, Birkhäuser Boston, Inc., Boston, MA, 2004.

[E.10] L. C. Evans, *Partial differential equations. The second edition*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 2010.

[F.18] Y. Fujita, *Lower estimates of L^∞-norm of gradients for Cauchy problems*, J. Math. Anal. Appl. **458** (2018), 910–924.

[G.06] Y. Giga, *Surface evolution equations: A level set approach*, Monographs in Mathematics **99**, Birkhäuser Verlag, Basel, 2006.

[H.19] N. Hamamuki, *An improvement of level set equations via approximation of a distance function*, Appl. Anal. **98** (2019), 1901–1915.

[HH.p] N. Hamamuki, K. Hirose, *A dynamical approach to lower gradient estimates for viscosity solutions of Hamilton-Jacobi equations*, to appear in SIAM J. Math. Anal.

[HK.p] N. Hamamuki, S. Kikkawa, *A lower spatially Lipschitz bound for solutions to fully nonlinear parabolic equations and its optimality*, to appear in Indiana Univ. Math. J.

[HN.16] N. Hamamuki, E. Ntovoris, *A rigorous setting for the reinitialization of first order level set equations*, Interfaces Free Bound. **18** (2016), 579–621.

[I.84] H. Ishii, *Uniqueness of unbounded viscosity solution of Hamilton–Jacobi equations*, Indiana Univ. Math. J. **33** (1984), 721–748.

[I.86] H. Ishii, *Existence and uniqueness of solutions of Hamilton-Jacobi equations*, Funkcial. Ekvac. **29** (1986), 167–188.

[I.07] H. Ishii, *Lecture notes on the weak KAM theorem*, `http://www.f.waseda.jp/hitoshi.ishii/`

[II.08] N. Ichihara, H. Ishii, *Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians*, Comm. Partial Differential Equations **33** (2008), 784–807.

[L.01] O. Ley, *Lower-bound gradient estimates for first-order Hamilton–Jacobi equations and applications to the regularity of propagating fronts*, Adv. Differential Equations **6** (2001), 547–576.

Bounds on the gradient of minimizers in variational denoising

Michał Łasica
Institute of Mathematics of the Polish Academy of Sciences
mlasica@impan.pl

1 Variational denoising and functions of bounded variation

In this talk, we will discuss a class of functionals originating in mathematical imaging. An *image* can be modeled as a function $w \in L^1(\Omega)^N$. Here Ω is a bounded, Lipschitz domain in $\mathbb{R}^d$. In the case of actual images such as photographs, $d = 2$. For monochrome images $N = 1$, for RGB color images $N = 3$. However, the same functionals are also of interest in the analysis of *signals* ($d = 1$) or any other type of data sets.

We consider a class of convex functionals $\mathcal{E}\colon L^1(\Omega)^N \to [0, \infty]$ given as a sum of *fidelity* $\mathcal{F}$ and *regularizer* $\mathcal{R}$,

$$\mathcal{E}(w) = \mathcal{F}(w - f) + \mathcal{R}(w).$$

Here f is a given *noisy image*. The fidelity term measures discrepancy between w and f. For simplicity, let us restrict to the case

$$\mathcal{F}(w - f) = \frac{1}{2}\int_\Omega |w - f|^2 \quad \text{with } f \in L^2(\Omega)^N, \tag{1}$$

where $|\cdot|$ is the Euclidean norm. We will assume that the regularizer is of form

$$\mathcal{R}(w) = \int_\Omega \rho(Dw), \tag{2}$$

where $\rho\colon \mathbb{R}^{N\times d} \to [0, \infty[$ is convex. Note that we can always extend our functionals to the whole $L^1(\Omega)^N$ by prescribing their value as ∞ outside their natural domain. Under broad assumptions on ρ there exists a minimizer u of $\mathcal{E}$. Owing to strict convexity of $\mathcal{F}$, it is unique. In the case of superlinear growth, such as $\rho(\xi) \sim |\xi|^p$ with $p > 1$, finiteness of $\mathcal{E}(w)$ implies Sobolev regularity of w. In particular, the minimizer u cannot have jump discontinuities. This is undesirable in applications to image processing, as images of good quality tend to have sharp contours.

In [15] (see also [7]), the authors proposed to use the total variation regularizer, formally given by

$$TV(w) = \int_\Omega |Dw| \tag{3}$$

for noise removal. Minimization of $\mathcal{E}$ with $\mathcal{R} = \lambda TV$ ($\lambda > 0$ — parameter chosen empirically) came to be known as the Rudin-Osher-Fatemi denoising model. The total variation is a prototypical example of a regularizer of linear growth, which under assumption (2) means that

$$\lambda|\xi| \le \rho(\xi) \le \Lambda(1 + |\xi|) \quad \text{with } \Lambda \ge \lambda > 0. \tag{4}$$

In this case, $\mathcal{E}$ generically does not admit a minimizer in the Sobolev space $W^{1,1}(\Omega)^N$; this is related to the lack of reflexivity of this space. Instead one needs to non-trivially extend the functional to a larger space

$$BV(\Omega)^N = \{w \in L^1(\Omega)^N \colon Dw \text{ is a vector Radon measure}\}.$$

For $w \in BV(\Omega)^N$, $Dw = D^a w + D^s w$ is the decomposition of Dw into absolutely continuous and singular part w. r. to the Lebesgue measure and $Dw = \nu_w\, |Dw|$ is the polar decomposition of Dw. Note that we use here for simplicity the "physicist's notation" where measures on Ω are treated as functions on Ω, in particular the symbol "d" is omitted and the Lebesgue measure is identified with the unity on Ω, and therefore also omitted. One then defines a measure $\rho(Dw)$ by

$$\rho(Dw) = \rho(D^a w) + \rho^\infty(\nu_w)|D^s w|,$$

where $\rho^\infty \colon \mathbb{R}^{N\times d} \to [0,\infty[$ given by $\rho^\infty(\xi) = \lim_{t\to\infty} \rho(t\xi)/t$ for $\xi \in \mathbb{R}^{N\times d}$ is the perspective function of ρ. With this understanding, extending $\mathcal{R}$ by ∞ outside $BV(\Omega)^N$, $\mathcal{E}$ forms a lower semicontinuous functional on $L^1(\Omega)^N$ [10, 9, 1]. By the Rellich-Kondrashov theorem, the embedding $BV(\Omega)^N \subset L^1(\Omega)^N$ is compact and so $\mathcal{E}$ indeed admits a (unique) minimizer u.

The space $BV(\Omega)^N$ is significantly larger than $W^{1,1}(\Omega)^N$; in particular it is not separable. For example, characteristic function of any Lipschitz proper subset U of Ω belongs to $BV(\Omega)$, but not to $W^{1,1}(\Omega)$. In this case, $D\mathbf{1}_U = -\nu^U \mathcal{H}^{d-1} \llcorner \partial U$, where ν^U is the outer normal vector of U and $\mathcal{H}^{d-1} \llcorner \partial U$ denotes the restriction of the $d-1$ dimensional Hausdorff measure to ∂U. In general, for any $w \in BV(\Omega)^N$ one can extract a *jump part* of Dw that is supported on a $d-1$-dimensional set. In order to introduce it, let us recall that, given a function $w \in L^1(\Omega)^N$, $x \in \Omega$ is called a *point of approximate continuity* of w if there exists $z \in \mathbb{R}^N$ such that

$$\lim_{r\to 0^+} \frac{1}{r^d} \int_{B_x(r)\cap\Omega} |w - z| = 0.$$

On the other hand, if there exist $\nu \in \mathbb{R}^d$, $w^-, w^+ \in \mathbb{R}^N$, $w^- \neq w^+$ such that

$$\lim_{r\to 0^+} \frac{1}{r^d} \int_{B^\pm_{x,\nu}(r)\cap\Omega} |w - w^\pm| = 0, \text{ where } B^\pm_{x,\nu}(r) = \{y \in B_x(r) \colon \pm\nu\cdot(y-x) > 0\},$$

then x is an *approximate jump point* of w. Denoting by S_w the set of points of approximate discontinuity of w (i. e. the complement of points of approximate continuity) and by J_w the jump set of w (i. e. the set of approximate jump points of w), we have $J_w \subset S_w$. The Federer-Vol'pert theorem [1] states that if $w \in BV(\Omega)^N$, then S_w is a countably $\mathcal{H}^{d-1}$-rectifiable set, $\mathcal{H}^{d-1}(S_w \setminus J_w) = 0$ and

$$D^j w := Dw \llcorner J_w = \nu_w \otimes (w^+ - w^-)\, \mathcal{H}^{d-1} \llcorner J_w.$$

Clearly $D^j w \perp \mathcal{L}^d$, so we can ultimately decompose Dw into the (Lebesgue) absolutely continuous part, the jump part, and the remaining *Cantor part* $D^c w = Dw \llcorner \Omega \setminus S_u$:

$$Dw = D^a w + D^j w + D^c w.$$

2 Bounds on the gradient in the scalar case

It is natural to ask whether the minimizer u of $\mathcal{E}$ does not actually belong to $W^{1,1}(\Omega)^N$; or if it does not, what can we say about the singular part of the gradient $D^s u$. Let us for now focus on the scalar case $N = 1$. Looking at examples in the simple case $\mathcal{R} = \lambda TV$, $d = 1$ (see Figure 1), we observe that if $f \in BV(\Omega) \setminus W^{1,1}(\Omega)$ has jump discontinuities, the minimizers (for small enough $\lambda > 0$) also exhibit jump discontinuities, corresponding to non-trivial singular gradient. However, those discontinuities occur only at jump points of f. In fact this is a consequence of a more general observation.

Theorem 1 (Briani, Chambolle, Novaga, Orlandi 2011 [4]; Bonforte, Figalli 2012 [3])**.** *Let $\mathcal{R} = \lambda TV$, $d = 1$ and $f \in BV(\Omega)$. Then*

$$|Du| \leq |Df| \quad \textit{as Borel measures.} \tag{5}$$

In other words, $|Du|(B) \leq |Df|(B)$ for any Borel set $B \subset \Omega$. This proposition can be demonstrated using density of step functions; if f is a step function, the minimizer can be explicitly characterized.

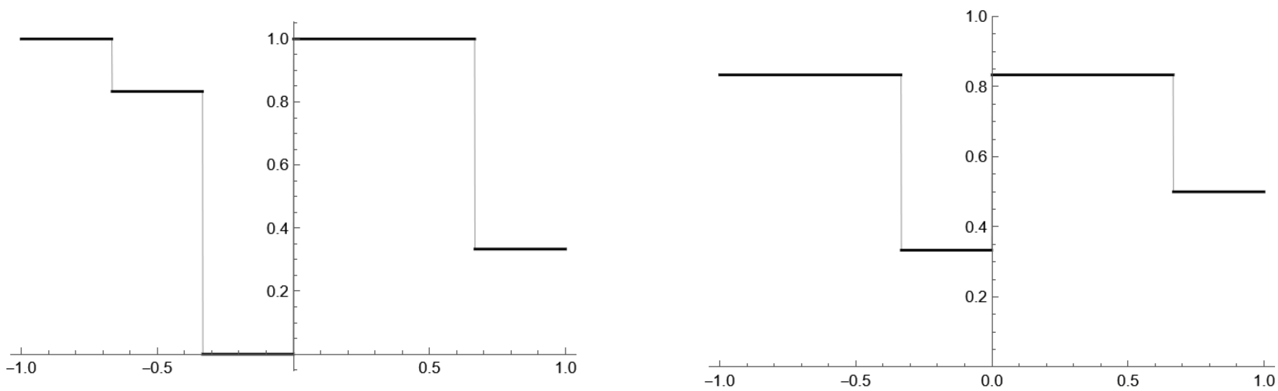


Figure 1: Example f and u in the case $d = N = 1$, $\mathcal{R} = \lambda TV$, $\lambda = 1/6$.

A pointwise estimate on the whole measure Du of form (5) does not hold if $d > 1$. In fact, it remains an open question even in the case of scalar TV regularizer whether

$$|D^s u| \leq |D^s f| \quad \text{as Borel measures.}$$

However, analogous estimate is known for the jump part of Du.

Theorem 2 (Caselles, Chambolle, Novaga 2007 [5]; Caselles, Jalalzai, Novaga 2013 [6])**.** *Let $\mathcal{R} = \lambda TV$, $d > 1$ and $f \in BV(\Omega) \cap L^\infty(\Omega)$. Then*

$$|D^j u| \leq |D^j f| \quad \textit{as Borel measures.} \tag{6}$$

The proof follows via analysis of level sets of the minimizer, which can be showed to solve a prescribed mean curvature problem. This technique, as it is, does not seem to be applicable if $N > 1$. However, the authors generalize it to a class of ρ different than the Euclidean norm, including other smooth norms.

If ρ is not differentiable, there are examples where (6) fails. A simple example like that in the case TV_1, where ρ is the ℓ^1 norm on $\mathbb{R}^d$, can be found in [14], see Figure 2. This is

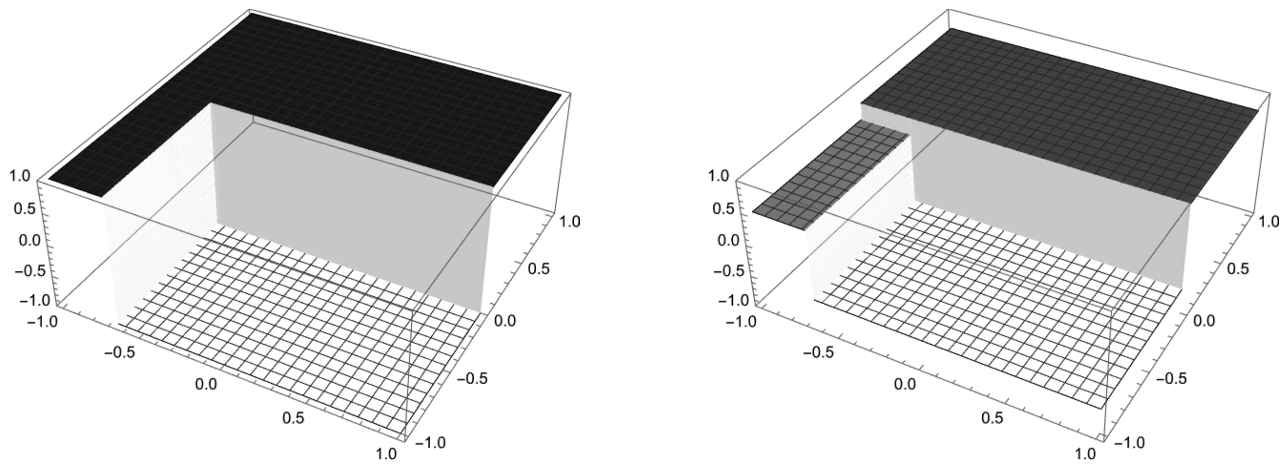


Figure 2: Example f and u in the case $d = 2$, $\mathcal{R} = \lambda TV_1$ with a suitable choice of λ.

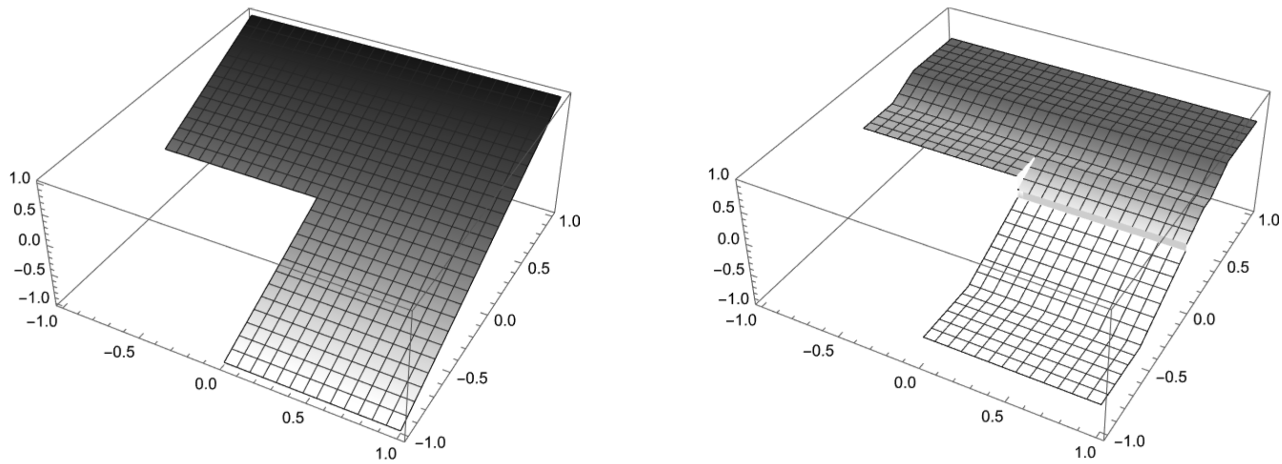


Figure 3: Example f and u in the case $d = 2$, $\mathcal{R} = \lambda TV_1$ with a suitable choice of λ.

related to the phenomenon of *facet breaking* observed in crystalline mean curvature flows, see e. g. [2]. More strikingly, the minimizer u can exhibit jump discontinuities even if f is smooth up to the boundary [12], see Figure 3. However, this can only happen if Ω is non-convex.

Theorem 3 (Ł., Rybka 2021 [13])**.** *Let $\mathcal{R}$ be of form (2), where ρ is any convex function of linear growth, and suppose that Ω is convex. If $f \in BV(\Omega)$, then*

$$\int_\Omega \rho^\infty(D^s u) \leq \int_\Omega \rho^\infty(D^s f). \tag{7}$$

In particular, if $f \in W^{1,1}(\Omega)$, then $u \in W^{1,1}(\Omega)$. Moreover, if $f \in W^{1,p}(\Omega)$ then $u \in W^{1,p}(\Omega)$ for $p > 1$.

In the proof, working with a smooth regularization of the problem, we obtain an estimate for solutions to the corresponding Euler-Lagrange equation

$$\operatorname{div} D_\xi \rho(Du) = u - f \text{ in } \Omega, \qquad \nu^\Omega \cdot D_\xi \rho(Du) = 0 \text{ on } \partial\Omega. \tag{8}$$

Here we have denoted by $D_\xi \rho$ the derivative of ρ with respect to its argument, to distinguish it from derivatives of functions on Ω. We differentiate the equation and test it with $D_\xi \psi(Du)$, where $\psi = \widetilde{\psi} \circ \rho$, $\widetilde{\psi}$ convex and non-decreasing. On the r. h. s. we get by convexity of ψ

$$\int_\Omega D_\xi \psi(Du) \cdot (Du - Df) \geq \int_\Omega \psi(Du) - \int_\Omega \psi(Df).$$

On the l. h. s., integrating the divergence by parts (we skip details of the calculation) we obtain

$$\int_\Omega D_\xi \psi(Du) \cdot D\mathrm{div}\, D_\xi \rho(Du)$$
$$= - \int_\Omega Tr(D^2_\xi \psi(Du)\, D^2 u\, D^2_\xi \rho(Du)\, D^2 u) + \int_{\partial\Omega} D_\xi \psi(Du) \cdot DD_\xi \rho(Du) \cdot \nu^\Omega. \quad (9)$$

The integrand in the first term is non-negative by convexity of ρ and ψ. As for the second term, we extend ν^Ω smoothly to a neighborhood of $\partial\Omega$ and calculate

$$D_\xi \psi(Du) \cdot DD_\xi \rho(Du) \cdot \nu^\Omega = \widetilde{\psi}'(\rho(Du)) D_\xi \rho(Du) \cdot DD_\xi \rho(Du) \cdot \nu^\Omega$$
$$= \widetilde{\psi}'(\rho(Du)) D_\xi \rho(Du) \cdot D(D_\xi \rho(Du) \cdot \nu^\Omega) - \widetilde{\psi}'(\rho(Du)) D_\xi \rho(Du) \cdot D\nu^\Omega \cdot D_\xi \rho(Du).$$

The first term on the r. h. s. vanishes owing to the Neumann boundary condition $D_\xi \rho(Du) \cdot \nu^\Omega = 0$. The second term can be rewritten as $\widetilde{\psi}'(\rho(Du)) \mathcal{A}^{\partial\Omega}(D_\xi \rho(Du), D_\xi \rho(Du))$, where $\mathcal{A}^{\partial\Omega}$ is the second fundamental form of $\partial\Omega$. By convexity of Ω it is non-negative. Summing up, we obtain

$$\int_\Omega \psi(Du) \leq \int_\Omega \psi(Df).$$

Passing to the limit with the regularization parameter (omitted in the calculations above), we obtain for $f \in BV(\Omega)$ and $\widetilde{\psi}$ of linear growth

$$\int_\Omega \psi(D^a u) + \int_\Omega \psi^\infty(\nu_u)\, |D^s u| \leq \int_\Omega \psi(D^a f) + \int_\Omega \psi^\infty(\nu_f)\, |D^s f|.$$

Choosing $\psi(\xi) = (\rho(\xi) - k)_+$, $k > 0$, we have $\psi^\infty = \rho^\infty$. Passing to the limit $k \to \infty$ we obtain (7).

On the other hand, choosing $\psi(\xi) = \rho(\xi)^p$, $p > 1$, we obtain for any $f \in W^{1,p}(\Omega)$

$$\int_\Omega \rho(Du)^p \leq \int_\Omega \rho(Df)^p,$$

thus proving the second part of the assertion.

3 Bounds on the gradient in the vectorial case

As in the case of Theorem 2, the proof of Theorem 3 only works in the scalar case $N = 1$. If $N > 1$, the first term on the r. h. s. of (9) does not necessarily have a definite sign. However, a similar technique can be used to obtain local estimates in the case $N > 1$, $d = 1$, generalizing Theorem 1.

Theorem 4 (Grochulska, Ł. preprint 2022 [11]). *Let $N > 1$, $d = 1$ and let $f \in BV(\Omega)^N$. Suppose that $\mathcal{R}$ is of form* (2)*. If ρ is a norm, then*

$$|Du| \leq |Df| \quad \textit{as Borel measures.}$$

If $\rho = \widetilde{\rho} \circ \varphi$, where φ is a norm on $\mathbb{R}^N$ and $\widetilde{\rho}$ is convex, increasing and of linear growth, then there exists $c \geq 1$ such that

$$|D^s u| \leq c|D^s f| \quad \textit{as Borel measures.} \tag{10}$$

If moreover φ is the Euclidean norm or ρ is strictly convex and differentiable, (10) *holds with $c = 1$.*

Let us give a sketch of proof of the first part of the theorem, where ρ is assumed to be a norm. We again work with a regularized problem, while still assuming for simplicity that the regularized ρ is a norm. We differentiate the Euler-Lagrange equation (8) and test it with $hDu/|Du|$, where h is a non-negative C_c^∞ function on Ω. On the r. h. s., recalling that $\xi/|\xi|$ is the derivative of the convex function $\xi \mapsto |\xi|$,

$$\int_\Omega h\frac{Du}{|Du|} \cdot (Du - Df) \geq \int_\Omega h|Du| - \int_\Omega h|Df|.$$

On the l. h. s., integrating by parts,

$$\begin{aligned}\int_\Omega h\frac{Du}{|Du|}\cdot D^2(D_\xi\rho(Du)) &= -\int_\Omega \left(Dh\frac{Du}{|Du|} + \frac{h}{|Du|}(D^2u - \frac{Du}{|Du|}\cdot D^2u\frac{Du}{|Du|})\right)\cdot D_\xi^2\rho(Du)D^2u \\ &= -\int_\Omega \frac{h}{|Du|}D^2u\cdot D_\xi^2\rho(Du)D^2u \leq 0,\end{aligned}$$

where we have used convexity of ρ and that by 1-homogeneity of ρ

$$D_\xi^2\rho(Du)\,Du = \frac{\mathrm{d}}{\mathrm{d}t}D_\xi\rho(tDu)\Big|_{t=1} = \frac{\mathrm{d}}{\mathrm{d}t}D_\xi\rho(Du)\Big|_{t=1} = 0.$$

Thus, we obtain for any non-negative $h \in C_c^\infty(\Omega)$

$$\int_\Omega h|Du| \leq \int_\Omega h|Df|.$$

This inequality passes to the limit with vanishing regularization parameter in unchanged form. By a standard result in measure theory, we conclude that $|Du| \leq |Df|$ as measures.

The proof of the second assertion follows along similar lines, however we need to use a more involved test function of form $h\, g_k(\varphi(Du))Du$. This form lets us exploit homogeneity of φ as before. A choice that works turns out to be

$$g_k(\sigma) = \frac{(\sigma - k)_+}{\sigma^2}\widetilde{\rho}'(\sigma).$$

The role of the parameter k is as in the proof of Theorem 3. However, $\xi \mapsto g_k(\varphi(\xi))\xi$ is not a derivative of convex function, which makes the limit passage in the proof more difficult. This can be circumvented by using equivalence of norms on $\mathbb{R}^N$, which leads to appearance of the constant c in (10). From the point of view of image processing context, where the minimization procedure is often iterated, such constant is undesirable. Alternatively, under the stronger assumption on ρ, we have stronger convergence of the approximating sequence, which leads to the optimal result with $c = 1$. One could ask whether we could use density of "regular" integrands ρ to prove the optimal result for general ρ. This is prevented by very weak continuity properties of the "singular part map" $u \mapsto D^s u$. In particular, it is not continuous with respect to the strict (or area-strict) convergence on $BV(\Omega)^N$.

As mentioned before, this type of methods has not yet produced results in the case $d > 1$, $N > 1$. On the other hand, it turns out it is possible to generalize Theorem 2 to the multi-dimensional vectorial case. A major step in this direction was made by T. Valkonen [16], who proposed a variational technique based on a special construction of competitors involving inner variations. He introduced an assumption of *double-Lipschitz comparability* of regularizers, under which his method works. This assumption is rather complicated and not easy to check. It is checked for scalar TV and so-called Huber-TV in [16], and for some more complicated regularizers in [17]. The case $N > 1$ has not been explicitly considered in those papers. Moreover, an estimate on the "vertical" size of jumps is not provided, only jump inclusion $J_u \subset J_f$.

It turns out that jump inclusion as well as bounds on jump size can be derived under a modest assumption of differentiability of the regularizer along inner variations at u, i. e.

$$\tau \mapsto \mathcal{R}(u_h^\tau) \text{ is differentiable at } 0 \text{ for any } h \in C_c^\infty(\Omega)^d, \text{ where } u_h^\tau(x) = u(x + \tau h(x)). \quad (11)$$

Theorem 5 (Chambolle, Ł. in preparation [8])**.** *Let $f \in BV(\Omega)^N \cap L^\infty(\Omega)^N$ and suppose that $\mathcal{E}$ has a minimizer $u \in BV(\Omega)^N \cap L^\infty(\Omega)^N$. If $\mathcal{R}$ is convex and* (11) *holds, then $J_u \subset J_f$ and*

$$|u^+ - u^+|^2 \leq (u^+ - u^-) \cdot (f^+ - f^-) \quad \mathcal{H}^{d-1}\text{-a. e. on } J_u. \quad (12)$$

In particular,

$$|D^j u| \leq |D^j f| \quad \text{as Borel measures.}$$

Note that here we do not assume anything about the structure of $\mathcal{R}$, instead requiring directly that $\mathcal{E}$ attains its minimum on $BV(\Omega)^N$. This is the case for any $\mathcal{R}$ of form (2) with ρ of linear growth, but there are many more examples. We also require that the minimizer u is bounded for bounded f. In the scalar case $N = 1$ this holds for $\mathcal{R}$ of form (2) with any ρ of linear growth. In the vectorial case this is not necessarily the case, but it does hold for most typical examples. Alternatively, it can always be enforced by adding a pointwise box constraint $w(x) \in K \subset \mathbb{R}^N$ (with K convex, bounded) to the regularizer. If a regularizer $\mathcal{R}$ is differentiable along inner variations, then its constrained version also has this property. In the case of $\mathcal{R}$ of form (2), the differentiability assumption (11) is satisfied for ρ such as coordinate-wise ℓ^p norms, $1 < p < \infty$, or more natural *p-Schatten norms* $A \mapsto (\mathrm{Tr}(A^T A)^{p/2})^{1/p}$ with $1 \leq p < \infty$, in particular for the *nuclear norm* (i. e. the 1-Schatten norm). On the other hand, it does not hold for the ∞-Schatten norm (i. e. the *operator norm*). However it does hold for the so-called *softmax function* $A \mapsto \log(\mathrm{Tr} \exp \sqrt{A^T A})$, whose recession function coincides with the operator norm. Moreover, any $\mathcal{R}$ of form (2) with convex ρ of linear growth such that ρ and ρ^∞ are differentiable outside 0 satisfies (11).

The basic premise of the proof is that while $\mathcal{R}$ is differentiable along inner variations at u, the fidelity term is not, in general. Since u is the minimizer, the difference between left and right limits of difference quotients gives some information about u at jump points. However considering pure inner variations only gives weak information. An important idea found in [16] is to use mixed variations of form

$$u_h^{\tau,\vartheta} = \vartheta u_h^\tau + (1 - \vartheta) u.$$

The regularizer is not necessarily differentiable along such variations, but by convexity of $\mathcal{R}$ we nevertheless have, denoting $R_h(\tau) = \mathcal{R}(u_h^\tau)$,

$$\tfrac{1}{\tau}(\mathcal{R}(u_h^{\pm\tau,\vartheta}) - \mathcal{R}(u)) \le \tfrac{\vartheta}{\tau}(\mathcal{R}(u_h^{\pm\tau}) - \mathcal{R}(u)) \to \pm\vartheta R_h'(0) \quad \text{as } \tau \to 0$$

for $\vartheta \in [0,1]$. Therefore, by minimality of u,

$$0 \le \pm\vartheta R_h'(0) + \liminf_{\tau\to 0^+} \tfrac{1}{\tau}(\mathcal{F}(u_h^{\pm\tau,\vartheta} - f) - \mathcal{F}(u-f))$$

and, summing these two inequalities,

$$\begin{aligned}
0 &\le \liminf_{\tau\to 0^+} \tfrac{1}{\tau}(\mathcal{F}(u_h^{\tau,\vartheta} - f) - \mathcal{F}(u-f)) + \liminf_{\tau\to 0^+} \tfrac{1}{\tau}(\mathcal{F}(u_h^{-\tau,\vartheta} - f) - \mathcal{F}(u-f)) \\
&= \liminf_{\tau\to 0^+} \tfrac{1}{2\tau}\int_\Omega |\vartheta u_h^\tau + (1-\vartheta)u - f|^2 - |u-f|^2 + \liminf_{\tau\to 0^+} \tfrac{1}{2\tau}\int_\Omega |\vartheta u_h^{-\tau} + (1-\vartheta)u - f|^2 - |u-f|^2 \\
&= \liminf_{\tau\to 0^+} \tfrac{\vartheta}{2\tau}\int_\Omega (\vartheta u_h^\tau + (2-\vartheta)u - 2f)\cdot(u_h^\tau - u) + \liminf_{\tau\to 0^+} \tfrac{\vartheta}{2\tau}\int_\Omega (\vartheta u_h^{-\tau} + (2-\vartheta)u - 2f)\cdot(u_h^{-\tau} - u).
\end{aligned} \tag{13}$$

Now we recall that a cylinder $Q_{x_0}(r)$ of small radius $r > 0$ and length $2r$ centered at an approximate jump point x_0 of u with axis $\nu_u(x_0)$ can be divided into half-cylinders where $u \approx u^\pm(x_0)$. If we take h approximating $\mathbf{1}_{Q_{x_0}(r)}\nu_u(x_0)$, then $u_h^\tau - u \approx 0$ except at a cylindrical strip of radius r and length τ, where $u_h^\tau \approx u^+$ and $u \approx u^-$. Similarly, $u_h^{-\tau} - u \approx 0$ except at a cylindrical strip where $u_h^{-\tau} \approx u^-$ and $u \approx u^+$. Thus, evaluating the limits in (13) and averaging over the base of the cylinder, we get roughly

$$0 \le \tfrac{\vartheta}{2}(\vartheta u^+ + (2-\vartheta)u^- - 2f^-)\cdot(u^+ - u^-) + \tfrac{\vartheta}{2}(\vartheta u^- + (2-\vartheta)u^+ - 2f^+)\cdot(u^- - u^+),$$

where $f^\pm$ are one-sided traces of f along the jump of u which are known to exist $\mathcal{H}^{d-1}$-a. e. [1]. Dividing the inequality by ϑ and passing to the limit $\vartheta \to 0^+$, we get

$$0 \le (u^- - f^-)\cdot(u^+ - u^-) + (u^+ - f^+)\cdot(u^- - u^+) = (f^+ - f^-)\cdot(u^+ - u^-) - (u^+ - u^-)\cdot(u^+ - u^-).$$

In particular $J_u \subset J_f$ and (12) holds. We also deduce

$$|u^+ - u^-| \le |f^+ - f^-|$$

and hence $|D^j u| \le |D^j f|$ as measures.

We note that Theorem 5 can be generalized in several directions. For example, we can consider fidelity terms of form

$$\mathcal{F}(w-f) = \int_\Omega \phi(w-f)$$

for any convex ϕ. If ϕ is strictly convex, then jump inclusion $J_u \subset J_f$ holds. We can also obtain bounds similar to (12), which however depend strongly on ϕ.

References

[1] Luigi Ambrosio, Nicola Fusco, Diego Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2000).

[2] Giovanni Bellettini, Matteo Novaga, Maurizio Paolini. Facet-breaking for three-dimensional crystals evolving by mean curvature. *Interfaces Free Bound.* 1 (1999) pp. 39–55. doi:10.4171/IFB/3.

[3] Matteo Bonforte, Alessio Figalli. Total variation flow and sign fast diffusion in one dimension. *J. Differential Equations* 252 (2012) pp. 4455–4480. doi:10.1016/j.jde.2012.01.003.

[4] Ariela Briani, Antonin Chambolle, Matteo Novaga, Giandomenico Orlandi. On the gradient flow of a one-homogeneous functional. *Confluentes Math.* 3 (2011) pp. 617–635. doi:10.1142/S1793744211000461.

[5] Vicent Caselles, Antonin Chambolle, Matteo Novaga. The discontinuity set of solutions of the TV denoising problem and some extensions. *Multiscale Model. Simul.* 6 (2007) pp. 879–894. doi:10.1137/070683003.

[6] Vicent Caselles, Khalid Jalalzai, Matteo Novaga. On the jump set of solutions of the total variation flow. *Rend. Semin. Mat. Univ. Padova* 130 (2013) pp. 155–168. doi: 10.4171/RSMUP/130-5.

[7] Antonin Chambolle, Pierre-Louis Lions. Image recovery via total variation minimization and related problems. *Numer. Math.* 76 (1997) pp. 167–188. doi:10.1007/s002110050258.

[8] Antonin Chambolle, Michał Łasica. Jump inclusion for minimizers in variational denoising (in preparation).

[9] Françoise Demengel, Roger Temam. Convex functions of a measure and applications. *Indiana University Mathematics Journal* 33 (1984) pp. 673–709.

[10] Casper Goffman, James Serrin. Sublinear functions of measures and variational integrals. *Duke Mathematical Journal* 31 (1964) pp. 159 – 178.

[11] Zofia Grochulska, Michał Łasica. Local estimates for vectorial Rudin-Osher-Fatemi type problems in one dimension. `arXiv:1406.1775` (2022).

[12] Michał Łasica, Salvador Moll, Piotr B. Mucha. Total variation denoising in l^1 anisotropy. *SIAM J. Imaging Sci.* 10 (2017) pp. 1691–1723. doi:10.1137/16M1103610.

[13] Michał Łasica, Piotr Rybka. Existence of $W^{1,1}$ solutions to a class of variational problems with linear growth on convex domains. *Indiana Univ. Math. J.* 70 (2021) pp. 2427–2450. doi:10.1512/iumj.2021.70.9479.

[14] Salvador Moll. The anisotropic total variation flow. *Math. Ann.* 332 (2005) pp. 177–218. doi:10.1007/s00208-004-0624-0.

[15] Leonid Rudin, Stanley Osher, Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena* 60 (1992) pp. 259–268.

[16] Tuomo Valkonen. The jump set under geometric regularization. Part 1: basic technique and first-order denoising. *SIAM J. Math. Anal.* 47 (2015) pp. 2587–2629. doi: 10.1137/140976248.

[17] Tuomo Valkonen. The jump set under geometric regularisation. Part 2: Higher-order approaches. *J. Math. Anal. Appl.* 453 (2017) pp. 1044–1085. doi: 10.1016/j.jmaa.2017.04.037.

Time decay estimates of L_p-L_q type for the Stokes semigroup arising from two-phase incompressible viscous flows

Hirokazu Saito (The University of Electro-Communications)*

1 Introduction

Let us consider a two-phase free boundary problem for two incompressible viscous fluids, fluid$_+$ and fluid$_-$, in the presence of a uniform gravitational field acting vertically downward in $\mathbf{R}^N$ for $N \geq 2$. At time $t \geq 0$, fluid$_+$ occupies

$$\Omega_+(t) = \{(x', x_N) : x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N > \eta(x', t)\},$$

while fluid$_-$ occupies

$$\Omega_-(t) = \{(x', x_N) : x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N < \eta(x', t)\},$$

where $\eta = \eta(x', t)$ is called the height function and needs to be determined as part of the problem. The interface $\Gamma(t)$ between fluid$_+$ and fluid$_-$ is given by

$$\Gamma(t) = \{(x', x_N) : x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}, x_N = \eta(x', t)\}.$$

Let $\rho_\pm$ be the densities of fluid$_\pm$, and let $\mu_\pm$ be the viscosity coefficients of fluid$_\pm$. In this talk, we assume that ρ_+, ρ_-, μ_+, and μ_- are positive constants and that surface tension is included on $\Gamma(t)$. We call fluid$_+$ the upper fluid and fluid$_-$ the lower fluid.

Define $\dot{\Omega}(t) = \Omega_+(t) \cup \Omega_-(t)$ for each $t \geq 0$. We denote the fluid velocity by $\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), \dots, u_N(x, t))^\mathsf{T}$, where the superscript T denotes the transpose, and the fluid pressure by $\mathfrak{p} = \mathfrak{p}(x, t)$ at $x \in \dot{\Omega}(t)$ with $t \geq 0$. The above two-phase free boundary problem is governed by the two-phase Navier-Stokes equations, see e.g. [3] or [8], and admits the trivial steady state $(\eta, \mathbf{u}, \mathfrak{p}) = (0, 0, c)$ with a constant c. We linearize the two-phase Navier-Stokes equations around the trivial steady state, and then we achieve the following two-phase Stokes equations:

$$\begin{cases} \partial_t \eta - u_N|_{x_N=0} = d & \text{on } \mathbf{R}^{N-1},\ t > 0, \\ \partial_t \mathbf{u} - \rho^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \mathbf{f} & \text{in } \dot{\mathbf{R}}^N,\ t > 0 \\ \operatorname{div} \mathbf{u} = g & \text{in } \dot{\mathbf{R}}^N,\ t > 0, \\ [\![(\mu \mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{e}_N]\!] + ([\![\rho]\!]\gamma_a + \sigma\Delta')\eta \mathbf{e}_N = \mathbf{h} & \text{on } \mathbf{R}^{N-1},\ t > 0, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^{N-1},\ t > 0, \\ \eta|_{t=0} = \eta_0 & \text{on } \mathbf{R}^{N-1}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot{\mathbf{R}}^N, \end{cases} \tag{1.1}$$

where

$$\dot{\mathbf{R}}^N = \mathbf{R}^N_+ \cup \mathbf{R}^N_-, \quad \mathbf{R}^N_\pm = \{(x', x_N) : x' = (x_1, \dots, x_{N-1}), \pm x_N > 0\}.$$

*e-mail: hsaito@uec.ac.jp

Here the right members d, $\mathbf{f}$, g, $\mathbf{h}$, η_0, and $\mathbf{u}_0$ are given functions; $u_N|_{x_N=0}$ is the trace of u_N on the hyperplane $x_N = 0$; ρ and μ are given by

$$\rho = \begin{cases} \rho_+ & \text{in } \mathbf{R}^N_+, \\ \rho_- & \text{in } \mathbf{R}^N_-, \end{cases} \quad \mu = \begin{cases} \mu_+ & \text{in } \mathbf{R}^N_+, \\ \mu_- & \text{in } \mathbf{R}^N_-; \end{cases} \tag{1.2}$$

$\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{e}_N = (0, \dots, 0, 1)^\mathsf{T}$; γ_a is the acceleration of gravity, while σ is the surface tension coefficient, where both γ_a and σ are positive constants.

Let $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ for $j = 1, \dots, N$. Then $\operatorname{div} \mathbf{u} = \sum_{j=1}^N \partial_j u_j$, $\Delta' \eta = \sum_{j=1}^{N-1} \partial_j^2 \eta$, and

$$\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^\mathsf{T} \quad \text{for } \nabla \mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \dots & \partial_N u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_N & \dots & \partial_N u_N \end{pmatrix}.$$

Furthermore, we set for a scalar-valued function $u = u(x)$

$$\nabla u = (\partial_1 u, \dots, \partial_N u)^\mathsf{T}, \quad \Delta u = \sum_{j=1}^N \partial_j^2 u$$

and set for a matrix-valued function $\mathbf{M} = (M_{ij}(x))_{1 \le i,j \le N}$

$$\operatorname{Div} \mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \dots, \sum_{j=1}^N \partial_j M_{Nj} \right)^\mathsf{T}.$$

This gives us

$$\operatorname{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \mu(\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u}) - \nabla \mathfrak{p} \quad \text{in } \dot{\mathbf{R}}^N,$$

where $\Delta \mathbf{u} = (\Delta u_1, \dots, \Delta u_N)$.

For $f = f(x) = f(x', x_N)$ defined on $\dot{\mathbf{R}}^N$, $[\![f]\!]$ stands for the jump of the quantity f across the flat interface $x_N = 0$, i.e.,

$$[\![f]\!] = [\![f]\!](x') = \lim_{x_N \to 0+} \left(f(x', x_N) - f(x', -x_N) \right),$$

where $\lim_{s \to a+} g(s)$ is the right-hand limit of $g(s)$ at $a \in \mathbf{R}$. In particular, $[\![\rho]\!] = \rho_+ - \rho_-$.

It is well-know that the trivial steady state is unstable due to the effect of gravity if the upper fluid is heavier than the lower one, i.e., if $\rho_+ > \rho_-$. This instability is called the Rayleigh-Taylor instability, see [1], [3]. On the other hand, we treat in this talk the following two cases: (i) the lower fluid is heavier than the upper one, i.e., $\rho_- > \rho_+$; (ii) the two fluids have equal density, i.e., $\rho_- = \rho_+$. In these two cases, we introduce time decay estimates of L_p-L_q type for an analytic C_0-semigroup associated with (1.1).

2 Semigroup setting

Let G be an open set in $\mathbf{R}^N$ and $1 \le p \le \infty$. We denote the Lebesgue spaces on G by $L_p(G)$ and the Sobolev spaces on G by $H_p^m(G)$ for $m \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $q \in (1, \infty)$. Define

$$\widehat{H}_q^1(\mathbf{R}^N) = \{u \in L_{1,\mathrm{loc}}(\mathbf{R}^N) : \partial_x^\alpha u \in L_q(\mathbf{R}^N) \text{ for } |\alpha| = 1\},$$

where $\partial_x^\alpha u = \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N} u$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}_0^N$. The Sobolev-Slobodeckij spaces on $\mathbf{R}^{N-1}$ are denoted by $W_q^s(\mathbf{R}^{N-1})$ for $s \in (0,\infty) \setminus \mathbf{N}$. For $\mathbf{a} = (a_1(x), \dots, a_N(x))^\mathsf{T}$ and $\mathbf{b} = (b_1(x), \dots, b_N(x))^\mathsf{T}$, we set

$$(\mathbf{a}, \mathbf{b})_{\dot{\mathbf{R}}^N} = \sum_{j=1}^N \int_{\dot{\mathbf{R}}^N} a_j(x) b_j(x)\, dx.$$

Let us consider the following weak problem: for a given $\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N$ find $u \in \widehat{H}_q^1(\mathbf{R}^N)$ such that

$$(\rho^{-1} \nabla u, \nabla \varphi)_{\dot{\mathbf{R}}^N} = (\mathbf{f}, \nabla \varphi)_{\dot{\mathbf{R}}^N} \quad \text{for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N), \tag{2.1}$$

where ρ is defined as (1.2) and $q' = q/(q-1)$. The next proposition is proved by [7].

Proposition 2.1. *Let $q \in (1,\infty)$ and $q' = q/(q-1)$. Then the following assertions hold.*

(1) **Existence**. *For any $\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N$, (2.1) admits a solution $u \in \widehat{H}_q^1(\mathbf{R}^N)$ satisfying $\|\nabla u\|_{L_q(\mathbf{R}^N)} \le C \|\mathbf{f}\|_{L_q(\dot{\mathbf{R}}^N)}$ with some positive constant C.*

(2) **Uniqueness**. *If $u \in \widehat{H}_q^1(\mathbf{R}^N)$ satisfies*

$$(\rho^{-1} \nabla u, \nabla \varphi)_{\dot{\mathbf{R}}^N} = 0 \quad \textit{for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N),$$

then $u = c$ for some constant c.

Let $\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)$ and $g \in W_q^{1-1/q}(\mathbf{R}^{N-1})$. Choose an extension $\widetilde{g} \in H_q^1(\mathbf{R}_+^N)$ of g such that $\widetilde{g}|_{x_N=0} = g$ on $\mathbf{R}^{N-1}$ and $\|\widetilde{g}\|_{H_q^1(\mathbf{R}_+^N)} \le C \|g\|_{W_q^{1-1/q}(\mathbf{R}^{N-1})}$. Let $\widetilde{g}_0$ be the zero extension of $\widetilde{g}$, i.e., $\widetilde{g}_0 = \widetilde{g}$ in $\mathbf{R}_+^N$ and $\widetilde{g}_0 = 0$ in $\mathbf{R}_-^N$. Proposition 2.1 tells us that there exists $\widetilde{u} \in \widehat{H}_q^1(\mathbf{R}^N)$ such that

$$(\rho^{-1} \nabla \widetilde{u}, \nabla \varphi)_{\dot{\mathbf{R}}^N} = (\mathbf{f} - \rho^{-1} \nabla \widetilde{g}_0, \nabla \varphi)_{\dot{\mathbf{R}}^N} \quad \text{for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N)$$

and

$$\|\nabla \widetilde{u}\|_{L_q(\mathbf{R}^N)} \le C \Big(\|\mathbf{f}\|_{L_q(\dot{\mathbf{R}}^N)} + \|g\|_{W_q^{1-1/q}(\mathbf{R}^{N-1})} \Big).$$

Define $u = \widetilde{u} + \widetilde{g}_0 \in \widehat{H}_q^1(\mathbf{R}^N) + H_q^1(\dot{\mathbf{R}}^N)$. Then u satisfies

$$\begin{aligned} &(\rho^{-1} \nabla u, \nabla \varphi)_{\dot{\mathbf{R}}^N} = (\mathbf{f}, \nabla \varphi)_{\dot{\mathbf{R}}^N} \quad \text{for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N), \\ &\|\nabla u\|_{L_q(\dot{\mathbf{R}}^N)} \le C \Big(\|\mathbf{f}\|_{L_q(\dot{\mathbf{R}}^N)} + \|g\|_{W_q^{1-1/q}(\mathbf{R}^{N-1})} \Big), \end{aligned} \tag{2.2}$$

and also it follows from $[\![\widetilde{u}]\!] = 0$ and $[\![\widetilde{g}_0]\!] = g$ that

$$[\![u]\!] = g. \tag{2.3}$$

Let $(\eta, \mathbf{u}) \in W_q^{3-1/q}(\mathbf{R}^{N-1}) \times H_q^2(\dot{\mathbf{R}}^N)^N$. Choose in (2.2) and (2.3)

$$\mathbf{f} = \rho^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{u})), \quad g = \mathbf{e}_N \cdot [\![\mu \mathbf{D}(\mathbf{u}) \mathbf{e}_N]\!] + ([\![\rho]\!] \gamma_a + \sigma \Delta') \eta,$$

where ρ, μ are defined as (1.2). Then the mapping

$$\mathcal{K} : W_q^{3-1/q}(\mathbf{R}^{N-1}) \times H_q^2(\dot{\mathbf{R}}^N)^N \ni (\eta, \mathbf{u}) \mapsto \mathcal{K}(\eta, \mathbf{u}) \in \widehat{H}_q^1(\mathbf{R}^N) + H_q^1(\dot{\mathbf{R}}^N)$$

can be defined in the following manner:

$$\begin{cases} (\rho^{-1}\nabla\mathcal{K}(\eta, \mathbf{u}), \nabla\varphi)_{\dot{\mathbf{R}}^N} = (\rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})), \nabla\varphi)_{\dot{\mathbf{R}}^N} \quad \text{for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N), \\ \qquad\qquad [\![\mathcal{K}(\eta, \mathbf{u})]\!] = \mathbf{e}_N \cdot [\![\mu\mathbf{D}(\mathbf{u})\mathbf{e}_N]\!] + ([\![\rho]\!]\gamma_a + \sigma\Delta')\eta \quad \text{on } \mathbf{R}^{N-1}. \end{cases}$$

Define the space of solenoidal vector fields by

$$J_q(\dot{\mathbf{R}}^N) = \{\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N : (\mathbf{f}, \nabla\varphi)_{\dot{\mathbf{R}}^N} = 0 \text{ for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N)\},$$

where $q \in (1, \infty)$ and $q' = q/(q-1)$. We set

$$X_q = W_q^{2-1/q}(\mathbf{R}^{N-1}) \times J_q(\dot{\mathbf{R}}^N),$$
$$\|(\eta, \mathbf{u})\|_{X_q} = \|\eta\|_{W_q^{2-1/q}(\mathbf{R}^{N-1})} + \|\mathbf{u}\|_{L_q(\dot{\mathbf{R}}^N)},$$

and we introduce the Stokes operator A_q as follows:

$$A_q(\eta, \mathbf{u}) = (u_N|_{x_N=0}, \rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) - \mathcal{K}(\eta, \mathbf{u})\mathbf{I}))$$

with the domain

$$\begin{aligned} D(A_q) = \{(\eta, \mathbf{u}) \in (W_q^{3-1/q}(\mathbf{R}^{N-1}) \times H_q^2(\dot{\mathbf{R}}^N)^N) \cap X_q : \\ [\![\mu\mathbf{D}(\mathbf{u})\mathbf{e}_N]\!] - (\mathbf{e}_N \cdot [\![\mu\mathbf{D}(\mathbf{u})\mathbf{e}_N]\!])\mathbf{e}_N = 0,\ [\![\mathbf{u}]\!] = 0 \text{ on } \mathbf{R}^{N-1}\}. \end{aligned}$$

Then A_q generates an analytic C_0-semigroup on X_q, see [5] for more details.

Proposition 2.2. *Suppose that $N \geq 2$ and that ρ_+, ρ_-, μ_+, μ_-, γ_a, and σ are positive constants. Let $q \in (1, \infty)$. Then the following assertions hold.*

(1) *A_q is densely defined closed operator on X_q.*

(2) *A_q generates an analytic C_0-semigroup $(S(t))_{t\geq 0}$ on X_q.*

Remark 2.3. (1) *We call $(S(t))_{t\geq 0}$ the Stokes semigroup in this talk.*

(2) *Let $(\eta, \mathbf{u}) = S(t)(\eta_0, \mathbf{u}_0)$ for $(\eta_0, \mathbf{u}_0) \in X_q$ with $q \in (1, \infty)$. Then $(\eta, \mathbf{u})$ is a unique solution to* (1.1) *with* $(d, \mathbf{f}, g, \mathbf{h}) = (0, 0, 0, 0)$.

3 Time decay estimates of the Stokes semigroup

Let P_1 and P_2 be the projections defined by

$$P_1 : X_q \to W_q^{2-1/q}(\mathbf{R}^{N-1}), \quad P_2 : X_q \to J_q(\dot{\mathbf{R}}^N),$$

where $q \in (1, \infty)$. For the Stokes semigroup $(S(t))_{t\geq 0}$, we set

$$\mathcal{H}(t)(\eta_0, \mathbf{u}_0) = P_1 S(t)(\eta_0, \mathbf{u}_0), \quad \mathcal{U}(t)(\eta_0, \mathbf{u}_0) = P_2 S(t)(\eta_0, \mathbf{u}_0).$$

We now introduce time decay estimates for $\mathcal{H}(t)(\eta_0, \mathbf{u}_0)$ and $\mathcal{U}(t)(\eta_0, \mathbf{u}_0)$. Let us start with the case $\rho_- > \rho_+$.

Theorem 3.1 ([6]). *Suppose that $N \geq 2$ and that γ_a, σ, ρ_+, ρ_-, μ_+, and μ_- are positive constants with $\rho_- > \rho_+$. Let $1 < p < 2 \leq q < \infty$ and*

$$(\eta_0, \mathbf{u}_0) \in (L_p(\mathbf{R}^{N-1}) \times L_p(\dot{\mathbf{R}}^N)^N) \cap X_q,$$

and let ε_1 be a constant satisfying

$$0 < \varepsilon_1 < \min\left(1, 2(N-1)\left(\frac{1}{p} - \frac{1}{2}\right)\right).$$

Let $j \in \mathbf{N}_0$, $\alpha' \in \mathbf{N}_0^{N-1}$ with $|\alpha'| \leq 2$, and $\beta \in \mathbf{N}_0^N$ with $|\beta| \leq 2$. Then there exist positive constants δ_1 and C, independent of η_0 and $\mathbf{u}_0$, such that for any $t \geq 1$

$$\begin{aligned}
&\|\partial_t^j \partial_{x'}^{\alpha'} \mathcal{H}(t)(\eta_0, \mathbf{u}_0)\|_{W_q^{1-1/q}(\mathbf{R}^{N-1})} \\
&\leq C\Big(t^{-\frac{4(N-1)}{5}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{5}j-\frac{4}{5}|\alpha'|}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ t^{-\frac{N-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-j-\frac{1}{2}|\alpha'|-\frac{3}{4}\varepsilon_1}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ e^{-\delta_1 t}\|(\eta_0, \mathbf{u}_0)\|_{X_q}\Big), \\
&\|\partial_t^j \partial_x^{\beta} \mathcal{U}(t)(\eta_0, \mathbf{u}_0)\|_{L_q(\dot{\mathbf{R}}^N)} \\
&\leq C\Big(t^{-\frac{4(N-1)}{5}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{4}{5}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{5}j-\frac{1}{5}|\beta|}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-j-\frac{1}{2}|\beta|}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ e^{-\delta_1 t}\|(\eta_0, \mathbf{u}_0)\|_{X_q}\Big).
\end{aligned}$$

We next introduce decay properties in the case $\rho_- = \rho_+$.

Theorem 3.2 ([6]). *Suppose that $N \geq 2$ and that γ_a, σ, ρ_+, ρ_-, μ_+, and μ_- are positive constants with $\rho_- = \rho_+$. Let $1 < p < 2 \leq q < \infty$ and*

$$(\eta_0, \mathbf{u}_0) \in (L_p(\mathbf{R}^{N-1}) \times L_p(\dot{\mathbf{R}}^N)^N) \cap X_q,$$

and let ε_2 be a constant satisfying

$$0 < \varepsilon_2 < \min\left(\frac{1}{3}, \frac{2N}{3}\left(\frac{1}{p} - \frac{1}{2}\right)\right).$$

Let $j \in \mathbf{N}_0$, $\alpha' \in \mathbf{N}_0^{N-1}$ with $|\alpha'| \leq 1$, and $\beta \in \mathbf{N}_0^N$ with $|\beta| \leq 2$. Then there exist positive constants δ_2 and C, independent of η_0 and $\mathbf{u}_0$, such that for any $t \geq 1$

$$\begin{aligned}
&\|\partial_t^j \partial_{x'}^{\alpha'} \nabla' \mathcal{H}(t)(\eta_0, \mathbf{u}_0)\|_{W_q^{1-1/q}(\mathbf{R}^{N-1})} \\
&\leq C\Big(t^{-\frac{4(N-1)}{7}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{4}{7}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{6}{7}j-\frac{4}{7}|\alpha'|}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ t^{-\frac{N-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-j-\frac{1}{2}|\alpha'|-\frac{1}{4}\varepsilon_2}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N} \\
&+ e^{-\delta_2 t}\|(\eta_0, \mathbf{u}_0)\|_{X_q}\Big), \\
&\|\partial_t^j \partial_x^{\beta} \mathcal{U}(t)(\eta_0, \mathbf{u}_0)\|_{L_q(\dot{\mathbf{R}}^N)} \\
&\leq C\Big(t^{-\frac{4N}{7}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{6}{7}j-\frac{3}{7}|\beta|}\|(\eta_0, \mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N}
\end{aligned}$$

$$
\begin{aligned}
&+ t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-j-\frac{1}{2}|\beta|}\|(\eta_0,\mathbf{u}_0)\|_{L_p(\mathbf{R}^{N-1})\times L_p(\dot{\mathbf{R}}^N)^N}\\
&+ e^{-\delta_2 t}\|(\eta_0,\mathbf{u}_0)\|_{X_q}\Big),
\end{aligned}
$$

where $\nabla' = (\partial_1, \dots, \partial_{N-1})^{\mathsf{T}}$.

4 Idea of the proof of main results

See [6] for the detailed proof of main results, Theorems 3.1 and 3.2. In what follows, we introduce briefly the reason why the decay rate $t^{-(4(N-1)/5)(1/p-1/q)}$ appears in Theorem 3.1 and the decay rate $t^{-(4(N-1)/7)(1/p-1/q)}$ appears in Theorem 3.2

The system (1.1) with $(d, \mathbf{f}, g, \mathbf{h}) = (0,0,0,0)$ leads us to the following resolvent problem independent of time t:

$$
\begin{cases}
\lambda\eta - u_N|_{x_N=0} = \eta_0 & \text{on } \mathbf{R}^{N-1},\\
\lambda\mathbf{u} - \rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \mathbf{u}_0 & \text{in } \dot{\mathbf{R}}^N,\\
\operatorname{div}\mathbf{u} = 0 & \text{in } \dot{\mathbf{R}}^N,\\
[\![(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{e}_N]\!] + ([\![\rho]\!]\gamma_a + \sigma\Delta')\eta\mathbf{e}_N = 0 & \text{on } \mathbf{R}^{N-1},\\
[\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^{N-1}.
\end{cases}
\tag{4.1}
$$

Solution formulas of this resolvent problem are obtained in [8] via the partial Fourier transform with respect to $x' = (x_1, \dots, x_{N-1})$ and its inverse transform. We focus on the boundary symbol, called also the Lopatinskii determinant, appearing in the solution formulas in what follows.

Let us define, for $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}$ and $\lambda \in \mathbf{C} \setminus (-\infty, -z_0|\xi'|^2]$ with $z_0 = \min(\mu_+/\rho_+, \mu_-/\rho_-)$,

$$
A = |\xi'|, \quad B_\pm = \sqrt{\frac{\rho_\pm}{\mu_\pm}\lambda + |\xi'|^2}, \quad D_\pm = \mu_\pm B_\pm + \mu_\mp A,
$$

where one has chosen a branch cut along the negative real axis and a branch of the square root so that $\Re\sqrt{z} > 0$ for $z \in \mathbf{C} \setminus (-\infty, 0]$. In [8], the boundary symbol of (4.1) is calculated as follows:

$$
L(A,\lambda) = \lambda F(A,\lambda) + A(-[\![\rho]\!]\gamma_a + \sigma A^2)(D_+ + D_-),
$$

where

$$
\begin{aligned}
F(A,\lambda) = &-(\mu_+ - \mu_-)^2A^3 + [(3\mu_+ - \mu_-)\mu_+B_+ + (3\mu_- - \mu_+)\mu_-B_-]A^2\\
&+ [(\mu_+B_+ + \mu_-B_-)^2 + \mu_+\mu_-(B_+ + B_-)^2]A\\
&+ (\mu_+B_+ + \mu_-B_-)(\mu_+B_+^2 + \mu_-B_-^2).
\end{aligned}
$$

The following lemma then holds.

Lemma 4.1. *Suppose that $N \geq 2$ and that γ_a, σ, ρ_+, ρ_-, μ_+, and μ_- are positive constants. Let $\xi' \in \mathbf{R}^{N-1}$ and $\lambda \in \mathbf{C} \setminus (-\infty, -z_0|\xi'|^2]$. Define*

$$
\alpha_1 = \frac{-[\![\rho]\!]\gamma_a}{\rho_+ + \rho_-}, \quad \alpha_2 = \frac{\sigma}{\rho_+ + \rho_-},
$$

and

$$
\mathcal{L}_A(\lambda) = \lambda^2 + \frac{4AD_+D_-}{(\rho_+ + \rho_-)(D_+ + D_-)}\lambda + \alpha_1A + \alpha_2A^3.
$$

Then $L(A,\lambda) = (\rho_+ + \rho_-)(D_+ + D_-)\mathcal{L}_A(\lambda)$.

Remark 4.2. *We observe that α_1 is negative when $\rho_+ > \rho_-$ and α_1 is positive when $\rho_- > \rho_+$. In addition, $\alpha_1 = 0$ when $\rho_- = \rho_+$.*

Concerning zeros of $\mathcal{L}_A(\lambda)$, we have the following lemma.

Lemma 4.3. *Suppose that $N \geq 2$ and that γ_a, σ, ρ_+, ρ_-, μ_+, and μ_- are positive constants. Then there exists a positive constant $A_0 \in (0,1)$ such that the following assertions hold for any $A = |\xi'| \in (0, A_0)$.*

(1) *Suppose $\rho_+ > \rho_-$. Then $\mathcal{L}_A(\lambda)$ has a simple zero λ_* with $\Re\lambda_* > 0$ and*

$$\lambda_* = |\alpha_1|^{1/2} A^{1/2} + o(A^{1/2}) \quad \text{as } A \to 0+.$$

(2) *Suppose $\rho_- \geq \rho_+$ and set*

$$\Theta = \begin{cases} 1/4 & \text{when } \rho_- > \rho_+, \\ 3/4 & \text{when } \rho_- = \rho_+, \end{cases} \qquad \alpha = \begin{cases} \alpha_1 & \text{when } \rho_- > \rho_+, \\ \alpha_2 & \text{when } \rho_- = \rho_+. \end{cases}$$

Then $\mathcal{L}_A(\lambda)$ has two simple zeros $\lambda_\pm$ with $\Re\lambda_\pm < 0$ and

$$\lambda_\pm = \pm i \alpha^{1/2} A^{2\Theta} - \sqrt{2}\alpha^{1/4}\beta(1 \pm i) A^{1+\Theta} + o(A^{1+\Theta}) \quad \text{as } A \to 0+,$$

where $i = \sqrt{-1}$ and β is a positive constant given by

$$\beta = \frac{\sqrt{\rho_+\mu_+}\sqrt{\rho_-\mu_-}}{(\rho_+ + \rho_-)(\sqrt{\rho_+\mu_+} + \sqrt{\rho_-\mu_-})}.$$

Suppose $\rho_- \geq \rho_+$ and define

$$\zeta_\pm = \pm i \alpha^{1/2} A^{2\Theta} - \sqrt{2}\alpha^{1/4}\beta(1 \pm i) A^{1+\Theta},$$

which yields

$$\lambda_\pm = \zeta_\pm + o(A^{1+\Theta}) \quad \text{as } A \to 0+.$$

Let $x' \cdot \xi' = \sum_{j=1}^{N-1} x_j \xi_j$ and

$$\widehat{u}(\xi') = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} u(x')\, dx', \quad \mathcal{F}^{-1}_{\xi'}[v(\xi')](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} v(\xi')\, d\xi'.$$

In the proof of Theorems 3.1 and 3.2, we treat the following type of operators to obtain time decay estimates:

$$T_\pm(t) f = \mathcal{F}^{-1}_{\xi'}[e^{\zeta_\pm t} \widehat{f}(\xi')](x') \quad \text{for } t > 0.$$

These operators cause the decay rates mentioned above. More precisely, we have

Proposition 4.4. *Suppose that $N \geq 2$ and that γ_a, σ, ρ_+, ρ_-, μ_+, and μ_- are positive constants with $\rho_- \geq \rho_+$. Let $1 \leq p \leq 2 \leq q \leq \infty$. Then there exists a positive constant C such that for any $f \in L_p(\mathbf{R}^{N-1})$ and for any $t > 0$*

$$\|T_\pm(t) f\|_{L_q(\mathbf{R}^{N-1})} \leq C t^{-\frac{N-1}{1+\Theta}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L_p(\mathbf{R}^{N-1})}.$$

Remark 4.5. *Since* $1/(1+\Theta) = 4/5$ *when* $\rho_- > \rho_+$ *and* $1/(1+\Theta) = 4/7$ *when* $\rho_- = \rho_+$*, we see from Proposition* 4.4 *that*

$$\|T_\pm(t)f\|_{L_q(\mathbf{R}^{N-1})} \le C\|f\|_{L_p(\mathbf{R}^{N-1})} \times \begin{cases} t^{-\frac{4(N-1)}{5}\left(\frac{1}{p}-\frac{1}{q}\right)} & \text{when } \rho_- > \rho_+, \\ t^{-\frac{4(N-1)}{7}\left(\frac{1}{p}-\frac{1}{q}\right)} & \text{when } \rho_- = \rho_+. \end{cases}$$

In the remaining part, we prove Proposition 4.4. To this end, we introduce

Lemma 4.6. *Let* $\theta > 0$ *and* $\nu > 0$*. Then the following assertions hold.*

(1) *Let* $1 \le r \le s \le \infty$*. Then for any* $t > 0$ *and* $\varphi \in L_r(\mathbf{R}^{N-1})$

$$\|\mathcal{F}_{\xi'}^{-1}[e^{-\nu t|\xi'|^\theta}\widehat{\varphi}(\xi')]\|_{L_s(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{\theta}\left(\frac{1}{r}-\frac{1}{s}\right)}\|\varphi\|_{L_r(\mathbf{R}^{N-1})},$$

where C *is a positive constant independent of* t *and* φ*.*

(2) *Let* $1 \le r \le 2$*. Then for any* $t > 0$ *and* $\varphi \in L_r(\mathbf{R}^{N-1})$

$$\|e^{-\nu t|\xi'|^\theta}\widehat{\varphi}\|_{L_2(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{\theta}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\varphi\|_{L_r(\mathbf{R}^{N-1})},$$

where C *is a positive constant independent of* t *and* φ*.*

Proof. (1) See e.g. [2, Lemma 3.1], [9, Lemma 2.5].

(2) It holds by Parseval's theorem that

$$\|e^{-\nu t|\xi'|^\theta}\widehat{\varphi}\|_{L_2(\mathbf{R}^{N-1})} = (2\pi)^{(N-1)/2}\|\mathcal{F}_{\xi'}^{-1}[e^{-\nu t|\xi'|^\theta}\widehat{\varphi}]\|_{L_2(\mathbf{R}^{N-1})},$$

which, combined with (1) for $s = 2$, furnishes the desired estimate. This completes the proof of Lemma 4.6. $\square$

Let us continue the proof of Proposition 4.4. We write $T_\pm(t)f$ as

$$T_\pm(t)f = \mathcal{F}_{\xi'}^{-1}[e^{(\Re\zeta_\pm/4)t} \cdot e^{-(\Re\zeta_\pm/2)t}e^{\zeta_\pm t} \cdot e^{(\Re\zeta_\pm/4)t}\widehat{f}(\xi')](x').$$

By Lemma 4.6 (1) with $(r,s) = (2,q)$, $\theta = 1+\Theta$, and $\nu = \sqrt{2}\alpha^{1/4}\beta/4$

$$\|T_\pm(t)f\|_{L_q(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{1+\Theta}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\mathcal{F}_{\xi'}^{-1}[e^{-(\Re\zeta_\pm/2)t}e^{\zeta_\pm t} \cdot e^{(\Re\zeta_\pm/4)t}\widehat{f}(\xi')]\|_{L_2(\mathbf{R}^{N-1})},$$

which, combined with Parseval's theorem, furnishes that

$$\|T_\pm(t)f\|_{L_q(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{1+\Theta}\left(\frac{1}{2}-\frac{1}{q}\right)}\|e^{-(\Re\zeta_\pm/2)t}e^{\zeta_\pm t} \cdot e^{(\Re\zeta_\pm/4)t}\widehat{f}(\xi')\|_{L_2(\mathbf{R}^{N-1})}.$$

It now holds that

$$|e^{-(\Re\zeta_\pm/2)t}e^{\zeta_\pm t}| = |e^{(\Re\zeta_\pm/2)t}| \le 1 \quad \text{for any } t > 0,$$

and thus

$$\|T_\pm(t)f\|_{L_q(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{1+\Theta}\left(\frac{1}{2}-\frac{1}{q}\right)}\|e^{(\Re\zeta_\pm/4)t}\widehat{f}(\xi')\|_{L_2(\mathbf{R}^{N-1})}.$$

Combining this with Lemma 4.6 (2) for $r = p$, $\theta = 1+\Theta$, and $\nu = \sqrt{2}\alpha^{1/4}\beta/4$ shows

$$\|T_\pm(t)f\|_{L_q(\mathbf{R}^{N-1})} \le Ct^{-\frac{N-1}{1+\Theta}\left(\frac{1}{2}-\frac{1}{q}\right)} \cdot t^{-\frac{N-1}{1+\Theta}\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L_p(\mathbf{R}^{N-1})},$$

which yields the desired inequality of Proposition 4.4. This completes the proof of Proposition 4.4.

References

[1] S. Chandrasekhar. *Hydrodynamic and hydromagnetic stability.* Dover, 1981.

[2] C. Miao, B. Yuan, and B. Zhang. Well-posedness of the Cauchy problem for the fractional power dissipative equations. *Nonlinear Anal.*, 68(3):461–484, 2008.

[3] J. Prüss and G. Simonett. On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations. *Indiana Univ. Math. J.*, 59(6):1853–1871, 2010.

[4] H. Saito. Time-decay estimates for linearized two-phase Navier-Stokes equations with surface tension and gravity. *Mathematics*, 9(7):Paper No. 761, 43 pp, 2021.

[5] H. Saito. On decay properties of solutions for the two-phase Stokes equations with surface tension and gravity. *RIMS Kôkyûroku*, to appear.

[6] H. Saito. Decay properties of the stokes semigroup for two-phase incompressible viscous flows (submitted).

[7] H. Saito and X. Zhang. Unique solvability of elliptic problems associated with two-phase incompressible flows in unbounded domains. *Discrete Contin. Dyn. Syst.*, 41(10):4609–4643, 2021.

[8] Y. Shibata and S. Shimizu. Maximal L_p-L_q regularity for the two-phase Stokes equations; model problems. *J. Differential Equations*, 251(2):373–419, 2011.

[9] M. Yamamoto and Y. Sugiyama. Asymptotic expansion of solutions to the drift-diffusion equation with fractional dissipation. *Nonlinear Anal.*, 141:57–87, 2016.

Boundary Conditions for Cylindrical Traveling Waves of Reaction-Diffusion Equations

Chih-Chiang Huang *

August 17, 2023

Abstract

In the talk, I would like to study the boundary effect of the Dirichlet or Neumann condition for traveling waves of reaction-diffusion equations in a cylinder. At the beginning, I will introduce some different properties for these two conditions in a bounded domain, such as symmetry and stability. Next, I would like construct traveling waves by a variational approach. In addition, as the diffusion of cross section for the Dirichlet problem approaches to zero, the traveling wave will be close to a planar wave in some sense. Based on a variational method, we discuss such a phenomenon and some related results.

1 Introduction

In this talk, I study the so-called the FitzHugh-Nagumo system(FHNS) in a cylinder Ω. Suppose $(x, y) \in \Omega := \mathbf{R}^1 \times \omega$ and ω is a bounded $C^{2,\alpha}$ domain in $\mathbf{R}^N$.

$$\begin{aligned} u_t &= d_1 u_{xx} + d_2 \Delta_y u + f(u) - v, \\ \tau v_t &= d_3 v_{xx} + d_4 \Delta_y v + u - \gamma v, \\ u|_{\partial\Omega} &= v|_{\partial\Omega} = 0 \text{ or } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \end{aligned}$$

where $\tau, d_1, d_2, d_3, d_4, \gamma > 0$, $f(u) = u(1-u)(u-\beta)$, $0 < \beta < \frac{1}{2}$ and ν is the outer unit normal of Ω.

In particular, as $\gamma = \infty$, the system will be reduced as a scalar equation, so-called Nagumo's equation (NE):

$$\begin{aligned} u_t &= d_1 u_{xx} + d_2 \Delta_y u + f(u), \\ u|_{\partial\Omega} &= 0 \text{ or } \frac{\partial u}{\partial \nu} = 0. \end{aligned}$$

*Department of Mathematics, National Chung Cheng University, Taiwan (cchuang@ccu.edu.tw).

2 Steady States of (NE) on Bounded Domains

Now, we study some properties of steady states for (NE) on a bounded domain.

2.1 Symmetry

In 1979, B. Gidas, W.-M. Ni, and L. Nirenberg [3] proved the following theorem.

Theorem 2.1. *Suppose f is locally Lipschitz continuous and $u > 0$ solves*

$$\begin{aligned} d_2 \Delta_y u + f(u) = 0 \ \textit{in} \ B_R(0), \\ u = 0 \ \textit{on} \ \partial B_R(0). \end{aligned}$$

Then u must be radially symmetric, i.e., $u(y) = u(|y|) =: u(r)$, and $u'(r) < 0$ for all $0 < r < R$.

Remark. There exists a non-symmetric solution for Neumann problems. For example: $d_2 u''(y) + u(1-u)(u-1/2) = 0$, $u'(\pm 1) = 0$ and $d_2 \ll 1$.

2.2 Stability

In 1978, Casten and Holland [2] proved the following result.

Theorem 2.2. *If ω is a convex domain and v is a stable solution for*

$$d_2 \Delta_y u + f(u) = 0 \ \textit{in} \ \omega, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \ \textit{on} \ \partial\omega, \tag{2.2}$$

then u must be a constant.

Remark 1. There exists a stable nonconstant steady state for (2.1)-(2.2) in a dumbbell-shaped domain.
Remark 2. There exists a stable steady state for Dirichlet Problems in a convex domain. For example: $d_2 u'' + u(1-u)(u-\beta) = 0, u(\pm 1) = 0$ and $d_2 \ll 1$.

2.3 Concentration and Boundary Layer

In general, the Dirichlet condition and Neumann condition are independent of the study on PDE. However, in view of calculus of variation, we can construct a connection of those two boundary conditions in some sense. Consider

$$d_2 \Delta_y u + f(u) = 0 \text{ in } \omega, \tag{2.3}$$

$$\frac{\partial v}{\partial \nu} = 0 \text{ or } v = 0 \text{ on } \partial\omega, \tag{2.4}$$

associated with the functional

$$E[u] = \int_{\omega} \left(\frac{d_2}{2} |\nabla_y u|^2 + F(u) \right) dy.$$

Here $F(u) = -\int_0^u f(s)ds$. We know that the admissible class of E for Dirichlet's condition (DC) is $H_0^1(\omega)$ and one for Nuemann's condition (NC) is $H^1(\omega)$, respectively. Since $H_0^1(\omega)$ is a subspace of $H^1(\omega)$, we can explain why solutions for (DC) is better than ones for (NC), such as Theorem 2.1 and Theorem 2.2 . Another viewpoint is concerned about the concentration for the minimizer of E. In fact, we have the following properties.

THEOREM 2.3. *Let u_{d_2} be the minimizer of E on $H_0^1(\omega)$. If $d_2 \ll 1$, then $0 > E[u_{d_2}] > \min_{H^1(\omega)} E = |\omega| F(1)$. In addition, $u_{d_2} \to 1$ locally in ω as $d_2 \to 0$ and $\min_{H_0^1(\omega)} E \to \min_{H^1(\omega)} E$ as $d_2 \to 0$.*

This theorem gives a connection between (DC) and (NC) on a bounded domain. Our main purpose of this talk is to generalize this property to traveling waves of (NE) and (FHNS).

3 Boundary Conditions for Cylindrical Traveling Waves of (NE)

Consider

$$u_t = d_1 u_{xx} + d_2 \Delta_y u + f(u), \tag{3.1}$$

$$u|_{\partial\Omega} = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0. \tag{3.2}$$

3.1 Construction of Traveling Waves

Based on the develop of construction for traveling waves in [4] and [5]. Letting the moving coordinate of traveling wave be $\xi = c(x - d_1 ct)$, we have that the wave profile $w(\xi, y) = u(x, y, t)$ satisfies

$$d_1 c^2 (w_{\xi\xi} + w_\xi) + d_2 \Delta_y w + f(w) = 0 \text{ in } \Omega, \tag{3.3}$$

$$w = 0 \text{ or } \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.4}$$

Consider

$$\Psi[w] := \frac{-\int_{-\infty}^{\infty} e^{\xi} E[w] d\xi}{\frac{d_1}{2} \int_{\Omega} e^{\xi} w_\xi^2 d\xi dy} \tag{3.5}$$

Let $H_0^1(e^\xi, \Omega)$ be the weight Sobolev space with the weight e^ξ in Ω. For Dirichlet's problem, we have the following result.

THEOREM 3.1. *If $d_2 \leq d^*$ for some small $d^* > 0$, then there exists $w_0 \in H_0^1(e^\xi, \Omega)$ and $c_0 > 0$ such that*

$$c_0^2 = \sup_{H_0^1(e^\xi,\Omega)} \Psi = \Psi[w_0]. \tag{3.6}$$

In addition, w_0 is a strictly decreasing.

3.2 Boundary Conditions for Traveling Waves

In the proof of [5], we know that if $w_0(\xi, y)$ satisfies $\sup_{H^1(e^\xi,\Omega)} \Psi = \Psi[w_0]$, then w_0 is independent of y, In other words, the traveling wave is just planar. Applying a test function argument, we can prove the wave speed for (DC) converges to the wave speed for (NC).

THEOREM 3.2. *If (w, c) solves*

$$\begin{aligned} d_1c^2(w_{\xi\xi} + w_\xi) + d_2\Delta_y w + f(w) = 0 &\quad \text{in } \Omega, \\ w(-\infty, y) = v^*(y), w(\infty, y) = 0 &\quad \text{in } \omega, \\ w = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

Then $c \to \theta$ as $d_2 \to 0$, where $(\phi(\xi), \theta)$ solves

$$\begin{aligned} &d_1\theta^2(\phi_{\xi\xi} + \phi_\xi) + f(\phi) = 0, \\ &\phi(-\infty) = 1, \phi(\infty) = 0. \end{aligned}$$

4 Traveling Waves of (FHNS)

Consider

$$u_t = d_1u_{xx} + d_2\Delta_y u + f(u) - v, \tag{4.1}$$

$$\tau v_t = d_3v_{xx} + d_4\Delta_y v + u - \gamma v, \tag{4.2}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \text{ or } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0. \tag{4.3}$$

Due to the limitation of our technique, we assume $\tau = \frac{d_3}{d_1}$. Letting the moving coordinate of traveling wave be $\xi = c(x - d_1ct)$, we have that the wave profiles satisfy

$$d_1c^2(u_{\xi\xi} + u_\xi) + d_2\Delta_y u + f(u) - v = 0, \tag{4.4}$$

$$d_3c^2(v_{\xi\xi} + v_\xi) + d_4\Delta_y v + u - \gamma v = 0. \tag{4.5}$$

By scaling with respect to c, we may assume $d_3 = 1$. Observing that (4.5) is a linear equation, we can formally solve v, expressed in term of u. Denote v by $B_c[u]$. Consequently, system (4.4)-(4.5) is reduced to a single equation:

$$d_1c^2(u_{\xi\xi} + u_\xi) + d_2\Delta_y u + f(u) - B_c[u] = 0. \tag{4.6}$$

Moreover, (4.6) is the Euler-Lagrange equation of the nonlocal functional $\Phi_c[u]$, defined by

$$\Phi_c[u] = \frac{1}{2}\int_\Omega e^\xi(d_1c^2u_x^2 + d_2|\nabla_y u|^2) + \int_\Omega e^\xi F(u) + \frac{1}{2}\int_\Omega e^\xi uB_c[u].$$

In 2016, C.-N. Chen, C.-C. Chen and C.C. Huang [1] study this problem with $d_1 = d_2 = d_4 = 1$, i.e.,

$$c^2(u_{\xi\xi} + u_\xi) + \Delta_y u + f(u) - v = 0, \tag{4.7}$$
$$c^2(v_{\xi\xi} + v_\xi) + \Delta_y v + u - \gamma v = 0 \tag{4.8}$$

THEOREM 4.1. *Suppose $\gamma > \frac{9}{2\beta^2-5\beta+2}$. Then*
(a) If ω contains a sufficiently large ball, then there exists a traveling wave solving (4.7)-(4.8) with (DC).
(b) The traveling wave solving (4.7)-(4.8) with (NC) is just planar.

Following the proof of [1], we can consider any ω and any $\gamma > 0$ for (4.4)-(4.5) with (NC) . For $u \in \mathbf{H} := H^1(e^\xi, \Omega)$, set

$$\tilde{B}_c[u] := \left(-c^2(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}) - d_4\Delta_y + \gamma\right)^{-1}[u], \tag{4.9}$$
$$\tilde{\Phi}_c[u] = \int_\Omega e^\xi\left(\frac{d_1}{2}u_\xi^2 + \frac{d_2}{2}|\nabla_y u|^2 + F(u) + \frac{1}{2}u\tilde{B}_c[u]\right)d\xi dy, \tag{4.10}$$

and

$$\tilde{\mu}_c := \inf_{\mathcal{B}} \tilde{\Phi}_c, \tag{4.11}$$

where $\mathcal{B} := \{\frac{1}{2}\int_\Omega e^\xi u_\xi^2 = 1\}$.

Based on the eigenfunction decomposition method, we can compare the value of nonlocal term for (ξ, y)- and ξ- direction in the following lemma.

LEMMA 4.2. *Assume that $\gamma > \sqrt{\frac{d_4}{d_2}}$. Let $c > 0$, $\mathcal{D} = -c^2(\frac{\partial^2}{\partial\xi^2} + \frac{\partial}{\partial\xi}) + \gamma$ and $\mathcal{L} = \mathcal{D}^{-1}$. Then*

$$\tilde{\Phi}_c[u] \ge \int_\omega\int_{-\infty}^{\infty} e^\xi(\frac{d_1c^2}{2}u_\xi^2 + F(u) + \frac{1}{2}u\mathcal{L}u)\,d\xi dy \tag{4.12}$$

and the equality holds if and only if u depends on ξ only.

Therefore, we can prove the main theorem.

THEOREM 4.3. *Let $\gamma > \sqrt{\frac{d_4}{d_2}}$. Suppose $u \in \mathcal{B}$ and $\tilde{\Phi}_c[u] = \tilde{\mu}_c = 0$, then $\nabla_y u \equiv 0$.*

References

[1] C.-N. Chen, C.-C. Chen and C.-C. Huang, *Traveling waves for the FitzHugh-Nagumo system on an infinite channel*, Journal of Differential Equations, 261(2016), p3010-3041.

[2] R. G. Casten and C. J. Holland, *Instability results for a reaction diffusion equation with Neumann boundary conditions.*, J. Differential Equations 27 (1978), 266-273.

[3] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the Maximum Principle.*, Comm. Math. Phys. 68 (1979), 209-243.

[4] Heinze, S., *A Variational Approach to Traveling Waves.*, Preprint 85, Max Planck Institute for Mathematical Sciences, 2001.

[5] Lucia, M.; Muratov, C. B. and Novaga, M., *Existence of traveling waves of invasion for Ginzburg-Landau-type problems in infinite cylinders*, Arch. Ration. Mech. Anal. 188 (2008), no. 3, 475-508.

D'ALEMBERT FORMULA APPROACH TO SEMI-LINEAR SYSTEMS OF WAVE EQUATIONS WITH OR WITHOUT THE NULL CONDITION

KUNIO HIDANO

Abstract. It is well known that in finite time, singularity generally occurs in solutions to the Cauchy problem for the *scalar* wave equation $\partial_t^2 u - \Delta u = (\partial_t u)^2$ in $n = 1, 2, 3$ space dimensions no matter how small and smooth initial data is. The first half of this talk is concerned with global existence of small solutions to *multiple-speed systems* of wave equations with quadratic nonlinear terms in **one** space dimension. Under a certain condition that creates nice cancellation, we obtain global solutions for small data. Building upon the d'Alembert formula, our proof employs a point-wise estimation approach. The weight functions of our sup norms are dependent on the propagation speeds, and they are useful in observing that gain of time decay occurs in the nonlinear interaction of two waves with *different* propagation speeds. In our argument, gain of time decay is observed also for the derivatives in the characteristic directions, and we can thereby allow for "null-form" nonlinear interactions of waves with *equal* propagation speed. Before the end of the first half of my talk, an unpublished result of Tartar is mentioned.

The second half of this talk is concerned with global existence of small solutions to multiple-speed systems of nonlinear wave equations in **three** space dimensions. We are interested in certain two-speed and three-component systems with quadratic nonlinear terms that so far have fallen beyond the capability of existing techniques. At the price of assuming radial symmetry, we resort to a point-wise estimation approach. Then the standard energy estimate is no longer necessary, and the proof of global existence is based on the iteration argument using only weighted sup norms. Again, our weight functions depend on the propagation speeds, and they are useful in observing that gain of time decay occurs in the product of derivatives of two components with different propagation speeds. This *gain* compensates for a small *loss* of decay of a certain component, and we can carry out the iteration globally in time. Gain of time decay is observed also for "the tangential derivatives", and we thereby allow for some null forms in the nonlinear terms.

1. 1-D semi-linear systems of wave equations

The first half of my talk is concerned with global existence of small solutions to semi-linear systems of wave equations in one space dimension of the form

$$\partial_t^2 u_i - c_i^2 \partial_x^2 u_i = \sum_{j,\,k=1}^{m} \sum_{\alpha,\,\beta=0}^{1} B_{ijk}^{\alpha\beta} (\partial_\alpha u_j)(\partial_\beta u_k), \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

This talk is based on a joint work with Kazuyoshi Yokoyama (Hokkaido University of Science).

subject to the initial condition

$$u_i(0) = f_i, \quad \partial_t u_i(0) = g_i, \tag{1.2}$$

where $i = 1, 2, \ldots, m(\in \mathbb{N})$, $c_i > 0$, and $B_{ijk}^{\alpha\beta} \in \mathbb{R}$. Without loss of generality, we assume $B_{ijk}^{\alpha\beta} = B_{ikj}^{\beta\alpha}$. We know by the result of Masuda [20] that even for small and smooth data, the *scalar* equation

$$\partial_t^2 u - \Delta u = (\partial_t u)^2, \quad t > 0,\, x \in \mathbb{R} \tag{1.3}$$

has solutions that become singular in finite time. Suppose that the coefficients $B_{ijk}^{\alpha\beta}$ satisfy the following: For $i = 1, 2, \ldots, m$ and for j, k with $c_j = c_k$, we have

$$\sum_{\alpha,\,\beta=0}^{1} B_{ijk}^{\alpha\beta} X_\alpha X_\beta = 0 \ \text{ for any } X = (X_0, X_1) \in \mathbb{R}^2 \text{ with } X_0^2 = c_j^2 X_1^2, \tag{1.4}$$

which implies that the nonlinear terms of the system (1.1) have the form of sum of constant-multiples of

$$(\partial_\alpha u_j)(\partial_\beta u_k) \quad \text{with } j,\, k \text{ satisfying } c_j \neq c_k, \tag{1.5}$$

$$(\partial_t u_j)(\partial_t u_k) - c_j^2(\partial_x u_j)(\partial_x u_k) \quad \text{with } j,\, k \text{ satisfying } c_j = c_k \tag{1.6}$$

or

$$(\partial_t u_j)(\partial_x u_k) - (\partial_x u_j)(\partial_t u_k) \quad \text{with } j,\, k\ (j \neq k) \text{ satisfying } c_j = c_k. \tag{1.7}$$

Assuming (1.4), we aim at showing that the system (1.1) admits a global C^2-solution for small, smooth, and decaying (sufficiently fast as $|x| \to \infty$) data.

Our primary interest lies in the case where there exist some indices, say j and k, such that $c_j \neq c_k$ and we aim at showing that this difference together with the condition (1.5) serves for suppressing the occurrence of singularity in small solutions to (1.1). Also, we attempt to show that the quadratic terms with the special form (1.6) or (1.7) play a similar role. Our main theorem reads as follows.

Theorem 1.1. *Suppose* (1.4). *Let* $(f_i, g_i) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. *Then, there exists a constant* $\varepsilon_0 > 0$ *such that if*

$$\sum_{i=1}^{m} \Big(\sup_{x\in\mathbb{R}} \langle x \rangle^\kappa |f_i'(x)| + \sup_{x\in\mathbb{R}} \langle x \rangle^\kappa |f_i''(x)| + \sup_{x\in\mathbb{R}} \langle x \rangle^\kappa |g_i(x)| + \sup_{x\in\mathbb{R}} \langle x \rangle^\kappa |g_i'(x)| \Big) < \varepsilon_0 \tag{1.8}$$

for some $\kappa > 1$, *then the Cauchy problem* (1.1)–(1.2) *admits a* C^2*-solution defined for all* $(t, x) \in (0, \infty) \times \mathbb{R}$. *It satisfies*

$$|\partial^\alpha u_i(t, x)| \le \frac{C\varepsilon_0}{\langle c_i t - |x| \rangle^\kappa}, \quad |\alpha| = 1, 2, \tag{1.9}$$

$$|(\partial_t + c_i\partial_x)\partial^\alpha u_i(t,x)| \le \frac{C\varepsilon_0}{\langle c_i t + x\rangle^\kappa}, \quad |\alpha| = 0,1, \tag{1.10}$$

$$|(\partial_t - c_i\partial_x)\partial^\alpha u_i(t,x)| \le \frac{C\varepsilon_0}{\langle c_i t - x\rangle^\kappa}, \quad |\alpha| = 0,1, \tag{1.11}$$

$(i = 1, 2, \dots, m)$ *for all* $(t,x) \in (0,\infty) \times \mathbb{R}$.

Here we have used the common notation $\langle a\rangle := \sqrt{1+a^2}$ $(a \in \mathbb{R})$. By the assumption (1.4), we mean to pose no restriction on the nonlinear interaction of two waves with **different** speeds. See (1.5). Meanwhile, (1.4) enters the nonlinear interaction of two waves with **equal** speed, or the nonlinear self-interaction of a single wave. See (1.6)–(1.7). To illustrate the role that (1.5), (1.6), and (1.7) play in stopping singularity from occurring in solutions in finite time, let us take two simple examples.

Example 1.2. Firstly, consider the 2-component system

$$\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 = (\partial_t u_1)(\partial_t u_2), & t > 0,\ x \in \mathbb{R}, \\ \partial_t^2 u_2 - c_2^2 \partial_x^2 u_2 = (\partial_t u_1)(\partial_t u_2), & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{1.12}$$

Here $c_2 > 0$. For $c_2 = 1$, the pair (u, u) of a solution to the scalar equation (1.3) is exactly a solution to the system (1.12) with the particular choice of data $u_1(0,x) = u_2(0,x)$, $\partial_t u_1(0,x) = \partial_t u_2(0,x)$. As mentioned above, singularity can occur in solutions to the equation (1.3) even for small and smooth data. This implies that the system (1.12) with $c_2 = 1$ has solutions that become singular in finite time even if they are initially small and smooth enough. On the other hand, if $c_2 \ne 1$, then the system (1.12) falls within the scope of Theorem 1.1 (see (1.5)) and hence we see that the difference of propagation speeds keeps singularity from occurring in solutions with small and smooth data.

Example 1.3. Secondly, we consider the 3-component and 2-speed system

$$\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 = (\partial_t u_1)(\partial_t u_2) - (\partial_x u_1)(\partial_x u_2), & t > 0,\ x \in \mathbb{R}, \\ \partial_t^2 u_2 - \partial_x^2 u_2 = \big((\partial_t u_1)^2 - (\partial_x u_1)^2\big) + (\partial_t u_2)(\partial_t u_3), & t > 0,\ x \in \mathbb{R}, \\ \partial_t^2 u_3 - c_3^2 \partial_x^2 u_3 = (\partial_t u_2)(\partial_t u_3), & t > 0,\ x \in \mathbb{R}. \end{cases} \tag{1.13}$$

Here $c_3 > 0$. If $c_3 \ne 1$, then it is obvious that this system falls *inside* the scope of Theorem 1.1, and it admits global C^2-solutions for small and smooth data.

Actually, in the L^1 setting, global existence result for (1.1) under the assumption (1.4) follows from an unpublished result of Tartar [27] which the present author learnt from Bianchini and Staffilani [3]. Indeed, by setting $v_i := (\partial_t - c_i\partial_x)u_i$ and $v_{m+i} := (\partial_t + c_i\partial_x)u_i$ $(i = 1, 2, \dots, m)$, we can transform the second-order system

(1.1) into the first-order system of the form

$$
\begin{cases}
(\partial_t + \tilde{c}_1 \partial_x) v_1 = \displaystyle\sum_{j,\,k=1}^{2m} A_{1,\,jk} v_j v_k, \\
\qquad \vdots \\
(\partial_t + \tilde{c}_m \partial_x) v_m = \displaystyle\sum_{j,\,k=1}^{2m} A_{m,\,jk} v_j v_k, \\
(\partial_t + \tilde{c}_{m+1} \partial_x) v_{m+1} = \displaystyle\sum_{j,\,k=1}^{2m} A_{m+1,\,jk} v_j v_k, \\
\qquad \vdots \\
(\partial_t + \tilde{c}_{2m} \partial_x) v_{2m} = \displaystyle\sum_{j,\,k=1}^{2m} A_{2m,\,jk} v_j v_k,
\end{cases}
\tag{1.14}
$$

$(t > 0,\ x \in \mathbb{R})$. Here, $\tilde{c}_i = c_i$ for $i = 1, 2, \ldots, m$, $\tilde{c}_i = -c_i$ for $i = m+1, \ldots, 2m$, $A_{i,\,jk} = A_{i,\,kj}$, and under the conditions (1.6)–(1.7), we have

$$
\begin{cases}
A_{i,\,jk} = 0 \text{ for all } i = 1, 2, \ldots, 2m \\
\text{if the indices } j \text{ and } k \text{ satisfy } \tilde{c}_j = \tilde{c}_k.
\end{cases}
\tag{1.15}
$$

Since the condition (1.8) implies f_i', $g_i \in L^1(\mathbb{R})$, we know by [3, Theorem 1] that if ε_0 (see (1.8)) is small enough, then the Cauchy problem (1.14) with data $v_i(0) = g_i - \tilde{c}_i f_i'$ admits a global solution. The main purpose of this talk is therefore to give an alternative approach to the proof of global existence result for the system (1.1).

When space dimensions $n \geq 2$ and the nonlinear term depends only on the first or the second derivatives of unknown functions, the proof of global existence of small solutions to wave equations usually relies upon the time decay estimates and the energy estimate. See, e.g., [12], [13], and [24]. In one space dimension, we can no longer expect uniform or L^p-type $(2 < p < \infty)$ decay of solutions, and hence the standard technique seems useless. We like to show that such *point-wise* decay properties as (1.9), (1.10), and (1.11), which are obtainable via the d'Alembert formula, work well in the global (in space and time) iteration argument using weighted sup (in space and time) norms, under the assumption (1.4). This approach can actually handle not only the sum of constant-multiples of the terms displayed in (1.5), (1.6), or (1.7), but also that of the $q(u, \partial u)$-multiples of them, if in addition we assume $\|f_i\|_{L^\infty} < \infty$ $(i = 1, 2, \ldots, m)$ and ε_0 is small enough. Here, by $q(u, \partial u) = q(u_1, \ldots, u_m, \partial_t u_1, \ldots, \partial_t u_m, \partial_x u_1, \ldots, \partial_x u_m)$, we mean any polynomial of $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m}, x_{2m+1}, \ldots, x_{3m})$. Also, our weighted sup norm method can give an alternative proof of the global existence result for the system (1.1) with

$c_1 = c_2 = \cdots = c_m$ that Luli, Yang, and Yu obtained by using a new energy-type weighted estimate [19]. Besides, we like to note that our approach is applicable to studying global existence of small, radially symmetric solutions to 3-d, multiple-speed systems of wave equations violating the standard null condition, as explained below.

2. 3-d semi-linear systems of wave equations

The second half of my talk is concerned with global existence of small solutions to the multiple-speed system of nonlinear wave equations in three space dimensions of the form

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_1)(\partial_t u_2), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = \big((\partial_t u_1)^2 - |\nabla u_1|^2\big) + (\partial_t u_1)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_3 - c_3^2\Delta u_3 = (\partial_t u_1)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3 \end{cases} \tag{2.1}$$

subject to the initial condition

$$u_i(0) = f_i, \quad \partial_t u_i(0) = g_i, \qquad i=1,2,3. \tag{2.2}$$

Here, $c_3 > 0$, $c_3 \neq 1$. Throughout my talk, the initial data are assumed to be sufficiently smooth and decay sufficiently fast as $|x| \to \infty$. We will later discuss two-speed and three-component systems with wider class of nonlinear terms (see (3.1)–(3.3) below), but **the most part of my talk will focus on several examples of simplified systems with fairly specified nonlinear terms, just for exposition**.

To explain our motivation for studying (2.1), we go over previous related results concerning the Cauchy problem for the system

$$\partial_t^2 u_i - c_i^2\Delta u_i = F_i(\partial u_1,\dots,\partial u_m), \qquad t>0,\ x\in\mathbb{R}^3 \tag{2.3}$$

($i=1,2,\dots,m$, $m\in\mathbb{N}$) with small data. Here, we mean $\partial u_i = (\partial_0 u_i, \partial_1 u_i, \partial_2 u_i, \partial_3 u_i)$, $\partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ ($j=1,2,3$), and F_i stands for a quadratic nonlinear term of the form

$$F_i(\partial u_1,\dots,\partial u_m) = \sum_{\alpha,\beta=0}^{3}\sum_{j,k=1}^{m} B_{ijk}^{\alpha\beta}(\partial_\alpha u_j)(\partial_\beta u_k), \tag{2.4}$$

where $B_{ijk}^{\alpha\beta}\in\mathbb{R}$, $B_{ijk}^{\alpha\beta} = B_{ikj}^{\beta\alpha}$. It is well known by the results due to John [9] and Sideris [25] for the *scalar* equation of the form $\partial_t^2 u - \Delta u = (\partial_t u)^2$ or $\partial_t^2 u - \Delta u = |\nabla u|^2$, that nonexistence of global solutions generally occurs even when initial data are small enough. (Actually, the purpose of Masuda [20] was to extend the 3-d result of John to $n(\le 3)$-d one. See (1.3) above.) It can also occurs for some systems of equal propagation speeds, such as $\partial_t^2 u_1 - \Delta u_1 = (\partial_t u_2)^2$, $\partial_t^2 u_2 - \Delta u_2 = |\nabla u_1|^2$. See Rammaha [23] and Deng [4].

On the other hand, we know some sufficient conditions on the coefficients $B^{\alpha\beta}_{ijk}$ (see (2.4)) leading to global solutions with small data. They are formulated separately for the three different cases:

(i) $c_1 = c_2 = \cdots = c_m$ (all the speeds are the same),

(ii) $m \geq 2$ and $c_i \neq c_j$ if $i \neq j$ (all the speeds are different from each other),

(iii) $m \geq 3$ and there exist non-empty, mutually disjoint subsets $I_1, I_2, \dots, I_l$, with $l < m$, of $\{1, 2, \dots, m\}$ such that $\{1, 2, \dots, m\} = I_1 \cup I_2 \cup \cdots \cup I_l$,
$c_i = c_j$ if $i, j \in I_p$ for some p,
$c_i \neq c_j$ if $i \in I_p$, $j \in I_q$ for some p, q with $p \neq q$ (some of the speeds are repeated).

Our main concern is global existence of small solutions to (2.1)–(2.2) with $c_3 \neq 1$, and hence it is closely related to seeking for a new sufficient condition under which the Cauchy problem for (2.3) admits global solutions for small data when some of the speeds are repeated as in (iii). **The main interest of this talk therefore lies in the case** (iii), but for the sake of convenience, we start with reviewing the sufficient condition in the case of (i) where all the speeds are the same. Let us assume $c_1 = c_2 = \cdots = c_m = 1$ without loss of generality. It is well known that Klainerman's null condition

$$\begin{cases} \displaystyle\sum_{\alpha,\,\beta=0}^{3} B^{\alpha\beta}_{ijk} X_\alpha X_\beta = 0 \quad \text{for any } X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \text{ with} \\ X_0^2 = X_1^2 + X_2^2 + X_3^2 \text{ and for every } i, j, k = 1, 2, \dots, m \end{cases} \tag{2.5}$$

is sufficient for the Cauchy problem (2.3) with small data to admit global solutions [14]. This condition is equivalent to the following: the nonlinear term has the form of sum of constant-multiples of

$$(\partial_t u_j)(\partial_t u_k) - (\nabla u_j) \cdot (\nabla u_k) \tag{2.6}$$

or

$$(\partial_\alpha u_j)(\partial_\beta u_k) - (\partial_\beta u_j)(\partial_\alpha u_k) \quad \text{with } j \neq k \text{ and } \alpha \neq \beta. \tag{2.7}$$

(We note that (2.6) and (2.7) are regarded as counterparts of (1.6) and (1.7) with $c_j = c_k = 1$, respectively).

Remark 2.1. To relax the null condition of Klainerman (2.5) and widen a class of quadratic nonlinear terms is related to the study of global existence of small solutions under the weak null condition that Lindblad and Rodnianski introduced in [17]. See [1], [11], [5], [6], [8], [21] for the study in this direction concerning systems with quadratic nonlinear terms depending on products of the first derivatives.

When all the speeds are distinct from each other as in (ii), it is Kovalyov [15] that first obtained a sufficient condition for the Cauchy problem for (2.3) to admit global, small solutions. The condition that Kovalyov assumed reads as

$$B_{ijj}^{\alpha\beta} = 0 \text{ for any } i,\, j = 1, 2, \ldots, m \text{ and any } \alpha,\, \beta = 0, 1, 2, 3, \tag{2.8}$$

which means that the nonlinear interaction of the form $(\partial_\alpha u_j)(\partial_\beta u_j)$ is entirely prohibited (the "null-form" $(\partial_t u_j)^2 - c_j^2|\nabla u_j|^2$ is hence prohibited), but that of the form

$$(\partial_\alpha u_j)(\partial_\beta u_k) \quad (j \neq k) \tag{2.9}$$

is permitted. (Note that, in the case (ii), $j \neq k \iff c_j \neq c_k$.) We note that (2.9) is regarded as a counterpart of (1.5). The condition (2.8) was later relaxed by Yokoyama [28]. It follows from [28, Theorem 1.1] that global existence of small solutions holds under the assumption: For every $i = 1, 2, \ldots, m$, we have

$$\begin{cases} \displaystyle\sum_{\alpha,\,\beta=0}^{3} B_{iii}^{\alpha\beta} X_\alpha X_\beta = 0 \ \text{ for any } X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \text{ with} \\ X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2). \end{cases} \tag{2.10}$$

Example 2.2. Since the Yokoyama condition enters only the coefficients $B_{iii}^{\alpha\beta}$, we know by [28, Theorem 1.1] that surprisingly enough, the Cauchy problem for

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_2)^2, & t > 0,\, x \in \mathbb{R}^3, \\ \partial_t^2 u_2 - c_2^2 \Delta u_2 = (\partial_t u_1)^2, & t > 0,\, x \in \mathbb{R}^3 \end{cases} \tag{2.11}$$

($c_2 > 0$) admits global solutions for small, smooth data, provided $c_2 \neq 1$. The existence of global, small solutions to (2.11) with $c_2 \neq 1$ forms a sharp contrast to the results for the 1-d system discussed in Section 1. See (1.5), which entirely prohibits nonlinear terms of the form $(\partial_\alpha u_j)(\partial_\beta u_k)$ with $c_j = c_k$.

As for (iii) that most concerns us in this talk, it follows from Lindblad, Nakamura, and Sogge [16, Theorem 1.1] that global existence of small solutions to (2.3) holds under the condition: For every i, j, and k with $c_i = c_j = c_k$, we have

$$\begin{cases} \displaystyle\sum_{\alpha,\,\beta=0}^{3} B_{ijk}^{\alpha\beta} X_\alpha X_\beta = 0 \ \text{ for any } X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \text{ with} \\ X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2). \end{cases} \tag{2.12}$$

(We note that Sideris and Tu also obtained a similar sufficient condition, when systems are quasilinear. See Remark following [26, Theorem 3.1].) Since the Lindblad-Nakamura-Sogge condition enters the coefficients $B_{ijk}^{\alpha\beta}$ only with i, j, and k such that $c_i = c_j = c_k$, it is regarded as a natural extension of the Yokoyama condition.

Example 2.3. The result of Lindblad, Nakamura, and Sogge [16] can be applied to the two-speed and three component system:

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_1)(\partial_t u_2) - (\nabla u_1)\cdot(\nabla u_2), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = \big((\partial_t u_1)^2 - |\nabla u_1|^2\big) + (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_3 - c_3^2\Delta u_3 = (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3 \end{cases} \tag{2.13}$$

($c_3>0$, $c_3\neq 1$), which is a 3-d analog of (1.13) and can be regarded as a "mixture" of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_1)(\partial_t u_2) - (\nabla u_1)\cdot(\nabla u_2), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = (\partial_t u_1)^2 - |\nabla u_1|^2, & t>0,\ x\in\mathbb{R}^3 \end{cases} \tag{2.14}$$

and

$$\begin{cases} \partial_t^2 u_2 - \Delta u_2 = (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_3 - c_3^2\Delta u_3 = (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3. \end{cases} \tag{2.15}$$

By the result due to Klainerman (see the case (i) above), we easily see that the Cauchy problem for (2.14) admits global solutions for small, smooth data. Also, we know by the result of Kovalyov (see the case (ii) above) that small and smooth data yield global solutions to (2.15). As one may expect, the "mixed" system (2.13), which is one of the typical examples satisfying the Lindblad-Nakamura-Sogge condition, admits global solutions for small, smooth data.

As remarked above (see Remark 2.1), a lot of authors have shown interest in relaxing the null condition of Klainerman (2.5), and one of the simplest examples that fail to satisfy the null condition but still admit global solutions for small, smooth data is the system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_1)(\partial_t u_2), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = (\partial_t u_1)^2 - |\nabla u_1|^2, & t>0,\ x\in\mathbb{R}^3. \end{cases} \tag{2.16}$$

Actually, this is one of the typical examples satisfying the weak null condition of Lindblad and Rodnianski [17]. At first sight one may think that the "null-from" $(\partial_t u_1)^2 - |\nabla u_1|^2$ yields more time decay thanks to nice cancellation and hence there exists no hurdle to handling this nonlinear term and closing estimates globally in time. As a matter of fact, because of the "ill" behavior of ∂u_1, the earlier ideas in [14], [26] are no longer useful in handling this null form and a new idea based on a certain weighted estimate (see, e.g., [2, Chapter 9], [18, p. 52]) plays an essential role in obtaining global solutions to (2.16). See, e.g., [5]. Then, it is natural to ask whether or not the system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_1)(\partial_t u_2), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = \big((\partial_t u_1)^2 - |\nabla u_1|^2\big) + (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_3 - c_3^2\Delta u_3 = (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3 \end{cases} \tag{2.17}$$

($c_3 \neq 1$), which is a "mixture" of (2.15) and (2.16), admits global solutions for small, smooth data. Obviously, the system (2.17) falls outside the scope of Lindblad, Nakamura, and Sogge [16]. In the direction toward finding a new condition different from Lindblad, Nakamura, and Sogge's, there exist two previous results, due to Pusateri and Shatah [22], Hidano, Yokoyama, and Zha [7]. We know by [22] that the Cauchy problem for the system

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_t u_2)^2 + (\partial_t u_1)(\partial_t u_3)^2, & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_2 - \Delta u_2 = \big((\partial_t u_2)^2 - |\nabla u_2|^2\big) + (\partial_t u_2)(\partial_t u_3), & t>0,\ x\in\mathbb{R}^3, \\ \partial_t^2 u_3 - c_3^2\Delta u_3 = (\partial_t u_2)(\partial_t u_3) + (\partial_t u_1)(\partial_t u_2)^2, & t>0,\ x\in\mathbb{R}^3 \end{cases} \tag{2.18}$$

($c_3 \neq 1$) admits global solutions for small data, with the first component having a weaker decay $\|\partial u_1(t)\|_{L^\infty(\mathbb{R}^3)} = O(t^{-1+\varepsilon})$ ($\varepsilon > 0$ is sufficiently small) as $t \to \infty$. In [7], the class of the nonlinear terms was extended, with ∂u_1 (which has a weaker decay) partially permitted to enter into the quadratic part of the nonlinear terms. It follows from [7, Theorem 1.1] that the system (2.17) admits global solutions for small, smooth data.

As the assumptions (1.10) and (1.11) in [7] tell, however, no product of the first derivatives of u_1 and u_3 of the form $(\partial_\alpha u_1)(\partial_\beta u_3)$ ($\alpha, \beta = 0, 1, 2, 3$) was permitted to appear in the quadratic part of the nonlinear terms of the second or the third equation of (2.17). In other words, the problem has been left open whether or not such a system as (2.1) with $c_3 \neq 1$ having an ill-behaving quadratic term $(\partial_t u_1)(\partial_t u_3)$ in the second or the third equation admits global solutions for small data. Though one may expect to benefit from the difference of the propagation speeds, this problem does not seem amenable to the space-time resonance method in [22] or the energy method involving a collection of generalized derivatives in [7]. This is the motivation for pursuing the problem of global existence of small solutions to (2.1).

The difficulty to get global solutions to (2.1) stems from the presence of the first derivatives of the ill-behaving component u_1 in the quadratic part of the nonlinear terms of the second or the third equation in the form of $(\partial_t u_1)(\partial_t u_3)$. One may expect that some gain of time decay occurs for the term $(\partial_t u_1)(\partial_t u_3)$ owing to the difference of the propagation speeds and such gain is sure to compensate for a weaker decay of ∂u_1. As for the system (2.1), however, to utilize such a helpful property for the purpose of closing the estimates seems beyond the current technology. In this talk, at the cost of limiting radially symmetric equations and data (and thus radially symmetric solutions), we invoke the weighted L^∞ approach that Sideris [25] introduced to get global radially symmetric solutions to the scalar equation

$$\partial_t^2 u - \Delta u = |\partial_t u|^p + |\nabla u|^p, \quad t > 0,\ x \in \mathbb{R}^3 \tag{2.19}$$

under the sharp condition $p > 2$. Sideris actually did more: He employed the weight function of the form $(1 + |t - |x||)^{p-1}$ in the definition of the norm of space of functions and investigated *global behavior* of radial solutions. Our approach in this talk is very much inspired by [25], and we employ speed-dependent weight functions, such as $(1 + |c_3 t - |x||)$, in the definition of the norms of space of functions where we carry out the iteration argument. Such weight functions are useful in observing that gain of time decay occurs in the term $(\partial_t u_1)(\partial_t u_3)$. This gain compensates for a small loss of decay of ∂u_1 and we can close the estimates globally in time. In our argument, with the help of John-type weighted estimates for inhomogeneous equations, we succeed in observing that gain of time decay occurs also for "the tangential derivatives". We thereby allow for some null forms in the nonlinear terms and handle systems whose nonlinear terms are more general than those of (2.1).

3. Global existence of radially symmetric solutions

This section is devoted to the problem of global existence of small solutions to the 3-component and 2-speed system (2.3) with $m = 3$, $c_1 = c_2 \neq c_3$. In what follows, we set $c_1 = c_2 = 1$ without loss of generality. Aiming at getting radially symmetric solutions for radially symmetric data, we make some assumptions on the nonlinear terms and the initial data.

$$\begin{aligned}
&\bullet\, F_1(\partial u) \text{ is a sum of constant-multiples of } (\partial_t u_1)^2 - |\nabla u_1|^2,\\
&(\partial_t u_i)(\partial_t u_j),\ (\nabla u_i)\cdot(\nabla u_j), \text{where } 1 \le i \le j \le 3,\ (i,j) \neq (1,1) \qquad (3.1)\\
&\bullet\, F_2(\partial u) \text{ is a sum of constant-multiples of } (\partial_t u_1)^2 - |\nabla u_1|^2,\\
&(\partial_t u_1)(\partial_t u_2) - (\nabla u_1)\cdot(\nabla u_2),\ (\partial_t u_2)^2 - |\nabla u_2|^2,\ (\partial_t u_i)(\partial_t u_j),\\
&(\nabla u_i)\cdot(\nabla u_j), \text{where } 1 \le i \le j \le 3,\ (i,j) \neq (1,1),(1,2),(2,2), \qquad (3.2)\\
&\bullet\, F_3(\partial u) \text{ is a sum of constant-multiples of } (\partial_t u_1)^2 - |\nabla u_1|^2,\\
&(\partial_t u_1)(\partial_t u_2) - (\nabla u_1)\cdot(\nabla u_2),\ (\partial_t u_3)^2 - c_3^2|\nabla u_3|^2,\ (\partial_t u_i)(\partial_t u_j),\\
&(\nabla u_i)\cdot(\nabla u_j), \text{where } 1 \le i \le j \le 3,\ (i,j) \neq (1,1),(1,2),(3,3). \qquad (3.3)
\end{aligned}$$

For the initial data f_i and g_i, we suppose that they are radially symmetric and hence written as

$$\varphi_i(r) = f_i(x) \text{ and } \psi_i(r) = g_i(x). \tag{3.4}$$

We suppose that the functions φ_i and ψ_i are actually defined for all $r \in \mathbb{R}$, satisfying

$$\varphi_i(-r) = \varphi_i(r), \quad \psi_i(-r) = \psi_i(r) \quad \text{for } r \ge 0, \tag{3.5}$$

$$\varphi_i \in C^1(\mathbb{R}),\ \psi_i \in C(\mathbb{R}),\ r\varphi_i \in C^2(\mathbb{R}),\ r\psi_i \in C^1(\mathbb{R}). \tag{3.6}$$

Moreover, we suppose that there exists a constant $\kappa > 1$ such that

$$\begin{aligned}\Lambda(\varphi_i, \psi_i) := &\|\langle r\rangle^\kappa \varphi_i\|_{L^\infty} + \|\langle r\rangle^{\kappa+1}\varphi_i'\|_{L^\infty} + \|\langle r\rangle^{\kappa+1}\psi_i\|_{L^\infty}\\ &+ \|\langle r\rangle^\kappa r\varphi_i''\|_{L^\infty} + \|\langle r\rangle^\kappa r\psi_i'\|_{L^\infty} < \infty. \end{aligned}\tag{3.7}$$

We consider the system of the integral equations

$$\begin{aligned}v_i(t,r) =&v_i^{(0)}(t,r)\\ &+\frac{1}{2r}\int_0^t d\tau\int_{r-c_i(t-\tau)}^{r+c_i(t-\tau)}\lambda F_i(\partial_t v(\tau,\lambda), \partial_r v(\tau,\lambda))d\lambda,\ t>0,\ r\in\mathbb{R}\end{aligned}\tag{3.8}$$

$(i = 1, 2, 3)$, where $\partial_t v := (\partial_t v_1, \partial_t v_2, \partial_t v_3)$, $\partial_r v := (\partial_r v_1, \partial_r v_2, \partial_r v_3)$,

$$v_i^{(0)}(t,r) = \begin{cases}\dfrac{(r+c_it)\varphi_i(r+c_it) + (r-c_it)\varphi_i(r-c_it)}{2r}\\ +\dfrac{1}{2r}\displaystyle\int_{r-c_it}^{r+c_it}\lambda\psi_i(\lambda)d\lambda, \quad r\in\mathbb{R}\setminus\{0\},\\ \varphi_i(t) + t\varphi_i'(t) + t\psi_i(t), \quad r = 0.\end{cases}\tag{3.9}$$

(Taking L'Hôpital's rule into account, we have set $v_i^{(0)}(t,0)$ as above.) As mentioned above, we mean $c_i = 1$ for $i = 1, 2$. Here, owing to (3.1), (3.2), and (3.3) we know

- $F_1(\partial_t v, \partial_r v)$ is a sum of constant-multiples of $(\partial_t v_1)^2 - (\partial_r v_1)^2$, $(\partial_t v_i)(\partial_t v_j)$, $(\partial_r v_i)(\partial_r v_j)$, where $1 \le i \le j \le 3$, $(i,j) \neq (1,1)$ (3.10)
- where $F_2(\partial_t v, \partial_r v)$ is a sum of constant-multiples of $(\partial_t v_1)^2 - (\partial_r v_1)^2$, $(\partial_t v_1)(\partial_t v_2) - (\partial_r v_1)(\partial_r v_2)$, $(\partial_t v_2)^2 - (\partial_r v_2)^2$, $(\partial_t v_i)(\partial_t v_j)$, $(\partial_r v_i)(\partial_r v_j)$, where $1 \le i \le j \le 3$, $(i,j) \neq (1,1), (1,2), (2,2)$, (3.11)
- where $F_3(\partial_t v, \partial_r v)$ is a sum of constant-multiples of $(\partial_t v_1)^2 - (\partial_r v_1)^2$, $(\partial_t v_1)(\partial_t v_2) - (\partial_r v_1)(\partial_r v_2)$, $(\partial_t v_3)^2 - c_3^2(\partial_r v_3)^2$, $(\partial_t v_i)(\partial_t v_j)$, $(\partial_r v_i)(\partial_r v_j)$, where $1 \le i \le j \le 3$, $(i,j) \neq (1,1), (1,2), (3,3)$. (3.12)

To solve (3.8) by iteration, we set the space of functions

$$\begin{aligned}\Sigma := \{(w_1, w_2, w_3) : w_i(t,-r) = w_i(t,r),&\\ &w_i(t,r)\in C^1([0,\infty)\times\mathbb{R}),\ rw_i(t,r)\in C^2([0,\infty)\times\mathbb{R}),\\ &\|(w_1,w_2,w_3)\|_\Sigma := \sum_{i=1}^3 N_i(w_i) < \infty\}\end{aligned}\tag{3.13}$$

where, for $i = 1, 2, 3$, we have set for scalar functions $\chi(t,x)$

$$N_i(\chi) := \sum_{|\alpha|=0}^1 \sup_{(t,r)\in[0,\infty)\times\mathbb{R}}\Phi_i(t,r)|\partial^\alpha\chi(t,r)| + \sup_{(t,r)\in[0,\infty)\times\mathbb{R}}\langle t+|r|\rangle\langle t\rangle^{-\delta}|\chi(t,r)|$$

$$+\sum_{|\alpha|=0}^{1} \sup_{(t,r)\in[0,\infty)\times\mathbb{R}} \Psi_{i,\pm}(t,r)|(\partial_t \pm c_i\partial_r)\partial^\alpha(r\chi(t,r))|, \tag{3.14}$$

where, $\delta > 0$ is small enough, and we mean $c_1 = c_2 = 1$ as before. Besides, we have set

$$\Phi_1(t,r) = \langle t-|r|\rangle\langle t\rangle^{-\delta}, \quad \Phi_2(t,r) = \langle t-|r|\rangle, \quad \Phi_3(t,r) = \langle c_3t-|r|\rangle,$$
$$\Psi_{1,\pm}(t,r) = \langle t\pm r\rangle\langle t\rangle^{-\delta}, \quad \Psi_{2,\pm}(t,r) = \langle t\pm r\rangle, \quad \Psi_{3,\pm}(t,r) = \langle c_3t\pm r\rangle.$$

Theorem 3.1. *Suppose $c_1 = c_2 = 1$ and $c_3 \neq 1$. There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that if $\Lambda(\varphi_1,\psi_1) + \Lambda(\varphi_2,\psi_2) + \Lambda(\varphi_3,\psi_3) < \varepsilon_0$, then the system of the integral equations* (3.8) *admits a unique solutions $(v_1,v_2,v_3) \in \Sigma$ verifying $\|(v_1,v_2,v_3)\|_\Sigma \le C\varepsilon_0$.*

Acknowledgments

The author is very grateful to all the organizers for inviting him to Sapporo Symposium on Partial Differential Equations. He was supported in part by the Grant-in-Aid for Scientific Research (C) (No. 21K03324), Japan Society for the Promotion of Science (JSPS).

References

1. S. Alinhac, *Semilinear hyperbolic systems with blowup at infinity.* Indiana Univ. Math. J. **55** (2006), 1209–1232.
2. S. Alinhac, "Geometric analysis of hyperbolic differential equations: an introduction". London Mathematical Society Lecture Note Series, **374**. Cambridge University Press, Cambridge, 2010.
3. R. Bianchini and G. Staffilani, *Revisitation of a Tartar's result on a semilinear hyperbolic system with null condition.* arXiv:2001.03688v2 [math.AP]
4. K. Deng, *Blow-up of solutions of some nonlinear hyperbolic systems.* Rocky Mountain J. Math. **29** (1999), 807–820.
5. K. Hidano and K. Yokoyama, *Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition.* Int. Math. Res. Not. IMRN **2020**, 39–70.
6. K. Hidano and K. Yokoyama, *Global existence and blow up for systems of nonlinear wave equations related to the weak null condition.* Discrete Contin. Dyn. Syst., **42** (2022), 4385–4414.
7. K. Hidano, K. Yokoyama, and D. Zha, *Global existence for a system of multiple-speed wave equations violating the null condition.* Tohoku Math. J. (2) **74** (2022), 597–626.
8. K. Hidano and D. Zha, *Remarks on a system of quasi-linear wave equations in 3D satisfying the weak null condition.* Commun. Pure Appl. Anal. **18** (2019), 1735–1767.
9. F. John, *Blow-up for quasilinear wave equations in three space dimensions.* Comm. Pure Appl. Math. **34** (1981), 29–51.
10. I. Kadar, *Small data nonlinear wave equation numerology: The role of asymptotics.* arXiv:2302.07312v1 [math.AP].
11. S. Katayama, T. Matoba, and H. Sunagawa, *Semilinear hyperbolic systems violating the null condition.* Math. Ann. **361** (2015), 275–312.
12. S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations.* Comm. Pure Appl. Math. **36** (1983), 133–141.

13. S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation.* Comm. Pure Appl. Math. **38** (1985), 321–332.
14. S. Klainerman, *The null condition and global existence to nonlinear wave equations.* Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326, Lectures in Appl. Math., **23**, Amer. Math. Soc., Providence, RI, 1986.
15. M. Kovalyov, *Resonance-type behaviour in a system of nonlinear wave equations.* J. Differential Equations **77** (1989), 73–83.
16. H. Lindblad, M. Nakamura, and C.D. Sogge, *Remarks on global solutions for nonlinear wave equations under the standard null conditions.* J. Differential Equations **254** (2013), 1396–1436.
17. H. Lindblad and I. Rodnianski, *The weak null condition for Einstein's equations.* C. R. Math. Acad. Sci. Paris **336** (2003), 901–906.
18. H. Lindblad and I. Rodnianski, *Global existence for the Einstein vacuum equations in wave coordinates.* Comm. Math. Phys. **256** (2005), 43–110.
19. G.K. Luli, S. Yang, and P. Yu, *On one-dimension semi-linear wave equations with null conditions.* Adv. Math. **329** (2018), 174–188.
20. K. Masuda, *Blow-up of solutions of quasilinear wave equations in two space dimensions.* Recent topics in nonlinear PDE (Hiroshima, 1983), 87–91, North-Holland Math. Stud., **98**, Lecture Notes Numer. Appl. Anal., **6**, North-Holland, Amsterdam, 1984.
21. J. Metcalfe and A. Stewart, *On a system of weakly null semilinear wave equations.* Anal. Math. Phys., **12** (2022), Paper No.125, 23 pp.
22. F. Pusateri and J. Shatah, *Space-time resonances and the null condition for first-order systems of wave equations.* Comm. Pure Appl. Math. **66** (2013), 1495–1540.
23. M.A. Rammaha, *Upper bounds for the life span of solutions to systems of nonlinear wave equations in two and three space dimensions.* Nonlinear Anal. **25** (1995), 639–654.
24. J. Shatah, *Global existence of small solutions to nonlinear evolution equations.* J. Differential Equations 46 (1982), no. 3, 409–425.
25. T.C. Sideris, *Global behavior of solutions to nonlinear wave equations in three dimensions.* Comm. Partial Differential Equations **8** (1983), 1291–1323.
26. T.C. Sideris and S.-Y. Tu, *Global existence for systems of nonlinear wave equations in* $3D$ *with multiple speeds.* SIAM J. Math. Anal. **33** (2001), 477–488.
27. L. Tartar, *Some existence theorems for semilinear hyperbolic systems in one space variable.* 1981, unpublished.
28. K. Yokoyama, *Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions.* J. Math. Soc. Japan **52** (2000), 609–632.

Department of Mathematics, Faculty of Education, Mie University, 1577 Kurimamachiya-cho Tsu, Mie Prefecture 514-8507, Japan
Email address: hidano@edu.mie-u.ac.jp

Characterization of concavity preserved by the Dirichlet heat flow

Asuka TAKATSU (Tokyo Metropolitan University)
asuka@tmu.ac.jp

1. Introduction

In 1976, Brascamp–Lieb [2] proved that the heat flow preserves log-concavity. To be precise, for the solution $e^{t\Delta_{\mathbb{R}^n}}\phi$ to

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial}{\partial t}u = \Delta u & \text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ u(0,\cdot) = \phi & \text{in} \quad \mathbb{R}^n, \end{cases}$$

where ϕ is a bounded nonnegative function ϕ on $\mathbb{R}^n$, $\log e^{t\Delta_{\mathbb{R}^n}}\phi$ is concave in $\mathbb{R}^n$ for any $t > 0$ if $\log \phi$ is concave in $\mathbb{R}^n$.

The proof relies on the fact that the function $x \mapsto \log K(t,x,y)$ is concave in $x \in \mathbb{R}^n$ for fixed $t > 0$ and $y \in \mathbb{R}^n$, where $K : (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ stands for the heat kernel on $\mathbb{R}^n$, that is,

$$K(t,x,y) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

This result means that when we draw the graph of a nonnegative function on $\mathbb{R}^n$ with logarithmic scale for the value, the concavity of the graph of an initial datum is preserved by the heat flow. Then the following question naturally arises:

Question.
Are there any scales of a graph where the concavity of an initial datum is preserved by the heat flow?

We have the following negative answer.

Answer.
If we discuss all bounded nonnegative functions on $\mathbb{R}^n$ as initial data for problem (P), the logarithmic scale is the only scale where the concavity of an initial datum is preserved by the heat flow.

To state the answer precisely, we introduce the notion of F-concavity in Section 2. In Section 3, we revisit the answer.

The talk and this abstract are based on joint work with Kazuhiro Ishige (The University of Tokyo) and Paolo Salani (Università di Firenze).

2. F-concavity

Recall that a function f on $\mathbb{R}^n$ is said *concave* in $\mathbb{R}^n$ if

$$f((1-\mu)x_0 + \mu x_1) \geq (1-\mu)f(x_0) + \mu f(x_1)$$

for $x_0, x_1 \in \mathbb{R}^n$ and $\mu \in (0,1)$. The aim of this section is to generalize the notion of concavity. Throughout this note, we adhere to the following natural convention:

$$\begin{array}{lll} -\infty + (-\infty) = -\infty, & c + (-\infty) = -\infty, & \infty - (-\infty) = \infty, \\ \mu \cdot (-\infty) = -\infty, & -\infty \leq -\infty, & \infty \leq \infty, \end{array}$$

and so on for $c \in \mathbb{R}$ and $\mu \in (0,1)$. We shall always use Ω to denote a convex domain of $\mathbb{R}^n$.

Definition 2.1. Let $a \in (0, \infty]$.

- Set
$$\mathcal{A}_\Omega([0,a)) := \{f : \Omega \to \mathbb{R} \mid f(\Omega) \subset [0,a)\}.$$
- A function $F : [0,a) \to [-\infty, \infty)$ is said *admissible* on $[0,a)$ if F is strictly increasing on $[0,a)$, $F \in C((0,a))$ and $F(0) = -\infty$. We denote by F^{-1} the inverse function of $F : [0,a) \to F([0,a))$.
- Let F be an admissible function on $[0,a)$. Given $f \in \mathcal{A}_\Omega([0,a))$, we say that f is F*-concave* in Ω if
$$f\left((1-\mu)x_0 + \mu x_1\right) \geq F^{-1}\Big((1-\mu)F\left(f(x_0)\right) + \mu F\left(f(x_1)\right)\Big)$$
for $x_0, x_1 \in \Omega$ and $\mu \in (0,1)$. We denote by $\mathcal{C}_\Omega[F]$ the set of F-concave functions in Ω.
- Let F and G be admissible on $[0, a_F)$ and $[0, a_G)$ with $a_F, a_G \in [a, \infty]$, respectively. We say that *F-concavity is weaker than G-concavity in $\mathcal{A}_\Omega([0,a))$* (or equivalently, *$G$-concavity is stronger than F-concavity in $\mathcal{A}_\Omega([0,a))$*) if
$$\mathcal{C}_\Omega[G] \cap \mathcal{A}_\Omega([0,a)) \subset \mathcal{C}_\Omega[F].$$
We also say that *F-concavity is strictly weaker than G-concavity in $\mathcal{A}_\Omega([0,a))$* (or equivalently, *$G$-concavity is strictly stronger than F-concavity in $\mathcal{A}_\Omega([0,a))$*) if
$$\mathcal{C}_\Omega[G] \cap \mathcal{A}_\Omega([0,a)) \subsetneq \mathcal{C}_\Omega[F].$$

The condition $F(0) = -\infty$ is appropriate to make $\mathcal{C}_{\mathbb{R}^n}[F]$ nontrivial. Indeed, for a function $F : [0,a) \to [-\infty, \infty)$, the set

$$\{f \in \mathcal{A}_{\mathbb{R}^n}([0,a)) \mid F \circ f \text{ is concave in } \mathbb{R}^n\}$$

includes other than constant functions if and only if $F(0) = -\infty$ (see [5, Theorem 3.1]).

For the hierarchy among F-concavities, the following criterion is known.

Proposition 2.2 ([7, Lemma 2.4])**.** *Let $a \in (0, \infty]$. Let F and G be admissible on $[0, a_F)$ and $[0, a_G)$ with $a_F, a_G \in [a, \infty]$, respectively. Then F-concavity is weaker than G-concavity in $\mathcal{A}_\Omega([0,a))$ if and only if $F \circ G^{-1}$ is concave in $G((0,a))$.*

This ensures that the hierarchy among F-concavities depends on intervals $[0, a)$ but not on convex domains Ω.

Let us give three examples of F-concavity.

Example 2.3 (Power concavity). For $\alpha \in \mathbb{R}$, define an admissible function Φ_α on $[0, \infty)$ by

$$\Phi_\alpha(r) := \int_1^r s^{-\alpha-1} ds = \begin{cases} \dfrac{1}{\alpha}(r^\alpha - 1) & \text{if } \alpha \neq 0, \\ \log r & \text{if } \alpha = 0, \end{cases}$$

for $r \in (0, \infty)$ and $\Phi_\alpha(0) := -\infty$. We usually refer to Φ_α-concavity as α-concavity. In particular, 0-concavity is referred as to *log-concavity*.

Due to Jensen's inequality, α-concavity enjoys the monotonicity property with respect to $\alpha \in \mathbb{R}$:

- For $\alpha, \beta \in \mathbb{R}$, α-concavity is strictly weaker than β-concavity in $\mathcal{A}_\Omega([0, \infty))$ if and only if $\alpha < \beta$.

By the monotonicity property, α-concavity can be extended in a natural way to the case of $\alpha = \pm\infty$ as follows: a nonnegative function f on Ω is said $-\infty$-*concave* (resp. ∞-*concave*) in Ω if

$$f((1-\mu)x_0 + \mu x_1) \geq \min\{f(x_0), f(x_1)\}$$
$$\Big(\text{resp. } f((1-\mu)x_0 + \mu x_1) \geq \max\{f(x_0), f(x_1)\}\Big)$$

for $x_0, x_1 \in \Omega$ and $\mu \in (0,1)$. We use *power concavity* as a generic term for α-concavity with $\alpha \in [-\infty, \infty]$.

The second example is a sort of hybrid between log-concavity and power concavity.

Example 2.4 (Power log-concavity). For $\alpha \in \mathbb{R}$, define an admissible function L_α on $[0, 1)$ by

$$L_\alpha(r) := -\Phi_\alpha(-\log r) = \begin{cases} -\dfrac{1}{\alpha}\left\{(-\log r)^\alpha - 1\right\} & \text{if } \alpha \neq 0, \\ -\log(-\log r) & \text{if } \alpha = 0, \end{cases}$$

for $r \in (0, \infty)$ and $L_\alpha(0) := -\infty$. We refer to L_α-concavity as α-log-concavity and, generically, as power log-concavity. Notice that $L_1(r) = \Phi_0(r) + 1$ for $r \in [0, 1)$ and

$$\mathcal{C}_\Omega[L_1] = \mathcal{C}_\Omega[\Phi_0] \cap \mathcal{A}_\Omega([0, 1)),$$

i.e. 1-log-concavity coincides with log-concavity in $\mathcal{A}_\Omega([0, 1))$.

The monotonicity property of power log-concavity follows from that of power-concavity:

- For $\alpha, \beta \in \mathbb{R}$, β-log-concavity is strictly weaker than α-log-concavity in $\mathcal{A}_\Omega([0, 1))$ if and only if $\alpha < \beta$.

Thus α-log-concavity is weaker (resp. stronger) than log-concavity in $\mathcal{A}_\Omega([0,1))$ if $\alpha \geq 1$ (resp. $\alpha \leq 1$). Furthermore, it can be roughly said that power log-concavity is a refinement of log-concavity. Indeed, for $\alpha \in \mathbb{R}$ and $r \in (0,1)$, set

$$\beta_\alpha(r) := \frac{1-\alpha}{-\log r}.$$

Then the following properties hold:

- Let $\alpha < 1$, i.e. α-log-concavity is strictly stronger than β-concavity in $\mathcal{A}_\Omega([0,1))$ for any $\beta \leq 0$. For $a \in (0,1)$, $\beta_\alpha(a) > 0$ and α-log-concavity is strictly weaker than $\beta_\alpha(a)$-concavity in $\mathcal{A}_\Omega([0,a))$.
- Let $\alpha > 1$, i.e. α-log-concavity is strictly weaker than β-concavity in $\mathcal{A}_\Omega([0,1))$ for any $\beta \geq 0$. For $a \in (0,1)$, $\beta_\alpha(a) < 0$ and α-log-concavity is strictly stronger than $\beta_\alpha(a)$-concavity in $\mathcal{A}_\Omega([0,a))$.

Notice that, by the monotonicity property, α-log-concavity can be extended to the case of $\alpha = \pm\infty$, where ∞-log-concavity corresponds to $-\infty$-concavity in $\mathcal{A}_\Omega([0,1))$ and $-\infty$-log-concavity corresponds to ∞-concavity in $\mathcal{A}_\Omega([0,1))$, respectively.

Power log-concavity plays an important role in answering Question. Indeed, it was proved in [7, Corollary 6.10] that α-log-concavity is preserved by the heat flow if and only if $\alpha \in [1/2, 1]$. The above observation suggests that the domains of admissible functions are crucial to answering Question.

The third example is used to answer to Question.

Example 2.5 (Hot-concavity). Let

$$h(z) := \left(e^{\Delta_{\mathbb{R}}}\mathbf{1}_{[0,\infty)}\right)(z) = (4\pi)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{|z-w|^2}{4}}\, dw \quad \text{for } z \in \mathbb{R}.$$

Then the function h is smooth in $\mathbb{R}$, $\lim_{z\to-\infty} h(z) = 0$, $\lim_{z\to\infty} h(z) = 1$, and $h' > 0$ in $\mathbb{R}$. Denote by h^{-1} the inverse function of $h : \mathbb{R} \to (0,1)$. For $a \in (0,\infty)$, we define an admissible function H_a on $[0,a)$ by

$$H_a(r) := h^{-1}(r/a)$$

for $r \in (0,a)$ and $H_a(0) := -\infty$. Then H_a-concavity also enjoys the the monotonicity property with respect to $a \in (0,\infty)$:

- For $a, b \in (0,\infty)$, H_b-concavity is strictly weaker than H_a-concavity in $\mathcal{A}_\Omega([0,a))$ if and only if $a < b$.

By the monotonicity property, H_a-concavity can be extended to the case of $a = \infty$, and it was proved in [7, Lemma 2.10] to be consistent if we define $H_\infty := \Phi_0$. This implies that log-concavity is strictly weaker than H_a-concavity in $\mathcal{A}_\Omega([0,a))$ for any $a \in (0,\infty)$. Moreover, it follows from [6, Proposition 4.2] and [7, Theorem 1.5, Corollary 6.10] that α-log-concavity is strictly weaker than H_1-concavity in $\mathcal{A}_\Omega([0,1))$ if and only if $\alpha \geq 1/2$. We use *hot-concavity* as a generic term for H_a-concavity with $a \in (0,\infty]$.

3. Main Results

Before answering Question, we clarity and generalize the statement of Question. Unless stated otherwise, we shall always assume that $a \in (0, \infty]$ and F is an admissible function on $[0, a)$ throughout this section.

Definition 3.1. We say that *F-concavity is preserved by the Dirichlet heat flow in Ω* if the solution $e^{t\Delta_\Omega}\phi$ to

$$\begin{cases} \dfrac{\partial}{\partial t}u = \Delta u & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0,\cdot) = \phi & \text{in} \quad \Omega, \end{cases}$$

is F-concave in Ω for every $t > 0$ if $\phi \in \mathcal{C}_\Omega[F] \cap L^\infty(\Omega)$.

Now we are ready to state the main result of this note.

Theorem 3.2 ([7, Theorems 1.5,1.6])**.**

(1) *H_a-concavity is preserved by the Dirichlet heat flow in Ω.*

(2) *Assume that F-concavity is preserved by the Dirichlet heat flow in Ω.*
   - (i) *F-concavity is weaker than H_a-concavity in $\mathcal{A}_\Omega([0, a))$.*
   - (ii) *If $n \geq 2$, then F-concavity is stronger than* log*-concavity in $\mathcal{A}_\Omega([0, a))$.*

Thus we have the following answer to Question.

Corollary 3.3 ([7, Corollary 1.7])**.** *Let $n \geq 2$. For an admissible function F on $[0, \infty)$, F-concavity is preserved by the Dirichlet heat flow in Ω if and only if $\mathcal{C}_\Omega[F] = \mathcal{C}_\Omega[H_\infty]$, i.e. F-concavity coincides with log-concavity in $\mathcal{A}_\Omega([0, \infty))$.*

The reason why the treatment of the case $n = 1$ differs from the case $n \geq 2$ is due to the following known result.

Proposition 3.4 ([1], [3, Theorem 1.1],[4, Theorem 4.1])**.**

(1) *For $n = 1$, $e^{t\Delta_\Omega}\phi$ is quasi-concave in Ω for all $t > 0$ if ϕ is nonnegative, bounded and quasi-concave in Ω. Namely, quasi-concavity is preserved in dimension* 1.

(2) *For $n \geq 2$, there exists $\phi \in C_0(\Omega)$ such that ϕ is α-concave in Ω for some $\alpha \in (-\infty, 0)$ and $e^{t\Delta_\Omega}\phi$ is not quasi-concave in Ω for some $t > 0$. Namely, quasi-concavity is in general not preserved in dimension $n \geq 2$:*

However, under a suitable regularity condition for admissible functions, a similar statement of Corollary 3.3 also holds for $n = 1$.

Theorem 3.5 ([7, Theorem 1.8])**.** *Assume $F \in C^2((0, a))$.*

(1) *F-concavity is preserved by the Dirichlet heat flow in Ω if and only if*

$$\lim_{r\downarrow 0} F(r) = -\infty, \quad F' > 0 \text{ in } (0,a) \quad \text{and} \quad (\log(F^{-1})')' \text{ is concave in } F((0,a)).$$

(2) *Theorem* 3.2 (2)(ii) *and Corollary* 3.3 *hold even for $n = 1$.*

References

[1] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96.

[2] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), 366–389.

[3] K. Ishige and P. Salani, *Is quasi-concavity preserved by heat flow?*, Arch. Math. (Basel) **90** (2008), 450–460.

[4] K. Ishige and P. Salani, *Convexity breaking of the free boundary for porous medium equations*, Interfaces Free Bound. **12** (2010), 75–84.

[5] K. Ishige, P. Salani, and A. Takatsu, *New characterizations of log-concavity via Dirichlet heat flow*, Ann. Mat. Pura Appl. (4) **201** (2022), no. 4, 1531–1552.

[6] K. Ishige, P. Salani, and A. Takatsu, *Hierarchy of deformations in concavity*, Inf. Geom., DOI 10.1007/s41884-022-00088-4.

[7] K. Ishige, P. Salani, and A. Takatsu, *Characterization of F-concavity preserved by the Dirichlet heat flow*, arXiv:2207.13449.