Topology of Complements of Real Space Line Arrangements and Linearly Embedded Graphs

(実直線配置及びグラフ線形埋め込みの補空間に関するトポロジー)

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CONTENTS

1. Introduction

In this thesis, we consider complements of real space line arrangements and linearly embedded graphs with half-lines.

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a subspace arrangement in $\mathbb{R}^n$ and $M(\mathcal{A}) = \mathbb{R}^n \setminus \bigcup_{i=1}^{m} A_i$ be the complement of the subspace arrangement $\mathcal{A}$. The homology of $M(\mathcal{A})$ is well known and it is determined by the intersection poset of the arrangement $\mathcal{P}(\mathcal{A})$ (Goresky, MacPherson [6]). Furthermore, de Longueville and Schultz give the ring structure of the integral cohomology of $M(\mathcal{A})$ [3]. In general, the homotopy type of $M(\mathcal{A})$ is not determined by the intersection poset of arrangements (Ziegler [20]).

Therefore, we focus on real space line arrangements and we find the homotopy (diffeomorphism) type of a complement of a real space line arrangement is determined by the intersection poset. More precisely, it is determined by the cardinality m and the number of multiple points $(p_2, \ldots, p_m)$ of the real space line arrangement. We obtain the following theorem:

Theorem 1.1. *(Theorem 5.4, Ishikawa, Oyama* [8]*) Let $\mathcal{B} = \{l_1, \ldots, l_m\}$ be a real space line arrangement in $\mathbb{R}^n$ of cardinality m. Let p_k denote the number of k-multiple points of $\mathcal{B}$. If $n \geq 3$, then $M(\mathcal{B})$ is diffeomorphic to the interior of the space obtained by attaching trivially $m + \sum_{k=2}^{m}(k-1)p_k$ pieces of n-dimensional $(n-2)$-handles to the n-dimensional closed ball.*

Let X be a topological space. X is called *minimal* if it is homotopy equivalent to a cell complex with as many i-cells as its i-th Betti number, for each $i \geq 0$. It is known that complements of complex hyperplane arrangements are minimal by Dimca, Papadima [4], Randell [18]. However, according to the Björner [2], it is known that there exists a subspace arrangement such that complements are not minimal. On the other hand, minimality of the complements of subspace arrangements in special cases has been studied by Mori, Salvetti [13] and Adiprasito [1]. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a hyperplane arrangement in $\mathbb{R}^n$ and d be a positive integer. The new subspace arrangement $\mathcal{A}^{(d)}$ in $(\mathbb{R}^n)^d$ consists of the subspace $H_i^{(d)} = \{(x^{(1)}, \ldots, x^{(d)}) \in (\mathbb{R}^n)^d | a_i \cdot x^{(1)} = 0, \ldots, a_i \cdot x^{(d)} = 0\}$ in $\mathcal{A}^{(d)}$ for any $H_i = \{x \in \mathbb{R}^n | a_i \cdot x = 0\} \in \mathcal{A}$, where $\cdot$ is the Euclidean inner product in $\mathbb{R}^n$. Mori and Salvetti showed the complement of $\mathcal{A}^{(d)}$ is minimal [13]. Let $c \geq 1$. A c-arrangement is a finite collection of distinct affine subspace of $\mathbb{R}^n$, all of codimension c, with the property that the codimension of the non-empty intersection of any subset of it is a multiple of

c. Adiprasito showed complements of *essential* c-arrangements are minimal [1]. Note that $\mathcal{A}$ is called essential subspace arrangement in $\mathbb{R}^n$ if there exist 0-dimensional intersections.

By Theorem 1.1, we find complements of real space line arrangements are minimal (Corollary 4.4).

Graph embeddings are generalizations of knots and have been studied by many researchers. In particular, graph linear embedding has application to the polymer chemistry ([15]). Furthermore, we can regard the unions of real space line arrangements as linearly embedded *graphs with half-lines*.

In this thesis, we studied differomorphism types for complements to linear embeddings of graphs with half-lines. Let V be an arbitrary set and it is called the set of vertices. Assume that $P(V,1) := \{\{u\} \mid u \in V\}$ and

$$P(V,2) := \{\{u,u\} \mid u \in V\} \sqcup \{\{u,v\} \mid u,v \in V, u \neq v\}.$$

Let E be a multiset of $P(V,2)$. It is called the set of edges. When $e \in E$ satisfies that there exists $\tilde{e} \in E \setminus \{e\}$ such that $e = \tilde{e}$, it is called a multiple edge. And when $e \in E$ satisfies that there exists $u \in V$ such that $e = \{u,u\}$, it is called a loop. Let E' be a multiset of $P(V,1)$. E' is called the set of half-lines. We call $G = (V,E,E')$ a graph with half-lines. A graph with half-lines $G = (V,E,E')$ is called *finite* if cardinalities of sets $V$and E, E' are finite respectively, and is called *simple* if (V,E) is a simple graph, and is called *connected* if a graph $\tilde{G} = (\tilde{V},\tilde{E})$ is a connected graph, where $\tilde{V} = V \cup \{v_\infty\}$ and $\tilde{E} = E \cup \{\{v,v_\infty\} | \{v\} \in E^{'}\}$ and $v_\infty \notin V$. Besides, let $G = (V,E,E')$ be a finite simple graph with half-lines. Let

$$\rho : V \to \mathbb{R}^n, \mu : E^{'} \to S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

be maps. Then, we call $f := (\rho,\mu)$ a *linear map* of a graph with half-lines G. It is denoted by $f : G \to \mathbb{R}^n$. And we define $f(G)$ as the union of $\rho(V)$ and $\bigcup_{\substack{e^{'} \in E^{'} \\ u \in e^{'}}} \mathrm{h}\ell(\rho(u),\mu(e^{'}))$ and $\bigcup_{\{u,v\} \in E} \overline{\rho(u),\rho(v)}$, where $a,c \in \mathbb{R}^n, b \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathrm{h}\ell(a,b) := \{a + sb \in \mathbb{R}^n \mid s \geq 0\}, \overline{a,c} := \{sa + (1-s)c \in \mathbb{R}^n \mid 0 \leq s \leq 1\}$. Let $f := (\rho,\mu)$ be a linear map of a graph with half-lines G. It is called a *linear embedding* if ρ is an injection and any two distinct elements of

$$\{\mathrm{h}\ell(\mu(u),\mu(e^{'})) \in \mathbb{R}^n \mid e^{'} \in E^{'}, u \in e^{'}\} \cup \{\overline{\rho(u),\rho(v)} \in \mathbb{R}^n \mid \{u,v\} \in E\}$$

intersect only at the common vertex. Let $G = (V,E,E')$ be a finite graph with half-lines. We define

$$\chi(G) := \mathrm{card}(V) - \mathrm{card}(E) - \mathrm{card}(E'),$$

where $\mathrm{card}(A)$ is the cardinality of the finite set A. Note that if G does not have half-lines, $\chi(G)$ is equal to the Euler characteristic of the graph G. We obtain the following theorem:

Theorem 1.2. *(Theorem 6.19) Let $G = (V, E, E')$ be a finite simple connected graph with half-lines. Let $f : G \to \mathbb{R}^n$ be a linear embedding and $n \geq 4$. Then, $M(G, f) = \mathbb{R}^n \setminus f(G)$ is diffeomorphic to the interior of the space obtained by attaching trivially $-\chi(G)$ pieces of n-dimensional $(n-2)$-handles to the n-dimensional closed ball.*

We obtain the following theorem (see Definition 6.12):

Theorem 1.3. *(Theorem 6.20) Let $G = (V, E, E')$ be a finite simple connected graph with half-lines. Let $f : G \to \mathbb{R}^3$ be a linear embedding. If there exists a linear embedding which has a complete ascending direction and is linear isotopic to f, then $\mathbb{R}^3 \setminus f(G)$ is diffeomorphic to the interior of the handle body which has genus $-\chi(G)$.*

Let $G = (V, E, E')$ be a finite graph with half-lines. Then, G is regarded as the topological space which consists the following way:

(1) First, any $u \in V$ is regarded as a 0-cell and V is regarded as a discrete points of which the number is equal to cardinallty of V. This topological space is denoted by G_0.
(2) Second, elements of E are regarded as a 1-cells and any edges are attached to G_0. This 1-dimensional cell complex is denoted by G_1.
(3) Finally, elements of $E^{'}$ are regareded as half-open intervals and any half-lines are attached to G_1. This space is regarded as a graph G, when we consider a graph embedding.

Note that when G does not have half-lines, G is regarded as a cell complex. Let M be a topological manifold. A continuous map $f : G \to M$ is called *embedding* if f is a topological embedding map.

Remark 1.4. Let $G = (V, E, E')$ be a finite simple graph with half-lines and $f := (\rho, \mu)$ be a linear embedding of a graph with half-lines G, where

$$\rho : V \to \mathbb{R}^n, \mu : E^{'} \to S^{n-1}.$$

Then, it is obvious that there exists an embedding $g : G \to \mathbb{R}^n$ such that $g(G) = f(G)$.

Furthermore, we study the existense of a graph embedding such that the fundamental group of complement is free group. In embedding in the 3-sphere S^3, there are previous works by Kobayashi [9], Endo, Otsuki [5]. Kobayashi [9], Endo, Otsuki [5] proved following properties with respect to *locally unknotted* graph embedding.

Definition 1.5. (Kobayashi [9]) Let G be a finite simple connected graph which does not have vertices with degree 1 and cut edges. Let $f : G \to S^3$ be an embedding. A spatial graph $f(G)$ is a locally unknotted if there are a base $\{x_1, \ldots, x_\gamma\}$ of $H_1(G : \mathbb{Z})$ and a map $\psi : \bigcup_{i=1}^{\gamma} D_i^2 \to S^3$ such that

(1) $\psi(\partial D_i^2) = C_i$, where C_i is a representation curve of x_i in $f(G)$ for $i = 1, 2, \ldots, \gamma$.
(2) $\psi(\bigcup_{i=1}^{\gamma} D_i^2) = f(G)$.
(3) $\psi|_{D_i^2}$ is an embedding for $i = 1, 2, \ldots, \gamma$.
(4) $\psi(\mathrm{int}(D_i^2)) \cap \psi(\mathrm{int}(D_j^2)) = \emptyset$ for $i \neq j$.
(5) $\psi(D_i^2) \cap f(G) = \psi(\partial D_i^2) \cap f(G) = C_i$ for $i = 1, 2, \ldots, \gamma$.

Theorem 1.6. *(Kobayashi* [9], [10]*) Assume that the graph G is finite, connected, simple and does not have vertices with degree 1 and cut edges. Let $f : G \to S^3$ is a embedding. If a spatial graph $f(G)$ is locally unknotted then the fundamental group of $S^3 \setminus f(G)$ is free group.*

Theorem 1.7. *(Endo, Otsuki* [5]*, Kobayashi* [10]*) Assume that the graph G is finite, connected, simple and does not have vertices with degree 1 and cut edges. Then any graph G has a graph embedding $f : G \to S^3$ such that $f(G)$ satisfies a locally unknotted spatial graph.*

By Theorem 1.6, 1.7, when we assume that the graph G is finite, connected, simple and does not have vertices with degree 1 and cut edges, it is obvious that an existense of a graph embedding such that the fundamental group of complement is free group. In embedding in $\mathbb{R}^3$, Huh, Lee proved the following theorem (see Remark 6.17):

Theorem 1.8. *(Huh, Lee* [7]*) If a linear embedding of a simple graph in $\mathbb{R}^3$ has a descending direction, then the fundamental group of the complement of this embedded graph is a free group.*

In embedding in S^3, we assume only finite connected graphs and give an another proof without going through a locally unknotted graph embedding.

Theorem 1.9. *(Theorem 6.23) If $G = (V, E)$ is a finite connected graph, then there exists graph embedding $f : G \to S^3$ suth that $S^3 \setminus f(G)$ is diffeomorphic to the interior of the handle body which has genus $1 - \chi(G)$, where $\chi(G)$ is the Euler characteristic of graphs.*

By Theorem 1.9, we obtain the following corollary:

Corollary 1.10. *(Corollary 6.24) If $G = (V, E)$ is a finite connected graph, then there exists a graph embedding $f : G \to S^3$ such that the fundamental group of $S^3 \setminus f(G)$ is a free group.*

In embedding in $\mathbb{R}^3$, we prove any finite connected simple graph has a linear embedding which has a descending direction.

Theorem 1.11. *(Theorem 6.25) If $G = (V, E)$ is a finite connected simple graph, then there exists a linear embedding $f : G \to \mathbb{R}^3$ which has a descending direction.*

By Theorem 1.8, 1.11, we obtain the following corollary:

Corollary 1.12. *(Corollary 6.26) If $G = (V, E)$ is a finite connected (simple) graph, then there exists a (linear) embedding $f : G \to \mathbb{R}^3$ such that the fundamental group of $\mathbb{R}^3 \setminus f(G)$ is a free group.*

2. Homology groups of complements of real space line arrangements in $\mathbb{R}^n$

2.1. **Homology groups of complements of subspace arrangements in $\mathbb{R}^n$.**

Definition 2.1. If $\mathcal{A} = \{A_1, \dots, A_m\}$ is a set of affine subspaces in the n-dimensional affine space $\mathbb{R}^n$ then it is called a *subspace arrangement* in $\mathbb{R}^n$.

Suppose that $A_1, \dots, A_m$ are distinct. The complement of the union of these subspaces is denoted by $M(\mathcal{A}) = \mathbb{R}^n \setminus \bigcup_{i=1}^{m} A_i$. Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a subspace arrengement in $\mathbb{R}^n$. Associated to this collection $\mathcal{A}$ of subspaces there is a partially ordered set $\mathcal{P}(\mathcal{A})$ whose element v corresponds to an affine subspace $v = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r} \neq \emptyset$ partially ordered by inclusion, with one maximal element T corresponding to the ambient space $\mathbb{R}^n$. We shall use the notation $v < w$ if v and w are distinct elements of $\mathcal{P}(\mathcal{A})$ such that v is contained in w. We define the *ranking function* d, whose value on the v is the dimension $d(v) = \dim_{\mathbb{R}}(v)$.

For any partially ordered set $\mathcal{P}$, we may consider the order complex $K(\mathcal{P})$. This is a simplicial complex with one vertex for each element $v \in \mathcal{P}$ and one k-simplex for each chain $v_0 < v_1 < \cdots < v_{k-1} < v_k$ of elements of $\mathcal{P}$. We define the following subsets

$$\mathcal{P}_{(v,w)} = \{x \in \mathcal{P} | v < x < w\}, \quad \mathcal{P}_{>v} = \{x \in \mathcal{P} | v < x\}$$

of $\mathcal{P}$. Let $C_p(K(\mathcal{P}))$ be the free abelian group generated by all the p-simplices of $K(\mathcal{P})$. We also define the cochain group $C^*(K(\mathcal{P}))$ and the coboundary map δ of $K(\mathcal{P})$ as follows:

$$C^*(K(\mathcal{P})) = \bigoplus_{p \in \mathbb{Z}} C^p(K(\mathcal{P})),$$

where $C^p(K(\mathcal{P}))$ is $Hom(C^p(K(\mathcal{P})), \mathbb{Z})$. The coboundary map $\delta : C^p(K(\mathcal{P})) \to C^{p+1}(K(\mathcal{P}))$ is defined by

$$\delta([v_0, \dots v_p]^*) := \sum_{v_0 < \cdots < v_{j-1} < w < v_j < \cdots < v_p} (-1)^j [v_0, \dots, v_{j-1}, w, v_j, \dots, v_p]^*,$$

where $[v_0, \dots v_p]^*$ is the dual basis to the basis $\{[v_0, \dots v_p]\} \subset C^p(K(\mathcal{P}))$ consisting of p-simplices of $K(\mathcal{P})$. The next theorem is introduced in [6].

Theorem 2.2. *(Goresky, MacPherson* [6]*) The homology of the complement $M(\mathcal{A})$ is given by*

$$H_i(M(\mathcal{A}); \mathbb{Z}) \cong \bigoplus_{v \in \mathcal{P}(\mathcal{A})} H^{n-d(v)-i-1}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)}); \mathbb{Z}),$$

where we make the convention that $H^{-1}(\emptyset, \emptyset; \mathbb{Z}) \cong \mathbb{Z}$ i.e. the top vertex $v = T$ contribute a copy of $\mathbb{Z}$ to the homology group $H_0(M(\mathcal{A}); \mathbb{Z})$.

2.2. **Homology groups of complements of real space line arrangements in $\mathbb{R}^n$.**

Definition 2.3. We call a subspace arrangement $\mathcal{A} = \{A_1, \dots, A_m\}$ a real space line arrangement if $\dim_{\mathbb{R}} A_i = 1$ $(i = 1, \dots, m)$.

Definition 2.4. Let $\mathcal{B} = \{l_1, \dots, l_m\}$ be a real space line arrangement in $\mathbb{R}^n$. Suppose $l_1, \dots, l_m$ are distinct. We call $x \in \bigcup_{1 \le i \le m} l_i$ a k-multiple point of $\mathcal{B}$ if the cardinality of $\{l_i \in \mathcal{B} | x \in l_i\}$ is equal to k.

Let $\mathcal{B} = \{l_1, \dots, l_m\}$ be a real space line arrangement of distinct m lines. Then we obtain the following theorem:

Theorem 2.5. *Let $\mathcal{B} = \{l_1, \dots, l_m\}$ be a real space line arrangement in $\mathbb{R}^n$ having p_k number of k-multiple points. Assume that $l_1, \dots, l_m$ are distinct. If $n \ge 3$, then the homology of the complement $M(\mathcal{B})$ is given by*

$$\begin{aligned} H_0(M(\mathcal{B}); \mathbb{Z}) &\cong \mathbb{Z}, \\ H_{n-2}(M(\mathcal{B}); \mathbb{Z}) &\cong \mathbb{Z}^{m + \sum_{k=2}^{m} (k-1) p_k}, \\ H_i(M(\mathcal{B}); \mathbb{Z}) &\cong 0, \quad (i \ne 0, n-2). \end{aligned}$$

If $n = 2$, Then

$$\begin{aligned} H_0(M(\mathcal{B}); \mathbb{Z}) &\cong \mathbb{Z}^{m+1+\sum_{k=2}^{m} (k-1) p_k}, \\ H_i(M(\mathcal{B}); \mathbb{Z}) &\cong 0 \quad (i \ne 0). \end{aligned}$$

Proof. Firstly, we consider the case $n \geq 3$. By the Theorem 2.2,

$$\begin{aligned} H_0(M(\mathcal{A});\mathbb{Z}) &\cong \bigoplus_{v\in\mathcal{P}(\mathcal{A})} H^{n-d(v)-1}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \\ &\cong H^{-1}(\emptyset,\emptyset;\mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

When $i = n-2$,

$$H_{n-2}(M(\mathcal{A});\mathbb{Z}) \cong \bigoplus_{v\in\mathcal{P}(\mathcal{A})} H^{1-d(v)}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}).$$

If $d(v) = 1$, then 0-cocycle is $\mathbb{Z}\overline{[T]^*}$. If $d(v) = 0$ and v is k-multiple point then 1-cocycle is $\bigoplus_{v\in l_{i_j}} \mathbb{Z}\overline{[l_{i_j}, T]^*}$ and 1-coboundary is

$$\delta(\overline{[T]^*}) = \overline{[l_{i_1}, T]^*} + \cdots + \overline{[l_{i_k}, T]^*} \quad (v \in l_{i_j}, 1 \leq j \leq k).$$

Therefore, $H^1(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \cong \mathbb{Z}^{k-1}$. From the above, $H_{n-2}(M(\mathcal{A});\mathbb{Z}) \cong \mathbb{Z}^{m+\sum_{k=2}^{m}(k-1)p_k}$. When $i \neq 0, n-2$, we have to consider

$$H_i(M(\mathcal{A});\mathbb{Z}) \cong \bigoplus_{v\in\mathcal{P}(\mathcal{A})} H^{n-d(v)-i-1}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}).$$

If $d(v) = 0$, then $H^{n-i-1}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \cong 0$. Because there is no $(n-i-1)$-cocycle. If $d(v) = 1$, then

$$H^{n-i-2}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \cong 0.$$

Clearly, if $v = T$, then $H^{-i-1}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \cong 0$. From the above $H_i(M(\mathcal{B});\mathbb{Z}) \cong 0$ $(i \neq 0, n-2)$.

Finally, we consider the case $n = 2$.

$$\begin{aligned} H_0(M(\mathcal{A});\mathbb{Z}) &\cong \bigoplus_{v\in\mathcal{P}(\mathcal{A})} H^{1-d(v)}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \\ &\cong \mathbb{Z}^{m+1+\sum_{k=2}^{m}(k-1)p_k}. \text{ (Since } H^{-1}(\emptyset,\emptyset;\mathbb{Z}) \cong \mathbb{Z}.) \end{aligned}$$

If $i \neq 0$,

$$H_i(M(\mathcal{A});\mathbb{Z}) \cong \bigoplus_{v\in\mathcal{P}(\mathcal{A})} H^{1-d(v)-i}(K(\mathcal{P}(\mathcal{A})_{>v}), K(\mathcal{P}(\mathcal{A})_{(v,T)});\mathbb{Z}) \cong 0.$$

Thus we have the theorem. □

3. WHITNEY STRATIFICATION AND STRATIFIED MORSE THEORY

We study the homotopy type of the complement of real space line arrangements in $\mathbb{R}^n$. We find that these have relation to the number of the multiple points and the dimension of real space. In order to prove our result, we use the Whitney stratification and stratified Morse theory.

3.1. $\mathcal{P}$-decomposition and Whitney stratification.

Definition 3.1. Let $\mathcal{P}$ denote a partially ordered set with order relation denoted by $<$. A $\mathcal{P}$-*decomposition* of a topological space Z is a locally finite collection of disjoint locally closed subsets called pieces, $S_i \subset Z$ (one for each $i \in \mathcal{P}$) such that

(1) $Z = \cup_{i\in\mathcal{P}} S_i$
(2) $S_i \cap \overline{S_j} \neq \emptyset \Leftrightarrow S_i \subset \overline{S_j} \Leftrightarrow i = j$ or $i < j$ and we write $S_i < S_j$.

Definition 3.2. Let Z be a closed subset of a smooth manifold M, and suppose that $Z = \bigcup_{i\in\mathcal{P}} S_i$ is a $\mathcal{P}$-decomposition of Z, where $\mathcal{P}$ is some partially ordered set. This decomposition and Z are respectively called a *Whitney stratification* of Z, a *Whitney stratified space* provided:

(1) Each pieces S_i is a locally closed smooth submanifold (which may or may not be connected) of M.
(2) Whenever $S_\alpha < S_\beta$, the pair (S_α, S_β) satisfies Whitney's conditions A and B: Suppose $x_i \in S_\beta$ is a sequence of points converging to some $y \in S_\alpha$. Suppose $y_i \in S_\alpha$ also converges to y, and suppose that (with respect to some local coordinate system on M) the secant lines $l_i = \overline{x_i y_i}$ converge to some limiting line l and $T_{x_i}S_\beta \to \tau$. Then
   (Whitney's condition A): $T_{x_i}S_\alpha \subset \tau$ and
   (Whitney's condition B): $l \subset \tau$.

Remark 3.3. By Mather [12], it is proved that (Whitney's condition B)⇒(Whitney's condition A).

Let $Z_1 = \bigcup_{i\in\mathcal{P}} S_i$ and $Z_2 = \bigcup_{i\in\mathcal{P}} S_i^{'}$ be Whitney stratified spaces. If a homeomorphism $f : Z_1 \to Z_2$ satisfies $f|_{S_i}$ is diffeomorphism and $f(S_i) = S_i^{'}$ for any $i \in \mathcal{P}$, then it is called a *stratum preserving homeomorphism*.

3.2. Morse function and Thom's first isotopy lemma with respect to Whitney stratification.

Suppose Z is a Whitney stratified space of a smooth manifold M. Let $\tilde{f} : M \to N$ be a smooth map such that

(1) $f = \tilde{f}|_Z$ is *proper*.
(2) for each stratum A of Z, the restriction $f|_A : A \to N$ is a submersion.

Such a map is called a *proper stratified submersion*. For each $t \in \mathbb{R}^n$, the set $Z \cap \tilde{f}^{-1}(t)$ is Whitney stratified by its intersection with the strata of Z.

Theorem 3.4. *(Thom's first isotopy lemma* [6], [11], [19]*) Let $\tilde{f} : M \to \mathbb{R}^n$ be a proper stratified submersion with respect to a Whitney stratified space $Z \subset M$. Then there is a stratum preserving homeomorphism,*

$$h : Z \to \mathbb{R}^n \times (Z \cap \tilde{f}^{-1}(0))$$

which is smooth on each stratum and commutes with the projection to $\mathbb{R}^n$. In particular the fibers of $f = \tilde{f}|_Z$ are homeomorpic by a stratum preserving homeomorphism.

Let Z be a Whitney stratified space of a smooth manifold M.

Definition 3.5. Suppose $p \in Z$. Let S be the stratum of Z which contains p. A *generalized tangent space* Q at the point p is any plane of the form

$$Q = \lim_{p_i \to p} T_{p_i} R$$

where $R \supset S$ is a stratum of Z and $p_i \in R$ is a sequence converging to p.

Definition 3.6. A *Morse function* $f : Z \to \mathbb{R}$ is the restriction of a smooth function $\tilde{f} : M \to \mathbb{R}$ such that

(1) $f = \tilde{f}|_Z$ is proper and the critical values of f are distinct.
(2) For each stratum S of Z, the critical points of $f|_S$ are nondegenerate (i.e., if $\dim(S) \geq 1$, the Hessian of $f|_S$ is nonzero at each critical point of $f|_S$).
(3) For every such critical point $p \in S$, and for each generalized tangent space Q at the point p, the following nondegeneracy condition holds: $d\tilde{f}(p)(Q) \neq 0$ except for $Q = T_p S$.

Suppose Z is a Whitney stratified space and $f : Z \to \mathbb{R}$ is a Morse function. Suppose $p \in Z$ is a critical point of $f|_S$ and $p \in S$.

Definition 3.7. The *Morse index* of $f : Z \to \mathbb{R}$ at the critical point $p \in Z$ with respect to this particular Whitney stratification of Z is defined by the number of negative eigenvalues of the Hessian matrix of f at the critical point p with respect to the stratum S of Z, where S contains the critical point p.

Example 3.8. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a subspace arrangement in $\mathbb{R}^n$. Let $\mathcal{P}$ denote the partially ordered set of the intersections of the affine spaces in $\mathbb{R}^n$ (see subsection 2.1). The arrangement $\mathcal{A}$ gives rise to a Whitney stratification of $\mathbb{R}^n$, with one stratum

$$S(v) = v \setminus \bigcup_{w<v} w$$

for each $v \in \mathcal{P}$.

In order to introduce the Morse function on the above example, we consider the squared distance function $f : \mathbb{R}^n \to \mathbb{R}$ from $q \in M(\mathcal{A})$ defined by $f(x) = \mathrm{distance}^2(q, x)$, where $\mathrm{distance}(y, z) = ||y - z||$.

Theorem 3.9. *([6], see also [16] for details) Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$. There exists point $q \in M(\mathcal{A})$ such that $f(x) = \mathrm{distance}^2(q, x)$ is a Morse function on $\mathbb{R}^n$ with respect to particular Whitney stratification of $\mathbb{R}^n$.*

3.3. **Upper halflinks.** Let Z be a Whitney stratified space of a Riemannian manifold M, let f be a function with a nondegenerate critical point $p \in Z$ and critical value $\alpha = f(p)$ and let S be the stratum of Z which contains p. Let $D^M_\delta(p)$ denote the closed disk of radius δ in M, which is centered at p. Suppose $0 < \varepsilon \ll \delta \ll 1$ are sufficiently small. Let N be a smooth submanifold of M containing p which is transverse to each stratum of Z and satisfies

$$\dim(S) + \dim(N) = \dim(M).$$

Definition 3.10. Suppose X is a union of strata of Z. The *upper halflink* of X at the point p (with respect to the function f) is the pair of spaces

$$(l^+_X, \partial l^+_X) = (N \cap X \cap D^M_\delta(p) \cap f^{-1}(\varepsilon+\alpha), N \cap X \cap \partial D^M_\delta(p) \cap f^{-1}(\varepsilon+\alpha)).$$

Furthermore, suppose $f : Z \to \mathbb{R}$ is a proper Morse function, and $[a, b] \subset \mathbb{R}$ is an interval which contains no critical values except for a single isolated critical value $v \in (a, b)$ which corresponds to a critical point p which lies in some stratum S of Z. Let λ be the Morse index of $f|_S$ at the point p.

Lemma 3.11. *(Goresky, MacPherson [6]) If $p \notin X$ then the space*

$$X_{\leq b} = \{x \in X | f(x) \leq b\}$$

has the homotopy type of a space obtained from $X_{\leq a}$ by attaching the pair

$$(D^\lambda \times l^+_X, (\partial D^\lambda \times l^+_X) \cup (D^\lambda \times \partial l^+_X)).$$

3.4. **Regular values.** Let $f : Z \to \mathbb{R}$ is a proper Morse function. Suppose X is a union of strata of Z.

Lemma 3.12. *(Goresky, MacPherson [6]) Suppose the interval $[a, b]$ contains no critical values of $f|_Z$. Then $X_{\leq a}$ is homeomorphic to $X_{\leq b}$.*

3.5. **Controlled vector fields.** Let M be a smooth manifold and E be a smooth vector bundle over M. Then E is called a *smooth inner product bundle*, if it has an inner product $\langle \cdot, \cdot \rangle_u$ on each fiber $E_u \subset E$, $u \in M$ and those inner products have the following property: if U is any open set in M and s_1, s_2 are two smooth sections of E above U then the mapping

$u \mapsto \langle s_1(u), s_2(u)\rangle_u$ is smooth. If $\pi : E \to M$ is an smooth inner product bundle over a smooth manifold, and $\tilde{\varepsilon}$ is a positive function on M, then the open $\tilde{\varepsilon}$-ball bundle $B_{\tilde{\varepsilon}}$ of E will be defined as the set of $e \in E$ such that $\|e\|_{\pi(e)} < \tilde{\varepsilon}(\pi(e))$, where $\|e\|_{\pi(e)}$ is defined as $\langle e, e\rangle_{\pi(e)}^{\frac{1}{2}}$.

Definition 3.13. Let $A \subset M$ be a submanifold. A *tubular neighborhood* T_A of A in M is a triple $(E, \tilde{\varepsilon}, \tilde{\varphi})$, where $\pi : E \to A$ is a smooth inner product bundle, $\tilde{\varepsilon}$ is a positive smooth function on A, and $\tilde{\varphi}$ is a diffeomorphism of $B_{\tilde{\varepsilon}}$ onto an open subset of M which commutes with the zero section ζ of E:

$$\begin{tikzcd} B_{\tilde{\varepsilon}} \arrow[dr, "\tilde{\varphi}"] & \\ A \arrow[u, "\zeta"] \arrow[r, "inclusion"'] & M. \end{tikzcd}$$

We set $|T_A| := \tilde{\varphi}(B_{\tilde{\varepsilon}})$. The map $\pi_A := \pi \circ \tilde{\varphi}^{-1} : |T_A| \to A$ is called the *projecton associated to* T_A. And we define the *tubular function associated to* T_A such that $\rho_A := \rho \circ \tilde{\varphi}^{-1} : |T_A| \to \mathbb{R}$, where $\rho(e) := \|e\|^2$ for all $e \in B_{\tilde{\varepsilon}}$.

Let M be a smooth manifold. Let $Z \subset M$ be a Whitney stratified space. Suppose that for each stratum A of Z we are given a tubular neighborhood T_A of A in M. Let $\pi_A : |T_A| \to A$ denote the projection associated to T_A and $\rho_A : |T_A| \to \mathbb{R}$ be the tubular function associated to T_A.

Definition 3.14. The family $\{T_A\}$ of neighborhoods will be called *control data* for Z if the following commutation relations are satisfied: if A and B are strata and $A < B$, then

$$\pi_A\pi_B(x) = \pi_A(x), \rho_A\pi_B(x) = \rho_A(x)$$

for any $x \in \{x \in |T_A| \cap |T_B| \mid \pi_B(x) \in |T_A|\}$.

Assume that M, P are smooth manifolds and $f : M \to P$ is a map. Let $Z \subset M$ be a Whitney stratified set. The family $\{T_A\}$ of tubular neighborhoods is called *compatible* with f if for any stratum $A \subset Z$ and any $x \in |T_A|$, we have $f\pi_A(x) = f(x)$.

Lemma 3.15. *(Mather* [12]*) If $f : M \to P$ is a smooth map and $f|_A$ is a submersion into P for each stratum $A \subset Z$, then there exists a family $\{T_A\}$ of control data for Z which is compatible with f.*

Assume that M are smooth manifold and $Z \subset M$ be a Whitney stratified set. Let T_A, π_A, ρ_A be a tubular neighborhood of A in Z and projection and tubular tunction associated to T_A, respectively, for any stratum $A \subset Z$. Let A, B be strata of Z. We define $T_{A,B} := |T_A| \cap B$, $\pi_{A,B} := \pi_A|_{T_{A,B}}$, $\rho_{A,B} := \rho_A|_{T_{A,B}}$.

Definition 3.16. By a *stratified vector field* η on Z, we mean a collection $\{\eta_A \mid A \text{ is a stratum of } Z\}$, where for each stratum A, we have that η_A is a smooth vector field on A.

Definition 3.17. A stratified vector field η on Z will be said to be *controlled vector field* if the following *control conditions* are satisfied: for any stratum A there exists a neighborhood $T_A^{'}$ of A in T_A such that for any second stratum $B > A$ and any $x \in |T_A^{'}| \cap B$, we have

$$\begin{aligned} \eta_B \rho_{A,B}(x) &= 0, \\ (\pi_{A,B})_* \eta_B(x) &= \eta_A(\pi_{A,B}(x)). \end{aligned}$$

Definition 3.18. Let P be a smooth manifold. A continuous map $f : Z \to P$ is called a *controlled submersion* if it is satisfies the following conditions:

(1) $f|_A : A \to P$ is a smooth submersion, for each stratum A of Z,
(2) For any stratum A, there is a neighborhood $T_A^{'}$ of A in T_A such that $f(x) = f\pi_A(x)$ for any $x \in |T_A^{'}|$.

Lemma 3.19. *If $f : Z \to P$ is a controlled submersion, then for any smooth vector field ξ on P, there is a controlled vector field η on Z such that $f_*\eta(x) = \xi(f(x))$ for any $x \in Z$.*

4. Homotopy type of complements of real space line arrangements in $\mathbb{R}^n$

Let $\mathcal{B} = \{l_1, ..., l_m\}$ be a real space line arrangement in $\mathbb{R}^n$ of cardinality m. Let p_k denote the number of k-multiple points of $\mathcal{B}$. We determine the homotopy type of $M(\mathcal{B})$.

Theorem 4.1. *If $n \geq 3$, then $M(\mathcal{B})$ is homotopy equivalent to the one point union of $m + \sum_{k=2}^{m}(k-1)p_k$ pieces of $(n-2)$-spheres,*

$$M(\mathcal{B}) \cong \bigvee^{m+\sum_{k=2}^{m}(k-1)p_k} S^{n-2}.$$

Proof. As Example 3.8, the real space line arrangement $\mathcal{B}$ in $\mathbb{R}^n$ gives rise to a Whitney stratification of $\mathbb{R}^n$, with one stratum

$$S(v) = v \setminus \bigcup_{w<v} w$$

for each $v \in \mathcal{P}(\mathcal{B})$. Let $X = M(\mathcal{B})$. By Theorem 3.9, we are able to take q in $M(\mathcal{B})$ such that $f(x) = \mathrm{distance}^2(q, x)$ is a proper Morse function on $\mathbb{R}^n$. We fix q in $M(\mathcal{B})$ and a single critical point p in some stratum $S(v) \subset v$ of the arrangement and set $\alpha = f(p)$. Let N be an affine subspace of $\mathbb{R}^n$

which meets v transversally at the point p and which is satisfies $\dim(v) + \dim(N) = n$. Choose $0 < \varepsilon \ll \delta \ll 1$ sufficiently small, i.e., first choose $\delta > 0$ so that the closed ball of radius δ, $D_\delta(p)$ intersects only those w for which $w \geq v$, and so that the boundary $\partial D_\delta(p)$ is transverse to $N \cap w$. Then choose $\varepsilon > 0$ so small that $f|_{(N \cap D_\delta(p))}$ has no critical values in the interval $[\alpha - \varepsilon, \alpha + \varepsilon]$, except for the single critical value α. The upper halflink of $M(\mathcal{B})$ and its boundary are defined by :

$$(l^+_{M(\mathcal{B})}, \partial l^+_{M(\mathcal{B})}) = (N \cap M(\mathcal{B}) \cap D_\delta(p) \cap f^{-1}(\varepsilon + \alpha), N \cap M(\mathcal{B}) \cap \partial D^M_\delta(p) \cap f^{-1}(\varepsilon + \alpha)).$$

Since the Morse index λ of $f|S(v)$ is 0, by Lemma 3.11, the space $M(\mathcal{B})_{\leq \alpha+\varepsilon}$ has the homotopy equivalent of a space obtained from $M(\mathcal{B})_{\leq \alpha - \varepsilon}$ by attaching the pair $(l^+_{M(\mathcal{B})}, \partial l^+_{M(\mathcal{B})})$. We consider following cases.

(1) When p is not multiple point, then $M(\mathcal{B})_{\leq \alpha+\varepsilon}$ is homotopy equivalent to a space $M(\mathcal{B})_{\leq \alpha-\varepsilon} \vee S^{n-2}$.

(2) When p is a k-multiple point, then $M(\mathcal{B})_{\leq \alpha+\varepsilon}$ is homotopy equivalent to a space $M(\mathcal{B})_{\leq \alpha-\varepsilon} \vee (\bigvee^{k-1} S^{n-2})$.

(3) When p is q in $S(\mathbb{R}^n)$, then $M(\mathcal{B})_{\leq \varepsilon}$ is homotopy equivalent to the one point space.

By (1), (2), (3), and Lemma 3.12, $M(\mathcal{B})$ is homotopy equivalent to an one point union of $m + \sum_{k=2}^{m} (k-1)p_k$ pieces of $(n-2)$-spheres. □

Theorem 4.2. *If $n = 2$, then $M(\mathcal{B})$ is homotopy equivalent to $m + 1 + \sum_{k=2}^{m} (k-1)p_k$ pieces of discrete points.*

Proof. As with proof of Theorem 3.9, we take q in $M(\mathcal{B})$ such that

$$f(x) = \text{distance}^2(q, x)$$

is a proper Morse function on $\mathbb{R}^2$. We fix q in $M(\mathcal{B})$ and a single critical point p in some stratum $S(v) \subset v$ of the arrangement and set $\alpha = f(p)$. Suppose $\varepsilon > 0$ is sufficiently small.

(1) When p is not multiple point, then the number of regions of $M(\mathcal{B})_{\leq \alpha+\varepsilon}$, $R(M(\mathcal{B})_{\leq \alpha+\varepsilon})$, is equal to $R(M(\mathcal{B})_{\leq \alpha-\varepsilon}) + 1$. Thus, $M(\mathcal{B})_{\leq \alpha+\varepsilon}$ is homotopy equivalent to a space $M(\mathcal{B})_{\leq \alpha-\varepsilon} \sqcup \{p_1\}$, where p_1 is a point.

(2) When p is a k-multiple point, then $R(M(\mathcal{B})_{\leq \alpha+\varepsilon})$ is equal to $R(M(\mathcal{B})_{\leq \alpha-\varepsilon}) + k - 1$. Thus $M(\mathcal{B})_{\leq \alpha+\varepsilon}$ is homotopy equivalent to a space $M(\mathcal{B})_{\leq \alpha-\varepsilon} \sqcup \{p_1, \ldots p_{k-1}\}$, where $\{p_1, \ldots p_{k-1}\}$ is a discrete set.

(3) When p is q in $S(\mathbb{R}^n)$, then $M(\mathcal{B})_{\leq \varepsilon}$ is homotopy equivalent to the one point space.

By (1), (2), (3), and Lemma 3.12, we have the theorem. □

In particular on the fundamental group of the complement of real space line arrangement in $\mathbb{R}^n$, Theorem 4.1 gives the following corollary:

Corollary 4.3. *Let $\mathcal{B} = \{l_1, ..., l_m\}$ be a real space line arrangement in $\mathbb{R}^3$ of cardinality m having p_k number of k-multiple points. The fundamental group $M(\mathcal{B})$ which is denoted by $\pi_1(M(\mathcal{B}))$ is isomorphic to $F_{m+\sum_{k=2}^m (k-1)p_k}$ which is the free group on a set of $m + \sum_{k=2}^m (k-1)p_k$ generators.*

Let X be a topological space. X is called *minimal* if it is homotopy equivalent to a cell complex with as many i-cells as its i-th Betti number, for each $i \geq 0$. Theorem 4.1 and Theorem 4.2 give the following corollary:

Corollary 4.4. *Let $\mathcal{B} = \{l_1, ..., l_m\}$ be a real space line arrangement in $\mathbb{R}^n$ and $n \geq 2$. Then $M(\mathcal{B})$ is minimal.*

5. Diffeomorphic types of complement of real space line arrangements

5.1. **Trivial handle attachments.** First we introduce *trivial handle attachments*.

Let $j < n$. Let $S^j \subset \mathbb{R}^n$ be the sphere defined by $x_1^2 + \cdots + x_j^2 + x_n^2 = 1, x_{j+1} = 0, \ldots, x_{n-1} = 0$, and $\partial(D^j) = S^{j-1} = S^j \cap \{x_n = 0\}$. Let $e_l \in \mathbb{R}^n$ be the vector defined by $(e_l)_i = \delta_{li}$. Then define an embedding $\tilde{\Phi} : D^{n-j} \times S^j \to \mathbb{R}^n$ by

$$\tilde{\Phi}(t_1, \ldots, t_{n-j-1}, t_{n-j}, x) := x + t_1 e_{n-1} + \cdots + t_{n-j-1} e_{j+1} + \frac{1}{2} t_{n-j} x,$$

which gives a tubular neighborhood of S^{j-1} in $\mathbb{R}^{n-1} = \{x_n = 0\}$, where D^{n-j} is the $(n-j)$-dimensional closed disk of which radius is 1. Set

$$\varphi_{st} := \tilde{\Phi}|_{D^{n-j} \times \partial(D^j)} : D^{n-j} \times S^{j-1} \to \mathbb{R}^{n-1} \subset \mathbb{R}^n,$$

which gives a tubular neighborhood of S^{j-1} in $\mathbb{R}^{n-1} = \{x_n = 0\}$. We call φ_{st} the *standard* attaching map of the n-dimensional handle of index j. Note that the embedding φ_{st} extends to the *standard* handle $\Phi : D^{n-j} \times D^j \to \mathbb{R}^n$, which is defined by

$$\begin{aligned}&\Phi(t_1, \ldots, t_{n-j-1}, t_{n-j}, x_1, \ldots, x_j)\\ &\qquad := \tilde{\Phi}(t_1, \ldots, t_{n-j-1}, t_{n-j}, x_1, \ldots, x_j, 0, \ldots, 0, \sqrt{1 - \sum\nolimits_{i=1}^j x_i^2})\end{aligned}$$

attached to $\{x_n \leq 0\}$ along φ_{st}.

Let M be a differentiable n-manifold with a *connected* boundary ∂M. Let $p \in \partial M$. A coordinate neighborhood $(U, \psi), \psi : U \to \psi(U) \subset \mathbb{R}^{n-1} \times \mathbb{R}$ around p in M is called *adapted* if $\psi : U \to \mathbb{R}^n$ is a homeomorphism of U and $\psi(U) \cap \{x_n \leq 0\}$ which maps $U \cap \partial M$ into $\mathbb{R}^{n-1} = \{x_n = 0\}$.

Now we consider an attaching of several handles of index j to M along ∂M. We call a handle attaching map $\varphi : \bigsqcup_{k=1}^{\ell} \left(D_k^{n-j} \times \partial(D_k^j)\right) \to \partial M$ *trivial* if there exist disjoint adapted coordinate neighborhoods $(U_1, \psi_1), \ldots, (U_\ell, \psi_\ell)$ on M such that $\varphi\left(D_k^{n-j} \times \partial(D_k^j)\right) \subset U_k$ and $\psi_k \circ \varphi : D_k^{n-j} \times \partial(D_k^j) \to \mathbb{R}^{n-1} \times \mathbb{R}$ is standard attachment for $k = 1, \ldots, \ell$.

Lemma 5.1. *Let $M^{'}$ be a differentiable n-manifold with connected boundary $\partial M^{'}$. Suppose $M^{'}$ is diffeomorphic to a space $M_1 := M \cup_\varphi \left(\sqcup_{k=1}^{\ell}(D_k^{n-j} \times D_k^j)\right)$ obtained, from a differentiable manifold M with connected boundary, by attaching ℓ number of trivial handles of index j. Then the space $M_2 := M^{'} \cup_{\varphi^{'}} \left(\sqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times D_k^j)\right)$ obtained from $M^{'}$ by attaching m number of trivial handles of index j is diffeomorphic to the space $M_3 := M \cup_{\varphi^{''}} \left(\sqcup_{k=1}^{\ell+m}(D_k^{n-j} \times D_k^j)\right)$ obtained from M by attaching $\ell + m$ number of trivial handles of index j.*

Proof. Let $f : M_1 \to M^{'}$ be a diffeomorphism. Then $f(\sqcup_{k=1}^{\ell}\left(D_k^{n-j} \times D_k^j)\right)$ is not contained in $\partial M^{'}$. Then we slide, up to isotopy, the attaching map $\varphi^{'} : \sqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M^{'}$ to $\varphi^{'''} : \sqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M^{'}$ such that

$$f\left(\varphi\left(\sqcup_{k=1}^{\ell}(D_k^{n-j} \times \partial D_k^j)\right)\right) \cap \varphi^{'''}\left(\sqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j)\right) = \emptyset.$$

Consider $\varphi^{''} := \varphi \sqcup f^{-1} \circ \varphi^{'''} : \sqcup_{k=1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M$. Then M_2 is diffeomorphic to M_3. $\square$

5.2. **Real affine space line arrangements.** Let $n \geq 2$. We consider real affine space line arrangements in $\mathbb{R}^n$ or more generally consider a subset X in $\mathbb{R}^n$ which is a union of finite number of closed line segments and half lines. Then X may be regarded as a finite graph (with compact and non-compact edges) embedded as a closed set in $\mathbb{R}^n$. Here we admit vertices of valency 1.

Take a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$ and define the height function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) := x \cdot v$ using the Euclidean inner product. Choose v so that

(1) v is neither perpendicular to any line segments nor half lines in X.
(2) For each $c \in \mathbb{R}$, the hyperplane $h(x) = c$ of level c contains at most one vertex of X.

Note that there exists a union Σ of finite number of great hyperspheres such that any unit vector in $S^{n-1} \setminus \Sigma$ satisfies the conditions (1) and (2).

After a rotation of $\mathbb{R}^n$, we may suppose $h(x) = x_n$. We write $x = (x^{'}, x_n)$, where $x^{'} = (x_1, \ldots, x_{n-1})$. Set $M = \mathbb{R}^n \setminus X$ and, for any $c \in \mathbb{R}$,

$$M_{\leq c} := \{x \in M | x_n \leq c\}, \quad M_{<c} := \{x \in M | x_n < c\}.$$

Let $V \subset X$ be the set of vertices of X. Set $V = \{u_1, u_2, \dots, u_r\}$, $c_i = h(u_i)$ and $C = h(V) = \{c_1, c_2, \dots, c_r\}$ with $c_1 < c_2 < \cdots < c_r$.

Lemma 5.2. *The homeomorphic type of $M_{\leq c}$ is constant on $c_i < c < c_{i+1}$ and the diffeomorphic type of $M_{<c}$ is constant on $c_i < c \leq c_{i+1}$, $i = 0, 1, \dots, r$, with $c_0 = -\infty$. Here $M_{<\infty}$ means M itself.*

Proof. First we treat the case $i < r$. Take a sufficiently large $R > 0$ such that

$$\{x = (x^{'}, x_n) \in X | c_i < x_n < c_{i+1}, \|x^{'}\| > R/2\} = \emptyset$$

Consider the cylinder

$$C := \{x \in \mathbb{R}^n | c_i < x_n < c_{i+1}, \|x^{'}\| \leq R\}.$$

Then $\mathscr{C} := \{\mathrm{int}\, C \setminus X, X \cap C, \partial C\}$ is a Whitney stratification of C. Since the function $h : C \to (c_i, c_{i+1})$ is proper and the restriction of h to each stratum is a submersion, by Lemma 3.15, we can take a control data for C which is compatible with h. Now we follow the standard method (the proof of Thom's first isotopy lemma) to show differentiable triviality of mappings. Note that the flow used in the proof of isotopy lemma is differentiable in each stratum. Assume that $c, c^{'} \in (c_i, c_{i+1}]$ satisfy $c_i < c < c^{'} \leq c_{i+1}$. For any $\varepsilon > 0$ which satisfies $c > c_i + \varepsilon$, take a smooth vector field η over (c_i, c_{i+1}) such that $\eta = 0$ on $(c_i, c_i + \varepsilon/2)$ and $\eta = \frac{\partial}{\partial y}$ on $(c_i + \varepsilon, c_{i+1})$, where y is the coordinate on $\mathbb{R}$. By Lemma 3.19, we take a controlled vector field ξ over C such that ξ tangents to each stratum and $h_* \xi(x) = \eta(h(x))$ for any $x \in C$. Suppose the retraction

$$\begin{array}{rccc} \pi : & \{x \in \mathbb{R}^n | c_i < x_n < c_{i+1}, \|x^{'}\| \geq R\} & \xrightarrow{\pi} & \partial C \\ & \cup & & \cup \\ & x = (x^{'}, x_n) & \longmapsto & (\frac{1}{\|x^{'}\|} R x^{'}, x_n). \end{array}$$

We take a vector field $\tilde{\xi}$ over $\{x \in \mathbb{R}^n | c_i + \varepsilon/2 < x_n < c_{i+1}, \|x^{'}\| \geq R\}$ such that $\tilde{\xi}$ satisfies $\pi_* \tilde{\xi}(x) = \xi(\pi(x))$ for any $x \in \{x \in \mathbb{R}^n | c_i + \varepsilon/2 < x_n < c_{i+1}, \|x^{'}\| \geq R\}$. We extend $\tilde{\xi}$ to $\{x \in \mathbb{R}^n | x_n < c_{i+1}\}$ such that $\tilde{\xi}(x) := \xi(x)$ for any $x \in \{x \in \mathbb{R}^n | c_i + \varepsilon/2 < x_n < c_{i+1}, \|x^{'}\| \leq R\}$ and $\tilde{\xi}(x) := 0$ for any $x \in \{x \in \mathbb{R}^n | x_n \leq c_i + \varepsilon/2\}$. By transforming $M_{\leq c}$ and $M_{<c}$ using the vector field $\tilde{\xi}$, we find $M_{<c}$ is diffeomorphic to $M_{<c'}$ and if $c^{'} \neq c_{i+1}$, $M_{\leq c}$ is homeomorphic to $M_{\leq c'}$.

Second we treat the case $i = r$. Consider the quadratic cone $\|x^{'}\|^2 - R x_n^2 = 0$ in $\mathbb{R}^n$. Supposing $c_r > 0$ after a translation along x_n-axis in necessary, and taking R sufficiently large, we have that $X \cap \{x \in \mathbb{R}^n | c_r < x_n\}$ lies inside of the cone $\|x^{'}\|^2 - R x_n^2 < 0$. Now set

$$D := \{x \in \mathbb{R}^n | c_r < x_n, \|x^{'}\|^2 - R x_n^2 \leq 0\},$$

and consider the proper map $h : D \to (c_r, \infty)$ with the Whitney stratification

$$\mathscr{D} := \{\operatorname{int} D \setminus X, X \cap D, \partial D\}.$$

Assume that $c, c' \in (c_r, c_{r+1}]$ satisfy $c_r < c < c' \leq c_{r+1}$. For any $\varepsilon > 0$ which satisfies $c > c_r + \varepsilon$, take a smooth vector field η over (c_r, ∞) such that $\eta = 0$ on $(c_r, c_r + \varepsilon/2)$ and $\eta = (1 + y^2)\partial/\partial y$ on $(c_r + \varepsilon, \infty)$. By Lemma 3.15, 3.19, we lift η to a controlled vector field ξ over D. Suppose the retraction

$$\begin{array}{ccc} \pi : \{x \in \mathbb{R}^n | c_r < x_n, \|x'\|^2 - Rx_n^2 \geq 0\} & \xrightarrow{\pi} & \partial D \\ \cup & & \cup \\ x = (x', x_n) & \longmapsto & (\frac{\|x_n\|\sqrt{R}}{\|x'\|}x', x_n). \end{array}$$

We take a vector field $\tilde{\xi}$ over $\{x \in \mathbb{R}^n \mid c_r + \varepsilon/2 < x_n, \|x'\|^2 - Rx_n^2 \geq 0\}$ such that $\tilde{\xi}$ satisfies $\pi_*\tilde{\xi}(x) = \xi(\pi(x))$ for any $x \in \{x \in \mathbb{R}^n \mid c_r + \varepsilon/2 < x_n, \|x'\|^2 - Rx_n^2 \geq 0\}$. We extend $\tilde{\xi}$ to $\mathbb{R}^n$ such that $\tilde{\xi}(x) := \xi(x)$ for any $x \in \{x \in \mathbb{R}^n | c_r + \varepsilon/2 < x_n, \|x'\|^2 - Rx_n^2 \leq 0\}$ and $\tilde{\xi}(x) := 0$ for any $x \in \{x \in \mathbb{R}^n | x_n \leq c_r + \varepsilon/2\}$. By transforming $M_{\leq c}$ and $M_{<c}$ using the vector field $\tilde{\xi}$, we find $M_{<c}$ is diffeomorphic to $M_{<c'}$ and if $c' \neq c_{r+1}$, $M_{\leq c}$ is homeomorphic to $M_{\leq c'}$. In particular we have that $M_{<c}$ for $c_{r+1} < c$ is diffeomorphic to M itself.

□

Lemma 5.3. *Let u be a vertex of X and let $c = h(u)$. Suppose $s = s(u)$ is the number of edges of X which are adjacent to u from above with respect to h. Suppose $t = t(u)$ is the number of edges of X which are adjacent to u from below with respect to h. Then, for a sufficiently small $\varepsilon > 0$, the open set $M_{<c+\varepsilon}$ is diffeomorphic to the interior of*

$$M_{\leq c-\varepsilon} \cup_\varphi \left(\bigsqcup_{i=1}^{s-1} (D_i^2 \times D_i^{n-2}) \right),$$

obtained by an attaching map

$$\varphi : \bigsqcup_{i=1}^{s-1} \left(D_i^2 \times \partial(D_i^{n-2})\right) \to h^{-1}(c - \varepsilon) \setminus X = \partial(M_{\leq c-\varepsilon}) \subset M_{\leq c-\varepsilon},$$

of $(s-1)$ number of trivial handles of index $n-2$, provided $s \geq 1$.

Proof. For sufficiently small $0 < \varepsilon < \varepsilon'$, $M_{<c-\varepsilon} \setminus M_{\leq c-\varepsilon'}$ is a space

$$\{x \in \mathbb{R}^n \mid c - \varepsilon' < h(x) < c - \varepsilon\}$$

deleted t-half-lines. We may suppose the intersection $X \cap h^{-1}(c - \varepsilon)$ lies on a line, up to a diffeomorphism of $M_{\leq c-\varepsilon}$. We delete t-small tubular neighborhoods of the half-lines from the half space, then still we have a diffeomorphic space to $M_{<c-\varepsilon} \setminus M_{\leq c-\varepsilon'}$. Then we connect the t-holes by boring a sequence of canals without changing the diffeomorphism type of complements. See Figures 7 and 2. The boring a canal means, in general dimension, to delete $D^1 \times D^{n-1}$ along the line segment connecting the holes.

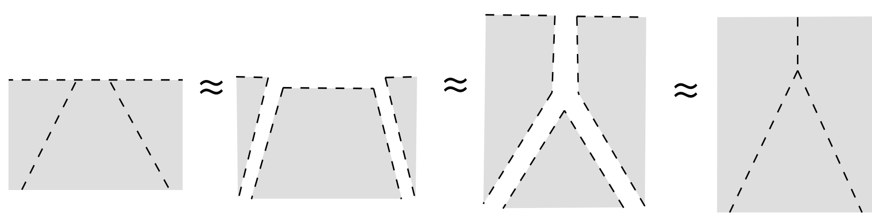

FIGURE 1. No topological changes of complements occur when $s = 1$.

FIGURE 2. Boring a canal does not change the topology of ground.

First let $s = 1$. Then the resulting space is diffeomorphic to $M_{<c+\varepsilon} \setminus M_{\leq c-\varepsilon'}$. The diffeomorphism is taken to be identity on $M_{\leq c-\varepsilon'}$ and it extends to a diffeomorphism between $M_{<c-\varepsilon}$ and $M_{<c+\varepsilon}$. This shows Lemma 5.3 in the case $s = 1$.

Next we treat the case $s = 2, t = 0$. The topological change from $M_{<c-\varepsilon}$ to $M_{<c+\varepsilon}$ is give by digging a tunnel, which is, equivalently, given by a handle attaching of index $n-2$. In fact, we examine the topological change of the complement to

$$\sqcup := \{(0, x_{n-1}, x_n) \in \mathbb{R}^n \mid (-2 \leq x_{n-1} \leq 2, x_n = 0) \\ \text{or } (x_{n-1} = -2, x_n \geq 0) \text{ or } (x_{n-1} = 2, x_n \geq 0)\},$$

in $\mathbb{R}^n$ when x_n goes across $x_n = c = 0$. Take the closed tube T of radius 1 of $\sqcup$. Then for the complement $M := \mathbb{R}^n \setminus T$, $M_{<\varepsilon}$ is diffeomorphic to the interior of the half space $\{x_n \leq 0\}$ attached the handle

$$H := \{x \in \mathbb{R}^n \mid -1 \leq x_{n-1} \leq 1, \frac{1}{2} \leq x_1^2 + \cdots + x_{n-2}^2 + x_n^2 \leq 2, x_n \geq 0\}$$

along

$$H \cap \{x_n \leq 0\} = \{x \in \mathbb{R}^n \mid -1 \leq x_{n-1} \leq 1, \frac{1}{2} \leq x_1^2 + \cdots + x_{n-2}^2 \leq 2\}.$$

The pair $(H, H \cap \{x_n \leq 0\})$ is diffeomorphic to the pair $(D^2 \times D^{n-2}, D^2 \times \partial D^{n-2})$, where the core $(0 \times D^{n-2}, \partial D^{n-2})$ correspond to

$$\{x_1^2 + \ldots x_{n-2}^2 + x_n^2 = 1, x_{n-1} = 0, x_n \geq 0\}$$

and

$$\{x_1^2 + \ldots x_{n-2}^2 = 1, x_{n-1} = 0, x_n = 0\}.$$

Note that the latter bounds an $(n-1)$-dimensional disk

$$\{x_1^2 + \cdots + x_{n-2}^2 \leq 1, x_{n-1} = 0, x_n = 0\},$$

which does not touch the boundary $\partial M_{<\varepsilon}$. See Figures 3 and 4.

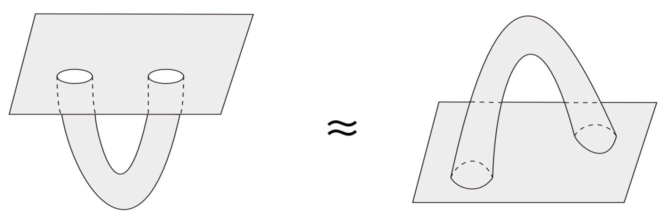

FIGURE 3. Digging a tunnel is same as bridging for the topology of ground.

The same argument works for any t. See Figure 4 for the case $s = 2, t = 2$. Note that complements to " X " and " H " are diffeomorphic. See Figures 4, 5 and 6.

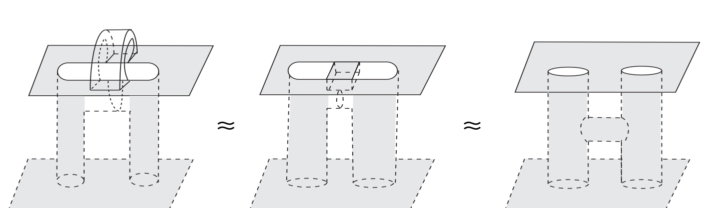

FIGURE 4. The case $s = 2, t = 2$.

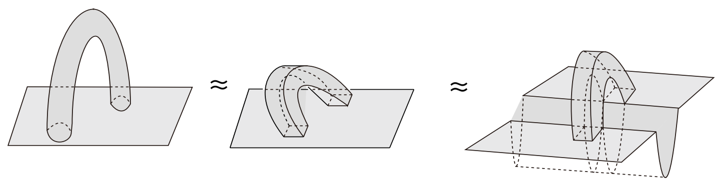

FIGURE 5. Trivial handle attachment and topological bifurcation.

In general, for any $s \geq 2$, the topological change is obtained by attaching trivial $s-1$ handles of index $n-2$. See Figure 6.

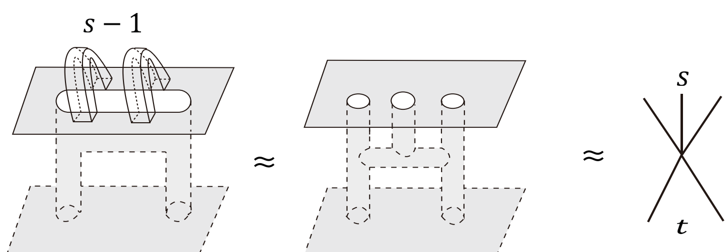

FIGURE 6. The case $s = 3, t = 2$.

When $n = 2$, the topological bifurcation occurs just as putting $s-1$ number of disjoint open disks. Thus we have Lemma 5.3. □

Theorem 5.4. *(Ishikawa, Oyama* [8]*) Let $\mathcal{B} = \{l_1, \ldots, l_m\}$ be a real space line arrangement in $\mathbb{R}^n$ of cardinality m. Let p_k denote the number of k-multiple points of $\mathcal{B}$. If $n \geq 3$, then $M(\mathcal{B})$ is diffeomorphic to the interior of the space obtained by attaching trivially $m + \sum_{k=2}^{m}(k-1)p_k$ pieces of n-dimensional $(n-2)$-handles to the n-dimensional closed ball.*

Proof. For a $c \in \mathbb{R}$ with $c \ll 0$, the space $M_{\leq c}$ (resp. $M< c$) is diffeomorphic to the half space $\{x_n \leq c\}$ (resp. $\{x_n < c\}$) deleted m number of half lines. By passing a multiple point of multiplicity k, for a sufficiently large c, the space $M_{\leq c}$ is obtained by attaching $k-1$ number of trivial handles of index $n-2$, by Lemma 5.3. After passing all multiple points, the space $M_{\leq c}$ is diffeomorphic to the space obtained by attaching $\sum_{i=2}^{m}(k-1)p_k$ number of trivial handles of index $n-2$ to the half space deleted m number of half lines. Then $M_{<c}$ is diffeomorphic to the interior of the space obtained by attaching trivially $m+\sum_{k=2}^{m}(k-1)p_k$ pieces of n-dimensional $(n-2)$-handles to the n-dimensional closed ball. By Lemma 5.2, for $c \in \mathbb{R}$ with $0 \ll c$, $M_{<c}$ is diffeomorphic to $M(\mathcal{B})$. Hence we have Theorem 5.4. $\square$

6. Diffeomorphism type of complements of linear embedding graphs with half-lines

Let V be an arbitrary set. It is called a set of vertices. Assume that $P(V,1) := \{\{u\} \mid u \in V\}$ and

$$P(V,2) := \{\{u,u\} \mid u \in V\} \sqcup \{\{u,v\} \mid u,v \in V, u \neq v\}.$$

Let E be a multiset of $P(V,2)$. It is called a set of edges. When $e \in E$ satisfies that there exists $\tilde{e} \in E \setminus \{e\}$ such that $e = \tilde{e}$, it is called a multiple edge. And when $e \in E$ satisfies that there exists $u \in V$ such that $e = \{u,u\}$, it is called a loop. Let E' be a multiset of $P(V,1)$. E' is called a set of half-lines. We call $G = (V,E,E')$ a graph with half-lines. A graph with half-lines $G = (V,E,E')$ is called *finite* if cardinalities of sets $V$and E, E' are finite respectively, and is called *simple* if (V,E) is a simple graph. A finite simple graph with half-lines $G = (V,E,E')$ is called *connected* if graph $\tilde{G} = (\tilde{V},\tilde{E})$ is a connected graph, where $\tilde{V} = V \cup \{v_\infty\}$ and $\tilde{E} = E \cup \{\{v, v_\infty\} | \{v\} \in E^{'}\}$ and $v_\infty \notin V$. Besides, let $G = (V,E,E')$ be a finite simple graph with half-lines. Let

$$\rho : V \to \mathbb{R}^n, \mu : E^{'} \to S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

be maps. Then, we call $f := (\rho,\mu)$ a *linear map* of a graph with half-lines G. It is denoted by $f : G \to \mathbb{R}^n$. And we define $f(G)$ as a union of $\rho(V)$ and $\bigcup_{\substack{e^{'} \in E^{'} \\ u \in e^{'}}} \mathrm{h}\ell(\rho(u),\mu(e^{'}))$ and $\bigcup_{\{u,v\}\in E} \overline{\rho(u),\rho(v)}$, where $a, c \in \mathbb{R}^n, b \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathrm{h}\ell(a,b) := \{a + sb \in \mathbb{R}^n \mid s \geq 0\}, \overline{a,c} := \{sa + (1-s)c \in \mathbb{R}^n \mid 0 \leq s \leq 1\}$. Let $f := (\rho,\mu)$ be a linear map of a graph with half-lines G. It is called a *linear embedding* if ρ is an injection and Any two distinct elements of

$$\{\mathrm{h}\ell(\mu(u),\mu(e^{'})) \in \mathbb{R}^n \mid e^{'} \in E^{'}, u \in e^{'}\} \cup \{\overline{\rho(u),\rho(v)} \in \mathbb{R}^n \mid \{u,v\} \in E\}$$

intersect only at the common vertex. Let $G = (V,E,E')$ be a finite graph with half-lines. We define

$$\chi(G) := \mathrm{card}(V) - \mathrm{card}(E) - \mathrm{card}(E'),$$

where $card(A)$ is the cardinality of set A. Note that if G does not have half-lines, $\chi(G)$ is equal to the Euler characteristic of the graph G.

Example 6.1. Let $n \geq 2$. Suppose $\mathcal{A}$ is a real space line arrangement in $\mathbb{R}^n$. Then there exist a finite simple connected graph with half-lines G and linear embedding $f : G \to \mathbb{R}^n$ which satisfy $f(G) = \bigcup_{\ell \in \mathcal{A}} \ell$.

Example 6.2. Let $n \geq 2$. If $\ell : \mathbb{R} \to \mathbb{R}^n$ be a embedding map which satisfies $\lim\limits_{x \to \pm\infty} \|\ell(x)\| = \infty$, then the image of ℓ is called a *pseudo-line* (similar to a long knot). A finite set of pseudo-lines $\mathcal{A}$ is a *pseudo-line arrangement* if the number of intersection two lines is at most 1 for any distinct two lines. Suppose $\mathcal{A}$ is pseudo-line arrangement in $\mathbb{R}^n$ such that each pseudo-line is consisting of a union of a finite number of line segments and half-lines. Then there exists a finite simple connected graph with half-lines G and linear embedding $f : G \to \mathbb{R}^n$ which satisfy $f(G) = \bigcup_{\ell \in \mathcal{A}} \ell$.

Example 6.3. Let $G = (V, E)$ be a finite simple connected graph. By removing an arbitrary vertex, We can decompose the finite simple connected graph with half-lines.

Note that linear map and linear embedding can be characterized by two maps $\rho : V \to \mathbb{R}^n$, $\mu : E^{'} \to S^{n-1} = \{x \in \mathbb{R}^n | \|x\| = 1\}$. Therefore we denote a linear map of a graph $f = (\rho, \mu)$. Assume $d := \mathrm{card}(V) \geq 1$ and $m := \mathrm{card}(E') \geq 0$. Assume that $V = \{v_1, \dots v_d\}$, $E^{'} = \{e_1^{'}, \dots, e_m^{'}\}$. Let $\mathrm{LM}(G, n)$ be the set of linear map with graph G from $\mathbb{R}^n$. Then, the following map

$$\begin{array}{ccc} \mathrm{LM}(G,n) & \xrightarrow{f} & (\mathbb{R}^n)^d \times (S^{n-1})^m \\ \cup & & \cup \\ f = (\rho, \mu) & \longmapsto & (\rho(v_1), \dots, \rho(v_d), \mu(e_1^{'}), \dots, \mu(e_m^{'})) \end{array}$$

is bijection. Therefore we can regard $\mathrm{LM}(G, n)$ as $(\mathbb{R}^n)^d \times (S^{n-1})^m$.

Definition 6.4. Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let f_0, f_1 be linear embedding from G to $\mathbb{R}^n$. Assume that these maps f_0, f_1 are denoted by $f_0 = (\rho_0, \mu_0)$, $f_1 = (\rho_1, \mu_1)$. Linear embedding maps f_0, f_1 are *linear isotopic* if those satisfy the the following property: There exists smooth maps

$$F : V \times \mathbb{R} \to \mathbb{R}^n$$
$$H : E^{'} \times \mathbb{R} \to S^{n-1}$$

such that the linear map $(F_t = F|_{V \times \{t\}}, H_t = |_{E' \times \{t\}})$ is linear embedding for any $t \in \mathbb{R}$ and the linear embedding (F_t, H_t) is equal to (ρ_t, μ_t), for any $t = 0, 1$.

This pair (F, H) is called a *linear isotopy*. Next, we prove that if two linear embedding maps are linear isotopic, then complements of these images are diffeomorphism by reference to Randell [17].

Lemma 6.5. *Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Suppose $E^{'}$ is non-empty set. Let $f_0 = (\rho_0, \mu_0)$, $f_1 = (\rho_1, \mu_1)$ linear embedding maps from G to $\mathbb{R}^n$. If f_0 and f_1 are linear isotopic then the space $\mathbb{R}^n \setminus f_0(G)$ is diffeomorphic to $\mathbb{R}^n \setminus f_1(G)$.*

Proof. We define a diffeomorphism $K : (\mathbb{R}^n \sqcup \{\infty\}) \times \mathbb{R} \to S^n \times \mathbb{R}$ such that

$$\begin{array}{ccc} K : (\mathbb{R}^n \sqcup \{\infty\}) \times \mathbb{R} & \xrightarrow{K} & S^n \times \mathbb{R} \\ \Uparrow & & \Uparrow \\ (x,t) & \longmapsto & (\kappa(x), t)\,, \end{array}$$

where $\kappa : \mathbb{R}^n \sqcup \{\infty\} \to S^n$ is a diffeomorphism. We consider the projection map from $S^n \times \mathbb{R}$ to $\mathbb{R}$. We denote this map $P_{\mathbb{R}}$. This map is proper since S^n is the compact. It is also a submersion. We will describe a Whitney stratification on the domain, constructed from the linear isotopy, so that the restriction of the projection is a submersion on each stratum. Then the results follow from Thom's first isotopy lemma. By the assumption, there exists a linear isotopy (F, H). We define maps $\tilde{F}_t = i_t \circ F_t$ and $\tilde{H}_t = i_t \circ H_t$, where F_t, F_t are restriction maps $F_t = F|_{V \times \{t\}}$, $H_t = |_{E' \times \{t\}}$ and i_t is inclusion map $x \mapsto (x, t)$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}$.

Let $v_\infty \notin V$. We define the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ suth as $\tilde{V} = V \sqcup \{v_\infty\}$ and $\tilde{E} = E \sqcup \left\{e^{'} \sqcup \{v_\infty\} | e^{'} \in E^{'}\right\}$. Let T be $T \notin \tilde{V} \cup \tilde{E}$. An order with respect to the set $\tilde{V} \cup \tilde{E} \cup \{T\}$ is defined by following method: If $v \in \tilde{V}$ and $e \in \tilde{E}$ satisfy $v \in e$, then we define an order $v < e$, where T is ordered by $u < T$ for any $u \in \tilde{V} \cup \tilde{E}$. This partially ordered set $\left(\tilde{V} \cup \tilde{E} \cup \{T\}, \leq\right)$ is denoted by $\mathcal{P}(\tilde{G})$.

Second, we construct $\mathcal{P}(\tilde{G})$-decomposition of $S^n \times \mathbb{R}$. For any $e \in E$, we define this space $S_e = \bigcup_{t \in \mathbb{R}} K\left(\left\{s\left(\tilde{F}_t(u)\right) + (1-s)\left(\tilde{F}_t(v)\right) | 0 < s < 1\right\}\right)$, where the edge e is equal to $\{u, v\}$. For any $e \in \{e^{'} \sqcup \{v_\infty\} | e^{'} \in E^{'}\}$, we define this space $S_e = \bigcup_{t \in \mathbb{R}} K(\{(\tilde{F}_t(u)) + s(\tilde{H}_t(e^{'})) | s > 0\})$, where the edge e is equal to $e^{'} \sqcup \{v_\infty\}$ and a half-line $e^{'}$ is equal to $\{u\}$. For any $v \in \tilde{V} \setminus \{v_\infty\}$, we define this space $S_v = \bigcup_{t \in \mathbb{R}} K(\{\tilde{F}_t(v)\})$. Forthermore, we define spaces $S_{v_\infty} = K(\{\infty\} \times \mathbb{R})$ and $S_T = S^n \times \mathbb{R} \setminus \bigcup_{u \in \tilde{V} \cup \tilde{E}} S_u$. It is obvious that $S_u \subset S^n \times \mathbb{R}$ is a locally closed smooth submanifold for any $u \in \mathcal{P}(\tilde{G})$. Furthermore, the family $\{S_u\}_{u \in \mathcal{P}(\tilde{G})}$ satisfies following conditions:

(1) $S^n \times \mathbb{R} = \bigcup_{u \in \mathcal{P}(\tilde{G})} S_u$

(2) $S_i \cap \mathrm{cl}(S_j) \neq \emptyset \Leftrightarrow S_i \subset \mathrm{cl}(S_j) \Leftrightarrow i = j \;\; or \;\; i < j$,

where $\mathrm{cl}(S_j)$ is the closure of $S_j \subset S^n \times \mathbb{R}$.

Third, we prove that $\{S_u\}_{u \in \mathcal{P}(\tilde{G})}$ satisfies the Whitney's condition B with the use of Randell's lemma.

Lemma 6.6. *(Randell* [17]*) Let x_i and y_i be sequences in M converging to a point $x \in M$ in a smooth manifold $M \subset \mathbb{R}^k$ so that $x_i \neq y_i$ and $\overline{x_i y_i}$ converges to ℓ. Then $\ell \in T_x M$.*

Since $\mathrm{cl}(S_T) = S^n \times \mathbb{R}$ is a smooth manifold, by the Lemma 6.6, if $u \in \mathcal{P}(\tilde{G}) \setminus \{T\}$, then (S_u, S_T) satisfy the Whitney's condition B. Furthermore, we consider the following case:

$$v \in \tilde{V} \text{ and } e \in \tilde{E} \text{ satisfy } S_v \subset \mathrm{cl}(S_e).$$

Then we define the the set $\mathscr{S}_e$ for any $e \in \tilde{E}$ in the following method: If $e \in E$

$$\mathscr{S}_e = \bigcup_{t \in \mathbb{R}} K(i_t(L(F_t(u), F_t(v) - F_t(u)))),$$

where $e = \{u, v\}$ and $L(x, y) \subset \mathbb{R}^n$ is a 1-dimensional affine subspace which has an element $x \in \mathbb{R}^n$ and has a direction vector $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. If $e \in \{e' \sqcup \{v_\infty\} | e' \in E'\}$

$$\mathscr{S}_e = \bigcup_{t \in \mathbb{R}} K(i_t(L(F_t(u), H_t(e')))),$$

where $e = e' \sqcup \{v_\infty\}$ and $e' = \{u\}$. Since $\mathrm{cl}(\mathscr{S}_e) \subset S^n \times \mathbb{R}$ is a smooth submanifold and $S_e \subset \mathscr{S}_e$ for any $e \in \tilde{E}$ and , by the Lemma 6.6, If $v \in \tilde{V}$ and $e \in \tilde{E}$ satisfy $S_v \subset \mathrm{cl}(S_e)$, then (S_v, S_e) satisfies the Whintey condition B. Therefore, we found this $\mathcal{P}(\tilde{G})$-decomposition $\{S_u\}_{u \in \mathcal{P}(G)}$ is a Whitney stratification of $S^n \times \mathbb{R}$.

It is obvious that the projection $P_{\mathbb{R}}$ is a submersion on each stratum. Since $\mathrm{card}(E') \geq 1$, $P_{\mathbb{R}}^{-1}(t) \cap S_T$ is diffeomorphic to $\mathbb{R}^n \setminus f_t(G)$ for $t \in \mathbb{R}$, where $\mathrm{card}(E')$ is a cardinality of a half-lines set E' and f_t is a graph embedding with respect to (F_t, H_t). Therefore, The theorem follows from Thom's first isotopy lemma (Theorem 3.4).

□

Next, in order to prove that if $n \geq 4$, then arbitrary two linear embedding maps are linear isotopic, we define the following definition. Let

$G = (V, E, E')$ be a finite simple graph with half-lines. A linear embedding $f = (\rho, \mu) : G \to \mathbb{R}^n$ is called *non-parallel linear embedding* if $\mu : E^{'} \to S^{n-1}$ is an injection.

Definition 6.7. Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let f_0, f_1 be non-parallel linear embedding from G to $\mathbb{R}^n$. Assume that these maps f_0, f_1 are denoted by $f_0 = (\rho_0, \mu_0)$, $f_1 = (\rho_1, \mu_1)$. Linear embedding maps f_0, f_1 are *non-parallel linear isotopic* if those satisfy the the following property: There exists smooth maps

$$F : V \times \mathbb{R} \to \mathbb{R}^n,$$
$$H : E^{'} \times \mathbb{R} \to S^{n-1}$$

such that the linear map $(F_t = F|_{V\times\{t\}}, H_t = |_{E^{'}\times\{t\}})$ is non-parallel linear embedding for any $t \in \mathbb{R}$ and the non-parallel linear embedding (F_t, H_t) is equal to (ρ_t, μ_t), for any $t = 0, 1$.

Lemma 6.8. *Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let $n \geq 4$. If f_0, f_1 be non-parallel linear embedding from G to $\mathbb{R}^n$, then f_0, f_1 are non-parallel linear isotopic.*

Proof. Let $V = \{v_1, \dots v_d\}$, $E^{'} = \{e^{'}_1, \dots, e^{'}_m\}$. We define the following subsets. When $i \neq j$, we define

$$Y_{v_i,v_j} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m | x_i = x_j\}.$$

When $\{v_i, v_j\} \cap \{v_k, v_\ell\} = \emptyset$, we define

$$Y_{\{v_i,v_j\}\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \neq x_j, x_k \neq x_\ell, \overline{x_i, x_j} \cap \overline{x_k, x_\ell} \neq \emptyset\},$$

where $\overline{x, y} = \{sx + (1-s)y \in \mathbb{R}^n | 0 \leq s \leq 1\}$. When $\{v_i, v_j\} \cap \{v_k, v_\ell\} \neq \emptyset$, we define

$$Y_{\{v_i,v_j\}\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \neq x_j, x_k \neq x_\ell, \operatorname{int} \overline{x_i, x_j} \cap \operatorname{int} \overline{x_k, x_\ell} \neq \emptyset\}.$$

When $\{v_j\} \cap \{v_k, v_\ell\} = \emptyset$ and $e^{'}_i = \{v_j\}$, we define

$$Y_{\{e^{'}_i,v_j\}\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_k \neq x_\ell, \mathrm{h}\ell(x_j, y_i) \cap \overline{x_k, x_\ell} \neq \emptyset\},$$

where $\mathrm{h}\ell(x, y) := \{x + sy \in \mathbb{R}^n | s \geq 0\}$. When $\{v_j\} \cap \{v_k, v_\ell\} \neq \emptyset$ and $e^{'}_i = \{v_j\}$, we define

$$Y_{\{e^{'}_i,v_j\}\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_k \neq x_\ell, \operatorname{int} \mathrm{h}\ell(x_j, y_i) \cap \operatorname{int} \overline{x_k, x_\ell} \neq \emptyset\}.$$

When $v_j \neq v_\ell$ and $e_i^{'} = \{v_j\}$, $e_k^{'} = \{v_\ell\}$, we define

$$Y_{\{e_i^{'},v_j\}\{e_k^{'},v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ \mid \mathrm{h}\ell(x_j, y_i) \cap \mathrm{h}\ell(x_\ell, y_k) \neq \emptyset\}.$$

When $v_j = v_\ell$ and $e_i^{'} = \{v_j\}$, $e_k^{'} = \{v_\ell\}$, we define

$$Y_{\{e_i^{'},v_j\}\{e_k^{'},v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ \mid \mathrm{int}\,\mathrm{h}\ell(x_j, y_i) \cap \mathrm{int}\,\mathrm{h}\ell(x_\ell, y_k) \neq \emptyset\}.$$

When $i \neq j$, we define

$$Y_{e_i^{'},e_j^{'}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \mid y_i = y_j\}.$$

Furthermore, we define

$$\begin{aligned} Y :=& \Big(\bigcup_{1\leq i<j\leq d} Y_{v_i,v_j}\Big) \cup \Big(\bigcup_{\substack{\{v_i,v_j\},\{v_k,v_\ell\}\in E,\\ \{v_i,v_j\}\neq\{v_k,v_\ell\}}} Y_{\{v_i,v_j\}\{v_k,v_\ell\}}\Big) \\ &\cup \Big(\bigcup_{\substack{e_i^{'}\in E^{'}, e_i^{'}=\{v_j\}\\ \{v_k,v_\ell\}\in E}} Y_{\{e_i^{'},v_j\}\{v_k,v_\ell\}}\Big) \\ &\cup \Big(\bigcup_{\substack{i\neq k\\ e_i^{'},e_k^{'}\in E^{'},\\ e_i^{'}=\{v_j\},e_k^{'}=\{v_\ell\}}} Y_{\{e_i^{'},v_j\}\{e_k^{'},v_\ell\}}\Big) \cup \Big(\bigcup_{1\leq i<j\leq m} Y_{e_i^{'},e_j^{'}}\Big). \end{aligned}$$

We denote the set of non-parallel linear embeddings from G to $\mathbb{R}^n$ by $\mathrm{NPLE}(G, n)$. Then we can regard $\mathrm{NPLE}(G, n)$ as $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y$ for the following bijection

$$\begin{array}{ccc} \mathrm{NPLE}(G, n) & \longrightarrow & (\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y \\ \uplus & & \uplus \\ f = (\rho, \mu) & \longmapsto & (\rho(v_1), \dots, \rho(v_d), \mu(e_1^{'}), \dots, \mu(e_m^{'})). \end{array}$$

In order to prove this lemma, it is enough to prove $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y$ is a connected smooth submanifold.

First, since $\mathrm{NPLE}(n, G)$ is a set of non-parallel linear embedding, it is obvious that $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is an open set.

Second, we prove $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y$ is connected. It is enough to proof $Y \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a finite disjoint union of submanifolds with boundary of which codimensions are 2 or more.

(1) When $i \neq j$, it is obvious that Y_{v_i,v_j}, $Y_{e_i^{'},e_j^{'}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ are submanifolds of which codimension are n and $n-1$, respectively.

(2) When $\{v_i, v_j\} \cap \{v_k, v_\ell\} = \emptyset$, it is obvious that $Y_{\{v_i,v_j\},\{v_k,v_\ell\}}$ is a disjoint union of the following two subsets

$$\begin{aligned} Y_{0,\{v_i,v_j\},\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \neq x_j, x_k \neq x_\ell, \overline{x_i, x_j} \cap \overline{x_k, x_\ell} \neq \emptyset, \\ L(x_i, x_j - x_i) \neq L(x_k, x_\ell - x_k)\}, \end{aligned}$$

$$\begin{aligned} Y_{1,\{v_i,v_j\},\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \neq x_j, x_k \neq x_\ell, \overline{x_i, x_j} \cap \overline{x_k, x_\ell} \neq \emptyset, \\ L(x_i, x_j - x_i) = L(x_k, x_\ell - x_k)\}, \end{aligned}$$

where $L(x, y) := \{x + sy \in \mathbb{R}^n | s \in \mathbb{R}\}$. The space $Y_{1,\{v_i,v_j\},\{v_k,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold with boundary of which codimension is $2n-2$. Because $Y_{1,\{v_i,v_j\},\{v_k,v_\ell\}}$ is a disjoint union of the following two sets:

$$\begin{aligned} \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n \setminus \{x_i\}, x_k \in \overline{x_i, x_j}, \\ x_\ell \in L(x_i, x_j - x_i) \setminus \{x_k\}\}, \end{aligned}$$

$$\begin{aligned} \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n \setminus \{x_i\}, \\ x_k \in L(x_i, x_j - x_i) \setminus \overline{x_i, x_j}, x_\ell \in \overline{x_i, x_j}\}. \end{aligned}$$

Furthermore, we find

$$Y_{0,\{v_i,v_j\},\{v_k,v_\ell\}} = Y_{0,n-2,\{v_i,v_j\},\{v_k,v_\ell\}} \sqcup Y_{0,n-1,\{v_i,v_j\},\{v_k,v_\ell\}},$$

where $Y_{0,n-2,\{v_i,v_j\},\{v_k,v_\ell\}}$ and $Y_{0,n-1,\{v_i,v_j\},\{v_k,v_\ell\}}$ are defined by

$$\begin{aligned} Y_{0,n-2,\{v_i,v_j\},\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n \setminus \{x_i\}, x_k \in \mathbb{R}^n \setminus L(x_i, x_j - x_i), \\ x_\ell \in x_k + \{u(x_j - x_i) + v(x_i - x_k) \in \mathbb{R}^n \\ | 1 \leq v, 0 \leq u \leq v\}\}, \end{aligned}$$

$$\begin{aligned} Y_{0,n-1,\{v_i,v_j\},\{v_k,v_\ell\}} := \{(x_1, \dots, x_d, y_1, \dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ | x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n \setminus \{x_i\}, x_k \in \overline{x_i, x_j}, \\ x_\ell \in \mathbb{R}^n \setminus L(x_i, x_j - x_i)\}. \end{aligned}$$

Since $Y_{0,n-2,\{v_i,v_j\},\{v_k,v_\ell\}}, Y_{0,n-1,\{v_i,v_j\},\{v_k,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ are submanifolds with boundary of which codimension are $n-2$ and $n-1$ respectively, the space $Y_{\{v_i,v_j\},\{v_k,v_\ell\}}$ is a disjoint union of submanifolds with boundary of which codimension $n-2$ and more.

(3) When $\{v_i, v_j\} \neq \{v_k, v_\ell\}$ and $\{v_i, v_j\} \cap \{v_k, v_\ell\} \neq \emptyset$, We may assume without loss of generality that $i = k$. Then we find

$$Y_{\{v_i,v_j\},\{v_i,v_\ell\}} = \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n \setminus \{x_i\}, x_\ell \in \operatorname{int} \mathrm{h}\ell(x_i, x_j - x_i)\},$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\mathrm{h}\ell(a, b) := \{a + sb \in \mathbb{R}^n | s \geq 0\}$. Therefore, we find $Y_{\{v_i,v_j\},\{v_i,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a smooth submanifold of codimension $n - 1$.

(4) When $\{v_j\} \cap \{v_k, v_\ell\} = \emptyset$ and $e_i^{'} = \{v_j\}$, it is obvious that $Y_{\{e_i^{'},v_j\},\{v_k,v_\ell\}}$ is a disjoint union of the following two subsets:

$$Y_{0,\{e_i',v_j\},\{v_k,v_\ell\}} := \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |x_k \neq x_\ell, \mathrm{h}\ell(x_j, y_i) \cap \overline{x_k, x_\ell} \neq \emptyset, L(x_j, y_i) \neq L(x_k, x_\ell - x_k)\}$$

and

$$Y_{1,\{e_i',v_j\},\{v_k,v_\ell\}} := \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |x_k \neq x_\ell, \mathrm{h}\ell(x_j, y_i) \cap \overline{x_k, x_\ell} \neq \emptyset, L(x_j, y_i) = L(x_k, x_\ell - x_k)\}.$$

We find $Y_{1,\{e_i',v_j\},\{v_k,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold with boundary of which codimension is $2n - 2$. Because $Y_{1,\{e_i',v_j\},\{v_k,v_\ell\}}$ is a disjoint union of the following two sets:

$$\{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, x_k \in \mathrm{h}\ell(x_j, y_i), \\ x_\ell \in L(x_j, y_i) \setminus \{x_k\}\},$$

$$\{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, \\ x_k \in L(x_j, y_i) \setminus \mathrm{h}\ell(x_j, y_i), x_\ell \in \mathrm{h}\ell(x_j, y_i)\}.$$

Furthermore, we find $Y_{0,\{e_i',v_j\},\{v_k,v_\ell\}}$ is a disjoint union of the following subsets:

$$Y_{0,n-2,\{e_i',v_j\},\{v_k,v_\ell\}} := \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, x_k \in \mathbb{R}^n \setminus L(x_j, y_i), \\ x_\ell \in x_k + \{u(x_j - x_k) + vy_i \in \mathbb{R}^n | u \geq 1, v \geq 0\}\},$$

$$Y_{0,n-1,\{e_i',v_j\},\{v_k,v_\ell\}} := \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, x_k \in \mathrm{h}\ell(x_j, y_i), x_\ell \in \mathbb{R}^n \setminus L(x_j, y_i)\}.$$

Since $Y_{0,n-2,\{e_i',v_j\},\{v_k,v_\ell\}}, Y_{0,n-1,\{e_i',v_j\},\{v_k,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ are submanifolds with boundary of which codimension are $n-2$ and $n-1$ respectively, the space $Y_{\{e_i',v_j\},\{v_k,v_\ell\}}$ is a finite disjoint union of submanifolds with boundary of which codimensions are $n-2$ and more.

(5) When $\{v_j\} \cap \{v_k, v_\ell\} \neq \emptyset$ and $e_i^{'} = \{v_j\}$, we may assume without loss of generality that $j = k$. Then we found

$$Y_{\{e_i',v_j\},\{v_j,v_\ell\}} := \{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, x_\ell \in \operatorname{int} \mathrm{h}\ell(x_j,y_i)\}.$$

Therefore, the space $Y_{\{e_i',v_j\},\{v_i,v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold of codimension $n-1$.

(6) When $j \neq \ell$ and $e_i^{'} = \{v_j\}$, $e_k^{'} = \{v_\ell\}$, it is obvious that $Y_{\{e_i',v_j\},\{e_k',v_\ell\}}$ is a disjoint union of the following subsets:

$$Y_{0,\{e_i',v_j\},\{e_k',v_\ell\}} := \{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |\, \mathrm{h}\ell(x_j,y_i) \cap \mathrm{h}\ell(x_\ell,y_k) \neq \emptyset, L(x_j,y_i) \neq L(x_\ell,y_k)\},$$

and

$$Y_{1,\{e_i',v_j\},\{e_k',v_\ell\}} := \{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |\, \mathrm{h}\ell(x_j,y_i) \cap \mathrm{h}\ell(x_\ell,y_k) \neq \emptyset, L(x_j,y_i) = L(x_\ell,y_k)\}.$$

We find $Y_{1,\{e_i',v_j\},\{e_k',v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold with boundary of codimension $2n-2$. Because $Y_{1,\{e_i',v_j\},\{e_k',v_\ell\}}$ is a disjoint union of the following two subsets:

$$\{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, y_k = \pm y_i, \\ x_\ell \in \mathrm{h}\ell(x_j,y_i)\},$$

$$\{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, \\ y_k = y_i, x_\ell \in L(x_j,y_i) \setminus \mathrm{h}\ell(x_j,y_i)\}.$$

Furthermore, we find $Y_{0,\{e_i',v_j\},\{e_k',v_\ell\}} \subset (\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold with boundary of which codimension is $n-2$. Because $Y_{0,\{e_i',v_j\},\{e_k',v_\ell\}}$ is equal to the following set:

$$\{(x_1,\dots,x_d,y_1,\dots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, y_k \in S^{n-1} \setminus \{\pm y_i\}, \\ x_\ell \in x_j + \{uy_i - vy_k \in \mathbb{R}^n | u \geq 0, v \geq 0\}\}.$$

Therefore, the space $Y_{\{e_i^{'},v_j\},\{e_k^{'},v_\ell\}}$ is a finite disjoint union of submanifolds with boundary of which codimensions are $n-2$ and more.

(7) When $i \neq k$ and $e_i^{'} = \{v_j\}$, $e_k^{'} = \{v_j\}$, it is obvious that

$$\begin{aligned} Y_{\{e_i^{'},v_j\},\{e_k^{'},v_j\}} = \{(x_1, \ldots, x_d, y_1, \ldots y_m) \in (\mathbb{R}^n)^d \times (S^{n-1})^m \\ |y_i \in S^{n-1}, x_j \in \mathbb{R}^n, y_k = y_i\}. \end{aligned}$$

Therefore, $Y_{\{e_i^{'},v_j\},\{e_k^{'},v_j\}}(\mathbb{R}^n)^d \times (S^{n-1})^m$ is a submanifold of codimension $n-1$.

From the above results, when $n \geq 4$, $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y$ is a connected smooth submanifold. Therefore, we obtain this lemma. $\square$

Remark 6.9. Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. We define a topology of the set of linear map from a graph with half-lines G to $\mathbb{R}^n$, $\mathrm{LM}(G, n)$ as the induced topology from the following bijective map

$$\begin{array}{ccc} \mathrm{LM}(G,n) & \xrightarrow{f} & (\mathbb{R}^n)^d \times (S^{n-1})^m \\ \cup & & \cup \\ f = (\rho, \mu) & \longmapsto & (\rho(v_1), \ldots, \rho(v_d), \mu(e_1^{'}), \ldots, \mu(e_m^{'})). \end{array}$$

Besides, the set of linear embedding from a graph with half-lines G to $\mathbb{R}^n$, $\mathrm{LE}(G, n)$ may be regarded as $(\mathbb{R}^n)^d \times (S^{n-1})^m \setminus Y^{'}$, where

$$Y^{'} := Y \setminus (\bigcup_{1 \leq i < j \leq m} Y_{e_i^{'},e_j^{'}}).$$

By using the same method as in the proof of Lemma 6.8, it is obvious that $Y^{'}$ is composed of a disjoint union of subsets of submanifolds with codimensions $n-2$ or more. As a result, If $n \geq 3$, then $\mathrm{LM}(G, n) \setminus \mathrm{LE}(G, n) \subset \mathrm{LM}(G, n)$ is a nowhere dense set.

Lemma 6.10. *Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let $n \geq 3$. Let $f = (\rho, \mu)$ be a linear embedding from G to $\mathbb{R}^n$. Then there exists some non-palallel linear embedding which satisfies linear isotopic to f.*

Proof. Suppose that $V = \{v_1, \ldots v_d\}$, $E^{'} = \{e_1^{'}, \ldots, e_m^{'}\}$. It is obvious that this theorem holds when f is a non-palallel linear embedding. So we shall assume that f is not a non-palallel linear embedding. Assume $e_i^{'} \in E^{'}$ satisfies $\mu(e_i^{'}) \in \mu(E^{'} \setminus \{e_i^{'}\})$ and $e_i^{'} = \{v_j\}$. We define $E_i := \{e \in E | e_i^{'} \subset e\}$ and $\tilde{G}_i := (V \setminus e_i^{'}, E \setminus E_i, \emptyset)$ and $\tilde{g}_i := (e_i^{'}, \emptyset, \{e_i^{'}\})$. It is satisfied with $f(\tilde{G}_i) \cap f(\tilde{g}_i) = \emptyset$. Suppose $0 \ll R$ is sufficiently large such that $f(\tilde{G}_i) \subset D_R^{\mathbb{R}^n}(\rho(v_j))$. Since $D_R^{\mathbb{R}^n}(\rho(v_j))$ is the normal space, there exists two open sets $U_1, U_2 \subset D_R^{\mathbb{R}^n}(\rho(v_j))$ such that $f(\tilde{G}_i) \subset U_1$ and $f(\tilde{g}_i) \cap D_R^{\mathbb{R}^n}(\rho(v_j)) \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Moreover, since $f(\tilde{g}_i) \cap D_R^{\mathbb{R}^n}(\rho(v_j))$

is compact, there exists an open neighborhood $\mu(e_i^{'}) \in U_{\tilde{G}_i} \subset S^{n-1}$ which satisfies $\mathrm{h}\ell(\rho(v_j), y) \cap f(\tilde{G}_i) = \emptyset$ for any $y \in U_{\tilde{G}_i}$. We define $E_i^{'} := \{e^{'} \in E^{'} | e^{'} = e_i^{'}$ as sets$\}$. Assume $e_k^{'} \in E^{'} \setminus E_i^{'}$ satisfies $e_k^{'} = \{v_\ell\}$. Then, $\mathrm{h}\ell(\rho(v_\ell), \mu(e_k^{'})) \cap \mathrm{h}\ell(\rho(v_j), y) \neq \emptyset$ if and only if $y \in \{\frac{s\mu(e_k^{'})+\rho(v_\ell)-\rho(v_j)}{\|s\mu(e_k^{'})+\rho(v_\ell)-\rho(v_j)\|} \in S^{n-1} | s \geq 0\} \subset S^{n-1}$. This subset

$$\{\frac{s\mu(e_k^{'})+\rho(v_\ell)-\rho(v_j)}{\|s\mu(e_k^{'})+\rho(v_\ell)-\rho(v_j)\|} \in S^{n-1} | s \geq 0\}$$

is denoted by P_k. When $\rho(v_j) \in L(\rho(v_\ell), \mu(e_k^{'})) \setminus \mathrm{h}\ell(\rho(v_\ell), \mu(e_k^{'}))$, it is obvious that $P_k = \{\mu(e_k^{'})\}$. When $\rho(v_j) \in \mathbb{R}^n \setminus L(\rho(v_\ell), \mu(e_k^{'}))$, it is obvious that P_k is an 1-dimensional connected submanifold with boundary and the closure of P_k is equal to $P_k \sqcup \{\mu(e_k^{'})\} \subset S^{n-1}$. Moreover, since S^{n-1} is the regular space, when $\mu(e_i^{'}) \neq \mu(e_k^{'})$, then there exists an open neighborhood $\mu(e_i^{'}) \in U_k \subset S^{n-1}$ which satisfies $\mathrm{h}\ell(\rho(v_\ell), \mu(e_k^{'})) \cap \mathrm{h}\ell(\rho(v_j), y) = \emptyset$ for any $y \in U_k$. We define

$$Q_i := \{\frac{\rho(v)-\rho(v_j)}{||\rho(v)-\rho(v_j)||} \in S^{n-1} | e_i^{'} \sqcup \{v\} \in E_i\} \cup (\mu(E_i^{'}) \setminus \{\mu(e_i^{'})\}).$$

Since any

$$P \in \{P_k \subset S^n \mid e_k^{'} \in \{e^{'} \in E^{'} \setminus E_i^{'} | \mu(e^{'}) = \mu(e_i^{'})\}\}$$

is an 1-dimensional connected submanifold with boundary and the closure of P is equal to $P \sqcup \{\mu(e_i^{'})\}$ and any elements

$$P, P^{'} \in \{P_k \subset S^n \mid e_k^{'} \in \{e^{'} \in E^{'} \setminus E_i^{'} | \mu(e^{'}) = \mu(e_i^{'})\}\}$$

satisfy only the following condition:

$$P \cap P^{'} = \emptyset \text{ or } P \subset P^{'} \text{ or } P^{'} \subset P,$$

it is obvious that

$$U_i := (U_{\tilde{G}_j} \cap (\bigcap_{e_k^{'} \in \{e^{'} \in E^{'} \setminus E_i^{'} | \mu(e^{'}) \neq \mu(e_i^{'})\}} U_k)) \setminus (Q_i \cup (\bigcup_{e_k^{'} \in \{e^{'} \in E^{'} \setminus E_i^{'} | \mu(e^{'}) = \mu(e_i^{'})\}} P_k))$$

is connected and has the element $\mu(e_i^{'})$. Since $U_i \setminus \{\mu(e_i^{'})\}$ is a submanifold of S^{n-1}, any point $y \in U_i$ has a smooth path from $\mu(e_i^{'})$ to y which is included by U_i. As the result, we found linear embeddings $\tilde{f} = (\tilde{\rho}, \tilde{\mu})$, $f = (\rho, \mu)$ are linear isotopic, where $\tilde{f} = (\tilde{\rho}, \tilde{\mu})$ satisfies $\tilde{\rho} = \rho$, $\tilde{\mu}(e^{'}) = \mu(e^{'})$ for any $e^{'} \in E^{'} \setminus \{e_i^{'}\}$ and $\tilde{\mu}(e_i^{'}) = y$, where $y \in U_i \setminus \{\mu(e_i^{'})\}$. Therefore, this lemma is proved by repeating the same method. □

By using Lemma 6.8, 6.10, the following lemma is proved.

Corollary 6.11. *Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let $n \geq 4$. If f_0, f_1 be linear embedding from G to $\mathbb{R}^n$, then f_0, f_1 are linear isotopic.*

Definition 6.12. Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let $n \geq 2$. Let f be a linear embedding from G to $\mathbb{R}^n$. A direction vector $u \in S^{n-1}$ is called a *complete ascending direction* of the linear embesdding $f : G \to \mathbb{R}^n$, if $u \in S^{n-1}$ satisfies the followig properties:

(i) u is neither perpendicular to any line segments nor half lines in $f(G)$.
(ii) For each $c \in \mathbb{R}$, the hyperplane $x \cdot u = c$ of level c contains at most one vertex of $f(G)$, where $\cdot$ is the Euclidean inner product.
(iii) For each $v \in V$, there exists $e \in E$ or $\{v\} \in E^{'}$ which satisfy $(\rho(w) - \rho(v)) \cdot u > 0$, $\mu(\{v\}) \cdot u > 0$, where $e = \{v, w\}$.

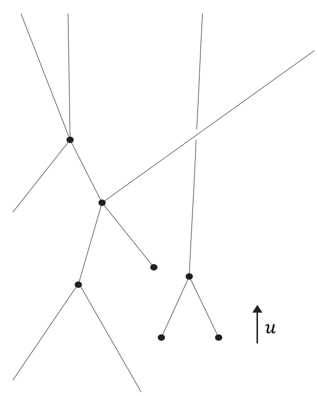

FIGURE 7. Linearly embedded graph with half-lines having a complete ascending direction u.

Example 6.13. Let $n \geq 2$ and $\mathcal{A}$ be a real space line arrangement in $\mathbb{R}^n$. Assume a finite simple connected graph with half-lines G and a linear embedding $f : G \to \mathbb{R}^n$ satisfy $f(G) = \bigcup_{\ell \in \mathcal{A}} \ell$. Then this map has a complete ascending direction.

Example 6.14. Let $s \geq 1$ and $K_s = (V, E)$ be a complete graph with s vertices. By removing a vertex $v \in V$, we obtain the finite simple connected graph with half-lines $\tilde{K}_s = (\tilde{V}, \tilde{E}, \tilde{E}^{'})$, where $\tilde{V} := V \setminus \{v\}$ and $\tilde{E} := \{e \in E | v \notin e\}$, $\tilde{E}^{'} := \{e \setminus \{v\} | e \in E, v \in e\}$. It is called a *complete graph with half-lines*. Let $s \geq 2$ and $n \geq 2$. Assume $\tilde{K}_s$ is a complete graph with half-lines and we give E's exements index such that $E^{'} = \{e^{'}_1, \ldots, e^{'}_{s-1}\}$. Suppose a linear embedding $f = (\rho, \mu) : \tilde{K}_s \to \mathbb{R}^n$ satisfies the following condition: if $1 \leq i < j \leq s-1$, then $\mu(e_i^{'}) \cdot \mu(e_j^{'}) > 0$. Then this map has a complete ascending direction.

Lemma 6.15. *Let $G = (V, E, E^{'})$ be a finite simple graph with half-lines. Let $n \geq 2$. If $G = (V, E, E^{'})$ is not a conneced graph with half-lines, then there does not exist a linear embedding which has a complete ascending direction.*

Proof. There uniquely exists a connected subgraph $G_1 = (V_1, E_1, E_1^{'})$ which satisfies the following property: If $\overline{G} = (\overline{V}, \overline{E}, \overline{E}^{'})$ is subgraph of G and $\operatorname{card} \overline{V} + \operatorname{card} \overline{E} + \operatorname{card} \overline{E}^{'} > \operatorname{card} V_1 + \operatorname{card} E_1 + \operatorname{card} E_1^{'}$, then $\overline{G}$ is not connected. We define $G_2 := (V \setminus V_1, E \setminus E_1, E^{'} \setminus E_1^{'})$. By the assumption, $V \setminus V_1 \neq \emptyset$ and $E^{'} \setminus E_1^{'} = \emptyset$, and $\{v\} \notin E^{'}$ for any $v \in V \setminus V_1$. Second, we assume $f : G \to \mathbb{R}^n$ is a linear embedding which has a complete ascending direction $u \in S^{n-1}$. Since $f(G_2)$ is a compact set, there exists a maximum values of $h|_{f(G_2)}$, where h is a continuous function which is defined by $h(x) := x \cdot u$. By conditions (i), (ii), a maximum point is determined by $f(v) \in f(V \setminus V_1)$, uniquely. Thus this contradicts the assumption. □

Lemma 6.16. *Let $G = (V, E, E^{'})$ be a finite simple connected graph with half-lines. Let $n \geq 3$. Then, there exists a linear embedding which has a complete ascending direction.*

Proof. Assume $v_\infty \notin V$ and $\tilde{G} = (V \sqcup \{v_\infty\}, E \sqcup \{\{v_\infty\} \sqcup e^{'} | e^{'} \in E^{'}\}$. Suppose $V_k \subset V$ is the set of vertices of which distanse from v_∞ with respect to $\tilde{G}$ is $k \geq 1$. Then we give V_k an index $\{v_{k_1}, v_{k_2}, \ldots, v_{k_{i_k - 1}}, v_{k_{i_k}}\}$. Suppose $d \geq 1$ is maximum value of distanse from v_∞. We take $\infty > c_{1_1} > c_{1_2} > \cdots > c_{1_{i_1 - 1}} > c_{1_{i_1}} > c_{2_1} > \cdots > c_{d_{i_d - 1}} > c_{d_{i_d}} > -\infty$, and $u \in S^{n-1}$. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a height function which is defined by $h(x) := x \cdot u$. We construct a linear map $f = (\rho, \mu)$ with respect to a graph G, by the following method: the map $\rho : V \to \mathbb{R}^n$ satisfies $\rho(v_j) \in h^{-1}(c_j)$, for any $j \in \{1_1, \ldots, d_{i_d}\}$ and $\mu : E^{'} \to \mathbb{R}^n$ satisfies $h(\mu(\{v_j\})) > 0$ for any $j \in \{1_1, \ldots, 1_{i_1}\}$. By the Remark 6.9, $LM(G, n) \setminus LE(G, n) \subset LM(G, n)$ is a nowhere dense set. Therefore, we obtain a linear embedding which has a complete ascending direction $u \in S^{n-1}$ by perturbing this linear map of the graph G. □

Remark 6.17. The complete ascending direction is similar concept to a *descending direction* defined in [7]. Let G be a finite simple connected graph. Let $f : G \to \mathbb{R}^n$ be a linear embedding. A unit vector $u \in S^{n-1}$ is called descending direction, if $u \in S^{n-1}$ satisfies the following properties:

(i) u is neither perpendicular to any line segments nor half lines in $f(G)$.

(ii) For each $v \in V$, except for only one, there exists $e \in E$ or $\{v\} \in E^{'}$ which satisfy $(\rho(w) - \rho(v)) \cdot u > 0$, $\mu(\{v\}) \cdot u > 0$, where $e = \{v, w\}$.

Huh, Lee proved the following theorem:

Theorem 6.18. *(Huh, Lee* [7]*) If a linear embedding of a simple graph into $\mathbb{R}^3$ has a descending direction, then the fundamental group of the complement of this embedded graph is a free group.*

By using Lemma 5.2, 5.3, 6.5, 6.16, Corollary 6.11, we obtain the following main theorem.

Theorem 6.19. *Let $G = (V, E, E')$ be a finite simple connected graph with half-lines. Let $n \geq 4$. If $f : G \to \mathbb{R}^n$ is a linear embedding, then $\mathbb{R}^n \setminus f(G)$ is diffeomorphic to the interior of the space obtained by attaching trivially $-\chi(G)$ pieces of n-dimensional $(n-2)$-handles to the n-dimensional closed ball, where $\chi(G) = \mathrm{card}(V) - \mathrm{card}(E) - \mathrm{card}(E')$ and $\mathrm{card}(A)$ means cardinality of set A.*

Proof. First, by using Corollary 6.11 and Lemma 6.16, there exists a linear embedding which has a ascending direction and is isotopic to f. Second, by using Lemma 5.2, 5.3, 6.5, we found $\mathbb{R}^n \setminus f(G)$ is diffeomorphic to the interior of the space trivially attached some n-dimensional $n-2$-handles on the n-dimensional closed ball. Finally, we consider how many this space has $n-2$-handles. By using Alexander duality theorem, we found the reduced homology of $\mathbb{R}^n \setminus f(G)$ is

$$\begin{cases} \tilde{H}_i(\mathbb{R}^n \setminus f(G); \mathbb{Z}) \simeq \mathbb{Z}^{-\chi(G)} & (i = n-2) \\ \tilde{H}_i(\mathbb{R}^n \setminus f(G); \mathbb{Z}) \simeq \mathbf{0}. \end{cases}$$

Therefore, we prove this main theorem. □

Theorem 6.18 is a theorem with respect to a sufficient condition for fundamental group of a complement of an embedded graph to be free group. We prove the following theorem which is similar to this theorem.

Theorem 6.20. *Let $G = (V, E, E')$ be a finite simple connected graph with half-lines. Let $f : G \to \mathbb{R}^3$ be a linear embedding. If there exists a linear embedding which has a complete ascending direction and is linear isotopic to f, then $\mathbb{R}^3 \setminus f(G)$ is diffeomorphic to the interior of the handle body which has genus $-\chi(G)$.*

Proof. By using Lemma 5.2, 5.3, 6.5, we obtain this theorem. □

Besides, Theorem 6.18 is a generalization of Nicholson's theorem.

Theorem 6.21. *(Nicholson* [14]*) Let K_s be the complete graph. Then fundamental group of complement of linear embedded of K_s is free group.*

In fact, it is clear that a linear embedding of a complete graph has a descending direction. However, in general, a linear embedding of a complete graph with half-lines may not have a linear isotopic map which has a complete ascending direction (see figure 8).

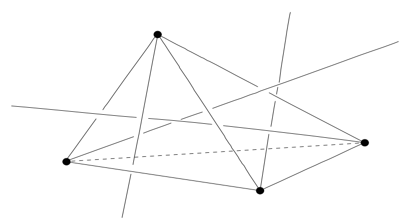

FIGURE 8. Linearly embedded complete graph with half-lines $\tilde{K}_5$.

By 1.6, 1.7, it is clear that we obtain the following corollary.

Corollary 6.22. *Assume that the graph G is finite, connected, simple and does not have vertices with degree 1 and cut edges. Then there exists a graph embedding $f : G \to S^3$ satisfies the following condition: the fundamental group of $S^3 \setminus f(G)$ is free group.*

We obtain a theorem that generalizes 6.22. Let G be a finite connected simple graph. Besides, we prove there exists a linear embedding of G which has a descending direction.

Theorem 6.23. *If $G = (V, E)$ is a finite connected graph, then there exists graph embedding $f : G \to S^3$ such that $S^3 \setminus f(G)$ is diffeomorphic to the interior of the handle body which has genus $1 - \chi(G)$,where $\chi(G)$ is the Euler characteristic of graphs.*

Proof. First, we construct a finite connected simple graph with half-lines from G. We do the following operations. When $e \in E$ is a multiple edge, we add to a one vertex and divide e into two edges. And when $e \in E$ is a loop, we add to two vertices and divide e into three edges. By performing this operation for all loops and multiple edges of G, we can construct the finite connected simple graph. This graph is denoted by $\tilde{G} = (\tilde{V}, \tilde{E})$. Let $v \in \tilde{V}$. We define $\overline{V} := \tilde{V} \setminus \{v\}$ and $\overline{E} := \{e \in \tilde{E} | v \notin e\}$, $\overline{E}' := \{e \setminus \{v\} | e \in \tilde{E}, v \in e\}$. It is clear that this graph $\overline{G} = (\overline{V}, \overline{E}, \overline{E}')$ is a finete connected simple graph with half-lines. By Lemma 6.16, Theorem 6.20, we find there exists linear embedding $f = (\rho, \mu) : \overline{G} \to \mathbb{R}^3$ which satisfies the following condition: $\mathbb{R}^3 \setminus f(\overline{G})$ is diffeomorphic to the interior of the handle body which has genus $-\chi(\overline{G})$, where $\chi(\overline{G}) = \mathrm{card}\,\overline{V} - \mathrm{card}\,\overline{E} - \mathrm{card}\,\overline{E}'$.

Let $f : \overline{G} \to \mathbb{R}^3$ be a linear embedding. As Remark 1.4, there exists a embedding $g : \overline{G} \to \mathbb{R}^3$ such that $g(\overline{G}) = f(\overline{G})$. Since we can regard $\overline{G}$ as $\tilde{G} \setminus \{v\}$, we define the map $\tilde{g} : \tilde{G} \to \mathbb{R}^3 \sqcup \{\infty\}$ such that $\tilde{g}|_{\tilde{G}\setminus\{v\}} := g$ and $\tilde{g}(v) := \infty$. Since there exist homeomorphisms $K : \mathbb{R}^3 \sqcup \{\infty\} \to S^3$ and $L : G \to \tilde{G}$, the map $K \circ \tilde{g} \circ L : G \to S^3$ is a embedding which satisfies $S^3 \setminus K \circ \tilde{g} \circ L(G)$ is homeomorphic to $R^3 \setminus f(\overline{G})$. Furthermore, since $\chi(\overline{G}) = \chi(G) - 1$, we obtain this theorem. $\square$

Corollary 6.24. *If $G = (V, E)$ is a finite connected graph, then there exists a graph embedding $f : G \to S^3$ such that the fundamental group of $S^3 \setminus f(G)$ is a free group.*

Furthermore, by the same proof method as Lemma 6.16, we can obtain the following theorem with respect to existence of linear embedding which has descending direction.

Theorem 6.25. *If $G = (V, E)$ is a finite connected simple graph, then there exists a linear embedding $f : G \to \mathbb{R}^3$ which has a descending direction.*

Proof. We take an arbitrary vertex $v \in V$ and fix this vertex. Suppose $V_k \subset V$ is the set of vertices of which distanse from v with respect to $\tilde{G}$ is $k \geq 0$. Then we give V_k an index $\{v_{k_1}, v_{k_2}, \dots, v_{k_{i_k-1}}, v_{k_{i_k}}\}$. Suppose $d \geq 0$ is maximum value of distanse from v. We take $-\infty < c_{0_1} < c_{1_1} < c_{1_2} < \cdots < c_{1_{i_1-1}} < c_{1_{i_1}} < c_{2_1} < \cdots < c_{d_{i_d-1}} < c_{d_{i_d}} < \infty$. and $u \in S^2$. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a height function which is defined by $h(x) := x \cdot u$. We construct a linear map $f = (\rho, \mu)$ with respect to a graph G, by the following method: the map $\rho : V \to \mathbb{R}^n$ satisfies $\rho(v_j) \in h^{-1}(c_j)$, for any $j \in \{0_1, 1_1, \dots, d_{i_d}\}$ and $\mu : \emptyset \to \mathbb{R}^n$. By the Remark 6.9, $LM(G, 3) \setminus LE(G, 3) \subset LM(G, 3)$ is a nowhere dense set. Therefore, we obtain a linear embedding which has a descending direction $u \in S^2$ by perturbing this linear map of the graph G. $\square$

By Theorem 6.18, 6.25, we obtain the following corollary.

Corollary 6.26. *If $G = (V, E)$ is a finite connected (simple) graph, then there exists a (linear) embedding $f : G \to \mathbb{R}^3$ such that the fundamental group of $\mathbb{R}^3 \setminus f(G)$ is a free group.*

Proof. First, we construct a finite connected simple graph with half-lines from G. We do the following operations. When $e \in E$ is a multiple edge, we add to a one vertex and divide e into two edges. And when $e \in E$ is a loop, we add to two vertices and divide e into three edges. By performing this operation for all loops and multiple edges of G, we can construct the finite connected simple graph. This graph is denoted by $\tilde{G} = (\tilde{V}, \tilde{E})$. By Theorem 6.25, there exists a linear embedding $f = (\rho, \mu) : \tilde{G} \to \mathbb{R}^3$ which has a descending direction. Since when $G = (V, E)$ is a finite connected simple graph, $\tilde{G}$ is equal to G, we obtain this theorem by using Theorem 6.18. Second, we consider the following case: $G = (V, E)$ is a finite connected graph. As Remark 1.4, there exists an embedding $g : \tilde{G} \to \mathbb{R}^3$ such that $g(\tilde{G}) = f(\tilde{G})$. Since there exists a homeomorphism $L : G \to \tilde{G}$, $g \circ L : G \to \mathbb{R}^3$ is an embedding which satisfies $g \circ L(G) = f(\tilde{G})$. Therefore, we obtain this theorem by using Theorem 6.18. $\square$

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