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# Feynman graphs and Hyperplane arrangements defined over $\mathbb{F}_1$

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## Abstract

Motivated by some computations of Feynman integrals and certain conjectures on mixed Tate motives, Bejleri and Marcolli posed questions about the  $\mathbb{F}_1$ -structure (in the sense of torification) on the complement of a hyperplane arrangement, especially for an arrangement defined in the space of cycles of a graph.

In this paper, we prove that an arrangement has an  $\mathbb{F}_1$ -structure if and only if it is Boolean. We also prove that the arrangement in the cycle space of a graph is Boolean if and only if the cycle space has a basis consisting of cycles such that any two of them do not share edges.

**Keywords:** Hyperplane arrangements, graphs, torifications.

## 1 Introduction

Theoretical physics, especially quantum field theory and Feynman integrals, raises many mathematical problems [1]. In [2], Bejleri and Marcolli studied certain algebraic varieties associated with Feynman integrals for graphs from the viewpoint of mixed Tate motives and Grothendieck ring of varieties. They also discuss the  $\mathbb{F}_1$ -structure (torification, see [3]) of these varieties, which is a more recent perspective.

The category of mixed Tate motives is conjectured to be generated by the objects defined by using hyperplane arrangements. Therefore, it is a natural question to ask whether a hyperplane arrangement has an  $\mathbb{F}_1$ -structure or not. In [2], a combinatorial necessary condition for an arrangement to have  $\mathbb{F}_1$ -structure was given.

The purpose of this paper is to answer questions posed in [2, Question 5.3, 5.4]. Namely, we discuss the  $\mathbb{F}_1$ -structure on the complement of a hyperplane arrangement, especially for an arrangement defined in the cycle space of a graph which is closely related to a variety appearing in the Feynman integral [2, §2.2, §2.3].

The plan of the present paper is as follows. In §2, we recall some basic notions of hyperplane arrangements, especially, the notions of the characteristic polynomial and the Boolean arrangement. In §3, we show that an arrangement has an  $\mathbb{F}_1$ -structure if and only if it is Boolean. We also show that, under an additional assumption, the combinatorial necessary condition formulated in [2] is sufficient for the  $\mathbb{F}_1$ -structure. In §4, we consider the arrangement defined by a graph ([2, §2.3.2]). We prove that the arrangement is Boolean (namely having  $\mathbb{F}_1$ -structure) if and only if the first homology group of the original graph has a basis consisting of simple cycles which do not share edges.

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## 2 Characteristic polynomial

In this section, we recall some basic notions of hyperplane arrangements. See [4] for details. Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of affine hyperplanes in a vector space  $V$  with  $\dim V = \ell$  over a field  $\mathbb{K}$ . The intersection poset is the set  $L(\mathcal{A}) = \{\cap S \mid S \subset \mathcal{A} \text{ with } \cap S \neq \emptyset\}$  of nonempty intersections of  $\mathcal{A}$ . The intersection poset  $L(\mathcal{A})$  is partially ordered by reverse inclusion, which has a unique minimal element  $\widehat{0} = V$ . The arrangement  $\mathcal{A}$  is said to be *central* if  $L(\mathcal{A})$  has a unique maximal element  $\cap \mathcal{A} \neq \emptyset$ , and is said to be *essential* if the maximal elements of  $L(\mathcal{A})$  are 0-dimensional subspaces. An arrangement  $\mathcal{A}$  is called a *Boolean arrangement* if  $L(\mathcal{A})$  is isomorphic to a Boolean lattice, i.e., the lattice of all subsets of a ground set. Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a Boolean arrangement. Then there exists a system of coordinates  $(x_1, \dots, x_\ell)$  of  $V$  such that  $H_i$  is equal to the coordinate hyperplane  $\{x_i = 0\}$  for  $1 \leq i \leq n$ . In particular, we have  $n \leq \ell$ . An arrangement  $\mathcal{A}$  is the essential Boolean arrangement if and only if  $\mathcal{A}$  is essential and  $n = \ell$ .

Next, we recall the definition of the characteristic polynomial  $\chi(\mathcal{A}, t)$ . The Möbius function  $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$  is defined by

$$\mu(X) = \begin{cases} 1, & \text{if } X = V, \\ -\sum_{V \leq Y < X} \mu(Y), & \text{if } X > V. \end{cases}$$

Then  $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$ . One of the most important properties of  $\chi(\mathcal{A}, t)$  is the following *deletion-restriction formula* ([4, Cor. 2.57]). Let us fix a hyperplane  $H \in \mathcal{A}$ . Then naturally the deletion  $\mathcal{A}' := \mathcal{A} \setminus \{H\}$  and restriction  $\mathcal{A}'' := H \cap \mathcal{A}'$  are defined. Note that  $\mathcal{A}''$  is a reduced arrangement in the space  $H$ . Then the following recursive formula holds.

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t).$$

Define the complement of  $\mathcal{A}$  by  $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$ . It is easily seen that  $M(\mathcal{A}') = M(\mathcal{A}) \sqcup M(\mathcal{A}'')$ . Therefore in the Grothendieck ring  $K_0(\mathcal{V}ar_{\mathbb{K}})$  of varieties over  $\mathbb{K}$ , we have

$$[M(\mathcal{A})] = \chi(\mathcal{A}, \mathbb{L}),$$

where  $\mathbb{L} = [\mathbb{A}_{\mathbb{K}}^1]$  is the class of the affine line. Suppose  $\mathcal{A}$  is defined over  $\mathbb{Z}$ . Then, for a prime power  $q = p^r$  with  $p \gg 0$ , the intersection poset  $L(\mathcal{A})$  is isomorphic to  $L(\mathcal{A} \otimes \mathbb{F}_q)$ . Therefore,  $|M(\mathcal{A} \otimes \mathbb{F}_q)| = \chi(\mathcal{A}, q)$ .

**Example 2.1.** Let  $\mathcal{A}$  be the Boolean arrangement defined by  $x_1 x_2 \dots x_n = 0$  in  $\mathbb{K}^\ell$  ( $n \leq \ell$ ). Then  $\chi(\mathcal{A}, t) = (t-1)^n t^{\ell-n}$ .

**Remark 2.2.** Let  $\mathcal{A}$  be an arrangement in  $\mathbb{K}^\ell$  which is not necessarily central. Then the coning  $c\mathcal{A}$  ([4, Definition 1.15]) is a central arrangement in  $\mathbb{K}^{\ell+1}$ . We have  $M(c\mathcal{A}) = M(\mathcal{A}) \times \mathbb{K}^\times$ . So from now on, we assume that all arrangements are central.

## 3 Arrangements with torified complements

A *torification* ([2, 3]) of a scheme  $X$  over  $\mathbb{Z}$  is a morphism of schemes  $e : T \rightarrow X$ , such that  $T = \bigsqcup_j \mathbb{G}_m^{d_j}$  is a disjoint union of split tori (where  $G_m = \text{Spec } \mathbb{Z}[t, t^{-1}]$ ), the restriction  $e|_{\mathbb{G}_m^{d_j}}$  is an isomorphism into a locally closed subscheme of  $X$ , and  $e(\mathbb{K}) : T(\mathbb{K}) \rightarrow X(\mathbb{K})$  is bijective for every field  $\mathbb{K}$ .

Suppose that

- (a)  $\mathcal{A}$  is the Boolean arrangement.

Then, there exists a coordinate system such that  $\mathcal{A}$  is defined as  $\{x_1 = 0\}, \dots, \{x_n = 0\}$  in  $\mathbb{K}^\ell$  and  $M(\mathcal{A}) \simeq (\mathbb{K}^\times)^n \times \mathbb{K}^{\ell-n}$ . This implies (see [3, §1.3.2])

- (b)  $M(\mathcal{A})$  has a torification, more precisely, there exists a torified scheme  $X$  such that  $X \otimes \mathbb{K}$  is isomorphic to  $M(\mathcal{A})$  as varieties over  $\mathbb{K}$ .

Suppose that  $M(\mathcal{A})$  has a torification. Then in the Grothendieck ring,  $[M(\mathcal{A})]$  is expressed as a linear combination of  $(\mathbb{L} - 1)^i$ ,  $i \geq 0$ , with nonnegative coefficients. Therefore, we have the following.

- (c) Suppose  $\chi(\mathcal{A}, t) = \sum_i c_i \cdot (t - 1)^i$  is the Taylor expansion of  $\chi(\mathcal{A}, t)$  at  $t = 1$ . Then  $c_i \geq 0$  for all  $i \geq 0$ .

Then, (a)  $\implies$  (b)  $\implies$  (c) hold ([2]). Now we discuss other implications.

**Lemma 3.1.** *Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement in  $V = \mathbb{K}^\ell$ . If (c) holds, then  $n \leq \ell$ .*

*Proof.* Recall that the  $\chi(\mathcal{A}, t)$  is a polynomial of the form  $t^\ell - nt^{\ell-1} + (\text{terms of deg} \leq \ell - 2)$ . It is equal to  $(t - 1)^\ell + (\ell - n)(t - 1)^{\ell-1} + (\text{terms of deg} \leq \ell - 2)$ . Hence (c) implies  $\ell - n \geq 0$ .  $\square$

**Theorem 3.2.** *Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement in  $\mathbb{K}^\ell$ . Then,*

- (1) (a) and (b) are equivalent.
- (2) Suppose  $\mathcal{A}$  is essential. Then (a), (b) and (c) are equivalent.

*Proof.* We first consider (2). If  $\mathcal{A}$  is essential, then by definition,  $n \geq \ell$ . Suppose (c) holds. Then Lemma 3.1 tells that  $n \leq \ell$ , hence  $n = \ell$ , which implies (a).

Next we consider (1). We need to prove (b)  $\implies$  (a). Let us assume that  $M(\mathcal{A})$  has a torification. Then there exists a dominant morphism  $(\mathbb{K}^\times)^\ell \hookrightarrow M(\mathcal{A})$ . Suppose that  $\mathcal{A}$  is not Boolean. Then there exist dependent hyperplanes. Namely, after a change of coordinates, we may assume  $\{x_1 = 0\}, \dots, \{x_r = 0\}, \{x_1 + x_2 + \dots + x_r = 0\}$  (with  $2 \leq r \leq n - 1$ ) are in  $\mathcal{A}$ . Then we have  $(\mathbb{K}^\times)^\ell \hookrightarrow M(\mathcal{A}) \subset \{x_1 x_2 \dots x_r (x_1 + \dots + x_r) \neq 0\} \subset \mathbb{K}^\ell$ . Taking the ring of functions, we have an embedding of  $\mathbb{K}$ -algebras.

$$\mathbb{K} \left[ x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_\ell, \frac{1}{x_1 + \dots + x_r} \right] \subset \mathbb{K}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}].$$

However, this is impossible. Because in the right hand side, the set of invertible elements is equal to the set of monomials, which are linearly independent over  $\mathbb{K}$ . However, in the left hand side,  $x_1, \dots, x_r, x_1 + \dots + x_r$  are dependent invertible elements.  $\square$

The following example shows that if  $\mathcal{A}$  is not essential, (c) is not sufficient for (a) or (b).

**Example 3.3.** Consider the arrangement  $\mathcal{A}$  defined by  $x_1 x_2 x_3 (x_1 + x_2 + x_3) = 0$  in  $\mathbb{K}^4$ . Then,  $\chi(\mathcal{A}, t) = t^4 - 4t^3 + 6t^2 - 3t = (t - 1)^4 + (t - 1)$ . Hence the coefficients of  $(t - 1)^i$  are nonnegative, however,  $\mathcal{A}$  is not Boolean.

## 4 The arrangement in the cycle space of a graph

Let  $\Gamma = (V, E)$  be a finite graph. Let  $H_1(\Gamma) = H_1(\Gamma, \mathbb{K})$  be the cycle space over  $\mathbb{K}$ .

Since  $\Gamma$  is a 1-dimensional CW-complex, any 1-cochain is automatically a cocycle. Therefore, any edge  $e \in E$  (equipped with an orientation) determines an element of  $H^1(\Gamma) = \text{Hom}(H_1(\Gamma), \mathbb{K})$ . We denote the element by  $\eta_e : H_1(\Gamma) \longrightarrow \mathbb{K}$ .

If  $\eta_e \neq 0$ , then  $H_e := \text{Ker } \eta_e$  defines a hyperplane in  $H_1(\Gamma) \simeq \mathbb{K}^{b_1(\Gamma)}$ . As in [2], define  $\mathcal{A}_\Gamma$  as  $\mathcal{A}_\Gamma := \{H_e \mid e \in E, \eta_e \neq 0\}$ . We note that  $e, e' \in E$  with  $e \neq e'$  may define the same hyperplane. We just forget the multiplicities and consider the reduced arrangement.

**Lemma 4.1.** *The arrangement  $\mathcal{A}_\Gamma$  is essential. In particular,  $|\mathcal{A}_\Gamma| \geq b_1(\Gamma)$ .*

*Proof.* Since  $\{e \in E\}$  spans the space of 1-cochains,  $\{\eta_e\}$  generates the space of 1-cocycles, hence  $H^1(\Gamma)$ . We have  $\cap_{e \in E} H_e = \{0\}$ .  $\square$

Let  $\mathcal{F}$  be a spanning forest of  $\Gamma$ . Let  $e \in E \setminus \mathcal{F}$ . Then the vertices of  $e$  are connected by the unique minimal path in  $\mathcal{F}$ . By adding  $e$  to this path, we have a simple cycle  $C_e \subset E$ . By putting suitable orientations, the set of such cycles  $\mathcal{B}(\mathcal{F}) := \{C_e \subset E \mid e \in E \setminus \mathcal{F}\}$  forms a basis of  $H_1(\Gamma)$ . Such a basis is called a fundamental basis [6].

**Definition 4.2.** Let  $\mathcal{B} \subset H_1(\Gamma)$  be a basis consisting of simple cycles (that is, each cycle passes an edge at most once). Then  $\mathcal{B}$  is said to be separated if any two cycles  $C, C' \in \mathcal{B}$  ( $C \neq C'$ ) do not share edges.

**Theorem 4.3.** Let  $\Gamma = (V, E)$  be a graph. Then the following are equivalent.

- (i)  $M(\mathcal{A}_\Gamma)$  has a torification.
- (ii)  $\mathcal{A}_\Gamma$  is Boolean.
- (iii)  $|\mathcal{A}_\Gamma| = b_1(\Gamma)$ .
- (iv) For any spanning forest  $\mathcal{F} \subset \Gamma$ , the fundamental basis  $\mathcal{B}(\mathcal{F})$  is separated.
- (v) There exists a spanning forest  $\mathcal{F} \subset \Gamma$  such that the fundamental basis  $\mathcal{B}(\mathcal{F})$  is separated.
- (vi) There exists a separated basis  $\mathcal{B}$ .

*Proof.* (i)  $\iff$  (ii)  $\iff$  (iii) is obtained from Theorem 3.2 and Lemma 4.1. Also (iv)  $\implies$  (v)  $\implies$  (vi) is obvious. Now we assume (vi). Let  $\mathcal{B} = \{C_1, \dots, C_b\}$  be a basis as in (vi). Then every spanning forest is obtained from  $\Gamma$  by removing an edge of  $C_i$  for each  $i$ . Thus  $\mathcal{B}$  is the fundamental basis for every spanning forest  $\mathcal{F}$ . This proves (vi)  $\implies$  (iv).

Now we prove (vi)  $\implies$  (ii). Let  $\mathcal{F}$  be a basis as in (vi). Then every edge  $e \in E$  is either contained in the unique cycle  $C \in \mathcal{B}$  or not contained in  $\bigcup_{C \in \mathcal{B}} C$ . In the latter case,  $\eta_e = 0$ , hence we do not need to consider. Hence  $\{\eta_e\} \subset H^1(\Gamma)$  forms a dual basis of  $\mathcal{B}$ . Thus  $\mathcal{A}_\Gamma$  is Boolean.

Finally, we prove (ii)  $\implies$  (iv). Suppose (ii) and there exists a spanning forest  $\mathcal{F}$  such that  $\mathcal{B}(\mathcal{F})$  is not separated. There exists an edge  $e_0 \in E$  such that  $\{C \in \mathcal{B}(\mathcal{F}) \mid C \ni e_0\}$  contains more than one cycles. Let  $E \setminus \mathcal{F} = \{e_1, \dots, e_b\}$ . Then  $\{H_{e_0}, H_{e_1}, \dots, H_{e_b}\} \subset \mathcal{A}_\Gamma$  forms  $b + 1$  distinct hyperplanes. This contradicts the fact that  $\mathcal{A}_\Gamma$  is Boolean. This completes the proof.  $\square$

**Remark 4.4.** The graph satisfying the conditions in Theorem 4.3 is obtained by finitely many repetitions of gluing a graph with  $b_1 \leq 1$  at a vertex.

**Remark 4.5.** There is another way to associate a hyperplane arrangement  $\mathcal{A}(\Gamma)$  to a graph  $\Gamma$ , which is called the *graphic arrangement* (see [4, §2.4] for details). The two arrangements  $\mathcal{A}_\Gamma$  and  $\mathcal{A}(\Gamma)$  are dual to each other in the sense of matroids (see [5]). In view of this interpretation, Theorem 4.3 characterizes those graphs  $\Gamma$  for which the simplification of the dual of the circuit matroid of  $\Gamma$  is the free matroid.

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