LOGARITHMIC A-HYPERGEOMETRIC SERIES II

GO OKUYAMA* AND MUTSUMI SAITO

Abstract. In this paper, following [6], we continue to develop the perturbing method of constructing logarithmic series solutions to a regular A-hypergeometric system.

Fixing a fake exponent of an A-hypergeometric system, we consider some spaces of linear partial differential operators with constant coefficients. Comparing these spaces, we construct a fundamental system of series solutions with the given exponent by the perturbing method. In addition, we give a sufficient condition for a given fake exponent to be an exponent. As important examples of the main results, we give fundamental systems of series solutions to Aomoto-Gel'fand systems and to Lauricella's F_C systems with special parameter vectors, respectively.

1. Introduction

Let $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$-matrix of rank d with coefficients in $\mathbb{Z}$. Throughout this paper, we assume the homogeneity of A, i.e., we assume that all $\boldsymbol{a}_j$ belong to one hyperplane off the origin in $\mathbb{Q}^d$. Let $\mathbb{N}$ be the set of nonnegative integers. Let I_A denote the toric ideal in the polynomial ring $\mathbb{C}[\partial_{\boldsymbol{x}}] = \mathbb{C}[\partial_1, \ldots, \partial_n]$, i.e.,

$$I_A = \langle \partial_{\boldsymbol{x}}^{\boldsymbol{u}} - \partial_{\boldsymbol{x}}^{\boldsymbol{v}} \mid A\boldsymbol{u} = A\boldsymbol{v},\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n \rangle \subseteq \mathbb{C}[\partial_{\boldsymbol{x}}].$$

Here and hereafter we use the multi-index notation; for example, $\partial_{\boldsymbol{x}}^{\boldsymbol{u}}$ means $\partial_1^{u_1} \cdots \partial_n^{u_n}$ for $\boldsymbol{u} = (u_1, \ldots, u_n)^T$. Given a column vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^T \in \mathbb{C}^d$, let $H_A(\boldsymbol{\beta})$ denote the left ideal of the Weyl algebra

$$D = \mathbb{C}\langle \boldsymbol{x}, \partial_{\boldsymbol{x}} \rangle = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by I_A and

$$\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i \qquad (i = 1, \ldots, d),$$

2020 *Mathematics Subject Classification.* Primary: 33C70.

Key words and phrases. A-hypergeometric systems; the method of Frobenius.

*Corresponding author. E-mail: gokuyama@hus.ac.jp.

where $\theta_j = x_j\partial_j$. The quotient $M_A(\boldsymbol{\beta}) = D/H_A(\boldsymbol{\beta})$ is called the *A-hypergeometric system with parameter* $\boldsymbol{\beta}$, and a formal series annihilated by $H_A(\boldsymbol{\beta})$ an *A-hypergeometric series with parameter* $\boldsymbol{\beta}$. The homogeneity of A is known to be equivalent to the regularity of $M_A(\boldsymbol{\beta})$ by Hotta [4] and Schulze, Walther [9].

Logarithm-free series solutions to $M_A(\boldsymbol{\beta})$ were constructed by Gel'fand et al. [2, 3] for a generic parameter $\boldsymbol{\beta}$, and more generally in [8].

Note that the logarithmic coefficients of A-hypergeometric series solutions are polynomials of $\log x^{\boldsymbol{b}}$ ($\boldsymbol{b} \in L$) [5, Proposition 5.2], where

$$L := \mathrm{Ker}_{\mathbb{Z}}(A) = \{\boldsymbol{u} \in \mathbb{Z}^n \mid A\boldsymbol{u} = \boldsymbol{0}\}.$$

To construct logarithmic series solutions, the second author [6] introduced a method of perturbation by a finite subset $B = \{\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)}\} \subset L$, and explicitly described logarithmic series solutions for a fake exponent and a set B that satisfy certain conditions [6, Theorems 5.4, 6.2 and Remarks 5.6, 6.3].

In this paper, following [6], we continue to develop the perturbing method of constructing logarithmic series solutions to a regular A-hypergeometric system.

Fixing a fake exponent of an A-hypergeometric system, we consider some spaces of linear partial differential operators with constant coefficients. Comparing these spaces, we construct a fundamental system of series solutions with the given exponent by the perturbing method. In addition, we give a sufficient condition for a given fake exponent to be an exponent. As important examples of the main results, we give fundamental systems of series solutions to Aomoto-Gel'fand systems and to Lauricella's F_C systems with special parameter vectors, respectively.

This paper is organized as follows. In Section 2, we first recall a power series to perturb from [6], associated with a fake exponent $\boldsymbol{v}$ and a linearly independent subset B of L. In particular, we discuss properties of each term $a_{\boldsymbol{u}}(\boldsymbol{s})$ appearing in the series (for the definition of $a_{\boldsymbol{u}}(\boldsymbol{s})$, see (1)), and modify the series by changing the range of the sum from $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ in [6] to $\mathcal{N}$ which incorporates B. We give a refinement of [6, Theorem 6.2] as Theorem 2.7.

In Section 3, for a fake exponent $\boldsymbol{v}$ of the A-hypergeometric ideal $H_A(\boldsymbol{\beta})$ with respect to a generic weight vector $\boldsymbol{w}$, we recall the structure of the ideal $Q_{\boldsymbol{v}}$ associated with the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ and that of its orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ defined in [8, Sections 2.3 and 3.6]. We introduce ideals $P_{\mathcal{N}}$ and P_B of $\mathbb{C}[\boldsymbol{s}]$, and their orthogonal complements $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$. Then we discuss relations among these ideals. Under a certain condition, we can derive $Q_{\boldsymbol{v}}^{\perp}$ as the image of a linear map from $P_{\mathcal{N}}^{\perp}$ (Proposition 3.11, Theorem 3.14).

In Section 4, we give a sufficient condition for a fake exponent $\boldsymbol{v}$ to be an exponent (Proposition 4.1). Then we construct a fundamental system of solutions with the exponent $\boldsymbol{v}$ (Theorem 4.4) by applying Theorem 2.7 and the results of Section 3 under the condition that B is a basis of L, which is the main theorem of this paper.

In Sections 5 and 6, we deal with the Aomoto-Gel'fand systems and Lauricella's F_C systems, which are important examples of $H_A(\boldsymbol{\beta})$. We discuss a fundamental system of solutions to $H_A(\mathbf{0})$ in each system. In each case, we have a unique fake exponent $\boldsymbol{v} = \mathbf{0}$. Taking a basis B of L, we can obtain a fundamental system of series solutions for $\boldsymbol{\beta} = \mathbf{0}$.

2. Refinement of [6, Theorem 6.2]

In this section, we refine [6, Theorem 6.2].

Recall that for $\boldsymbol{v} = (v_1, \dots, v_n)^T \in \mathbb{C}^n$ its support $\operatorname{supp}(\boldsymbol{v})$ and its *negative support* $\operatorname{nsupp}(\boldsymbol{v})$ are defined as

$$\operatorname{supp}(\boldsymbol{v}) := \{j \in \{1, \dots, n\} \mid v_j \neq 0\},$$
$$\operatorname{nsupp}(\boldsymbol{v}) := \{j \in \{1, \dots, n\} \mid v_j \in \mathbb{Z}_{<0}\},$$

respectively.

For $\boldsymbol{v} \in \mathbb{C}^n$ and $\boldsymbol{u} \in \mathbb{N}^n$, set

$$[\boldsymbol{v}]_{\boldsymbol{u}} := \prod_{j=1}^{n} v_j(v_j - 1) \cdots (v_j - u_j + 1).$$

Here recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Note that we can uniquely write $\boldsymbol{u} \in \mathbb{Z}^n$ as the sum $\boldsymbol{u} = \boldsymbol{u}_+ - \boldsymbol{u}_-$ with $\boldsymbol{u}_+, \boldsymbol{u}_- \in \mathbb{N}^n$ and $\operatorname{supp}(\boldsymbol{u}_+) \cap \operatorname{supp}(\boldsymbol{u}_-) = \emptyset$.

Let $B = \{\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)}\} \subset L$. We write the same symbol B for the $n \times h$ matrix $(\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)})$.

Set

$$\operatorname{supp}(B) := \bigcup_{k=1}^{h} \operatorname{supp}(\boldsymbol{b}^{(k)}) \subset \{1, \dots, n\},$$

which means the set of all labels for nonzero rows in B.

Let $\boldsymbol{s} = (s_1, \dots, s_h)^T$ be indeterminates, and let

$$(B\boldsymbol{s})_j := \sum_{k=1}^{h} b_j^{(k)} s_k \in \mathbb{C}[\boldsymbol{s}] := \mathbb{C}[s_1, \dots, s_h]$$

for $j = 1, \dots, n$. Set

$$(B\boldsymbol{s})^J := \prod_{j \in J} (B\boldsymbol{s})_j \in \mathbb{C}[\boldsymbol{s}]$$

for $J \subset \{1,\dots,n\}$. Note that $(B\boldsymbol{s})_j = 0$ if $j \notin \mathrm{supp}(B)$, hence we have $(B\boldsymbol{s})^J = 0$ if $J \not\subset \mathrm{supp}(B)$.

Lemma 2.1. *Let $B = \{\boldsymbol{b}^{(1)},\dots,\boldsymbol{b}^{(h)}\} \subset L$, $\boldsymbol{u}, \boldsymbol{u}' \in L$ and $\boldsymbol{v} \in \mathbb{C}^n$. Let $\boldsymbol{s} = (s_1,\dots,s_h)^T$ be indeterminates. Then $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} \neq 0$ if and only if $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}-\boldsymbol{u}') \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$. In particular, $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}_+} \neq 0$ if and only if $\mathrm{nsupp}(\boldsymbol{v}) \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$.*

Proof. Note that

$$[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = \prod_{j;u'_j>0} (v_j + (B\boldsymbol{s})_j + u_j)\cdots(v_j + (B\boldsymbol{s})_j + u_j - u'_j + 1).$$

Hence, $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ if and only if there exists j such that $v_j + u_j - u'_j \in \mathbb{Z}_{<0}$, $v_j + u_j \in \mathbb{N}$, and $b_j^{(k)} = 0$ for all k.

Hence $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ if and only if $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}-\boldsymbol{u}') \not\subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$. □

Let $\boldsymbol{w}$ be a generic weight. Recall that $\boldsymbol{v}$ is called a *fake exponent* of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$ if $A\boldsymbol{v} = \boldsymbol{\beta}$ and $[\boldsymbol{v}]_{\boldsymbol{u}_+} = 0$ for all $\boldsymbol{u} \in L$ with $\boldsymbol{u}_+ \cdot \boldsymbol{w} > \boldsymbol{u}_- \cdot \boldsymbol{w}$, where $\boldsymbol{u}\cdot\boldsymbol{w} = \sum_{j=1}^n u_j w_j$.

Throughout this paper, fix a generic weight $\boldsymbol{w}$, a fake exponent $\boldsymbol{v}$ of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$.

We abbreviate $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$ to $I_{\boldsymbol{u}}$ for $\boldsymbol{u} \in L$. In particular, $I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v})$. Then the condition $\mathrm{nsupp}(\boldsymbol{v}) \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$ in Lemma 2.1 can be rewritten as

$$I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}.$$

For $\boldsymbol{u} \in L$ with $I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$, let

$$a_{\boldsymbol{u}}(\boldsymbol{s}) := \frac{[\boldsymbol{v}+B\boldsymbol{s}]_{\boldsymbol{u}_-}}{[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}_+}}. \tag{1}$$

Note that the denominator is nonzero by Lemma 2.1.

Lemma 2.2. *Let $\boldsymbol{u}, \boldsymbol{u}' \in L$. Assume that $\boldsymbol{u}$ satisfies $I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$. Then the following hold.*

(i) *$a_{\boldsymbol{u}}(\boldsymbol{s}) \neq 0$ if and only if $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, if and only if $\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.*

(ii) *If $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, then $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$.*

Proof. (i) We have $[\boldsymbol{v}+B\boldsymbol{s}]_{\boldsymbol{u}_-} = 0$ if and only if there exists j such that $[v_j + \sum_{k=1}^h s_k b_j^{(k)}]_{-u_j} = [v_j]_{-u_j} = 0$, namely $v_j \in \mathbb{N}, v_j + u_j \in \mathbb{Z}_{<0}$, and $b_j^{(k)} = 0$ for all k. Hence it is equivalent to saying that there exists j such that $j \in I_{\boldsymbol{u}} \setminus (\mathrm{supp}(B) \cup I_{\boldsymbol{0}})$, or $I_{\boldsymbol{u}} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.

By the assumption, the inclusion $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$ is equivalent to the equality $\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.

(ii) Suppose that $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$. Note that

$$\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = a_{\boldsymbol{u}}(\boldsymbol{s})[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{u}'_+}.$$

Hence, if $I_{\boldsymbol{u}} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, then $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$ by (i).

If $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, that is, $I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus (\mathrm{supp}(B) \cup I_{\boldsymbol{u}}) \neq \emptyset$, then we have $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ and $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$ by Lemma 2.1. □

We recall the definitions related to $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ for a fake exponent $\boldsymbol{v}$ from [6] and modify them.

Let

$$\mathcal{G} := \left\{ \partial^{\boldsymbol{g}_+^{(i)}} - \partial^{\boldsymbol{g}_-^{(i)}} \,\middle|\, i = 1, \ldots, m \right\}$$

denote the reduced Gröbner basis of I_A with respect to $\boldsymbol{w}$ with $\partial^{\boldsymbol{g}_+^{(i)}} \in \mathrm{in}_{\boldsymbol{w}}(I_A)$ for all i. Note that the $\mathcal{G}$ in [6, Section 4] should be *the reduced* Gröbner basis. Set

$$C(\boldsymbol{w}) := \sum_{i=1}^{m} \mathbb{N}\boldsymbol{g}^{(i)}.$$

A collection $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ of negative supports $I_{\boldsymbol{u}}$ ($\boldsymbol{u} \in L$) is defined by

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) := \{ I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L. \text{ If } I_{\boldsymbol{u}} = I_{\boldsymbol{u}'} \text{ for } \boldsymbol{u}' \in L, \text{ then } \boldsymbol{u}' \in C(\boldsymbol{w}). \}.$$

In addition, define

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c := \{ I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L \} \setminus \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}).$$

We modify the definition of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$. To do this, in this paper we make the following assumption.

Assumption 2.3. A subset $B = \{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(h)}\} \subset L$ is linearly independent, hence $\mathrm{rank}(B) = h$, and satisfies

$$I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$$

for any $\boldsymbol{u} \in L$.

Remark 2.4. (1) In the proofs of Lemma 2.6 and Theorem 2.7 below, we need the condition $I_{\boldsymbol{0}} \subset I_{\boldsymbol{u}-\boldsymbol{u}'} \cup \mathrm{supp}(B)$ for any $\boldsymbol{u}, \boldsymbol{u}' \in L$, which Assumption 2.3 guarantees.

(2) If a linearly independent set B satisfies $I_{\boldsymbol{0}} \subset \mathrm{supp}(B)$, then it satisfies Assumption 2.3. For example, this condition holds for each of the following cases:

(i) $\mathrm{supp}(B) = \{1, \dots, n\}$,
(ii) $\mathrm{nsupp}(\boldsymbol{v}) = \emptyset$.

(3) If B is a basis of L, then B satisfies

$$\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$$

for all $\boldsymbol{u} \in L$. Indeed, since B is a basis of L, we see that if $j \notin \mathrm{supp}(B)$ then $\boldsymbol{u}_j = 0$ for all $\boldsymbol{u} \in L$. This implies that $I_{\boldsymbol{u}} \setminus \mathrm{supp}(B) = I_{\boldsymbol{0}} \setminus \mathrm{supp}(B)$ for all $\boldsymbol{u} \in L$.

Define a subset $\mathcal{N}$ of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ as a modification of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ by

$$\mathcal{N} := \{I_{\boldsymbol{u}} \mid \boldsymbol{u} \in L, \mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}\} \cap \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}), \tag{2}$$

and set

$$\mathcal{N}^c := \{I_{\boldsymbol{u}} \mid \boldsymbol{u} \in L, \mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}\} \cap \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c.$$

Consider the subset L' of L defined by

$$L' := \{\boldsymbol{u} \in L \mid I_{\boldsymbol{u}} \in \mathcal{N}\}. \tag{3}$$

By definition, we see that $L' \subset C(\boldsymbol{w})$.

Let

$$K_{\mathcal{N}} := \bigcap_{I \in \mathcal{N}} I,$$

and define the homogeneous ideal $P_{\mathcal{N}}$ of $\mathbb{C}[\boldsymbol{s}]$ for $\mathcal{N}$ as

$$P_{\mathcal{N}} := \left\langle (B\boldsymbol{s})^{I \cup J \setminus K_{\mathcal{N}}} \,\middle|\, I \in \mathcal{N}, J \in \mathcal{N}^c \right\rangle. \tag{4}$$

In addition, we define the orthogonal complement $P^{\perp}$ for a homogeneous ideal $P \subset \mathbb{C}[\boldsymbol{s}]$ as

$$\begin{aligned} P^{\perp} &:= \{q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}] \mid (q(\partial_{\boldsymbol{s}}) \bullet h(\boldsymbol{s}))|_{\boldsymbol{s}=\boldsymbol{0}} = 0 \text{ for all } h(\boldsymbol{s}) \in P\} \\ &= \{q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}] \mid q(\partial_{\boldsymbol{s}}) \bullet P \subset \langle s_1, \dots, s_h \rangle\}, \end{aligned} \tag{5}$$

where $\mathbb{C}[\partial_{\boldsymbol{s}}] := \mathbb{C}[\partial_{s_1}, \dots, \partial_{s_h}]$, and the symbol $\bullet$ denotes the natural action of $\mathbb{C}[\partial_{\boldsymbol{s}}]$ on polynomials of $\mathbb{C}[\boldsymbol{s}]$. Since P and $\langle s_1, \dots, s_h \rangle$ are both homogeneous, $P^{\perp}$ is homogeneous with respect to the usual total ordering.

Example 2.5. (cf. [6, Examples 3.3, 4.8, 6.4]) Let $n = 5$, $d = 3$, and

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

Let $\boldsymbol{\beta} = (1,0,0)^T$ and $\boldsymbol{w} = (1,1,1,1,0)$. Then $\boldsymbol{v} = (0,0,0,0,1)^T$ is the unique exponent, and

$$\mathcal{G} = \{\underline{\partial_{x_1}\partial_{x_3}} - \partial_{x_5}^2, \underline{\partial_{x_2}\partial_{x_4}} - \partial_{x_5}^2\},$$

where the underlined terms are the leading ones. Put $\boldsymbol{g}^{(1)} := (1,0,1,0,-2)^T$ and $\boldsymbol{g}^{(2)} := (0,1,0,1,-2)^T$. Recall that

$$\begin{aligned}
\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) &= \{\emptyset = I_{\mathbf{0}}, \{5\}\},\\
\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c &= \{\{1,3\},\{2,4\},\{1,3,5\},\{2,4,5\},\{1,2,3,4\}\}.
\end{aligned}$$

Let $B := \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then we have $\mathrm{supp}(B) = \{1,2,3,4,5\}$, and

$$\begin{aligned}
\mathcal{N} &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\\
\mathcal{N}^c &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c,\\
K_{\mathcal{N}} &= \emptyset.
\end{aligned}$$

The homogeneous ideal $P_{\mathcal{N}} \subset \mathbb{C}[\boldsymbol{s}] = \mathbb{C}[s_1, s_2]$ and the vector space $P_{\mathcal{N}}^{\perp} \subset \mathbb{C}[\partial_{\boldsymbol{s}}] = \mathbb{C}[\partial_{s_1}, \partial_{s_2}]$ are given as

$$\begin{aligned}
P_{\mathcal{N}} &= \langle (B\boldsymbol{s})^{\{1,3\}}, (B\boldsymbol{s})^{\{2,4\}} \rangle = \langle s_1^2, s_2^2 \rangle,\\
P_{\mathcal{N}}^{\perp} &= \{q(\partial_{s_1}, \partial_{s_2}) \in \mathbb{C}[\partial_{s_1}, \partial_{s_2}] \mid q(\partial_{s_1}, \partial_{s_2}) \bullet \langle s_1^2, s_2^2 \rangle \subset \langle s_1, s_2 \rangle\}\\
&= \mathbb{C}1 + \mathbb{C}\partial_{s_1} + \mathbb{C}\partial_{s_2} + \mathbb{C}\partial_{s_1}\partial_{s_2}.
\end{aligned}$$

We consider another case. Let $B_1 = \{\boldsymbol{g}^{(1)}\}$. Then we have $\mathrm{supp}(B_1) = \{1,3,5\}$ and

$$\begin{aligned}
\mathcal{N}_1 &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\emptyset = I_{\mathbf{0}}, \{5\}\},\\
\mathcal{N}_1^c &= \{\{1,3\},\{1,3,5\}\},\\
K_{\mathcal{N}_1} &= \emptyset.
\end{aligned}$$

The homogeneous ideal $P_{\mathcal{N}_1} \subset \mathbb{C}[s]$ and the vector space $P_{\mathcal{N}_1}^{\perp} \subset \mathbb{C}[\partial_s]$ are given as

$$\begin{aligned}
P_{\mathcal{N}_1} &= \langle (B_1 s)^{\{1,3\}} \rangle = \langle s^2 \rangle,\\
P_{\mathcal{N}_1}^{\perp} &= \{q(\partial_s) \in \mathbb{C}[\partial_s] \mid q(\partial_s) \bullet \langle s^2 \rangle \subset \langle s \rangle\} = \mathbb{C}1 + \mathbb{C}\partial_s.
\end{aligned}$$

Throughout this paper, put

$$m(\boldsymbol{s}) := (B\boldsymbol{s})^{I_{\mathbf{0}} \setminus K_{\mathcal{N}}}.$$

The following lemma guarantees that we may plug $\boldsymbol{s} = \mathbf{0}$ into the series appearing in Theorem 2.7.

Lemma 2.6. *Let $\mathcal{N}$ be the set defined by* (2)*, and let $\boldsymbol{u}, \boldsymbol{u}' \in L$. Then, under Assumption 2.3, each term of the power series for $m(\boldsymbol{s}) \cdot a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+}$ in the indeterminates $\boldsymbol{s}$ is divided by $(B\boldsymbol{s})^{I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus K_{\mathcal{N}}}$.*

Proof. By [6, Lemma 6.1], there exists a formal power series $g(\boldsymbol{y})$ in the indeterminates $\boldsymbol{y} = (y_1, \dots, y_n)$ such that

$$\begin{aligned}
&a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\
&\quad = \frac{[\boldsymbol{v} + B\boldsymbol{s}]_{\boldsymbol{u}_-}}{[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}_+}} \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\
&\quad = \left(\frac{(B\boldsymbol{s})^{I_{\boldsymbol{u}} \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus I_{\boldsymbol{u}}}} \cdot (B\boldsymbol{s})^{I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus I_{\boldsymbol{u}}} \right) \cdot g((B\boldsymbol{s})_1, \dots, (B\boldsymbol{s})_n) \\
&\quad = \frac{(B\boldsymbol{s})^{(I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'}) \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus (I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'})}} \cdot g((B\boldsymbol{s})_1, \dots, (B\boldsymbol{s})_n).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&m(\boldsymbol{s}) \cdot a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\
&\quad = (B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus K_{\mathcal{N}}} \cdot \frac{(B\boldsymbol{s})^{(I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'}) \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus (I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'})}} \cdot g((B\boldsymbol{s})_1, \dots, (B\boldsymbol{s})_n) \\
&\quad = (B\boldsymbol{s})^{I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus K_{\mathcal{N}}} \cdot g((B\boldsymbol{s})_1, \dots, (B\boldsymbol{s})_n),
\end{aligned}$$

and the assertion holds. □

We can refine the main results [6, Theorem 5.4, Theorem 6.2] as follows.

Theorem 2.7. *Let $\mathcal{N}$ be the set defined by* (2)*. Set*

$$F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) := \sum_{\boldsymbol{u} \in L'} a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}},$$

and

$$\widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) := m(\boldsymbol{s}) F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}),$$

where L' is defined by (3)*.*

Then $(q(\partial_s) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ are solutions to $M_A(\boldsymbol{\beta})$ for any $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$.

Proof. Let $\boldsymbol{u}' \in L$ and $\boldsymbol{u} \in L'$. If $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) \neq 0$, then we have $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \subset \operatorname{supp}(B) \cup I_{\boldsymbol{0}}$ by Lemma 2.2, and hence $\operatorname{supp}(B) \cup I_{\boldsymbol{u}-\boldsymbol{u}'} = \operatorname{supp}(B) \cup I_{\boldsymbol{0}}$ by Assumption 2.3. Thus $I_{\boldsymbol{u}-\boldsymbol{u}'} \notin \mathcal{N}$ implies $I_{\boldsymbol{u}-\boldsymbol{u}'} \in \mathcal{N}^c$.

Similar to the arguments in the proofs of [6, Theorem 5.4, Theorem 6.2], we see that

$$\begin{aligned}(\partial^{\boldsymbol{u}'_+} - \partial^{\boldsymbol{u}'_-}) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) \\ &= \sum_{\boldsymbol{u}\in L', I_{\boldsymbol{u}-\boldsymbol{u}'}\in\mathcal{N}^c} m(\boldsymbol{s})\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) \\ &\quad - \sum_{\boldsymbol{u}\in L', I_{\boldsymbol{u}+\boldsymbol{u}'}\in\mathcal{N}^c} m(\boldsymbol{s})\partial^{(-\boldsymbol{u})'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}).\end{aligned}$$

Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$. Then the series $(q(\partial_s) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ is a solution to $M_A(\boldsymbol{\beta})$ if

$$\left(q(\partial_s) \bullet \left(m(\boldsymbol{s})\partial^{\boldsymbol{u}'_+}a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}\right)\right)_{|\boldsymbol{s}=\boldsymbol{0}} = 0$$

for any $\boldsymbol{u} \in L'$ and $\boldsymbol{u}' \in L$ with $I_{\boldsymbol{u}-\boldsymbol{u}'} \in \mathcal{N}^c$.

By Lemma 2.6, each coefficient of

$$\begin{aligned}m(\boldsymbol{s})(\partial^{\boldsymbol{u}'_+}a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) \\ &= m(\boldsymbol{s})a_{\boldsymbol{u}}(\boldsymbol{s})[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+}x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{u}'_+}\end{aligned}$$

in the indeterminates $\boldsymbol{s}$ is divided by $(B\boldsymbol{s})^{I_{\boldsymbol{u}}\cup I_{\boldsymbol{u}-\boldsymbol{u}'}\setminus K_{\mathcal{N}}}$, hence belongs to $P_{\mathcal{N}}$. By the definition of $P_{\mathcal{N}}^{\perp}$, the assertion holds. □

3. Relations between $P_{\mathcal{N}}^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$

In this section, we recall $Q_{\boldsymbol{v}}$ and its orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ defined in [8, Section 2.3], and discuss relations between $P_{\mathcal{N}}^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$. For the definitions of $P_{\mathcal{N}}$ and $P_{\mathcal{N}}^{\perp}$, see (4) and (5).

Consider the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$:

$$\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta})) := \langle A\theta_{\boldsymbol{x}} - \boldsymbol{\beta}\rangle + \widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A) \subset \mathbb{C}[\theta_{\boldsymbol{x}}] := \mathbb{C}[\theta_1, \dots, \theta_n].$$

Here $\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A)$ is the distraction of the initial ideal $\mathrm{in}_{\boldsymbol{w}}(I_A)$ with respect to $\boldsymbol{w}$ (cf. [8, Section 3.1]). Related to the reduced Gröbner basis $\mathcal{G} = \{\partial^{\boldsymbol{g}_+^{(i)}} - \partial^{\boldsymbol{g}_-^{(i)}} \mid i = 1, \dots, m\}$ of I_A with respect to $\boldsymbol{w}$ with $\partial^{\boldsymbol{g}_+^{(i)}} \in \mathrm{in}_{\boldsymbol{w}}(I_A)$ for all i, define

$$G^{(i)} := I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} = \{j \in \{1, \dots, n\} \mid v_j \in \mathbb{N}, g_j^{(i)} - v_j > 0\}$$

for $i = 1, \dots, m$. Since

$$\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A) = \left\langle [\theta_{\boldsymbol{x}}]_{\boldsymbol{g}_+^{(i)}} := \prod_{j;\, g_j^{(i)}>0} \prod_{\nu=0}^{g_j^{(i)}-1} (\theta_j - \nu) \,\middle|\, i = 1, \dots, m \right\rangle$$

by [8, Theorem 3.2.2], we see that its primary component at a fake exponent $\boldsymbol{v}$ is

$$\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A)_{\boldsymbol{v}} = \left\langle ([\theta]_{\boldsymbol{g}_+^{(i)}})_{\boldsymbol{v}} := \prod_{j\in G^{(i)}} (\theta_j - v_j) \,\middle|\, i = 1,\dots,m \right\rangle. \tag{6}$$

We obtain the homogeneous ideal $Q_{\boldsymbol{v}}$ of $\mathbb{C}[\theta_{\boldsymbol{x}}]$ from $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$ by replacing $\theta_j \mapsto \theta_j + v_j$ for $j = 1,\dots,n$ (cf. [8, Section 2.3]). Namely,

$$Q_{\boldsymbol{v}} = \langle\, A\theta_{\boldsymbol{x}} \,\rangle + \left\langle \prod_{j\in G^{(i)}} \theta_j \,\middle|\, i = 1,\dots,m \right\rangle. \tag{7}$$

The orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ of $Q_{\boldsymbol{v}}$ is defined by

$$Q_{\boldsymbol{v}}^{\perp} := \{ f \in \mathbb{C}[\boldsymbol{x}] \,|\, \phi(\partial_{\boldsymbol{x}})(f) = 0 \text{ for all } \phi = \phi(\theta_{\boldsymbol{x}}) \in Q_{\boldsymbol{v}} \}.$$

Note that $Q_{\boldsymbol{v}}^{\perp}$ is a graded $\mathbb{C}$-vector space with the usual grading.

Proposition 3.1. *Let $f(\boldsymbol{x})$ be a polynomial. Then $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ if and only if $f(\boldsymbol{x})$ satisfies the following conditions:*

(i) $f(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}G] := \mathbb{C}[\boldsymbol{x}\boldsymbol{g}^{(1)},\dots,\boldsymbol{x}\boldsymbol{g}^{(m)}]$.
(ii) $\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet f(\boldsymbol{x}) = 0$ *for all* $i = 1,\dots,m$.

Here

$$\boldsymbol{x}G := (\boldsymbol{x}\boldsymbol{g}^{(1)},\dots,\boldsymbol{x}\boldsymbol{g}^{(m)}) = \left(\sum_{j=1}^{n} g_j^{(1)} x_j, \dots, \sum_{j=1}^{n} g_j^{(m)} x_j \right)$$

for $\boldsymbol{x} = (x_1,\dots,x_n)$.

Proof. By [8, Theorem 2.3.11], the function $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\beta))$ if and only if $f(\boldsymbol{x}) \in Q_{\boldsymbol{v}}^{\perp}$. From $f(\boldsymbol{x}) \in \langle A\partial_{\boldsymbol{x}}\rangle^{\perp}$, we see (i) [5, Lemma 5.1]. (ii) follows from Equation (6). □

Example 3.2 (Continuation of Example 2.5)**.** Note that $\boldsymbol{v} - \boldsymbol{g}^{(1)} = (-1,0,-1,0,3)^T$ and $\boldsymbol{v} - \boldsymbol{g}^{(2)} = (0,-1,0,-1,3)^T$. Thus we see that

$$G^{(1)} = I_{-\boldsymbol{g}^{(1)}} \setminus I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(1)}) \setminus \mathrm{nsupp}(\boldsymbol{v}) = \{1,3\},$$
$$G^{(2)} = I_{-\boldsymbol{g}^{(2)}} \setminus I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(1)}) \setminus \mathrm{nsupp}(\boldsymbol{v}) = \{2,4\}.$$

The ideal $Q_{\boldsymbol{v}} \subset \mathbb{C}[\theta_{\boldsymbol{x}}] = \mathbb{C}[\theta_1,\theta_2,\theta_3,\theta_4,\theta_5]$ is given as

$$Q_{\boldsymbol{v}} = \langle \theta_1+\theta_2+\theta_3+\theta_4+\theta_5, -\theta_1+\theta_2+\theta_3-\theta_4, -\theta_1-\theta_2+\theta_3+\theta_4, \theta_1\theta_3, \theta_2\theta_4 \rangle.$$

In addition, we see that

$$Q_{\boldsymbol{v}}^{\perp} = \mathbb{C}\cdot 1 + \mathbb{C}\cdot \boldsymbol{x}\boldsymbol{g}^{(1)} + \mathbb{C}\cdot \boldsymbol{x}\boldsymbol{g}^{(2)} + \mathbb{C}\cdot (\boldsymbol{x}\boldsymbol{g}^{(1)})\cdot(\boldsymbol{x}\boldsymbol{g}^{(2)}).$$

To compare $Q_{\boldsymbol{v}}$ with $P_{\mathcal{N}}$, we consider the graded ring homomorphism $\Phi_B : \mathbb{C}[\theta_{\boldsymbol{x}}] \to \mathbb{C}[\boldsymbol{s}]$ defined by $\theta_j \mapsto (B\boldsymbol{s})_j$ for $j = 1, \ldots, n$. By the linear independence of B, we see that Φ_B is surjective. Define $P_B := \Phi_B(Q_{\boldsymbol{v}})$. By the ring isomorphism theorem, Φ_B induces the ring isomorphism

$$\widetilde{\Phi}_B : \mathbb{C}[\theta_{\boldsymbol{x}}]/\Phi_B^{-1}(P_B) \simeq \mathbb{C}[\boldsymbol{s}]/P_B.$$

Since $\langle A\theta_{\boldsymbol{x}} \rangle$ is vanished by Φ_B, we have

$$P_B = \left\langle (B\boldsymbol{s})^{G^{(i)}} \,\middle|\, i = 1, \ldots, m \right\rangle. \tag{8}$$

Proposition 3.3. *Let $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. Then $G^{(i)} \subset J \setminus I_{\boldsymbol{0}}$ for some i.*

Proof. By definition and [6, Lemma 4.2], we see that there exists $\boldsymbol{u} \in L \setminus C(\boldsymbol{w})$ such that $J = I_{\boldsymbol{u}}$ and $\partial^{\boldsymbol{u}_+} \notin \mathrm{in}_{\boldsymbol{w}}(I_A)$. Hence $\partial^{\boldsymbol{u}_-} = \mathrm{in}_{\boldsymbol{w}}(\partial^{\boldsymbol{u}_-} - \partial^{\boldsymbol{u}_+})$ is divided by some $\partial^{\boldsymbol{g}_+^{(i)}}$. Let $j \in G^{(i)} = I_{-\boldsymbol{g}_+^{(i)}} \setminus I_{\boldsymbol{0}}$. Then $v_j \in \mathbb{N}$ and $v_j - g_j^{(i)} \in \mathbb{Z}_{<0}$. Since $g_j^{(i)} \in \mathbb{Z}_{>0}$, we see that $g_j^{(i)} \leq -u_j$ and $v_j + u_j \leq v_j - g_j^{(i)} < 0$. Thus we have $j \in I_{\boldsymbol{u}} \setminus I_{\boldsymbol{0}} = J \setminus I_{\boldsymbol{0}}$. □

Three ideals $Q_{\boldsymbol{v}}$, $P_{\mathcal{N}}$, and P_B are related as follows.

Proposition 3.4. *Let $Q_{\boldsymbol{v}}$, $P_{\mathcal{N}}$, and P_B be the ones in* (7), (4), *and* (8)*, respectively. Then, the following hold.*

(i) *$m(\boldsymbol{s}) \cdot P_B \subset P_{\mathcal{N}} \subset P_B$. In particular, if $K_{\mathcal{N}} = I_0$, then $P_{\mathcal{N}} = P_B$.*
(ii) *If B is a basis of L, then $\Phi_B^{-1}(P_B) = Q_{\boldsymbol{v}}$.*

Proof. (i) Let $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$. Since $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ and $K_{\mathcal{N}} \subset I_{\boldsymbol{0}}$, $I \cup J \setminus K_{\mathcal{N}}$ contains some $G^{(i)}$ by Proposition 3.3. Hence the inclusion $P_{\mathcal{N}} \subset P_B$ holds.

For any $i = 1, \ldots, m$, since $-\boldsymbol{g}^{(i)} \notin C(\boldsymbol{w})$ we see that $I_{-\boldsymbol{g}^{(i)}} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. If $I_{-\boldsymbol{g}^{(i)}} \notin \mathcal{N}^c$, then

$$\mathrm{supp}(B) \cup I_{-\boldsymbol{g}^{(i)}} \neq \mathrm{supp}(B) \cup I_{\boldsymbol{0}}. \tag{9}$$

By Assumption 2.3, (9) implies that $G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} \not\subset \mathrm{supp}(B)$. Hence we have $(B\boldsymbol{s})^{G^{(i)}} = 0$. If $I_{-\boldsymbol{g}^{(i)}} \in \mathcal{N}^c$, then

$$m(\boldsymbol{s})(B\boldsymbol{s})^{G^{(i)}} = (B\boldsymbol{s})^{I_{\boldsymbol{0}} \cup I_{-\boldsymbol{g}^{(i)}} \setminus K_{\mathcal{N}}} \in P_{\mathcal{N}}.$$

Hence we have $m(\boldsymbol{s}) \cdot P_B \subset P_{\mathcal{N}}$.

(ii) Since B is a basis of L, we have $\mathrm{Ker}(\Phi_B) = \langle A\theta_{\boldsymbol{x}} \rangle$. Thus the assertion ${\Phi_B}^{-1}(P_B) = Q_{\boldsymbol{v}}$ holds from (7). □

Example 3.5 (Continuation of Example 2.5 and 3.2)**.** Consider the case where $B = \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then we have $m(\boldsymbol{s}) = (B\boldsymbol{s})^{\emptyset} = 1$, and

$$P_B = \langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle = \langle s_1^2, s_2^2 \rangle = P_{\mathcal{N}}.$$

Furthermore, since B is a basis of L, we see that

$$\Phi_B^{-1}(P_B) = \Phi_B^{-1}(\langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle) = \langle \theta_1\theta_3, \theta_2\theta_4 \rangle + \langle A\theta_{\boldsymbol{x}} \rangle = Q_{\boldsymbol{v}}.$$

Consider the other case where $B_1 = \{\boldsymbol{g}^{(1)}\}$. Then we have $m(s) = (B_1 s)^{\emptyset} = 1$, and

$$P_{B_1} = \langle (B_1 s)^{G^{(1)}} \rangle = \langle s^2 \rangle = P_{\mathcal{N}_1}.$$

We see that B_1 does not span L and that

$$\Phi_B^{-1}(P_{B_1}) = \Phi_{B_1}^{-1}(\langle (B_1 s)^{G^{(1)}} \rangle) = \langle \theta_1\theta_3 \rangle + \langle A\theta_{\boldsymbol{x}} \rangle \subsetneq Q_{\boldsymbol{v}}.$$

We consider relations between $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$, and between $P_B^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$. Recall the construction of a basis of orthogonal complements in [8, Section 2.3].

Let P be a homogeneous ideal of $\mathbb{C}[\boldsymbol{s}]$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\mathcal{H} \subset \mathbb{C}[\boldsymbol{s}]$ be the reduced Gröbner basis of P with respect to $\prec$. For any $\boldsymbol{\mu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)$, there exist unique $c_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \mathbb{C}$ for $\boldsymbol{\nu} \in \mathbb{N}^h$ with $|\boldsymbol{\nu}| = |\boldsymbol{\mu}|$ and $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)$ such that

$$p_{\boldsymbol{\mu}}(\boldsymbol{s}) := \boldsymbol{s}^{\boldsymbol{\mu}} - \sum_{\substack{\boldsymbol{\nu} \in \mathbb{N}^h;\, |\boldsymbol{\nu}| = |\boldsymbol{\mu}|, \\ \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)}} c_{\boldsymbol{\mu},\boldsymbol{\nu}} \boldsymbol{s}^{\boldsymbol{\nu}} \in P.$$

We obtain $p_{\boldsymbol{\mu}}(\boldsymbol{s})$ by taking the normal form modulo $\mathcal{H}$ for the monomial $\boldsymbol{s}^{\boldsymbol{\mu}}$. For $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)$, define the homogeneous polynomial $q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}})$ of degree $|\boldsymbol{\nu}|$ by

$$q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) := \frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{s}}^{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\mu} \in \mathbb{N}^h;\, |\boldsymbol{\mu}| = |\boldsymbol{\nu}|, \\ \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)}} \frac{c_{\boldsymbol{\mu},\boldsymbol{\nu}}}{\boldsymbol{\mu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} \in \mathbb{C}[\partial_{\boldsymbol{s}}].$$

Lemma 3.6. *Let P be a homogeneous ideal of $\mathbb{C}[\boldsymbol{s}]$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\mathcal{H} \subset \mathbb{C}[\boldsymbol{s}]$ be the reduced Gröbner basis of P with respect to $\prec$. Then*

$$\{p_{\boldsymbol{\mu}}(\boldsymbol{s}) \mid \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)\}$$

and

$$\{q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \mid \boldsymbol{\nu} \in \mathbb{N}^h \text{ with } \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)\}$$

form $\mathbb{C}$-bases of P and $P^{\perp}$, respectively.

Proof. This is similar to [8, Proposition 2.3.13]. □

Lemma 3.7. *Let P and $\widetilde{P}$ be homogeneous ideals of $\mathbb{C}[\boldsymbol{s}]$. Then $P \subset \widetilde{P}$ if and only if $P^{\perp} \supset \widetilde{P}^{\perp}$.*

Proof. Assume that $P \subset \widetilde{P}$. By definition, $P^{\perp} \supset \widetilde{P}^{\perp}$ is clear.

Conversely, assume that $P^{\perp} \supset \widetilde{P}^{\perp}$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\widetilde{\mathcal{H}}$ be the reduced Gröbner basis of $\widetilde{P}$.

Let $\{\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \,|\, \boldsymbol{\nu} \in \mathbb{N}^h \text{ with } \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(\widetilde{P})\}$ be the $\mathbb{C}$-basis of $\widetilde{P}^{\perp}$ as in Lemma 3.6. Let $p \in P$. Applying the division algorithm with respect to $\widetilde{\mathcal{H}}$ to p, we can express p as

$$p = \widetilde{p} + \sum_{\boldsymbol{\lambda};\boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_{\prec}(\widetilde{P})} d_{\boldsymbol{\lambda}} s^{\boldsymbol{\lambda}}$$

with some $\widetilde{p} \in \widetilde{P}$ and $d_{\boldsymbol{\lambda}} \in \mathbb{C}$. For each $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(\widetilde{P})$, since $[\partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} \bullet \boldsymbol{s}^{\boldsymbol{\lambda}}]_{|\boldsymbol{s}=\boldsymbol{0}} = \boldsymbol{\mu}! \delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ for any $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, where $\delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ denotes the Kronecker delta, we have

$$\begin{aligned}
&[\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet p]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet \left(\widetilde{p} + \sum_{\boldsymbol{\lambda};\boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_{\prec}(\widetilde{P})} d_{\boldsymbol{\lambda}} s^{\boldsymbol{\lambda}}\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \sum_{\boldsymbol{\lambda};\boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_{\prec}(\widetilde{P})} d_{\boldsymbol{\lambda}} \left[\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet \boldsymbol{s}^{\boldsymbol{\lambda}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \sum_{\boldsymbol{\lambda};\boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_{\prec}(\widetilde{P})} d_{\boldsymbol{\lambda}} \left[\left(\frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{s}}^{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\mu} \in \mathbb{N}^h;\, |\boldsymbol{\mu}| = |\boldsymbol{\nu}|, \\ \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(\widetilde{P})}} \frac{c_{\boldsymbol{\mu},\boldsymbol{\nu}}}{\boldsymbol{\mu}!}\partial_{\boldsymbol{s}}^{\boldsymbol{\mu}}\right) \bullet \boldsymbol{s}^{\boldsymbol{\lambda}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = d_{\boldsymbol{\nu}}.
\end{aligned}$$

It follows from the assumption $P^{\perp} \supset \widetilde{P}^{\perp}$ that $d_{\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(\widetilde{P})$. Hence, we have $p \in \widetilde{P}$. □

Let $\mathbb{C}[\partial_{\boldsymbol{z}}] := \mathbb{C}[\partial_{z_1}, \dots, \partial_{z_h}]$ be the ring of partial differential operators with constant coefficients in indeterminates $\boldsymbol{z} = (z_1, \dots, z_h)$. To describe relations between $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$, and between $P_B^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$, we define an action of $\mathbb{C}[\partial_{\boldsymbol{z}}]$ on $\mathbb{C}[\partial_{\boldsymbol{s}}]$ and a ring homomorphism Ψ_B from $\mathbb{C}[\partial_{\boldsymbol{s}}]$ to $\mathbb{C}[\boldsymbol{x}]$.

For $U(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$ and $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, we define a $\mathbb{C}$-linear operation $U(\partial_{z_1}, \dots, \partial_{z_h}) \star q(\partial_{\boldsymbol{s}})$ by

$$U(\partial_{z_1}, \dots, \partial_{z_h}) \star q(\partial_{\boldsymbol{s}}) := (U(\partial_{\boldsymbol{z}}) \bullet q(\boldsymbol{z}))_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}} \in \mathbb{C}[\partial_{\boldsymbol{s}}].$$

Lemma 3.8. *The following hold for the $\star$-operation.*

(i) *Let $k = 1, \dots, h$ and $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$\partial_{z_k} \star q(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}})s_k - s_k q(\partial_{\boldsymbol{s}}) \in \mathbb{C}\langle \boldsymbol{s}, \partial_{\boldsymbol{s}}\rangle.$$

(ii) *Let $U(\partial_z), U'(\partial_z) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$, and $q(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$U(\partial_{\boldsymbol{z}}) \star (U'(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) = (U(\partial_{\boldsymbol{z}})U'(\partial_{\boldsymbol{z}})) \star q(\partial_s).$$

(iii) *Let $U(\partial_{\boldsymbol{z}}) = \prod_{\nu=1}^N l_\nu(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$ be the product of non-zero linear homogeneous polynomials $l_\nu(\partial_{\boldsymbol{z}})$, and let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then there exists $r(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that $U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}})$.*

Proof. (i) For any $k = 1, \dots, h$, and $\boldsymbol{\mu} \in \mathbb{N}^h$, we have

$$\partial_{z_k} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} = \mu_k \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}-\boldsymbol{e}_k} = \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} s_k - s_k \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}}.$$

(ii) It suffices to show that the equality holds for $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}}$, $U'(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$, and $q(\partial_{\boldsymbol{s}}) = \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}$ with $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{N}^h$. We see that

$$\begin{aligned}
\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star (\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}) &= \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star ((\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} \bullet \boldsymbol{z}^{\boldsymbol{\nu}})|_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}}) \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-\boldsymbol{\mu}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} (\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \bullet \boldsymbol{z}^{\boldsymbol{\nu}-\boldsymbol{\mu}})_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} [\boldsymbol{\nu}-\boldsymbol{\mu}]_{\boldsymbol{\lambda}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-\boldsymbol{\mu}-\boldsymbol{\lambda}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\lambda}+\boldsymbol{\mu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-(\boldsymbol{\lambda}+\boldsymbol{\mu})} \\
&= (\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}+\boldsymbol{\mu}} \bullet \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}})_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}} \\
&= \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}+\boldsymbol{\mu}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}},
\end{aligned}$$

and hence the assertion holds.

(iii) We show the statement by induction on N. First, let $U(\partial_{\boldsymbol{z}})$ be a non-zero linear homogeneous polynomial, and let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. By changing coordinates, we may assume that $U(\partial_{\boldsymbol{z}}) = \partial_{z_1}$. Put

$$q(\partial_{\boldsymbol{s}}) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^h} d_{\boldsymbol{\nu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}.$$

Then

$$r(\partial_{\boldsymbol{s}}) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^h} \frac{d_{\boldsymbol{\nu}}}{\nu_1 + 1} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}+\boldsymbol{e}_1}$$

satisfies $U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}})$.

Next, fix $N > 1$, and let $U(\partial_{\boldsymbol{z}}) = \prod_{\nu=1}^N l_\nu(\partial_{\boldsymbol{z}})$ such that $l_\nu(\partial_{\boldsymbol{z}})$ are non-zero linear homogeneous polynomials. Assume that the assertion holds for any product of non-zero linear homogeneous polynomial of degree less than N. By the induction hypothesis, there exist

$r(\partial_{\boldsymbol{s}}), \widetilde{r}(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that

$$l_1(\partial_{\boldsymbol{z}}) \star \widetilde{r}(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}}), \qquad \left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right) \star r(\partial_{\boldsymbol{s}}) = \widetilde{r}(\partial_{\boldsymbol{s}}).$$

By (ii), we have

$$\begin{aligned} U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) &= \left(l_1(\partial_{\boldsymbol{z}}) \left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right)\right) \star r(\partial_{\boldsymbol{s}}) \\ &= l_1(\partial_{\boldsymbol{z}}) \star \left(\left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right) \star r(\partial_{\boldsymbol{s}})\right) \\ &= l_1(\partial_{\boldsymbol{z}}) \star \widetilde{r}(\partial_{\boldsymbol{s}}) \\ &= q(\partial_{\boldsymbol{s}}), \end{aligned}$$

and hence the assertion holds. □

Lemma 3.9. *Let* $U(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, *and* $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. *Then*

$$[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} = [(U(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}}.$$

Proof. We show that the statement holds for any monomial operator $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$ by induction on $|\boldsymbol{\mu}|$. Firstly, the assertion is clear for $\boldsymbol{\mu} = \boldsymbol{0}$.

Secondly, assume that $\boldsymbol{\mu} = \boldsymbol{e}_k$ for $k = 1, \dots, k$, hence $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} = \partial_{z_k}$. Note that $[s_k q(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0$ for any $k = 1 \dots, h$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, and $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. Thus, by Lemma 3.8, we see that

$$\begin{aligned} &[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(q(\partial_{\boldsymbol{s}}) s_k) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(s_k q(\partial_{\boldsymbol{s}}) + \partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [s_k (q(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} + [(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}}. \end{aligned}$$

Hence the assertion holds for $|\boldsymbol{\mu}| = 1$.

Finally, fix $\boldsymbol{\mu} \in \mathbb{N}^h$ with $|\boldsymbol{\mu}| > 1$. Let $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, and $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. Assume that the assertion holds for any $\widetilde{U}(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\widetilde{\boldsymbol{\mu}}}$ with $|\widetilde{\boldsymbol{\mu}}| < |\boldsymbol{\mu}|$. Then there exists k such that $\boldsymbol{\mu}_k > 0$. Applying the induction hypothesis to the operators ∂_{z_k} and $\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}-\boldsymbol{e}_k}$, respectively, we see from Lemma 3.8 (ii) that

$$\begin{aligned}
&[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s})f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[q(\partial_{\boldsymbol{s}}) \bullet \left(s_k \cdot \boldsymbol{s}^{\boldsymbol{\mu}-\boldsymbol{e}_k} f(\boldsymbol{s})\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet \left(\boldsymbol{s}^{\boldsymbol{\mu}-\boldsymbol{e}_k} f(\boldsymbol{s})\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[\left(\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \star (\partial_{z_k} \star q(\partial_{\boldsymbol{s}}))\right) \bullet f(\boldsymbol{s})\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = [(U(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}}\,.
\end{aligned}$$

Hence the assertion holds. □

We define a ring homomorphism $\Psi_B : \mathbb{C}[\partial_{\boldsymbol{s}}] \to \mathbb{C}[\boldsymbol{x}]$ as

$$\Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x}) := q(\boldsymbol{x}B) = q\left(\sum_{j=1}^{n} b_j^{(1)} x_j, \ldots, \sum_{j=1}^{n} b_j^{(h)} x_j\right) \tag{10}$$

for $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Note that Ψ_B is injective by the linear independence of B.

Proposition 3.10. *Let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$\left[q(\partial_{\boldsymbol{s}}) \bullet \left(m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}}\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} = x^{\boldsymbol{v}} \Psi_B \left(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})\right)(\log \boldsymbol{x}),$$

where $\log \boldsymbol{x} := (\log x_1, \ldots, \log x_n)$.

Proof. Note that we can regard $x^{\boldsymbol{v}+B\boldsymbol{s}}$ as the formal series $x^{\boldsymbol{v}} e^{(\log \boldsymbol{x})B\boldsymbol{s}}$ in $\boldsymbol{s}$, where

$$(\log \boldsymbol{x})B\boldsymbol{s} := \sum_{j=1}^{n} \sum_{k=1}^{h} (\log x_j) b_j^{(k)} s_k.$$

Put $r(\partial_{\boldsymbol{s}}) := m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})$. Then, by Lemma 3.9,

$$\begin{aligned}
\left[q(\partial_{\boldsymbol{s}}) \bullet \left(m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}}\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} &= \left[(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet x^{\boldsymbol{v}+B\boldsymbol{s}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= \left[r(\partial_{\boldsymbol{s}}) \bullet x^{\boldsymbol{v}+B\boldsymbol{s}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= [r((\log \boldsymbol{x})B) x^{\boldsymbol{v}+B\boldsymbol{s}}]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= x^{\boldsymbol{v}} \Psi_B(r(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}).
\end{aligned}$$

□

Proposition 3.11. *The following hold.*

(i) $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} \subset P_B^{\perp} \subset P_{\mathcal{N}}^{\perp}$. *In particular, if $K_{\mathcal{N}} = I_{\boldsymbol{0}}$, then $P_{\mathcal{N}}^{\perp} = P_B^{\perp}$.*
(ii) $m(\boldsymbol{s}) \in P_{\mathcal{N}}$ *if and only if* $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = \{0\}$.
(iii) *If $P_{\mathcal{N}} = m(\boldsymbol{s}) \cdot P_B$, then $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = P_B^{\perp}$.*

Proof. (i) $P_B^\perp \subset P_{\mathcal{N}}^\perp$ is clear by Lemma 3.4 (i) and Lemma 3.7. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Then, for any $f(\boldsymbol{s}) \in P_B$, Lemma 3.9 shows that

$$[(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = [q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s})f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}}.$$

It follows from Lemma 3.4 (i) that the right hand side is 0. Hence we have $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp \subset P_B^\perp$.

(ii) Assume that $m(\boldsymbol{s}) \in P_{\mathcal{N}}$. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Put $m(\partial_{\boldsymbol{z}}) \star q(\partial_s) = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}$, where $a_{\boldsymbol{\nu}} \in \mathbb{C}$. Then, by Lemma 3.9, we have

$$\begin{aligned}\boldsymbol{\nu}! a_{\boldsymbol{\nu}} &= [(m(\partial_{\boldsymbol{z}}) \star q(\partial_s)) \bullet \boldsymbol{s}^{\boldsymbol{\nu}}]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s})\boldsymbol{s}^{\boldsymbol{\nu}})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0\end{aligned}$$

for any $\boldsymbol{\nu} \in \mathbb{N}^h$. Hence, we have $m(\partial_{\boldsymbol{z}}) \star q(\partial_s) = 0$.

Conversely, assume that $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp = \{0\}$. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Then, Lemma 3.9 shows that

$$[q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s})f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} = [(m(\partial_{\boldsymbol{z}}) \star q(\partial_s)) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0$$

for any $f(\boldsymbol{s}) \in \mathbb{C}[\boldsymbol{s}]$. Thus we have $q(\partial_{\boldsymbol{s}}) \in \langle m(\boldsymbol{s})\rangle^\perp$, that is, $P_{\mathcal{N}}^\perp \subset \langle m(\boldsymbol{s})\rangle^\perp$. By Lemma 3.7, $m(\boldsymbol{s}) \in P_{\mathcal{N}}$.

(iii) In (i), we have seen $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp \subset P_B^\perp$. We show its reverse inclusion. Let $q(\partial_{\boldsymbol{s}}) \in P_B^\perp$. By Lemma 3.8 (iii), there exists $r(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that $q(\partial_{\boldsymbol{s}}) = m(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}})$. It suffices to show that $r(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Let $f(\boldsymbol{s}) \in P_{\mathcal{N}}$. By the assumption, we have $f(\boldsymbol{s}) = m(\boldsymbol{s})g(\boldsymbol{s})$ for some $g(\boldsymbol{s}) \in P_B$. By Lemma 3.9, we see that

$$\begin{aligned}[r(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} &= [r(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s})g(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [(m(\partial_{\boldsymbol{z}}) \star r(\partial_s)) \bullet g(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [q(\partial_s) \bullet g(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0.\end{aligned}$$

Hence we have the assertion. □

Example 3.12. (cf. [6, Examples 3.2 and 4.7]) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$ and let $\boldsymbol{w} = (3, 1, 0, 0)$. Then the reduced Gröbner basis of I_A is

$$\mathcal{G} = \{\underline{\partial_{x_1}\partial_{x_3}^2} - \partial_{x_2}^2\partial_{x_4}, \underline{\partial_{x_2}\partial_{x_4}^2} - \partial_{x_3}^3, \underline{\partial_{x_1}^2\partial_{x_3}} - \partial_{x_2}^3, \underline{\partial_{x_1}\partial_{x_4}} - \partial_{x_2}\partial_{x_3}\}.$$

Here underlined terms are the leading ones. Thus we have

$$\mathrm{in}_{\boldsymbol{w}}(I_A) = \langle \partial_{x_1}\partial_{x_3}^2, \partial_{x_2}\partial_{x_4}^2, \partial_{x_1}^2\partial_{x_3}, \partial_{x_1}\partial_{x_4}\rangle.$$

Put

$$\begin{aligned}\boldsymbol{g}^{(1)} &= (1, -2, 2, -1)^T, \quad \boldsymbol{g}^{(2)} = (0, 1, -3, 2)^T, \\ \boldsymbol{g}^{(3)} &= (2, -3, 1, 0)^T, \qquad \boldsymbol{g}^{(4)} = (1, -1, -1, 1)^T.\end{aligned}$$

Let $\boldsymbol{\beta} = (-2, -1)^T$, and let

$$B = (\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Note that $\operatorname{supp}(B) = \{1, 2, 3, 4\}$. Take $\boldsymbol{v} = (0, -2, -1, 1)^T$ as a fake exponent. Then we have

$$\begin{aligned} \mathcal{N} &= \{\{2\}, \{3\}, \{2, 3\} = I_{\boldsymbol{0}}\}, \\ \mathcal{N}^c &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}, \\ K_{\mathcal{N}} &= \emptyset. \end{aligned}$$

Furthermore, we have

$$G^{(1)} = I_{-\boldsymbol{g}^{(1)}} \setminus I_{\boldsymbol{0}} = \{1\}, \quad G^{(2)} = I_{-\boldsymbol{g}^{(2)}} \setminus I_{\boldsymbol{0}} = \{4\},$$

$$G^{(3)} = I_{-\boldsymbol{g}^{(3)}} \setminus I_{\boldsymbol{0}} = \{1\}, \quad G^{(4)} = I_{-\boldsymbol{g}^{(4)}} \setminus I_{\boldsymbol{0}} = \{1\}.$$

Thus the ideals $P_{\mathcal{N}}$ and P_B are

$$\begin{aligned} P_{\mathcal{N}} &= \langle (B\boldsymbol{s})^{\{1,2\}}, (B\boldsymbol{s})^{\{1,3\}}, (B\boldsymbol{s})^{\{2,4\}} \rangle \\ &= \langle s_1(-2s_1 + s_2), s_1(2s_1 - 3s_2), (-2s_1 + s_2)(-s_1 + 2s_2) \rangle \\ &= \langle s_1^2, s_1 s_2, s_2^2 \rangle \end{aligned}$$

and

$$P_B = \langle (B\boldsymbol{s})^{\{1\}}, (B\boldsymbol{s})^{\{4\}} \rangle = \langle s_1, -s_1 + 2s_2 \rangle = \langle s_1, s_2 \rangle,$$

respectively. The orthogonal complements $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$ are

$$P_{\mathcal{N}}^{\perp} = \mathbb{C}1 + \mathbb{C}\partial_{s_1} + \mathbb{C}\partial_{s_2}$$

and

$$P_B^{\perp} = \mathbb{C}1,$$

respectively. In this case, note that

$$m(\boldsymbol{s}) = (B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus K_{\mathcal{N}}} = (B\boldsymbol{s})^{\{2,3\}} = (-2s_1 + s_2)(2s_1 - 3s_2) \in P_{\mathcal{N}}.$$

Hence, by Proposition 3.11, we have

$$m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = \{0\}.$$

Lemma 3.13. *Let $q(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \dots, z_h]$ be a homogeneous polynomial in indeterminates $\boldsymbol{z}$ of degree r. Then*

$$q(\partial_{\boldsymbol{s}}) \bullet \left(\frac{1}{r!} (\boldsymbol{x} B \boldsymbol{s})^r \right) = q(\boldsymbol{x} B) = \Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x}).$$

Here,

$$\boldsymbol{x}B\boldsymbol{s} := \sum_{j=1}^{n}\sum_{k=1}^{h} x_j b_j^{(k)} s_k$$

denotes the quadratic form associated with B*.*

Proof. We show the assertion by induction on r. In the case of $r=1$, the assertion is clear. Fix $r>1$ and assume that the assertion holds for any homogeneous polynomial of degree less than r. Let $\boldsymbol{\mu}\in\mathbb{N}^h$ with $|\boldsymbol{\mu}|=r$ and $\mu_k>0$. Then, by the chain rule and the induction hypothesis, we see that

$$\begin{aligned}
\partial_s{}^{\boldsymbol{\mu}} \bullet \left(\frac{1}{r!}(\boldsymbol{x}B\boldsymbol{s})^r\right) &= \partial_s{}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \bullet \left(\partial_{s_k} \bullet \left(\frac{1}{r!}(\boldsymbol{x}B\boldsymbol{s})^r\right)\right)\\
&= (\boldsymbol{x}B)_k \cdot \partial_{\boldsymbol{s}}{}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \bullet \left(\frac{1}{(r-1)!}(\boldsymbol{x}B\boldsymbol{s})^{r-1}\right)\\
&= (\boldsymbol{x}B)_k(\boldsymbol{x}B)^{\boldsymbol{\mu}-\boldsymbol{e}_k} = (\boldsymbol{x}B)^{\boldsymbol{\mu}} = \Psi_B(\partial_{\boldsymbol{s}}^{\boldsymbol{\mu}})(\boldsymbol{x}).
\end{aligned}$$

□

Two vector spaces $Q_{\boldsymbol{v}}^{\perp}$ and $P_B^{\perp}$ are related as follows.

Theorem 3.14. *Let* Ψ_B *be the homomorphism in* (10)*. Then,* $P_B^{\perp} = \Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})$ *and* $\dim_{\mathbb{C}}(P_B^{\perp}) \le \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp})$*. Furthermore, if* B *is a basis of* L*, then* $\Psi_B(P_B^{\perp}) = Q_{\boldsymbol{v}}^{\perp}$ *and* $\dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp}) = \dim_{\mathbb{C}}(P_B^{\perp})$*.*

Proof. Let $\deg(q(\boldsymbol{z})) = r$. Then, it follows from Lemma 3.13 that

$$\begin{aligned}
\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet q(\boldsymbol{x}B) &= \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left\{ q(\partial_{\boldsymbol{s}}) \bullet \left(\frac{1}{r!} (\boldsymbol{x}B\boldsymbol{s})^r \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left(\frac{1}{r!} (\boldsymbol{x}B\boldsymbol{s})^r \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left(\frac{1}{r!} \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r} \frac{r!}{\boldsymbol{\mu}!} \boldsymbol{x}^{\boldsymbol{\mu}} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r} \frac{\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}} \frac{\boldsymbol{x}^{\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}}}}{(\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}})!} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right\} \\
&= \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}} \frac{\boldsymbol{x}^{\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}}}}{(\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}})!} \{ q(\partial_{\boldsymbol{s}}) \bullet (B\boldsymbol{s})^{\boldsymbol{\mu}} \}
\end{aligned}$$

for any i. Here $\boldsymbol{e}_{G^{(i)}} := \sum_{j \in G^{(i)}} \boldsymbol{e}_j$ denotes the indicator vector of $G^{(i)}$. Since $\partial_{\boldsymbol{s}}^{\boldsymbol{p}} \bullet \boldsymbol{s}^{\boldsymbol{q}} = \boldsymbol{p}! \delta_{\boldsymbol{p},\boldsymbol{q}}$ for any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^h$ with $|\boldsymbol{p}| = |\boldsymbol{q}|$, by Proposition 3.1 we have

$$\begin{aligned}
q(\boldsymbol{x}B) \in Q_{\boldsymbol{v}}^{\perp} &\iff \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet q(\boldsymbol{x}B) = 0 \text{ for all } i = 1, \dots, m \\
&\iff q(\partial_{\boldsymbol{s}}) \bullet (B\boldsymbol{s})^{\boldsymbol{\mu}} = 0 \\
&\qquad \text{for all } i \text{ and all } \boldsymbol{\mu} \in \mathbb{N}^n \text{ with } |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)} \\
&\iff q(\partial_{\boldsymbol{s}}) \in P_B^{\perp}.
\end{aligned}$$

Here, the second equivalence follows from the linear independence of the monomials $\boldsymbol{x}^{\boldsymbol{\mu}}$. The third equivalence follows from the linear independence of B, because it yields that the ideal P_B is spanned as a vector space by the polynomials whose terms are of the form $(B\boldsymbol{s})^{\boldsymbol{\mu}}$ with $\mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}$ for some i. Thus we have $P_B^{\perp} = \Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})$. Moreover, we have the inequality $\dim_{\mathbb{C}}(P_B^{\perp}) \leq \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp})$ because $\Psi_B({P_B}^{\perp}) = \Psi_B\left(\Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})\right) \subset Q_{\boldsymbol{v}}^{\perp}$ and Ψ_B is injective.

Assume that B is a basis of L. Note that each $\boldsymbol{x}\boldsymbol{g}^{(k)} = \sum_{j=1}^n g_j^{(k)} x_j$ can be represented by a linear combination of $(\boldsymbol{x}B)_1, \dots, (\boldsymbol{x}B)_h$. Let $f(\boldsymbol{x}) \in Q_{\boldsymbol{v}}^{\perp}$. Then, by Proposition 3.1 and the above result, there exists $q(\partial_{\boldsymbol{s}}) \in P_B^{\perp}$ such that $f(\boldsymbol{x}) = q(\boldsymbol{x}B)$. Thus we have $f(\boldsymbol{x}) = \Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x})$. □

4. Fundamental systems of solutions

In this section, we construct a fundamental system of series solutions with a given exponent to $M_A(\boldsymbol{\beta})$. We recall that the homogeneity of A yields the regular holonomicity of $M_A(\boldsymbol{\beta})$. This means that, for a fixed generic weight $\boldsymbol{w}$, the solution space to $M_A(\boldsymbol{\beta})$ has a basis consisting of canonical series with starting monomial $x^{\boldsymbol{v}}(\log \boldsymbol{x})^{\boldsymbol{b}}$ for some exponent $\boldsymbol{v}$ and $\boldsymbol{b} \in \mathbb{N}^n$. Note that each $x^{\boldsymbol{v}}(\log \boldsymbol{x})^{\boldsymbol{b}}$ is derived as the initial monomial of a solution to the indicial ideal $\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$, or of an element of $Q_{\boldsymbol{v}}^{\perp}$. For the detail, see [8, Sections 2.3, 2.4, and 2.5] and Proposition 3.1.

Throughout this section, we assume that B is a basis of L. Since B satisfies Assumption 2.3 (see Remark 2.4), we have the following homomorphisms by Propositions 3.1, 3.10, and Theorem 3.14:

$$\begin{array}{ccccc} P_{\mathcal{N}}^{\perp} & \to & P_B^{\perp} & \simeq & \mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}), \\ q(\partial_{\boldsymbol{s}}) & \mapsto & m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}) & \leftrightarrow & x^{\boldsymbol{v}}\Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}). \end{array} \tag{11}$$

Here, $\mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})$ denotes the solution space of the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$.

Proposition 4.1. *If $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$, then $\boldsymbol{v}$ is an exponent.*

Proof. Assume that $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$. Then, by Proposition 3.11 (ii), there exists $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$ such that $m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}) \neq 0$. We see from Theorem 2.7 that

$$\begin{aligned} &(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = \sum_{\boldsymbol{u} \in L'} \big(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}})\big)_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = (q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}})_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\qquad\qquad + \sum_{\boldsymbol{u} \in L' \setminus \{0\}} \big(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}})\big)_{|\boldsymbol{s}=\boldsymbol{0}} \end{aligned}$$

is a solution to $M_A(\boldsymbol{\beta})$. By Proposition 3.10, we have

$$(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}})_{|\boldsymbol{s}=\boldsymbol{0}} = x^{\boldsymbol{v}}\Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}),$$

hence this solution has a *non-zero* starting term. Hence $\boldsymbol{v}$ is an exponent. □

Example 4.2 (Continuation of Example 3.12)**.** Let A and $\boldsymbol{v}$ be the ones in Example 3.12. Recall that $m(\boldsymbol{s}) \in P_{\mathcal{N}}$, which is a necessary condition for the fake exponent $\boldsymbol{v}$ not to be an exponent. We see that $\boldsymbol{v}$ is not an exponent from the following calculation. Note that

$$\theta_1 - 2\theta_3 - 3\theta_4 + 1 \in \langle A\theta_{\boldsymbol{x}} - \boldsymbol{\beta} \rangle.$$

Then we have

$$\begin{aligned} 0 &\equiv \theta_1\theta_3(\theta_1 - 2\theta_3 - 3\theta_4 + 1) \\ &= \theta_1^2\theta_3 - 2\theta_1\theta_3^2 - 3\theta_1\theta_3\theta_4 + \theta_1\theta_3 \\ &= \theta_1(\theta_1 - 1)\theta_3 - 2\theta_1\theta_3(\theta_3 - 1) - 3\theta_1\theta_3\theta_4 \\ &= x_1^2x_3\partial_{x_1}^2\partial_{x_3} - 2x_1x_3^2\partial_{x_1}\partial_{x_3}^2 - 3x_1x_3x_4\partial_{x_1}\partial_{x_3}\partial_{x_4} \\ &\equiv x_1^2x_3\partial_{x_2}^3 - 2x_1x_3^2\partial_{x_2}^2\partial_{x_4} - 3x_1x_3x_4\partial_{x_2}\partial_{x_3}^2 \end{aligned}$$

modulo $H_A(\boldsymbol{\beta})$. Hence

$$x_1^2x_3\partial_{x_2}^3 - 2x_1x_3^2\partial_{x_2}^2\partial_{x_4} - 3x_1x_3x_4\partial_{x_2}\partial_{x_3}^2 \in H_A(\boldsymbol{\beta}),$$

and

$$x_1x_3^2\partial_{x_2}^2\partial_{x_4} \in \mathrm{in}_{(-\boldsymbol{w},\boldsymbol{w})}(H_A(\beta)).$$

Since

$$x_1x_3^2\partial_{x_2}^2\partial_{x_4} \bullet x^{\boldsymbol{v}} = x_1x_3^2\partial_{x_2}^2\partial_{x_4} \bullet x_2^{-2}x_3^{-1}x_4 \neq 0,$$

$\boldsymbol{v}$ is not an exponent.

Corollary 4.3. *Assume that B is a basis of L. If $|I \cup J| > |I_{\mathbf{0}}|$ for any $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$, then $\boldsymbol{v}$ is an exponent.*

Proof. For any $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$, we see that

$$|I \cup J \setminus K_{\mathcal{N}}| = |I \cup J| - |K_{\mathcal{N}}| > |I_{\mathbf{0}}| - |K_{\mathcal{N}}| = |I_{\mathbf{0}} \setminus K_{\mathcal{N}}|$$

because both of I and $I_{\mathbf{0}}$ contain $K_{\mathcal{N}}$. Since the degree of $m(\boldsymbol{s}) = (B\boldsymbol{s})^{I_{\mathbf{0}}\setminus K_{\mathcal{N}}}$ is less than that of any $(B\boldsymbol{s})^{I\cup J\setminus \mathcal{N}}$, $m(\boldsymbol{s})$ cannot belong to $P_{\mathcal{N}}$. By Proposition 4.1, $\boldsymbol{v}$ is an exponent. □

Theorem 4.4. *Assume that B is a basis of L, and that $P_{\mathcal{N}} = m(\boldsymbol{s})\cdot P_B$. Then $\boldsymbol{v}$ is an exponent, and the set*

$$\{(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F_{\mathcal{N}}}(x, \boldsymbol{s}))_{|\boldsymbol{s}=\mathbf{0}} \mid q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}\}.$$

spans the space of series solutions in the direction of $\boldsymbol{w}$ to $M_A(\boldsymbol{\beta})$ with exponent $\boldsymbol{v}$. In particular, for $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$, the solution $(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F}_{\mathcal{N}}(x, \boldsymbol{s}))_{|\boldsymbol{s}=\mathbf{0}}$ has the starting term $x^{\boldsymbol{v}}\Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x})$.

Proof. By definition, $P_B \neq \mathbb{C}[\boldsymbol{s}]$. It follows from the assumption that $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$. Hence, by Proposition 4.1, $\boldsymbol{v}$ is an exponent. Moreover, by Proposition 3.11, the homomorphism (11) is surjective. Hence we have

$$\begin{aligned} \dim_{\mathbb{C}}(P_{\mathcal{N}}^{\perp}) &\geq \dim_{\mathbb{C}}(P_B^{\perp}) = \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp}) = \dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})) \\ &\geq \dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})). \end{aligned}$$

By [8, Proposition 2.3.6, Theorem 2.5.1, and Corollary 2.5.11], the regularity of $M_A(\boldsymbol{\beta})$ indicates that the dimension of the space of series solutions with the exponent $\boldsymbol{v}$ coincides with $\dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}))$. Hence, by Theorem 2.7, we have the former half of the assertion.

The latter half of the assertion follows from Proposition 3.10. □

Example 4.5 (Continuation of Examples 2.5, 3.2, and 3.5)**.** Consider the case where $B = \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then, B satisfies the assumption in Theorem 4.4. Recall that $m(\boldsymbol{s}) = (B\boldsymbol{s})^{\emptyset} = 1$, and

$$P_B = \langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle = \langle s_1^2, s_2^2 \rangle = P_{\mathcal{N}}.$$

Hence, we see by Proposition 4.1 that $\boldsymbol{v}$ is an exponent, and that

$$\{1, \partial_{s_1}, \partial_{s_2}, \partial_{s_1}\partial_{s_2}\}$$

is a basis of $P_B^{\perp}$. Hence $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$ if and only if

$$f \in \langle 1, \boldsymbol{x}\boldsymbol{g}^{(1)}, \boldsymbol{x}\boldsymbol{g}^{(2)}, (\boldsymbol{x}\boldsymbol{g}^{(1)}) \cdot (\boldsymbol{x}\boldsymbol{g}^{(2)}) \rangle_{\mathbb{C}}.$$

By the uniqueness of an exponent, the above space coincides with the space of solutions to $M_A(\boldsymbol{\beta})$. Note that the holonomic rank of $M_A(\boldsymbol{\beta})$ is four (cf. [8, Example 3.5.2]).

Example 4.6. [8, Examples 3.6.3, 3.6.11, 3.6.16] Let $d = 3$, $n = 9$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

Let $\boldsymbol{w} = (2, 0, 0, 0, -1, 0, 0, 0, 2)$, $\boldsymbol{\beta} = (1, 1, 1)^T = \boldsymbol{a}_5$. Consider an exponent $\boldsymbol{v} = (0, 0, 0, 0, 1, 0, 0, 0, 0)^T$. Then $K_{\mathcal{N}_B} = I_{\boldsymbol{0}} = \emptyset$. The reduced Gröbner basis consists of the following twenty binomials

$$\{\underline{\partial^{\boldsymbol{g}_+^{(i)}}} - \partial^{\boldsymbol{g}_-^{(i)}} \mid i = 1, 2, \ldots, 20\},$$

where

$$\begin{aligned} \boldsymbol{g}^{(1)} &:= (0, 1, -1, 0, -1, 1, 0, 0, 0)^T, \boldsymbol{g}^{(2)} := (0, 0, 0, 1, -1, 0, -1, 1, 0)^T, \\ \boldsymbol{g}^{(3)} &:= (0, 0, 0, 1, -2, 1, 0, 0, 0)^T, \boldsymbol{g}^{(4)} := (0, 1, 0, 0, -2, 0, 0, 1, 0)^T, \\ \boldsymbol{g}^{(5)} &:= (1, -1, 0, -1, 1, 0, 0, 0, 0), \boldsymbol{g}^{(6)} := (0, 0, 0, 0, 1, -1, 0, -1, 1), \\ \boldsymbol{g}^{(7)} &, \ldots, \boldsymbol{g}^{(20)}. \end{aligned}$$

Hence we have

$$\begin{aligned}
&\{G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} \mid i = 1, \dots, 20\} \\
&\quad = \{G^{(1)} = \{2,6\}, G^{(2)} = \{4,8\}, G^{(3)} = \{4,6\}, G^{(4)} = \{2,8\}, \\
&\qquad\quad G^{(5)} = \{1\}, G^{(6)} = \{9\}, \{1,8\}, \{2,7\}, \{1,3\}, \{1,6\}, \{3,4\}, \\
&\qquad\quad \{1,9\}, \{3,7\}, \{4,9\}, \{6,7\}, \{7,9\}, \{2,9\}, \{3,8\}, \{3,9\}, \{1,7\}\}.
\end{aligned}$$

Let

$$B = [\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}, \boldsymbol{g}^{(3)}, \boldsymbol{g}^{(4)}, \boldsymbol{g}^{(5)}, \boldsymbol{g}^{(6)}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & -1 & -2 & -2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned}
P_{\mathcal{N}} &= P_B \\
&= \langle (s_1 + s_4 - s_5)(s_1 + s_3 - s_6), (s_2 + s_3 - s_5)(s_2 + s_4 - s_6), \\
&\qquad (s_2 + s_3 - s_5)(s_1 + s_3 - s_6), (s_1 + s_4 - s_5)(s_2 + s_4 - s_6), \\
&\qquad s_5, s_6, s_2(s_1 + s_4 - s_5), s_1(s_2 + s_3 - s_5), \\
&\qquad s_1 s_2, s_2(s_1 + s_3 - s_6), s_1(s_2 + s_4 - s_6)\rangle \\
&= \langle s_1 s_2, s_1 s_3, s_1 s_4, s_2 s_3, s_2 s_4, s_3^2, s_4^2, s_1^2 + s_3 s_4, s_2^2 + s_3 s_4, s_5, s_6\rangle,
\end{aligned}$$

where the last generator set gives the reduced Gröbner basis with respect to the lexicographic order $<$ with $s_1 > s_2 > s_3 > s_4 > s_5 > s_6$. We see that

$$\{\boldsymbol{\nu} \in \mathbb{N}^6 \mid \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_<(P_B)\} = \{\boldsymbol{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_3 + \boldsymbol{e}_4\},$$

and hence we immediately have

$$q_{\boldsymbol{0}} = 1, q_{\boldsymbol{e}_1} = \partial_{s_1}, q_{\boldsymbol{e}_2} = \partial_{s_2}, q_{\boldsymbol{e}_3} = \partial_{s_3}, q_{\boldsymbol{e}_4} = \partial_{s_4}.$$

As to the last generator $q_{\boldsymbol{e}_3+\boldsymbol{e}_4}$, since

$$c_{\boldsymbol{\mu}, \boldsymbol{e}_3+\boldsymbol{e}_4} = \begin{cases} -1 & (\text{if } \boldsymbol{\mu} = 2\boldsymbol{e}_1, 2\boldsymbol{e}_2) \\ 0 & (\text{otherwise}) \end{cases},$$

we have

$$q_{\boldsymbol{e}_3+\boldsymbol{e}_4} = \partial_{s_3}\partial_{s_4} + \sum_{\boldsymbol{\mu}; |\boldsymbol{\mu}|=2} c_{\boldsymbol{\mu}, \boldsymbol{e}_3+\boldsymbol{e}_4} \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} = \partial_{s_3}\partial_{s_4} - \frac{1}{2}\partial_{s_1}^2 - \frac{1}{2}\partial_{s_2}^2.$$

Hence a $\mathbb{C}$-basis of $Q_{\boldsymbol{v}}^{\perp}$ is given by

$$\{1, \boldsymbol{x}\boldsymbol{g}^{(1)}, \boldsymbol{x}\boldsymbol{g}^{(2)}, \boldsymbol{x}\boldsymbol{g}^{(3)}, \boldsymbol{x}\boldsymbol{g}^{(4)}, (\boldsymbol{x}\boldsymbol{g}^{(3)})\cdot(\boldsymbol{x}\boldsymbol{g}^{(4)}) - \frac{1}{2}(\boldsymbol{x}\boldsymbol{g}^{(1)})^2 - \frac{1}{2}(\boldsymbol{x}\boldsymbol{g}^{(2)})^2\}.$$

5. Aomoto-Gel'fand systems

In this section, let

$$A = \{\boldsymbol{a}_{i,j} \,|\, 1 \leq i \leq m,\, m+1 \leq j \leq m+l\},$$

where $\boldsymbol{a}_{i,j} = \boldsymbol{e}_i + \boldsymbol{e}_j$, and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{l+m}\}$ is the standard basis of $\mathbb{Z}^{l+m}$.

Then $\mathbb{Z}A = \{\boldsymbol{a} \in \mathbb{Z}^{l+m} \mid \sum_{i=1}^{m} a_i = \sum_{j=m+1}^{m+l} a_j\}$, $\operatorname{rank}(A) = m+l-1$, and $\operatorname{rank}(L) = ml - (m+l-1) = (m-1)(l-1)$, where

$$L = \{[c_{ij}]_{1\leq i\leq m,\, m+1\leq j\leq m+l} \in M_{m\times l}(\mathbb{Z}) \mid \sum_{i,j} c_{ij}\boldsymbol{a}_{ij} = \boldsymbol{0}\}.$$

Since A is normal, (that is, $\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_{\geq 0}A$), I_A is a Cohen-Macaulay ideal and hence $\operatorname{rank}(M_A(\boldsymbol{\beta})) = \operatorname{vol}(A)$ for any $\boldsymbol{\beta}$ (see [3]).

Take a weight vector $\boldsymbol{w}$ satisfying $w_{i,j} > w_{p,q}$ whenever $(i,j) \neq (p,q)$, $i \leq p$, and $j \leq q$.

Then the reduced Gröbner basis of I_A with respect to $\boldsymbol{w}$ equals

$$\mathcal{G} = \{\underline{\partial^{(\boldsymbol{g}^{(i,j)}_{(p,q)})_+}} - \partial^{(\boldsymbol{g}^{(i,j)}_{(p,q)})_-} \,|\, i < p,\, j < q\},$$

and

$$\operatorname{in}_{\boldsymbol{w}}(I_A) = \langle \partial_{i,j}\partial_{p,q} \,|\, i < p,\, j < q\rangle,$$

where $\boldsymbol{g}^{(i,j)}_{(p,q)} := E_{i,j} + E_{p,q} - E_{i,q} - E_{p,j} \in L$ and $E_{i,j}$ are matrix units.

The weight $\boldsymbol{w}$ induces a staircase regular triangulation, which is unimodular (cf. [10, Example 8.12]); for example, let $m = 2, l = 4$, the standard pairs are

$$\begin{bmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & * & * \\ * & * & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & * & * \\ * & * & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix},$$

where let the row-numbers be $1, \dots, m$ and the column-numbers $m+1, \dots, m+l$.

For general l, m, the standard pairs correspond to the paths from the southwest corner to the northeast corner going only northward or eastward. In this way, we see

$$\operatorname{vol}(A) = \binom{m+l-2}{m-1}.$$

Let

$$B := \{\boldsymbol{b}^{(i,j)} := \boldsymbol{g}^{(i,j)}_{(i+1,j+1)} \,|\, 1 \leq i < m,\, m+1 \leq j < m+l\}.$$

Then B is a basis of L and $\mathrm{supp}(B) = \{1, \ldots, m+l\}$, hence B satisfies Assumption 2.3. Let $\boldsymbol{s} = (s_{(i,j)})_{1\le i<m, m+1\le j<m+l}$ be indeterminates such that $s_{(i,j)}$ corresponds to $\boldsymbol{b}^{(i,j)}$. For convenience, set $s_{(i,j)} := 0$ unless $(i,j) \in \{1, \ldots, m-1\} \times \{m+1, \ldots, m+l-1\}$. Then

$$(B\boldsymbol{s})_{(\mu,\nu)} = \sum_{\substack{1\le i<m \\ m+1\le j<m+l}} s_{(i,j)} (\boldsymbol{g}^{(i,j)}_{(i+1,j+1)})_{(\mu,\nu)}$$
$$= s_{(\mu,\nu)} - s_{(\mu,\nu-1)} - s_{(\mu-1,\nu)} + s_{(\mu-1,\nu-1)}.$$

Lemma 5.1. *Let $\boldsymbol{\beta} = \boldsymbol{0}$. Then $\boldsymbol{v} = \boldsymbol{0}$ is a unique exponent.*

Proof. Since $\mathrm{in}_{\boldsymbol{w}}(I_A)$ is square-free, every fake exponent is an exponent by Theorem 3.6.6 in [8].

Let $\boldsymbol{v}$ be an exponent. Then there exists a standard pair $(\boldsymbol{a}, \sigma) = (\boldsymbol{0}, \sigma)$ corresponding to $\boldsymbol{v}$ such that

$$v_j = 0 \quad (j \notin \sigma), \qquad A\boldsymbol{v} = \boldsymbol{\beta} = \boldsymbol{0}.$$

Since the submatrix $A_\sigma = (\boldsymbol{a}_j)_{j\in\sigma}$ is invertible and satisfies $A_\sigma \boldsymbol{v}_\sigma = \boldsymbol{0}$, we have $\boldsymbol{v} = \boldsymbol{0}$. □

From now on, let $\boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{v} = \boldsymbol{0}$. Hence $I_{\boldsymbol{0}} = \emptyset = K_{\mathcal{N}_B}$.

Lemma 5.2. (i) $\{(ij), (pq)\} \in \mathcal{N}^c = \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ *for $i < p$, $j < q$.*

(ii)) *Let $J \in \mathcal{N}^c = \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. Then there exist $i < p$, $j < q$ such that $J \supset \{(ij), (pq)\}$.*

Proof. (i) This follows from $G^{(ij)}_{(pq)} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(ij)}_{(pq)}) = \mathrm{nsupp}(-\boldsymbol{g}^{(ij)}_{(pq)}) = \{(ij), (pq)\}$.

(ii) This is immediate from Proposition 3.3. □

Proposition 5.3.

$$P_B = \langle (B\boldsymbol{s})^{\{(i,j),(p,q)\}} \mid i < p, j < q \rangle = \langle s_{(i,j)} s_{(p,q)} \mid i \le p,\ j \le q \rangle.$$

Proof. The first equality follows from Lemma 5.2. We show the second equality. Since, for any i, j, p, q with $i < p$ and $j < q$,

$$(B\boldsymbol{s})^{\{(i,j),(p,q)\}} = (s_{(i,j)} - s_{(i,j-1)} - s_{(i-1,j)} + s_{(i-1,j-1)})$$
$$\times (s_{(p,q)} - s_{(p,q-1)} - s_{(p-1,q)} + s_{(p-1,q-1)}) \in \mathrm{RHS},$$

we only need to show the reverse inclusion. We introduce the total order $<$ on $\mathcal{X} := \{((i,j),(p,q)) \mid 1 \le i \le p < m, m+1 \le j \le q < m+l\}$

defined by

$$
\begin{aligned}
&((i,j),(p,q)) < ((i',j'),(p',q')) \\
&\qquad \iff \begin{cases} i < i', \\ \text{or } [i = i' \text{ and } j < j'], \\ \text{or } [(i,j) = (i',j') \text{ and } p > p'], \\ \text{or } [(i,j,p) = (i',j',p') \text{ and } q > q']. \end{cases}
\end{aligned}
$$

We prove by induction on the totally ordered set $(\mathcal{X}, <)$.

First, for the minimum element $((1, m+1), (m-1, m+l-1))$, we have

$$s_{(1,m+1)} s_{(m-1,m+l-1)} = (B\boldsymbol{s})^{\{(1,m+1),(m,m+l)\}} \in \text{LHS}.$$

Next, fix $((i,j),(p,q)) \neq ((1,m+1),(m-1,m+l-1))$ with $1 \leq i \leq p < m$ and $m+1 \leq j \leq q < m+l$. Suppose that $s_{(i',j')} s_{(p',q')} \in \text{LHS}$ for all $((i',j'),(p',q')) < ((i,j),(p,q))$. Then the leading monomial in

$$
\begin{aligned}
(\text{LHS} \ni)\ \ (B\boldsymbol{s})^{\{(i,j),(p+1,q+1)\}} &= \left(s_{(i,j)} - s_{(i,j-1)} - s_{(i-1,j)} + s_{(i-1,j-1)}\right) \\
&\quad \times \left(s_{(p+1,q+1)} - s_{(p+1,q)} - s_{(p,q+1)} + s_{(p,q)}\right)
\end{aligned}
$$

is $s_{(i,j)} s_{(p,q)}$. It follows from the induction hypothesis that $s_{(i,j)} s_{(p,q)} \in$ LHS. Hence we have the assertion. □

Corollary 5.4.

$$
\begin{aligned}
P_B^{\perp} &= \left\langle \partial_{\boldsymbol{s}}^{\boldsymbol{p}} \,\middle|\, \boldsymbol{s}^{\boldsymbol{p}} \notin \langle s_{(i,j)} s_{(p,q)} \,|\, i \leq p,\ j \leq q \rangle \right\rangle_{\mathbb{C}} \\
&= \langle \partial_{s_{(i_1,j_1)}} \cdots \partial_{s_{(i_r,j_r)}} \,|\, i_1 < \cdots < i_r, j_1 > \cdots > j_r \rangle_{\mathbb{C}}.
\end{aligned}
$$

Furthermore,

$$\dim_{\mathbb{C}}(P_B^{\perp}) = \sum_{r=0}^{\min\{l-1,m-1\}} \binom{l-1}{r}\binom{m-1}{r} = \binom{l+m-2}{m-1}.$$

Proof. The first part follows from Proposition 5.3. For the second part, compare the coefficients of z^{m-1} in the following:

$$
\begin{aligned}
\sum_{k=0}^{l+m-2} \binom{l+m-2}{k} z^k &= (1+z)^{l+m-2} = (1+z)^{l-1}(1+z)^{m-1} \\
&= (\sum_{p=0}^{l-1} \binom{l-1}{p} z^p)(\sum_{q=0}^{m-1} \binom{m-1}{q} z^{m-1-q}) \\
&= \sum_{p,q} \binom{l-1}{p}\binom{m-1}{q} z^{m-1-q+p}.
\end{aligned}
$$

□

Corollary 5.5.

$$\{((\partial_{s_{(i_1,j_1)}}\cdots\partial_{s_{(i_r,j_r)}})\bullet F_{\mathcal{N}}(\boldsymbol{x},\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}\,|\,i_1<\cdots<i_r, j_1>\cdots>j_r\}$$

forms a fundamental system of solutions to $M_A(\mathbf{0})$.

Proposition 5.6.

$$Q_{\mathbf{0}}^{\perp}=\left\langle \prod_{k=1}^{r}(\boldsymbol{x}\boldsymbol{g}^{(i_k,j_k)}_{(i_k+1,j_k+1)}) \;\middle|\; \begin{matrix} i_1<i_2<\cdots<i_r \\ j_1>j_2>\cdots>j_r \end{matrix} \right\rangle_{\mathbb{C}}.$$

Proof. This is immediate from Theorem 3.14 and Corollary 5.4. □

Example 5.7. Let $m=2$. This case corresponds to the Lauricella's F_D (e.g. see [1, §3.1.3]). Then $\mathrm{vol}(A)=\binom{l+m-2}{m-1}=l$, and

$$P_B^{\perp}=\langle 1,\partial_{s_{(1,3)}},\partial_{s_{(1,4)}},\ldots,\partial_{s_{(1,l+1)}}\rangle_{\mathbb{C}}.$$

Example 5.8. Let $l=m=3$. Then $\mathrm{vol}(A)=\binom{l+m-2}{m-1}=\binom{4}{2}=6$, and

$$P_B^{\perp}=\langle 1,\partial_{s_{(1,4)}},\partial_{s_{(1,5)}},\partial_{s_{(2,4)}},\partial_{s_{(2,5)}},\partial_{s_{(1,5)}}\partial_{s_{(2,4)}}\rangle_{\mathbb{C}}.$$

6. Lauricella's F_C

Let

$$A:=\{\boldsymbol{a}_i:=\boldsymbol{e}_0+\boldsymbol{e}_i,\ \boldsymbol{a}_{-i}:=\boldsymbol{e}_0-\boldsymbol{e}_i\,|\,i=1,2,\ldots,m\}.$$

In this case, A is normal, and the A-hypergeometric systems correspond to Lauricella's F_C [7]. Then

$$\mathbb{Z}A=\{\boldsymbol{a}\in\mathbb{Z}^{m+1}\,|\,\sum_{i=0}^{m}a_i\in 2\mathbb{Z}\},$$

and

$$L=\{\boldsymbol{l}\in\mathbb{Z}^{2m}\,|\,\sum_{i=\pm1,\ldots,\pm m}l_i=0,\ l_i-l_{-i}=0\ \ (1\le i\le m)\}.$$

We have $\mathrm{rank}(L)=2m-(m+1)=m-1$.

Take a weight $\boldsymbol{w}$ so that

$$w_1+w_{-1}>w_2+w_{-2}>\cdots>w_m+w_{-m}.$$

Then

$$\mathrm{in}_{\boldsymbol{w}}(I_A)=\langle\partial_1\partial_{-1},\partial_2\partial_{-2},\ldots,\partial_{m-1}\partial_{-(m-1)}\rangle,$$

and the reduced Gröbner basis $\mathcal{G}$ is given by

$$\mathcal{G}=\{\underline{\partial^{\boldsymbol{g}_+^{(i)}}}-\partial^{\boldsymbol{g}_-^{(i)}}\,|\,i=1,2,\ldots,m-1\},$$

where $\boldsymbol{g}^{(i)} = \boldsymbol{e}_i + \boldsymbol{e}_{-i} - \boldsymbol{e}_m - \boldsymbol{e}_{-m}$. Set $B := \{\boldsymbol{g}^{(i)} \,|\, i = 1, 2, \ldots, m-1\}$. Since B is a basis of the free $\mathbb{Z}$-module L, it satisfies Assumption 2.3. Note that $\mathrm{supp}(B) = \{\pm 1, \ldots, \pm m\}$. Let $\boldsymbol{s} = (s_i)_{1 \le i \le m-1}$ be indeterminates such that s_i corresponds to $\boldsymbol{b}^{(i)} := \boldsymbol{g}^{(i)}$. The standard pairs are pairs of $*$-place $\{\epsilon(i)i \,|\, i \in [1, m-1]\} \cup \{\pm m\}$ ($\epsilon : [1, m-1] \to \{\pm 1\}$) and 0-place its complement.

Hence $\mathrm{vol}(A) = 2^{m-1}$.

Lemma 6.1. *Let $\boldsymbol{\beta} = \boldsymbol{0}$. Then $\boldsymbol{v} = \boldsymbol{0}$ is a unique exponent.*

Proof. The proof is similar to Lemma 5.1. □

Proposition 6.2.

$$P_B = \langle (B\boldsymbol{s})^{\{\pm i\}} \,|\, i = 1, \ldots, m-1 \rangle = \langle s_i^2 \,|\, 1 \le i \le m-1 \rangle.$$

Proof. We have $G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} = \{\pm i\}$ and

$$(B\boldsymbol{s})^{\{\pm i\}} = \left(\sum_{\nu=1}^{m-1} s_\nu \boldsymbol{g}^{(\nu)}_{+i} \right) \left(\sum_{\nu=1}^{m-1} s_\nu \boldsymbol{g}^{(\nu)}_{-i} \right) = s_i^2.$$

□

Proposition 6.3.

$$\begin{aligned} P_B^{\perp} &= \left\langle \partial_{\boldsymbol{s}}^{\boldsymbol{p}} \,|\, \boldsymbol{s}^{\boldsymbol{p}} \notin \langle s_i^2 \,|\, 1 \le i \le m-1 \rangle \right\rangle_{\mathbb{C}} \\ &= \langle \partial_{\boldsymbol{s}}^{I} = \prod_{i \in I} \partial_{s_i} \,|\, I \subset \{1, \ldots, m-1\} \rangle_{\mathbb{C}}, \end{aligned}$$

and

$$Q_{\boldsymbol{0}}^{\perp} = \langle \prod_{i \in I} (\boldsymbol{x}\boldsymbol{g}^{(i)}) \,|\, I \subset \{1, \ldots, m-1\} \rangle_{\mathbb{C}}.$$

Furthermore,

$$\dim_{\mathbb{C}}(Q_{\boldsymbol{0}}^{\perp}) = 2^{m-1}.$$

Proof. This is immediate from Lemma 3.6, Theorem 3.14, and Proposition 6.2. □

Corollary 6.4. $\{((\prod_{i \in I} \partial_{s_i}) \bullet F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} \,|\, I \subset [1, m-1]\}$ *forms a basis of solutions of* $M_A(\boldsymbol{0})$.

Acknowledgment

The authors thank the referee for careful reading and helpful comments.

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(Go Okuyama)
Higher Education Support Center
Hokkaido University of Science
Sapporo, 006-8585, Japan.

Email address: `gokuyama@hus.ac.jp`

(Mutsumi Saito)
Department of Mathematics, Faculty of Science
Hokkaido University
Sapporo, 060-0810, Japan.

Email address: `saito@math.sci.hokudai.ac.jp`