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Doctoral Thesis

Proof-Theoretic Study of Distributed Knowledge

Ryo Murai

Supervisor: Katsuhiko Sano

Graduate School of Humanities and Human Sciences

Hokkaido University

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Contents

Chapter 1

Introduction

1.1 Background

Epistemic logic was invented in the 1950s and 1960s as a form of modal logic by von Wright [60] and Hintikka [21]. In the 1980s and 1990s, it was studied extensively by several computer scientists [8, 29]. From the point of view of logic, multi-agent epistemic logic is a type of modal logic with a modal operator K_a parameterized by agent a, where the expression $K_a\varphi$ is interpreted as "an agent a knows that a proposition φ is the case." In multi-agent epistemic logic, $K_a\varphi$ is interpreted in a relational structure called the Kripke frame. A Kripke frame is a pair of a set of states (possible worlds) and a family $(R_a)_{a\in\mathsf{Agt}}$ of binary relations on the set, indexed by an agent where Agt is a set of agents. Each state represents a possible situation and wR_av means that a state v is epistemically possible (indistinguishable) for an agent a in a state w. Under this setting, "$K_a\varphi$ is true in a state w." is defined as "φ is true in a state v for every state v satisfying wR_av," that is, "φ is true in a state v for every state v that is epistemically possible for an agent a in a state w." By the nature of knowledge as understood by computer scientists, the operator K_a is usually required to satisfy the following axioms:

- (K) $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$ (knowledge is deductively closed or closed under modus ponens).
- (T) $K_a\varphi \to \varphi$ (a known proposition is a fact).
- (4) $K_a\varphi \to K_aK_a\varphi$ (if someone knows something, he/she knows that he/she knows it).
- (5) $\neg K_a\varphi \to K_a\neg K_a\varphi$ (if someone does not know something, he/she knows that he/she does not know it).

These are collectively referred to as **S5** axioms. The reason why (5) is considered valid in computer science is described as follows:

> [F]or some artificial agents, dealing with *finite* information, like only a finite set $\mathcal{P}$ of propositional atoms and a finite set of formulas that it knows, the truth of this axiom [the axiom (5)] may be argued (informally) like this: if the artificial agent does not know a formula, then this formula does not follow from the agent's finite information, and the agent is able to detect this, so that it knows that it does not know the formula. Also, in some cases, the validity of the axiom follows directly from the special kind of models that is used in applications - as in the case of using epistemic logic in distributed systems, cf. Halpern and Moses (1990) [the paper [17]] and Meyer and van der Hoek (1995) [the book [29]]. ([28, p.188])

As there are multiple agents, knowledge ascribed to a group of agents, not a single agent, is conceivable in multi-agent epistemic logic, and has been an important subject for the researchers in the field [8, 29]. The simplest group knowledge is the one called "*everyone knows*." This concept is represented by the modal operator E_G where G is a group of agents. For example, the expression $E_{\{a,b,c\}}\varphi$ indicates that $K_a\varphi$, $K_b\varphi$, and $K_c\varphi$ are true. Another notion of group knowledge is *common knowledge*, which is the most well-studied. The concept is represented by the modal operator C_G; intuitively, the meaning of $C_G\varphi$ is an infinite conjunction "$E_G\varphi \wedge E_GE_G\varphi \wedge E_GE_GE_G\varphi \wedge \cdots$". That is, common knowledge is knowledge publicly known to all the members of the group.

Another notion of group knowledge, which is the subject of this thesis, is that of *distributed knowledge*. In the context of epistemic logic, the notion of distributed knowledge became known by Halpern and Moses' [17]. Roughly speaking, distributed knowledge is knowledge potentially held by a group. For example, suppose that an agent a knows that if p then q and suppose that an agent b knows that p. Then, the group $\{a, b\}$ can be said to potentially know q, which is the distributed knowledge of $\{a, b\}$. According to Ågotnes and Wáng [1, Section 1], "distributed knowledge is the knowledge of a third party, someone 'outside the system' who somehow has access to the epistemic states of all the group members". Fagin *et al.* [8, p. 3] provides an intuitive description of distributed knowledge: "a group has distributed knowledge of a fact φ if the knowledge of φ is distributed among its members, so that by pooling their knowledge together the members of the group can deduce φ." At first glance, this seems clearer than the explanation by Ågotnes and Wáng described above. Ågotnes *et al.* [1] state, however, that the aforementioned intuitive description is inappropriate in an illustrative example given in [1, Section

1].

Formally, distributed knowledge is expressed as a modal operator D_G, parameterized by a group of agents. $D_G\varphi$ holds at a state w, by the standard definition, if and only if: φ holds at all states v such that v can be reached in a single step from w for all agents in G, that is, $wR_a v$ for all agents $a \in G$.

Contrary to what the name suggests, distributed knowledge is generally not distributed in a group G under the aforementioned definition, in the sense that a proposition φ cannot always be deduced from pieces of knowledge owned by each member of G, even when $D_G\varphi$ is true. This is expressed by van der Hoek *et al.*'s [58], in their own words, as "the principle of full communication does not hold for distributed knowledge." van der Hoek *et al.* and Gerbrandy [58, 14] present sufficient conditions for the Kripke model for the principle of full communication to be valid, and Roelofsen [44] presents the necessary and sufficient condition.

For the completeness theorem, which is an important type of theorem in logic which assures that the notion of validity in semantics and the notion of provability in proof theory are logically equivalent, the literature [7, 56, 18, 57, 8, 29, 14, 61] are known. Recently, Wáng and Ågotnes [62] provide a new detailed proof of the completeness of epistemic logic with distributed knowledge.

We describe more direct background for the content of this thesis below. So far, the study of distributed knowledge is mainly model-theoretic [1, 44, 14, 58]; proof-theoretic studies have not been pursued actively. In modern logic, the study of proof theory is conducted mainly using the proof system called "sequent calculus." In sequent calculus, one can express an inference between "sequents". A sequnt is a pair of finite multisets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$," which reads as "if all formulas in Γ hold then some formulas in Δ hold." To the best of the author's knowledge, the existing sequent calculi for logic with distributed knowledge are presented only in [16, 39, 15]. The first one by Hakli and Negri [16] contains a natural G3-style (without structural rules) formalization, in which each formula has a label designating the state where the formula holds. The second one by Pliuškevičius and Pliuškevičienė [39] contains a Gentzen-style sequent calculus for **S4** distributed knowledge logic which is simpler than the one we are interested in, in that the operator is not parameterized by group G. The third one by Giedra [15] contains Gentzen-style and Kanger-style sequent calculi for **S5** distributed knowledge logic with the same type of operator as the second one.

Moreover, epistemic logic as a whole has been studied mainly in the classical setting, whereas several types of intuitionistic epistemic logics have been proposed from different perspectives. Several philosophical logicians have proposed intuitionistic epistemic logics

[63, 42, 3] to analyze Fitch's knowability paradox [9], from the verificationist perspective. Another type of intuitionistic epistemic logic [22] is proposed for the analysis of distributed computing in the sense of [20, 45]. The intuitionistic aspect of the logic is required to describe the property of asynchronous communication among agents in distributed computing.

Jäger and Marti [23] formulate intuitionistic epistemic logic with distributed knowledge for the first time, to the best of the author's knowledge, and prove the semantic completeness of Hilbert systems of intuitionistic **K** and **KT** with distributed knowledge. A formula of the logic is interpreted in the Kripke model with a preorder $\preceq$ for the intuitionistic aspect of the logic and relations $(R_i)_{1 \leq i \leq l}$ for agents $ag_1 \cdots ag_l$, where a condition $\preceq; R_i \subseteq R_i$ is satisfied for each i. Here, the composition $R;S$ of relations R and S is defined as $R;S := \{(x,z) \mid \text{there exists } y \text{ such that } xRy \text{ and } ySz\}$. In the proof of the semantic completeness, a notion of "pseudo-model," where distributed knowledge operators are interpreted as plain modal operators, is used as in the standard proof of the semantic completeness of the epistemic logic with distributed knowledge based on classical logic. However, the way a pseudo-model is transformed into an intended model is different from the standard proof where the transformation is conducted via an operation of "tree unraveling". Instead of tree unraveling, Jäger and Marti introduce an operation called "strict extension" [23, Definition 4.4] which transforms a given pseudo-model into another pseudo-model by indexing the set of states of the original pseudo-model by the relations of the original pseudo-model. Relations in the strict extension are defined such that the intended model, equivalent to the strict extension, can be constructed from the strict extension. Aside from [23], Su *et al.* [50] also develop intuitionistic epistemic logic with distributed knowledge, although their logic is based on the system **IEL** [3], which is quite different in its sprit from the basic epistemic logic introduced in Section 1.3.

Recently, dynamic epistemic logic has played a significant role in the study of epistemic logic. It is used to study changes in agents' knowledge caused by an event or action. Public announcement logic [38], through which one can express the change in agents' knowledge caused by the truthful public announcement of a proposition, is the simplest dynamic epistemic logic [59]. It can be used to analyze problems in which the influence of publicly conducted announcements or publicly observed actions on agents' knowledge matters, such as in the Muddy Children Puzzle [59, Section 4.10].

Since [38], many expansions and variants of public announcement logic have been studied. Public announcement logic based on intuitionistic logic [25, 4, 32] is one of such studies. According to [25], intuitionistic public announcement logic can be useful when dealing with changes in *constructive* knowledge. They expand the intuitionistic modal logic **IK** [47, 48] and **MIPC** [41] with public announcement operators. Public announcement logic expanded with distributed knowledge has also been studied [19, 61, 10]. [61] develops a public announcement logic with distributed knowledge $\mathcal{PAD}$ and a public announcement logic with distributed knowledge and common knowledge $\mathcal{PACD}$, which are based on **S5** epistemic logic, establishes the completeness of the logics, and examines the expressivity and computational complexity of the logics.

However, public announcement logic with distributed knowledge based on intuitionistic logic has not yet been studied.

1.2 Contributions

1.2.1 Sequent Calculus of Distributed Knowledge

In Chapter 2, Gentzen-style sequent calculi (without label) are proposed for five kinds of multi-agent epistemic propositional logics with distributed knowledge operators, parameterized by groups, which are reasonable generalization of sequent calculi for basic epistemic (modal) logic. The cut elimination theorem for four of them is further proved. Using the method described in [26], the Craig interpolation theorem is also established for the four systems, in which not only the condition of propositional variables but also that of agents is considered. This is a new result for logic on distributed knowledge, as far as the author knows. The Craig interpolation theorem does not hold for some expansions of basic modal logic [53]. Thus, the result suggests that the logics with distributed knowledge are "good" expansions of basic modal logic in this sense.

1.2.2 Distributed Knowledge over Intuitionistic Logic

In Chapter 3, *intuitionistic* epistemic logics with distributed knowledge are developed based on [23]. Our logics are different from the one in [23] in the following respects: First, in our logics, the distributed knowledge operator is parameterized by a group, that is, a subset of all agents, whereas [23] deals with only the distributed knowledge of all agents. Second, we handle more axioms than in [23] in proposing intuitionistic **K**, **KT**, **KD**, **K4**, **K4D**, and **S4** with distributed knowledge. One point to note here is that axioms (K), (T), and (4) in our logics are simply a D_G-versions of the respective axioms in the basic modal logic. However, our axiom (D) is restricted to a single agent (i.e., $\neg D_{\{a\}}\bot$). This is because the seriality for each R_a is generally not preserved under taking teh intersection among a group (refer to [2]), whereas reflexivity and transitivity are always preserved. For proof of the semantic completeness, the method based on the concept of tree unraveling (of a pseudo-model) is adopted, unlike [23]. This is because it is not obvious if the method using "strict extension" in [23] can be generalized to the logic with distributed knowledge that is parameterized by group. Our method is similar to the one in [62], in that there is no "folding" step which is included in the completeness proofs in e.g., [7, 61]. Moreover, we also show the semantic completeness with respect to more restricted classes of Kripke frames than the ordinary one, which are characterized by the notion of "stability." A Kripke frame $F = (W, \leqslant, (R_a)_{a \in \mathsf{Agt}})$, where a preorder $\leqslant$ is included for the intuitionistic aspect of the logic in question, is called *stable* if $\leqslant; R_a \subseteq R_a$ and $R_a; \leqslant \subseteq R_a$ hold for any $a \in \mathsf{Agt}$ (this is equivalent to $\leqslant; R_a; \leqslant \subseteq R$). This notion of stability is adopted from [46, 49] by Sano and Stell, and Stell *et al.* In their work, stability is introduced in the context of defining a binary relation on hypergraphs. The same notion is also studied in Wolter and Zakharyaschev's [64] as $\Box$-frame, in order to define a natural Gödel transformation from an intuitionistic modal logic to a classical-logical poly-modal logic. In our study, the notion of stability is introduced because Kripke frame should be stable in order for the intuitionistic public announcement logic with distributed knowledge introduced in Chapter 4 to be sound. To show this, first we prove the strong completeness with respect to the suitable class of frames including nonstable ones, by constructing a model called "tree unraveling." Then, we make the tree unraveling model stable by the operation called stabilization. Notably, the operation of stabilization is compatible with tree unraveling by certain property of it (Proposition 3.42). In addition, cut-free sequent calculi are proposed for our logics, based on the idea introduced in Chapter 2 and the Craig interpolation theorem is proven using Maehara's method [26, 35]. The decidability of the sequent calculi are further established by the standard argument [11, 12] on a cut-free derivation of a sequent, whereas [23] does not show this for their Hilbert systems.

1.2.3 Public Announcement with Intuitionistic Distributed Knowledge

In Chapter 4, intuitionistic public announcement logics with distributed knowledge are developed based on the intuitionistic epistemic logics with distributed knowledge developed in Chapter 3. The intuitionistic epistemic logics with distributed knowledge developed in Chapter 3 are expanded with a public announcement operator except the ones having the axiom (D) and its semantic strong completeness is proved by a standard argument using a reduction axiom [59]. Note that a reduction axiom for the distributed knowledge is *not* sound for the class of all frames, which are defined to enjoy the condition $\leqslant; R_a \subseteq R_a$. This means that we must restrict our attention to a subclass of frames. As mentioned earlier, the condition for the restriction is called "stability."

By naturally transforming the reduction axioms into inference rules, sequent calculi of the logics are also developed, and the cut-elimination and Craig interpolation theorems are proved. The inductive proof of the cut-elimination theorem is made possible by using the complexity function for a formula, which is introduced in the argument of the completeness proof, as a measure for cut formula, instead of the ordinary complexity measure used in the cut-elimination theorem for other logics.

In the following three sections, the commonly known definitions and facts that form the basis of this study are summarized.

1.3 Basic Epistemic Logic

In this section, the standard multi-agent epistemic logic is explained. The logic introduced is essentially the same as basic modal logic with an operator $\Box$ because we only consider knowledge of single agent here. The description is based on [6, 8, 59].

1.3.1 Language

We denote by Agt a finite set of agents. We call a *nonempty* subset of Agt "group" and denote it by G, H, etc. Let Prop be a countable set of propositional variables and Form be the set of formulas defined inductively by the following clauses:

$$\mathsf{Form} \ni \varphi ::= p \in \mathsf{Prop} \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid K_a\varphi.$$

It is noted that $\wedge$ and $\vee$ are defined in the same way as in the classical propositional logic. That is, $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$ and $\varphi \vee \psi := \neg\varphi \to \psi$. We also define $\top$ as $\bot \to \bot$. We define $\bigwedge \Gamma := \bigwedge_{\varphi\in\Gamma} \varphi$ for a finite nonempty set Γ of formulas, and $\bigwedge \emptyset := \top$. We define $\bigvee \Gamma := \bigvee_{\varphi\in\Gamma} \varphi$ for a finite nonempty set Γ of formulas, and $\bigvee \emptyset := \bot$.

1.3.2 Kripke Semantics

We introduce Kripke semantics for multi-agent epistemic logic here. Let W be a possibly countable set of states, $(R_a)_{a\in\mathsf{Agt}}$ be a family of binary relations on W, indexed by agents, and V be a valuation function $\mathsf{Prop} \to \mathcal{P}(W)$. We call a pair $F = (W, (R_a)_{a\in\mathsf{Agt}})$ a *frame* and a tuple $M = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ a *model*. For a model $M = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model*. *Satisfaction relation* $M, w \models \varphi$ on pointed models and formulas are defined recursively as follows:

$$\begin{array}{lll} M, w \models p & \text{iff} & w \in V(p), \\ M, w \models \bot & & \text{Never}, \\ M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi, \\ M, w \models \varphi \to \psi & \text{iff} & M, w \not\models \varphi \text{ or } M, w \models \psi, \\ M, w \models K_a\varphi & \text{iff} & \text{for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \models \varphi. \end{array}$$

Given a frame $F = (W, (R_a)_{a \in \mathsf{Agt}})$, we say that a formula φ is *valid* in F (notation: $F \models \varphi$) if $(F, V), w \models \varphi$ for every valuation function V and every $w \in W$. Moreover, a formula φ is valid in a class $\mathbb{F}$ of frames (notation: $\mathbb{F} \Vdash \varphi$) if $F \models \varphi$ for every $F \in \mathbb{F}$. Let us say that a set Γ of formulas *defines* a class $\mathbb{F}$ of frames if, for every frame F, $F \in \mathbb{F}$ is equivalent to: $F \models \varphi$ for all $\varphi \in \Gamma$. Based on the notion of satisfaction, the relation of semantic consequence is defined as follows.

Definition 1.1 ([6, Definition 1.35])**.** A formula φ is a *semantic consequence* of Γ in a frame class $\mathbb{F}$ if for all frame $F \in \mathbb{F}$, a valuation V on F, a state $w \in |F|$, if $(F, V), w \models \Gamma$, then $(F, V), w \models \varphi$, where "$(F, V), w \models \Delta$" means that $(F, V), w \models \psi$ for all $\psi \in \Delta$. We write it as "$\Gamma \models_{\mathbb{F}} \varphi$".

1.3.3 Hilbert Systems

We review the known Hilbert system for epistemic logics with K_a operators. Hilbert system $\mathsf{H}(\mathbf{K})$ is defined as in the following table.

Hilbert System $\mathsf{H}(\mathbf{K})$	
(Taut)	all instantiations of propositional tautologies
(K)	$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
(MP)	From $\varphi \to \psi$ and φ infer ψ
(Nec)	From φ infer $K_a\varphi$
Additional Axiom Schemes	
(T)	$K_a\varphi \to \varphi$
(D)	$\neg K_a\bot$
(4)	$K_a\varphi \to K_aK_a\varphi$
(5)	$\neg K_a\varphi \to K_a\neg K_a\varphi$

Definition 1.2. Let X be a set and $R \subseteq X \times X$. The binary relation R is:

- *reflexive* if wRw for any $w \in X$.

- *serial* if for any $w \in X$, there exists $v \in X$ such that wRv.
- *transitive* if wRv and vRu jointly imply wRu for any $w, v, u \in X$.
- *Euclidean* if wRv and wRu jointly imply vRu for any $w, v, u \in X$.

For additional axioms schemes, we note that (T), (D), (4) and (5) define the class of reflexive, serial, transitive and Euclidean frames, respectively (here, e.g., a "reflexive" frame means that R_a is reflexive for all agents $a \in \mathsf{Agt}$). Hilbert systems $\mathsf{H}(\mathbf{KT})$, $\mathsf{H}(\mathbf{KD})$, $\mathsf{H}(\mathbf{K4})$, $\mathsf{H}(\mathbf{K4D})$, $\mathsf{H}(\mathbf{S4})$, and $\mathsf{H}(\mathbf{S5})$ are defined as axiomatic expansions of $\mathsf{H}(\mathbf{K})$ with (T), (D), (4), (4) and (D), (T) and (4), and (T) and (5), respectively). In any axiom system $\mathsf{H}(\mathbf{X})$ ($\mathbf{X} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}, \mathbf{S5}\}$), (MP) and (Nec) is called "inference rule" and the rest are called "axioms". The notion of proof and provability is defined as follows.

Definition 1.3 ([59, Definition 2.17])**.** Let φ be a formula. A *proof for* φ *within* $\mathsf{H}(\mathbf{X})$ is a finite sequence $(\varphi_i)_1^m$ of formulas such that

1. $\varphi_m = \varphi$.
2. every φ_i in the sequence is
   (a) either an instance of one of the axioms
   (b) or else the result of the application of one of the inference rules to one or more formulas in the sequence that appear before φ_i.

A formula φ *is provable in* $\mathsf{H}(\mathbf{X})$ (notation: $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$) if there exists a proof for φ within $\mathsf{H}(\mathbf{X})$.

We also define derivability relation between a set Γ of formulas and a formula φ as below.

Definition 1.4 ([6, p.36])**.** A formula φ is *derivable* from Γ in a logic $\mathbf{X}$ if $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \to \varphi$ for some finite set Γ' which is a subset of Γ. We write it as "$\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$".

We introduce a class of frames corresponding to each logic, in order to state soundness of our axiomatization.

Definition 1.5. A class of frames $\mathbb{F}(\mathbf{X})$ is defined as follows:

- $\mathbb{F}(\mathbf{K})$ is the class of all frames.
- $\mathbb{F}(\mathbf{KT})$ is the class of all frames such that R_a is reflexive ($a \in \mathsf{Agt}$).

- $\mathbb{F}(\mathbf{KD})$ is the class of all frames such that R_a is serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{K4})$ is the class of all frames such that R_a is transitive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{K4D})$ is the class of all frames such that R_a is transitive and serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{S4})$ is the class of all frames such that R_a is reflexive and transitive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{S5})$ is the class of all frames such that R_a is reflexive and Euclidean ($a \in \mathsf{Agt}$).

For Hilbert systems $\mathsf{H}(\mathbf{K})$, $\mathsf{H}(\mathbf{KT})$, $\mathsf{H}(\mathbf{KD})$, $\mathsf{H}(\mathbf{K4})$, $\mathsf{H}(\mathbf{K4D})$, $\mathsf{H}(\mathbf{S4})$, and $\mathsf{H}(\mathbf{S5})$, the following soundness and completeness results hold.

Theorem 1.6 (soundness, [6, p.193])**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, $\mathbf{S4}$, *and* $\mathbf{S5}$. *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, *then* $\mathbb{F}(\mathbf{X}) \models \varphi$.

Theorem 1.7 (strong completeness, [6, Theorem 4.23, 4.27, 4.28, 4.29])**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, $\mathbf{S4}$, *and* $\mathbf{S5}$. *Then, if* $\Gamma \models_{\mathbb{F}(\mathbf{X})} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$.

1.3.4 Sequent Calculi

We describe the sequent calculus of the basic epistemic logic, obtained by expanding the sequent calculus **LK** [11, 12] of classical propositional logic. We refer to [35, 52, 40].

A *sequent* is a pair of finite multi-sets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$", whose reading is "If all formulas in Γ hold then some formulas in Δ hold." If $\Gamma = \emptyset$, "$\Rightarrow \Delta$" means "Some formulas in Δ hold (without assumptions)." If $\Delta = \emptyset$, "$\Gamma \Rightarrow$" means "Assuming that all formulas in Γ hold leads to a contradiction." Let $\mathbf{X}$ be any of $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, $\mathbf{S4}$, and $\mathbf{S5}$. We denote by $\mathsf{G}(\mathbf{X})$, the sequent calculus consisting of axioms, structural rules, propositional logical rules, and logical rules for K_a of $\mathbf{X}$ in Table 1.1 [52]. We note that when $n = 0$, e.g., in the rule (K) of Table 1.1, the multi-set is regarded as the empty multi-set and that $K_a\Gamma := \{K_a\varphi \mid \varphi \in \Gamma\}$.

A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in each calculus $\mathsf{G}(\mathbf{X})$ if there exists a finite tree of sequents, whose root is $\Gamma \Rightarrow \Delta$ and each node of which is inferred by some rule (including axioms) in $\mathsf{G}(\mathbf{X})$. We write it as $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$. It is well-known that for each logic $\mathbf{X}$, the provability of formula (or sequent) is equivalent between $\mathsf{H}(\mathbf{X})$ and $\mathsf{G}(\mathbf{X})$.

Theorem 1.8 (Equipollence)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, $\mathbf{S4}$, *and* $\mathbf{S5}$. *Then, the following hold.*

1. *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, *then* $\vdash_{\mathsf{G}(\mathbf{X})} \Rightarrow \varphi$.
2. *If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma \to \bigvee \Delta$, *where* $\bigwedge \emptyset := \top$ *and* $\bigvee \emptyset := \bot$.

Table 1.1: Sequent Calculi for **K**, **KT**, **KD**, **K4**, **K4D**, **S4**, and **S5**

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (Id) \quad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow\to) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

Logical Rules for K_a of K

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (K)$$

Logical Rules for K_a of KT

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (K) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{K_a\varphi, \Gamma \Rightarrow \Delta}\ (K \Rightarrow)$$

Logical Rules for K_a of KD

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (K) \quad \frac{\Gamma \Rightarrow}{K_a\Gamma \Rightarrow}\ (K_{\mathbf{KD}})$$

Logical Rules for K_a of K4

$$\frac{\varphi_1, \ldots, \varphi_n, K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (\Rightarrow K_{\mathbf{K4}})$$

Logical Rules for K_a of K4D

$$\frac{\varphi_1, \ldots, \varphi_n, K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (\Rightarrow K_{\mathbf{K4}}) \quad \frac{\Gamma, K_a\Gamma \Rightarrow}{K_a\Gamma \Rightarrow}\ (\Rightarrow K_{\mathbf{K4D}})$$

Logical Rules for K_a of S4

$$\frac{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow \psi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi}\ (\Rightarrow K_{\mathbf{S4}}) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{K_a\varphi, \Gamma \Rightarrow \Delta}\ (K \Rightarrow)$$

Logical Rules for K_a of $\mathbf{S5}_D$

$$\frac{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi_1, \ldots, K_a\psi_m, \chi}{K_a\varphi_1, \ldots, K_a\varphi_n \Rightarrow K_a\psi_1, \ldots, K_a\psi_m, K_a\chi}\ (\Rightarrow K_{\mathbf{S5}})$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{K_a\varphi, \Gamma \Rightarrow \Delta}\ (K \Rightarrow)$$

For the logics except **S5**, we can show the cut-elimination theorem, which is regarded as an important theorem concerning sequent calculi.

Theorem 1.9 (Cut-Elimination, [52, Corollary 2.4]). *Let* $\mathbf{X}$ *be any of* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, *and* $\mathbf{S4}$. *Then, the following holds: If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{G}^{-}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *where* $\mathsf{G}^{-}(\mathbf{X})$ *denotes a system "*$\mathsf{G}(\mathbf{X})$ *minus the cut rule".*

The cut elimination theorem does not hold for $\mathsf{G}(\mathbf{S5})$, because the application of (Cut) rule in the following derivation cannot be eliminated, as pointed out by Onishi and Matsumoto in [33, footnote 3 on p.116].

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$p \Rightarrow p$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg p, p \Rightarrow$}
\RightLabel{$(K \Rightarrow)$}
\UnaryInfC{$K_a \neg p, p \Rightarrow$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$p \Rightarrow \neg K_a \neg p$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$K_a \neg p \Rightarrow K_a \neg p$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\Rightarrow \neg K_a \neg p, K_a \neg p$}
\RightLabel{$(\Rightarrow K_{\mathbf{S5}_D})$}
\UnaryInfC{$\Rightarrow K_a \neg K_a \neg p, K_a \neg p$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg K_a \neg p \Rightarrow K_a \neg K_a \neg p$}
\RightLabel{(Cut)}
\BinaryInfC{$p \Rightarrow K_a \neg K_a \neg p$}
\RightLabel{$(\Rightarrow \rightarrow)$}
\UnaryInfC{$\Rightarrow p \rightarrow K_a \neg K_a \neg p$}
\end{prooftree}
$$

Based on the cut-elimination theorem, Craig interpolation theorem can be shown by the method developed by Maehara [26, 35].

Theorem 1.10 (Craig Interpolation Theorem, cf. [35, Thorem 41]). *Let* $\mathbf{X}$ *be any of* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K4}$, $\mathbf{K4D}$, *and* $\mathbf{S4}$. *Given that* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$, *there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \chi \Rightarrow \psi$.

2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

1.4 Public Announcement Logic over Basic Epistemic Logic

We describe the standard public announcement logic here. The description is based on [59, 5].

1.4.1 Language

We expand the syntax of the basic epistemic logic with the public announcement operator and define the set of all formulas of the expanded syntax as:

$$\mathsf{Form}^{+} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid K_a\varphi \mid [\varphi]\varphi,$$

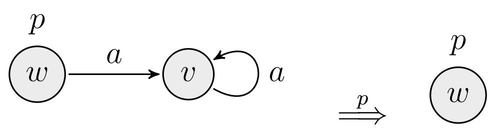

Figure 1.1: M_{srl} and M_{srl}^{p}

where $p \in \mathsf{Prop}$.

1.4.2 Kripke Semantics

Definition 1.11 ([59, Definition 4.7]). Let $M = (W, (R_a)_{a \in \mathsf{Agt}}, V)$ and $\varphi, \psi \in \mathsf{Form}^+$. The satisfaction relation $M, w \models \varphi$ is defined as the basic epistemic logic except:

$$M, w \models [\varphi]\psi \quad \text{iff} \quad M, w \models \varphi \text{ implies } M^{\varphi}, w \models \psi,$$

where $M^{\varphi} := (\llbracket \varphi \rrbracket_M, (R_a^{\varphi})_{a \in \mathsf{Agt}}, V^{\varphi})$ (*a model updated from* M *by* φ) is defined as follows:

- $\llbracket \varphi \rrbracket_M := \{w \in W \mid M, w \models \varphi\}$,
- $R_a^{\varphi} := R_a \cap (\llbracket \varphi \rrbracket_M \times \llbracket \varphi \rrbracket_M)$,
- $V^{\varphi}(p) := V(p) \cap \llbracket \varphi \rrbracket_M$.

Intuively speaking, the operation $(-)^{\varphi}$ means "the publicly conducted act of eliminating the possibility of φ's being false". Therefore, by this operation, agents can know more. It is easy to show the following.

Proposition 1.12 ([5, Public Announcement Closure Theorem]). *Let* $M = (W, (R_a)_{a \in \mathsf{Agt}}, V)$ *and* $\varphi \in \mathsf{Form}^+$. *If* R_a *is reflexive (transitive, or Euclidean), then so is* R_a^{φ}.

This proposition assures us that the operation $(-)^{\varphi}$ is well-defined on reflexive (transitive, or Euclidean) model.

Remark 1.13. Seriality is not preserved under $(-)^{\varphi}$. A counterexample is depicted in Figure 1.1. The model M_{srl} on the left is defined as: $M_{srl} := (\{w, v\}, R_a, V)$, where $\mathsf{Agt} = \{a\}$, $\mathsf{Prop} = \{p\}$, $R_a := \{(w, v), (v, v)\}$, and $V(p) = \{w\}$. The model M_{srl} is serial, but M_{srl}^{p} on the right is not because $R_a^{p} = \emptyset$. Hence, the corresponding axiom (D) is not in consideration below.

The notion of semantic consequence is defined the same as the basic epistemic logic.

1.4.3 Hilbert Systems

The axioms in Table 1.2 are for the expansion with the public announcement operator. For $\mathbf{X} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$, we call the axiom system expanded from $\mathsf{H}(\mathbf{X})$ by all the axioms in Table 1.2, $\mathsf{H}(\mathbf{X})^+$. The notions of proof, provability, and derivability are defined in the same way as $\mathsf{H}(\mathbf{X})$.

Table 1.2: Axioms for Public Announcement Operator

$([]p)$	$[\varphi]p \leftrightarrow (\varphi \to p)$	$([]\bot)$	$[\varphi]\bot \leftrightarrow (\varphi \to \bot)$
$([]\neg)$	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	$([] \to)$	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$
$([]K)$	$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	$([][])$	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$

Proposition 1.14 ([59, Proposition 4.22])**.** *The axioms in Table 1.2 are valid with respect to the class of all frames.*

Hence, we have the soundness theorem for $\mathsf{H}(\mathbf{X})^+$.

Theorem 1.15 (soundness, [59, Theorem 4.51])**.** *Let* $\varphi \in \mathsf{Form}^+$. *If* $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$, *then* $\Vdash_{\mathbb{F}(\mathbf{X})} \varphi$.

Proof. Obvious from Theorem 1.6 and Proposition 1.14. □

Theorem 1.16 (strong completeness, cf. [59, Theorem 7.26])**.** *Let* $\mathbf{X}$ *be* $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{K4}$, *or* $\mathbf{S4}$ *and* $\Gamma \cup \{\varphi\} \subseteq \mathsf{Form}^+$. *If* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X})} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$.

Proof. Obvious from the reduction technique described in [59, Section 7.4] and Theorem 1.7. □

1.5 Intuitionistic Logic

We describe intuitionistic propositional logic here. The description is based on [55, 35].

1.5.1 Language

Let Prop be a countable set of propositional variables and Form be the set of formulas defined inductively by the following clauses:

$$\mathsf{Form} \ni \varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi,$$

where $p \in \mathsf{Prop}$. We define $\neg\varphi$ as $\varphi \to \bot$ and $\top$ as $\bot \to \bot$.

1.5.2 Kripke Semantics

We introduce Kripke semantics for intuitionistic logic.

Definition 1.17 (frame, model, satisfaction relation)**.** A tuple $F = (W, \leqslant)$ is a *frame* if: W is a set of states, and $\leqslant$ is a preorder on W. A pair $M = (F, V)$ is a *model* if F is a frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$. We denote an underlying set of states of a frame F or a model M by $|F|$ or $|M|$. For a model $M = (W, \leqslant, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model. Satisfaction relation* $M, w \Vdash \varphi$ on pointed models and formulas is defined recursively as follows:

$$
\begin{array}{lll}
M, w \Vdash p & \text{iff} & w \in V(p), \\
M, w \Vdash \bot & & \text{Never}, \\
M, w \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \in W, \text{ if } w \leqslant v \text{ then } M, v \nVdash \varphi \text{ or } M, v \Vdash \psi, \\
M, w \Vdash \varphi \wedge \psi & \text{iff} & M, w \Vdash \varphi \text{ and } M, w \Vdash \psi \\
M, w \Vdash \varphi \vee \psi & \text{iff} & M, w \Vdash \varphi \text{ or } M, w \Vdash \psi
\end{array}
$$

The property of heredity is carried over to any formula.

Proposition 1.18 (heredity, [55, Lemma 6.3.4])**.** *If $M, w \Vdash \varphi$ and $w \leqslant v$, then $M, v \Vdash \varphi$.*

Given a frame $F = (W, \leqslant)$, we say that a formula φ is *valid* in F (notation: $F \Vdash \varphi$) if $(F, V), w \Vdash \varphi$ for every valuation function V and every $w \in W$. Moreover, a formula φ is valid (notation: $\Vdash \varphi$) if $F \Vdash \varphi$ for any frame F.

Definition 1.19 (semantic consequence, [55, p.168])**.** A formula φ is a *semantic consequence* of Γ if for all frame F, a valuation V on F, a state $w \in |F|$, if $(F, V), w \Vdash \Gamma$, then $(F, V), w \Vdash \varphi$. We write it as "$\Gamma \Vdash \varphi$".

1.5.3 Hilbert System

Table 1.3: Axioms and Rules for $\mathsf{H}(\mathbf{Int})$

Axioms and Rules for Intuitionistic Logic			
$(\mathbf{k})$	$\varphi \to (\psi \to \varphi)$	$(\wedge\mathbf{e}_1)$	$(\varphi \wedge \psi) \to \varphi$
$(\mathbf{s})$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	$(\wedge\mathbf{e}_2)$	$(\varphi \wedge \psi) \to \psi$
$(\vee\mathbf{i}_1)$	$\varphi \to (\varphi \vee \psi)$	$(\wedge\mathbf{i})$	$\varphi \to (\psi \to (\varphi \wedge \psi))$
$(\vee\mathbf{i}_2)$	$\psi \to (\varphi \vee \psi)$	$(\bot)$	$\bot \to \varphi$
$(\vee\mathbf{e})$	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$	$(\mathbf{MP})$	From φ and $\varphi \to \psi$, infer ψ

The Hilbert system $\mathsf{H}(\mathbf{Int})$ for propositional intuitionistic logic is constructed from axioms and rules shown in Table 1.3. Derivability is defined as follows.

Definition 1.20. Let $\Gamma \cup \{\varphi\}$ be a set of formulas. A *proof for* φ *from* Γ *within* $\mathsf{H}(\mathbf{Int})$ is a finite sequence $(\varphi_i)_1^m$ of formulas such that

1. $\varphi_m = \varphi$.
2. every φ_i in the sequence is
   (a) either an instance of one of the axioms
   (b) or a member of Γ
   (c) or else the result of the application of the inference rule (**MP**) to two formulas in the sequence that appear before φ_i.

A formula φ *is derivable from* Γ *in* $\mathsf{H}(\mathbf{Int})$ (notation: $\Gamma \vdash_{\mathsf{H}(\mathbf{Int})} \varphi$) if there exists a proof for φ from Γ within $\mathsf{H}(\mathbf{Int})$.

We have the following theorems.

Theorem 1.21 (soundness, [55, Theorem 6.3.6])**.** *If* $\vdash_{\mathsf{H}(\mathbf{Int})} \varphi$*, then* $\Vdash \varphi$*.*

Theorem 1.22 (strong completeness, [55, Theorem 6.3.10])**.** *If* $\Gamma \Vdash \varphi$*, then* $\Gamma \vdash_{\mathsf{H}(\mathbf{Int})} \varphi$*.*

1.5.4 Sequent Calculus

A *sequent* of the sequent calculus **LJ** [11, 12] for propositional intuitionistic logic is a pair of finite multisets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$", where $\#\Delta \leq 1$. Here $\#\Sigma$ denotes the number of elements in the multiset Σ. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in **LJ** if there exists a finite tree of sequents, whose root is $\Gamma \Rightarrow \Delta$ and each node of which is inferred by some rule (including axioms) in **LJ**. We write it as $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \Delta$.

We note that $\mathsf{H}(\mathbf{Int})$ and **LJ** are equipollent in the following sense.

Theorem 1.23 (Equipollence, [54, Theorem 2.4.2, 3.3.1])**.** *1. If* $\vdash_{\mathsf{H}(\mathbf{Int})} \varphi$*, then* $\vdash_{\mathbf{LJ}} \Rightarrow \varphi$*. 2. If* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{H}(\mathbf{Int})} \bigwedge \Gamma \to \bigvee \Delta$*, where* $\bigwedge \varnothing := \top$ *and* $\bigvee \varnothing := \bot$*.*

We have the cut-elimination theorem for **LJ**.

Theorem 1.24 (Cut-Elimination, [13, 'Hauptsatz' in p.38], [35, Theorem 2])**.** *If* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{LJ}^-} \Gamma \Rightarrow \Delta$*, where* $\mathbf{LJ}^-$ *denotes a system "***LJ** *minus the cut rule".*

We also have the Craig interpolation theorem by the Maehara method [26, 35].

Table 1.4: Sequent Calculus **LJ**

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (Id) \quad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to) \quad \frac{\Gamma_1 \Rightarrow \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^1) \quad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^2)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^1) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^2) \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

Theorem 1.25 (Craig Interpolation Theorem, [35, Theorem 35])**.** *Given that* $\vdash_{\mathbf{LJ}} \varphi \Rightarrow \psi$*, there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathbf{LJ}} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathbf{LJ}} \chi \Rightarrow \psi$.
2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

Chapter 2

Proof Theory of Distributed Knowledge

This chapter is organized as follows. Section 2.1 provides the necessary preliminaries of distributed epistemic logics. We fix our language and give semantic definition of a distributed knowledge operator D_G. We also introduce the known Hilbert system of epistemic logics with distributed knowledge. In Section 2.2, we propose Gentzen-style sequent calculi for the logics defined in Section 2.1. We also establish the equipollence results between the sequent calculi and the Hilbert systems introduced in Section 2.1 (Theorem 2.3). In Section 2.3, we prove our main technical results, Cut Elimination Theorem (Theorem 2.4) and Craig Interpolation Theorem (Theorem 2.9).

The author's contribution is as follows. The sequent calculus for $\mathbf{K}_D$ was constructed by the author's supervisor, Katsuhiko Sano. The author constructed the sequent calculi for $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$ under the advice of his supervisor and the sequent calculus for $\mathbf{S5}_D$ by himself. The author proved the cut elimination theorem and Craig interpolation theorem for $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$.

The content of this chapter is based on [30].

2.1 Overview of Epistemic Logics with Distributed Knowledge Operators

2.1.1 Language

We denote a finite set of agents by Agt. We call a *nonempty* subset of Agt "group" and denote it by G, H, etc. Let Prop be a countable set of propositional variables and Form

be the set of formulas defined inductively by the following clauses:

$$\mathsf{Form} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid D_G\varphi,$$

where $p \in \mathsf{Prop}$ and G is a group. It is noted that $\wedge$ and $\vee$ are defined in the same way as in the classical propositional logic. That is, $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$ and $\varphi \vee \psi := \neg\varphi \to \psi$. We also define $\top$ as $\bot \to \bot$. We also define the epistemic operator $K_a\varphi$ (read "agent a knows that φ") as $D_{\{a\}}\varphi$. As noted above, an expression of the form $D_\emptyset\varphi$ is *not* a well-formed formula, since we have excluded $\emptyset$ from our definition of groups.

2.1.2 Kripke Semantics

We introduce the ordinary Kripke semantics for multi-agent epistemic logic here. Let W be a possibly countable set of states, $(R_a)_{a\in\mathsf{Agt}}$ be a family of binary relations on W, indexed by agents, and V be a valuation function $\mathsf{Prop} \to \mathcal{P}(W)$. We call a pair $F = (W, (R_a)_{a\in\mathsf{Agt}})$ a *frame* and a tuple $M = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ a *model*. For a model $M = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model*. *Satisfaction relation* $M, w \models \varphi$ on pointed models and formulas are defined recursively as follows:

$$\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p), \\
M, w \models \bot & & \text{Never}, \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi, \\
M, w \models \varphi \to \psi & \text{iff} & M, w \not\models \varphi \text{ or } M, w \models \psi, \\
M, w \models D_G\varphi & \text{iff} & \text{for all } v \in W, \text{ if } (w, v) \in \bigcap_{a\in G} R_a \text{ then } M, v \models \varphi.
\end{array}$$

It is noted from our definition of $K_a\varphi := D_{\{a\}}\varphi$ that the satisfaction of $K_a\varphi$ at a state w of a model M is given as follows:

$$M, w \models K_a\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \models \varphi.$$

Given a frame $F = (W, (R_a)_{a\in\mathsf{Agt}})$, we say that a formula φ is *valid* in F (notation: $F \models \varphi$) if $(F, V), w \models \varphi$ for every valuation function V and every $w \in W$. Moreover, a formula φ is valid in a class $\mathbb{F}$ of frames if $F \models \varphi$ for every $F \in \mathbb{F}$. Let us say that a set Γ of formulas *defines* a class $\mathbb{F}$ of frames if, for every frame F, $F \in \mathbb{F}$ is equivalent to: $F \models \varphi$ for all $\varphi \in \Gamma$.

2.1.3 Hilbert Systems

We review the known Hilbert system for epistemic logics with D_G operators (cf. [8]). Hilbert system $\mathsf{H}(\mathbf{K}_D)$ is defined as in the following table.

Hilbert System $\mathsf{H}(\mathbf{K}_D)$	
(Taut)	all instantiations of propositional tautologies
(Incl)	$D_G\varphi \to D_H\varphi \quad (G \subseteq H)$
(K)	$D_G(\varphi \to \psi) \to (D_G\varphi \to D_G\psi)$
(MP)	From $\varphi \to \psi$ and φ infer ψ
(Nec)	From φ infer $D_G\varphi$
Additional Axiom Schemes	
(T)	$D_G\varphi \to \varphi$
(4)	$D_G\varphi \to D_G D_G\varphi$
(5)	$\neg D_G\varphi \to D_G\neg D_G\varphi$

For additional axioms schemes, we note that (T), (4) and (5) define the class of reflexive, transitive and Euclidean frames, respectively (here, e.g., a "reflexive" frame means that R_a is reflexive for all agents $a \in \mathsf{Agt}$). Hilbert systems $\mathsf{H}(\mathbf{KT}_D)$,$\mathsf{H}(\mathbf{K4}_D)$, $\mathsf{H}(\mathbf{S4}_D)$, and $\mathsf{H}(\mathbf{S5}_D)$ are defined as axiomatic expansions of $\mathsf{H}(\mathbf{K}_D)$ with (T), (4), (T) and (4), and (T) and (5), respectively). Given any Hilbert system $\mathbf{X}$ above, the notion of provability is defined the same as Definition 1.3.

For Hilbert systems $\mathsf{H}(\mathbf{K}_D)$, $\mathsf{H}(\mathbf{KT}_D)$, $\mathsf{H}(\mathbf{S4}_D)$ and $\mathsf{H}(\mathbf{S5}_D)$, the following soundness and completeness results are known [8, 62] (we cannot find any explicit reference on $\mathsf{H}(\mathbf{K4}_D)$, private communication by Thomas Ågotnes).

Fact 2.1 ([8, Theorem 3.4.1], [62, Theorem 15]). *Each of Hilbert systems $\mathsf{H}(\mathbf{K}_D)$, $\mathsf{H}(\mathbf{KT}_D)$, $\mathsf{H}(\mathbf{S4}_D)$ and $\mathsf{H}(\mathbf{S5}_D)$ is sound and complete with regard to the class of frames defined by additional axiom schemes.*

2.2 Sequent Calculi

A *sequent* is a pair of finite multi-sets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$", whose reading is "if all formulas in Γ hold then some formulas in Δ hold." We now propose our sequent calculi for the logics for distributed knowledge as in Table 2.1. Axioms, structural rules and propositional logical rules are common to **LK** [11, 12] and the rest are new. We note that when $n = 0$, e.g., in the rule (D) of Table 2.1, the multi-sets $\{\varphi_1, \dots, \varphi_n\}$ and $\{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n\}$ are regarded as the empty multi-set and $\bigcup_{i=1}^{n} G_i = \bigcup \emptyset = \emptyset$. A

Table 2.1: Sequent Calculi for $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, $\mathbf{S4}_D$, and $\mathbf{S5}_D$

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (Id) \quad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow\to) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

Logical Rules for D_G of $\mathbf{K}_D$

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D)$$

Logical Rules for D_G of $\mathbf{KT}_D$

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

Logical Rules for D_G of $\mathbf{K4}_D$

$$\frac{\varphi_1, \ldots, \varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D^{\mathbf{K4}_D})$$

Logical Rules for D_G of $\mathbf{S4}_D$

$$\frac{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D^{\mathbf{S4}_D}) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

Logical Rules for D_G of $\mathbf{S5}_D$

$$\frac{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m, \chi \quad (\bigcup_{i=1}^{n} G_i \cup \bigcup_{j=1}^{m} H_j \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m, D_G\chi}\ (\Rightarrow D^{\mathbf{S5}_D})$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

sequent $\Gamma \Rightarrow \Delta$ is *derivable* in each calculus $\mathsf{G}(\mathbf{X})$ if there exists a finite tree of sequents, whose root is $\Gamma \Rightarrow \Delta$ and each node of which is inferred by some rule in $\mathsf{G}(\mathbf{X})$. We write it as $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$. We introduce a notion of "principal formula" for a proof described later. A principal formula is defined for each inference rule, except for the axioms and (Cut) rule and is informally expressed as "a formula, on which the inference rule acts". A principal formula of the structural rules, the rules for $\to$ and the rule $(D \Rightarrow)$ is a formula appearing in the lower sequent, which is not contained in $\Gamma, \Delta, \Gamma_1, \Gamma_2, \Delta_1$, or Δ_2. A principal formula of the rules for D_G operator other than $(D \Rightarrow)$ is every formula in the lower sequent.

Remark 2.2. The idea underlying the rule (D) is similar to that of an inference rule called "$R12$" described in [43, section 4]. Sequent calculi $\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, $\mathsf{G}(\mathbf{S4}_D)$, and $\mathsf{G}(\mathbf{S5}_D)$ are constructed based on the known sequent calculi for $\mathbf{KT}, \mathbf{K4}, \mathbf{S4}$, and $\mathbf{S5}$, respectively (the reader is referred to subsection 1.3.4).

We note that for any epistemic logic $\mathbf{X}$ with distributed knowledge under consideration, $\mathsf{H}(\mathbf{X})$ and $\mathsf{G}(\mathbf{X})$ are equipollent in the following sense, and hence that each system $\mathsf{G}(\mathbf{X})$ deserves its own name.

Theorem 2.3 (Equipollence)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, $\mathbf{S4}_D$, *and* $\mathbf{S5}_D$. *Then, the following hold.*

1. *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, *then* $\vdash_{\mathsf{G}(\mathbf{X})} \Rightarrow \varphi$.
2. *If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma \to \bigvee \Delta$, *where* $\bigwedge \emptyset := \top$ *and* $\bigvee \emptyset := \bot$.

Proof. We show the case of $\mathbf{K}_D$ and $\mathbf{S5}_D$ because the case of $\mathbf{K}_D$ is the simplest and the case of $\mathbf{S5}_D$ is the most non-trivial. The other cases can be similarly shown. Here we focus on item 2 alone because item 1 is trivial in both cases.

($\mathbf{X} = \mathbf{K}_D$) We show item 2 by induction on the structure of the derivation for the sequent $\Gamma \Rightarrow \Delta$. We deal with the case for the rule (D) only, since the other cases are well-known to be true. Suppose we have a derivation

$$\dfrac{\dfrac{\mathcal{D}}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

We show $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to D_G\psi$. We have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} \varphi_i \to \psi$ as the induction hypothesis for the derivation $\mathcal{D}$. From this, we can infer by necessitation $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^{n} \varphi_i \to \psi)$. By this and axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^{n} \varphi_i) \to D_G\psi$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})}$

$\bigwedge_{i=1}^{n} D_G\varphi_i \to D_G\psi$. Therefore, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to \bigwedge_{i=1}^{n} D_G\varphi_i$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to D_G\varphi_i$ for any $i \in \{1,\dots,n\}$. This is evident because we have a propositional tautology $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to D_{G_i}\varphi_i$ and the axiom (Incl) $\vdash_{\mathsf{H}(\mathbf{X})} D_{G_i}\varphi_i \to D_G\varphi_i$.

($\mathbf{X} = \mathbf{S5}_D$) We show item 2 by induction on the structure of the derivation for the sequent $\Gamma \Rightarrow \Delta$. We deal with the case for the rules $(\Rightarrow D^{\mathbf{S5}_D})$ and $(D \Rightarrow)$ only, since the other cases are well-known to be true.

First, we show the case of $(\Rightarrow D^{\mathbf{S5}_D})$. Suppose we have a derivation

$$\dfrac{\dfrac{\mathcal{D}}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_{H_1}\psi_1,\dots,D_{H_m}\psi_m,\chi} \quad (\bigcup_{i=1}^{n} G_i \cup \bigcup_{j=1}^{m} H_j \subseteq G)}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_{H_1}\psi_1,\dots,D_{H_m}\psi_m, D_G\chi} \,(\Rightarrow D^{\mathbf{S5}_D}).$$

We show $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to \bigvee_{j=1}^{m} D_{H_j}\psi_j \vee D_G\chi$. We have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to \bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi$ as the induction hypothesis for the derivation $\mathcal{D}$. From this, we can infer by necessitation $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to \bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi)$. By this and axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^{n} D_{G_i}\varphi_i) \to D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi)$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_G D_{G_i}\varphi_i \to D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi)$. Therefore, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to \bigwedge_{i=1}^{n} D_G D_{G_i}\varphi_i$ and $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \to \bigvee_{j=1}^{m} D_{H_j}\psi_j \vee D_G\chi$. First, we show the former. This is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{G_i}\varphi_i \to D_G D_{G_i}\varphi_i$ for any $i \in \{1,\dots,n\}$. This is easily shown by $\vdash_{\mathsf{H}(\mathbf{X})} D_{G_i}\varphi_i \to D_{G_i}D_{G_i}\varphi_i$ and the axiom (Incl) $\vdash_{\mathsf{H}(\mathbf{X})} D_{G_i}D_{G_i}\varphi_i \to D_G D_{G_i}\varphi_i$. Next, we show the latter, i.e., $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \to \bigvee_{j=1}^{m} D_{H_j}\psi_j \vee D_G\chi$. This is easily shown to be equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j \to D_G\chi$ through propositional tautologies. Then, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j \to D_G((\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j)$ and $\vdash_{\mathsf{H}(\mathbf{X})} D_G((\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j) \to D_G\chi$. The former is easily obtained through the axiom (K) from $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j \to D_G(\bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j)$, which is shown by the axioms (5), (Incl), and (K). The latter, i.e., $\vdash_{\mathsf{H}(\mathbf{X})} D_G((\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j) \to D_G\chi$ is easily obtained through the axiom (K) from $\vdash_{\mathsf{H}(\mathbf{X})} (\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j \to \chi$, which is the case because the antecedent $(\bigvee_{j=1}^{m} D_{H_j}\psi_j \vee \chi) \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j$ is equivalent to $\chi \wedge \bigwedge_{j=1}^{m} \neg D_{H_j}\psi_j$ through propositional tautologies.

Second, we show the case of $(D \Rightarrow)$. Suppose we have a derivation

$$\dfrac{\dfrac{\mathcal{D}}{\varphi, \Gamma \Rightarrow \Delta}}{D_G\varphi, \Gamma \Rightarrow \Delta} \,(D \Rightarrow).$$

We show $\vdash_{\mathsf{H}(\mathbf{X})} D_G\varphi \wedge \bigwedge\Gamma \to \bigvee\Delta$. We have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \bigwedge\Gamma \to \bigvee\Delta$ as the induction hypothesis for the derivation $\mathcal{D}$. Then, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} D_G\varphi\wedge\bigwedge\Gamma \to \varphi\wedge\bigwedge\Gamma$, which is obvious by the axiom (T) $\vdash_{\mathsf{H}(\mathbf{X})} D_G\varphi \to \varphi$. $\square$

2.3 Main Proof-Theoretic Results

2.3.1 Cut-Elimination

The cut elimination theorem does not hold for $\mathsf{G}(\mathbf{S5}_D)$, because the application of (Cut) rule in the following derivation cannot be eliminated [33].

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,(Id)}{\neg p, p \Rightarrow}\,(\neg\Rightarrow)}{D_{\{a\}}\neg p, p \Rightarrow}\,(D\Rightarrow)}{p \Rightarrow \neg D_{\{a\}}\neg p}\,(\Rightarrow\neg) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{D_{\{a\}}\neg p \Rightarrow D_{\{a\}}\neg p}\,(Id)}{\Rightarrow \neg D_{\{a\}}\neg p, D_{\{a\}}\neg p}\,(\Rightarrow\neg)}{\Rightarrow D_{\{a\}}\neg D_{\{a\}}\neg p, D_{\{a\}}\neg p}\,(\Rightarrow D^{\mathbf{S5}_D})}{\neg D_{\{a\}}\neg p \Rightarrow D_{\{a\}}\neg D_{\{a\}}\neg p}\,(\neg\Rightarrow)}{p \Rightarrow D_{\{a\}}\neg D_{\{a\}}\neg p}\,(Cut)}{\Rightarrow p \to D_{\{a\}}\neg D_{\{a\}}\neg p}\,(\Rightarrow\to)
$$

Therefore, we establish the cut elimination theorem for our sequent calculi except for $\mathsf{G}(\mathbf{S5}_D)$.

Theorem 2.4 (Cut-Elimination). *Let* $\mathbf{X}$ *be any of* $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, *and* $\mathbf{S4}_D$. *Then, the following holds: If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$, *where* $\mathsf{G}^-(\mathbf{X})$ *denotes a system "*$\mathsf{G}(\mathbf{X})$ *minus the cut rule".*

Following [24, Section 9.3] and [36, Section 2.2], we consider a system $\mathsf{G}^*(\mathbf{X})$, in which the cut rule is replaced by the "extended" cut rule defined as:

$$
\dfrac{\Gamma \Rightarrow \Delta, \varphi^n \quad \varphi^m, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (ECut),
$$

where φ^n denotes the multi-set of n-copies of φ and $n, m \geq 0$. Since $(ECut)$ is the same as (Cut) when we set $n = m = 1$, it is obvious that if $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, so it suffices to show the following.

Lemma 2.5. *Let* $\mathbf{X}$ *be any of* $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, *and* $\mathbf{S4}_D$. *Then, the following holds: If* $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$.

Proof. Let $\mathbf{X}$ be any of $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$. Suppose $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$ and fix one derivation for the sequent. To obtain an $(ECut)$-free derivation of $\Gamma \Rightarrow \Delta$, it is enough to concentrate on a derivation whose root is derived by $(ECut)$ and which has no other application of $(ECut)$. Let us suppose that $\mathcal{D}$ has the following structure:

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \Delta, \varphi^n}\ (\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (ECut),$$

where the derivations $\mathcal{L}$ and $\mathcal{R}$ has no application of $(ECut)$ and $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are meta-variables for the name of rule applied there. Let the number of logical symbols (including D_G) appearing in φ be $c(\mathcal{D})$ and the number of sequents in $\mathcal{L}$ and $\mathcal{R}$ be $w(\mathcal{D})$. We show the lemma by double induction on $(c(\mathcal{D}), w(\mathcal{D}))$. If $n = 0$ or $m = 0$, we can derive the root sequent of $\mathcal{D}$ without using $(ECut)$ by weakening rules. So, in what follows we assume $n, m > 0$. Then, it is sufficient to consider the following four cases following [37, proof of Theorem 2.3], [24, Section 9.3], and [36, Section 2.2]: [1]

1. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is an axiom.

2. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a structural rule.

3. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a logical rule and a cut formula φ is not principal (in the sense we have specified in Section 2.2) for that rule.

4. $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are both logical rules (including (D)) for the same logical symbol and a cut formula φ is principal for each rule.

We omit case 1 and case 2 in the proof since this is well-known in the proof for **LK** (cf. [36, pp.28-29, Theorem 2.2]). For the same reason, the cases concerning propositional logical rules are also not treated. Therefore, we treat case 3 and case 4 involving D_G for each logic **X**.

$(\mathbf{X} = \mathbf{K}_D)$ There is no case to be considered in case 3. As case 4, we consider the case where both $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are rules (D). In that case, the given derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \quad \dfrac{\dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^m H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (D)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow D_H\chi}\ (ECut)$$

[1] In case 4, we assume the condition for both rule applications, because if the one of the two rule applications does not satisfy the condition, the whole derivation should be categorized into one of the rest cases.

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$: [2]

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1,\dots,\varphi_n \Rightarrow \psi} \quad \dfrac{\mathcal{R}'}{\psi^m,\psi_1,\dots,\psi_m \Rightarrow \chi}}{\varphi_1,\dots,\varphi_n,\psi_1,\dots,\psi_m \Rightarrow \chi}\,(ECut) \quad (\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H)}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n,D_{H_1}\psi_1,\dots,D_{H_m}\psi_m \Rightarrow D_H\chi}\,(D).$$

We call $\mathcal{E}'$ its subderivation whose root sequent is $\varphi_1,\dots,\varphi_n,\psi_1,\dots,\psi_m \Rightarrow \chi$. The derivation $\mathcal{E}'$ have no application of $(ECut)$ and $c(\mathcal{E}') < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}'$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}}'$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n,D_{H_1}\psi_1,\dots,D_{H_m}\psi_m \Rightarrow D_H\chi$ as required.

($\mathbf{X} = \mathbf{KT}_D$) As case 3, the case involving the rule $(D \Rightarrow)$ should be considered. If $\mathbf{rule}_{\mathcal{L}} = (D \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\psi,\Gamma \Rightarrow \Delta,\varphi^n}}{D_G\psi,\Gamma \Rightarrow \Delta,\varphi^n}\,(D \Rightarrow) \quad \dfrac{\mathcal{R}}{\varphi^m,\Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{D_G\psi,\Gamma,\Sigma \Rightarrow \Delta,\Theta}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\psi,\Gamma \Rightarrow \Delta,\varphi^n} \quad \dfrac{\mathcal{R}}{\varphi^m,\Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{\psi,\Gamma,\Sigma \Rightarrow \Delta,\Theta}\,(ECut)}{D_G\psi,\Gamma,\Sigma \Rightarrow \Delta,\Theta}\,(D \Rightarrow)$$

The subderivation $\mathcal{E}'$ whose root is $\psi,\Gamma,\Sigma \Rightarrow \Delta,\Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}'$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}}'$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_G\psi,\Gamma,\Sigma \Rightarrow \Delta,\Theta$.

If $\mathbf{rule}_{\mathcal{R}} = (D \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \Delta,\varphi^n}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\dfrac{\mathcal{R}'}{\varphi^m,\psi,\Sigma \Rightarrow \Theta}}{\varphi^m,D_G\psi,\Sigma \Rightarrow \Theta}\,(D \Rightarrow)}{\Gamma,D_G\psi,\Sigma \Rightarrow \Delta,\Theta}\,(ECut)$$

[2] Note that the condition $\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H$ in $\mathcal{E}$ can be obtained by the conditions $\bigcup_{i=1}^n G_i \subseteq G$ and $G \cup \bigcup_{j=1}^m H_j \subseteq H$ in $\mathcal{D}$ through "cutting" G.

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \Delta, \varphi^n}\,(\mathbf{rule}_{\mathcal{L}}) \qquad \dfrac{\mathcal{R}'}{\varphi^m, \psi, \Sigma \Rightarrow \Theta}}{\Gamma, \psi, \Sigma \Rightarrow \Delta, \Theta}\,(ECut)}{\Gamma, D_G\psi, \Sigma \Rightarrow \Delta, \Theta}\,(D \Rightarrow)$$

The subderivation $\mathcal{E}'$ whose root is $\Gamma, \psi, \Sigma \Rightarrow \Delta, \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, D_G\psi, \Sigma \Rightarrow \Delta, \Theta$.

In case 4, the derivation $\mathcal{D}$ has either the same structure as the case where $\mathbf{X} = \mathbf{K}_D$ or the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D) \qquad \dfrac{\dfrac{\mathcal{R}'}{\psi, (D_G\psi)^{m-1}, \Sigma \Rightarrow \Theta}}{(D_G\psi)^m, \Sigma \Rightarrow \Theta}\,(D \Rightarrow)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \qquad \dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}'}{\psi, (D_G\psi)^{m-1}, \Sigma \Rightarrow \Theta}}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \psi, \Sigma \Rightarrow \Theta}\,(ECut)}{\varphi_1, \dots, \varphi_n, D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}\,(ECut)}{\vdots}\,(D \Rightarrow)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}\,(D \Rightarrow)}{\dfrac{\vdots}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}\,(c \Rightarrow)}\,(c \Rightarrow)$$

The subderivation $\mathcal{E}_1$ whose root is $D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \psi, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_1}$ having the same root sequent. Name $\mathcal{E}_2$, the derivation obtained by replacing the derivation $\mathcal{E}_1$ by $\tilde{\mathcal{E}_1}$ in the subderivation whose root is $\varphi_1, \dots, \varphi_n, D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta$. The derivation $\mathcal{E}_2$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_2) < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_2}$ having the same root sequent. Thus, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta$.

($\mathbf{X} = \mathbf{K4}_D$) There is no case to be considered in case 3. In case 4, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}}{(D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow D_H\chi}\ (ECut),$$

where $\mathcal{L}$ and $\mathcal{R}$ are of the following form.

$$\mathcal{L} \equiv \dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D^{\mathbf{K4}_D})$$

$$\mathcal{R} \equiv \dfrac{\dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m, (D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^m H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (\Rightarrow D^{\mathbf{K4}_D})$$

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi} \quad \dfrac{\mathcal{E}_1}{\psi^m, \psi_1, \ldots, \psi_m, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow \chi}}{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow \chi}\ (ECut)}{}\ (c \Rightarrow)}}{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow \chi}\ (c \Rightarrow)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (\Rightarrow D^{\mathbf{K4}_D}),$$

where

$$\mathcal{E}_1 \equiv \dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m, (D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow \chi}}{\psi^m, \psi_1, \ldots, \psi_m, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow \chi}\ (ECut).$$

Note that $\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H$ is satisfied at the last rule application in $\mathcal{E}$. The subderivation $\mathcal{E}_1$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_1}$ having the same root sequent. Name $\mathcal{E}_2$, the derivation obtained by replacing the derivation $\mathcal{E}_1$ by $\tilde{\mathcal{E}_1}$ in the subderivation whose root is $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow \chi$. The derivation $\mathcal{E}_2$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_2) < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_2}$ having the same root sequent. Thus, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi$.

($\mathbf{X} = \mathbf{S4}_D$) We omit case 3 since the case to be considered is the same as $\mathbf{KT}_D$. In case 4, one of the possible structures of the derivation $\mathcal{D}$ is the following.

$$
\dfrac{\dfrac{\dfrac{\mathcal{L}'}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi}(\Rightarrow D^{\mathbf{S4}_D}) \quad \dfrac{\dfrac{\mathcal{R}'}{(D_G\psi)^m, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^m H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_m \Rightarrow D_H\chi}(\Rightarrow D^{\mathbf{S4}_D})}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_n \Rightarrow D_H\chi}(ECut)
$$

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$:

$$
\dfrac{\dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}'}{(D_G\psi)^m, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_m \Rightarrow \chi}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_n \Rightarrow \chi}(ECut) \quad (\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_m \Rightarrow D_H\chi}(\Rightarrow D^{\mathbf{S4}_D}).
$$

The subderivation $\mathcal{E}'$ whose root sequent is $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_n \Rightarrow \chi$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{H_1}\psi_1,\ldots,D_{H_m}\psi_m \Rightarrow D_H\chi$.

The other possible structure of the derivation $\mathcal{D}$ is the following:

$$
\dfrac{\dfrac{\dfrac{\mathcal{L}'}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi}(\Rightarrow D^{\mathbf{S4}_D}) \quad \dfrac{\dfrac{\mathcal{R}'}{\psi, (D_G\psi)^{m-1}, \Sigma \Rightarrow \Theta}}{(D_G\psi)^m, \Sigma \Rightarrow \Theta}(D \Rightarrow)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}(ECut)
$$

This can be transformed into the following derivation $\mathcal{E}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{L}'}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow \psi} \quad \dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}'}{\psi, (D_G\psi)^{m-1}, \Sigma \Rightarrow \Theta}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \psi, \Sigma \Rightarrow \Theta}(ECut)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}(ECut)}{\vdots}(c \Rightarrow)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta}(c \Rightarrow)
$$

The subderivation $\mathcal{E}_1$ whose root is $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \psi, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_1}$ having the same root sequent. Name $\mathcal{E}_2$, the derivation obtained by replacing the derivation $\mathcal{E}_1$ by $\tilde{\mathcal{E}_1}$ in the subderivation whose root is $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta$. The derivation $\mathcal{E}_2$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_2) < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}_2$ having the same root sequent. Thus, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, \Sigma \Rightarrow \Theta$. $\square$

2.3.2 Craig Interpolation Theorem

As an application of the cut elimination theorem, Craig interpolation theorem can be derived, using a Maehara method described in [26] (application of the method to basic modal logic can also be found in [35]). To state a main lemma for proving Craig Interpolation Theorem, some definitions are needed.

Definition 2.6 (Partition)**.** A partition for a sequent $\Gamma \Rightarrow \Delta$ is defined as a tuple $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$, such that the multi-set union of Γ_1 and Γ_2 (Δ_1 and Δ_2) is equal to Γ (Δ, respectively).

Definition 2.7. For a formula φ, $\mathsf{Prop}(\varphi)$ is defined as the set of propositional variables appearing in φ. For a multi-set of formulas Γ, $\mathsf{Prop}(\Gamma)$ is defined as $\bigcup_{\varphi\in\Gamma}\mathsf{Prop}(\varphi)$. Similarly, $\mathsf{Agt}(\varphi)$ is defined as the set of agents appearing in φ and $\mathsf{Agt}(\Gamma)$ as $\bigcup_{\varphi\in\Gamma}\mathsf{Agt}(\varphi)$

The following is a key lemma for Craig Interpolation Theorem.

Lemma 2.8. *Let $\mathbf{X}$ be any of $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$. Suppose $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$. Then, for any partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ for the sequent $\Gamma \Rightarrow \Delta$, there exists a formula φ called "interpolant", satisfying the following:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma_1 \Rightarrow \Delta_1, \varphi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi, \Gamma_2 \Rightarrow \Delta_2$.
2. $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$.
3. $\mathsf{Agt}(\varphi) \subseteq \mathsf{Agt}(\Gamma_1, \Delta_1) \cap \mathsf{Agt}(\Gamma_2, \Delta_2)$.

Proof. We prove the case of $\mathbf{K}_D$ by induction on the structure of a derivation for $\Gamma \Rightarrow \Delta$, because the other cases can be shown by the same idea. Fix the derivation and name it $\mathcal{D}$. By the cut-elimination theorem (Theorem 2.4), we can assume that $\mathcal{D}$ is cut-free. We treat only the case of (D) below (for other cases, the reader is referred to [35]). Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{E}}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

There are the following two partitions of $D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi$:

(a) a partition $\langle(D_{G_1}\varphi_1, \dots, D_{G_k}\varphi_k : \emptyset); (D_{G_{k+1}}\varphi_{k+1}, \dots, D_{G_n}\varphi_n : D_G\psi)\rangle$.

(b) a partition $\langle(D_{G_1}\varphi_1, \dots, D_{G_k}\varphi_k : D_G\psi); (D_{G_{k+1}}\varphi_{k+1}, \dots, D_{G_n}\varphi_n : \emptyset)\rangle$.

For case (a), induction hypothesis on $\mathcal{E}$ for a partition $\langle(\varphi_1,\ldots,\varphi_k : \emptyset);(\varphi_{k+1},\ldots,\varphi_n : \psi)\rangle$ is used. That is, we have derivations for $\varphi_1,\ldots,\varphi_k \Rightarrow \chi$ and $\chi,\varphi_{k+1},\ldots,\varphi_n \Rightarrow \psi$ for some formula χ. If $k > 0$, we can choose $D_{\bigcup_{i=1}^k G_i}\chi$ as a required interpolant, because we have the following derivations:

$$\dfrac{\dfrac{\text{I.H.}}{\varphi_1,\ldots,\varphi_k \Rightarrow \chi} \quad (\bigcup_{i=1}^k G_i \subseteq \bigcup_{i=1}^k G_i)}{D_{G_1}\varphi_1,\ldots,D_{G_k}\varphi_k \Rightarrow D_{\bigcup_{i=1}^k G_i}\chi}\,(D)$$

$$\dfrac{\dfrac{\text{I.H.}}{\chi,\varphi_{k+1},\ldots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^k G_i \cup \bigcup_{i=k+1}^n G_i = \bigcup_{i=1}^n G_i \subseteq G)}{D_{\bigcup_{i=1}^k G_i}\chi, D_{G_{k+1}}\varphi_{k+1},\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D)$$

Furthermore, the interpolant enjoys the condition 2 and 3 by the induction hypothesis and simple calculation. If $k = 0$, we can choose χ (equivalent to $\top$) as an interpolant.

Next, consider case (b). By induction hypothesis on $\mathcal{E}$ for a partition $\langle(\varphi_1,\ldots,\varphi_k : \psi);(\varphi_{k+1},\ldots,\varphi_n : \emptyset)\rangle$, we have derivations for $\varphi_1,\ldots,\varphi_k \Rightarrow \psi,\chi$ and $\chi,\varphi_{k+1},\ldots,\varphi_n \Rightarrow$ for some formula χ. If $k < n$, we can choose $\neg D_{\bigcup_{i=k+1}^n G_i}\neg\chi$ as a required interpolant, because we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\varphi_1,\ldots,\varphi_k \Rightarrow \psi,\chi}}{\neg\chi,\varphi_1,\ldots,\varphi_k \Rightarrow \psi}\,(\neg\Rightarrow) \quad (\bigcup_{i=k+1}^n G_i \cup \bigcup_{i=1}^k G_i = \bigcup_{i=1}^n G_i \subseteq G)}{D_{\bigcup_{i=k+1}^n G_i}\neg\chi, D_{G_1}\varphi_1,\ldots,D_{G_k}\varphi_k \Rightarrow D_G\psi}\,(D)}{D_{G_1}\varphi_1,\ldots,D_{G_k}\varphi_k \Rightarrow D_G\psi, \neg D_{\bigcup_{i=k+1}^n G_i}\neg\chi}\,(\Rightarrow\neg)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\chi,\varphi_{k+1},\ldots,\varphi_n \Rightarrow}}{\varphi_{k+1},\ldots,\varphi_n \Rightarrow \neg\chi}\,(\Rightarrow\neg) \quad (\bigcup_{i=k+1}^n G_i \subseteq \bigcup_{i=k+1}^n G_i)}{D_{G_{k+1}}\varphi_{k+1},\ldots,D_{G_n}\varphi_n \Rightarrow D_{\bigcup_{i=k+1}^n G_i}\neg\chi}\,(D)}{\neg D_{\bigcup_{i=k+1}^n G_i}\neg\chi, D_{G_{k+1}}\varphi_{k+1},\ldots,D_{G_n}\varphi_n \Rightarrow}\,(\neg\Rightarrow)$$

Furthermore, the interpolant enjoys the condition 2 and 3 by the induction hypothesis and a simple calculation. If $k = n$, we can choose χ (equivalent to $\bot$) as an interpolant. □

Theorem 2.9 (Craig Interpolation Theorem)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{K}_D$*,* $\mathbf{KT}_D$*,* $\mathbf{K4}_D$*, and* $\mathbf{S4}_D$*. Given that* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$*, there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \chi \Rightarrow \psi$.

2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

3. $\mathsf{Agt}(\chi) \subseteq \mathsf{Agt}(\varphi) \cap \mathsf{Agt}(\psi)$.

We note that not only the condition for propositional variables but also the condition for agents can be satisfied. By the third condition, it is assured, for example, that there exists an interpolant without any D_G operator for a pair of formulas φ and ψ such that $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$ and $\mathsf{Agt}(\varphi) \cap \mathsf{Agt}(\psi) = \emptyset$.

Proof. When we set $\Gamma := \varphi$ and $\Delta := \psi$, and take a partition $\langle(\varphi : \emptyset); (\emptyset : \psi)\rangle$, Lemma 2.8 proves Craig Interpolation Theorem. $\square$

Chapter 3

Intuitionistic Epistemic Logic with Distributed Knowledge

This chapter is organized as follows. In Section 3.1, we introduce syntax and semantics for intuitionistic epistemic logic with distributed knowledge to be made dynamic and the notion of stability of Kripke frame. Section 3.2 defines Hilbert systems of the logics, and states soundness results. In Section 3.3, strong completeness of the Hilbert systems of the logics with respect to the suitable classes of stable frames is shown, via a notion of "pseudo-model". In Section 3.4, we introduce sequent calculi for the logics and prove the cut-elimination theorem, Craig interpolation theorem, and decidability.

The author's contribution is as follows. The definition of tree unraveling in the case of **IK** was determined through discussions with the author's supervisor. The definition of tree unravelings in the case of the logics other than **IK** was found by the author on his own, based on the case of **IK**. The author proved the strong completeness theorem with respect to the class of stable Kripke frames, using the notion of stabilization presented by the supervisor. The sequent calculi are defined by the author based on the suggestion by the supervisor. The cut elimination theorem, Craig interpolation theorem, and decidability are shown by the author.

The content of this chapter is based on [31].

3.1 Syntax and Semantics

We denote a finite set of agents by Agt. We call a *nonempty* subset of Agt "group" and denote it by G, H, etc. We denote by Grp the set of all groups, i.e., the set $\wp(\mathsf{Agt}) \setminus \{\emptyset\}$ of all non-empty subset of Agt. Let Prop be a countable set of propositional variables and

Form be the set of formulas defined inductively by the following clauses:

$$\mathsf{Form} \ni \varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid D_G\varphi,$$

where $p \in \mathsf{Prop}$ and $G \in \mathsf{Grp}$. We read $D_G\varphi$ as "φ is distributed knowledge among a group G". We define $\neg\varphi$ as $\varphi \to \bot$, $\varphi \leftrightarrow \psi$ as $\varphi \to \psi \wedge \psi \to \varphi$, $\top$ as $\bot \to \bot$, and the epistemic operator $K_a\varphi$ (read "agent a knows that φ") as $D_{\{a\}}\varphi$. As noted above, an expression of the form $D_\varnothing\varphi$ is *not* a well-formed formula, since we have excluded $\varnothing$ from our definition of groups.

We introduce Kripke semantics for intuitionistic multi-agent epistemic logic with distributed knowledge, along the lines of [23]. [1]

Definition 3.1 (frame, model)**.** A tuple $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$ is a *frame* if: W is a set of states, $\leqslant$ is a preorder on W, $(R_a)_{a\in\mathsf{Agt}}$ is a family of binary relations on W indexed by agents, and $\leqslant; R_a \subseteq R_a$ (for all $a \in \mathsf{Agt}$), where $S_1; S_2 := \{(x, z) \mid$ there exists y such that xS_1y and $yS_2z\}$.

A pair $M = (F, V)$ is a *model* if F is a frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$. We denote an underlying set of states of a frame F or a model M by $|F|$ or $|M|$.

For a model $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model.*

The following notion of stability is needed in intuitionistic public announcement logic with distributed knowledge introduced later.

Definition 3.2 (stable)**.** Let $\leq$ be a preorder on a set X. A relation $R \subseteq X \times X$ is called *stable with regard to* $\leq$ if $\leqslant; R \subseteq R$ and $R; \leqslant \subseteq R$. We say a frame $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$ is *stable* if each R_a is stable with regard to $\leqslant$. We name the class of all stable frames $\mathbb{ST}$.

Proposition 3.3. *Let X be a set, R be a binary relation on X, and $\leq$ be a preorder on X.*

1. *R is stable with regard to $\leq$ iff $\leq; R; \leq \subseteq R$.*
2. *If R is reflexive and transitive, then $\leq; R \subseteq R$ implies $R; \leq \subseteq R$.*

Proof.
- (item 1) Left-to-right: It is evident that $(\leq; R); \leq \subseteq R; \leq \subseteq R$. Right-to-left: It is evident that $\leq; R \subseteq \leq; R; \leq \subseteq R$ and that $R; \leq \subseteq \leq; R; \leq \subseteq R$.

[1] There are other possible ways to build a model with a time-like relation like $\leqslant$ in our definition and a relation for modalities like R_a in our definition. For example, in the system $\mathcal{LORA}$ in [65], a time-like relation T is internalized in a possible world.

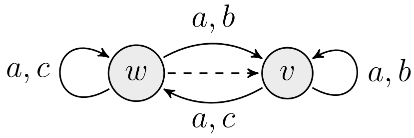

Figure 3.1: Example of frame

- (item 2) By reflexivity and transitivity of R, we have $R; \leq \subseteq R; \leq; R \subseteq R; R \subseteq R$. $\square$

Due to item 1, we can show that for a certain tuple $(W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$, R_G is stable with regard to $\leqslant$ for any $G \in \mathsf{Grp}$ by checking whether $\leqslant; R_G; \leqslant \subseteq R_G$ for any $G \in \mathsf{Grp}$. *Satisfaction relation* $M, w \Vdash \varphi$ on pointed models and formulas is defined recursively as follows:

$$\begin{array}{lll} M, w \Vdash p & \text{iff} & w \in V(p), \\ M, w \Vdash \bot & & \text{Never}, \\ M, w \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \in W, \text{ if } w \leqslant v \text{ then } M, v \nVdash \varphi \text{ or } M, v \Vdash \psi, \\ M, w \Vdash \varphi \wedge \psi & \text{iff} & M, w \Vdash \varphi \text{ and } M, w \Vdash \psi \\ M, w \Vdash \varphi \vee \psi & \text{iff} & M, w \Vdash \varphi \text{ or } M, w \Vdash \psi \\ M, w \Vdash D_G\varphi & \text{iff} & \text{for all } v \in W, \text{if } (w, v) \in \bigcap_{a \in G} R_a \text{ then } M, v \Vdash \varphi. \end{array}$$

It is noted from our definition of $K_a\varphi := D_{\{a\}}\varphi$ that the satisfaction of $K_a\varphi$ at a state w of a model M is given as follows:

$$M, w \Vdash K_a\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \Vdash \varphi.$$

As is the case with ordinary intuitionistic logic, we have the following heredity property for a formula.

Proposition 3.4 (heredity)**.** *If $M, w \Vdash \varphi$ and $w \leqslant v$, then $M, v \Vdash \varphi$*

Proof. By induction on φ. For the case where $\varphi \equiv D_G\psi$, it is noted that the condition $\leqslant; R_a \subseteq R_a$ of a frame implies that $\leqslant; \bigcap_{a \in G} R_a \subseteq \bigcap_{a \in G} R_a$. For the rest cases, the reader is referred to [55, Lemma 6.3.4]. $\square$

Example 3.5. Figure 3.1 is an example of a frame. The preorder is depicted by a dotted arrow. Note that we omit reflexive arrows for the preorder. If a valuation is defined by, for example, $V(p) = \{v\}$ for any $p \in \mathsf{Prop}$, V satisfies the heredity condition. In this model, it can be seen that different groups have different distributed knowledge even at the same state. Indeed, $D_{\{a,b\}}p$ is true at w, but $D_{\{a,c\}}p$ is false at w. We can see

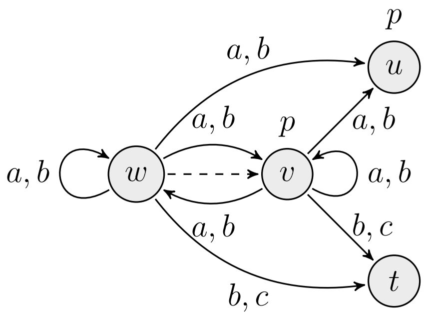

Figure 3.2: Example of stable model M_{stable}

that seriality for each agent's relation is not always preserved under taking intersection among a group. Namely, R_b and R_c are serial but $R_b \cap R_c$ is not in the example. This is why we should restrict (D) axiom to $\neg D_{\{a\}}\bot$, as defined in Table 3.1. Figure 3.2 is an example of a stable model named M_{stable} used later in the explanation of the extension to public announcement logic. From this model too, it can be seen that different groups have different distributed knowledge even at the same state. Indeed, $D_{\{b,c\}}\neg p$ is true at v, but $D_{\{a,b\}}\neg p$ is false at v.

Given a frame $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$, we say that a formula φ is *valid* in F (notation: $F \Vdash \varphi$) if $(F, V), w \Vdash \varphi$ for every valuation function V and every $w \in W$. Moreover, a formula φ is valid in a class $\mathbb{F}$ of frames (notation: $\mathbb{F} \Vdash \varphi$) if $F \Vdash \varphi$ for every $F \in \mathbb{F}$.

Definition 3.6. A formula φ is a *semantic consequence* of Γ in a frame class $\mathbb{F}$ if for all frame $F \in \mathbb{F}$, a valuation V on F, a state $w \in |F|$, if $(F, V), w \Vdash \Gamma$, then $(F, V), w \Vdash \varphi$. We write it as "$\Gamma \Vdash_{\mathbb{F}} \varphi$".

3.2 Hilbert Systems

Hilbert systems for intuitionistic epistemic logics with D_G operators are constructed from axioms and rules shown in Table 3.1.
A Hilbert system $\mathsf{H}(\mathbf{IK})$ consists of axioms and rules for intuitionistic logic, axioms (Incl) and (K), and a rule (Nec). Hilbert systems $\mathsf{H}(\mathbf{IKT})$, $\mathsf{H}(\mathbf{IKD})$, $\mathsf{H}(\mathbf{IK4})$, $\mathsf{H}(\mathbf{IK4D})$, and $\mathsf{H}(\mathbf{IS4})$ are defined as axiomatic expansions of $\mathsf{H}(\mathbf{IK})$ with (T), (D), (4), (4) and (D), and (T) and (4), respectively. Let $\mathbf{X}$ be any of $\mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IKD}$, $\mathbf{IK4}$, $\mathbf{IK4D}$, and $\mathbf{IS4}$ in what follows. The notion of provability in each system is defined the same as Definition 1.3, and the fact that a formula φ is provable in $\mathsf{H}(\mathbf{X})$ is denoted by "$\vdash_{\mathsf{H}(\mathbf{X})} \varphi$". We also define derivability relation between a set Γ of formulas and a formula φ as below.

Definition 3.7. A formula φ is *derivable* from Γ in a logic $\mathbf{X}$ if $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \to \varphi$ for some finite set Γ' which is a subset of Γ. We write it as "$\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$".

Table 3.1: Axioms and Rules for Hilbert Systems

Axioms and Rules for Intuitionistic Logic			
(**k**)	$\varphi \to (\psi \to \varphi)$	($\wedge$**e**$_1$)	$(\varphi \wedge \psi) \to \varphi$
(**s**)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	($\wedge$**e**$_2$)	$(\varphi \wedge \psi) \to \psi$
($\vee$**i**$_1$)	$\varphi \to (\varphi \vee \psi)$	($\wedge$**i**)	$\varphi \to (\psi \to (\varphi \wedge \psi))$
($\vee$**i**$_2$)	$\psi \to (\varphi \vee \psi)$	($\bot$)	$\bot \to \varphi$
($\vee$**e**)	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$	(**MP**)	From φ and $\varphi \to \psi$, infer ψ

Axioms and Rules for $\mathsf{H}(\mathbf{IK})$			
(Incl)	$D_G\varphi \to D_H\varphi$ $(G \subseteq H)$	(K)	$D_G(\varphi \to \psi) \to (D_G\varphi \to D_G\psi)$
(Nec)	From φ, infer $D_G\varphi$		

Additional Axioms for D_G operators					
(T)	$D_G\varphi \to \varphi$	(D)	$\neg D_{\{a\}}\bot$	(4)	$D_G\varphi \to D_G D_G\varphi$

We introduce a class of frames corresponding to each logic, in order to state soundness of our axiomatization.

Definition 3.8. A class of frames $\mathbb{F}(\mathbf{X})$ is defined as follows:

- $\mathbb{F}(\mathbf{IK})$ is the class of all frames.
- $\mathbb{F}(\mathbf{IKT})$ is the class of all frames such that R_a is reflexive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IKD})$ is the class of all frames such that R_a is serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IK4})$ is the class of all frames such that R_a is transitive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IK4D})$ is the class of all frames such that R_a is transitive and serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IS4})$ is the class of all frames such that R_a is reflexive and transitive ($a \in \mathsf{Agt}$).

Reflexivity, seriality, and transitivity are defined in Definition 1.2.

We can prove the following soundness theorem by induction on φ. Note that axioms (T) and (4) are valid in reflexive and transitive frames, respectively, because if R_a is reflexive or transitive for any $a \in G$, $\bigcap_{a \in G} R_a$ is also reflexive or transitive, respectively.

Theorem 3.9 (soundness)**.** *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$*, then* $\mathbb{F}(\mathbf{X}) \Vdash \varphi$.

Proof. It is eveident from the proofs of Theorem 1.6, Theorem 1.21, and Fact 2.1. We show the validity of the axiom (Incl) in the case of $\mathbf{X} = \mathbf{IK}$, for example. Suppose $G \subseteq H$. We show $\mathbb{F}(\mathbf{IK}) \Vdash D_G\varphi \to D_H\varphi$. Take any frame F, and any valuation V on F

and any state $w \in |F|$. Suppose $(F, V), v \Vdash D_G\varphi$ for any v such that $w \leqslant v$. We show $(F, V), v \Vdash D_H\varphi$. Take any u such that $(v, u) \in \bigcap_{a\in H} R_a$. Since $\bigcap_{a\in H} R_a \subseteq \bigcap_{a\in G} R_a$, we have $(F, V), u \Vdash \varphi$ by the supposition, as required.

We remark on the other cases. To show the validity of the axiom (T) ((D), and (4)), the property of reflexivity (seriality, and transitivity, respectively) is used. □

The soundness theorem with regard to the corresponding class of stable frames also trivially holds.

Corollary 3.10. *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, *then* $\mathbb{F}(\mathbf{X}) \cap \mathbb{ST} \Vdash \varphi$.

Proof. It is obvious from the fact that $\mathbb{F} \Vdash \varphi$ implies $\mathbb{F} \cap \mathbb{ST} \Vdash \varphi$ for any class $\mathbb{F}$ of frames and any formula φ. □

3.3 Semantic Completeness

In the present section, we explain a proof of two kinds of the strong completeness theorems of our logic. Let Γ be a set of formulas and φ be a formula. We firstly show the ordinary strong completeness theorem stated as follows.

Theorem 3.11. *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Then, if* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X})} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$.

Then, we also show the completeness with regard to the corresponding class of stable frames for the sake of the completeness proof for the **PAL** extension:

Theorem 3.12. *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Then, if* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X})\cap\mathbb{ST}} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$.

We show this based on the proof of Theorem 3.11, by way of the method called stabilization. Note that any frame $F \in \mathbb{F}(\mathbf{IS4})$ is stable due to item 2 of Proposition 3.3, that is, $\mathbb{F}(\mathbf{IS4}) = \mathbb{F}(\mathbf{IS4}) \cap \mathbb{ST}$, which means that Theorems 3.11 and 3.12 are equivalent for **IS4**.

As in [7], we show Theorem 3.11 via the notion of "pseudo-model", which is Kripke model with relations for a group, not a single agent. The reason why the notion of pseudo-model is introduced is that the ordinary canonical model with relations for a single agent only does not work when proving the completeness of a logic with distributed knowledge, which is a consequence of the fact that a logic with distributed knowledge is more expressive than the basic epistemic logic introduced in Section 1.3 (the reader is referred to [44, Section 4]).

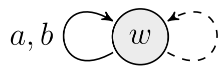

Figure 3.3: Example of pseudo-frame

Definition 3.13 (pseudo-frame, pseudo-model)**.** A tuple $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$ is a *pseudo-frame* if:

1. $\leqslant; R_G \subseteq R_G$ for any $G \in \mathsf{Grp}$, and

2. $R_H \subseteq R_G$ if $G \subseteq H$ (called "inclusion condition").

A pair $M = (F, V)$ is a *pseudo-model* if F is a pseudo-frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$.

Definition 3.14 (pseudo-satisfaction relation)**.** For a pseudo-model M, a state $w \in |M|$, and a formula φ, a *pseudo-satisfaction relation* $M, w \Vdash^{ps} \varphi$ is defined the same as the satisfaction relation $\Vdash$, except for the clause for $D_G\varphi$: that is,

$$M, w \Vdash^{ps} D_G\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in R_G \text{ then } M, v \Vdash^{ps} \varphi.$$

Namely, in a pseudo-model, an operator D_G is treated like a primitive box operator, parameterized by a group.

Considering the definition of satisfaction relation for $D_G\varphi$, a pseudo-frame can be seen as a frame in the sense of Definition 3.1 if the following "intersection condition" is satisfied.

Definition 3.15 (intersection condition)**.** Let $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$ be a pseudo-frame. The condition "$R_G = \bigcap_{a\in G} R_{\{a\}}$ for any group G" is called *intersection condition*.

See that this condition is not always satisfied by way of the following example:

Example 3.16. Figure 3.3 is an example of a pseudo-frame. We name it F_{ex}. Here, we set $\mathsf{Agt} := \{a, b\}$. Note that $\{a\}$ is written as "a" and $R_{\{a,b\}}$ is defined as $\emptyset$ here. Since $R_{\{a,b\}} = \emptyset$, the condition of "$R_H \subseteq R_G$ if $G \subseteq H$" is self-evidently satisfied, i.e., $R_{\{a,b\}} \subseteq R_{\{a\}}$ and $R_{\{a,b\}} \subseteq R_{\{b\}}$. Note that the intersection condition is false for a group $\{a, b\}$, because $R_{\{a\}} \cap R_{\{b\}} \not\subseteq R_{\{a,b\}}$. Any frame can be regarded as a pseudo-frame with only relations for singleton groups, as in F_{ex}.

In order to show Theorem 3.11, we first construct a canonical pseudo-model satisfying truth lemma (subsection 3.3.1), and then transform it into another pseudo-model enjoying the intersection condition without changing satisfaction, by a method called "tree unraveling" (subsection 3.3.2).

3.3.1 Canonical Pseudo-Model

We define a canonical pseudo-model of our logics and state some properties of it in the present subsection. Since D_G operators are interpreted as primitive box-like operators indexed by a group in a pseudo-model, a canonical pseudo-model defined here is essentially the same as the canonical model of intuitionistic epistemic logics without distributed knowledge, which is described in detail e.g., in [27, Chapter 1]. Let **X** be any of **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, and **IS4** below.

Definition 3.17 (consistent, prime, theory)**.** A set Γ of formulas is:

- **X**-*consistent* if $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$.
- *prime* if $\varphi_1 \vee \varphi_2 \in \Gamma$ implies $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$.
- an **X**-theory if $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ implies $\varphi \in \Gamma$.

The following are useful and well-known properties of a consistent and prime theory.

Lemma 3.18. *Let a set Γ of formulas be an* **X***-theory.*

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ *iff* $\varphi \in \Gamma$.
2. *If* $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$*, then* $\psi \in \Gamma$.
3. $\bot \notin \Gamma$*, if* Γ *is* **X***-consistent.*
4. $\varphi \wedge \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
5. $\varphi \vee \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$*, if* Γ *is prime.*

Proof. (item 1) The left-to-right is evident since Γ is **X**-theory. We show the right-to-left. Suppose $\varphi \in \Gamma$. Then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ is the case because $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \varphi$.

(item 2) Suppose $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$. Then, by item 1, we have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ and $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$, that is, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma_1 \to \varphi$ and $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma_2 \to (\varphi \to \psi)$ for some finite sets $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Then we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to \varphi$ and $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to (\varphi \to \psi)$, which jointly entail $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to (\varphi \wedge (\varphi \to \psi))$. Since we have $\vdash_{\mathsf{H}(\mathbf{X})} (\varphi \wedge (\varphi \to \psi)) \to \psi$ as an intuitionistic theorem, we obtain $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to \psi$, which means that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Since Γ is $\mathbf{X}$-theory, $\psi \in \Gamma$.

(item 3) Suppose $\bot \in \Gamma$ for contradiction. Then, by item 1, $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \bot$. However, this contradicts with $\mathbf{X}$-consistency of Γ.

(item 4) First, we show the left-to-right. Suppose $\varphi \wedge \psi \in \Gamma$. By item 1, it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ and $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. It is the case that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$, because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi \to \varphi$ as an axiom. $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$ can be similarly shown. Next, we show the right-to-left. Suppose $\varphi \in \Gamma$ and $\psi \in \Gamma$. By item 1, it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi$, which is the case because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi \to \varphi \wedge \psi$ as an intuitionistic theorem.

(item 5) The left-to-right is the case by primeness of Γ. We show the right-to-left. Suppose $\varphi \in \Gamma$ or $\psi \in \Gamma$. By item 1, it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$. First, assume $\varphi \in \Gamma$. Then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$ is the case because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \varphi \vee \psi$ as an axiom. Similarly, $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$ is the case by $\vdash_{\mathsf{H}(\mathbf{X})} \psi \to \varphi \vee \psi$ when assuming $\psi \in \Gamma$. □

Lemma 3.19 (Lindenbaum, [27, Lemma 1.16])**.** *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. If $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$, then there is an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Γ^+ such that $\Gamma \subseteq \Gamma^+$ and $\Gamma^+ \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$.*

Definition 3.20. A *canonical pseudo-model* $M^{\mathbf{X}} = (W^{\mathbf{X}}, \leqslant^{\mathbf{X}}, (R^{\mathbf{X}}_G)_{G \in \mathsf{Grp}}, V^{\mathbf{X}})$ is defined as follows:

- $W^{\mathbf{X}} := \{\Gamma \in \mathcal{P}(\mathsf{Form}) \mid \Gamma$ is an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory$\}$.
- $\Gamma \leqslant^{\mathbf{X}} \Delta$ iff $\Gamma \subseteq \Delta$.
- $\Gamma R^{\mathbf{X}}_G \Delta$ iff $D_G^{-1}\Gamma \subseteq \Delta$, where $D_G^{-1}\Gamma := \{\varphi \in \mathsf{Form} \mid D_G\varphi \in \Gamma\}$.
- $V^{\mathbf{X}}(p) := \{\Gamma \in W^{\mathbf{X}} \mid p \in \Gamma\}$.

The definition is well-defined and a canonical pseudo-model is always stable:

Proposition 3.21. $M^{\mathbf{X}}$ is *a stable pseudo-model.*

Proof. First, we show the stability of $R^{\mathbf{X}}_G$, that is, $\leqslant^{\mathbf{X}}; R^{\mathbf{X}}_G; \leqslant^{\mathbf{X}} \subseteq R^{\mathbf{X}}_G$ (cf. item 1 of Proposition 3.3). Suppose that $\Gamma(\leqslant^{\mathbf{X}}; R^{\mathbf{X}}_G; \leqslant^{\mathbf{X}})\Delta$. Then, there are sets Θ and Π such that $\Gamma \subseteq \Theta$, $\Theta R^{\mathbf{X}}_G \Pi$, and $\Pi \subseteq \Delta$. To show that $\Gamma R^{\mathbf{X}}_G \Delta$, assume that $\varphi \in D_G^{-1}\Gamma$. Since $\Gamma \subseteq \Theta$,

$\varphi \in D_G^{-1}\Theta$. Then, $\varphi \in \Delta$ since $\Theta R_G^{\mathbf{X}}\Pi$ and $\Pi \subseteq \Delta$. Next, we show that $R_H^{\mathbf{X}} \subseteq R_G^{\mathbf{X}}$ if $G \subseteq H$. Assume that $\Gamma R_H^{\mathbf{X}}\Delta$ and $\varphi \in D_G^{-1}\Gamma$, i.e., $D_G\varphi \in \Gamma$. We show that $\varphi \in \Delta$ Since $D_G\varphi \to D_H\varphi$ is an axiom in any $\mathsf{H}(\mathbf{X})$ and hence $D_G\varphi \to D_H\varphi \in \Gamma$, we have $D_H\varphi \in \Gamma$ by item 2 of Lemma 3.18. Then, by $\Gamma R_H^{\mathbf{X}}\Delta$, $\varphi \in \Delta$. It is obvious that $V^{\mathbf{X}}$ satisfies the heredity condition. □

Lemma 3.22. *Let a set Γ of formulas be an $\mathbf{X}$-theory. Then, we have:*

1. *If $\varphi \to \psi \notin \Gamma$, then $\Gamma \cup \{\varphi\} \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.*
2. *If $D_G\psi \notin \Gamma$, then $D_G^{-1}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.*

Proof. (item 1) We show the contraposition. Suppose $\Gamma \cup \{\varphi\} \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Then, there exists a finite subset Γ' of Γ such that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \to \psi$. First, assume $\varphi \notin \Gamma'$. Then, it turns out that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Since we also have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi \to (\varphi \to \psi)$, $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$ is obtained. Then, $\varphi \to \psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory. Next, assume $\varphi \in \Gamma'$. Put $\Delta \coloneqq \Gamma' - \{\varphi\}$. Since $\vdash_{\mathsf{H}(\mathbf{X})} (\bigwedge \Gamma' \to \psi) \leftrightarrow (\bigwedge \Delta \to \varphi \to \psi)$, we have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$. Then, $\varphi \to \psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory.

(item 2) We show the contraposition. $D_G^{-1}\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Then, there exists a finite subset Δ of $D_G^{-1}\Gamma$ such that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Delta \to \psi$. By axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge D_G\Delta \to D_G\psi$, which means that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} D_G\psi$. Then, $D_G\psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory. □

Lemma 3.23 (Truth Lemma)**.** *Let a set Γ of formulas be an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory. Then, $\varphi \in \Gamma$ if and only if $M^{\mathbf{X}}, \Gamma \Vdash^{ps} \varphi$.*

Proof. By induction on φ.

(the case of $\varphi \equiv p$) Obvious by the definition.

(the case of $\varphi \equiv \bot$) Obvious by item 3 of Lemma 3.18.

(the case of $\varphi \equiv \psi_1 \to \psi_2$) First, we show the left-to-right. Assume $\psi_1 \to \psi_2 \in \Gamma$. To show $M^{\mathbf{X}}, \Gamma \Vdash^{ps} \psi_1 \to \psi_2$, fix any Δ such that $\Gamma \leqslant^{\mathbf{X}} \Delta$, i.e., $\Gamma \subseteq \Delta$ and suppose that $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi_1$. The goal is to show that $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi_2$. By the induction hypothesis, we have $\psi_1 \in \Delta$. Further, we have $\psi_1 \to \psi_2 \in \Delta$ by the assumption and $\Gamma \subseteq \Delta$. Then, we use item 2 of Lemma 3.18 to obtain $\psi_2 \in \Delta$, which is equivalent to the goal by the induction hypothesis.

Next, we show the right-to-left by contraposition. Assume $\psi_1 \to \psi_2 \notin \Gamma$. The goal is to show that $M^{\mathbf{X}}, \Gamma \nVdash^{ps} \psi_1 \to \psi_2$, that is, to find an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $\Gamma \leqslant^{\mathbf{X}} \Delta$, i.e., $\Gamma \subseteq \Delta$, $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi_1$, and $M^{\mathbf{X}}, \Delta \nVdash^{ps} \psi_2$. Applying item 1 of

Lemma 3.22 to the assumption, we have $\Gamma \cup \{\psi_1\} \nvdash_{\mathsf{H}(\mathbf{X})} \psi_2$. Then, by Lemma 3.19, we can find an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $\Gamma \cup \{\psi_1\} \subseteq \Delta$ and $\Delta \nvdash_{\mathsf{H}(\mathbf{X})} \psi_2$. We show Δ satisfies the three desired conditions. First, $\Gamma \subseteq \Delta$ is obvious. Second, we show $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi_1$. Clearly, we have $\psi_1 \in \Delta$, which gives $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi_1$ by the induction hypothesis. Third, we show $M^{\mathbf{X}}, \Delta \nVdash^{ps} \psi_2$. By $\Delta \nvdash_{\mathsf{H}(\mathbf{X})} \psi_2$ and item 1 of Lemma 3.18, we have $\psi_2 \notin \Delta$, which gives $M^{\mathbf{X}}, \Delta \nVdash^{ps} \psi_2$ by the induction hypothesis.

(the case of $\varphi \equiv \psi_1 \vee \psi_2$) Obvious by item 5 of Lemma 3.18.

(the case of $\varphi \equiv \psi_1 \wedge \psi_2$) Obvious by item 4 of Lemma 3.18.

(the case of $\varphi \equiv D_G\psi$) First, we show the left-to-right. Assume $D_G\psi \in \Gamma$ and fix any $\Delta \in W^{\mathbf{X}}$ such that $\Gamma R^{\mathbf{X}}_G \Delta$, i.e., $D_G^{-1}\Gamma \subseteq \Delta$. Clearly, $\psi \in \Delta$, and by the induction hypothesis, we have $M^{\mathbf{X}}, \Delta \Vdash^{ps} \psi$.

Next, we show the contraposition of the right-to-left. Assume $D_G\psi \notin \Gamma$. By item 2 of Lemma 3.22 and Lemma 3.19, there is an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $D_G^{-1}\Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathsf{H}(\mathbf{X})} \psi$. By item 1 of Lemma 3.18 and induction hypothesis, we have $M^{\mathbf{X}}, \Delta \nVdash^{ps} \psi$, which shows $M^{\mathbf{X}}, \Gamma \nVdash^{ps} D_G\psi$. □

For each axiom, the canonical pseudo-model satisfies the corresponding property on relations for D_G.

Proposition 3.24. *1. If* $\mathbf{X}$ *has the axiom* (T), $R^{\mathbf{X}}_G$ *is reflexive in* $M^{\mathbf{X}}$.

2. If $\mathbf{X}$ *has the axiom* (D), $R^{\mathbf{X}}_{\{a\}}$ *is serial in* $M^{\mathbf{X}}$.

3. If $\mathbf{X}$ *has the axiom* (4), $R^{\mathbf{X}}_G$ *is transitive in* $M^{\mathbf{X}}$.

Proof. Items 1 and 3 can be shown the same as in the case of the basic classical-logical modal logic (the reader is referred to [6, Theorems 4.27 and 4.28]). We show item 2. Fix any $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Γ. The aim is to find an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $D_{\{a\}}^{-1}\Gamma \subseteq \Delta$. By Lemma 3.19, it suffices to show $D_{\{a\}}^{-1}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$. Assuming the contrary, we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} \varphi_i \to \bot$ for some $\varphi_i \in D_{\{a\}}^{-1}\Gamma$. By (Nec), (K), and intuitionistic propositional tautologies, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{\{a\}}\varphi_i \to D_{\{a\}}\bot$. Since $D_{\{a\}}\varphi_i \in \Gamma$, it means $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} D_{\{a\}}\bot$. However, we also have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \neg D_{\{a\}}\bot$ by the assumption, which leads to contradiction by items from 1 to 3 of Lemma 3.18. □

3.3.2 Tree Unraveling

If the canonical pseudo-model satisfies the intersection condition, we can regard it as a model, and the completeness proof is done. However, the canonical pseudo-model does not satisfy the intersection condition. A counterexample in the classical-logical setting is described in [57, proof for Φ_{3a} of Proposition 3.13]. The counterexample is induced from a certain (classical) Kripke model, so the counterexample works in our setting too, by defining a preorder in the model as a mere identity relation.

Hence, we introduce a method called "tree unraveling", which transforms a pseudo-model into another pseudo-model satisfying the intersection condition $\bigcap_{a\in G} R_{\{a\}} = R_G$ (i.e., a model in the sense of Definition 3.1). Our definitions below are intuitionistic generalizations of definitions proposed in [7] over classical logic.

Definition 3.25. Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ be a pseudo-model. A pseudo-model $M' = (W', \leqslant', (R'_G)_{G\in\mathsf{Grp}}, V')$ is a *generated submodel* of M if:

- $W' \subseteq W$.
- $\leqslant' = \leqslant \cap (W' \times W')$.
- $R'_G = R_G \cap (W' \times W')$.
- If $w \in W'$ and $w \leqslant w'$ then $w' \in W'$.
- If $w \in W'$ and $w R_G w'$ then $w' \in W'$.
- $V'(p) = V(p) \cap W'$ for any $p \in \mathsf{Prop}$.

For $X \subseteq |M|$, we define M_X as the smallest generated submodel containing X. If $M = M_X$, we say that M is generated by X.

Proposition 3.26. *Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ be a pseudo-model and $M' = (W', \leqslant', (R'_G)_{G\in\mathsf{Grp}}, V')$ be a generated submodel of M. Then, for any $w \in W'$ and formula φ, $M', w \Vdash^{ps} \varphi$ iff $M, w \Vdash^{ps} \varphi$.*

Proof. By induction on the structure of φ. The cases of $\varphi \equiv \psi \to \chi$ and $\varphi \equiv D_G\psi$ are shown. The other cases are obvious.

(the case of $\varphi \equiv \psi \to \chi$) First, we show the left-to-right. Assume $M', w \Vdash^{ps} \psi \to \chi$. Take any $v \in W$ satisfying $w \leqslant v$ and suppose $M, v \Vdash^{ps} \psi$. We show $M, v \Vdash^{ps} \chi$. Since M' is a generated submodel of M, we have $v \in W'$. Hence, by the induction hypothesis, $M', v \Vdash^{ps} \psi$, which entails $M', v \Vdash^{ps} \chi$ under the assumption. Then, again by the induction hypothesis, we have $M, v \Vdash^{ps} \chi$ as required.

Next, we show the right-to-left. Assume $M, w \Vdash^{ps} \psi \to \chi$. Take any $v \in W'$ satisfying $(w, v) \in \leqslant \cap (W' \times W')$ and suppose $M', v \Vdash^{ps} \psi$. $M', v \Vdash^{ps} \chi$ By the induction hypothesis, $M, v \Vdash^{ps} \psi$ is obtained, which entails $M, v \Vdash^{ps} \chi$ under the assumption. Then, again by the induction hypothesis, we have $M', v \Vdash^{ps} \chi$ as required.

(the case of $\varphi \equiv D_G\psi$) First, we show the left-to-right. Assume $M', w \Vdash^{ps} D_G\psi$. Take any $v \in W$ satisfying wR_Gv. We show $M, v \Vdash^{ps} \psi$. Since M' is a generated submodel of M, we have $v \in W'$. Hence, by the assumption, $M', v \Vdash^{ps} \psi$, which entails $M, v \Vdash^{ps} \psi$ by the induction hypothesis, as required.

Next, we show the right-to-left. Assume $M, w \Vdash^{ps} D_G\psi$. Take any $v \in W'$ satisfying $(w, v) \in R_G \cap (W' \times W')$. We show $M', v \Vdash^{ps} \psi$. By the the assumption, $M, v \Vdash^{ps} \psi$ is obtained, which entails $M', v \Vdash^{ps} \psi$ by the induction hypothesis, as required. $\square$

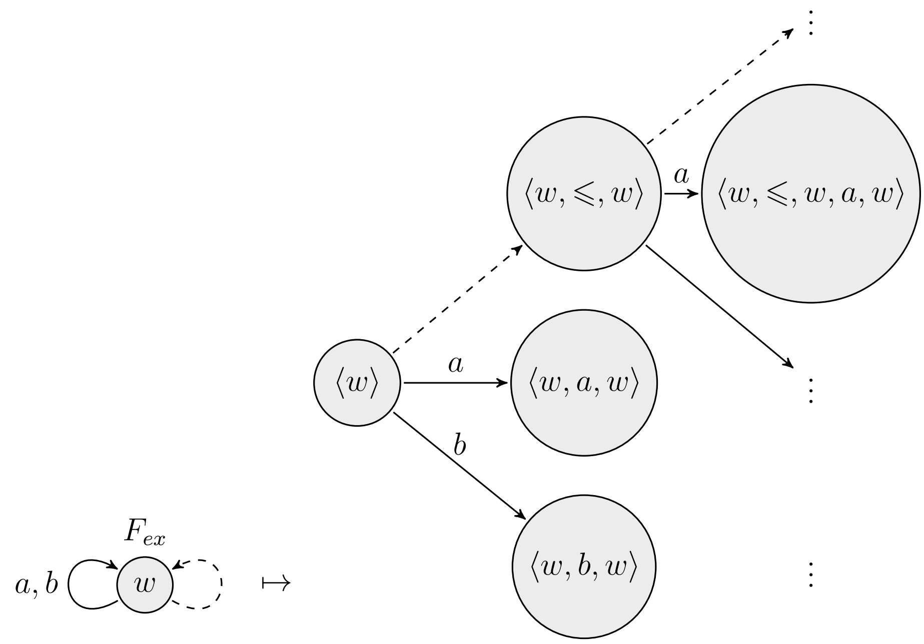

Figure 3.4: Tree unraveling of F_{ex}

Definition 3.27. Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$.

- We put $w_0 := w$ and define Finpath(F, w) as

$$\{\langle w_0, L_1, w_1, L_2, \cdots, L_n, w_n \rangle \mid n \geq 0, L_i \in \{\leqslant, R_G\}_{G \in \mathsf{Grp}}, w_{i-1} L_i w_i \text{ for all } 1 \leq i \leq n\}.$$

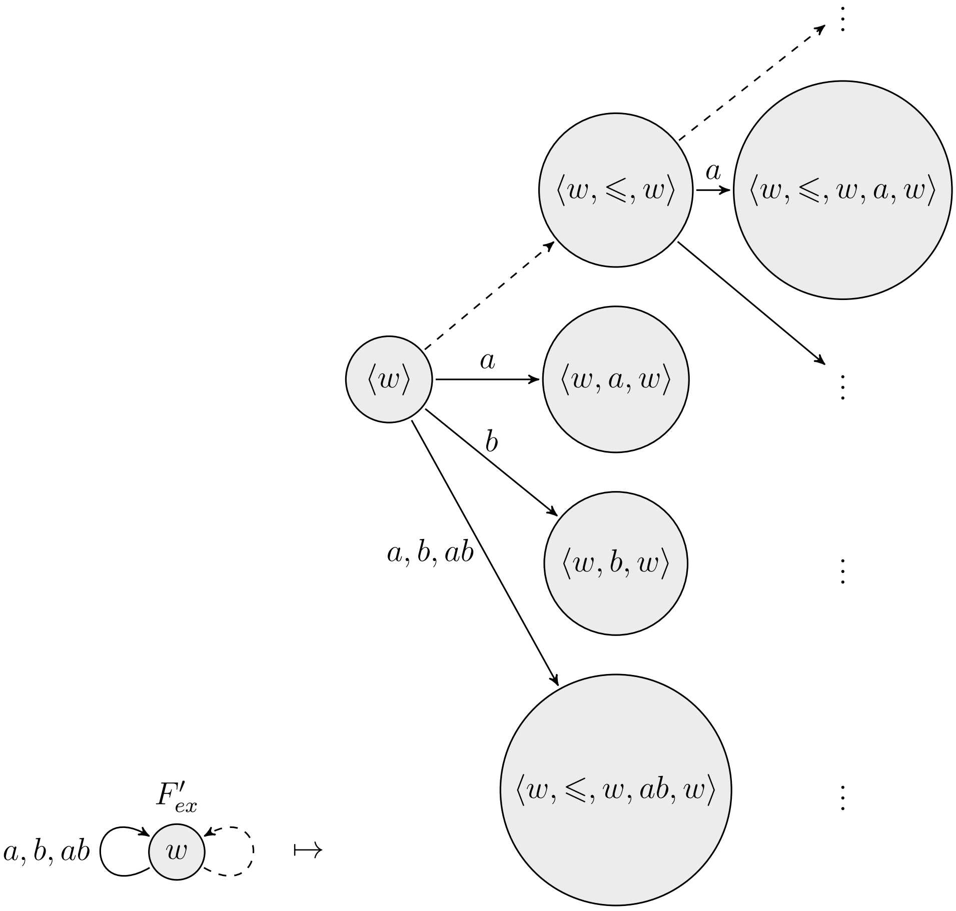

Figure 3.5: Tree unraveling of F'_{ex}

We call an element of $\mathrm{Finpath}(F, w)$ "a path (from a state w)" and denote it by $\overrightarrow{u}, \overrightarrow{v}$, etc.

- For $\overrightarrow{u} = \langle w_0, L_1, w_1, L_2, \cdots, L_{n-1}, w_{n-1}, L_n, w_n\rangle \in \mathrm{Finpath}(F, w)$,

$$\mathsf{body}(\overrightarrow{u}) := \langle w_0, L_1, w_1, L_2, \cdots, L_{n-1}, w_{n-1}\rangle \text{ and } \mathsf{tail}(\overrightarrow{u}) := w_n.$$

- We say that paths $\overrightarrow{u}, \overrightarrow{v} \in \mathrm{Finpath}(F, w)$ satisfy a relation $\overrightarrow{u} \preccurlyeq \overrightarrow{v}$ if and only if $\overrightarrow{v} \equiv \overrightarrow{u} ^\frown \langle \leqslant, w'\rangle$, where $^\frown$ means concatenation of two tuples.

- We say that paths $\overrightarrow{u}, \overrightarrow{v} \in \mathrm{Finpath}(F, w)$ satisfy a relation $\overrightarrow{u} \mathcal{R}_G \overrightarrow{v}$ if and only if $\overrightarrow{v} \equiv \overrightarrow{u} ^\frown \langle R_H, w'\rangle$ and $G \subseteq H$.

- A *valuation function* $\mathcal{V} \colon \mathsf{Prop} \to \mathcal{P}(\mathrm{Finpath}(F, w))$ is defined by:

$$\mathcal{V}(p) = \{\overrightarrow{u} \in \mathrm{Finpath}(F, w) \mid \mathsf{tail}(\overrightarrow{u}) \in V(p)\}.$$

Take F_{ex} in Figure 3.3 as an example. The set $\mathrm{Finpath}(F_{ex}, w)$ of paths on F_{ex} and $\preccurlyeq$ and $\mathcal{R}_G$ on this set are drawn in Figure 3.4. The point is that the a-arrow and b-arrow on w in F_{ex} are transformed into two arrows with different destinations, so that the condition "$R_{\{a\}} \cap R_{\{b\}} = R_{\{a,b\}}$" is not satisfied in F_{ex} but becomes satisfied in $\mathrm{Finpath}(F_{ex}, w)$. However, as it is, $(\mathrm{Finpath}(F_{ex}, w), \preccurlyeq, (\mathcal{R}_G)_{G\in\mathsf{Grp}})$ is not a pseudo-frame, since $\preccurlyeq$ itself is not a preorder and the condition "$\leqslant; R_G \subseteq R_G$" is not satisfied because, for example, there is no a-arrow from $\langle w\rangle$ to $\langle w, \leqslant, w, a, w\rangle$. Therefore, a preorder and relations for D_G on $\mathrm{Finpath}(F, w)$ in general should be defined as follows.

A pseudo-frame F'_{ex} and its tree unraveling in Figure 3.5 is another example. Here, we set $\mathsf{Agt} := \{a, b\}$, and $R_{\{a,b\}}$ is represented by "ab". It is noted that there are an a-arrow, a b-arrow, and an ab-arrow from $\langle w\rangle$ to $\langle w, \leqslant, w, ab, w\rangle$. It can be seen that the definition of $\mathcal{R}_G$ assures the inclusion condition defined in Definition 3.13.

Definition 3.28. Let $R \subseteq X \times X$ be a binary relation on X. The reflexive (transitive, or reflexive and transitive) closure of R is defined and denoted as follows, respectively:

- $R^\circ := R \cup Id$, where $Id := \{(x, x) \mid x \in X\}$,

- $R^+ := \bigcup_{n\geq 1} R^n$,

- $R^* := \bigcup_{n\geq 0} R^n$,

where $R^n := \overbrace{R; \cdots ; R}^{n}$ for $n \geq 1$ and $R^0 := Id$.

Definition 3.29 (Tree Unraveling)**.** Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$. *Tree unravelings* of a pointed pseudo-model (M, w) are defined as follows:

1. $\mathrm{Tree}(M, w) \coloneqq (\mathrm{Finpath}(F, w), \preccurlyeq^*, (\preccurlyeq^*; \mathcal{R}_G)_{G \in \mathsf{Grp}}, \mathcal{V})$.

2. $\mathrm{Tree}^\circ(M, w) \coloneqq (\mathrm{Finpath}(F, w), \preccurlyeq^*, (\preccurlyeq^*; \mathcal{R}^\circ_G)_{G \in \mathsf{Grp}}, \mathcal{V})$.

3. $\mathrm{Tree}^+(M, w) \coloneqq \left(\mathrm{Finpath}(F, w), \preccurlyeq^*, \left((\preccurlyeq^*; \mathcal{R}^+_G)^+\right)_{G \in \mathsf{Grp}}, \mathcal{V}\right)$.

4. $\mathrm{Tree}^*(M, w) \coloneqq \left(\mathrm{Finpath}(F, w), \preccurlyeq^*, ((\preccurlyeq^*; \mathcal{R}^*_G)^*)_{G \in \mathsf{Grp}}, \mathcal{V}\right)$.

The following is an useful proposition for proving several properties of tree unraveling.

Proposition 3.30. *Let $R, R_1, \cdots, R_n$ be binary relations on a set X.*

1. *If $R_1 \subseteq R_2$, then $R; R_1 \subseteq R; R_2$.*

2. *If $R_1 \subseteq R_2$, then $R_1; R \subseteq R_2; R$.*

3. *If $R_1 \subseteq R_2$, then $R_1^n \subseteq R_2^n$ for all $n \in \mathbb{N}$, in particular, $R_1^+ \subseteq R_2^+$ and $R_1^* \subseteq R_2^*$.*

4. *$R; \bigcap_{i \in I} R_i \subseteq \bigcap_{i \in I}(R; R_i)$.*

5. *$(\bigcap_{i \in I} R_i); R \subseteq \bigcap_{i \in I}(R_i; R)$.*

6. *$\left(\bigcap_{i \in I} R_i\right)^n \subseteq \bigcap_{i \in I} R_i^n$ for all $n \in \mathbb{N}$. In particular, $\left(\bigcap_{i \in I} R_i\right)^+ \subseteq \bigcap_{i \in I} R_i^+$ and $\left(\bigcap_{i \in I} R_i\right)^* \subseteq \bigcap_{i \in I} R_i^*$.*

Proof. 1. Suppose $R_1 \subseteq R_2$ and $x(R; R_1)y$. we have xRz and zR_1y for some $z \in X$. From the latter and the supposition, we have zR_2y, which entails $x(R; R_2)y$ in conjunction with xRz.

2. The same as item 1.

3. Assume $R_1 \subseteq R_2$ and show $R_1^n \subseteq R_2^n$ by induction on $n \in \mathbb{N}$. If $n = 0$, it is obvious. We show $R_1^{k+1} \subseteq R_2^{k+1}$, assuming $R_1^k \subseteq R_2^k$. From I.H. and item 1, we have $R_1^{k+1} \subseteq R_1; R_2^k$. By the assumption and item 2, we have $R_1; R_2^k \subseteq R_2^{k+1}$. Then, we obtain $R_1^{k+1} \subseteq R_2^{k+1}$.

4. Assume $x(R; \bigcap_{i \in I} R_i)y$. Then, we can find $z \in X$ such that xRz and zR_iy for any $i \in I$. The element $z \in X$ obviously makes "$x\left(\bigcap_{i \in I}(R; R_i)\right)y$" true.

5. The same as item 4.

6. We show $\left(\bigcap_{i\in I} R_i\right)^n \subseteq \bigcap_{i\in I} R_i^n$ by induction on n. If $n = 0$, it is obvious. For the step case, we have the following: $\left(\bigcap_{i\in I} R_i\right)^{n+1} = (\bigcap_{i\in I} R_i)^n; \bigcap_{i\in I} R_i \overset{\text{(I.H. and 2)}}{\subseteq} \left(\bigcap_{i\in I} R_i^n\right); \bigcap_{i\in I} R_i \overset{(5)}{\subseteq} \bigcap_{i\in I} \left(R_i^n; \bigcap_{j\in I} R_j\right) \overset{(4)}{\subseteq} \bigcap_{i\in I} \bigcap_{j\in I} (R_i^n; R_j) = \bigcap_{i\in I} \left(R_i^{n+1} \cap \bigcap_{j\neq i} (R_i^n; R_j)\right) \subseteq \bigcap_{i\in I} R_i^{n+1}$. □

Proposition 3.31. *All the tree unravelings of a pointed pseudo-model (M, w) defined in Definition 3.29 are pseudo-models.*

Proof. The condition 1 "$\leqslant; R_G \subseteq R_G$ for any G" of pseudo-frame is obvious by the transitivity of $\preccurlyeq^*$. That is, by $\preccurlyeq^*; \preccurlyeq^* \subseteq \preccurlyeq^*$, we have $\preccurlyeq^*; (\preccurlyeq^*; \mathcal{R}_G) = (\preccurlyeq^*; \preccurlyeq^*); \mathcal{R}_G \subseteq \preccurlyeq^*; \mathcal{R}_G$ for Tree(M, w).

It is seen from items 1 and 3 of Proposition 3.30 that it suffices to show "$\mathcal{R}_H \subseteq \mathcal{R}_G$, when $G \subseteq H$", in order to make sure the condition 2 "$R_H \subseteq R_G$ if $G \subseteq H$" of pseudo-frame. Suppose $\overrightarrow{u}\mathcal{R}_H\overrightarrow{v}$, i.e., $\overrightarrow{v}$ is of the form $\overrightarrow{u}\frown\langle R_{H'}, w'\rangle$ and $H \subseteq H'$. We thus have $G \subseteq H'$ by the assumption, and hence $\overrightarrow{u}\mathcal{R}_G\overrightarrow{v}$. We show $\mathcal{V}$ is hereditary. Take any $p \in \mathsf{Prop}$ and suppose $\overrightarrow{u} \in \mathcal{V}(p)$ and $\overrightarrow{u} \preccurlyeq^* \overrightarrow{v}$. By the former, we have $\mathsf{tail}(\overrightarrow{u}) \in V(p)$. By the latter, it is easily seen that $\mathsf{tail}(\overrightarrow{u}) \leqslant \mathsf{tail}(\overrightarrow{v})$. Since V is hereditary, it turns out that $\mathsf{tail}(\overrightarrow{v}) \in V(p)$, which means $\overrightarrow{v} \in \mathcal{V}(p)$. □

Also, note that Tree$^\bullet(M, w)$ is: a reflexive pseudo-model if $\bullet = \circ$, a transitive pseudo-model if $\bullet = +$, and a reflexive and transitive pseudo-model if $\bullet = *$, which is easily seen from the definition of the relation for D_G in each tree unraveling. Moreover, seriality is inherited from an original pseudo-model:

Proposition 3.32. *$\preccurlyeq^*; \mathcal{R}_{\{a\}}$ and $\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}{}^+\right)^+$ are serial if $R_{\{a\}}$ is serial.*

Proof. Assume that $R_{\{a\}}$ is serial. It is sufficient to show that $\mathcal{R}_{\{a\}}$ is serial, since $\preccurlyeq^*$ is reflexive. Take $\overrightarrow{u} \in$ Finpath(F, w). Since $R_{\{a\}}$ is serial, $\mathsf{tail}(\overrightarrow{u})R_{\{a\}}x$ for some $x \in W$. Therefore, $\left(\overrightarrow{u}, \overrightarrow{u}\frown\langle R_{\{a\}}, x\rangle\right) \in \mathcal{R}_{\{a\}}$. □

The following two propositions are used in the proof of Proposition 3.35.

Proposition 3.33. *Let $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$ be a pseudo-frame and take $w \in W$. We refer to $\mathcal{R}_G$ and $\preccurlyeq$ on* Finpath(F, w) *collectively as "tree relation" and denote such relation by $\mathcal{R}, \mathcal{S}, \mathcal{R}_1, \mathcal{R}_2$, etc. The following hold for tree relations.*

1. *If $\overrightarrow{u}\mathcal{R}\overrightarrow{v}$, then $\overrightarrow{u} = \mathsf{body}(\overrightarrow{v})$.*

2. *If $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m); (\mathcal{S}_1; \cdots; \mathcal{S}_n)$ $(m \geq 0, n \geq 1)$, then the only $\mathsf{body}^n(\overrightarrow{v})$ satisfies the conditions $(\overrightarrow{u}, \mathsf{body}^n(\overrightarrow{v})) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m)$ and $(\mathsf{body}^n(\overrightarrow{v}), \overrightarrow{v}) \in (\mathcal{S}_1; \cdots; \mathcal{S}_n)$, where $\mathsf{body}^n(\overrightarrow{v})$ means $\overbrace{\mathsf{body}(\cdots(\mathsf{body}}^{n}(\overrightarrow{v})))$.*

3. *If* $(\mathcal{R}_1; \cdots; \mathcal{R}_m) \cap (\mathcal{S}_1; \cdots; \mathcal{S}_n) \neq \emptyset$*, then* $n = m$*.*

4. $\mathcal{R}_1; \cdots; \mathcal{R}_m; \bigcap_{i \in I}(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}) = \bigcap_{i \in I}\left(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}\right)$

5. $\bigcap_{i \in I}\left(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}\right); \mathcal{R}_1; \cdots; \mathcal{R}_m = \bigcap_{i \in I}\left(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}; \mathcal{R}_1; \cdots; \mathcal{R}_m\right)$

Proof.

- (item 1) Item 1 is obvious by the definition of $\mathcal{R}_G$ and $\preccurlyeq$.

- (item 2) We show item 2 by induction on n. If $n = 1$, there exists $\overrightarrow{t}$ such that $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_n)\overrightarrow{t}$ and $\overrightarrow{t}\mathcal{S}_1\overrightarrow{v}$. By item 1, $\overrightarrow{t} = \mathsf{body}(\overrightarrow{v})$.
  Since $(\mathcal{R}_1; \cdots; \mathcal{R}_m); (\mathcal{S}_1; \cdots; \mathcal{S}_n) = (\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_1); (\mathcal{S}_2; \cdots; \mathcal{S}_n)$,
  $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_1)\mathsf{body}^{n-1}(\overrightarrow{v})$ and $\mathsf{body}^{n-1}(\overrightarrow{v})(\mathcal{S}_2; \cdots; \mathcal{S}_n)\overrightarrow{v}$ hold by I.H. Then,
  $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_m)\mathsf{body}^{n}(\overrightarrow{v})$ is obtained from the former.

- (item 3) Item 3 is obvious from item 2.

- (item 4) We show item 4. By item 3, if n_i is not constant, the equation is equivalent to $\emptyset = \emptyset$. So, we assume n_i is constant and call it n. The left-to-right is obvious from item 4 of Proposition 3.30. We show the converse. Asuume $(\overrightarrow{u}, \overrightarrow{v}) \in \bigcap_{i \in I}\left(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)}\right)$, i.e., $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)}\right)$ for all i. By item 2, $(\overrightarrow{u}, \mathsf{body}^n(\overrightarrow{v})) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m)$ and $(\mathsf{body}^n(\overrightarrow{v}), \overrightarrow{v}) \in (\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)})$ for all i. Then, we have $(\overrightarrow{u}, \overrightarrow{v}) \in \mathcal{R}_1; \cdots; \mathcal{R}_m; \bigcap_{i \in I}(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)})$.

- (item 5) Item 5 is shown similarly to item 4. □

Proposition 3.34. *Let* $G_1, G_2 \in \mathsf{Grp}$*. Then,* $\mathcal{R}_{G_1} \cap \mathcal{R}_{G_2} = \mathcal{R}_{G_1 \cup G_2}$*.*

Proof. The right-to-left is obvious by the fact that Tree(M, w) is a pseudo-model, so we concentrate on the left-to-right. Assume $(\overrightarrow{u}, \overrightarrow{v}) \in \mathcal{R}_{G_1} \cap \mathcal{R}_{G_2}$, i.e. $\overrightarrow{v} \equiv \overrightarrow{u}{}^\frown\langle R_H, w'\rangle$, where $G_i \subseteq H$ for $i = 1, 2$. Here, $G_1 \cup G_2 \subseteq H$ holds, so we have $\overrightarrow{u}\mathcal{R}_{G_1 \cup G_2}\overrightarrow{v}$. □

Proposition 3.35. *All the tree unravelings of a pointed pseudo-model* (M, w) *defined in Definition 3.29 satisfy the intersection condition defined in Definition 3.15. That is:*

1. *In* Tree(M, w)*,* $\bigcap_{a \in G}(\preccurlyeq^*; \mathcal{R}_{\{a\}}) = \preccurlyeq^*; \mathcal{R}_G$ *holds for any* $G \in \mathsf{Grp}$*.*

2. *In* Tree$^\circ(M, w)$*,* $\bigcap_{a \in G}(\preccurlyeq^*; \mathcal{R}^\circ_{\{a\}}) = \preccurlyeq^*; \mathcal{R}^\circ_G$ *holds for any* $G \in \mathsf{Grp}$*.*

3. *In* Tree$^+(M, w)$*,* $\bigcap_{a \in G}\left(\preccurlyeq^*; \mathcal{R}^+_{\{a\}}\right)^+ = \left(\preccurlyeq^*; \mathcal{R}^+_G\right)^+$ *holds for any* $G \in \mathsf{Grp}$*.*

4. *In* Tree$^*(M, w)$*,* $\bigcap_{a \in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^* = (\preccurlyeq^*; \mathcal{R}^*_G)^*$ *holds for any* $G \in \mathsf{Grp}$*.*

Proof. First, we show that $(\mathcal{R}_G)_{G\in\mathsf{Grp}}$ satisfies the intersection condition, i.e., $\bigcap_{a\in G}\mathcal{R}_{\{a\}} = \mathcal{R}_G$ for any $G \in \mathsf{Grp}$. We show by induction on the cardinarity $\sharp G$ of a group G. If $\sharp G = 1$, the equation trivially holds. Let $\sharp G = 2$ and suppose $G = \{a, b\}$. Then, by Proposition 3.34, $\mathcal{R}_{\{a\}} \cap \mathcal{R}_{\{b\}} = \mathcal{R}_{\{a,b\}}$. Let $\sharp G > 2$ and suppose $G = \{a, b\} \cup G'$. Then, by Proposition 3.34 and I.H., $\bigcap_{c\in G}\mathcal{R}_{\{c\}} = \mathcal{R}_{\{a\}} \cap \mathcal{R}_{\{b\}} \cap \bigcap_{c\in G'}\mathcal{R}_{\{c\}} = \mathcal{R}_{\{a,b\}} \cap \mathcal{R}_{G'} = \mathcal{R}_{\{a,b\}\cup G'} = \mathcal{R}_G$. Thus, we obtain $\bigcap_{a\in G}\mathcal{R}_{\{a\}} = \mathcal{R}_G$ for any $G \in \mathsf{Grp}$. We show each item based on this equation.

- (item 1) From this equation, we have $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}} = \preccurlyeq^*; \mathcal{R}_G$. So, it suffices to show $\bigcap_{a\in G}(\preccurlyeq^*; \mathcal{R}_{\{a\}}) = \preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}$. The right-to-left is the case by 4 of Proposition 3.30. We show the left-to-right. Suppose $\overrightarrow{u} \bigcap_{a\in G}(\preccurlyeq^*; \mathcal{R}_{\{a\}})\overrightarrow{v}$. That is, we have $\overrightarrow{u}(\preccurlyeq^*; \mathcal{R}_{\{a\}})\overrightarrow{v}$ for all $a \in G$. Then, we can take $\overrightarrow{t_a}$ such that $\overrightarrow{u} \preccurlyeq^* \overrightarrow{t_a}$ and $\overrightarrow{t_a}\mathcal{R}_{\{a\}}\overrightarrow{v}$, depending on $a \in G$. Actually, however, according to 1 of Proposition 3.33, we have $\overrightarrow{t_a} = \mathsf{body}(\overrightarrow{v})$, regardless of $a \in G$. Therefore, $\mathsf{body}(\overrightarrow{v})$ satisfies $\overrightarrow{u} \preccurlyeq^* \mathsf{body}(\overrightarrow{v})$ and $\mathsf{body}(\overrightarrow{v})\mathcal{R}_{\{a\}}\overrightarrow{v}$ for all $a \in G$, which entails $\overrightarrow{u}(\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}})\overrightarrow{v}$.

- (item 2) From "$\bigcap_{a\in G}\mathcal{R}_{\{a\}} = \mathcal{R}_G$", we have $\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^\circ = \mathcal{R}_G^\circ$. It is easy to see that $\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^\circ = \bigcap_{a\in G}\mathcal{R}_{\{a\}}^\circ$. Then, we obtain $\bigcap_{a\in G}\mathcal{R}_{\{a\}}^\circ = \mathcal{R}_G^\circ$, which entails $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^\circ = \preccurlyeq^*; \mathcal{R}_G^\circ$. Therefore, our goal is to show that $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^\circ = \bigcap_{a\in G} \preccurlyeq^*; \mathcal{R}_{\{a\}}^\circ$. We have $\mathcal{R}_{\{a\}}^\circ = \mathcal{R}_{\{a\}} \cup Id$, where $Id := \{(\overrightarrow{u}, \overrightarrow{u}) \mid \overrightarrow{u} \in \mathrm{Finpath}(F, w)\}$, and, for any binary relations R and S, $R; (S_1 \cup S_2) = R; S_1 \cup R; S_2$, as easily checked. Using these and $\bigcap_{a\in G}(\preccurlyeq^*; \mathcal{R}_{\{a\}}) = \preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}$ shown in the proof of item 1, we have:

$$
\begin{aligned}
\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^\circ &= \preccurlyeq^*; \bigcap_{a\in G}(\mathcal{R}_{\{a\}} \cup Id) \\
&= \preccurlyeq^*; (\bigcap_{a\in G}\mathcal{R}_{\{a\}} \cup Id) \\
&= (\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}) \cup (\preccurlyeq^*; Id) \\
&= (\bigcap_{a\in G} \preccurlyeq^*; \mathcal{R}_{\{a\}}) \cup (\preccurlyeq^*; Id) \\
&= \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}} \cup \preccurlyeq^*; Id\right) \\
&= \bigcap_{a\in G}\left(\preccurlyeq^*; (\mathcal{R}_{\{a\}} \cup Id)\right) \\
&= \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^\circ\right).
\end{aligned}
$$

- (item 3) By applying the same operations to the both sides of the equation $\bigcap_{a\in G}\mathcal{R}_{\{a\}} =$

$\mathcal{R}_G$, we have $\left(\preccurlyeq^*; (\bigcap_{a\in G}\mathcal{R}_{\{a\}})^+\right)^+ = (\preccurlyeq^*; \mathcal{R}_G^+)^+$. So, the goal is to show $\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)^+ = \left(\preccurlyeq^*; (\bigcap_{a\in G}\mathcal{R}_{\{a\}})^+\right)^+$. We divide the goal into three parts: $(\bigcap_{a\in G}\mathcal{R}_{\{a\}})^+ = \bigcap_{a\in G}\mathcal{R}_{\{a\}}^+$, $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^+ = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)$, and $\left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)\right)^+ = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)^+$. We firstly prove $(\bigcap_{a\in G}\mathcal{R}_{\{a\}})^+ = \bigcap_{a\in G}\mathcal{R}_{\{a\}}^+$. It suffices to show that $(\bigcap_{a\in G}\mathcal{R}_{\{a\}})^n = \bigcap_{a\in G}\mathcal{R}_{\{a\}}^n$ for all n. We show it by induction on n. The case of $n=0$ is already treated in the proof for item 2. As for the step case, we have the following sequence of equations:

$$
\begin{aligned}
\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^{n+1} &= \left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^{n}; \bigcap_{b\in G}\mathcal{R}_{\{b\}} \\
&\overset{\text{(I.H.)}}{=} \left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}^n\right); \bigcap_{b\in G}\mathcal{R}_{\{b\}} \\
&\overset{\text{(5 of Prop. 3.33)}}{=} \bigcap_{a\in G}\left(\mathcal{R}_{\{a\}}^n; \bigcap_{b\in G}\mathcal{R}_{\{b\}}\right) \\
&\overset{\text{(4 of Prop. 3.33)}}{=} \bigcap_{a\in G}\left(\bigcap_{b\in G}(\mathcal{R}_{\{a\}}^n; \mathcal{R}_{\{b\}})\right) \\
&= \bigcap_{a\in G}\left(\mathcal{R}_{\{a\}}^{n+1} \cap \bigcap_{b\neq a}(\mathcal{R}_{\{a\}}^n; \mathcal{R}_{\{b\}})\right) \\
&= \bigcap_{a\in G}\mathcal{R}_{\{a\}}^{n+1} \cap \bigcap_{a\in G}\bigcap_{b\neq a}(\mathcal{R}_{\{a\}}^n; \mathcal{R}_{\{b\}}).
\end{aligned}
$$

We show $\bigcap_{a\in G}\mathcal{R}_{\{a\}}^{n+1} \cap \bigcap_{a\in G}\bigcap_{b\neq a}\left(\mathcal{R}_{\{a\}}^n; \mathcal{R}_{\{b\}}\right) = \bigcap_{a\in G}\mathcal{R}_{\{a\}}^{n+1}$ to complete the step case. The left-to-right is obvious. Suppose $\overrightarrow{u}\bigcap_{a\in G}\mathcal{R}_{\{a\}}^{n+1}\overrightarrow{v}$. Then, by item 2 of Proposition 3.33, $\overrightarrow{u}\mathcal{R}_{\{a\}}^n\mathsf{body}(\overrightarrow{v})$ and $\mathsf{body}(\overrightarrow{v})\mathcal{R}_{\{a\}}\overrightarrow{v}$ for all $a \in G$. The path $\mathsf{body}(\overrightarrow{v})$ obviously makes "$\overrightarrow{u}\left(\bigcap_{a\in G}\bigcap_{b\neq a}\left(\mathcal{R}_{\{a\}}^n; \mathcal{R}_{\{b\}}\right)\right)\overrightarrow{v}$" true. Second, we show $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^+ = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)$. The left-to-right is evident. We show the opposite. Assume $(\overrightarrow{u}, \overrightarrow{v}) \in \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)$, i.e., $\overrightarrow{u}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)\overrightarrow{v}$ for all $a \in G$. By the definition of $\preccurlyeq$ and $\mathcal{R}_G$, it can be seen that $\overrightarrow{t_a}$ satisfying $\overrightarrow{u}\preccurlyeq^*\overrightarrow{t_a}$ and $\overrightarrow{t_a}\mathcal{R}_{\{a\}}^+\overrightarrow{v}$ is constant regardless of $a \in G$, and determined from the form of $\overrightarrow{v}$. We name it $\overrightarrow{t}$. This path $\overrightarrow{t}$ obviously establishes $\overrightarrow{u}\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}_{\{a\}}^+\overrightarrow{v}$. Third, we show $\left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)\right)^+ = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)^+$. The left-to-right is the case by item 6 of Proposition 3.30. We show the right-to-left. Suppose $(\overrightarrow{u}, \overrightarrow{v}) \in \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)^+$, i.e., $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\preccurlyeq^*; \mathcal{R}_{\{a\}}^+\right)^+$ for all $a \in G$. By the definition of $\preccurlyeq$ and $\mathcal{R}_G$, $\overrightarrow{v}$ should be of the following form:

$$\begin{array}{rl}\overrightarrow{v} \equiv \overrightarrow{u}^\frown\langle & \leqslant, w_{1,1}, \cdots, w_{1,l_1-1}, \leqslant, w_{1,l_1}, H_{1,1}, u_{1,1}, \cdots, u_{1,n_1-1}, H_{1,n_1}, u_{1,n_1}, \\ & \vdots \\ & \leqslant, w_{i,1}, \cdots, w_{i,l_i-1}, \leqslant, w_{i,l_i}, H_{i,1}, u_{i,1}, \cdots, u_{i,n_i-1}, H_{i,n_i}, u_{i,n_i}, \\ & \vdots \\ & \leqslant, w_{m,1}, \cdots, w_{m,l_m-1}, \leqslant, w_{m,l_m}, H_{m,1}, u_{m,1}, \cdots, u_{m,n_m-1}, H_{m,n_m}, u_{m,n_m}\rangle \end{array} \tag{3.1}$$

It is noted that $R_{H_{i,j}}$ is denoted by $H_{i,j}$ for the sake of simplicity, that l_i (the number of $\leqslant$ in in the i-th section) can be 0 for any $i \in \{1, \cdots, m\}$, and that the symbols $w_{i,j}$ and $u_{i,k}$ ($j \in \{1, \cdots, l_i\}$, $k \in \{1, \cdots, n_i\}$, $i \in \{1, \cdots, m\}$) denote states in $|M|$. By the definition of $\mathcal{R}_{\{a\}}$, $a \in H_{i,j}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n_i$) for any $a \in G$. Hence, it is obvious by the form of $\overrightarrow{v}$ that $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^+_{\{a\}}\right)\right)^m$.

- (item 4) By applying the same operations to the both sides of the equation $\bigcap_{a\in G}\mathcal{R}_{\{a\}} = \mathcal{R}_G$, we have $\left(\preccurlyeq^*; \left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^*\right)^* = (\preccurlyeq^*; \mathcal{R}^*_G)^*$. So, the goal is to show $\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^* = \left(\preccurlyeq^*; \left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^*\right)^*$. We divide the goal into three parts: $\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^* = \bigcap_{a\in G}\mathcal{R}^*_{\{a\}}$, $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^*_{\{a\}} = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)$, and $\left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)\right)^* = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^*$. First, we have $\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^* = \bigcap_{a\in G}\mathcal{R}^*_{\{a\}}$ because it is shown in the proof for item 3 that $\left(\bigcap_{a\in G}\mathcal{R}_{\{a\}}\right)^n = \bigcap_{a\in G}\mathcal{R}^n_{\{a\}}$. Second, we show $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^*_{\{a\}} = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)$. Under the equation $R^*_{\{a\}} = R^+_{\{a\}} \cup Id$, the equation $R; (S_1 \cup S_2) = R; S_1 \cup R; S_2$ used in the proof for item 2 and the equation $\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^+_{\{a\}} = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^+_{\{a\}}\right)$ shown in the proof for item 3, we have:

$$\begin{aligned}\preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^*_{\{a\}} &= \preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^+_{\{a\}} \cup Id \\ &= \preccurlyeq^*; \bigcap_{a\in G}\mathcal{R}^+_{\{a\}} \cup \preccurlyeq^*; Id \\ &= \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^+_{\{a\}}\right) \cup \preccurlyeq^*; Id \\ &= \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^+_{\{a\}} \cup \preccurlyeq^*; Id\right) \\ &= \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right).\end{aligned}$$

Third, we show $\left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)\right)^* = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^*$. This is equivalent to $\left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)\right)^+ = \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^+$ because Id is a subset of the both side. The left-to-right is the case by item 6 of Proposition 3.30. We show the right-to-left.

Suppose $(\overrightarrow{u}, \overrightarrow{v}) \in \bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^+$, i.e., $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)^+$ for all $a \in G$. Then, as in the proof for item 3, $\overrightarrow{v}$ should be of the the same form as (3.1), except that each n_i can be 0. Since $a \in H_{i,j}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n_i$) for any $a \in G$, it is obvious by the form of $\overrightarrow{v}$ that $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\bigcap_{a\in G}\left(\preccurlyeq^*; \mathcal{R}^*_{\{a\}}\right)\right)^m$. $\square$

Definition 3.36 (bounded morphism)**.** Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ and $M' = (W', \leqslant', (R'_G)_{G\in\mathsf{Grp}}, V')$ be pseudo-models. A function $f\colon W \to W'$ is called a bounded morphism from M to M' if and only if any of the following is satisfied:

- if $w \leqslant v$ then $f(w) \leqslant' f(v)$.
- if $f(w) \leqslant' v'$ then there is $v \in W$ such that $w \leqslant v$ and $v' = f(v)$.
- if wR_Gv then $f(w)R'_Gf(v)$.
- if $f(w)R'_Gv'$ then there is $v \in W$ such that wR_Gv and $v' = f(v)$.
- $w \in V(p)$ iff $f(w) \in V'(p)$.

Proposition 3.37. *Let f be a bounded morphism from a pseudo-model $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ to a pseudo-model $M' = (W', \leqslant', (R'_G)_{G\in\mathsf{Grp}}, V')$. Then, for any $w \in W$ and formula φ,*

$$M, w \Vdash^{ps} \varphi \text{ iff } M', f(w) \Vdash^{ps} \varphi.$$

Proof. By induction on φ. We deal with only the case of $\varphi \equiv \psi \to \chi$ and $\varphi \equiv D_G\psi$.

- (the case of $\varphi \equiv \psi \to \chi$) First, we show the left-to-right. Fix any v such that $f(w) \leqslant' v'$ and assume $M', v' \Vdash^{ps} \psi$. We show $M', v' \Vdash^{ps} \chi$. Since f is a bounded morphism, there exists $v \in W$ such that $w \leqslant v$ and $v' = f(v)$. Then, we have $M', f(v) \Vdash^{ps} \psi$, which entails, by I.H., $M, v \Vdash^{ps} \psi$. Since we have $M, w \Vdash^{ps} \psi \to \chi$ and $w \leqslant v$, we obtain $M, v \Vdash^{ps} \chi$. By I.H., we get $M', v'(= f(v)) \Vdash^{ps} \chi$. Second, we show the right-to-left. Fix any v such that $w \leqslant v$ and assume $M, v \Vdash^{ps} \psi$. We show $M, v \Vdash^{ps} \chi$. Since f is a bounded morphism, $f(w) \leqslant f(v)$. By I.H., we have $M', f(v) \Vdash^{ps} \psi$. Hence, $M', f(v) \Vdash^{ps} \chi$ is obtained by the assumption. We thus have $M, v \Vdash^{ps} \chi$ again by I.H.

- (the case of $\varphi \equiv D_G\psi$) First, we show the left-to-right. Fix any v' such that $f(w)R'_Gv'$. We show $M', v' \Vdash^{ps} \psi$. Since f is a bounded morphism, there exists $v \in W$ such that wR_Gv and $v' = f(v)$. By the assumption, we have $M, v \Vdash^{ps} \psi$, from which it follows, by I.H., that $M, f(v)(= v') \Vdash^{ps} \psi$, as desired. Second, we show the right-to-left. Fix any v such that wR_Gv. We show $M, v \Vdash^{ps} \psi$. Since f

is a bounded morphism, $f(w)R'_G f(v)$. Hence, by the assumption, $M', f(v) \Vdash^{ps} \psi$. Then, by I.H., we obtain $M, v \Vdash^{ps} \psi$. $\square$

Theorem 3.38. *Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$.*

1. *There is a surjective bounded morphism from* $\mathrm{Tree}(M, w)$ *to M.*

2. *There is a surjective bounded morphism from* $\mathrm{Tree}^{\circ}(M, w)$ *to M, if R_G is reflexive.*

3. *There is a surjective bounded morphism from* $\mathrm{Tree}^{+}(M, w)$ *to M, if R_G is transitive.*

4. *There is a surjective bounded morphism from* $\mathrm{Tree}^{*}(M, w)$ *to M, if R_G is reflexive and transitive.*

Proof. In any item, we define a function f from $|\mathrm{Tree}^{\bullet}(M, w)|$ to $|M|$ as one which maps $\overrightarrow{u} \in \mathrm{Finpath}(F, w)$ to $\mathsf{tail}(\overrightarrow{u})$ ($\bullet \in \{\epsilon, \circ, +, *\}$). Surjectivity is evident from the definition of $\mathrm{Finpath}(F, w)$ and the assumption that M is generated by w. We show that f is a bounded morphism. The condition for a valuation is obviously satisfied by the definition of $\mathcal{V}$. We check the condition for preorder. Suppose $\overrightarrow{u} \preccurlyeq^{*} \overrightarrow{v}$. Then, it is obvious by the definition of $\preccurlyeq$ and the transitivity of $\leqslant$ that $f(\overrightarrow{u}) \leqslant f(\overrightarrow{v})$. Suppose $f(\overrightarrow{u}) \leqslant v$. Put $\overrightarrow{v} \coloneqq \overrightarrow{u} {}^\frown \langle \leqslant, v \rangle$, which clearly gives $\overrightarrow{u} \preccurlyeq \overrightarrow{v}$ and $f(\overrightarrow{v}) = v$. We check the condition for R_G relation. The back condition is easier. Suppose $f(\overrightarrow{u}) R_G v$. Put $\overrightarrow{v} \coloneqq \overrightarrow{u} {}^\frown \langle R_G, v \rangle$, which clearly gives $\overrightarrow{u} \mathcal{R}_G \overrightarrow{v}$ and $v = f(\overrightarrow{v})$. Since $\mathcal{R}_G$ is a subset of $\preccurlyeq^{*}; \mathcal{R}_G, \preccurlyeq^{*}; \mathcal{R}_G^{\circ}, \left(\preccurlyeq^{*}; \mathcal{R}_G^{+}\right)^{+}$, and $(\preccurlyeq^{*}; \mathcal{R}_G^{*})^{*}$, the back condition turns out to be satisfied in all items by $\overrightarrow{v}$. We check the forth condition for each item below.

- (item 1) Suppose $\overrightarrow{u} (\preccurlyeq^{*}; \mathcal{R}_G) \overrightarrow{v}$. By definition, $\overrightarrow{v} \equiv \overrightarrow{u} {}^\frown \langle \leqslant, w_1, \cdots, w_{n-1}, \leqslant, w_n, R_H, v \rangle$ ($n \geq 0$ and $G \subseteq H$). Therefore, $f(\overrightarrow{u})(\leqslant; R_H) f(\overrightarrow{v})$, which entails $f(\overrightarrow{u}) R_H f(\overrightarrow{v})$. Since $R_H \subseteq R_G$, we have $f(\overrightarrow{u}) R_G f(\overrightarrow{v})$.

- (item 2) Suppose $\overrightarrow{u} (\preccurlyeq^{*}; \mathcal{R}_G^{\circ}) \overrightarrow{v}$, i.e. $\overrightarrow{u} (\preccurlyeq^{*}; \mathcal{R}_G) \overrightarrow{v}$ or $\overrightarrow{u} \preccurlyeq^{*} \overrightarrow{v}$. In the former case, the same argument as item 1 can be applied. In the latter case, we have $f(\overrightarrow{u}) \leqslant f(\overrightarrow{v})$. Since R_G is reflexive, $\leqslant \subseteq \leqslant; R_G (\subseteq R_G)$. Hence, $f(\overrightarrow{u}) R_G f(\overrightarrow{v})$.

- (item 3) Suppose $\overrightarrow{u} \left(\preccurlyeq^{*}; \mathcal{R}_G^{+}\right)^{+} \overrightarrow{v}$. Then, $\overrightarrow{v}$ is of the same form as (3.1) in the proof of Proposition 3.35. Note that $G \subseteq H_{i,j}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n_i$). Under $R_{H_{i,j}} \subseteq R_G$, we obtain $f(\overrightarrow{u})(\leqslant; R_G^{n_1}; \cdots; \leqslant; R_G^{n_m}) f(\overrightarrow{v})$. Since R_G is transitive, $\leqslant; R_G^{n_1}; \cdots; \leqslant; R_G^{n_m} \subseteq (\leqslant; R_G)^{m} \subseteq R_G^{m} \subseteq R_G$. Thus, $f(\overrightarrow{u}) R_G f(\overrightarrow{v})$.

- (item 4) Suppose $\overrightarrow{u}\,(\preccurlyeq^*;\mathcal{R}_G^*)^*\,\overrightarrow{v}$. If $\overrightarrow{u}\equiv\overrightarrow{v}$, then it is evident from the reflexivity of R_G that $f(\overrightarrow{u})R_Gf(\overrightarrow{v})$. So, we assume $\overrightarrow{u}\not\equiv\overrightarrow{v}$. Then, $\overrightarrow{v}$ is of the same form as (3.1) in the proof of Proposition 3.35, except that n_i can be 0. Also, $G\subseteq H_{i,j}$ ($i=1,\cdots,m$ and $j=1,\cdots,n_i$). Under $R_{H_{i,j}}\subseteq R_G$, we obtain $f(\overrightarrow{u})(\leqslant;R_G^{n_1};\cdots;\leqslant;R_G^{n_m})f(\overrightarrow{v})$. Since R_G is reflexive and transitive, $\leqslant;R_G^{n_1};\cdots;\leqslant;R_G^{n_m}\subseteq(\leqslant;R_G)^m\subseteq R_G^m\subseteq R_G$. Thus, $f(\overrightarrow{u})R_Gf(\overrightarrow{v})$. $\square$

We prove Theorem 3.11. The statement is "*Let* **X** *be any of* **IK***,* **IKT***,* **IKD***,* **IK4***,* **IK4D***, and* **IS4***. Then, if* $\Gamma\Vdash_{\mathbb{F}(\mathbf{X})}\varphi$*, then* $\Gamma\vdash_{\mathsf{H}(\mathbf{X})}\varphi$*.*"

Proof. We show the contraposition. Assume $\Gamma\nvdash_{\mathsf{H}(\mathbf{X})}\varphi$. By Lemma 3.19, We can find an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Γ^+ such that $\Gamma\subseteq\Gamma^+$ and $\Gamma^+\nvdash_{\mathsf{H}(\mathbf{X})}\varphi$. Since $\Gamma\subseteq\Gamma^+$, $M^{\mathbf{X}},\Gamma^+\Vdash^{ps}\Gamma$ by the left-to-right of Lemma 3.23. On the other hand, $M^{\mathbf{X}},\Gamma^+\nVdash^{ps}\varphi$ by the right-to-left of Lemma 3.23 and item 1 of Lemma 3.18. We take an appropriate tree unraveling depending on $\mathbf{X}$.

- ($\mathbf{X}=\mathbf{IK},\mathbf{IKD}$) We can take Tree$\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$, because, by Proposition 3.21, $M^{\mathbf{X}}_{\Gamma^+}$ is a pseudo-model generated by Γ^+. Since Tree$\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$ can be seen as a model in the sense of Definition 3.1 by Proposition 3.35, it suffices to show that $(M^{\mathbf{X}},\Gamma^+)$ satisfies exactly the same formulas as $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\right)$ in the case of **IK**. First, $(M^{\mathbf{X}},\Gamma^+)$ satisfies exactly the same formulas as $(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+)$ by Proposition 3.26. Next, $(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+)$ satisfies exactly the same formulas as $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\right)$ by Proposition 3.37, because $f(\langle\Gamma^+\rangle)=\Gamma^+$ for the bounded morphism f, which is shown to exist in Theorem 3.38. Hence, Tree$\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\Vdash\Gamma$ but Tree$\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\nVdash\varphi$. In the case of **IKD**, we need to additionally show that Tree$\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$ is serial. This is true, because of item 2 of Proposition 3.24, the obvious fact that seriality is preserved under taking generated submodel, and Proposition 3.32.

- ($\mathbf{X}=\mathbf{IKT}$) Take Tree$^\circ\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$. We can show that $\left(\text{Tree}^\circ\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\right)$, which can be seen as a model by Proposition 3.35, satisfies exactly the same formulas as $(M^{\mathbf{X}},\Gamma^+)$ by the same argument as the case of **IK**. Note that Tree$^\circ\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$ is reflexive.

- ($\mathbf{X}=\mathbf{IK4},\mathbf{IK4D}$) Take Tree$^+\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$. We can show $\left(\text{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right),\langle\Gamma^+\rangle\right)$, which can be seen as a model by Proposition 3.35, satisfies exactly the same formulas as $(M^{\mathbf{X}},\Gamma^+)$ by the same argument as the case of **IK**. Note that Tree$^+\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$ is transitive. For the case of **IK4D**, we can show that Tree$^+\left(M^{\mathbf{X}}_{\Gamma^+},\Gamma^+\right)$ is serial by the same argument as the case of **IKD**.

- ($\mathbf{X} = \mathbf{IS4}$) Take $\mathrm{Tree}^*\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$. We can show $\left(\mathrm{Tree}^*\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right), \langle \Gamma^+ \rangle\right)$, which can be seen as a model by Proposition 3.35, satisfies exactly the same formulas as $(M^{\mathbf{X}}, \Gamma^+)$ by the same argument as the case of **IK**. Note that $\mathrm{Tree}^*\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$ is reflexive and transitive. □

Proof for Theorem 3.12

For Theorem 3.12, we introduce an operation which turns a pseudo-model into a stable one without changing satisfaction on any state. Moreover, the operation preserves reflexivity, transitivity, and seriality, and for tree unravelings, also preserves the intersection condition. Using these properties, we prove Theorem 3.12.

Definition 3.39 (stabilization)**.** Let $M = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}}, V)$ be a pseudo-model. $M^{\mathrm{st}} := (W, \leqslant, (R_G; \leqslant)_{G \in \mathsf{Grp}}, V)$ is called "*the stabilization of* M".

Proposition 3.40. *Let $M = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}}, V)$ be a pseudo-model. Then, the stabilization $M^{\mathrm{st}} := (W, \leqslant, (R_G; \leqslant)_{G \in \mathsf{Grp}}, V)$ of M is a stable pseudo-model. Moreover, $M, w \Vdash^{ps} \varphi$ iff $M^{\mathrm{st}}, w \Vdash^{ps} \varphi$ for any $\varphi \in \mathsf{Form}$ and $w \in W$.*

Proof. First, we show that M^{st} is a stable pseudo-model. We show $\leqslant; (R_G; \leqslant); \leqslant \subseteq R_G; \leqslant$. Using $\leqslant; R_G \subseteq R_G$ and the transitivity of $\leqslant$, we have $\leqslant; R_G; \leqslant; \leqslant \subseteq R_G; \leqslant; \leqslant \subseteq R_G; \leqslant$. Further, if $G \subseteq H$, we have $R_H; \leqslant \subseteq R_G; \leqslant$ by $R_H \subseteq R_G$.

Next, we show the latter by induction on φ. Only the case $\varphi \equiv D_G\psi$ is treated. (left-to-right) Assume that $M, w \Vdash^{ps} D_G\psi$. We show that $M^{\mathrm{st}}, w \Vdash^{ps} D_G\psi$. Fix v such that $w(R_G; \leqslant)v$. Then, there is u such that wR_Gu and $u \leqslant v$. From the former and the assumption, we have $M, u \Vdash^{ps} \psi$. Then, by the latter and the heredity, $M, v \Vdash^{ps} \psi$. (right-to-left) Obvious by $R_G \subseteq R_G; \leqslant$. □

Proposition 3.41. *Let $M = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}}, V)$ be a pseudo-model. If R_G is reflexive (transitive, or serial), then $R_G; \leqslant$ is also reflexive (transitive, or serial, respectively).*

Proof. (reflexivity) Obvious by $Id \subseteq \leqslant$, where $Id := \{(w, w) \mid w \in W\}$. (transitivity) Under $\leqslant; R_G \subseteq R_G$ and the transitivity of R_G, we have: $R_G; \leqslant; R_G; \leqslant \subseteq R_G; R_G; \leqslant \subseteq R_G; \leqslant$. (seriality) Obvious by $R_G \subseteq R_G; \leqslant$. □

However, in general, the intersection condition $\left(\bigcap_{a \in G} R_{\{a\}} = R_G\right)$ is not preserved under the stabilization. That is, there is a pseudo-model which satisfies $\bigcap_{a \in G} R_{\{a\}} = R_G$ for any $G \in \mathsf{Grp}$, but does not satisfy "$\bigcap_{a \in G}(R_{\{a\}}; \leqslant) = R_G; \leqslant$ for any $G \in \mathsf{Grp}$". The

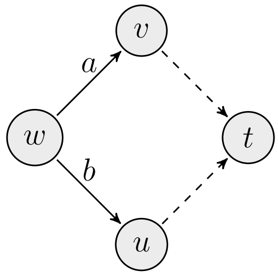

Figure 3.6: Counter pseudo-model M to preservation of intersection condition under stabilization

pseudo-model M depicted in Figure 3.6 is such a model. Let $\mathsf{Agt} = \{a, b\}$. The pseudo-model M is defined as $(\{w, v, u, t\}, \leqslant, (R_{\{a\}}, R_{\{b\}}, R_{\{a,b\}}), V)$, where

$$\leqslant = \{(w, w), (v, v), (v, t), (u, u), (u, t), (t, t)\}, R_{\{a\}} = \{(w, v)\}, R_{\{b\}} = \{(w, u)\}, \text{ and } R_{\{a,b\}} = \emptyset.$$

The valuation V can be any. The solid line stands for the relations for groups and the dotted arrow stands for the preorder. Reflexive arrows for the preorder is omitted in the figure. We can easily see that $\bigcap_{a \in G} R_{\{a\}} = R_G$ for any $G \in \mathsf{Grp}$. However, $\bigcap_{c \in G}(R_{\{c\}}; \leqslant) = R_G; \leqslant$ is not true when $G = \{a, b\}$, because $(w, t) \in \big(R_{\{a\}}; \leqslant\big) \cap \big(R_{\{b\}}; \leqslant\big)$ but $(w, t) \notin R_{\{a,b\}}; \leqslant = \emptyset$.

As can be guessed from the counterexample, "$(R_{G_1}; \leqslant) \cap (R_{G_2}; \leqslant) \subseteq R_{G_1 \cup G_2}; \leqslant$" is a sufficient condition for the preservation of the intersection condition.

Proposition 3.42. *Let M be a pseudo-model satisfying the intersection condition and "$(R_{G_1}; \leqslant) \cap (R_{G_2}; \leqslant) \subseteq R_{G_1 \cup G_2}; \leqslant$" for any $G_1, G_2 \in \mathsf{Grp}$. Then, the intersection condition holds in M^{st}, too.*

Proof. "$R_G; \leqslant \subseteq \bigcap_{a \in G} \big(R_{\{a\}}; \leqslant\big)$" always holds, so we show $\bigcap_{a \in G} \big(R_{\{a\}}; \leqslant\big) \subseteq R_G; \leqslant$ by induction on $\sharp G$. If $\sharp G = 1$, the equation trivially holds. Let $\sharp G = 2$ and suppose $G = \{a, b\}$. Then, by the assumption, $\big(R_{\{a\}}; \leqslant\big) \cap \big(R_{\{b\}}; \leqslant\big) \subseteq R_{\{a,b\}}; \leqslant$. Let $\sharp G > 2$ and suppose $G = \{a, b\} \cup G'$. Then, by the assumption and I.H., $\bigcap_{c \in G} \big(R_{\{c\}}; \leqslant\big) = \big(R_{\{a\}}; \leqslant\big) \cap \big(R_{\{b\}}; \leqslant\big) \cap \bigcap_{c \in G'} \big(R_{\{c\}}; \leqslant\big) \subseteq \big(R_{\{a,b\}}; \leqslant\big) \cap \bigcap_{c \in G'} \big(R_{\{c\}}; \leqslant\big) \subseteq \big(R_{\{a,b\}}; \leqslant\big) \cap R_{G'}; \leqslant \subseteq R_G; \leqslant$. □

We show that the tree unravelings satisfy the very condition.

Proposition 3.43. *The tree unravelings of a pointed pseudo-model (M, w) satisfy the condition "$(R_{G_1}; \leqslant) \cap (R_{G_2}; \leqslant) \subseteq R_{G_1 \cup G_2}; \leqslant$" for any $G_1, G_2 \in \mathsf{Grp}$.*

Proof.
- (Tree(M, w)) Assume $(\overrightarrow{u}, \overrightarrow{v}) \in (\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*) \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*)$. We show that $(\overrightarrow{u}, \overrightarrow{v}) \in \preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*$. Then, by the definitions of $\preccurlyeq$ and $\mathcal{R}_G$,

$\overrightarrow{v} \equiv \overrightarrow{u}^\frown\langle \leqslant, w_1, \cdots, w_{n-1}, \leqslant, w_n, R_H, u_0, \leqslant, u_1, \cdots, u_{m-1}, \leqslant, u_m \rangle$, where $n, m \geq 0$ and $G_i \subseteq H$ for $i = 1, 2$. Since $G_1 \cup G_2 \subseteq H$, we also have $(\overrightarrow{u}, \overrightarrow{v}) \in \preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*$.

- (Tree$^\circ(M, w)$) Since $\left(\preccurlyeq^*; \mathcal{R}^\circ_{G_1}; \preccurlyeq^*\right) \cap \left(\preccurlyeq^*; \mathcal{R}^\circ_{G_2}; \preccurlyeq^*\right) = ((\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*) \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*)) \cup \preccurlyeq^*$ and $\left(\preccurlyeq^*; \mathcal{R}^\circ_{G_1 \cup G_2}; \preccurlyeq^*\right) = (\preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*) \cup \preccurlyeq^*$, the condition is obviously satisfied by the argument of the case of Tree(M, w).

- (Tree$^+(M, w)$) Assume $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\left(\preccurlyeq^*; \mathcal{R}^+_{G_1}\right)^+; \preccurlyeq^*\right) \cap \left(\left(\preccurlyeq^*; \mathcal{R}^+_{G_2}\right)^+; \preccurlyeq^*\right)$. We show $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\preccurlyeq^*; \mathcal{R}^+_{G_1 \cup G_2}\right)^+; \preccurlyeq^*$. By the definitions of $\preccurlyeq$ and $\mathcal{R}_G$,

$$\begin{aligned}\overrightarrow{v} \equiv \overrightarrow{u}^\frown\langle\ & \leqslant, w_{1,1}, \cdots, w_{1,l_1-1}, \leqslant, w_{1,l_1}, H_{1,1}, u_{1,1}, \cdots, u_{1,n_1-1}, H_{1,n_1}, u_{1,n_1}, \\ & \vdots \\ & \leqslant, w_{i,1}, \cdots, w_{i,l_i-1}, \leqslant, w_{i,l_i}, H_{i,1}, u_{i,1}, \cdots, u_{i,n_i-1}, H_{i,n_i}, u_{i,n_i}, \\ & \vdots \\ & \leqslant, w_{m,1}, \cdots, w_{m,l_m-1}, \leqslant, w_{m,l_m}, H_{m,1}, u_{m,1}, \cdots, u_{m,n_m-1}, H_{m,n_m}, u_{m,n_m}, \\ & \leqslant, w_{m+1,1}, \cdots, w_{m+1,l_{m+1}-1}, \leqslant, w_{m+1,l_{m+1}}\rangle,\end{aligned}$$

where $R_{H_{i,j}}$ is denoted by $H_{i,j}$ for the sake of simplicity, l_i (≥ 0) denote the number of $\leqslant$ in in the i-th section for any $i \in \{1, \cdots, m+1\}$, $n_i > 0$ for any $i \in \{1, \cdots, m\}$, and $\mathcal{R}_G$, $G_k \subseteq H_{i,j}$ $(i = 1, \cdots, m$, and $j = 1, \cdots, n_i)$ for $k = 1, 2$. Since $G_1 \cup G_2 \subseteq H_{i,j}$, we have $(\overrightarrow{u}, \overrightarrow{v}) \in \left(\preccurlyeq^*; \mathcal{R}^+_{G_1 \cup G_2}\right)^+; \preccurlyeq^*$. $\square$

We end the present section by proving Theorem 3.12. The statement is "*Let* **X** *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Then, if* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$. "

Proof. We show the contraposition. Assume $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$. By the same argument as the proof of Theorem 3.11, we have $M^{\mathbf{X}}, \Gamma^+ \Vdash^{ps} \Gamma$ and $M^{\mathbf{X}}, \Gamma^+ \nVdash^{ps} \varphi$.

- ($\mathbf{X} = \mathbf{IK}, \mathbf{IKD}$) First, by Proposition 3.40, $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}, \langle\Gamma^+\rangle\right)$ satisfies the same formulas as $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right), \langle\Gamma^+\rangle\right)$, which is shown in the proof of Theorem 3.11 to satisfy the same formulas as $(M^{\mathbf{X}}, \Gamma^+)$. Second, Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}$ also can be seen as a model by Proposition 3.42 and 3.43. Therefore, Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}, \langle\Gamma^+\rangle \Vdash \Gamma$ and Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}, \langle\Gamma^+\rangle \nVdash \varphi$. For the case of **IKD**, we have to show that Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}$ is serial. This is true by the fact shown in the proof of Theorem 3.11 that Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$ is serial and Proposition 3.41.

- ($\mathbf{X} = \mathbf{IKT}$) By the same argument as the case of **IK**, Tree$^\circ$ $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}, \langle\Gamma^+\rangle \Vdash \Gamma$ and Tree$^\circ$ $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}, \langle\Gamma^+\rangle \nVdash \varphi$. Note that Tree$^\circ$ $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\text{st}}$ is reflexive by Proposition 3.41.

- ($\mathbf{X} = \mathbf{IK4}, \mathbf{IK4D}$) By the same argument as the case of **IK**, $\mathrm{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\mathrm{st}}, \langle\Gamma^+\rangle \Vdash \Gamma$ and $\mathrm{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\mathrm{st}}, \langle\Gamma^+\rangle \nVdash \varphi$. Note that $\mathrm{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\mathrm{st}}$ is transitive by Proposition 3.41. If $\mathbf{X} = \mathbf{IK4D}$, $\mathrm{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)^{\mathrm{st}}$ is also serial, because of the fact shown in the proof of Theorem 3.11 that $\mathrm{Tree}^+\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$ is serial and Proposition 3.41.

- ($\mathbf{X} = \mathbf{IS4}$) Theorem 3.11 is sufficient because $\mathbb{F}(\mathbf{IS4}) = \mathbb{F}(\mathbf{IS4}) \cap \mathbb{ST}$ by item 2 of Proposition 3.3. □

3.4 Sequent Calculi

3.4.1 Equipollence and Cut-Elimination

A *sequent* is a pair of finite multisets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$", where $\#\Delta \leq 1$. The multiset Γ is called an"antecedent" of a sequent $\Gamma \Rightarrow \Delta$, and Δ a "succedent". A sequent is intuitively interpreted as "if all formulas in Γ hold, then a formula in Δ holds." The reason why the number of Δ is restricted is that we build our calculus on the basis of Gentzen's **LJ** [11, 12] for intuitionistic propositional logic. Our sequent calculi for the intuitionistic epistemic logics with distributed knowledge are presented in Table 3.2. Axioms, structural rules, and propositional logical rules are common to **LJ**. The other rules are the same as the ones in [30], except that rules for (D) axiom, i.e., $(D_{\mathbf{IKD}})$ and $(D_{\mathbf{IK4D}})$ are added, in order to construct calculi for the logics **IKD** and **IK4D**.

We note that when $n = 0$, e.g., in the rule (D) of Table 3.2, the multiset is regarded as the empty multiset and thus $\bigcup_{i=1}^{n} G_i$ is regarded as $\varnothing$. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in each calculus $\mathsf{G}(\mathbf{X})$ if there exists a finite tree of sequents, whose root is $\Gamma \Rightarrow \Delta$ and each node of which is inferred by some rule (including axioms) in $\mathsf{G}(\mathbf{X})$. We write it as $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$.

We note that for any logic $\mathbf{X}$ under consideration, $\mathsf{H}(\mathbf{X})$ and $\mathsf{G}(\mathbf{X})$ are equipollent in the following sense.

Theorem 3.44 (Equipollence)**.** *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Then, the following hold. 1. If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, *then* $\vdash_{\mathsf{G}(\mathbf{X})} \Rightarrow \varphi$. *2. If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma \to \bigvee \Delta$, *where* $\bigwedge \varnothing := \top$ *and* $\bigvee \varnothing := \bot$.

Proof. We show the case of **IK**. The idea for proof is common to the rest. Here we focus on item 2 alone. We show item 2 by induction on the structure of the derivation for the sequent $\Gamma \Rightarrow \Delta$. We deal with the case for the rule (D) only. Suppose we have a

Table 3.2: Sequent Calculi for **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, and **IS4**

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (Id) \quad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to) \quad \frac{\Gamma_1 \Rightarrow \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^1) \quad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^2)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^1) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^2) \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

Logical Rules for D_G of IK

$$\frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D)$$

Logical Rules for D_G of IKT

$$\frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

Logical Rules for D_G of IKD

$$\frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \quad \frac{\Gamma \Rightarrow}{D_{\{a\}}\Gamma \Rightarrow}\ (D_{\mathbf{IKD}})$$

Logical Rules for D_G of IK4

$$\frac{\varphi_1, \dots, \varphi_n, D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IK4}})$$

Logical Rules for D_G of IK4D

$$\frac{\varphi_1, \dots, \varphi_n, D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IK4}}) \quad \frac{\Gamma, D_{\{a\}}\Gamma \Rightarrow}{D_{\{a\}}\Gamma \Rightarrow}\ (\Rightarrow D_{\mathbf{IK4D}})$$

Logical Rules for D_G of IS4

$$\frac{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IS4}}) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

derivation

$$\dfrac{\dfrac{\mathcal{D}}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \ (D).$$

We show $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_G\psi$. We have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n \varphi_i \to \psi$ as the induction hypothesis for the derivation $\mathcal{D}$. From this, we can infer by necessitation $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^n \varphi_i \to \psi)$. By this and axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^n \varphi_i) \to D_G\psi$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_G\varphi_i \to D_G\psi$. Therefore, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to \bigwedge_{i=1}^n D_G\varphi_i$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_G\varphi_i$ for any $i \in \{1, \ldots, n\}$. This is evident because we have a theorem in intuitionistic propositional logic $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_{G_i}\varphi_i$ and the axiom (Incl) $\vdash_{\mathsf{H}(\mathbf{X})} D_{G_i}\varphi_i \to D_G\varphi_i$. $\square$

We have the cut-elimination theorem for all of the logics in consideration. First, we introduce a notion of "principal formula". A principal formula is defined for each inference rule, except for the axioms and (Cut) rule and is informally expressed as "a formula, on which the inference rule acts."

Definition 3.45. A *principal formula* of the structural rules, the propositional logical rules, and the rule $(D \Rightarrow)$ is a formula appearing in the lower sequent, which is not contained in Γ, Γ_1, Γ_2, or Δ. A *principal formula* of the rules for D_G operator other than $(D \Rightarrow)$ is every formula in the lower sequent.

Theorem 3.46 (Cut-Elimination). *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Then, the following holds: If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$, *where* $\mathsf{G}^-(\mathbf{X})$ *denotes a system "*$\mathsf{G}(\mathbf{X})$ *minus the cut rule".*

Proof. Following [24, Section 9.3] and [36, Section 2.2], we consider a system $\mathsf{G}^*(\mathbf{X})$, in which the cut rule is replaced by the "extended" cut rule defined as:

$$\frac{\Gamma \Rightarrow \varphi^n \quad \varphi^m, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Theta} \ (ECut),$$

where φ^n denotes the multi-set of n-copies of φ and $n = 0, 1$ and $m \geq 0$. Since $(ECut)$ is the same as (Cut) when we set $n = m = 1$, it is obvious that if $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, so it suffices to show that if $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$.

Suppose $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$ and fix one derivation for the sequent. To obtain an $(ECut)$-free derivation of $\Gamma \Rightarrow \Delta$, it is enough to concentrate on a derivation whose root is derived by $(ECut)$ and which has no other application of $(ECut)$. In what follows, we let $\mathbf{X}$ be **IKT**. The cases of the other logics can be similarly shown. Let us suppose that $\mathcal{D}$ has the following structure:

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi^n}\ (\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{\Gamma, \Sigma \Rightarrow \Theta}\ (ECut),$$

where the derivations $\mathcal{L}$ and $\mathcal{R}$ has no application of $(ECut)$ and $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are meta-variables for the name of rule applied there. Let the number of logical symbols (including D_G) appearing in φ be $c(\mathcal{D})$ and the number of sequents in $\mathcal{L}$ and $\mathcal{R}$ be $w(\mathcal{D})$. We show the lemma by double induction on $(c(\mathcal{D}), w(\mathcal{D}))$. If $n = 0$ or $m = 0$, we can derive the root sequent of $\mathcal{D}$ without using $(ECut)$ by weakening rules. So, in what follows we assume $n = 1$ and $m > 0$. Then, it is sufficient to consider the following four cases following [37, proof of Theorem 2.3], [24, Section 9.3], and [36, Section 2.2]: [2]

1. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is an axiom.
2. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a structural rule.
3. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a logical rule and a cut formula φ is not principal (in the sense we have specified above) for that rule.
4. $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are both logical rules (including (D)) for the same logical symbol and a cut formula φ is principal for each rule.

We omit case 1 and case 2 in the proof since this is well-known in a proof for **LJ** (the reader is referred to [36, p.28]). For the same reason, the cases concerning propositional logical rules are also not treated. Therefore, we treat case 3 and case 4 involving D_G.

(Case 3) If $\mathbf{rule}_{\mathcal{L}} = (D \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\psi, \Gamma \Rightarrow \varphi}}{D_G\psi, \Gamma \Rightarrow \varphi}\ (D \Rightarrow) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{D_G\psi, \Gamma, \Sigma \Rightarrow \Theta}\ (ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\psi, \Gamma \Rightarrow \varphi} \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{\psi, \Gamma, \Sigma \Rightarrow \Theta}\ (ECut)}{D_G\psi, \Gamma, \Sigma \Rightarrow \Theta}\ (D \Rightarrow)$$

[2] In case 4, we assume the condition for both rule applications, because if the one of the two rule applications does not satisfy the condition, the whole derivation should be categorized into one of the rest cases.

The subderivation $\mathcal{E}'$ whose root is $\psi, \Gamma, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_G\psi, \Gamma, \Sigma \Rightarrow \Theta$.

If $\mathbf{rule}_{\mathcal{R}} = (D \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\ (\mathbf{rule}_{\mathcal{L}}) \qquad \dfrac{\dfrac{\mathcal{R}'}{\varphi^m, \psi, \Sigma \Rightarrow \Theta}}{\varphi^m, D_G\psi, \Sigma \Rightarrow \Theta}\ (D \Rightarrow)}{\Gamma, D_G\psi, \Sigma \Rightarrow \Theta}\ (ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\ (\mathbf{rule}_{\mathcal{L}}) \qquad \dfrac{\mathcal{R}'}{\varphi^m, \psi, \Sigma \Rightarrow \Theta}}{\Gamma, \psi, \Sigma \Rightarrow \Theta}\ (ECut)}{\Gamma, D_G\psi, \Sigma \Rightarrow \Theta}\ (D \Rightarrow)$$

The subderivation $\mathcal{E}'$ whose root is $\Gamma, \psi, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, D_G\psi, \Sigma \Rightarrow \Theta$.

(Case 4) One of the possible structures of the derivation $\mathcal{D}$ is the following.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \qquad \dfrac{\dfrac{\mathcal{R}'}{\psi^m, \psi_1, \dots, \psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^m H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1, \dots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (D)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \dots, D_{H_m}\psi_n \Rightarrow D_H\chi}\ (ECut)$$

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \quad \dfrac{\mathcal{R}'}{\psi^m, \psi_1, \dots, \psi_m \Rightarrow \chi}}{\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \Rightarrow \chi}\ (ECut) \qquad (\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \dots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (D).$$

We call $\mathcal{E}'$ its subderivation whose root sequent is $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \Rightarrow \chi$. The derivation $\mathcal{E}'$ have no application of $(ECut)$ and $c(\mathcal{E}') < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \dots, D_{H_m}\psi_m \Rightarrow D_H\chi$ as required.

The other possible structure of the derivation $\mathcal{D}$ is the following:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1,\ldots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D) \qquad \dfrac{\dfrac{\mathcal{R}'}{\psi,(D_G\psi)^{m-1},\Sigma \Rightarrow \Theta}}{(D_G\psi)^{m},\Sigma \Rightarrow \Theta}\,(D \Rightarrow)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1,\ldots,\varphi_n \Rightarrow \psi} \qquad \dfrac{\dfrac{\mathcal{L}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}'}{\psi,(D_G\psi)^{m-1},\Sigma \Rightarrow \Theta}}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\psi,\Sigma \Rightarrow \Theta}\,(ECut)}{\varphi_1,\ldots,\varphi_n,D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta}\,(ECut)}{\vdots}\,(D\Rightarrow)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta}\,(D\Rightarrow)}{\vdots}\,(c\Rightarrow)}{D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta}\,(c\Rightarrow)$$

The subderivation $\mathcal{E}_1$ whose root is $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\psi,\Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_1}$ having the same root sequent. Name $\mathcal{E}_2$, the derivation obtained by replacing the derivation $\mathcal{E}_1$ by $\tilde{\mathcal{E}_1}$ in the subderivation whose root is $\varphi_1,\ldots,\varphi_n,D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta$. The derivation $\mathcal{E}_2$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_2) < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_2}$ having the same root sequent. Thus, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1,\ldots,D_{G_n}\varphi_n,\Sigma \Rightarrow \Theta$. □

The following subformula property is an important corollary of the cut-elimination theorem, and later used in a proof of decidability.

Corollary 3.47 (Subformula Property)**.** *Let* **X** *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4** *and suppose* $\vdash_{\mathsf{G(X)}} \Gamma \Rightarrow \Delta$. *Then, there exists a derivation of* $\Gamma \Rightarrow \Delta$ *satisfying a condition that any formula occurring in the derivation is a subformula of certain formula in* Γ *or* Δ.

Proof. A cut-free derivation of $\Gamma \Rightarrow \Delta$ satisfies the condition, because any formula in the upper sequent is a subformula of certain formula in the lower sequent in every inference rules of our calculi except (Cut). □

3.4.2 Craig Interpolation Theorem and Decidability

In many logics, the Craig interpolation theorem can be derived as an application of the cut-elimination theorem, using a Maehara method originally described in [26]. An application of the method to basic modal logic can also be found in [35]. Unlike [30], the concept of 'partition' is simplified, because we do not allow multiple formulas to appear in the succedent of a sequent.

Definition 3.48 (Partition)**.** A *partition* for a sequent $\Gamma \Rightarrow \Delta$ is defined as a tuple $\langle \Gamma_1; \Gamma_2 \rangle$, such that $\Gamma = \Gamma_1, \Gamma_2$.

Definition 3.49. For a formula φ, $\mathsf{Prop}(\varphi)$ is defined as the set of all propositional variables appearing in φ. For a multiset Γ of formulas, $\mathsf{Prop}(\Gamma)$ is defined as $\bigcup_{\varphi \in \Gamma} \mathsf{Prop}(\varphi)$. Similarly, $\mathsf{Agt}(\varphi)$ is defined as the set of agents appearing in φ and $\mathsf{Agt}(\Gamma)$ as $\bigcup_{\varphi \in \Gamma} \mathsf{Agt}(\varphi)$

The following is a key lemma for Craig Interpolation Theorem.

Lemma 3.50. *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Suppose* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$. *Then, for any partition* $\langle \Gamma_1; \Gamma_2 \rangle$ *for the sequent* $\Gamma \Rightarrow \Delta$, *there exists a formula* φ *called "interpolant", satisfying the following:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma_1 \Rightarrow \varphi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi, \Gamma_2 \Rightarrow \Delta$.
2. $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \Delta)$.
3. $\mathsf{Agt}(\varphi) \subseteq \mathsf{Agt}(\Gamma_1) \cap \mathsf{Agt}(\Gamma_2, \Delta)$.

Proof. We prove the case of **IKT** by induction on the structure of a derivation for $\Gamma \Rightarrow \Delta$. Fix the derivation and name it $\mathcal{D}$. By Theorem 3.46, we can assume that $\mathcal{D}$ is cut-free. We treat the case for axioms and the case involving D_G below (for other cases, the reader is referred to [35]).

Suppose $\mathcal{D}$ is of the form $\dfrac{}{\varphi \Rightarrow \varphi}\,(Id)$. For the partition $\langle \varphi; \rangle$, φ is clearly an interpolant of the sequent $\varphi \Rightarrow \varphi$. For the partition $\langle ; \varphi \rangle$, $\bot \to \bot$ is an interpolant of the sequent $\varphi \Rightarrow \varphi$, because we have the following derivations:

$$\dfrac{\dfrac{}{\bot \Rightarrow \bot}\,(Id)}{\Rightarrow \bot \to \bot}\,(\Rightarrow\to) \quad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(Id)}{\bot \to \bot, \varphi \Rightarrow \varphi}\,(w \Rightarrow).$$

Items 2 and 3 are clearly satisfied because $\mathsf{Prop}(\bot) = \mathsf{Agt}(\bot) = \emptyset$.

Suppose $\mathcal{D}$ is of the form $\dfrac{}{\bot \Rightarrow}\,(\bot)$. For the partition $\langle \bot; \rangle$, $\bot$ is clearly an interpolant of the sequent $\bot \Rightarrow$. For the partition $\langle ; \bot \rangle$, $\bot \to \bot$ is an interpolant of the sequent $\bot \Rightarrow$, because we have the following derivations:

$$\dfrac{\dfrac{}{\bot \Rightarrow \bot}\,(Id)}{\Rightarrow \bot \to \bot}\,(\Rightarrow\to) \quad \dfrac{\dfrac{}{\bot \Rightarrow}\,(\bot)}{\bot \to \bot, \bot \Rightarrow}\,(w\Rightarrow).$$

Item 2 and 3 are clearly satisfied because $\mathsf{Prop}(\bot) = \mathsf{Agt}(\bot) = \mathsf{Prop}(\bot \to \bot) = \mathsf{Agt}(\bot \to \bot) = \emptyset$.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D).$$

A partition of $D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi$ is of the following form:

$$\langle D_{G_1}\varphi_1, \ldots, D_{G_k}\varphi_k; D_{G_{k+1}}\varphi_{k+1}, \ldots, D_{G_n}\varphi_n \rangle.$$

As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \varphi_1, \ldots, \varphi_k; \varphi_{k+1}, \ldots, \varphi_n \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \varphi_1, \ldots, \varphi_k \Rightarrow \chi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \chi, \varphi_{k+1}, \ldots, \varphi_n \Rightarrow \psi$ for some formula χ. If $k > 0$, we can choose $D_{\bigcup_{i=1}^{k} G_i}\chi$ as a required interpolant, because we have the following derivations:

$$\dfrac{\dfrac{\text{I.H.}}{\varphi_1, \ldots, \varphi_k \Rightarrow \chi} \quad (\bigcup_{i=1}^{k} G_i \subseteq \bigcup_{i=1}^{k} G_i)}{D_{G_1}\varphi_1, \ldots, D_{G_k}\varphi_k \Rightarrow D_{\bigcup_{i=1}^{k} G_i}\chi}\,(D)$$

$$\dfrac{\dfrac{\text{I.H.}}{\chi, \varphi_{k+1}, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{k} G_i \cup \bigcup_{i=k+1}^{n} G_i = \bigcup_{i=1}^{n} G_i \subseteq G)}{D_{\bigcup_{i=1}^{k} G_i}\chi, D_{G_{k+1}}\varphi_{k+1}, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D)$$

The interpolant $D_{\bigcup_{i=1}^{k} G_i}\chi$ satisfies item 2, because $\mathsf{Prop}\left(D_{\bigcup_{i=1}^{k} G_i}\chi\right) = \mathsf{Prop}(\chi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\varphi_1, \ldots, \varphi_k) \cap \mathsf{Prop}(\varphi_{k+1}, \ldots, \varphi_n, \psi) =$
$\mathsf{Prop}(D_{G_1}\varphi_1, \ldots, D_{G_k}\varphi_k) \cap \mathsf{Prop}(D_{G_{k+1}}\varphi_{k+1}, \ldots, D_{G_n}\varphi_n, D_G\psi)$. Item 3 is also satisfied. As the induction hypothesis, we have $\mathsf{Agt}(\chi) \subseteq \mathsf{Agt}(\varphi_1, \ldots, \varphi_k) \cap \mathsf{Agt}(\varphi_{k+1}, \ldots, \varphi_n, \psi)$. Then, $\mathsf{Agt}\left(D_{\bigcup_{i=1}^{k} G_i}\chi\right) = \bigcup_{i=1}^{k} G_i \cup \mathsf{Agt}(\chi) \subseteq \bigcup_{i=1}^{k} G_i \cup (\mathsf{Agt}(\varphi_1, \ldots, \varphi_k) \cap \mathsf{Agt}(\varphi_{k+1}, \ldots, \varphi_n, \psi)) = (\bigcup_{i=1}^{k} G_i \cup \mathsf{Agt}(\varphi_1, \ldots, \varphi_k)) \cap (\bigcup_{i=1}^{k} G_i \cup \mathsf{Agt}(\varphi_{k+1}, \ldots, \varphi_n, \psi)) \subseteq (\bigcup_{i=1}^{k} G_i \cup \mathsf{Agt}(\varphi_1, \ldots, \varphi_k)) \cap (G \cup \mathsf{Agt}(\varphi_{k+1}, \ldots, \varphi_n, \psi)) = \mathsf{Agt}(D_{G_1}\varphi_1, \ldots, D_{G_k}\varphi_k) \cap \mathsf{Agt}(D_{G_{k+1}}\varphi_{k+1}, \ldots, D_{G_n}\varphi_n, D_G\psi)$.
If $k = 0$, we can choose χ as an interpolant, since we have the following derivations:

$$\dfrac{\text{I.H.}}{\Rightarrow \chi} \quad \dfrac{\dfrac{\dfrac{\mathcal{D}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D)}{\chi, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(w\Rightarrow).$$

Item 2 is satisfied, because $\mathsf{Prop}(\chi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\varphi_1, \ldots, \varphi_n, \psi) = \mathsf{Prop}(D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_G\psi)$. Item 3 is satisfied, because $\mathsf{Agt}(\chi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Agt}(\varphi_1, \ldots, \varphi_n, \psi) \subseteq \mathsf{Agt}(D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_G\psi)$.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\varphi, \Gamma \Rightarrow \Delta}}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

There are two types of partition for the sequent $D_G\varphi, \Gamma \Rightarrow \Delta$ depending on whether $D_G\varphi$ belongs to the left or the right of a partition. First, fix the partition $\langle \Gamma_1, D_G\varphi; \Gamma_2 \rangle$. As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \Gamma_1, \varphi; \Gamma_2 \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1, \varphi \Rightarrow \psi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi, \Gamma_2 \Rightarrow \Delta$ for some formula ψ. The formula ψ is an interpolant for the sequent $\Gamma, D_G\varphi \Rightarrow \Delta$, too, because we have the following derivation:

$$\dfrac{\dfrac{\text{I.H.}}{\Gamma_1, \varphi \Rightarrow \psi}}{\Gamma_1, D_G\varphi \Rightarrow \psi}\ (D \Rightarrow).$$

The interpolant ψ also satisfies item 2, because $\mathsf{Prop}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_1, \varphi) \cap \mathsf{Prop}(\Gamma_2, \Delta) = \mathsf{Prop}(\Gamma_1, D_G\varphi) \cap \mathsf{Prop}(\Gamma_2, \Delta)$. Item 3 is also satisfied, because $\mathsf{Agt}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Agt}(\Gamma_1, \varphi) \cap \mathsf{Agt}(\Gamma_2, \Delta) \subseteq \mathsf{Agt}(\Gamma_1, D_G\varphi) \cap \mathsf{Agt}(\Gamma_2, \Delta)$.

Next, fix the partition $\langle \Gamma_1; \Gamma_2, D_G\varphi \rangle$. As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \Gamma_1; \Gamma_2, \varphi \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1 \Rightarrow \psi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi, \Gamma_2, \varphi \Rightarrow \Delta$ for some formula ψ. The formula ψ is an interpolant for the sequent $\Gamma, D_G\varphi \Rightarrow \Delta$, too, because we have the following derivation:

$$\dfrac{\dfrac{\text{I.H.}}{\psi, \Gamma_2, \varphi \Rightarrow \Delta}}{\psi, \Gamma_2, D_G\varphi \Rightarrow \Delta}\ (D \Rightarrow).$$

The interpolant ψ also satisfies item 2, because $\mathsf{Prop}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \varphi, \Delta) = \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, D_G\varphi, \Delta)$. Item 3 is also satisfied, because $\mathsf{Agt}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Agt}(\Gamma_1) \cap \mathsf{Agt}(\Gamma_2, \varphi, \Delta) \subseteq \mathsf{Agt}(\Gamma_1) \cap \mathsf{Agt}(\Gamma_2, D_G\varphi, \Delta)$. □

Theorem 3.51 (Craig Interpolation Theorem)**.** *Let* **X** *be any of* **IK**, **IKT**, **IKD**, **IK4**, **IK4D**, *and* **IS4**. *Given that* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$, *there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \chi \Rightarrow \psi$.

2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

3. $\mathsf{Agt}(\chi) \subseteq \mathsf{Agt}(\varphi) \cap \mathsf{Agt}(\psi)$.

We note that not only the condition for propositional variables but also the condition for agents can be satisfied.

Proof. When we set $\Gamma := \varphi$ and $\Delta := \psi$, and take a partition $\langle \Gamma; \varnothing \rangle$, Lemma 3.50 proves Craig Interpolation Theorem. $\square$

Further, decidability of the logics we investigate also follows from the cut-elimination theorem (Theorem 3.46). To show decidability, we introduce a notion of "(1-)reduced sequent".

Definition 3.52. A sequent $\Gamma \Rightarrow \Delta$ is called *reduced* if every formula occurs at most three times in Γ. A sequent $\Gamma \Rightarrow \Delta$ is called *1-reduced* if every formula occurs at most once in Γ.

Definition 3.53. For any sequent $\Gamma \Rightarrow \Delta$, a sequent $\Gamma^* \Rightarrow \Delta$ is a *1-reduced contraction* of $\Gamma \Rightarrow \Delta$ if $\Gamma^* \Rightarrow \Delta$ can be derived from $\Gamma \Rightarrow \Delta$ by applying $(c \Rightarrow)$ to $\Gamma \Rightarrow \Delta$ and is 1-reduced. Clearly, a 1-reduced contraction is determined uniquely.

Proposition 3.54. $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$ *if and only if* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma^* \Rightarrow \Delta$.

Proof. By definition of the 1-reduced contraction, the left-to-right is obvious. The right-to-left is also easily shown by applying $(w \Rightarrow)$ to $\Gamma^* \Rightarrow \Delta$. $\square$

Lemma 3.55. *Suppose that* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$. *Then, there exists a derivation of* $\Gamma^* \Rightarrow \Delta$ *such that the derivation is cut-free and has only reduced sequents.*

Proof. Thanks to Theorem 3.46, we can take a cut-free derivation of $\Gamma \Rightarrow \Delta$. We name it $\mathcal{D}$. We show by induction on the height of $\mathcal{D}$. We treat only the case where the last rule application of $\mathcal{D}$ is (D). That is, suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\varphi_1, \dots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

By induction hypothesis, we have a derivation $\mathcal{E}'$ of $(\varphi_1, \dots, \varphi_n)^* \Rightarrow \psi$ such that $\mathcal{E}'$ is cut-free and has only reduced sequents. Applying the rule (D) to $\mathcal{E}'$, we obtain the desired derivation of $(D_{G_1}\varphi_1, \dots, D_{G_n}\varphi_n)^* \Rightarrow D_G\psi$. $\square$

Remark 3.56. We admit three occurrences of the same formula in a reduced sequent, because if we only allow at most two occurrences, induction fails in the case of $(\rightarrow\Rightarrow)$ in the proof of this lemma.

Theorem 3.57 (Decidability)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK4}$*,* $\mathbf{IK4D}$*, and* $\mathbf{IS4}$*. A logic* $\mathbf{X}$ *is decidable, that is, there is an algorithm checking whether each sequent* $\Gamma \Rightarrow \Delta$ *has a derivation in* $\mathsf{G}(\mathbf{X})$ *or not.*

Proof. We describe a rough sketch of the proof, based on [35, p. 228]. By Proposition 3.54, it suffices to check whether $\Gamma^* \Rightarrow \Delta$ has a derivation. In what follows, by "tree (of $\Sigma \Rightarrow \Theta$)", we mean a tree of sequents (ending with $\Sigma \Rightarrow \Theta$), whose leaves are axioms, or sequents, to which no rule can be applied. Without any restriction, there are infinitely many trees of $\Gamma^* \Rightarrow \Delta$. Therefore, in order to execute a brute-force search, we impose three restrictions on the trees. In general, if a derivation exists, Lemma 3.55 allows us to find a derivation such that (i) it is cut-free and (ii) it has only reduced sequents. By Corollary 3.47, it has subformula property. Therefore, there are finitely many reduced sequents that can be a part of the derivation. Moreover, we can safely assume that (iii) for each path in the derivation from the root sequent to an initial sequent, each sequent in the path occurs exactly once, because, if there are multiple occurrences of the same sequent, we can always eliminate the redundant occurrences by grafting the subderivation for the uppermost occurrence onto the lowermost occurrence. From the above argument, if we impose the conditions (i) to (iii) on the trees of $\Gamma^* \Rightarrow \Delta$, the number of trees becomes finite and we can construct an algorithm enumerating all of them which also checks whether each tree is a derivation or not. If the algorithm does not find any derivation, we can conclude that $\Gamma^* \Rightarrow \Delta$ has no derivation. $\square$

Chapter 4

Intuitionistic Public Announcement Logic with Distributed Knowledge

This chapter is organized as follows. In Section 4.1, we expand the intuitionistic epistemic logic of Chapter 3 with reduction axioms for a public announcement operator, and prove its completeness by reducing to the completeness of static one via translation from a formula possibly with public announcement operators to a formula without any public announcement operators. In Section 4.3, we introduce sequent calculi for the logics and prove the cut-elimination theorem and Craig interpolation theorem.

The author's contribution is as follows. Regarding the Hilbert systems defined by the author, the author found that the soundness does not hold for the corresponding class of frames without any condition. The author confirmed that the problem can be solved by imposing the condition of stability defined in [64, 49, 46] on Kripke frames. The solution was a suggestion by the supervisor. The author proved the completeness theorem on his own. The sequent calculi were defined through discussions with the supervisor, and the author proved the cut elimination theorem and the Craig interpolation theorem.

4.1 Syntax, Semantics, and Hilbert System

We expand our syntax with the public announcement operator and define the set of all formulas of the expanded syntax as:

$$\mathsf{Form}^{+} \ni \varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid D_G\varphi \mid [\varphi]\varphi,$$

where $p \in \mathsf{Prop}$ and $G \in \mathsf{Grp}$. We define $\neg\varphi$ as $\varphi \to \bot$, $\varphi \leftrightarrow \psi$ as $\varphi \to \psi \wedge \psi \to \varphi$, $\top$ as $\bot \to \bot$, and the epistemic operator $K_a\varphi$ (read "agent a knows that φ") as $D_{\{a\}}\varphi$.

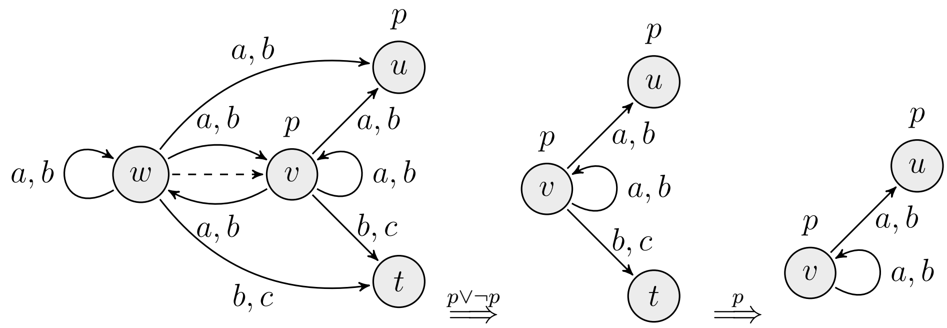

Figure 4.1: $M_{\text{stable}}, M^{p\vee\neg p}_{\text{stable}}, (M^{p\vee\neg p}_{\text{stable}})^p$

Definition 4.1. Let $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and $\varphi, \psi \in \mathsf{Form}^+$. The satisfaction relation $M, w \Vdash \varphi$ is defined as before except:

$$M, w \Vdash [\varphi]\psi \quad \text{iff} \quad \text{for all } v \in W, w \leqslant v \text{ and } M, v \Vdash \varphi \text{ jointly imply } M^\varphi, v \Vdash \psi,$$

where $M^\varphi := (\llbracket\varphi\rrbracket_M, \leqslant^\varphi, (R_a^\varphi)_{a\in\mathsf{Agt}}, V^\varphi)$ (*a model updated from M by φ*) is defined as follows:

- $\llbracket\varphi\rrbracket_M := \{w \in W \mid M, w \Vdash \varphi\}$,
- $\leqslant^\varphi := \leqslant \cap(\llbracket\varphi\rrbracket_M \times \llbracket\varphi\rrbracket_M)$,
- $R_a^\varphi := R_a \cap (\llbracket\varphi\rrbracket_M \times \llbracket\varphi\rrbracket_M)$,
- $V^\varphi(p) := V(p) \cap \llbracket\varphi\rrbracket_M$.

It is easy to show the following.

Proposition 4.2. *Let $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and $\varphi \in \mathsf{Form}^+$.*

1. *M^φ satisfies all the conditions of a model.*
2. *If R_a is reflexive (or transitive), then so is R_a^φ.*
3. *If M is stable, then so is M^φ.*

Remark 4.3. Seriality is not preserved under $(-)^\varphi$, as explained in Remark 1.13. Hence, the corresponding axiom (D) is not in consideration below.

Example 4.4. In Figure 4.1, we consider updates of the model M_{stable} discussed in Example 3.5 (recall Figure 3.2). If we update the first (leftmost) model by $p \vee \neg p$, we obtain the second model, $M^{p\vee\neg p}_{\text{stable}}$. This amounts to making the situation classical-logical. That is, the updated model can be seen as a classical Kripke model since no proper pair is ordered by $\leqslant^{p\vee\neg p}$. By this update, p becomes a distributed knowledge at v of a group $\{a,b\}$, that is, $M_{\text{stable}}, v \Vdash \neg D_{\{a,b\}}p$ but $M^{p\vee\neg p}_{\text{stable}}, v \Vdash D_{\{a,b\}}p$. Next, we update $M^{p\vee\neg p}_{\text{stable}}$ by p to obtain the model $(M^{p\vee\neg p}_{\text{stable}})^p$. It is easy to see that $(M^{p\vee\neg p}_{\text{stable}})^p$ and M^{p}_{stable} are the same. By this update, p becomes a knowledge at v of an agent b, that is, $M^{p\vee\neg p}_{\text{stable}}, v \Vdash \neg D_{\{b\}}p$ but $(M^{p\vee\neg p}_{\text{stable}})^p, v \Vdash D_{\{b\}}p$.

Under these definitions, the heredity condition still holds.

Proposition 4.5. *If $M, w \Vdash \varphi$ and $w \leqslant v$, then $M, v \Vdash \varphi$.*

Proof. We show the case where $\varphi \equiv [\psi]\chi$. For the rest cases, the reader is referred to Proposition 3.4. Suppose $M, w \Vdash [\psi]\chi$ and $w \leqslant v$. To show $M, v \Vdash [\psi]\chi$, fix u such that $v \leqslant u$ and assume $M, u \Vdash \psi$. Then, from $M, w \Vdash [\psi]\chi$ and $w \leqslant u$, we have $M^{\psi}, u \Vdash \chi$ as required. □

Note that this would not hold if we were to adopt "$M, w \Vdash \varphi$ implies $M^{\varphi}, w \Vdash \psi$" as the definition of $M, w \Vdash [\varphi]\psi$.

The axioms in Table 4.1 are for the PAL extension. We call the axiom system expanded from $\mathsf{H}(\mathbf{X})$ by all the axioms in Table 4.1, $\mathsf{H}(\mathbf{X})^+$, where $\mathbf{X} = \mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IK4}$, and $\mathbf{IS4}$. The notion of semantic consequence and derivability is defined in the same way as $\mathsf{H}(\mathbf{X})$.

Table 4.1: Axioms for Public Announcement Operator

$([]p)$	$[\varphi]p \leftrightarrow (\varphi \to p)$	$([]\bot)$	$[\varphi]\bot \leftrightarrow (\varphi \to \bot)$
$([]\to)$	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$	$([]\vee)$	$[\varphi](\psi \vee \chi) \leftrightarrow (\varphi \to [\varphi]\psi \vee [\varphi]\chi)$
$([]\wedge)$	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	$([]D)$	$[\varphi]D_G\psi \leftrightarrow (\varphi \to D_G[\varphi]\psi)$
$([][])$	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$		

In what follows, we establish that all the axioms in Table 4.1 are valid with respect to the class of all *stable* frames. First, we deal with composition of two announcements. Recall from Example 4.4 that $(M^{p\vee\neg p}_{\text{stable}})^p$ and M^{p}_{stable} of Figure 4.1 are the same. This can be understood as an example of the following lemma, because $(p \vee \neg p) \wedge [p \vee \neg p]p$ and p are equivalent.

Lemma 4.6. *Let $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ be a model and $\varphi, \psi \in \mathsf{Form}^+$. Then, $(M^{\varphi})^{\psi} = M^{\varphi\wedge[\varphi]\psi}$.*

Proof. It suffices to show that $|(M^{\varphi})^{\psi}| = |M^{\varphi\wedge[\varphi]\psi}|$. Assume $w \in |M^{\varphi\wedge[\varphi]\psi}|$, which is equivalent to $M, w \Vdash \varphi \wedge [\varphi]\psi$. This is equivalent to $M, w \Vdash \varphi$ and "$M, v \Vdash \varphi$ implies $M^{\varphi}, v \Vdash \psi$ for any v such that $w \leqslant v$". By instantiating v with w, we can infer "$M, w \Vdash \varphi$

implies $M^\varphi, w \Vdash \psi$" from the latter. Then, we have $M^\varphi, w \Vdash \psi$ by modus ponens, which means $w \in |(M^\varphi)^\psi|$. For the left-to-right, let us assume $M^\varphi, w \Vdash \psi$. We have to show $M, w \Vdash \varphi$ and "$M, v \Vdash \varphi$ implies $M^\varphi, v \Vdash \psi$ for any v such that $w \leqslant v$". The assumption presupposes $w \in |M^\varphi|$, which means $M, w \Vdash \varphi$, the former goal. For the latter implication, fix a v satisfying $w \leqslant v$. Then, by the heredity, $M, v \Vdash \varphi$, so $w \leqslant^\varphi v$. Then, again by the heredity, $M^\varphi, v \Vdash \psi$. □

Proposition 4.7. *The axioms in Table 4.1 except ($[]D$) are valid with respect to the class of all frames.*

Proof. We show the validity of ($[]\vee$) and ($[][]$) alone. First, we show the validity of ($[]\vee$). For any model M and $w \in |M|$, we show $M, w \Vdash [\varphi](\psi_1 \vee \psi_2) \to \varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)$. Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$. To show $M, v \Vdash \varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)$, fix u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M, u \Vdash [\varphi]\psi_1 \vee [\varphi]\psi_2$. From $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$, $v \leqslant u$, and $M, u \Vdash \varphi$, we have $M^\varphi, u \Vdash \psi_1 \vee \psi_2$. Then, it suffices to show that $M^\varphi, u \Vdash \psi_i$ implies $M, u \Vdash [\varphi]\psi_i$ for $i = 1, 2$. Assume $M^\varphi, u \Vdash \psi_i$ and fix any t such that $u \leqslant t$ and $M, t \Vdash \varphi$. Obviously, we have $u \leqslant^\varphi t$. Then, by the assumption and the heredity, $M^\varphi, t \Vdash \psi_i$. Next, we show $M, w \Vdash (\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)) \to [\varphi](\psi_1 \vee \psi_2)$. Fix any v such that $w \leqslant v$ and $M, v \Vdash \varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)$. To show $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$, fix u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M^\varphi, u \Vdash \psi_1 \vee \psi_2$. Since $v \leqslant u$ and $M, u \Vdash \varphi$, we have $M, u \Vdash [\varphi]\psi_1 \vee [\varphi]\psi_2$. Then, it suffices to show that $M, u \Vdash [\varphi]\psi_i$ implies $M^\varphi, u \Vdash \psi_i$ for $i = 1, 2$. This is obviously true because $M, u \Vdash \varphi$.

Second, we show the validity of ($[][]$). First, for any model M and $w \in |M|$, we show $M, w \Vdash [\varphi][\psi]\chi \to [\varphi \wedge [\varphi]\psi]\chi$. Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi][\psi]\chi$. To show $M, v \Vdash [\varphi \wedge [\varphi]\psi]\chi$, fix u such that $v \leqslant u$ and $M, u \Vdash \varphi \wedge [\varphi]\psi$. Since $M, u \Vdash \varphi$ and $M, v \Vdash [\varphi][\psi]\chi$, we have $M^\varphi, u \Vdash [\psi]\chi$. Also, by $M, u \Vdash [\varphi]\psi$, we have $M^\varphi, u \Vdash \psi$. Then, by these two, we obtain $(M^\varphi)^\psi, u \Vdash \chi$, which is equivalent to $M^{\varphi \wedge [\varphi]\psi}, u \Vdash \chi$ by Lemma 4.6. To show the right-to-left, Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi \wedge [\varphi]\psi]\chi$. Fix any u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M^\varphi, u \Vdash [\psi]\chi$. Fix any t such that $u \leqslant^\varphi t$ and $M^\varphi, t \Vdash \psi$. Instead of $(M^\varphi)^\psi, t \Vdash \chi$, we use Lemma 4.6 to show $M^{\varphi \wedge [\varphi]\psi}, t \Vdash \chi$. To show this, we show $v \leqslant t$ and $M, t \Vdash \varphi \wedge [\varphi]\psi$. The former is obvious. Also, $M, t \Vdash \varphi$ since $t \in |M^\varphi|$. Further, we have $M, t \Vdash [\varphi]\psi$. For this, take s such that $t \leqslant s$ and $M, s \Vdash \varphi$. Then, $M^\varphi, s \Vdash \psi$, since $M^\varphi, t \Vdash \psi$ and $t \leqslant^\varphi s$. □

Proposition 4.8. *The axiom ($[]D$) is not valid with respect to the class of all frames.*

Proof. The model M depicted in Figure 4.2 is a counterexample. Let $\mathsf{Agt} = \{a\}$. The model M is defined as $(\{w, v, u\}, \leqslant, R_a, V)$, where

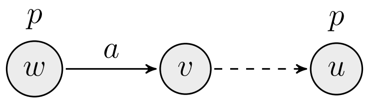

Figure 4.2: Counter model M to validity of $([]D)$

$$\leqslant\; = \{(w,w),(v,v),(v,u),(u,u)\},$$

$R_a = \{(w,v)\}$, and V is such that $V(p) = \{w,u\}$ and $V(q) = \emptyset$. The solid line stands for the relations for agents and the dotted arrow stands for the preorder. Reflexive arrows for the preorder is omitted in the figure. The condition "$\leqslant; R_a \subseteq R_a$" is easily checked. Also, the condition "$R_a; \leqslant \subseteq R_a$" is easily seen not to be satisfied, since $w(R_a;\leqslant)u$ holds but wR_au fails. Therefore, M is not a stable model.

To show that the axiom $([]D)$ is not valid with respect to the class of all frames, we show $[p]K_aq \to (p \to K_a[p]q)$ is not satisfied at the state w in M. First, $M,w \Vdash [p]K_aq$, because $M^p, w \Vdash K_aq$ is vacuously true by the model update which eliminates v, and w is the only world accessible from w by $\leqslant$. Second, however, $M,w \nVdash p \to K_a[p]q$. The antecedent is obviously true at w. To reject the consequent, we focus on v, the only world accessible from w by R_a, to show $M,v \nVdash [p]q$. This is true because we can find u, which is ahead of v, satisfies p, but does not q. □

By restricting our attention to the class of all *stable* frames, we can recover the validity of $([]D)$ as follows.

Proposition 4.9. *The axiom $([]D)$ is valid with respect to the class of all stable frames.*

Proof. For any stable model M and $w \in |M|$, we show $M,w \Vdash [\varphi]D_G\psi \to (\varphi \to D_G[\varphi]\psi)$. Fix any v such that $w \leqslant v$ and $M,v \Vdash [\varphi]D_G\psi$. We show $M,v \Vdash \varphi \to D_G[\varphi]\psi$. Fix any u such that $v \leqslant u$ and $M,u \Vdash \varphi$. To show $M,u \Vdash D_G[\varphi]\psi$, fix any t such that $(u,t) \in \bigcap_{a\in G} R_a$. We show $M,t \Vdash [\varphi]\psi$. Fix any s such that $t \leqslant s$ and $M,s \Vdash \varphi$. We show $M^\varphi, s \Vdash \psi$. By the stability of the underlying frame, we have $\bigcap_{a\in G} R_a;\leqslant = \bigcap_{a\in G} R_a$. Hence, $(u,s) \in \bigcap_{a\in G} R_a$. Then, we have $M^\varphi, s \Vdash \psi$, because $M^\varphi, u \Vdash D_G\psi$ holds by $M,v \Vdash [\varphi]D_G\psi$, $v \leqslant u$, and $M,u \Vdash \varphi$ and we have $s \in |M^\varphi|$. Next, we show $M,w \Vdash (\varphi \to D_G[\varphi]\psi) \to [\varphi]D_G\psi$. Fix any v such that $w \leqslant v$ and $M,v \Vdash \varphi \to D_G[\varphi]\psi$. To show $M,v \Vdash [\varphi]D_G\psi$, fix any u such that $v \leqslant u$ and $M,u \Vdash \varphi$. Take any t such that $(u,t) \in \bigcap_{a\in G} R_a^\varphi$. We show $M^\varphi, t \Vdash \psi$. By $M,v \Vdash \varphi \to D_G[\varphi]\psi$, $v \leqslant u$, and $M,u \Vdash \varphi$, we have $M,u \Vdash D_G[\varphi]\psi$. Then, $M,t \Vdash [\varphi]\psi$. Since $M,t \Vdash \varphi$, we obtain $M^\varphi, t \Vdash \psi$. □

Theorem 4.10 (soundness)**.** *Let $\varphi \in \mathsf{Form}^+$. If $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$, then $\Vdash_{\mathbb{F}(\mathbf{X})\cap\mathbb{ST}} \varphi$.*

Proof. Obvious by the soundness of the static part of the axiom system and Proposition 4.7. □

Note that $\mathsf{H}(\mathbf{X})^+$ is sound with respect to the class of all *stable* frames due to the axiom $([]D)$.

4.2 Semantic Completeness

We show the strong completeness of $\mathsf{H}(\mathbf{X})^+$ with respect to the class of stable frames using a translation $\mathsf{Form}^+ \to \mathsf{Form}$ as in [59].

Definition 4.11 (translation)**.** The *translation function* t on Form^+ is defined as follows:

- $t(p) = p$
- $t(\bot) = \bot$
- $t(\varphi \bullet \psi) = t(\varphi) \bullet t(\psi)$ $(\bullet \in \{\to, \vee, \wedge\})$
- $t(D_G\varphi) = D_G t(\varphi)$
- $t([\varphi]p) = t(\varphi \to p)$
- $t([\varphi]\bot) = t(\varphi \to \bot)$
- $t([\varphi](\psi \star \chi)) = t([\varphi]\psi \star [\varphi]\chi)$ $(\star \in \{\to, \wedge\})$
- $t([\varphi](\varphi \vee \chi)) = t(\varphi \to [\varphi]\psi \vee [\varphi]\chi)$
- $t([\varphi]D_G\psi) = t(\varphi \to D_G[\varphi]\psi)$
- $t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)$

To show that t is Form-valued, a measure called complexity of a formula is introduced.

Definition 4.12 (complexity)**.** The *complexity function* $c\colon \mathsf{Form}^+ \to \mathbb{N}$ is defined inductively as follows:

- $c(p) = 1$
- $c(\bot) = 1$
- $c(\varphi \bullet \psi) = 1 + \max(c(\varphi), c(\psi))$ $(\bullet \in \{\to, \vee, \wedge\})$
- $c(D_G\varphi) = 1 + c(\varphi)$
- $c([\varphi]\psi) = (2 + c(\varphi)) \cdot c(\psi)$

Note that $c(\varphi) \geq 1$ for any formula φ, as easily shown.

Lemma 4.13. *Let $\varphi, \psi, \chi \in \mathsf{Form}^+$. The following holds:*

1. *$c(\psi) > c(\varphi)$ if $\varphi \in Sub(\psi)$ and $\varphi \neq \psi$, where $Sub(\psi)$ denotes the set of subformulas of ψ.*
2. *$c([\varphi]p) > c(\varphi \to p)$.*
3. *$c([\varphi]\bot) > c(\varphi \to \bot)$.*
4. *$c([\varphi](\psi \star \chi)) > c([\varphi]\psi \star [\varphi]\chi)$. $(\star \in \{\to, \wedge\})$*
5. *$c([\varphi](\psi \vee \chi)) > c(\varphi \to [\varphi]\psi \vee [\varphi]\chi)$.*
6. *$c([\varphi]D_G\psi) > c(\varphi \to D_G[\varphi]\psi)$.*
7. *$c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$.*

Proof.

- (item 1) By induction on the structure of ψ. If $\psi \equiv p$ or $\bot$, item 1 is trivially true because there is no proper subformula of ψ. Suppose $\psi \equiv \psi_1 \bullet \psi_2$ ($\bullet \in \{\to, \vee, \wedge\}$). By the assumption, $\varphi \in Sub(\psi_i)$ ($i = 1$ or 2). Then, by I.H., $c(\psi_i) \geq c(\varphi)$. Thus, $c(\psi) = 1 + \max(c(\psi_1), c(\psi_2)) > 1 + c(\psi_i) > c(\varphi)$. Suppose $\psi \equiv D_G\chi$. By the assumption, $\varphi \in Sub(\chi)$. Then, by I.H., $c(\chi) \geq c(\varphi)$. Thus, $c(\psi) = 1 + c(\chi) > c(\varphi)$. Suppose $\psi \equiv [\psi_1]\psi_2$. By the assumption, $\varphi \in Sub(\psi_i)$ ($i = 1$ or 2). Then, by I.H., $c(\psi_i) \geq c(\varphi)$. If $i = 1$, $c(\psi) = (2 + c(\psi_1)) \cdot c(\psi_2) \geq 2 + c(\psi_1) > c(\varphi)$. If $i = 2$, $c(\psi) = (2 + c(\psi_1)) \cdot c(\psi_2) \geq 3 \cdot c(\psi_2) > c(\varphi)$.

- (item 2) We have $c([\varphi]p) = (2 + c(\varphi)) \cdot c(p) = 2 + c(\varphi) > 1 + c(\varphi) = 1 + \max(c(\varphi), c(p)) = c(\varphi \to p)$.

- (item 3) The same as item 2.

- (item 4) We may assume that $c(\psi) \geq c(\chi)$ without loss of generality. Then, we have $c([\varphi](\psi \star \chi)) = (2 + c(\varphi)) \cdot c(\psi \star \chi) = (2 + c(\varphi)) \cdot (1 + c(\psi))$. On the other hand, $c([\varphi]\psi \star [\varphi]\chi) = 1 + \max(c([\varphi]\psi), c([\varphi]\chi)) = 1 + c([\varphi]\psi) = 1 + (2 + c(\varphi)) \cdot c(\psi)$. Thus, $c([\varphi](\psi \star \chi)) - c([\varphi]\psi \star [\varphi]\chi) = 1 + c(\varphi) > 0$.

- (item 5) We may assume that $c(\psi) \geq c(\chi)$ without loss of generality. Then, $c([\varphi](\psi \vee \chi)) = (2 + c(\varphi)) \cdot (1 + c(\psi))$. On the other hand, $c(\varphi \to [\varphi]\psi \vee [\varphi]\chi) = 1 + \max(c(\varphi), 1 + c([\varphi]\psi)) \overset{\text{(item1)}}{=} 1 + 1 + c([\varphi]\psi) = 2 + (2 + c(\varphi)) \cdot c(\psi)$. Then, $c([\varphi](\psi \vee \chi)) - c(\varphi \to [\varphi]\psi \vee [\varphi]\chi) = c(\varphi) > 0$.

- (item 6) We have $c([\varphi]D_G\psi) = (2 + c(\varphi)) \cdot c(D_G\psi) = (2 + c(\varphi)) \cdot (1 + c(\psi))$. On the other hand, $c(\varphi \to D_G[\varphi]\psi) = 1 + \max(c(\varphi), c(D_G[\varphi]\psi)) \overset{\text{(item1)}}{=} 1 + c(D_G[\varphi]\psi) = 1+(1+c([\varphi]\psi)) = 2+(2+c(\varphi))\cdot c(\psi)$. Thus, $c([\varphi]D_G\psi) - c(\varphi \to D_G[\varphi]\psi) = c(\varphi) > 0$.

- (item 7) We have $c([\varphi][\psi]\chi) = (2 + c(\varphi)) \cdot c([\psi]\chi) = (2 + c(\varphi)) \cdot (2 + c(\psi)) \cdot c(\chi)$. On the other hand, $c([\varphi \wedge [\varphi]\psi]\chi) = (2+c(\varphi \wedge [\varphi]\psi))\cdot c(\chi) = (2+1+\max(c(\varphi), c([\varphi]\psi)))\cdot c(\chi) \overset{\text{(item1)}}{=} (3+c([\varphi]\psi))\cdot c(\chi) = (3+(2+c(\varphi))\cdot c(\psi))\cdot c(\chi)$. Then, $c([\varphi][\psi]\chi) - c([\varphi \wedge [\varphi]\psi]\chi) = c(\chi)\cdot((2+c(\varphi))\cdot(2+c(\psi)) - (3+(2+c(\varphi))\cdot c(\psi))) = c(\chi)\cdot(1+2\cdot c(\varphi)) > 0$. □

Lemma 4.14. *The translation function t is indeed a* Form*-valued function, i.e., $t(\varphi)$ does not include a public announcement operator for any formula $\varphi \in \mathsf{Form}^+$.*

Proof. By induction on $c(\varphi)$. It is obvious if $\varphi \equiv p$ or $\bot$. If $\varphi \equiv \psi_1 \bullet \psi_2$ ($\bullet \in \{\to, \wedge, \vee\}$), $c(\varphi) > c(\psi_i)$ by item 1 of Lemma 4.13. Hence, by I.H., $t(\psi_1 \bullet \psi_2) = t(\psi_1) \bullet t(\psi_2) \in \mathsf{Form}$. The same argument applies to the case that $\varphi \equiv D_G\psi$. Suppose that $\varphi \equiv [\psi]\chi$. Depending on the form of χ, the corresponding item from among items 2 to 7 of Lemma 4.13 assures that $t(\varphi) \in \mathsf{Form}$. □

Lemma 4.15. *Let* $\mathbf{X} = \mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IK4}$, *or* $\mathbf{IS4}$ *and* $\varphi \in \mathsf{Form}^+$. *Then,* $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi \leftrightarrow t(\varphi)$.

Proof. By induction on $c(\varphi)$. If $\varphi \equiv p$ or $\bot$, it is obvious. Suppose $\varphi \equiv \psi_1 \bullet \psi_2$ ($\bullet \in \{\to, \wedge, \vee\}$). By I.H., $\psi_i \leftrightarrow t(\psi_i)$. Then, by intuitionistic tautologies, we have $\vdash_{\mathsf{H}(\mathbf{X})^+} \psi_1 \bullet \psi_2 \leftrightarrow t(\psi_1) \bullet t(\psi_2)$. Suppose $\varphi \equiv D_G\psi$. By I.H. and the axiom (K), $\vdash_{\mathsf{H}(\mathbf{X})^+} D_G\psi \leftrightarrow D_G t(\psi)$. Suppose that $\varphi \equiv [\psi]\chi$. Depending on the form of χ, the corresponding item from among items 2 to 7 of Lemma 4.13 assures that $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi \leftrightarrow t(\varphi)$. □

Theorem 4.16 (strong completeness)**.** *Let* $\mathbf{X} = \mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IK4}$, *or* $\mathbf{IS4}$ *and* $\Gamma \cup \{\varphi\} \subseteq \mathsf{Form}^+$. *If* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$.

Proof. Assume that $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$. By Theorem 4.10 and Lemma 4.15, we have $t[\Gamma] \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} t(\varphi)$, where $t[\Delta] := \{t(\psi) \mid \psi \in \Delta\}$. By Lemma 4.14, $t[\Gamma \cup \{\varphi\}] \subseteq \mathsf{Form}$. Then, by Theorem 3.12, $t[\Gamma] \vdash_{\mathsf{H}(\mathbf{X})} t(\varphi)$. Then, for some finite set $\Gamma' \subseteq \Gamma$, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge t[\Gamma'] \to t(\varphi)$. Since $\mathsf{H}(\mathbf{X})$ is a subsystem of $\mathsf{H}(\mathbf{X})^+$, we also have $\vdash_{\mathsf{H}(\mathbf{X})^+} \bigwedge t[\Gamma'] \to t(\varphi)$. Then, by Lemma 4.15, $\vdash_{\mathsf{H}(\mathbf{X})^+} \bigwedge \Gamma' \to \varphi$, which means that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$. □

4.3 Sequent Calculi

Our sequent calculi $\mathsf{G}(\mathbf{X})^+$ for the intuitionistic public announcement logic with distributed knowledge are obtained by adding the rules in Table 4.2 to $\mathsf{G}(\mathbf{X})$ defined in Table 3.2.

Table 4.2: Additional Logical Rules for $\mathsf{G}(\mathbf{X})^+$

$$\dfrac{\Gamma, \varphi \Rightarrow p}{\Gamma \Rightarrow [\varphi]p}\ (\Rightarrow []p) \qquad \dfrac{\Gamma_1 \Rightarrow \varphi \quad p, \Gamma_2 \Rightarrow \Delta}{[\varphi]p, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ ([]p \Rightarrow)$$

$$\dfrac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow [\varphi]\bot}\ (\Rightarrow []\bot) \qquad \dfrac{\Gamma \Rightarrow \varphi}{[\varphi]\bot, \Gamma \Rightarrow}\ ([]\bot \Rightarrow)$$

$$\dfrac{\Gamma, [\varphi]\psi \Rightarrow [\varphi]\chi}{\Gamma \Rightarrow [\varphi](\psi \to \chi)}\ (\Rightarrow [] \to) \qquad \dfrac{\Gamma_1 \Rightarrow [\varphi]\psi \quad [\varphi]\chi, \Gamma_2 \Rightarrow \Delta}{[\varphi](\psi \to \chi), \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ ([] \to\Rightarrow)$$

$$\dfrac{\Gamma, \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}{\Gamma \Rightarrow [\varphi](\psi \vee \chi)}\ (\Rightarrow []\vee) \qquad \dfrac{\Gamma_1 \Rightarrow \varphi \quad [\varphi]\psi, \Gamma_2 \Rightarrow \Delta \quad [\varphi]\chi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, [\varphi](\psi \vee \chi) \Rightarrow \Delta}\ ([]\vee \Rightarrow)$$

$$\dfrac{\Gamma \Rightarrow [\varphi]\psi \quad \Gamma \Rightarrow [\varphi]\chi}{\Gamma \Rightarrow [\varphi](\psi \wedge \chi)}\ (\Rightarrow []\wedge)$$

$$\dfrac{\Gamma, [\varphi]\psi \Rightarrow \Delta}{\Gamma, [\varphi](\psi \wedge \chi) \Rightarrow \Delta}\ ([]\wedge \Rightarrow^1) \qquad \dfrac{\Gamma, [\varphi]\chi \Rightarrow \Delta}{\Gamma, [\varphi](\psi \wedge \chi) \Rightarrow \Delta}\ ([]\wedge \Rightarrow^2)$$

$$\dfrac{\Gamma, \varphi \Rightarrow D_G[\varphi]\psi}{\Gamma \Rightarrow [\varphi]D_G\psi}\ (\Rightarrow []D) \qquad \dfrac{\Gamma_1 \Rightarrow \varphi \quad D_G[\varphi]\psi, \Gamma_2 \Rightarrow \Delta}{[\varphi]D_G\psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ ([]D \Rightarrow)$$

$$\dfrac{\Gamma \Rightarrow [\varphi \wedge [\varphi]\psi]\chi}{\Gamma \Rightarrow [\varphi][\psi]\chi}\ (\Rightarrow [][]) \qquad \dfrac{[\varphi \wedge [\varphi]\psi]\chi, \Gamma \Rightarrow \Delta}{[\varphi][\psi]\chi, \Gamma \Rightarrow \Delta}\ ([][] \Rightarrow)$$

Since the rules are constructed naturally from the reduction axioms for the public announcement operator, the following equipollence theorem is easy to prove.

Theorem 4.17 (Equipollence)**.** *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IK4**, *and* **IS4**. *Then, the following hold. 1. If* $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$, *then* $\vdash_{\mathsf{G}(\mathbf{X})^+} \Rightarrow \varphi$. *2. If* $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{H}(\mathbf{X})^+} \bigwedge \Gamma \to \bigvee \Delta$, *where* $\bigwedge \varnothing := \top$ *and* $\bigvee \varnothing := \bot$.

Proof. We show a part of (item 1). The axiom ($[]p$) can be derivable in $\mathsf{G}(\mathbf{X})^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}(Id) \quad \dfrac{}{p \Rightarrow p}(Id)}{[\varphi]p, \varphi \Rightarrow p}([]p \Rightarrow)}{[\varphi]p \Rightarrow \varphi \to p}(\Rightarrow\to)}{\Rightarrow [\varphi]p \to (\varphi \to p)}(\Rightarrow\to) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}(Id) \quad \dfrac{}{p \Rightarrow p}(Id)}{\varphi \to p, \varphi \Rightarrow p}(\to\Rightarrow)}{\varphi \to p \Rightarrow [\varphi]p}(\Rightarrow []p)}{\Rightarrow (\varphi \to p) \to [\varphi]p}(\Rightarrow\to)}{\Rightarrow ([\varphi]p \to (\varphi \to p)) \wedge ((\varphi \to p) \to [\varphi]p)}(\Rightarrow \wedge).$$

The axiom ($[]\bot$) can be derivable in $\mathsf{G}(\mathbf{X})^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}(Id)}{[\varphi]\bot, \varphi \Rightarrow}([]\bot \Rightarrow)}{[\varphi]\bot, \varphi \Rightarrow \bot}(\Rightarrow w)}{[\varphi]\bot \Rightarrow \varphi \to \bot}(\Rightarrow\to)}{\Rightarrow [\varphi]\bot \to (\varphi \to \bot)}(\Rightarrow\to) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}(Id) \quad \dfrac{}{\bot \Rightarrow}(\bot)}{\varphi \to \bot, \varphi \Rightarrow}(\to\Rightarrow)}{\varphi \to \bot \Rightarrow [\varphi]\bot}(\Rightarrow []\bot)}{\Rightarrow (\varphi \to \bot) \to [\varphi]\bot}(\Rightarrow\to)}{\Rightarrow ([\varphi]\bot \to (\varphi \to \bot)) \wedge ((\varphi \to \bot) \to [\varphi]\bot)}(\Rightarrow \wedge).$$

The axiom $([]\rightarrow)$ can be derivable in $\mathsf{G}(\mathbf{X})^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\varphi]\psi \Rightarrow [\varphi]\psi}\,(Id) \quad \dfrac{}{[\varphi]\chi \Rightarrow [\varphi]\chi}\,(Id)}{[\varphi](\psi \to \chi), [\varphi]\psi \Rightarrow [\varphi]\chi}\,([]\to\Rightarrow)}{[\varphi](\psi \to \chi) \Rightarrow [\varphi]\psi \to [\varphi]\chi}\,(\Rightarrow\to)}{\Rightarrow [\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)}\,(\Rightarrow\to) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{[\varphi]\psi \Rightarrow [\varphi]\psi}\,(Id) \quad \dfrac{}{[\varphi]\chi \Rightarrow [\varphi]\chi}\,(Id)}{[\varphi]\psi \to [\varphi]\chi, [\varphi]\psi \Rightarrow [\varphi]\chi}\,(\to\Rightarrow)}{[\varphi]\psi \to [\varphi]\chi \Rightarrow [\varphi](\psi \to \chi)}\,(\Rightarrow []p)}{\Rightarrow ([\varphi]\psi \to [\varphi]\chi) \to [\varphi](\psi \to \chi)}\,(\Rightarrow\to)}{\Rightarrow ([\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)) \wedge (([\varphi]\psi \to [\varphi]\chi) \to [\varphi](\psi \to \chi))}\,(\Rightarrow\wedge).$$

The axiom $([]\vee)$ can be derivable in $\mathsf{G}(\mathbf{X})^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(Id) \quad \dfrac{}{[\varphi]\psi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}\,(Id) \quad \dfrac{}{[\varphi]\chi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}\,(Id)}{[\varphi](\psi \vee \chi), \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}\,([]\vee\Rightarrow)}{[\varphi](\psi \vee \chi) \Rightarrow \varphi \to [\varphi]\psi \vee [\varphi]\chi}\,(\Rightarrow\to)}{\Rightarrow [\varphi](\psi \vee \chi) \to (\varphi \to [\varphi]\psi \vee [\varphi]\chi)}\,(\Rightarrow\to) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(Id) \quad \dfrac{}{[\varphi]\psi \vee [\varphi]\chi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}\,(Id)}{\varphi \to [\varphi]\psi \vee [\varphi]\chi, \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}\,(\to\Rightarrow)}{\varphi \to [\varphi]\psi \vee [\varphi]\chi \Rightarrow [\varphi](\psi \vee \chi)}\,(\Rightarrow []\vee)}{\Rightarrow (\varphi \to [\varphi]\psi \vee [\varphi]\chi) \to [\varphi](\psi \vee \chi)}\,(\Rightarrow\to)}{\Rightarrow ([\varphi](\psi \vee \chi) \to (\varphi \to [\varphi]\psi \vee [\varphi]\chi)) \wedge ((\varphi \to [\varphi]\psi \vee [\varphi]\chi) \to [\varphi](\psi \vee \chi))}\,(\Rightarrow\wedge).$$

$\square$

4.3.1 Cut-Elimination

We show the cut-elimination theorem for $\mathsf{G}(\mathbf{X})^+$. The proof is different from Theorem 3.46, in that we use the complexity function for a formula, which is introduced in the argument of the completeness proof, as a measure for cut formula, not the complexity measure used in Theorem 3.46. First, we introduce a notion of "principal formula" as in Definition 3.45. A principal formula is defined for each inference rule, except for the axioms and (Cut) rule and is informally expressed as "a formula, on which the inference rule acts."

Definition 4.18. A *principal formula* of the structural rules, the propositional logical rules, the rule $(D \Rightarrow)$, and the rules for public announcement operator is a formula appearing in the lower sequent, which is not contained in Γ, Γ_1, Γ_2, or Δ. A *principal formula* of the rules for D_G operator other than $(D \Rightarrow)$ is every formula in the lower sequent.

Theorem 4.19 (Cut-Elimination)**.** *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IK4**, *and* **IS4**. *Then, the following holds: If* $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathsf{G}^-(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, *where* $\mathsf{G}^-(\mathbf{X})^+$ *denotes a system "*$\mathsf{G}(\mathbf{X})^+$ *minus the cut rule".*

Proof. Following [24, Section 9.3] and [36, Section 2.2], we consider a system $\mathsf{G}^*(\mathbf{X})^+$, in which the cut rule is replaced by the "extended" cut rule defined as:

$$\dfrac{\Gamma \Rightarrow \varphi^n \quad \varphi^m, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Theta}\,(ECut),$$

where φ^n denotes the multi-set of n-copies of φ and $n = 0, 1$ and $m \geq 0$. Since $(ECut)$ is the same as (Cut) when we set $n = m = 1$, it is obvious that if $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^*(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, so it suffices to show that if $\vdash_{\mathsf{G}^*(\mathbf{X})^+} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^-(\mathbf{X})^+} \Gamma \Rightarrow \Delta$.

Suppose $\vdash_{\mathsf{G}^*(\mathbf{X})^+} \Gamma \Rightarrow \Delta$ and fix one derivation for the sequent. To obtain an $(ECut)$-free derivation of $\Gamma \Rightarrow \Delta$, it is enough to concentrate on a derivation whose root is derived by $(ECut)$ and which has no other application of $(ECut)$. In what follows, let $\mathbf{X}$ be $\mathbf{IK}$. Let us suppose that $\mathcal{D}$ has the following structure:

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi^n}\ (\mathbf{rule}_{\mathcal{L}}) \qquad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{\Gamma, \Sigma \Rightarrow \Theta}\ (ECut),$$

where the derivations $\mathcal{L}$ and $\mathcal{R}$ has no application of $(ECut)$ and $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are meta-variables for the name of rule applied there. We define $c(\mathcal{D})$. $c(\mathcal{D}) := 0$ if $n = m = 0$ and $c(\mathcal{D}) := c(\varphi)$ (the complexity of φ defined in Definition 4.12) otherwise. Let the number of sequents in $\mathcal{L}$ and $\mathcal{R}$ be $w(\mathcal{D})$. We show the lemma by double induction on $(c(\mathcal{D}), w(\mathcal{D}))$. If $n = 0$ or $m = 0$, we can derive the root sequent of $\mathcal{D}$ without using $(ECut)$ by weakening rules. So, in what follows we assume $n = 1$ and $m > 0$. Then, it is sufficient to consider the following four cases following [37, proof of Theorem 2.3], [24, Section 9.3], and [36, Section 2.2]: [1]

1. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is an axiom.

2. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a structural rule.

3. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a logical rule and a cut formula φ is not principal (in the sense we have specified above) for that rule.

4. $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are both logical rules (including (D)) for the same logical symbol and a cut formula φ is principal for each rule.

We concentrate on the case where D_G or $[\varphi]$ is involved. Among the rules for public announcement operator, the rules for $[\varphi]p$ and $[\varphi](\psi \vee \chi)$ are treated.

[1] In case 4, we assume the condition for both rule applications, because if the one of the two rule applications does not satisfy the condition, the whole derivation should be categorized into one of the rest cases.

(Case 3) Consider the case where the logical rule is $(\Rightarrow []p)$. The derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\dfrac{\mathcal{R}'}{\varphi^m, \Sigma, \psi \Rightarrow p}}{\varphi^m, \Sigma \Rightarrow [\psi]p}\,(\Rightarrow []p)}{\Gamma, \Sigma \Rightarrow [\psi]p}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}'}{\varphi^m, \Sigma, \psi \Rightarrow p}}{\Gamma, \Sigma, \psi \Rightarrow p}\,(ECut)}{\Gamma, \Sigma \Rightarrow [\psi]p}\,(\Rightarrow []p) \quad .$$

The subderivation $\mathcal{E}'$ whose root is $\Gamma, \Sigma, \psi \Rightarrow p$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}'$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}}'$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, \Sigma \Rightarrow [\psi]p$.

Consider the case where the logical rule is $([]p \Rightarrow)$. If $\mathbf{rule}_{\mathcal{L}} = ([]p \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}_1}{\Gamma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{L}_2}{p, \Gamma_2 \Rightarrow \varphi}}{\Gamma_1, \Gamma_2, [\psi]p \Rightarrow \varphi}\,([]p \Rightarrow) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{\Gamma_1, \Gamma_2, [\psi]p, \Sigma \Rightarrow \Theta}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\mathcal{L}_1}{\Gamma_1 \Rightarrow \psi} \quad \dfrac{\dfrac{\mathcal{L}_2}{p, \Gamma_2 \Rightarrow \varphi} \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{p, \Gamma_2, \Sigma \Rightarrow \Theta}\,(ECut)}{\Gamma_1, \Gamma_2, [\psi]p, \Sigma \Rightarrow \Theta}\,([]p \Rightarrow) \quad .$$

The subderivation $\mathcal{E}'$ whose root is $p, \Gamma_2, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}') = c(\mathcal{D})$, and $w(\mathcal{E}') < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}'$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}}'$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma_1, \Gamma_2, [\psi]p, \Sigma \Rightarrow \Theta$.

If $\mathbf{rule}_{\mathcal{R}} = ([]p \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\dfrac{\mathcal{R}_1}{\varphi^{m_1}, \Sigma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{R}_2}{p, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\varphi^m, [\psi]p, \Sigma_1, \Sigma_2, \Rightarrow \Theta}\,([]p \Rightarrow)}{\Gamma, [\psi]p, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(ECut)$$

Here, $m = m_1 + m_2$. This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}_1}{\varphi^{m_1}, \Sigma_1 \Rightarrow \psi}}{\Gamma, \Sigma_1 \Rightarrow \psi}\,(ECut) \quad \dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}_2}{p, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, p, \Sigma_2 \Rightarrow \Theta}\,(ECut)}{\dfrac{\dfrac{\Gamma, \Gamma, [\psi]p, \Sigma_1, \Sigma_2 \Rightarrow \Theta}{\vdots}\,(c \Rightarrow)}{\Gamma, [\psi]p, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(c \Rightarrow)}\,([]p \Rightarrow)$$

The subderivation $\mathcal{E}_1$ whose root is $\Gamma, \Sigma_1 \Rightarrow \psi$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. The subderivation $\mathcal{E}_2$ whose root is $\Gamma, p, \Sigma_2 \Rightarrow \Theta$ also satisfies the same condition. Hence, by induction hypothesis, there exist $(ECut)$-free derivations $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ having the same root sequent as $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Replacing the derivation $\mathcal{E}_i$ by $\tilde{\mathcal{E}}_i$ in $\mathcal{E}$ $(i = 1, 2)$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, [\psi]p, \Sigma_1, \Sigma_2 \Rightarrow \Theta$.

The case where the logical rule is $(\Rightarrow []\vee)$ is is similar to the case for $(\Rightarrow []p)$, so we skip it. Consider the case where the logical rule is $([]\vee \Rightarrow)$. If $\mathbf{rule}_{\mathcal{L}} = ([]\vee \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}_1}{\Gamma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{L}_2}{[\psi]\chi, \Gamma_2 \Rightarrow \varphi} \quad \dfrac{\mathcal{L}_3}{[\psi]\rho, \Gamma_2 \Rightarrow \varphi}}{\Gamma_1, \Gamma_2, [\psi](\chi \vee \rho) \Rightarrow \varphi}\,([]\vee \Rightarrow) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{\Gamma_1, \Gamma_2, [\psi](\chi \vee \rho), \Sigma \Rightarrow \Theta}\,(ECut)$$

This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\mathcal{L}_1}{\Gamma_1 \Rightarrow \psi} \quad \dfrac{\dfrac{\mathcal{L}_2}{[\psi]\chi, \Gamma_2 \Rightarrow \varphi} \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{[\psi]\chi, \Gamma_2, \Sigma \Rightarrow \Theta}\,(ECut) \quad \dfrac{\dfrac{\mathcal{L}_3}{[\psi]\rho, \Gamma_2 \Rightarrow \varphi} \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\,(\mathbf{rule}_{\mathcal{R}})}{[\psi]\rho, \Gamma_2, \Sigma \Rightarrow \Theta}\,(ECut)}{\Gamma_1, \Gamma_2, [\psi](\chi \vee \rho), \Sigma \Rightarrow \Theta}\,([]\vee \Rightarrow) \; .$$

The subderivation $\mathcal{E}_1$ whose root is $[\psi]\chi, \Gamma_2, \Sigma \Rightarrow \Theta$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. The subderivation $\mathcal{E}_2$ whose root is $[\psi]\rho, \Gamma_2, \Sigma \Rightarrow \Theta$ also satisfies the same condition. Hence, by induction hypothesis, there exist $(ECut)$-free derivations $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ having the same root sequent as $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Replacing the derivation $\mathcal{E}_i$ by $\tilde{\mathcal{E}}_i$ in $\mathcal{E}$ $(i = 1, 2)$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma_1, \Gamma_2, [\psi](\chi \vee \rho), \Sigma \Rightarrow \Theta$.

If $\mathbf{rule}_{\mathcal{R}} = ([]\vee \Rightarrow)$, the derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \qquad \dfrac{\dfrac{\mathcal{R}_1}{\varphi^{m_1}, \Sigma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{R}_2}{[\psi]\chi, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta} \quad \dfrac{\mathcal{R}_3}{[\psi]\rho, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\varphi^{m}, [\psi](\chi \vee \rho), \Sigma_1, \Sigma_2, \Rightarrow \Theta}\,([]\vee \Rightarrow)}{\Gamma, [\psi](\chi \vee \rho), \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(ECut)$$

Here, $m = m_1 + m_2$. This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}_1}{\varphi^{m_1}, \Sigma_1 \Rightarrow \psi}}{\Gamma, \Sigma_1 \Rightarrow \psi}\,(ECut) \quad \dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}_2}{[\psi]\chi, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, [\psi]\chi, \Sigma_2 \Rightarrow \Theta}\,(ECut) \quad \dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi}\,(\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}_3}{[\psi]\rho, \varphi^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, [\psi]\rho, \Sigma_2 \Rightarrow \Theta}\,(ECut)}{\Gamma, \Gamma, [\psi](\chi \vee \rho), \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,([]\vee \Rightarrow)}{\vdots}\,(c \Rightarrow)}{\Gamma, [\psi](\chi \vee \rho), \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(c \Rightarrow)$$

The subderivation $\mathcal{E}_1$ whose root is $\Gamma, \Sigma_1 \Rightarrow \psi$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. The subderivation $\mathcal{E}_2$ whose root is $\Gamma, [\psi]\chi, \Sigma_2 \Rightarrow \Theta$ and The subderivation $\mathcal{E}_3$ whose root is $\Gamma, [\psi]\rho, \Sigma_2 \Rightarrow \Theta$ also satisfy the same condition. Hence, by induction hypothesis, there exists $(ECut)$-free derivation $\tilde{\mathcal{E}}_i$ having the same root sequent as $\mathcal{E}_i$ $(i = 1, 2, 3)$. Replacing the derivation $\mathcal{E}_i$ by $\tilde{\mathcal{E}}_i$ in $\mathcal{E}$ $(i = 1, 2, 3)$, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, [\psi](\chi \vee \rho), \Sigma_1, \Sigma_2 \Rightarrow \Theta$.

(Case 4) Consider the case where the cut formula is $[\psi]p$. The derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\Gamma, \psi \Rightarrow p}}{\Gamma \Rightarrow [\psi]p}\,(\Rightarrow []p) \qquad \dfrac{\dfrac{\mathcal{R}_1}{([\psi]p)^{m_1}, \Sigma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{R}_2}{p, ([\psi]p)^{m_2}, \Sigma_2 \Rightarrow \Theta}}{([\psi]p)^{m}, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,([]p \Rightarrow)}{\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(ECut)$$

Here, $m = m_1 + m_2 + 1$. This can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow [\psi]p} \quad \dfrac{\mathcal{R}_1}{([\psi]p)^{m_1}, \Sigma_1 \Rightarrow \psi}}{\Gamma, \Sigma_1 \Rightarrow \psi}\,(ECut) \qquad \dfrac{\dfrac{\mathcal{L}'}{\Gamma, \psi \Rightarrow p} \quad \dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow [\psi]p} \quad \dfrac{\mathcal{R}_2}{p, ([\psi]p)^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, p, \Sigma_2 \Rightarrow \Theta}\,(ECut)}{\Gamma, \Gamma, \psi, \Sigma_2 \Rightarrow \Theta}\,(ECut)}{\Gamma, \Gamma, \Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(ECut)}{\dfrac{\vdots}{\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}\,(c \Rightarrow)}\,(c \Rightarrow)$$

The subderivation $\mathcal{E}_1$ whose root is $\Gamma, \Sigma_1 \Rightarrow \psi$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}}_1$ having the same root sequent. For the subderivation

$\mathcal{E}_2$ whose root is $\Gamma, p, \Sigma_2 \Rightarrow \Theta$, we have an $(ECut)$-free derivation $\tilde{\mathcal{E}_2}$ having the same root sequent, by the same argument as $\mathcal{E}_1$. Name $\mathcal{E}_3$, the derivation obtained by replacing the derivation $\mathcal{E}_2$ by $\tilde{\mathcal{E}_2}$ in the subderivation whose root is $\Gamma, \Gamma, \psi, \Sigma_2 \Rightarrow \Theta$. The derivation $\mathcal{E}_3$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_3) < c(\mathcal{D})$ by item 1 of Lemma 4.13. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_3}$ having the same root sequent. Name $\mathcal{E}_4$, the derivation inferring $\Gamma, \Gamma, \Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta$ from $\tilde{\mathcal{E}_1}$ and $\tilde{\mathcal{E}_3}$. We have an $(ECut)$-free derivation $\tilde{\mathcal{E}_4}$ having the same root sequent, by the same argument as $\mathcal{E}_3$. Thus, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta$.

Consider the case where the cut formula is $[\psi](\chi \vee \rho)$. The derivation $\mathcal{D}$ has the following structure.

$$
\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\Gamma, \psi \Rightarrow [\psi]\chi \vee [\psi]\rho}}{\Gamma \Rightarrow [\psi](\chi \vee \rho)}(\Rightarrow []\vee) \quad \dfrac{\dfrac{\mathcal{R}_1}{([\psi](\chi \vee \rho))^{m_1}, \Sigma_1 \Rightarrow \psi} \quad \dfrac{\mathcal{R}_2}{[\psi]\chi, ([\psi](\chi \vee \rho))^{m_2}, \Sigma_2 \Rightarrow \Theta} \quad \dfrac{\mathcal{R}_3}{[\psi]\rho, ([\psi](\chi \vee \rho))^{m_2}, \Sigma_2 \Rightarrow \Theta}}{([\psi](\chi \vee \rho))^{m}, \Sigma_1, \Sigma_2 \Rightarrow \Theta}([]\vee \Rightarrow)}{\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}(ECut)
$$

Here, $m = m_1 + m_2 + 1$. This can be transformed into the following derivation $\mathcal{E}$:

$$
\dfrac{\dfrac{\dfrac{\mathcal{E}_1}{\Gamma, \Sigma_1 \Rightarrow \psi}(ECut) \quad \dfrac{\dfrac{\mathcal{L}'}{\Gamma, \psi \Rightarrow [\psi]\chi \vee [\psi]\rho} \quad \dfrac{\dfrac{\mathcal{E}_2}{\Gamma, [\psi]\chi, \Sigma_2 \Rightarrow \Theta}(ECut) \quad \dfrac{\mathcal{E}_3}{\Gamma, [\psi]\rho, \Sigma_2 \Rightarrow \Theta}(ECut)}{[\psi]\chi \vee [\psi]\rho, \Gamma, \Sigma_2 \Rightarrow \Theta}(\vee \Rightarrow)}{\Gamma, \Gamma, \psi, \Sigma_2 \Rightarrow \Theta}(ECut)}{\dfrac{\Gamma, \Gamma, \Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}{\vdots}(c \Rightarrow)}(ECut)
$$
$$
\dfrac{\vdots}{\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta}(c \Rightarrow)
$$

where $\mathcal{E}_1$ is the derivation

$$
\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow [\psi](\chi \vee \rho)} \quad \dfrac{\mathcal{R}_1}{([\psi](\chi \vee \rho))^{m_1}, \Sigma_1 \Rightarrow \psi}}{\Gamma, \Sigma_1 \Rightarrow \psi}(ECut),
$$

$\mathcal{E}_2$ is the derivation

$$
\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow [\psi](\chi \vee \rho)} \quad \dfrac{\mathcal{R}_2}{[\psi]\chi, ([\psi](\chi \vee \rho))^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, [\psi]\chi, \Sigma_2 \Rightarrow \Theta}(ECut),
$$

and $\mathcal{E}_3$ is the derivation

$$
\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow [\psi](\chi \vee \rho)} \quad \dfrac{\mathcal{R}_3}{[\psi]\rho, ([\psi](\chi \vee \rho))^{m_2}, \Sigma_2 \Rightarrow \Theta}}{\Gamma, [\psi]\rho, \Sigma_2 \Rightarrow \Theta}(ECut).
$$

The subderivation $\mathcal{E}_1$ has no application of $(ECut)$ except the lowermost one, $c(\mathcal{E}_1) = c(\mathcal{D})$, and $w(\mathcal{E}_1) < w(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_1}$ having the same root sequent. For the subderivation $\mathcal{E}_2$ and $\mathcal{E}_3$, we have $(ECut)$-free derivations $\tilde{\mathcal{E}_2}$ and $\tilde{\mathcal{E}_3}$ having the same root sequent as $\mathcal{E}_2$ and $\mathcal{E}_3$, respectively, by the same argument as $\mathcal{E}_1$. Name $\mathcal{E}_4$, the derivation obtained by replacing the derivations $\mathcal{E}_2$ and $\mathcal{E}_3$ by $\tilde{\mathcal{E}_2}$ and $\tilde{\mathcal{E}_3}$, respectively, in the subderivation whose root is $\Gamma, \Gamma, \psi, \Sigma_2 \Rightarrow \Theta$. The derivation $\mathcal{E}_4$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}_4) < c(\mathcal{D})$ by items 1 and 5 of Lemma 4.13. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}_4}$ having the same root sequent. Name $\mathcal{E}_5$, the derivation inferring $\Gamma, \Gamma, \Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta$ from $\tilde{\mathcal{E}_1}$ and $\tilde{\mathcal{E}_4}$. We have an $(ECut)$-free derivation $\tilde{\mathcal{E}_5}$ having the same root sequent, by the same argument as $\mathcal{E}_4$. Thus, we obtain an $(ECut)$-free derivation for the sequent $\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \Theta$.

Consider the case where the cut formula is $D_G\psi$. The derivation $\mathcal{D}$ has the following structure.

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\,(D) \qquad \dfrac{\dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^{m} H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\,(D)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_n \Rightarrow D_H\chi}\,(ECut)$$

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad \dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m \Rightarrow \chi}}{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \Rightarrow \chi}\,(ECut) \qquad (\bigcup_{i=1}^{n} G_i \cup \bigcup_{j=1}^{m} H_j \subseteq H)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\,(D).$$

The subderivation $\mathcal{E}'$ whose root sequent is $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \Rightarrow \chi$ has no application of $(ECut)$ except the lowermost one and $c(\mathcal{E}') < c(\mathcal{D})$ by item 1 of Lemma 4.13. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi$. □

4.3.2 Craig Interpolation Theorem

Now, we show Craig interpolation theorem for $\mathsf{G}(\mathbf{X})^+$ by the similar method to $\mathsf{G}(\mathbf{X})$.

Definition 4.20 (Partition)**.** A *partition* for a sequent $\Gamma \Rightarrow \Delta$ is defined as a tuple $\langle \Gamma_1; \Gamma_2 \rangle$, such that $\Gamma = \Gamma_1, \Gamma_2$.

Definition 4.21. For a formula φ, $\mathsf{Prop}(\varphi)$ is defined as the set of all propositional variables appearing in φ. For a multiset Γ of formulas, $\mathsf{Prop}(\Gamma)$ is defined as $\bigcup_{\varphi\in\Gamma}\mathsf{Prop}(\varphi)$. Similarly, $\mathsf{Agt}(\varphi)$ is defined as the set of agents appearing in φ and $\mathsf{Agt}(\Gamma)$ as $\bigcup_{\varphi\in\Gamma}\mathsf{Agt}(\varphi)$

The following is a key lemma for Craig Interpolation Theorem.

Lemma 4.22. *Let* **X** *be any of* **IK**, **IKT**, **IK4**, *and* **IS4**. *Suppose* $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma \Rightarrow \Delta$. *Then, for any partition* $\langle \Gamma_1; \Gamma_2\rangle$ *for the sequent* $\Gamma \Rightarrow \Delta$, *there exists a formula* φ *called "interpolant", satisfying the following:*

1. $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1 \Rightarrow \varphi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})^+} \varphi, \Gamma_2 \Rightarrow \Delta$.
2. $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \Delta)$.
3. $\mathsf{Agt}(\varphi) \subseteq \mathsf{Agt}(\Gamma_1) \cap \mathsf{Agt}(\Gamma_2, \Delta)$.

Proof. We prove by induction on the structure of a derivation for $\Gamma \Rightarrow \Delta$. Fix the derivation and name it $\mathcal{D}$. By Theorem 4.19, we can assume that $\mathcal{D}$ is cut-free. We only concentrate on the cases involving public announcement operator. Among the rules for public announcement operator, the rules for $[\varphi]p$, $[\varphi](\psi \vee \chi)$, and $[\varphi]D_G\psi$ are treated. For the other cases, the reader is referred to Lemma 3.50.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\Gamma_1, \Gamma_2, \varphi \Rightarrow p}}{\Gamma_1, \Gamma_2 \Rightarrow [\varphi]p}\,(\Rightarrow []p).$$

Fix the partition $\langle \Gamma_1; \Gamma_2\rangle$ for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi]p$. As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \Gamma_1; \Gamma_2, \varphi\rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1 \Rightarrow \psi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi, \Gamma_2, \varphi \Rightarrow p$ for some formula ψ. The formula ψ is an interpolant for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi]p$, too, because we have the following derivation:

$$\dfrac{\dfrac{\text{I.H.}}{\psi, \Gamma_2, \varphi \Rightarrow p}}{\psi, \Gamma_2 \Rightarrow [\varphi]p}\,(\Rightarrow []p).$$

The interpolant ψ also satisfies item 2, because $\mathsf{Prop}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \varphi, p) = \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, [\varphi]p)$. Item 3 is also similarly satisfied.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_{a_1}, \Gamma_{a_2} \Rightarrow \varphi} \quad \dfrac{\mathcal{D}_2}{p, \Gamma_{b_1}, \Gamma_{b_2} \Rightarrow \Delta}}{\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]p \Rightarrow \Delta}\,([]p \Rightarrow)$$

There are two types of partition for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]p \Rightarrow \Delta$ depending on whether $[\varphi]p$ belongs to the left or the right of a partition. First, fix the partition

$\langle \Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p; \Gamma_{a_2}, \Gamma_{b_2} \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_2}; \Gamma_{a_1} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_2} \Rightarrow \psi_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi_1, \Gamma_{a_1} \Rightarrow \varphi$ for some formula ψ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle p, \Gamma_{b_1}; \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} p, \Gamma_{b_1} \Rightarrow \psi_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi_2, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ψ_2. The formula $\psi_1 \to \psi_2$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]p \Rightarrow \Delta$, because we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\text{I.H.}}{\psi_1, \Gamma_{a_1} \Rightarrow \varphi} \quad \dfrac{\text{I.H.}}{p, \Gamma_{b_1} \Rightarrow \psi_2}}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p, \psi_1 \Rightarrow \psi_2}\,([]p\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p \Rightarrow \psi_1 \to \psi_2}\,(\Rightarrow\to) \quad ,$$

$$\dfrac{\dfrac{\text{I.H.}}{\Gamma_{a_2} \Rightarrow \psi_1} \quad \dfrac{\text{I.H.}}{\psi_2, \Gamma_{b_2} \Rightarrow \Delta}}{\psi_1 \to \psi_2, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,(\to\Rightarrow)$$

We show the interpolant $\psi_1 \to \psi_2$ satisfies item 2. First, $\mathsf{Prop}(\psi_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Second, $\mathsf{Prop}(\psi_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(p, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(p, \Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Thus, $\mathsf{Prop}(\psi_1 \to \psi_2) = \mathsf{Prop}(\psi_1) \cup \mathsf{Prop}(\psi_2) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]p) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Item 3 is also similarly satisfied.

Next, fix the partition $\langle \Gamma_{a_1}, \Gamma_{b_1}; \Gamma_{a_2}, \Gamma_{b_2}, [\varphi]p \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_1}; \Gamma_{a_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_1} \Rightarrow \psi_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi_1, \Gamma_{a_2} \Rightarrow \varphi$ for some formula ψ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle \Gamma_{b_1}; p, \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{b_1} \Rightarrow \psi_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi_2, p, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ψ_2. The formula $\psi_1 \wedge \psi_2$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]p \Rightarrow \Delta$, because we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\Gamma_{a_1} \Rightarrow \psi_1}}{\vdots}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \psi_1}\,(w\Rightarrow) \quad \dfrac{\dfrac{\dfrac{\text{I.H.}}{\Gamma_{b_1} \Rightarrow \psi_2}}{\vdots}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \psi_2}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \psi_1 \wedge \psi_2}\,(\Rightarrow\wedge)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\psi_1, \Gamma_{a_2} \Rightarrow \varphi}}{\psi_1 \wedge \psi_2, \Gamma_{a_2} \Rightarrow \varphi}\,(\wedge\Rightarrow^1) \quad \dfrac{\dfrac{\text{I.H.}}{p, \Gamma_{b_2}, \psi_2 \Rightarrow \Delta}}{p, \Gamma_{b_2}, \psi_1 \wedge \psi_2 \Rightarrow \Delta}\,(\wedge\Rightarrow^2)}{\psi_1 \wedge \psi_2, \psi_1 \wedge \psi_2, [\varphi]p, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,([]p\Rightarrow)}{\psi_1 \wedge \psi_2, [\varphi]p, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,(c\Rightarrow)$$

We show the interpolant $\psi_1 \wedge \psi_2$ satisfies item 2. First, $\mathsf{Prop}(\psi_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]p, \Delta)$. Second, $\mathsf{Prop}(\psi_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{b_1}) \cap \mathsf{Prop}(p, \Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(p, \Gamma_{a_2}, \Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]p, \Delta)$. Thus, $\mathsf{Prop}(\psi_1 \wedge \psi_2) = \mathsf{Prop}(\psi_1) \cup \mathsf{Prop}(\psi_2) \subseteq$

$\mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]p, \Delta)$. Item 3 is also similarly satisfied.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\Gamma_1, \Gamma_2, \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}}{\Gamma_1, \Gamma_2 \Rightarrow [\varphi](\psi \vee \chi)} \ (\Rightarrow []\vee).$$

Fix the partition $\langle \Gamma_1; \Gamma_2 \rangle$ for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi](\psi \vee \chi)$. As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \Gamma_1; \Gamma_2, \varphi \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1 \Rightarrow \psi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \psi, \Gamma_2, \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi$ for some formula ψ. The formula ψ is an interpolant for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi](\psi \vee \chi)$, too, because we have the following derivation:

$$\dfrac{\dfrac{\text{I.H.}}{\psi, \Gamma_2, \varphi \Rightarrow [\varphi]\psi \vee [\varphi]\chi}}{\psi, \Gamma_2 \Rightarrow [\varphi](\psi \vee \chi)} \ (\Rightarrow []\vee).$$

The interpolant ψ also satisfies item 2, because $\mathsf{Prop}(\psi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \varphi, [\varphi]\psi \vee [\varphi]\chi) = \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, [\varphi](\psi \vee \chi))$. Item 3 is also similarly satisfied.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_{a_1}, \Gamma_{a_2} \Rightarrow \varphi} \quad \dfrac{\mathcal{D}_2}{[\varphi]\psi, \Gamma_{b_1}, \Gamma_{b_2} \Rightarrow \Delta} \quad \dfrac{\mathcal{D}_3}{[\varphi]\chi, \Gamma_{b_1}, \Gamma_{b_2} \Rightarrow \Delta}}{\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi](\psi \vee \chi) \Rightarrow \Delta} \ ([]\vee \Rightarrow).$$

There are two types of partition for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi](\psi \vee \chi) \Rightarrow \Delta$ depending on whether $[\varphi](\psi \vee \chi)$ belongs to the left or the right of a partition. First, fix the partition $\langle \Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi); \Gamma_{a_2}, \Gamma_{b_2} \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_2}; \Gamma_{a_1} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_2} \Rightarrow \rho_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_1, \Gamma_{a_1} \Rightarrow \varphi$ for some formula ρ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle [\varphi]\psi, \Gamma_{b_1}; \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} [\varphi]\psi, \Gamma_{b_1} \Rightarrow \rho_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_2, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_2. Similarly, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} [\varphi]\chi, \Gamma_{b_1} \Rightarrow \rho_3$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_3, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_3, as the induction hypothesis for $\mathcal{D}_3$. The formula $\rho_1 \to \rho_2 \vee \rho_3$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi](\psi \vee \chi) \Rightarrow \Delta$, because we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\text{I.H.}}{\rho_1, \Gamma_{a_1} \Rightarrow \varphi} \quad \dfrac{\dfrac{\text{I.H.}}{[\varphi]\psi, \Gamma_{b_1} \Rightarrow \rho_2}}{[\varphi]\psi, \Gamma_{b_1} \Rightarrow \rho_2 \vee \rho_3} \ (\Rightarrow \vee^1) \quad \dfrac{\dfrac{\text{I.H.}}{[\varphi]\chi, \Gamma_{b_1} \Rightarrow \rho_3}}{[\varphi]\chi, \Gamma_{b_1} \Rightarrow \rho_2 \vee \rho_3} \ (\Rightarrow \vee^2)}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi), \rho_1 \Rightarrow \rho_2 \vee \rho_3} \ ([]\vee \Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi) \Rightarrow \rho_1 \to \rho_2 \vee \rho_3} \ (\Rightarrow\to),$$

$$\dfrac{\dfrac{\text{I.H.}}{\Gamma_{a_2} \Rightarrow \rho_1} \qquad \dfrac{\dfrac{\text{I.H.}}{\rho_2, \Gamma_{b_2} \Rightarrow \Delta} \quad \dfrac{\text{I.H.}}{\rho_3, \Gamma_{b_2} \Rightarrow \Delta}}{\rho_2 \vee \rho_3, \Gamma_{b_2} \Rightarrow \Delta}\,(\vee \Rightarrow)}{\rho_1 \to \rho_2 \vee \rho_3, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,(\to\Rightarrow)$$

We show the interpolant $\rho_1 \to \rho_2 \vee \rho_3$ satisfies item 2. First, $\mathsf{Prop}(\rho_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi)) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Second, $\mathsf{Prop}(\rho_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}([\varphi]\psi, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}([\varphi]\psi, \Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi)) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Similarly, we have $\mathsf{Prop}(\rho_3) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi)) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Thus, $\mathsf{Prop}(\rho_1 \to \rho_2 \vee \rho_3) = \mathsf{Prop}(\rho_1) \cup \mathsf{Prop}(\rho_2) \cup \mathsf{Prop}(\rho_3) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi](\psi \vee \chi)) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Item 3 is also similarly satisfied.

Next, fix the partition $\langle \Gamma_{a_1}, \Gamma_{b_1}; \Gamma_{a_2}, \Gamma_{b_2}, [\varphi](\psi \vee \chi) \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_1}; \Gamma_{a_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_1} \Rightarrow \rho_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_1, \Gamma_{a_2} \Rightarrow \varphi$ for some formula ρ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle \Gamma_{b_1}; [\varphi]\psi, \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{b_1} \Rightarrow \rho_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_2, [\varphi]\psi, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_2. Similarly, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{b_1} \Rightarrow \rho_3$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_3, [\varphi]\chi, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_3, as the induction hypothesis for $\mathcal{D}_3$. The formula $\rho_1 \wedge (\rho_2 \wedge \rho_3)$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi](\psi \vee \chi) \Rightarrow \Delta$, because we have the following derivations:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\text{I.H.}}{\Gamma_{a_1} \Rightarrow \rho_1}\,(w\Rightarrow)\\ \vdots\end{array}}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_1}\,(w\Rightarrow) \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\text{I.H.}}{\Gamma_{b_1} \Rightarrow \rho_2}\,(w\Rightarrow)\\ \vdots\end{array}}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_2}\,(w\Rightarrow) \quad \dfrac{\begin{array}{c}\dfrac{\text{I.H.}}{\Gamma_{b_1} \Rightarrow \rho_3}\,(w\Rightarrow)\\ \vdots\end{array}}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_3}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_2 \wedge \rho_3}\,(\Rightarrow\wedge)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_1 \wedge (\rho_2 \wedge \rho_3)}\,(\Rightarrow\wedge)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{I.H.}}{\rho_1, \Gamma_{a_2} \Rightarrow \varphi}}{\rho_1 \wedge (\rho_2 \wedge \rho_3), \Gamma_{a_2} \Rightarrow \varphi}\,(\wedge\Rightarrow^1) \quad \dfrac{\dfrac{\dfrac{\text{I.H.}}{[\varphi]\psi, \Gamma_{b_2}, \rho_2 \Rightarrow \Delta}}{[\varphi]\psi, \Gamma_{b_2}, \rho_2 \wedge \rho_3 \Rightarrow \Delta}\,(\wedge\Rightarrow^1)}{[\varphi]\psi, \Gamma_{b_2}, \rho_1 \wedge (\rho_2 \wedge \rho_3) \Rightarrow \Delta}\,(\wedge\Rightarrow^2) \quad \dfrac{\dfrac{\dfrac{\text{I.H.}}{[\varphi]\chi, \Gamma_{b_2}, \rho_3 \Rightarrow \Delta}}{[\varphi]\chi, \Gamma_{b_2}, \rho_2 \wedge \rho_3 \Rightarrow \Delta}\,(\wedge\Rightarrow^2)}{[\varphi]\chi, \Gamma_{b_2}, \rho_1 \wedge (\rho_2 \wedge \rho_3) \Rightarrow \Delta}\,(\wedge\Rightarrow^2)}{\rho_1 \wedge (\rho_2 \wedge \rho_3), \rho_1 \wedge (\rho_2 \wedge \rho_3), [\varphi](\psi \vee \chi), \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,([]\vee\Rightarrow)}{\rho_1 \wedge (\rho_2 \wedge \rho_3), [\varphi](\psi \vee \chi), \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,(c\Rightarrow)$$

We show the interpolant $\rho_1 \wedge (\rho_2 \wedge \rho_3)$ satisfies item 2. First, $\mathsf{Prop}(\rho_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi](\psi \vee \chi), \Delta)$. Second, $\mathsf{Prop}(\rho_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{b_1}) \cap \mathsf{Prop}([\varphi]\psi, \Gamma_{b_2}, \Delta) \subseteq$
$\mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}([\varphi]\psi, \Gamma_{a_2}, \Gamma_{b_2}, \Delta) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi](\psi \vee \chi), \Delta)$. Similarly, we have $\mathsf{Prop}(\rho_3) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi](\psi \vee \chi), \Delta)$. Thus, $\mathsf{Prop}(\rho_1 \wedge (\rho_2 \wedge \rho_3)) = \mathsf{Prop}(\rho_1) \cup \mathsf{Prop}(\rho_2) \cup \mathsf{Prop}(\rho_3) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi](\psi \vee \chi), \Delta)$. Item 3 is also similarly satisfied.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\Gamma_1, \Gamma_2, \varphi \Rightarrow D_G[\varphi]\psi}}{\Gamma_1, \Gamma_2 \Rightarrow [\varphi]D_G\psi}\ (\Rightarrow []D).$$

Fix the partition $\langle \Gamma_1; \Gamma_2 \rangle$ for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi]D_G\psi$. As the induction hypothesis for $\mathcal{D}'$ with the partition $\langle \Gamma_1; \Gamma_2, \varphi \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_1 \Rightarrow \chi$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \chi, \Gamma_2, \varphi \Rightarrow D_G[\varphi]\psi$ for some formula χ. The formula χ is an interpolant for the sequent $\Gamma_1, \Gamma_2 \Rightarrow [\varphi]D_G\psi$, too, because we have the following derivation:

$$\dfrac{\dfrac{\text{I.H.}}{\chi, \Gamma_2, \varphi \Rightarrow D_G[\varphi]\psi}}{\chi, \Gamma_2 \Rightarrow [\varphi]D_G\psi}\ (\Rightarrow []D).$$

The interpolant χ also satisfies item 2, because $\mathsf{Prop}(\chi) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \varphi, D_G[\varphi]\psi) = \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, D_G[\varphi]\psi) = \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, [\varphi]D_G\psi)$. Since $\mathsf{Agt}(D_G[\varphi]\psi) = \mathsf{Agt}([\varphi]D_G\psi) = G \cup \mathsf{Agt}(\varphi) \cup \mathsf{Agt}(\psi)$, item 3 is also similarly satisfied.

Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_{a_1}, \Gamma_{a_2} \Rightarrow \varphi} \quad \dfrac{\mathcal{D}_2}{D_G[\varphi]\psi, \Gamma_{b_1}, \Gamma_{b_2} \Rightarrow \Delta}}{\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]D_G\psi \Rightarrow \Delta}\ ([]D \Rightarrow)$$

There are two types of partition for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]D_G\psi \Rightarrow \Delta$ depending on whether $[\varphi]D_G\psi$ belongs to the left or the right of a partition. First, fix the partition $\langle \Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi; \Gamma_{a_2}, \Gamma_{b_2} \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_2}; \Gamma_{a_1} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_2} \Rightarrow \rho_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_1, \Gamma_{a_1} \Rightarrow \varphi$ for some formula ρ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle D_G[\varphi]\psi, \Gamma_{b_1}; \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} D_G[\varphi]\psi, \Gamma_{b_1} \Rightarrow \rho_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_2, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_2. The formula $\rho_1 \to \rho_2$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]D_G\psi \Rightarrow \Delta$, because we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\text{I.H.}}{\rho_1, \Gamma_{a_1} \Rightarrow \varphi} \quad \dfrac{\text{I.H.}}{D_G[\varphi]\psi, \Gamma_{b_1} \Rightarrow \rho_2}}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi, \rho_1 \Rightarrow \rho_2}\ ([]D \Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi \Rightarrow \rho_1 \to \rho_2}\ (\Rightarrow\to) \qquad \dfrac{\dfrac{\text{I.H.}}{\Gamma_{a_2} \Rightarrow \rho_1} \quad \dfrac{\text{I.H.}}{\rho_2, \Gamma_{b_2} \Rightarrow \Delta}}{\rho_1 \to \rho_2, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\ (\to\Rightarrow)$$

We show the interpolant $\rho_1 \to \rho_2$ satisfies item 2. First, $\mathsf{Prop}(\rho_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}) \cap \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$.

Second, $\mathsf{Prop}(\rho_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(D_G[\varphi]\psi, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{b_2}, \Delta) \subseteq$ $\mathsf{Prop}(D_G[\varphi]\psi, \Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta) = \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Thus, $\mathsf{Prop}(\rho_1 \to \rho_2) = \mathsf{Prop}(\rho_1) \cup \mathsf{Prop}(\rho_2) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}, [\varphi]D_G\psi) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \Delta)$. Item 3 is also similarly satisfied.

Next, fix the partition $\langle \Gamma_{a_1}, \Gamma_{b_1}; \Gamma_{a_2}, \Gamma_{b_2}, [\varphi]D_G\psi \rangle$. As the induction hypothesis for $\mathcal{D}_1$ with the partition $\langle \Gamma_{a_1}; \Gamma_{a_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{a_1} \Rightarrow \rho_1$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_1, \Gamma_{a_2} \Rightarrow \varphi$ for some formula ρ_1. As the induction hypothesis for $\mathcal{D}_2$ with the partition $\langle \Gamma_{b_1}; D_G[\varphi]\psi, \Gamma_{b_2} \rangle$, we have $\vdash_{\mathsf{G}(\mathbf{X})^+} \Gamma_{b_1} \Rightarrow \rho_2$ and $\vdash_{\mathsf{G}(\mathbf{X})^+} \rho_2, D_G[\varphi]\psi, \Gamma_{b_2} \Rightarrow \Delta$ for some formula ρ_2. The formula $\rho_1 \wedge \rho_2$ is an interpolant for the sequent $\Gamma_{a_1}, \Gamma_{a_2}, \Gamma_{b_1}, \Gamma_{b_2}, [\varphi]D_G\psi \Rightarrow \Delta$, because we have the following derivations:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\text{I.H.}}{\Gamma_{a_1} \Rightarrow \rho_1}}{\vdots}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_1}\,(w\Rightarrow)
\qquad
\dfrac{\dfrac{\dfrac{\text{I.H.}}{\Gamma_{b_1} \Rightarrow \rho_2}}{\vdots}\,(w\Rightarrow)}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_2}\,(w\Rightarrow)
}{\Gamma_{a_1}, \Gamma_{b_1} \Rightarrow \rho_1 \wedge \rho_2}\,(\Rightarrow\wedge)
$$

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\text{I.H.}}{\rho_1, \Gamma_{a_2} \Rightarrow \varphi}}{\rho_1 \wedge \rho_2, \Gamma_{a_2} \Rightarrow \varphi}\,(\wedge\Rightarrow^1)
\qquad
\dfrac{\dfrac{\text{I.H.}}{D_G[\varphi]\psi, \Gamma_{b_2}, \rho_2 \Rightarrow \Delta}}{D_G[\varphi]\psi, \Gamma_{b_2}, \rho_1 \wedge \rho_2 \Rightarrow \Delta}\,(\wedge\Rightarrow^2)
}{\rho_1 \wedge \rho_2, \rho_1 \wedge \rho_2, [\varphi]D_G\psi, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,([]D\Rightarrow)}{\rho_1 \wedge \rho_2, [\varphi]D_G\psi, \Gamma_{a_2}, \Gamma_{b_2} \Rightarrow \Delta}\,(c\Rightarrow)
$$

We show the interpolant $\rho_1 \wedge \rho_2$ satisfies item 2. First, $\mathsf{Prop}(\rho_1) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{a_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, \varphi) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]D_G\psi, \Delta)$. Second, $\mathsf{Prop}(\rho_2) \overset{\text{(I.H.)}}{\subseteq} \mathsf{Prop}(\Gamma_{b_1}) \cap \mathsf{Prop}(D_G[\varphi]\psi, \Gamma_{b_2}, \Delta) \subseteq$ $\mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(D_G[\varphi]\psi, \Gamma_{a_2}, \Gamma_{b_2}, \Delta) = \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]D_G\psi, \Delta)$. Thus, $\mathsf{Prop}(\rho_1 \wedge \rho_2) = \mathsf{Prop}(\rho_1) \cup \mathsf{Prop}(\rho_2) \subseteq \mathsf{Prop}(\Gamma_{a_1}, \Gamma_{b_1}) \cap \mathsf{Prop}(\Gamma_{a_2}, \Gamma_{b_2}, [\varphi]D_G\psi, \Delta)$. Item 3 is also similarly satisfied. $\square$

Theorem 4.23 (Craig Interpolation Theorem)**.** *Let* $\mathbf{X}$ *be any of* **IK**, **IKT**, **IK4**, *and* **IS4**. *Given that* $\vdash_{\mathsf{G}(\mathbf{X})^+} \varphi \Rightarrow \psi$, *there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathsf{G}(\mathbf{X})^+} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})^+} \chi \Rightarrow \psi$.

2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

3. $\mathsf{Agt}(\chi) \subseteq \mathsf{Agt}(\varphi) \cap \mathsf{Agt}(\psi)$.

We note that not only the condition for propositional variables but also the condition for agents can be satisfied.

Proof. When we set $\Gamma \coloneqq \varphi$ and $\Delta \coloneqq \psi$, and take a partition $\langle \Gamma; \varnothing \rangle$, Lemma 4.22 proves Craig Interpolation Theorem. $\square$

Chapter 5

Conclusion

5.1 Conclusion of Thesis

In Chapter 2, we propose Gentzen-style sequent calculi $\mathsf{G}(\mathbf{K}_D)$, $\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, $\mathsf{G}(\mathbf{S4}_D)$, and $\mathsf{G}(\mathbf{S5}_D)$ for multiagent epistemic propositional logics with distributed knowledge operators, parameterized by groups, which are reasonable generalization of sequent calculi for basic epistemic (modal) logic. We show the equipollence of the sequent calculi to the known Hilbert systems (Theorem 2.3) and prove the cut-elimination theorem for $\mathsf{G}(\mathbf{K}_D)$, $\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, and $\mathsf{G}(\mathbf{S4}_D)$ (Theorem 2.4). Using a method described in [26], Craig interpolation theorem is also established for $\mathsf{G}(\mathbf{K}_D)$, $\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, and $\mathsf{G}(\mathbf{S4}_D)$, in which not only condition of propositional variables but also that of agents is taken into account (Theorem 2.9).

In Chapter 3, we develop intuitionistic **K**, **KT**, **KD**, **K4**, **K4D**, and **S4** with distributed knowledge operator, parameterized by group G, based on [23]. We show the semantic completeness of the Hilbert systems $\mathsf{H}(\mathbf{IK})$, $\mathsf{H}(\mathbf{IKT})$, $\mathsf{H}(\mathbf{IKD})$, $\mathsf{H}(\mathbf{IK4})$, $\mathsf{H}(\mathbf{IK4D})$, and $\mathsf{H}(\mathbf{IS4})$ (Theorem 3.11), where we adopt a more standard method via the concept of "pseudo-model" and "tree unraveling" than [23]. Moreover, we also show the semantic completeness with respect to more restricted classes of Kripke frames than the ordinary one, which are characterized by the notion of 'stability' (Definition 3.2, Theorem 3.12). This is because Kripke frame should be stable in order for the intuitionistic public announcement logic with distributed knowledge introduced in Chapter 4 to be sound. To show this, first we prove the strong completeness with respect to the suitable class of frames including nonstable ones (Theorem 3.11) by constructing a model called "tree unraveling". Then, we make the tree unraveling model stable by the operation called stabilization (Definition 3.39). It is noted that the operation of stabilization is compatible with tree unraveling by certain property of it (Proposition 3.42). We also propose cut-free sequent calculi $\mathsf{G}(\mathbf{IK})$, $\mathsf{G}(\mathbf{IKT})$, $\mathsf{G}(\mathbf{IKD})$, $\mathsf{G}(\mathbf{IK4})$, $\mathsf{G}(\mathbf{IK4D})$, and $\mathsf{G}(\mathbf{IS4})$ for our logics (Theorem 3.44, Theorem 3.46), based on the idea introduced in Chapter 2 and prove Craig interpolation theorem (Theorem 3.51) by Maehara's method [26, 35]. In addition, we establish decidability (Theorem 3.57) of the sequent calculi by the standard argument [11, 12] on a cut-free derivation of a sequent, while [23] does not show it for their Hilbert systems.

In Chapter 4, we develop intuitionistic public announcement logics with distributed knowledge based on the logics developed in Chapter 3. We expand the logics developed in Chapter 3 except the ones having the axiom (D) with a public announcement operator to obtain $\mathsf{H}(\mathbf{IK})^+$, $\mathsf{H}(\mathbf{IKT})^+$, $\mathsf{H}(\mathbf{IK4})^+$, and $\mathsf{H}(\mathbf{IS4})^+$ and prove its semantic strong completeness (Theorem 4.16) by a standard argument using reduction axiom [59]. Note that a reduction axiom for the distributed knowledge is *not* sound for the class of all frames, which are defined to enjoy the condition $\leqslant; R_a \subseteq R_a$. This means that we need to restrict our attention to a subclass of frames, called a class of stable frames. We also develop sequent calculi $\mathsf{G}(\mathbf{IK})^+$, $\mathsf{G}(\mathbf{IKT})^+$, $\mathsf{G}(\mathbf{IK4})^+$, and $\mathsf{G}(\mathbf{IS4})^+$ of the logics by naturally transforming the reduction axioms into inference rules and prove the cut-elimination theorem (Theorem 4.19) and Craig interpolation theorem (Theorem 4.23). The inductive proof of the cut-elimination theorem is made possible by using the complexity function (Definition 4.12) for a formula, which is introduced in the argument of the completeness proof, as a measure for cut formula, not the ordinary complexity measure used in cut-elimination theorem for other logics.

5.2 Further Direction

We mention possible directions of further research of the work in Chapter 2. First, we may provide sequent calculi $\mathbf{KD}_D$, $\mathbf{K4D}_D$ or $\mathbf{KD45}_D$ to establish the cut-elimination and Craig interpolation theorems. Second, we may establish the subformula property for $\mathbf{S5}_D$ along the line of [51, 52], which proves the property on a sequent calculus for $\mathbf{S5}$ though the calculus is not cut-free. We note that even if the sequent calculus for $\mathbf{S5}$ is not cut-free but still we can apply Maehara method to establish Craig interpolation theorem (cf [35]). Third, it is interesting to see if we can construct a cut-free sequent calculus for $\mathbf{S5}_D$ on the basis of one of the known cut-free calculi for $\mathbf{S5}$ (with label or with the notion of hypersequent). Fourth, we may check what follows from Craig interpolation theorem. It is known that Craig interpolation theorem entails Beth definability theorem or Robinson consistency theorem in many systems. It is interesting to see whether these hold for our logics. Finally, we may establish completeness results on epistemic logics with distributed knowledge other than the ones mentioned in Fact 2.1 (if it has not been done).

Next, we mention possible directions of further research of the work in Chapters 3 and 4. The first direction is to simplify our semantic completeness argument of $\mathsf{H}(\mathbf{X})$ via a similar method given in [62] for classical epistemic logic with distributed knowledge. One of the merits of the method is that the notion of pseudo- (or pre-) model is not necessary. The second direction is to add **S5**-type axioms to our intuitionistic epistemic logic with distributed knowledge. Since Ono [34] showed that there are at least four distinct **S5**-type axioms over the intuitionistic modal logic **S4**, it would be interesting to study the corresponding **S5**-type axioms in our setting. The third direction is to expand our syntax with the common knowledge operator (cf. [59]). This amounts to investigating the intuitionistic counterpart of [62]. The final direction is to consider another dynamic expansions of our syntax. In order to formalize changes of agents' constructive knowledge caused by communication among a group, we may add resolution operators [1].

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