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TRACE INEQUALITIES OF THE SOBOLEV TYPE AND NONLINEAR DIRICHLET PROBLEMS

TAKANOBU HARA

Abstract. We discuss the solvability of nonlinear Dirichlet problems of the type $-\Delta_{p,w} u = \sigma$ in Ω; $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^n$, $\Delta_{p,w}$ is a weighted (p, w)-Laplacian and σ is a nonnegative locally finite Radon measure on Ω. We do not assume the finiteness of $\sigma(\Omega)$. We revisit this problem from a potential theoretic perspective and provide criteria for the existence of solutions by $L^p(w)$-$L^q(\sigma)$ trace inequalities or capacitary conditions. Additionally, we apply the method to the singular elliptic problem $-\Delta_{p,w} u = \sigma u^{-\gamma}$ in Ω; $u = 0$ on $\partial\Omega$ and derive connection with the trace inequalities.

1. Introduction

1.1. **Main results.** Let Ω be a bounded domain in $\mathbb{R}^n$, and let $1 < p < \infty$. We consider the existence problem of positive solutions to quasilinear elliptic equations of the type

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$ is a weighted (p, w)-Laplacian type elliptic operator, w is a p-admissible weight on $\mathbb{R}^n$ (see Sect. 2 for details) and σ is a nonnegative (locally finite) Radon measure on Ω. We do not assume the global finiteness of σ. The precise assumptions on $\mathcal{A} \colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are as follows: For each $z \in \mathbb{R}^n$, $\mathcal{A}(\cdot, z)$ is measurable, for each $x \in \Omega$, $\mathcal{A}(x, \cdot)$ is continuous, and there exist $0 < \alpha \le \beta < \infty$ such that

$$\mathcal{A}(x, z) \cdot z \ge \alpha w(x) |z|^p, \tag{1.2}$$

$$|\mathcal{A}(x, z)| \le \beta w(x) |z|^{p-1}, \tag{1.3}$$

$$(\mathcal{A}(x, z_1) - \mathcal{A}(x, z_2)) \cdot (z_1 - z_2) > 0, \tag{1.4}$$

$$\mathcal{A}(x, tz) = t|t|^{p-2} \mathcal{A}(x, z) \tag{1.5}$$

for all $x \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ and $t \in \mathbb{R}$. For the standard theory of quasilinear Dirichlet problems with finite signed measure data, we refer to [12, 36, 8, 48, 13, 23, 62]. See also [36, 37, 22, 10, 35] for the definitions of local solutions ($\mathcal{A}$-superharmonic functions or locally renormalized solutions) and their properties.

Most studies of the quasilinear measure data problem (1.1) assume the finiteness of σ to ensure the existence of solutions satisfying the Dirichlet boundary condition.

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However, the global finiteness of σ is not a necessary condition, and this existence problem has been stated as an open problem in a paper by Bidaut-Véron (see, [9, Problem 2]). If we recall classical potential theory (see, e.g., [41, 6]), the solution u to the Poisson equation $-\Delta u = \sigma$ in Ω; $u = 0$ on $\partial\Omega$ is given by the Green potential

$$u(x) = \int_\Omega G_\Omega(x, y)\, d\sigma(y).$$

Since the Green function $G_\Omega(\cdot, \cdot)$ vanishes on the boundary of Ω, the integral may be finite even if σ is not finite. In addition, the pointwise estimate of the Green function yields more concrete existence results if the boundary of Ω is C^2; $u \not\equiv +\infty$ if and only if $\int_\Omega \operatorname{dist}(x, \partial\Omega)\, d\sigma(x) < \infty$ (see, e.g., [44]). However, finding the estimate is another problem, and furthermore, the method is completely useless for nonlinear equations. Note that the use of the Green function is one of the methods to solve the problem and that there are rich examples of solutions even for nonlinear equations with infinite measure data.

One natural desire is to apply the theory of finite measure data problems to infinite measures, but it is actually not enough, because the above examples do not necessarily satisfy the Dirichlet boundary condition in the traditional sense. Therefore, we should adopt a two-step strategy; (i) find a local solution that satisfies the equation in a generalized sense, and (ii) confirm the boundary condition.

Variational methods (more generally, the theory of monotone operators) and comparison principles are effective for good measures. Hence, we find the solution u to (1.1) by

$$u(x) = \sup\left\{ v(x) \in H_0^{1,p}(\Omega; w) \cap \mathcal{SH}(\Omega) \colon 0 \le -\operatorname{div} \mathcal{A}(x, \nabla v) \le \sigma \right\},$$

where $H_0^{1,p}(\Omega; w)$ is a weighted Sobolev space and $\mathcal{SH}(\Omega)$ is the set of all $\mathcal{A}$-superharmonic functions in Ω. Perron's method for $\mathcal{A}$-superharmonic functions have been well studied (see, e.g., [29, 34]), and the relation between them and their Riesz measures is also known ([36, 59]). Furthermore, if σ is absolutely continuous with respect to the (p, w)-capacity (below, we denote this condition by $\sigma \in \mathcal{M}_0^+(\Omega)$), then the above set contains sufficiently many $\mathcal{A}$-superharmonic functions. As a result, u is a local solution if it is not identically infinite. In [18], Cao and Verbitsky proved a comparison principle leading to the minimality of u (see Theorem 3.5 below for a refinement of it). From this, u can be considered to satisfy the boundary condition in a very weak sense. Therefore, the remaining problem is to present a sufficient condition for u to not be identically infinite.

In previous work [32], the author provided an existence theorem for equations of the type (1.1). The sufficient condition was given by the $L^p(w)$-$L^q(\sigma)$ trace inequality of Sobolev type

$$\|f\|_{L^q(\Omega;\sigma)} \le C_1 \|\nabla f\|_{L^p(\Omega;w)}, \quad \forall f \in C_c^\infty(\Omega), \tag{1.6}$$

where $0 < q < p$. Note that if (1.6) holds, then σ must be absolutely continuous with respect to the (p, w)-capacity. Our first existence theorem is as follows.

Theorem 1.1. *Assume that $\mathcal{A}$ satisfies (1.2)-(1.5). Let $\sigma \in \mathcal{M}_0^+(\Omega)$, and let $0 < q < p$. Assume that (1.6) holds, and let C_1 be the best constant. Then, there exists a minimal nonnegative $\mathcal{A}$-superharmonic solution u to (1.1) satisfying*

$$\frac{(\alpha/\beta)^p}{\alpha} C_1^q \le \|\nabla u^{\frac{p-1}{p-q}}\|_{L^p(\Omega;w)}^{p-q} \le \frac{1}{q}\left(\frac{p-1}{p-q}\right)^{p-1} \frac{1}{\alpha} C_1^q \tag{1.7}$$

and

$$\frac{(\alpha/\beta)}{\alpha^{\frac{1}{p}}} C_1 \le \left(\int_\Omega u^{\frac{q(p-1)}{p-q}} \, d\sigma \right)^{\frac{p-q}{qp}} \le \frac{1}{q^{\frac{1}{p}}} \left(\frac{p-1}{p-q} \right)^{\frac{p-1}{p}} \frac{1}{\alpha^{\frac{1}{p}}} C_1. \tag{1.8}$$

In particular, u satisfies the Dirichlet boundary condition in the sense that

$$u^{\frac{p-1}{p-q}} \in H_0^{1,p}(\Omega; w). \tag{1.9}$$

Conversely, if there exists a nonnegative $\mathcal{A}$-superharmonic solution u to (1.1) *satisfying* (1.9)*, then* (1.6) *holds.*

Unlike [32], we deal with the variable coefficient equation (1.1). This assumption is natural when considering flatification of Lipschitz boundaries. We verify that results in [32] can be extended to (1.1) with small changes and give a direct proof of Theorem 1.1. In addition, we present a more concrete existence result using Hardy-type inequalies (Corollary 4.4). We also discuss what occurs when condition (1.6) is relaxed (Theorem 5.1 and Proposition 5.4).

Condition (1.9) has already appeared in the study of elliptic equations with boundary blow-up type nonlinearity since the work by Boccardo and Orsina [14]. In Sect. 6, we apply our framework to the singular elliptic problem (Corollary 6.5) and prove the following two theorems.

Theorem 1.2. *Let $0 < q < 1$. Suppose that $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$. Then there exists a unique finite energy weak solution $u \in H_0^{1,p}(\Omega; w)$ to*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

if and only if (1.6) *holds.*

Theorem 1.3. *Let $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$. Let $h\colon (0, \infty) \to (0, \infty)$ be a continuously differentiable nonincreasing function. Assume that there exists a continuous weak supersolution $v \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to* (1.1)*. Then there exists a unique continuous weak solution $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma h(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

Conversely, if there exists a continuous weak supersolution $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to (1.11)*, then there exists a continuous weak solution $v \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to* (1.1)*.*

1.2. **Related works.** We consider weighted equations using a framework in [34]. In particular, we directly connects the solvability of (1.6) with Hardy-type inequalities of the form (4.3) without reduction to ordinary differential equations. All results of the paper seem to be new even if $w \equiv 1$.

This work is strongly inspired by prior studies for L^p-L^q trace inequalities. General theory for (1.6) was started by Adams [1]. The first characterization of (1.6) with $0 < q < p$ was given by Maz'ya and Netrusov [46, 47] using a capacitary condition. Cascante, Ortega and Verbitsky [20] and Verbitsky [61] studied non-capacitary characterizations for inequalities of the type (1.6). Claims that are equivalent to

Theorem 1.1 were given by Seesanea and Verbitsky [56] for linear uniformly elliptic operators. In [32], their arguments were applied to (p, w)-Laplace operators and Theorem 1.1 was proved for $0 < q < p$ with a different formulation. The results were applied to elliptic problems of the form (1.10) for the case $1 < q < p$. See [19, 18, 30, 56, 33] and the above references.

When $q < 1$, the right hand side of (1.10) have a singularity at $u = 0$. General existence results of classical solutions to elliptic equations with singular nonlinearity were established by Crandall, Rabinowitz and Tartar [21]. Boccardo and Orsina [14] established existence results of generalized solutions for L^m-integrable data. For further information, we refer to [42, 40, 43, 65, 50, 14, 24, 53, 15, 17, 52, 4] and the references therein. Our approach to singular nonlinearity is not much different from the prior studies. New proposal is to consider connection with (1.6) and utilization of analogs of Green potentials. Also, we use only (1.2)-(1.5) and prove Theorems 1.2 and 1.3 for $H_0^{1,p}(\Omega; w)$- or $H_{\mathrm{loc}}^{1,p}(\Omega; w) \cap C(\overline{\Omega})$-solutions.

Theorem 1.2 corresponds to the case $0 < q < 1$ in [32, Theorems 1.2 and 1.3]. Many authors have dealt with the existence problem of finite energy solutions (see, e.g., [24, 52, 7] and the references therein). However, the necessary and sufficient condition via (1.6) was not been discussed for a long time. After the first draft of this paper, the same equivalence was asserted in [26] under the condition $\sigma \in L^1(\Omega)$. We confirm that this restriction is not necessary.

Theorem 1.3 is a nonlinear version of [43, Theorem 6]. In [43], Mâagli and Zribi proved an existence theorem of continuous solutions for the Laplace operator and σ in the Kato class and presented the bounds (6.1) and (6.6) using the Green potential of σ. We replace the Green potential with the solution to (1.1). Similar criteria for more specified linear and quasilinear operators can be found in [58, 60, 63, 54, 50].

1.3. **Organization of the paper.** In Sect. 2, we present auxiliary results from nonlinear potential theory. In Sect. 3 we provide a framework to solve (1.1). In Sect. 4, we prove Theorem 1.1. In Sect. 5, we extend Theorem 1.1 using capacitary conditions. This section is independent of Sect. 6. In Sect. 6, we apply the framework in Sect. 3 to Eq. (1.11) and prove Theorems 1.2 and 1.3 as a consequence.

Notation. We use the following notation. Let Ω be a domain (connected open subset) in $\mathbb{R}^n$.

- $\mathbf{1}_E(x) :=$ the indicator function of a set E.
- $C_c^\infty(\Omega) :=$ the set of all infinitely-differentiable functions with compact support in Ω.
- $\mathcal{M}^+(\Omega) :=$ the set of all nonnegative Radon measures on Ω.
- $L^p(\Omega; \mu) :=$ the L^p space with respect to $\mu \in \mathcal{M}^+(\Omega)$.

For simplicity, we often write $L^p(\Omega; \mu)$ as $L^p(\mu)$. For a ball $B = B(x, R)$ and $\lambda > 0$, $\lambda B := B(x, \lambda R)$. For measures μ and ν, we denote $\nu \leq \mu$ if $\mu - \nu$ is a nonnegative measure. For a sequence of extended real valued functions $\{f_j\}_{j=1}^\infty$, we denote $f_j \uparrow f$ if $f_{j+1} \geq f_j$ for all $j \geq 1$ and $\lim_{j \to \infty} f_j = f$. The letters c and C denote various constants with and without indices. The notation $A \approx_{\alpha,\beta} B$ means that there is a positive constant C depending only on α and β such that $C^{-1}B \leq A \leq CB$.

2. Preliminaries

2.1. **Weighted Sobolev spaces.** First, we recall basics of nonlinear potential theory from [34]. Let $1 < p < \infty$ be a fixed constant. A Lebesgue measurable function w on $\mathbb{R}^n$ is said to be the *weight* on $\mathbb{R}^n$ if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n; dx)$ and $w(x) > 0$ dx-a.e. We write $w(E) = \int_E w\,dx$ for a Lebesgue measurable set $E \subset \mathbb{R}^n$. We always assume that w is *p-admissible*, that is, positive constants C_D, C_P and $\lambda \geq 1$ exist, such that

$$w(2B) \leq C_D w(B)$$

and

$$⨍_B |f - f_B|\,dw \leq C_P \operatorname{diam}(B) \left(⨍_{\lambda B} |\nabla f|^p\,dw \right)^{\frac{1}{p}}, \quad \forall f \in C_c^\infty(\mathbb{R}^n),$$

where B is an arbitrary ball in $\mathbb{R}^n$, $⨍_B = w(B)^{-1} \int_B$ and $f_B = ⨍_B f\,dw$. For the basic properties of p-admissible weights, see [11, Chapter A.2], [34, Chapter 20] and the references therein. Every Muckenhoupt A_p-weight is p-admissible. One important property of p-admissible weights is the Sobolev inequality. In particular, the following form of the Poincaré inequality holds:

$$\int_B |f|^p\,dw \leq C \operatorname{diam}(B)^p \int_B |\nabla f|^p\,dw, \quad \forall f \in C_c^\infty(B),$$

where C is a constant depending only on p, C_D, C_P and λ.

Let Ω be a bounded domain in $\mathbb{R}^n$. The weighted Sobolev space $H^{1,p}(\Omega; w)$ is the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^{1,p}(\Omega;w)} := \left(\int_\Omega |u|^p + |\nabla u|^p\,dw \right)^{\frac{1}{p}}.$$

The corresponding local space $H^{1,p}_{\mathrm{loc}}(\Omega; w)$ is defined in the usual manner. We denote by $H^{1,p}_0(\Omega; w)$ the closure of $C_c^\infty(\Omega)$ in $H^{1,p}(\Omega; w)$. Since Ω is bounded, we can take $\|\nabla \cdot\|_{L^p(\Omega;w)}$ as the norm of $H^{1,p}_0(\Omega; w)$ by the Poincaré inequality.

Let $\Omega \subset \mathbb{R}^n$ be open and let $K \subset \Omega$ be compact. The *(variational)* (p,w)*-capacity* $\operatorname{cap}_{p,w}(K, \Omega)$ of the condenser (K, Ω) is defined by

$$\operatorname{cap}_{p,w}(K, \Omega) := \inf \left\{ \|\nabla u\|^p_{L^p(\Omega;w)} \colon u \geq 1 \text{ on } K,\ u \in C_c^\infty(\Omega) \right\}.$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, $\operatorname{cap}_{p,w}(E, \Omega) = 0$ if and only if $C_{p,w}(E) = 0$, where $C_{p,w}(\cdot)$ is the (Sobolev) capacity of E. We say that a property holds *quasieverywhere* (q.e.) if it holds except on a set of (p,w)-capacity zero. An extended real valued function u on Ω is called as *quasicontinuous* if for every $\epsilon > 0$ there exists an open set G such that $C_{p,w}(G) < \epsilon$ and $u|_{\Omega \setminus G}$ is continuous. Every $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$ has a quasicontinuous representative $\tilde{u}$ such that $u = \tilde{u}$ a.e.

We denote by $\mathcal{M}_0^+(\Omega)$ the set of all Radon measures μ that are absolutely continuous with respect to the (p,w)-capacity. If $\mu \in \mathcal{M}_0^+(\Omega)$ is finite, then the integral $\int_\Omega f\,d\mu$ is well-defined for any (p,w)-quasicontinuous function f on Ω.

2.2. **$\mathcal{A}$-superharmonic functions.** Assume that $\mathcal{A}$ satisfies (1.2)-(1.5). For $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$, we define the $\mathcal{A}$-Laplace operator $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$ by

$$\langle -\operatorname{div} \mathcal{A}(x, \nabla u), \varphi \rangle = \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi\,dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

If $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$ satisfies

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = (\geq) \, 0, \quad \forall \varphi \in C_c^\infty(\Omega), \ \varphi \geq 0,$$

then it is called a *weak solution (supersolution)* to $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in Ω.

A function $u \colon \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$*-superharmonic* if u is lower semicontinuous in Ω, is not identically infinite, and satisfies the comparison principle on each subdomain $D \Subset \Omega$; if $h \in H^{1,p}_{\mathrm{loc}}(D; w) \cap C(\overline{D})$ is a continuous weak solution to $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in D and if $u \geq h$ on ∂D, then $u \geq h$ in D.

If u is an $\mathcal{A}$-superharmonic function in Ω, then for any $k > 0$, $\min\{u, k\}$ is a weak supersolution to $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in Ω. Conversely, if u is a weak supersolution to $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in Ω, then its *lsc-regularization*

$$u^*(x) := \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u$$

is $\mathcal{A}$-superharmonic in Ω. If u and v are $\mathcal{A}$-superharmonic in Ω and $u(x) \leq v(x)$ for a.e. $x \in \Omega$, then $u \leq v$ in the pointwise sense.

A Radon measure $\mu = \mu[u]$ is called the *Riesz measure* of u if

$$\lim_{k \to \infty} \int_\Omega \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in C_c^\infty(\Omega).$$

It is known that every $\mathcal{A}$-superharmonic function has a unique Riesz measure.

The following weak continuity result was given by Trudinger and Wang [59].

Theorem 2.1 ([59, Theorem 3.1])**.** *Suppose that $\{u_k\}_{k=1}^\infty$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in Ω. Assume that $u_k \to u$ a.e. in Ω and that u is $\mathcal{A}$-superharmonic in Ω. Let $\mu[u_k]$ and $\mu[u]$ be the Riesz measures of u_k and u, respectively. Then $\mu[u_k]$ converges to $\mu[u]$ weakly, that is,*

$$\int_\Omega \varphi \, d\mu[u_k] \to \int_\Omega \varphi \, d\mu[u], \quad \forall \varphi \in C_c^\infty(\Omega).$$

The Harnack-type convergence theorem follows from combining Theorem 2.1 and [34, Lemma 7.3]: If $\{u_k\}_{k=1}^\infty$ is a nondecreasing sequence of $\mathcal{A}$-superharmonic functions in Ω and if $u := \lim_{k \to \infty} u_k \not\equiv \infty$, then u is $\mathcal{A}$-superharmonic in Ω and $\mu[u_k]$ converges to $\mu[u]$ weakly.

3. Minimal $\mathcal{A}$-superharmonic solution to (1.1)

Next, we introduce classes of smooth measures. For detail, see [32] and the references therein.

Definition 3.1. For $\mu \in (H_0^{1,p}(\Omega))^* \cap \mathcal{M}^+(\Omega)$, we denote by $\mathcal{W}^0_{\mathcal{A}} \mu$ the lsc-regularization of the weak solution $u \in H_0^{1,p}(\Omega; w)$ to

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \langle \mu, \varphi \rangle, \quad \forall \varphi \in H_0^{1,p}(\Omega; w).$$

Furthermore, we define a class of smooth measures $S_c[\mathcal{A}](\Omega)$ by

$$S_c[\mathcal{A}](\Omega) := \left\{ \mu \in (H_0^{1,p}(\Omega))^* \cap \mathcal{M}^+(\Omega) \colon \sup_\Omega \mathcal{W}^0_{\mathcal{A}} \mu < \infty \text{ and } \operatorname{supp} \mu \Subset \Omega \right\}.$$

By the two-sided Wolff potential estimate for $\mathcal{A}$-superharmonic functions due to Kilpeläinen and Malý (see [37, 48]), if $u \geq 0$ is $\mathcal{A}$-superharmonic in $B(x, 2R)$ and if μ is the Riesz measure of u, then,

$$\frac{1}{C}\mathbf{W}^{R}_{1,p,w}\mu(x) \leq u(x) \leq C\left(\inf_{B(x,R)} u + \mathbf{W}^{2R}_{1,p,w}\mu(x)\right),$$

where $C = C(p, C_D, C_P, \lambda)$ and $\mathbf{W}^{R}_{1,p,w}\mu$ is the *truncated Wolff potential* of μ, which is defined by

$$\mathbf{W}^{R}_{1,p,w}\mu(x) := \int_0^R \left(r^p \frac{\mu(B(x,r))}{w(B(x,r))}\right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{3.1}$$

Using this estimate twice, we can prove that $S_c[\mathcal{A}](\Omega) = S_c[w(x)|\nabla \cdot|^{p-2}\nabla \cdot](\Omega)$. Below, we write $S_c[\mathcal{A}](\Omega)$ as $S_c(\Omega)$ for simplicity.

Theorem 3.2 ([32])**.** *Let $\mu \in \mathcal{M}^+(\Omega)$. Then $\mu \in \mathcal{M}^+_0(\Omega)$ if and only if there exists an increasing sequence of compact sets $\{F_k\}_{k=1}^{\infty}$ such that $\mu_k := \mathbf{1}_{F_k}\mu \in S_c(\Omega)$ for all $k \geq 1$ and $\mu\left(\Omega \setminus \bigcup_{k=1}^{\infty} F_k\right) = 0$.*

Definition 3.3 ([18])**.** Let μ be a nonnegative Radon measure on Ω. We say that a function u is an *$\mathcal{A}$-superharmonic solution (supersolution)* to $-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu$ in Ω, if u is $\mathcal{A}$-superharmonic in Ω and $\mu[u] = \mu$ ($\mu[u] \geq \mu$), where $\mu[u]$ is the Riesz measure of u. We say that a nonnegative solution u is *minimal* if $v \geq u$ in Ω whenever v is a nonnegative supersolution to the same equation.

Definition 3.4. For $\mu \in \mathcal{M}^+_0(\Omega)$, we define

$$\mathcal{W}_{\mathcal{A}}\mu(x) := \sup\left\{\mathcal{W}^0_{\mathcal{A}}\nu(x) \colon \nu \in S_c(\Omega) \text{ and } \nu \leq \mu\right\}.$$

If $\mu \in (H^{1,p}_0(\Omega))^* \cap \mathcal{M}^+(\Omega)$, then $\mathcal{W}_{\mathcal{A}}\mu = \mathcal{W}^0_{\mathcal{A}}\mu$. If $\mathcal{W}_{\mathcal{A}}\mu \not\equiv \infty$, then $u = \mathcal{W}_{\mathcal{A}}\mu$ is the minimal nonnegative $\mathcal{A}$-superharmonic solution to $-\operatorname{div}\mathcal{A}(x, \nabla u) = \mu$ in Ω. From the argument in [32, Theorem 3.1], the following comparison principle holds.

Theorem 3.5. *Let Ω be a bounded domain. Let v be a nonnegative $\mathcal{A}$-superharmonic function in Ω with the Riesz measure ν. Assume that $\mu \in \mathcal{M}^+_0(\Omega)$ and that $\mu \leq \nu$. Then $\mathcal{W}_{\mathcal{A}}\mu \leq v$ in Ω.*

By the minimality, we can regard u as a solution to (1.1). We will discuss sufficient conditions for $\mathcal{W}_{\mathcal{A}}\mu \not\equiv \infty$ below.

4. Strong-type inequality

In [61], the following theorem was proved: If $\Omega = \mathbb{R}^n$, $w = 1$ and $0 < q < p$, then the best constant C_1 in (1.6) satisfies

$$\frac{1}{c}\, C_1 \leq \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-q}}\, d\sigma\right)^{\frac{p-q}{qp}} \leq c\, C_1,$$

where $c = c(n, p, q)$ and $\mathbf{W}_{1,p}\sigma = \mathbf{W}^{\infty}_{1,p,1}\sigma$ is the Wolff potential of σ. Following the strategy of the proof, we prove a logarithmic Caccioppoli type estimate. For (p, w)-Laplace operators, it follows from the Picone-type inequality (see [3, 16]).

Lemma 4.1. *Let $\sigma \in S_c(\Omega)$, and let $u = \mathcal{W}_{\mathcal{A}}\sigma$. Then*

$$\int_\Omega |f|^p \frac{d\sigma}{u^{p-1}} \leq \beta^p \alpha^{1-p} \int_\Omega |\nabla f|^p\, dw, \quad \forall f \in C^\infty_c(\Omega).$$

Proof. Without loss of generality we may assume that $f \geq 0$. Let $v = u + \epsilon$, where ϵ is a positive constant. Since $v \geq \epsilon > 0$, we have $f^p v^{1-p} \in H_0^{1,p}(\Omega; w)$; hence,

$$\int_\Omega f^p \frac{d\sigma}{v^{p-1}} = \int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla (f^p v^{1-p}) \, dx.$$

By (1.2) and (1.3), for a.e. $x \in \Omega$,

$$\begin{aligned}\mathcal{A}(x, \nabla v(x)) \cdot \nabla (f(x)^p v(x)^{1-p}) &= p\mathcal{A}(x, \nabla v(x)) \cdot \nabla f(x) f(x)^{p-1} v(x)^{1-p} \\ &\quad + (1-p)\mathcal{A}(x, \nabla v(x)) \cdot \nabla v(x) f(x)^p v(x)^{-p} \\ &\leq p\beta w(x) |\nabla v(x)|^{p-1} |\nabla f(x)| f(x)^{p-1} v(x)^{1-p} \\ &\quad + (1-p)\alpha w(x) |\nabla v(x)|^p f(x)^p v(x)^{-p}.\end{aligned}$$

Then, Young's inequality $ab \leq \frac{1}{p} a^p + \frac{p-1}{p} b^{\frac{p}{p-1}}$ $(a, b \geq 0)$ yields

$$\mathcal{A}(x, \nabla v(x)) \cdot \nabla (f(x)^p v(x)^{1-p}) \leq \beta^p \alpha^{1-p} |\nabla f(x)|^p w(x).$$

Therefore,

$$\int_\Omega |f|^p \frac{d\sigma}{(u+\epsilon)^{p-1}} \leq \beta^p \alpha^{1-p} \int_\Omega |\nabla f|^p \, dw, \quad \forall f \in C_c^\infty(\Omega).$$

The desired inequality follows from the monotone convergence theorem. □

Proof of Theorem 1.1. We first prove the existence part. Take $\{\sigma_k\}_{k=1}^\infty \subset S_c(\Omega)$ such that $\sigma_k = \mathbf{1}_{F_k} \sigma$ and $\mathbf{1}_{F_k} \uparrow \mathbf{1}_\Omega$ σ-a.e. Set $u_k = \mathcal{W}_\mathcal{A} \sigma_k \in H_0^{1,p}(\Omega; w) \cap L^\infty(\Omega)$. By (1.5), for each $\epsilon > 0$, we have

$$\begin{aligned}&\frac{1}{q}\left(\frac{p-1}{p-q}\right)^{p-1} \int_\Omega (u_k^{\frac{q(p-1)}{p-q}} - \epsilon^{\frac{q(p-1)}{p-q}})_+ \, d\sigma_k \\ &= \int_{\{x \in \Omega : \, u_k(x) > \epsilon\}} \mathcal{A}(x, \nabla u_k^{\frac{p-1}{p-q}}) \cdot \nabla u_k^{\frac{p-1}{p-q}} \, dx.\end{aligned}$$

Take the limit $\epsilon \to 0$. By the monotone convergence theorem and (1.2),

$$\begin{aligned}\alpha \int_\Omega |\nabla u_k^{\frac{p-1}{p-q}}|^p \, dw &\leq \int_\Omega \mathcal{A}(x, \nabla u_k^{\frac{p-1}{p-q}}) \cdot \nabla u_k^{\frac{p-1}{p-q}} \, dx \\ &= \frac{1}{q}\left(\frac{p-1}{p-q}\right)^{p-1} \int_\Omega u_k^{\frac{q(p-1)}{p-q}} \, d\sigma_k.\end{aligned}$$

The right-hand side is finite, and thus, $u_k^{\frac{p-1}{p-q}} \in H_0^{1,p}(\Omega; w)$. By (1.6) and density,

$$\int_\Omega u_k^{\frac{q(p-1)}{p-q}} \, d\sigma_k \leq C_1^q \left(\int_\Omega |\nabla u_k^{\frac{p-1}{p-q}}|^p \, dw \right)^{\frac{q}{p}}.$$

Combining the two inequalities, we obtain

$$\left(\int_\Omega |\nabla u_k^{\frac{p-1}{p-q}}|^p \, dw \right)^{\frac{p-q}{p}} \leq \frac{1}{q}\left(\frac{p-1}{p-q}\right)^{p-1} \frac{1}{\alpha} C_1^q.$$

By the Poincaré inequality, $u_k \uparrow u \not\equiv \infty$, so $u = \mathcal{W}_\mathcal{A} \sigma$ is $\mathcal{A}$-superharmonic in Ω and satisfies (1.1) by Theorem 2.1 and the monotone convergence theorem. By the uniqueness of the limit, $u^{\frac{p-1}{p-q}}$ satisfies the latter inequality in (1.7).

Conversely, assume the existence of u. Take $\{\sigma_k\}_{k=1}^{\infty} \subset S_c(\Omega)$ such that $\sigma_k = \mathbf{1}_{F_k}\sigma$ and $\mathbf{1}_{F_k} \uparrow \mathbf{1}_\Omega$ σ-a.e. by using Theorem 3.2, and set $u_k = \mathcal{W}_{\mathcal{A}}\sigma_k$. By Lemma 4.1,

$$
\begin{aligned}
\int_\Omega |f|^q \, d\sigma_k &= \int_\Omega |f|^q u_k^{p-1} \frac{d\sigma_k}{u_k^{p-1}} \\
&\le \left(\int_\Omega |f|^p \frac{d\sigma_k}{u_k^{p-1}} \right)^{\frac{q}{p}} \left(\int_\Omega u_k^{\frac{p(p-1)}{p-q}} \frac{d\sigma_k}{u_k^{p-1}} \right)^{\frac{p-q}{p}} \\
&\le \left(\beta^p \alpha^{1-p} \int_\Omega |\nabla f|^p \, dw \right)^{\frac{q}{p}} \left(\int_\Omega u_k^{\frac{q(p-1)}{p-q}} \, d\sigma_k \right)^{\frac{p-q}{p}} .
\end{aligned} \tag{4.1}
$$

Note that $v := u^{\frac{p-1}{p-q}} \ge u_k^{\frac{p-1}{p-q}}$ by Theorem 3.5. Since $v \in H_0^{1,p}(\Omega; w)$, by density,

$$
\int_\Omega u_k^{\frac{q(p-1)}{p-q}} \, d\sigma_k \le \int_\Omega v^q \, d\sigma_k \le \left(\beta^p \alpha^{1-p} \int_\Omega |\nabla v|^p \, dw \right)^{\frac{q}{p}} \left(\int_\Omega u_k^{\frac{q(p-1)}{p-q}} \, d\sigma_k \right)^{\frac{p-q}{p}} .
$$

Using (4.1) again, we obtain

$$
\int_\Omega |f|^q \, d\sigma_k \le \beta^p \alpha^{1-p} \left(\int_\Omega |\nabla f|^p \, dw \right)^{\frac{q}{p}} \left(\int_\Omega |\nabla v|^p \, dw \right)^{\frac{p-q}{p}} .
$$

The former inequality in (1.7) follows from the monotone convergence theorem. □

If $\mathcal{A}$ and $\mathcal{A}'$ satisfy the same structure conditions, then

$$
\left(\int_\Omega (\mathcal{W}_{\mathcal{A}}\sigma)^{\frac{q(p-1)}{p-q}} \, d\sigma \right)^{\frac{p-q}{qp}} \approx_{\alpha,\beta} \left(\int_\Omega (\mathcal{W}_{\mathcal{A}'}\sigma)^{\frac{q(p-1)}{p-q}} \, d\sigma \right)^{\frac{p-q}{qp}} .
$$

Note that we do not have a global pointwise estimate between solutions. The boundary behavior of two solutions may not be comparable.

Finally, we observe examples of σ satisfying (1.6). If $w \equiv 1$, $p < n$ and $\sigma \in L^m(\Omega; dx)$ with

$$
m = \left(\frac{p^*}{q} \right)' = \left(\frac{np}{q(n-p)} \right)', \tag{4.2}
$$

then (1.6) follows from Sobolev's inequality and Hölder's inequality.

Other type examples can be found from Hardy type inequalities. For one-dimensional cases, we refer to [46, Section 1.3.3] and [57]. To present multi-dimensional sufficient conditions, we add an assumption to the boundary of Ω. We say that a bounded domain Ω is *Lipschitz* if for each $y \in \partial\Omega$, there exist a local Cartesian coordinate system $(x_1, \dots, x_n) = (x', x_n)$, an open neighborhood $U = U_y$ and a Lipschitz function $h = h_y$ such that $\Omega \cap U = \{(x', x_n) \colon x_n > h(x')\} \cap U$. If Ω is bounded and Lipschitz, then by the Hardy inequality in [51, Theorem 1.6],

$$
\int_\Omega |f|^p \delta^{t-p} \, dx \le C \int_\Omega |\nabla f|^p \delta^t \, dx, \quad \forall f \in C_c^\infty(\Omega), \tag{4.3}
$$

where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$, $t < p - 1$ and C is a constant independent of f.

Proposition 4.2. *Let Ω be a bounded Lipschitz domain. Set $w = \delta^t$ and $d\sigma = \delta^{-s} dx$, where $-1 < t < p - 1$ and $1 \le s \le p - t$. Then, for any $q \in (\frac{p(s-1)}{p-1-t}, p)$, the trace inequality* (1.6) *holds.*

Proof. By the inner cone property of Lipschitz graphs, for each $r > -1$,

$$\int_{\Omega} \delta^r \, dx < \infty.$$

(For further information about the integrability of δ, see [31, Theorem 6] and the references therein.) Thus, by the Hardy inequality (4.3),

$$\begin{aligned}\int_{\Omega} |f|^q \delta^{-s} \, dx &= \int_{\Omega} \frac{|f|^q}{\delta^q} \delta^{q-s-t} \delta^t \, dx \\ &\le \left(\int_{\Omega} \frac{|f|^p}{\delta^p} \delta^t \, dx \right)^{\frac{q}{p}} \left(\int_{\Omega} \delta^{\frac{p(q-s-t)}{p-q}} \delta^t \, dx \right)^{\frac{p-q}{p}} \\ &\le C \left(\int_{\Omega} |\nabla f|^p \delta^t \, dx \right)^{\frac{q}{p}}, \quad \forall f \in C_c^\infty(\Omega).\end{aligned}$$

This completes the proof. $\square$

Remark 4.3. The boundary of a bounded Lipschitz domain is $(n-1)$-regular. Thus, by [25, Lemma 3.3] (see also [2] and [11, Chapter A.3]), $w = \delta^t$ is an A_p-weight (and hence a p-admissible weight) on $\mathbb{R}^n$ for $-1 < t < p-1$.

Corollary 4.4. *Let Ω be a bounded Lipschitz domain and let $w = \delta^t$, where $-1 < t < p-1$. Let $1 \le s < p-t$. Assume that $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $0 \le \sigma(x) \le C\delta(x)^{-s}$ for a.e. $x \in \Omega$. Then, there exists a minimal nonnegative $\mathcal{A}$-superharmonic solution u to* (1.1).

Proof. Combine Proposition 4.2, Remark 4.3 and Theorem 1.1. $\square$

Remark 4.5. Corollary 4.4 is not trivial even if $\operatorname{div} \mathcal{A}(x, \nabla u) = \Delta u$. In fact, the Green function of a polygon is not comparable to δ near the corners. This result asserts that the threshold of s still be $2\,(= p)$ even if such a case. We refer to an excellent work on barriers by Ancona [5] for further results for linear operators.

Remark 4.6. Note that $L^p(w)$-$L^p(\sigma)$ trace inequalities do not yield the existence of (bounded) solutions to (1.1). Let $\Omega = B(0,1)$ and $d\sigma = \delta^{-2} \, dx$. Then, the $L^2(dx)$-$L^2(d\sigma)$ trace inequality holds by (4.3), but the Green potential of σ does not exist.

5. Weak-type inequality and capacitary condition

In [20], the following form of a weak-type trace inequality was studied:

$$\|f\|_{L^{q,\infty}(\sigma)} \le C_2 \|\nabla f\|_{L^p(w)}, \quad \forall f \in C_c^\infty(\Omega). \tag{5.1}$$

Here, $L^{q,\infty}(\sigma)$ is the Lorentz space with respect to σ (see, e.g., [28, Chapter 1]). By a truncation argument, (5.1) implies

$$\sigma(K)^{\frac{p}{q}} \le C_2^p \operatorname{cap}_{p,w}(K, \Omega), \quad \forall K \Subset \Omega.$$

Hence, by Maz'ya's capacitary inequality (see, e.g., [45], [11, Lemma 6.22]), (5.1) is equivalent to the embedding into $L^{q,p}(\sigma)$. In particular, the condition (5.1) is weaker than (1.6) because $L^q(\sigma) = L^{q,q}(\sigma) \subsetneq L^{q,p}(\sigma)$ for $q < p$.

Theorem 5.1. *Assume that* $0 < q < p$. *Let* $\sigma \in \mathcal{M}_0^+(\Omega)$ *and let* C_2 *be the best constant of* (5.1). *Then,*

$$\frac{(\alpha/\beta)}{\alpha^{\frac{1}{p}}} C_2 \leq \|\mathcal{W}_{\mathcal{A}}\sigma\|_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)}^{\frac{p-1}{p}} \leq 4^{\frac{p-1}{p-q}} \frac{1}{\alpha^{\frac{1}{p}}} C_2. \tag{5.2}$$

To prove the lower bound in (5.2), we consider the counterparts of Lemma 5.10 and Proposition 6.1 in [55].

Lemma 5.2. *Let* $0 < q < \infty$. *Assume that* $\sigma \in \mathcal{M}_0^+(\Omega)$ *satisfies the following weighted norm inequality:*

$$\|\mathcal{W}_{\mathcal{A}}\nu\|_{L^{\frac{q(p-1)}{p},\infty}(\sigma)}^{p-1} \leq C_2' \nu(\Omega), \quad \forall \nu \in \mathcal{M}_0^+(\Omega). \tag{5.3}$$

Then, (5.1) *holds with* $C_2 = \left(\beta^p \alpha^{1-p} C_2'\right)^{\frac{1}{p}}$.

Proof. Let K be any compact subset of Ω. Let u be the $\mathcal{A}$-potential of the condenser (K, Ω), and let ν be the Riesz measure of u. Then, by [48, Corollary 4.8],

$$\nu(\Omega) \leq \beta^p \alpha^{1-p} \mathrm{cap}_{p,w}(K, \Omega).$$

Since $u = \mathcal{W}_{\mathcal{A}}\nu \geq 1$ on K,

$$\sigma(K)^{\frac{p}{q}} \leq \|\mathcal{W}_{\mathcal{A}}\nu\|_{L^{\frac{q(p-1)}{p},\infty}(\sigma)}^{p-1} \leq C_2' \nu(\Omega) \leq \beta^p \alpha^{1-p} C_2' \mathrm{cap}_{p,w}(K, \Omega).$$

Therefore,

$$\begin{aligned}\|f\|_{L^{q,\infty}(\sigma)}^p &= \sup_{k \geq 0} k^p \, \sigma(\{|f| \geq k\})^{\frac{p}{q}} \\ &\leq C_2^p \sup_{k \geq 0} k^p \, \mathrm{cap}_{p,w}(\{|f| \geq k\}, \Omega) \leq C_2^p \|\nabla f\|_{L^p(w)}^p.\end{aligned}$$

This completes the proof. □

Lemma 5.3. *Let* $\nu, \sigma \in \mathcal{M}_0^+(\Omega)$. *Then,*

$$\left\| \frac{\mathcal{W}_{\mathcal{A}}\nu}{\mathcal{W}_{\mathcal{A}}\sigma} \right\|_{L^{p-1,\infty}(\sigma)} \leq \nu(\Omega)^{\frac{1}{p-1}}.$$

Proof. Without loss of generality, we may assume that $\sigma, \nu \in S_c(\Omega)$. Set $u = \mathcal{W}_{\mathcal{A}}\sigma$ and $v = \mathcal{W}_{\mathcal{A}}\nu_t$, where $\nu_t = t^{1-p}\nu$. For $k > 0$, set $I_k(v-u) = k^{-1}\min\{(v-u)_+, k\}$. By (1.4),

$$\begin{aligned}&\int_\Omega I_k(v-u) \, d\nu_t - \int_\Omega I_k(v-u) \, d\sigma \\ &= \frac{1}{k} \int_{\{x \in \Omega : 0 < v(x) - u(x) < k\}} (\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)) \cdot \nabla(v-u) \, dx \geq 0.\end{aligned}$$

Passing to the limit $k \to 0$ yields

$$\nu_t(\Omega) \geq \sigma(\{x \in \Omega : v(x) > u(x)\}).$$

By (1.5), $v = t^{-1}\mathcal{W}_{\mathcal{A}}\nu$, and thus,

$$t^{p-1}\sigma\left(\left\{x \in \Omega : \frac{\mathcal{W}_{\mathcal{A}}\nu(x)}{\mathcal{W}_{\mathcal{A}}\sigma(x)} > t\right\}\right) \leq \nu(\Omega).$$

Taking the supremum over $t > 0$, we obtain the desired inequality. □

Proof of Theorem 5.1. We first prove the latter inequality for $\sigma \in S_c(\Omega)$. Let $u = \mathcal{W}_{\mathcal{A}}\sigma$. Then,

$$\sup_{j\in\mathbb{Z}} 2^{j\frac{q(p-1)}{p-q}}\sigma(E_j) \le \|u\|^{\frac{q(p-1)}{p-q}}_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)} \le \sup_{j\in\mathbb{Z}} 2^{(j+1)\frac{q(p-1)}{p-q}}\sigma(E_j),$$

where $E_j = \{x \in \Omega \colon u(x) > 2^j\}$. By (5.1), we have

$$\sigma(E_j) \le C_2^q \mathrm{cap}_{p,w}(E_j,\Omega)^{\frac{q}{p}} \le C_2^q \mathrm{cap}_{p,w}(E_j,E_{j-1})^{\frac{q}{p}}.$$

Let $U_j = \min\{(u-2^{j-1})_+, 2^{j-1}\}$. Then $U_j \in H_0^{1,p}(E_{j-1};w)$, $0 \le U_j \le 2^{j-1}$ in E_{j-1} and $U_j \equiv 2^{j-1}$ on E_j. By (1.2),

$$\begin{aligned}\mathrm{cap}_{p,w}(E_j,E_{j-1}) &\le 2^{(1-j)p}\int_\Omega |\nabla U_j|^p \, dw \le \frac{2^{(1-j)p}}{\alpha}\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla U_j \, dx \\ &= \frac{2^{(1-j)p}}{\alpha}\int_\Omega U_j \, d\sigma \le \frac{2^{p-1}2^{j(1-p)}}{\alpha}\sigma(E_{j-1}).\end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned}\|u\|^{\frac{q(p-1)}{p-q}}_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)} &\le 2^{\frac{q(p-1)}{p-q}}\frac{C_2^q}{\alpha^{\frac{q}{p}}}\left(2^{p-1}\sup_{j\in\mathbb{Z}} 2^{j\frac{q(p-1)}{p-q}}\sigma(E_{j-1})\right)^{\frac{q}{p}} \\ &\le 4^{\frac{q(p-1)}{p-q}}\frac{C_2^q}{\alpha^{\frac{q}{p}}}\left(\|u\|^{\frac{q(p-1)}{p-q}}_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)}\right)^{\frac{q}{p}}.\end{aligned}$$

Hence the desired inequality holds.

Let us prove the existence. Take $\{\sigma_k\}_{k=1}^\infty \subset S_c(\Omega)$ such that $\sigma_k = \mathbf{1}_{F_k}\sigma$ and $\mathbf{1}_{F_k} \uparrow \mathbf{1}_\Omega$ σ-a.e. By the monotone convergence theorem,

$$\|\mathcal{W}_{\mathcal{A}}\sigma\|^{\frac{q(p-1)}{p}}_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)} \le \lim_{k\to\infty}\|\mathcal{W}_{\mathcal{A}}\sigma_k\mathbf{1}_{F_k}\|^{\frac{q(p-1)}{p}}_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)} \le \frac{4^{\frac{q(p-1)}{p-q}}}{\alpha^{\frac{q}{p}}}C_2^q.$$

Thus $\mathcal{W}_{\mathcal{A}}\sigma \not\equiv \infty$ and is $\mathcal{A}$-superharmonic in Ω.

Conversely, assume that $\|\mathcal{W}_{\mathcal{A}}\sigma\|_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)}$ is finite. By Hölder's inequality for Lorentz spaces, Lemma 5.3 yields

$$\begin{aligned}\|\mathcal{W}_{\mathcal{A}}\nu\|_{L^{\frac{q(p-1)}{p},\infty}(\sigma)} &\le \|\mathcal{W}_{\mathcal{A}}\sigma\|_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)}\left\|\frac{\mathcal{W}_{\mathcal{A}}\nu}{\mathcal{W}_{\mathcal{A}}\sigma}\right\|_{L^{p-1,\infty}(\sigma)} \\ &\le \|\mathcal{W}_{\mathcal{A}}\sigma\|_{L^{\frac{q(p-1)}{p-q},\infty}(\sigma)}\nu(\Omega)^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}_0^+(\Omega).\end{aligned}$$

From Lemma 5.2, the desired lower bound follows. $\square$

To treat the more general $\sigma \in \mathcal{M}_0^+(\Omega)$, we consider the following relaxed capacitary condition. A typical example of such a measure is the sum of a finite measure in $\mathcal{M}_0^+(\Omega)$ and a measure satisfying (1.6). Note that it is not clear whether $\mathcal{W}_{\mathcal{A}}(\sigma_1+\sigma_2) \not\equiv \infty$ even if $\mathcal{W}_{\mathcal{A}}\sigma_i \not\equiv \infty$ for $i = 1, 2$.

Proposition 5.4. *Let $\sigma \in \mathcal{M}_0^+(\Omega)$. Assume that there exists a constant $C_3 > 0$ such that*

$$\sigma(K) \le C_3\left(\mathrm{cap}_{p,w}(K,\Omega)^{\frac{q}{p}} + 1\right), \quad \forall K \Subset \Omega, \tag{5.4}$$

where $0 < q < p$ and $C_3 > 0$ is a constant. Then, there exists a minimal nonnegative $\mathcal{A}$-superharmonic solution u to (1.1).

Proof. We first claim that if $\sigma \in S_c(\Omega)$ satisfies (5.4), then

$$\|(u-1)_+\|_{L^{p-1,\infty}(w)} \leq C, \tag{5.5}$$

where $C = C(p, q, w, \alpha, C_3)$. Let $E_j = \{x \in \Omega \colon u(x) > 2^j\}$. As in the proof of Theorem 5.1,

$$\sigma(E_j) \leq C \operatorname{cap}_{p,w}(E_j, \Omega)^{\frac{q}{p}} + C \leq C \left(2^{j(1-p)} \sigma(E_{j-1})\right)^{\frac{q}{p}} + C.$$

Thus,

$$2^{j\frac{q(p-1)}{p-q}} \sigma(E_j) \leq C \left(2^{(j-1)\frac{q(p-1)}{p-q}} \sigma(E_{j-1})\right)^{\frac{q}{p}} + 2^{j\frac{q(p-1)}{p-q}} C,$$

and hence

$$\max\left\{\sigma(E_0), \sup_{j \leq -1} 2^{j\frac{q(p-1)}{p-q}} \sigma(E_j)\right\} \leq C \left(\sup_{j \leq -1} 2^{j\frac{q(p-1)}{p-q}} \sigma(E_j)\right)^{\frac{q}{p}} + C.$$

By Young's inequality, $\sigma(E_0) \leq C$. Using the test function $\min\{(u-1)_+, l\}$ $(l > 0)$, we obtain

$$\begin{aligned} \alpha \|\nabla \min\{(u-1)_+, l\}\|_{L^p(w)}^p &\leq \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \min\{(u-1)_+, l\} \, dx \\ &= \int_\Omega \min\{(u-1)_+, l\} \, d\sigma \leq l \sigma(E_0). \end{aligned}$$

Then, the Poincaré inequality yields

$$l^p w(\{x \in \Omega : (u-1)_+(x) \leq l\}) \leq Cl.$$

This result implies (5.5).

Take $\{\sigma_k\}_{k=1}^\infty \subset S_c(\Omega)$ such that $\sigma_k = \mathbf{1}_{F_k} \sigma$ and $\mathbf{1}_{F_k} \uparrow \mathbf{1}_\Omega$ σ-a.e. Then $\mathcal{W}_A \sigma_k \uparrow \mathcal{W}_A \sigma \not\equiv \infty$ by (5.5). Hence, $u := \mathcal{W}_A \sigma$ is the desired minimal $\mathcal{A}$-superharmonic solution. □

6. Applications to singular elliptic problems

We first prove the following general existence result.

Theorem 6.1. *Let $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$. Assume that there exists a nonnegative $\mathcal{A}$-superharmonic supersolution v to* (1.1). *Let $h \colon (0, \infty) \to (0, \infty)$ be a continuously differentiable nonincreasing function. Then there exists a nonnegative $\mathcal{A}$-superharmonic solution u to* (1.11) *satisfying the Dirichlet boundary condition in the sense that*

$$0 < g(u)(x) \leq v(x), \quad \forall x \in \Omega, \tag{6.1}$$

where

$$g(u) := \int_0^u \frac{1}{h(t)^{\frac{1}{p-1}}} \, dt.$$

Remark 6.2. By assumption, g is a convex increasing function. In particular, $\lim_{t \to \infty} g(t) = \infty$. Use of this type transformation can be found in [64, 43, 27].

As in prior studies, we use the following approximating problems:

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_k) = \sigma_k h\left(u_k + \dfrac{1}{k}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

where $k \in \mathbb{N}$, $\sigma_k := \mathbf{1}_{F_k}\sigma$ and $\{F_k\}_{k=1}^\infty$ is a sequence of compact sets in Theorem 3.2. We may assume that $\sigma_1 \neq 0$ without loss of generality.

If $\{u_k\}_{k=1}^\infty \subset H_0^{1,p}(\Omega; w)$ is a sequence of weak solutions to (6.2), then $u_{k+1} \geq u_k$ a.e. in Ω for all $k \geq 1$. In fact, using the test function $(u_k - u_{k+1})_+ \in H_0^{1,p}(\Omega; w)$, we have

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x, \nabla u_k) - \mathcal{A}(x, \nabla u_{k+1})) \cdot \nabla (u_k - u_{k+1})_+ \, dx \\
&= \int_\Omega (u_k - u_{k+1})_+ \, h\left(u_k + \frac{1}{k}\right) d\sigma_k - \int_\Omega (u_k - u_{k+1})_+ \, h\left(u_k + \frac{1}{k+1}\right) d\sigma_{k+1} \\
&\leq \int_\Omega (u_k - u_{k+1})_+ \left\{ h\left(u_k + \frac{1}{k+1}\right) - h\left(u_{k+1} + \frac{1}{k+1}\right) \right\} d\sigma_{k+1} \leq 0.
\end{aligned}$$

By (1.4), $\nabla u_k = \nabla u_{k+1}$ a.e. in $\{x \in \Omega \colon u_k(x) > u_{k+1}(x)\}$, and thus, $(u_k - u_{k+1})_+ = 0$ in $H_0^{1,p}(\Omega; w)$.

It is well-known that the singular problem has a convex structure (see e.g., [40, 14]), so we use the Minty-Browder theorem (e.g., [38, Corollary III.1.8] and [49]). The same approach can be found in [39].

Lemma 6.3. *There exists a nonnegative weak solution $u_k \in H_0^{1,p}(\Omega; w)$ to* (6.2).

Proof. Set $V = H_0^{1,p}(\Omega; w)$. For $u \in V$, we define

$$A(u) := -\operatorname{div} \mathcal{A}(x, \nabla u) - \sigma_k h\left(u_+ + \frac{1}{k}\right).$$

We apply the Minty-Browder theorem to A. Since $\sigma_k \in S_c(\Omega) \subset (H_0^{1,p}(\Omega))^*$,

$$\left| \int_\Omega \varphi \, h\left(u_+ + \frac{1}{k}\right) d\sigma_k \right| \leq h\left(\frac{1}{k}\right) \int_\Omega |\varphi| \, d\sigma_k \leq h\left(\frac{1}{k}\right) \|\sigma_k\|_{V^*} \|\nabla \varphi\|_{L^p(\Omega)}$$

for all $\varphi \in V$. Thus, A is a bounded operator from V to the dual V^* of V. Moreover,

$$\frac{\langle A(u), u \rangle}{\|u\|_V} \to \infty \quad \text{as} \quad \|u\|_V \to \infty.$$

Since h is nonincreasing,

$$\int_\Omega (u - v) \left\{ h\left(u_+ - \frac{1}{k}\right) - h\left(v_+ - \frac{1}{k}\right) \right\} d\sigma_k \leq 0, \quad \forall u, v \in V,$$

and hence,

$$\langle A(u) - A(v), u - v \rangle \geq 0, \quad \forall u, v \in V.$$

Finally, we claim that if $\{u_i\}_{i=1}^\infty \subset V$ and $u_i \to u$ in V, then for any $\varphi \in V$,

$$\int_\Omega \mathcal{A}(x, \nabla u_i) \cdot \nabla \varphi \, dx \to \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \tag{6.3}$$

and

$$\int_\Omega \varphi \, h\left((u_i)_+ + \frac{1}{k}\right) d\sigma_k \to \int_\Omega \varphi \, h\left(u_+ + \frac{1}{k}\right) d\sigma_k. \tag{6.4}$$

The proof of (6.3) is standard (see [34, Proposition 17.2]). Let $\{u_{i_j}\}_{j=1}^\infty$ be any subsequence of $\{u_i\}_{i=1}^\infty$. Since $\sigma_k \in S_c(\Omega)$, the embedding $V \hookrightarrow L^1(\Omega; \sigma_k)$ is continuous, and hence $u_{i_j} \to u$ in $L^1(\Omega; \sigma_k)$. We choose a subsequence $\{u_{i_{j'}}\}_{j'=1}^\infty$

of $\{u_{i_j}\}_{j=1}^{\infty}$ such that $u_{i_{j'}} \to u$ σ_k-a.e. Since h is continuous and nonincreasing, by the dominated convergence theorem,

$$\int_\Omega \varphi\, h\left((u_{i_{j'}})_+ + \frac{1}{k}\right) d\sigma_k \to \int_\Omega \varphi\, h\left(u_+ + \frac{1}{k}\right) d\sigma_k.$$

The right hand side is independent of the choice of $\{u_{i_j}\}_{j=1}^{\infty}$, and hence (6.4) holds. Consequently, the map $A\colon V \to V^*$ is onto. In particular, there exists a unique $u \in V$ such that $A(u) = 0$. Since u is a supersolution to $-\operatorname{div}\mathcal{A}(x, \nabla u) = 0$ in Ω, $u \geq 0$ q.e. in Ω. Thus, $u = u_+$ σ_k-a.e. in Ω and satisfies (6.2). $\square$

Lemma 6.4. *Let $u_k \in H_0^{1,p}(\Omega; w)$ be an lsc-regularized weak solution to* (6.2). *Then $0 < g(u_k) \leq \mathcal{W}_\mathcal{A}\sigma_k$ in Ω.*

Proof. By the comparison principle for weak solutions,

$$0 \leq u_k(x) \leq \mathcal{W}_\mathcal{A}\left(h\left(\frac{1}{k}\right)\sigma_k\right)(x), \quad \forall x \in \Omega.$$

Since $\sigma_k \in S_c(\Omega)$, $u_k \in H_0^{1,p}(\Omega; w) \cap L^\infty(\Omega)$. Fix a nonnegative function $\varphi \in C_c^\infty(\Omega)$. By assumption, the function $t \mapsto \left(\frac{1}{h(t)} - \epsilon\right)_+$ $(\epsilon > 0)$ is nondecreasing and locally Lipschitz. Consider the test function $\left(\frac{1}{h(u_k)} - \epsilon\right)_+ \varphi \in H_0^{1,p}(\Omega; w)$. Since

$$\int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\left(\frac{1}{h(u_k)} - \epsilon\right)_+ \varphi\, dx \geq 0,$$

we have

$$\begin{aligned} &\int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi \left(\frac{1}{h(u_k)} - \epsilon\right)_+ dx \\ &\leq \int_\Omega \left(\frac{1}{h(u_k)} - \epsilon\right)_+ \varphi\, h\left(u_k + \frac{1}{k}\right) d\sigma_k \leq \int_\Omega \varphi\, d\sigma_k. \end{aligned}$$

By (1.5) and the dominated convergence theorem,

$$\int_\Omega \mathcal{A}(x, \nabla g(u_k)) \cdot \nabla\varphi\, dx = \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi\, \frac{1}{h(u_k)}\, dx \leq \int_\Omega \varphi\, d\sigma_k.$$

By the comparison principle for weak solutions, $0 < g(u_k) \leq \mathcal{W}_\mathcal{A}\sigma_k$ a.e. in Ω. In other words, $0 < u_k \leq g^{-1}(\mathcal{W}_\mathcal{A}\sigma_k)$ a.e. in Ω, where g^{-1} is the inverse function of g. Since g^{-1} is concave and increasing, $g^{-1}(\mathcal{W}_\mathcal{A}\sigma_k)$ is $\mathcal{A}$-superharmonic in Ω, and thus, the same inequality holds for all $x \in \Omega$. $\square$

Proof of Theorem 6.1. Let $\{u_k\}_{k=1}^{\infty}$ be the sequence of lsc-regularized weak solutions to (6.2). By Lemma 6.4,

$$0 < g(u_k) \leq \mathcal{W}_\mathcal{A}\sigma_k \leq \mathcal{W}_\mathcal{A}\sigma \leq v \quad \text{in } \Omega.$$

Thus, $u(x) := \lim_{k\to\infty} u_k(x)$ is not identically infinite. By Theorem 2.1, $\mu[u_k]$ converges to $\mu[u]$ weakly. Fix $\varphi \in C_c^\infty(\Omega)$. By the weak Harnack inequality, there exists a constant c such that $u_k \geq u_1 \geq c > 0$ on $\operatorname{supp}\varphi$. By the dominated convergence theorem,

$$\lim_{k\to\infty} \int_\Omega \varphi\, h\left(u_k + \frac{1}{k}\right) d\sigma_k = \int_\Omega \varphi\, h(u)\, d\sigma.$$

Thus, u satisfies (1.11) in the sense of $\mathcal{A}$-superharmonic solutions. $\square$

The following existence result was established by Boccardo and Orsina [14] for unweighted equations. The necessity part seems to be new.

Corollary 6.5. *Suppose that* $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$*. Set* $h(u) = u^{-\gamma}$*, where* $0 < \gamma < \infty$*. Then there exists an* $\mathcal{A}$*-superharmonic solution* u *to* (1.11) *satisfying* $u^{\frac{p-1+\gamma}{p}} \in H_0^{1,p}(\Omega; w)$ *if and only if* σ *is finite.*

Proof. Assume that σ is finite. Then $\mathcal{W}_{\mathcal{A}}\sigma \not\equiv \infty$ by [48, Theorem 6.6]. Thus, the existence of $u = \mathcal{W}_A(u^{-\gamma}\sigma)$ follows from Theorem 6.1. Set $\mu = u^{-\gamma}\sigma$. By (1.7) and (1.8),

$$\int_\Omega d\sigma = \int_\Omega u^\gamma u^{-\gamma} \, d\sigma = \int_\Omega (\mathcal{W}_{\mathcal{A}}\mu)^\gamma \, d\mu \approx_{\alpha,\beta} \int_\Omega |\nabla u^{\frac{p-1+\gamma}{p}}|^p \, dw.$$

The necessity part follows from this two-sided estimate directly. □

Remark 6.6. If there exists a finite energy solution $u \in H_0^{1,p}(\Omega; w)$ to (1.11), then

$$\begin{aligned} \int_\Omega \varphi \, h(u) \, d\sigma &= \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx \\ &\le \beta \left(\int_\Omega |\nabla u|^p \, dw \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla \varphi|^p \, dw \right)^{\frac{1}{p}}, \quad \forall \varphi \in C_c^\infty(\Omega). \end{aligned} \tag{6.5}$$

Thus, the embedding $H_0^{1,p}(\Omega; w) \hookrightarrow L^1(\Omega; h(u)\,\sigma)$ is continuous. Furthermore, such a solution is unique in $H_0^{1,p}(\Omega; w)$. In fact, if $v \in H_0^{1,p}(\Omega; w)$ is another solution, then

$$\int_\Omega (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)) \cdot \nabla(u - v) \, dx = \int_\Omega (u - v) \, (h(u) - h(v)) \, d\sigma \le 0.$$

Hence $\nabla u = \nabla v$ a.e. in Ω and $u = v$ in $H_0^{1,p}(\Omega; w)$.

Proof of Theorem 1.2. Assume that (1.6) holds. Let $h(u) = u^{q-1}$, and let $\{u_k\}_{k=1}^\infty$ be the sequence of lsc-regularized weak solutions to (6.2). Then

$$\begin{aligned} \alpha \int_\Omega |\nabla u_k|^p \, dw &\le \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla u_k \, dx = \int_\Omega u_k \left(u_k + \frac{1}{k} \right)^{q-1} \, d\sigma_k \\ &\le \int_\Omega u_k^q \, d\sigma \le C_1^q \left(\int_\Omega |\nabla u_k|^p \, dw \right)^{\frac{q}{p}}. \end{aligned}$$

Therefore, $u(x) = \lim_{k\to\infty} u_k(x)$ belongs to $H_0^{1,p}(\Omega; w)$, and

$$\|\nabla u\|_{L^p(\Omega;w)} \le \liminf_{k\to\infty} \|\nabla u_k\|_{L^p(\Omega;w)} \le \alpha^{\frac{-1}{p-q}} C_1^{\frac{q}{p-q}}.$$

Meanwhile, by the argument in the proof of Theorem 6.1, u satisfies (1.11) in the sense of weak solutions. The uniqueness follows from Remark 6.6.

Conversely, assume the existence of u. By (6.5) and Hölder's inequality,

$$\begin{aligned} \int_\Omega \varphi^q \, d\sigma &= \int_\Omega u^{q(1-q)} \left(\varphi u^{q-1} \right)^q \, d\sigma \\ &\le \left(\int_\Omega u^q \, d\sigma \right)^{1-q} \left(\int_\Omega \varphi u^{q-1} \, d\sigma \right)^q \\ &\le \beta \left(\int_\Omega |\nabla u|^p \, dw \right)^{\frac{p-q}{p}} \left(\int_\Omega |\nabla \varphi|^p \, dw \right)^{\frac{q}{p}}, \quad \forall \varphi \in C_c^\infty(\Omega), \ \varphi \ge 0. \end{aligned}$$

Therefore, (1.6) holds with $C_1 \leq \beta^{\frac{1}{q}} \|\nabla u\|_{L^p(\Omega;w)}^{\frac{p-q}{q}}$. □

Proof of Theorem 1.3. Assume the existence of v. Then, Theorem 6.1 gives a bounded (p,w)-superharmonic solution u to (1.11). By [34, Theorem 7.25], $u \in H^{1,p}_{\mathrm{loc}}(\Omega;w) \cap L^\infty(\Omega)$. By the weak Harnack inequality, $h(u)$ is locally bounded in Ω. Fix $x_0 \in \Omega$. By [37, Theorem 4.20] (see also [48, Corollary 3.17] for weighted equations), u is continuous at x_0 if and only if

$$\lim_{R\to 0} \sup_{x \in B(x_0,R)} \mathbf{W}^R_{p,w}(h(u)\sigma)(x) = 0.$$

On the other hand, since v is continuous at x_0, by the same reason,

$$\lim_{R\to 0} \sup_{x \in B(x_0,R)} \mathbf{W}^R_{p,w}\sigma(x) = 0.$$

Therefore, u is continuous in Ω. The boundary continuity of u follows from (6.1).

Let $u, v \in H^{1,p}_{\mathrm{loc}}(\Omega;w) \cap C(\overline{\Omega})$ be continuous weak solutions to (1.11). Assume that there exists $x \in \Omega$ such that $u(x) > v(x)$. Then the open set $D = \{x \in \Omega \colon u(x) > v(x) + \epsilon\}$ is not empty for $\epsilon > 0$ small. Recall that u vanishes on $\partial\Omega$ continuously and that $v \geq 0$ in D. Thus $\overline{D} \Subset \Omega$ and $u, v \in H^{1,p}(D;w)$. Using the test function $(u - v - \epsilon) \in H^{1,p}_0(D;w)$, we get

$$\begin{aligned}
&\int_D (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla v)) \cdot \nabla(u-v)\,dx \\
&= \int_D (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla v)) \cdot \nabla(u-v-\epsilon)\,dx \\
&= \int_D (u-v-\epsilon)\,(h(u) - h(v))\,d\sigma \leq 0.
\end{aligned}$$

Thus, $\nabla u = \nabla v$ a.e. in D and $(u - v - \epsilon) = 0$ in D. This contradicts the assumption.

Conversely, assume the existence of u. By Theorem 3.5,

$$\begin{aligned}
u(x) &\geq \mathcal{W}_{\mathcal{A}}\,(h(u)\sigma)\,(x) \\
&\geq \mathcal{W}_{\mathcal{A}}\left(h(\sup_\Omega u)\sigma\right)(x) = h(\sup_\Omega u)^{\frac{1}{p-1}}\mathcal{W}_{\mathcal{A}}\sigma(x), \quad \forall x \in \Omega.
\end{aligned}$$

Therefore, $v = \mathcal{W}_{\mathcal{A}}\sigma$ is a bounded weak solution to (1.1), and

$$v(x) \leq \frac{u(x)}{h(\sup_\Omega u)^{\frac{1}{p-1}}}, \quad \forall x \in \Omega. \tag{6.6}$$

The interior regularity of v follows from the same argument as above. □

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References

[1] D. R. Adams. A trace inequality for generalized potentials. *Studia Math.*, 48:99–105, 1973.

[2] H. Aimar, M. Carena, R. Durán, and M. Toschi. Powers of distances to lower dimensional sets as Muckenhoupt weights. *Acta Math. Hungar.*, 143(1):119–137, 2014.

[3] W. Allegretto and Y. X. Huang. A Picone's identity for the p-Laplacian and applications. *Nonlinear Anal.*, 32(7):819–830, 1998.

[4] C. O. Alves, C. A. Santos, and T. W. Siqueira. Uniqueness in $W^{1,p(x)}_{loc}(\Omega)$ and continuity up to portions of the boundary of positive solutions for a strongly-singular elliptic problem. *J. Differential Equations*, 269(12):11279–11327, 2020.

[5] A. Ancona. On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 34(2):274–290, 1986.

[6] D. H. Armitage and S. J. Gardiner. *Classical potential theory.* Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001.

[7] K. Bal and P. Garain. Weighted and anisotropic Sobolev inequality with extremal. *Manuscripta Math.*, 168(1-2):101–117, 2022.

[8] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[9] M.-F. Bidaut-Véron. Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian. In *Proceedings of the 2001 Luminy Conference on Quasilinear Elliptic and Parabolic Equations and System*, volume 8 of *Electron. J. Differ. Equ. Conf.*, pages 23–34. Southwest Texas State Univ., San Marcos, TX, 2002.

[10] M. F. Bidaut-Véron. Removable singularities and existence for a quasilinear equation with absorption or source term and measure data. *Adv. Nonlinear Stud.*, 3(1):25–63, 2003.

[11] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics.* European Mathematical Society (EMS), Zürich, 2011.

[12] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.

[13] L. Boccardo, T. Gallouët, and L. Orsina. Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(5):539–551, 1996.

[14] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, 2010.

[15] B. Bougherara, J. Giacomoni, and J. Hernández. Some regularity results for a singular elliptic problem. *Discrete Contin. Dyn. Syst.*, (Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl.):142–150, 2015.

[16] L. Brasco and G. Franzina. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.*, 37(3):769–799, 2014.

[17] A. Canino, B. Sciunzi, and A. Trombetta. Existence and uniqueness for p-Laplace equations involving singular nonlinearities. *NoDEA Nonlinear Differential Equations Appl.*, 23(2):Art. 8, 18, 2016.

[18] D. Cao and I. Verbitsky. Nonlinear elliptic equations and intrinsic potentials of Wolff type. *J. Funct. Anal.*, 272(1):112–165, 2017.

[19] D. T. Cao and I. E. Verbitsky. Finite energy solutions of quasilinear elliptic equations with sub-natural growth terms. *Calc. Var. Partial Differential Equations*, 52(3-4):529–546, 2015.

[20] C. Cascante, J. M. Ortega, and I. E. Verbitsky. Trace inequalities of Sobolev type in the upper triangle case. *Proc. London Math. Soc. (3)*, 80(2):391–414, 2000.

[21] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, 1977.

[22] G. Dal Maso and A. Malusa. Some properties of reachable solutions of nonlinear elliptic equations with measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):375–396 (1998), 1997. Dedicated to Ennio De Giorgi.

[23] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

[24] J. I. Díaz, J. Hernández, and J. M. Rakotoson. On very weak positive solutions to some semilinear elliptic problems with simultaneous singular nonlinear and spatial dependence terms. *Milan J. Math.*, 79(1):233–245, 2011.

[25] R. G. Durán and F. López García. Solutions of the divergence and analysis of the Stokes equations in planar Hölder-α domains. *Math. Models Methods Appl. Sci.*, 20(1):95–120, 2010.
[26] P. Garain and A. Ukhlov. Mixed local and nonlocal Sobolev inequalities with extremal and associated quasilinear singular elliptic problems. *Nonlinear Anal.*, 223:Paper No. 113022, 35, 2022.
[27] D. Giachetti, P. J. Martínez-Aparicio, and F. Murat. Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(4):1395–1442, 2018.
[28] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.
[29] S. Granlund, P. Lindqvist, and O. Martio. Note on the PWB-method in the nonlinear case. *Pacific J. Math.*, 125(2):381–395, 1986.
[30] A. Grigor'yan and I. Verbitsky. Pointwise estimates of solutions to nonlinear equations for nonlocal operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 20(2):721–750, 2020.
[31] P. Hajłasz and P. Koskela. Isoperimetric inequalities and imbedding theorems in irregular domains. *J. London Math. Soc. (2)*, 58(2):425–450, 1998.
[32] T. Hara. Quasilinear elliptic equations with sub-natural growth terms in bounded domains. *NoDEA Nonlinear Differential Equations Appl.*, 28(6):Paper No. 62, 2021.
[33] T. Hara and A. Seesanea. Existence of minimal solutions to quasilinear elliptic equations with several sub-natural growth terms. *Nonlinear Anal.*, 197:111847, 21, 2020.
[34] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.
[35] T. Kilpeläinen, T. Kuusi, and A. Tuhola-Kujanpää. Superharmonic functions are locally renormalized solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(6):775–795, 2011.
[36] T. Kilpeläinen and J. Malý. Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):591–613, 1992.
[37] T. Kilpeläinen and J. Malý. The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.*, 172(1):137–161, 1994.
[38] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.
[39] T. Klimsiak. Semilinear elliptic equations with Dirichlet operator and singular nonlinearities. *J. Funct. Anal.*, 272(3):929–975, 2017.
[40] A. V. Lair and A. W. Shaker. Classical and weak solutions of a singular semilinear elliptic problem. *J. Math. Anal. Appl.*, 211(2):371–385, 1997.
[41] N. S. Landkof. *Foundations of modern potential theory*. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy.
[42] A. C. Lazer and P. J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3):721–730, 1991.
[43] H. Mâagli and M. Zribi. Existence and estimates of solutions for singular nonlinear elliptic problems. *J. Math. Anal. Appl.*, 263(2):522–542, 2001.
[44] M. Marcus and L. Véron. *Nonlinear second order elliptic equations involving measures*, volume 21 of *De Gruyter Series in Nonlinear Analysis and Applications*. De Gruyter, Berlin, 2014.
[45] V. Maz'ya. Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings. *J. Funct. Anal.*, 224(2):408–430, 2005.
[46] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.
[47] V. Maz'ya and Y. Netrusov. Some counterexamples for the theory of Sobolev spaces on bad domains. *Potential Anal.*, 4(1):47–65, 1995.
[48] P. Mikkonen. On the Wolff potential and quasilinear elliptic equations involving measures. *Ann. Acad. Sci. Fenn. Math. Diss.*, (104):71, 1996.
[49] G. J. Minty. On the generalization of a direct method of the calculus of variations. *Bull. Amer. Math. Soc.*, 73:315–321, 1967.

[50] A. Mohammed. Positive solutions of the p-Laplace equation with singular nonlinearity. *J. Math. Anal. Appl.*, 352(1):234–245, 2009.

[51] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 16:305–326, 1962.

[52] F. Oliva and F. Petitta. Finite and infinite energy solutions of singular elliptic problems: existence and uniqueness. *J. Differential Equations*, 264(1):311–340, 2018.

[53] L. Orsina and F. Petitta. A Lazer-McKenna type problem with measures. *Differential Integral Equations*, 29(1-2):19–36, 2016.

[54] K. Perera and E. A. B. Silva. On singular p-Laplacian problems. *Differential Integral Equations*, 20(1):105–120, 2007.

[55] S. Quinn and I. E. Verbitsky. A sublinear version of Schur's lemma and elliptic PDE. *Anal. PDE*, 11(2):439–466, 2018.

[56] A. Seesanea and I. E. Verbitsky. Solutions to sublinear elliptic equations with finite generalized energy. *Calc. Var. Partial Differential Equations*, 58(1):Art. 6, 21, 2019.

[57] G. Sinnamon and V. D. Stepanov. The weighted Hardy inequality: new proofs and the case $p = 1$. *J. London Math. Soc. (2)*, 54(1):89–101, 1996.

[58] S. D. Taliaferro. A nonlinear singular boundary value problem. *Nonlinear Anal.*, 3(6):897–904, 1979.

[59] N. S. Trudinger and X.-J. Wang. On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.*, 124(2):369–410, 2002.

[60] H. Usami. On a singular elliptic boundary value problem in a ball. *Nonlinear Anal.*, 13(10):1163–1170, 1989.

[61] I. E. Verbitsky. Nonlinear potentials and trace inequalities. In *The Maz′ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 323–343. Birkhäuser, Basel, 1999.

[62] L. Véron. *Local and global aspects of quasilinear degenerate elliptic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Quasilinear elliptic singular problems.

[63] J. Wang and W. Gao. A singular boundary value problem for the one-dimensional p-Laplacian. *J. Math. Anal. Appl.*, 201(3):851–866, 1996.

[64] Z. Zhang. A remark on the existence of entire solutions of a singular semilinear elliptic problem. *J. Math. Anal. Appl.*, 215(2):579–582, 1997.

[65] Z. Zhang and J. Cheng. Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems. *Nonlinear Anal.*, 57(3):473–484, 2004.

Email address: `takanobu.hara.math@gmail.com`

Graduate School of Information Science and Technology, Kita 14, Nishi 9, Kita-ku, Sapporo, Hokkaido, 060-0814, Japan