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A convergence result for a minimizing movement scheme for mean curvature flow with prescribed contact angle in a curved domain

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Abstract

We consider a minimizing movement scheme of Chambolle type for the mean curvature flow equation with prescribed contact angle condition in a smooth bounded domain in $\mathbb{R}^d$ ($d \geq 2$). We prove that an approximate solution constructed by the proposed scheme converges to the level-set mean curvature flow with prescribed contact angle provided that the domain is convex and that the contact angle is away from zero under some control of derivatives of given prescribed angle. We actually prove that an auxiliary function corresponding to the scheme uniformly converges to a unique viscosity solution to the level-set equation with an oblique derivative boundary condition corresponding to the prescribed boundary condition.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^d$ with $d \geq 2$. We consider the mean curvature flow equation for an evolving family $\{\Gamma_t\}_{t\geq 0}$ of hypersurfaces in $\overline{\Omega}$ touching the boundary $\partial\Omega$ of Ω with prescribed angle. Namely, we consider its initial value problem of the form:

$$\begin{cases} V = -\operatorname{div}_{\Gamma_t} \nu \quad \text{on} \quad \Gamma_t, \quad t>0, \\ \angle(\nu, \nu_\Omega) = \theta \quad \text{on} \quad \partial\Omega, \\ \Gamma_0 = \Gamma, \end{cases} \tag{1.1}$$

where Γ is a given hypersurface in $\overline{\Omega}$. Here, ν denotes a unit vector field of Γ_t, and V denotes the normal velocity of each point on Γ_t in the direction of ν; $\operatorname{div}_{\Gamma_t}$ denotes the surface divergence of ν on Γ_t so that $-\operatorname{div}_{\Gamma_t} \nu$ equals the (($d-1$)-times) mean curvature of Γ_t in the direction of ν. Thus, the first equation of (1.1) is nothing but the mean curvature flow equation. The second equation of (1.1) is the boundary condition. The symbol $\angle(\nu, \nu_\Omega)$ denotes the angle between ν and ν_Ω, where ν_Ω denotes the outward unit normal vector field of $\partial\Omega$. The function $\theta : \partial\Omega \to (0, \pi)$ is given and prescribes the contact angle $\angle(\nu, \nu_\Omega)$.

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The authors [19] extended Chambolle's scheme [9] to construct an approximate solution to (1.1). In the sequel, the proposed scheme will be referred as a capillary Chambolle type scheme. We briefly review the proposed scheme together with some literature. Given an initial data $E_0 \subset \mathbb{R}^d$ and a time step $h > 0$, Almgren, Taylor and Wang [1] introduced the following energy functional:

$$\mathcal{A}(F) := \int_{\mathbb{R}^d} |\nabla \chi_F| + \frac{1}{h} \int_{F \Delta E_0} \mathrm{dist}(\cdot, \partial E_0)\, \mathrm{d}\mathcal{L}^d, \tag{1.2}$$

where χ_F denotes the characteristic function of F, i.e., $\chi_F(x) = 1$ if $x \in F$ and $\chi_F(x) = 0$ if $x \notin F$, and $\mathrm{dist}(x, A)$ denotes the distance between a point x and a set A; h is a positive parameter, and $F\Delta E_0$ denotes the symmetric difference of F and E_0, namely $F\Delta E_0 := (F\backslash E_0) \cup (E_0\backslash F)$; the first term of (1.2) denotes the total variation of χ_F while in the second term, $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. They minimized $\mathcal{A}(F)$ among all Caccioppoli sets F, and its minimizer $T_h(E_0)$ was regarded as a candidate for the next set to E_0. Repeating this process, an approximate solution of the mean curvature flow was defined by $T_h^n(E_0) := T_h(T_h^{n-1}(E_0))$ for $n \in \mathbb{N}$ with $T_h^0(E_0) := E_0$. Although they showed its convergence to the smooth mean curvature flow in L^1-setting, it is not clear whether a minimizer of $\mathcal{A}(F)$ is unique or not. Later, Chambolle [9] proposed another energy functional defined by:

$$E_h(u) := \int_\Omega |\nabla u| + \frac{1}{2h} \int_\Omega (u - d_{E_0})^2 \, \mathrm{d}\mathcal{L}^d,$$

where d_{E_0} denotes the signed distance function to E_0. The energy $E_h(u)$ was minimized over all $u \in L^2(\Omega) \cap BV(\Omega)$. Since it is lower semi-continuous and strictly convex, the minimizer $w_{E_0}^h$ is unique. Then, the set $T_h(E_0)$ was defined by the zero sub-level set of $w_{E_0}^h$. Chambolle [9] showed that his $T_h(E)$ is a minimizer of $\mathcal{A}(F)$, and the approximate solution tends to the mean curvature flow in L^1-setting if the corresponding level-set equation with the initial condition $u_0 := \chi_{\Omega\backslash E_0} - \chi_{E_0}$ has a unique viscosity solution where χ_{E_0} is the characteristic function of E_0. This scheme worked only if E does not touch the boundary $\partial\Omega$ hence a contact angle condition cannot be treated.

To cope with the contact angle problem, the authors [19] proposed a capillary Chambolle type scheme to construct an approximate solution to the mean curvature flow with prescribed contact angle condition. For $\beta \in L^\infty(\partial\Omega)$ with $\|\beta\|_\infty \leq 1$, they alternately solved the following variational problem:

$$\min_u E_h^\beta(u) \qquad \text{with} \qquad E_h^\beta(u) := C_\beta(u) + \frac{1}{2h} \int_\Omega (u - d_{\Omega,E_0})^2 \, \mathrm{d}\mathcal{L}^d. \tag{1.3}$$

Here, the minimum (1.3) should be taken over $L^2(\Omega) \cap BV(\Omega)$; d_{Ω,E_0} denotes the signed geodesic distance function to E_0 in Ω (see e.g. [19, Definition 4]), and

$$C_\beta(u) := \int_\Omega |\nabla u| + \int_{\partial\Omega} \beta\gamma u \, d\mathcal{H}^{d-1},$$

where $\gamma : BV(\Omega) \to L^1(\partial\Omega)$ is the trace operator and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. For (1.1), we take $\beta = \cos\theta$ and $\theta \in (0, \pi)$ which imply $|\beta(x)| < 1$. Since C_β is lower semi-continuous (see [34, Proposition 1.2] and [19, Proposition 4]) and the quantity to be minimized in (1.3) is strictly convex in $L^2(\Omega)$, the problem (1.3) admits a unique minimizer $w_{E_0}^h \in L^2(\Omega) \cap BV(\Omega)$. Then, the next set $T_h(E_0)$ to E_0 is defined as the zero sub-level set of $w^h_{E_0}$, namely $T_h(E_0) := \{w^h_{E_0} \leq 0\}$. They established well-posedness of the proposed scheme and some consistency with a capillary Almgren–Taylor–Wang type scheme. However, the convergence of this scheme as h tends to zero was not discussed in [19]. The aim of this paper is to show the convergence of this scheme in a suitable topology.

Let us state our main results. Since the level-set formulation provides a unique global-in-time solution (up to fattening) as shown in [16, 22] for $\Omega = \mathbb{R}^d$ (see also [23]), we consider the level-set formulation of (1.1). Namely, we consider the initial-boundary value problem for its level-set equation of the form:

$$\begin{cases} u_t = |\nabla u| \operatorname{div} \nabla\phi(\nabla u) \ \text{ in } \ \Omega \times (0,T), \\ \langle \nabla u, \nu_\Omega \rangle + \beta |\nabla u| = 0 \ \text{ on } \ \partial\Omega \times (0,T), \\ u(0,\cdot) = u_0 \ \text{ in } \ \overline{\Omega}, \end{cases} \tag{1.4}$$

where $T > 0$ is a time horizon; $u_0 : \overline{\Omega} \to \mathbb{R}$ is given as an initial condition and $\phi(p) := |p|$ for $p \in \mathbb{R}^d$. It is natural to consider such an oblique derivative boundary problem because the second condition of (1.4) implies that the hypersurface $\{u = 0\}$ intersects the boundary $\partial\Omega$ with the angle $\arccos \beta$. This condition readily corresponds to the second one of (1.1). Let $F : (\mathbb{R}^d \backslash \{\mathbf{0}\}) \times \mathbb{S}^d \to \mathbb{R}$ and $B : \partial\Omega \times \mathbb{R}^d \to \mathbb{R}$ be defined by

$$F(p,X) := -\operatorname{tr}\left(\left(I_d - \frac{p \otimes p}{|p|^2}\right) X\right) \quad \text{for} \ \ (p,X) \in (\mathbb{R}^d \backslash \{\mathbf{0}\}) \times \mathbb{S}^d, \tag{1.5}$$

$$B(x,p) := \langle p, \nu_\Omega(x) \rangle + \beta(x)|p|, \tag{1.6}$$

where $\mathbb{S}^d$ denotes the set of all symmetric matrices in $\mathbb{R}^{d\times d}$; $I_d \in \mathbb{R}^{d\times d}$ denotes the identity matrix. Then, the problem (1.4) can be expressed as

$$\begin{cases} u_t + F(\nabla u, \nabla^2 u) = 0 \qquad \text{in} \quad \Omega \times (0,T), \\ B(\cdot, \nabla u) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 \qquad \text{in} \quad \overline{\Omega}. \end{cases} \tag{1.7}$$

In this study, we adopt the notion of viscosity solutions and regard an evolving set by mean curvature as the level set of an auxiliary function as discussed in [23]. Its well-posedness is by now well known by [3, 31]. As in [20, 21], for each $u \in UC(\overline{\Omega})$ and a time step $h > 0$, we define a function operator S_h by

$$S_h u(x) := \sup\{\lambda \in \mathbb{R} \mid x \in T_h(\{u \geq \lambda\})\}, \tag{1.8}$$

where $UC(\overline{\Omega})$ denotes the space of all uniformly continuous functions in $\overline{\Omega}$. In terms of S_h, an approximate solution $u^h : [0,T] \times \overline{\Omega} \to \mathbb{R}$ to (1.4) is defined by

$$u^h(t,x) := S_h^{\lfloor \frac{t}{h} \rfloor} u(x),$$

where $\lfloor k \rfloor$ denotes the largest integer which does not exceed $k \in (0,\infty)$. Then, our main theorem reads as follows:

Theorem 1.1. *Assume that Ω is a bounded convex set in $\mathbb{R}^d$ whose boundary is sufficiently regular so that the comparison principle holds for (1.4). Suppose that $\beta \in C^1(\partial\Omega)$ and $\|\beta\|_\infty < 1$. Assume that $|\nabla_{\partial\Omega}\beta(x)| \leq k(x)$ for all $x \in \partial\Omega$. Here, $k(x)$ denotes the minimal nonnegative principal (inward) curvature of $\partial\Omega$ at the point x. Then, u^h uniformly converges to the unique viscosity solution to (1.4) as $h \to 0$.*

By a comparison principle for (1.4) known by [3] (see Theorem 2.1), we immediately obtain the following corollary:

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a $C^{2,1}$ bounded domain. Suppose that Ω and β satisfy the hypotheses in Theorem 1.1. Then, u^h uniformly converges to the unique viscosity solution to* (1.4) *as $h \to 0$.*

A key step of the proof for Theorem 1.1 is to confirm that the function operator S_h fulfills the following properties:

[**Monotonicity**]

$$S_h u \le S_h v \ \text{ if } \ u \le v. \tag{1.9}$$

[**Translation invariance**]

$$\begin{aligned} &S_h(u+c) = S_h u + c \ \text{ for all } \ c \in \mathbb{R}, \\ &S_h(0) = 0. \end{aligned} \tag{1.10}$$

[**Consistency**]
For every $\varphi \in C^2(\overline{\Omega})$, and $z \in \Omega$ either $\nabla\varphi(z) \neq \mathbf{0}$ or $\nabla\varphi(z) = \mathbf{0}$ and $\nabla^2\varphi(z) = O$, and $z \in \partial\Omega$ with $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| > 0$, it holds that

$$\operatorname*{limsup^*}_{h\to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} \le -F_*(\nabla\varphi(z), \nabla^2\varphi(z)). \tag{1.11}$$

Moreover, for every $\varphi \in C^2(\overline{\Omega})$, and $z \in \Omega$ either $\nabla\varphi(z) \neq \mathbf{0}$ or $\nabla\varphi(z) = \mathbf{0}$ and $\nabla^2\varphi(z) = O$, and $z \in \partial\Omega$ with $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| < 0$, it holds that

$$\operatorname{liminf}_{*h\to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} \ge -F^*(\nabla\varphi(z), \nabla^2\varphi(z)). \tag{1.12}$$

Here, for a function F_h on $\overline{\Omega}$ which is parametrized by $h > 0$, we have used the notation that for $x \in \overline{\Omega}$,

$$\begin{aligned} \operatorname*{limsup^*}_{h\to 0} F_h(x) &:= \lim_{h\to 0} \sup\{F_h(y) \mid |x-y| < \delta,\ 0 < \delta < h\}, \\ \operatorname{liminf}_{*h\to 0} F_h(x) &:= \lim_{h\to 0} \inf\{F_h(y) \mid |x-y| < \delta,\ 0 < \delta < h\}. \end{aligned}$$

Moreover, we define the upper(resp, lower) semi-continuous envelope F^* (resp, F_*) of F by

$$\begin{aligned} F^*(p, X) &:= \lim_{\varepsilon\to 0} \sup\{F(q, Y) \mid |p-q| < \varepsilon,\ \|X - Y\|_2 < \varepsilon\}, \\ F_*(p, X) &:= \lim_{\varepsilon\to 0} \inf\{F(q, Y) \mid |p-q| < \varepsilon,\ \|X - Y\|_2 < \varepsilon\}, \end{aligned}$$

where for a matrix $X = (x_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$, $\|X\|_2$ denotes the Hilbert–Schmidt norm of X, i.e., $\|X\|_2 := \sqrt{\sum_{i,j=1}^d x_{ij}^2}$.

If the limit equation (1.4) has a comparison principle and S_h satisfies the conditions (1.9)–(1.12), a general theory for the monotone scheme [4, Theorem 2.1] yields the desired result. In our case, we know (1.4) has a comparison principle (Theorem 2.1) so the main task is to prove (1.9)–(1.12) for S_h (Theorem 4.1). It is not difficult to prove (1.9) if we take the geodesic distance in (1.3). The property (1.10) is easy to confirm from the definition of S_h. Main efforts for the convergence result of our scheme are devoted to prove (1.11) and (1.12).

To this end, we first establish a relation between S_h and T_h (Lemma 4.1) to interpret the formulae (1.11) and (1.12) in that for T_h. More explicitly, we show

$$\{S_h u \geq \lambda\} = T_h(\{u \geq \lambda\}) \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{1.13}$$

The relation (1.13) is expected to hold because of the definition of S_h. One of important criteria to ensure (1.13) is the continuity of T_h (Lemma 3.6) which is defined by

$$\bigcap_{n=1}^{\infty} T_h(E_n) = T_h(E) \qquad \text{as} \quad n \to \infty, \tag{1.14}$$

where $\{E_n\}_{n\in\mathbb{N}}$ is an arbitrary non-increasing sequence of sets in $\overline{\Omega}$ with $E = \bigcap_{n=1}^{\infty} E_n$. Thanks to the monotonicity of T_h, we easily see that the left-hand side of (1.14) includes the right-hand side. To show the converse inclusion, we seek a subsequence $\{w^h_{E_n}\}_n$ which uniformly converges to a function w in $\overline{\Omega}$, and observe that $w = w^h_E$ in $\overline{\Omega}$. Since $w^h_{E_n}$ is uniformly bounded with respect to n by a maximum principle, the existence of such a subsequence can be proved by the Ascoli–Arzelà theorem provided that $w^h_{E_n}$ is equi-continuous with respect to $n \in \mathbb{N}$. We eventually know that the limit function w must equal w^h_E by the Lipschitz continuity of the map $L^2(\Omega) \ni g \mapsto w^h_g \in L^2(\Omega)$ (Proposition 3.1) and the uniform convergence of d_{Ω,E_n} to $d_{\Omega,E}$.

To derive the equi-continuity of $w^h_{E_n}$, we show that the gradient $\nabla w^h_{E_n}$ is bounded by a constant which is independent of $n \in \mathbb{N}$. For this, we adopt Bernstein's method. Namely, we are led to show that $\frac{1}{2}|\nabla w^h_{E_n}|^2$ is a subsolution to an elliptic problem with an oblique derivative boundary condition and to apply a comparison principle (Lemma 3.2) which is available for the problem. This procedure involving Bernstein's method requires the convexity of Ω and the assumption that the contact angle function β is continuously differentiable on $\partial\Omega$ and its gradient is bounded by the minimal nonnegative principal (inward) curvature of $\partial\Omega$ at each point.

Using the relation between S_h and T_h that we have obtained so far, we represent the formulae (1.11) and (1.12) in terms of the super level sets $E^\varphi_\mu := \{\varphi \geq \mu\}$ with $\mu = \varphi(z)$, and intend to prove that the following locally uniform limit in $\overline{\Omega}$ (Proposition 4.2):

$$\left| \frac{w^h_{E^\varphi_\mu} - d_{\Omega,E^\varphi_\mu}}{h} + \kappa_{E^\varphi_\mu} \right| \to 0 \qquad \text{as} \quad h \to 0. \tag{1.15}$$

For the case when $z \in \Omega$, (1.15) indicates that $w^h_{E^\varphi_\mu}$ approximately solves the inclusion:

$$\frac{w - d_{\Omega,E^\varphi_\mu}}{h} + \partial C_0(w) \ni 0 \qquad \text{in} \quad \Omega,$$

which is nothing but a discretization of the first equation of (1.4). Meanwhile, if $z \in \partial\Omega$, we need the assumption that $\|\beta\|_\infty < 1$ to construct a viscosity super(resp, sub)solution to (3.10) which approximates the following inclusion:

$$\frac{w - d_{\Omega,E^\varphi_\mu}}{h} + \partial C_\beta(w) \ni 0 \qquad \text{in} \quad \overline{\Omega}. \tag{1.16}$$

In fact, we deduce from the characterization of $\partial C_\beta(u)$ (see [19, Theorem 2]) that (1.16) is equivalent to that w solves (3.10). In [20, 21], a viscosity super(resp, sub)solutions were constructed by a ball $B(0,R)$ which is expected to approximate E^φ_μ nearby z. This is because Chambolle's scheme yields another ball $B(0,\widetilde{R})$, and this $\widetilde{R}$ can be explicitly computed. Due to the oblique derivative boundary condition, we require another type of subsets in $\overline{\Omega}$ to approximate E^φ_μ. Here, we notice that the characterization of $\partial C_\beta(u)$ again gives rigorous solutions to (3.10) in a short time when β is constant (Lemma 4.2). These rigorous solutions are so-called translating solitons in the literature. In this research, we compare E^φ_μ with these solitons instead of balls. By geometry, this soliton is available not only for $z\in\Omega$ but also for $z\in\partial\Omega$ whenever $\|\beta\|_\infty<1$. This is the basic idea to prove the main theorem.

Let us review existing works related to an energy minimizing scheme for the mean curvature flow. For the case when interfaces do not touch the boundary, Almgren, Taylor and Wang [1] derived several properties of a limit of their approximate solution and called it a *flat Φ curvature flow*. They proved its convergence to a smooth curvature flow up to the time when the latter exists. Later, Luckhaus and Sturzenhecker [33] showed its convergence to a distributional solution χ to the mean curvature flow under *no mass loss condition*:

$$\int_0^T\int_{\mathbb{R}^d}|\nabla\chi_h|\to\int_0^T\int_{\mathbb{R}^d}|\nabla\chi|\qquad\text{as}\quad h\to 0,$$

where χ_h denotes the characteristic function of a minimizer of $\mathcal{A}(F)$. Philippis and Laux [17] proved that this assumption is not necessary for the convergence result whenever the initial data E_0 is *outward minimizing*, i.e., it holds that

$$\int_{\mathbb{R}^d}|\nabla\chi_{E_0}|\le\int_{\mathbb{R}^d}|\nabla\chi_F|\qquad\text{if}\quad E_0\subset F. \tag{1.17}$$

For instance, if E_0 is mean convex, then the condition (1.17) is satisfied. For a bounded initial data E_0 and $\Omega\subset\mathbb{R}^d$ strictly including E_0, Chambolle [9] showed that his approximate solution constructed by the zero level set of the unique minimizer of $E_h(u)$ converges (in L^1 sense) to a *level-set flow* (up to fattening). It is known that $T_h(E_0)$ remains convex if E_0 is convex (see Caselles and Chambolle [8]). For a simple proof of the convergence (also in Hausdorff distance sense), we refer the reader to Chambolle and Novaga [15, Proposition 2.1, Proposition 4.1]. For an unbounded initial data E_0, the authors and Ishii [20, 21] showed the convergence (up to fattening in the sense of Hausdorff distance) of their approximate solution constructed by $E_h(u)$. Therein, they translated the set operator T_h into a function operator S_h as in (1.8) and showed that T_h is a morphological operator (see e.g., [7, Definition 4.4]). They utilized a sup-inf representation for S_h to obtain its generator as in (1.11) and (1.12). In particular, if E_0 is the complement of a bounded set, its treatment was explained in [13, §6.2]. Chambolle, Gennaro and Morini [11] considered such a scheme called a *minimizing movement scheme* for the mean curvature flow with a time-dependent spatially inhomogeneous driving force $f(x,t)$, an inhomogeneous anisotropic interfacial energy density ϕ and a mobility $\psi(x,\nu(x))$, namely they studied the equation:

$$\begin{cases}V=\psi(x,\nu(x))\{-\operatorname{div}_{\Gamma_t}\nabla_p\phi(x,\nu(x))+f(x,t)\}\qquad\text{for}\quad x\in\Gamma_t,\ t>0,\\ \Gamma_0=\Gamma,\end{cases} \tag{1.18}$$

where Γ is an initial data, and $p\mapsto\phi(x,p)$ is an anisotropy, i.e., a convex, positively 1-homogeneous function with respect to the variable $p\in\mathbb{R}^d$. They showed that a limit of approximate solutions constructed by their minimizing movement scheme is a *flat flow*, and it is a distributional solution (BV-solution) to the equation (1.18) under no mass loss condition (see [11, Theorem 1.2]). Note that their anisotropy for this result includes crystalline anisotropy, i.e., $\phi(x,p)$ need not be C^1 in p on $\{|p|=1\}$, typically piecewise linear. They also proved that their approximate solution converges to the level-set flow, and it is also a way to construct a solution of the corresponding level-set flow equation (see [11, Theorem 1.4]). However, crystalline anisotropy was excluded in this result. For spatially homogeneous cases, i.e., ϕ, ψ and f are independent of x, a level-set crystalline mean curvature flow equation is well-studied, and its well-posedness was established by [14, 12, 24, 25] (see also a review paper [27]). However, the case of spatially inhomogeneous crystalline anisotropy is not yet studied through a perturbed argument by [12] which may lead the existence of a solution. For homogeneous case, we note that Ishii [32] proved that the minimizing movement scheme converges to the level-set flow for crystalline mean curvature flow provided that the corresponding level-set flow equation is well-posed. The solution in [24, 25] was constructed by a solution of an approximate equation, while the solution in [14, 12] was constructed by a minimizing movement scheme. In [26], spatially inhomogeneous driving force term $f(x,t)$ was allowed. The method of [24, 25, 26] so far needs to assume that ϕ is piecewise linear but allows nonlinear dependency on the curvature term in V.

For the case when interfaces touch the boundary $\partial\Omega$, Bellettini and Kholmatov [6] considered a minimizing problem for a variant energy of (1.2) defined by

$$\mathcal{A}_\beta(F) := \int_{\mathbb{R}^d_+} |\nabla \chi_F| + \int_{\partial\mathbb{R}^d_+} \beta\gamma\chi_F d\mathcal{H}^{d-1} + \frac{1}{h}\int_{\mathbb{R}^d_+\cap(F\Delta E)} \mathrm{dist}(\cdot,\partial E)\,\mathrm{d}\mathcal{L}^d, \tag{1.19}$$

where $\mathbb{R}^d_+ := \mathbb{R}^{d-1}\times(0,\infty)$, and the energy (1.19) was minimized among all Caccioppoli sets in $\mathbb{R}^d_+$. They adopted a set theoretic approach and proved that a minimizing sequence for (1.19) converges to a generalized minimizing movement (GMM) (see [6, Theorem 7.1]). The GMM was shown to be a distributional solution to the mean curvature flow equation with a contact angle condition provided that the $(d-1)$-dimensional Hausdorff measure of discrete solutions converges to that of the GMM (see [6, Theorem 8.6]). They also showed regularity of the GMM up to the boundary provided that β is Lipschitz continuous on $\partial\Omega$ (see [6, Theorem 5.3]). In fact, the study [19] was inspired by their work to introduce the definition of the capillary Chambolle type energy (1.3). For a study of the GMM for a partition of $\mathbb{R}^d$ with mobility and driving force, we refer the reader to Bellettini, Chambolle and Kholmatov [5].

In [19, §7], the authors implemented the proposed scheme using the split Bregman method. Therein, they calculated the first variation of $E^\beta_h(u)$ with respect to ∇u and u separately and obtained a Neumann boundary problem in a strip $\Omega\subset\mathbb{R}^2$. This problem was solved by the finite difference method. They gave two examples of open curves with one end on $\{0\}\times\mathbb{R}$ and the other end on $\{2\}\times\mathbb{R}$ where $\Omega=(0,2)\times\mathbb{R}$. In the first example, we set $\beta\equiv\cos\frac{\pi}{4}$. In the second example, we set $\beta=\cos\frac{3\pi}{4}$ at $x=0$ and $\beta=\cos\frac{\pi}{4}$ at $x=2$. However, the discrete definition of β in the second example was missing. Here, we give it for the reader's convenience:

$$\beta_{i,j} := \begin{cases} -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4} & \text{if} \quad j=1, \\ \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} & \text{if} \quad j=N_x, \\ 0 \quad \text{otherwise,} \end{cases}$$

where the strip Ω is discretized by mesh points (x_j, y_i) with $1 \le i \le N_y$ and $1 \le j \le N_x$, say the discretized $\overline{\Omega}$ equals $\bigcup_{i=1}^{N_y} \bigcup_{j=1}^{N_x} \{(x_j, y_i)\}$; the symbol $\beta_{i,j}$ denotes the value of β at (x_j, y_i).

This paper is organized as follows. In Section 2, we collect basic definitions, notations and facts of convex analysis and viscosity solutions. In Section 3, we recall a capillary Chambolle type scheme which was proposed in [19]. Therein, we continue to explore its properties which are crucial to derive main results in this study. Section 4 is devoted to give a proof for convergence of the proposed scheme under some assumptions on the domain Ω and the contact angle function β. Finally, we conclude with summarizing consequence of this paper in Section 5.

A preliminary version of this paper was included in the first author's phD thesis.

2 Preliminaries

2.1 Convex analysis

In this section, we recall basic facts from the convex analysis. Throughout this section, let X be a normed space and X^* be the dual space of X. Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$.

Definition 2.1 (Subdifferential). *For $u \in X$, the subdifferential $\partial f(u) \subset X^*$ is defined by*

$$p \in \partial f(u) :\Longleftrightarrow f(v) \ge f(u) + \langle p, v - u \rangle \qquad \text{for all} \quad v \in X.$$

Definition 2.2 (Conjugate function). *The conjugate function $f^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ of f is defined by*

$$f^*(p) := \sup_{u \in X} \{\langle p, u \rangle - f(u)\} \qquad \text{for} \quad p \in X^*.$$

Proposition 2.1 (Fenchel identity). *Suppose that f is lower semi-continuous and convex. Then, it holds that for $u \in X$ and $p \in X^*$,*

$$p \in \partial f(u) \Longleftrightarrow u \in \partial f^*(p) \Longleftrightarrow f(u) + f^*(p) = \langle p, u \rangle$$

Proof. See [18, Proposition 5.1, Corollary 5.2]. □

Remark 2.1 (Characterization of conjugate functions). *Suppose that f is positively homogeneous of degree 1 and $f(0) = 0$. Then, it is easy to see that*

$$f^*(p) = \begin{cases} 0 & \text{if } p \in K, \\ \infty & \text{if } p \notin K, \end{cases}$$

where $K = \partial f(0)$.

2.2 Viscosity solution

In this section, we briefly recall the notion of viscosity solutions which is a kind of weak solutions to degenerate parabolic equations. Partial differential equations under consideration is of the form:

$$\begin{cases} u_t + F(x, t, u, \nabla u, \nabla^2 u) = 0 & \text{in} \quad \Omega \times (0, T), \\ B(x, t, u, \nabla u, \nabla^2 u) = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(0, \cdot) = u_0 & \text{in} \quad \overline{\Omega}, \end{cases} \tag{2.1}$$

where F and B are respectively functions defined on dense subsets in $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and $\partial\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$. Here, $\mathbb{S}^d$ denotes the set of all symmetric matrices in $\mathbb{R}^{d\times d}$. In this setting, let us define a viscosity sub- and supersolution to the problem (2.1).

Definition 2.3 (Viscosity solution). *A function $u : \overline{\Omega} \times (0,T) \to \mathbb{R}$ is called a viscosity subsolution to* (2.1) *provided that $u^*(x,t) < \infty$ for all $(x,t) \in \overline{\Omega} \times (0,T)$ and for any $\varphi \in C^2(\overline{\Omega} \times (0,T))$ and $(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,T)$ such that $u^* - \varphi$ takes a local maximum at $(\hat{x},\hat{t})$,*

$$\begin{cases} \varphi_t(\hat{x},\hat{t}) + F_*(\hat{x},\hat{t},u^*(\hat{x},\hat{t}),\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t})) \le 0 & \text{if } \nabla\varphi(\hat{x},\hat{t}) \neq 0, \\ \varphi_t(\hat{x},\hat{t}) \le 0 \quad \text{if } \nabla\varphi(\hat{x},\hat{t}) = 0 \quad \text{and} \quad \nabla^2\varphi(\hat{x},\hat{t}) = O \end{cases} \tag{2.2}$$

holds if $\hat{x} \in \Omega$ and either (2.2) *or $B_*(\hat{x},\hat{t},u^*(\hat{x},\hat{t}),\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t})) \le 0$ if $\hat{x} \in \partial\Omega$. A function u is called a viscosity supersolution to* (2.1) *provided that $u_*(x,t) > -\infty$ for all $(x,t) \in \overline{\Omega} \times (0,T)$ and for any $\varphi \in C^2(\overline{\Omega} \times (0,T))$ and $(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,T)$ such that $u_* - \varphi$ takes a local minimum at $(\hat{x},\hat{t})$,*

$$\begin{cases} \varphi_t(\hat{x},\hat{t}) + F^*(\hat{x},\hat{t},u_*(\hat{x},\hat{t}),\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t})) \ge 0 & \text{if } \nabla\varphi(\hat{x},\hat{t}) \neq 0, \\ \varphi_t(\hat{x},\hat{t}) \ge 0 \quad \text{if } \nabla\varphi(\hat{x},\hat{t}) = 0 \quad \text{and} \quad \nabla^2\varphi(\hat{x},\hat{t}) = O \end{cases} \tag{2.3}$$

holds if $\hat{x} \in \Omega$ and either (2.3) *or $B^*(\hat{x},\hat{t},u_*(\hat{x},\hat{t}),\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t})) \ge 0$ if $\hat{x} \in \partial\Omega$. A function u is called a viscosity solution if u is a sub- and supersolution of* (2.1).

Definition 2.4 (Degenerate ellipticity). *$F : \overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is said to be degenerate elliptic if for any $(x,t,r,p) \in \overline{\Omega} \times [0,T] \times \mathbb{R} \times (\mathbb{R}^d \setminus \{\mathbf{0}\})$, it holds that*

$$F(x,t,r,p,X) \le F(x,t,r,p,Y) \quad \text{for } X, Y \in \mathbb{S}^d \quad \text{with} \quad X \ge Y,$$

where $X \le Y$ means that $\langle X\xi, \xi\rangle \le \langle Y\xi, \xi\rangle$ for every $\xi \in \mathbb{R}^d$.

Let us recall a comparison principle for the problem (2.1) from [3, Theorem 3.1]:

Theorem 2.1 (Comparison principle). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{2,1}$ boundary, and let $u_0 \in C(\overline{\Omega})$. Let u and v be respectively a bounded upper semi-continuous subsolution and a bounded lower semi-continuous supersolution of* (2.1)*. Suppose that F and B fulfill the following conditions:*

(F1) *For every $R > 0$, there exists a constant $C_R \in \mathbb{R}$ such that for every $x \in \overline{\Omega}$, $t \in [0,T]$, $-R \le v \le u \le R$, $p \in \mathbb{R}^d$ and $X \in \mathbb{S}^d$, it holds that*

$$F(x,t,u,p,X) - F(x,t,v,p,X) \ge C_R(u - v).$$

(F2) *For any $R, K > 0$, there exists a function $\omega_{R,K} : [0,\infty) \to \mathbb{R}$ such that $\omega_{R,K}(s) \to 0$ as $s \downarrow 0$ and for all $\eta > 0$, it holds that*

$$F(y,t,u,q,Y) - F(x,t,u,p,X) \le \omega_{R,K}\left(\eta + |x-y|(1 + |p| \vee |q|) + \frac{|x-y|^2}{\varepsilon^2}\right)$$

for any $x, y \in \overline{\Omega}$, $t \in [0,T]$, $u \in [-R,R]$, $p, q \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $X, Y \in \mathbb{S}^d$ satisfying

$$-\frac{K\eta}{\varepsilon^2} I_{2d} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \le \frac{K\eta}{\varepsilon^2} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + K\eta I_{2d}, \tag{2.4}$$
$$|p - q| \le K\varepsilon(|p| \wedge |q|) \qquad \text{and} \qquad |x - y| \le K\eta\varepsilon,$$

where $|p| \vee |q| := \max\{|p|,|q|\}$ and $|p| \wedge |q| := \min\{|p|,|q|\}$.

(F3) *$F \in C(\overline{\Omega} \times [0,T] \times \mathbb{R} \times (\mathbb{R}^d \backslash \{\mathbf{0}\}) \times \mathbb{S}^d)$ and $F^*(x,t,u,\mathbf{0},O) = F_*(x,t,u,\mathbf{0},O)$ for every $x \in \overline{\Omega}$, $t \in [0,T]$ and $u \in \mathbb{R}$. In other words, $F(x,t,u,\cdot,\cdot)$ is continuous at $(\mathbf{0},O)$.*

(B1) *For any $R > 0$, there exists $C_R > 0$ such that for all $\lambda > 0$, $x \in \partial\Omega$, $t \in [0,T]$, $-R \le v \le u \le R$ and $p \in \mathbb{R}^d$, it holds that*

$$B(x,t,u,p+\lambda\nu_\Omega(x)) - B(x,t,v,p) \ge C_R\lambda.$$

(B2) *There exists a constant $C > 0$ such that for any $x, y \in \overline{\Omega}$, $t \in [0,T]$, $u \in \mathbb{R}$ and $p, q \in \mathbb{R}^d$, it holds that*

$$|B(x,t,u,p) - B(y,t,u,q)| \le C\{(|p|+|q|)|x-y| + |p-q|\}.$$

Then, it holds that $u \le v$ in $\overline{\Omega} \times [0,T]$.

Under regularity assumptions on Ω and the contact angle function β, we confirm that the problem (1.4) satisfies the comparison principle:

Theorem 2.2. *Suppose that Ω is uniformly C^2, thus ν_Ω is uniformly C^1. Assume that $\|\beta\|_\infty < 1$ and $\|\nabla_{\partial\Omega}\beta\|_\infty < \infty$. Then, the function F and B defined by (1.5) and (1.6) satisfy the hypotheses of Theorem 2.1. In particular, if Ω is $C^{2,1}$, then the comparison principle is available for the problem (1.4).*

Proof. Since F is independent of u, the condition (F1) is clearly fulfilled by setting $C_R = 0$. To show the condition (F2), fix any $K, R, \eta, \varepsilon > 0$ and let $X, Y \in \mathbb{S}^d$ be such that (2.4). Let $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_d\} \subset \mathbb{R}^d$ be the standard basis of $\mathbb{R}^d$, i.e., for each $1 \le i \le d$, the j-th element of $\boldsymbol{e}_i$ equals δ_{ij}, where δ_{ij} denotes Kronecker's delta. Letting $\xi := (\boldsymbol{e}_i, \boldsymbol{e}_i) \in \mathbb{R}^{2d}$ and applying ξ to (2.4), we have

$$-\frac{2k\eta}{\varepsilon^2} \le -y_{ii} + x_{ii} \le 2k\eta. \tag{2.5}$$

Summing up (2.5) through $1 \le i \le d$, we obtain

$$-\frac{2K\eta d}{\varepsilon^2} \le -\operatorname{tr}(Y) + \operatorname{tr}(X) \le 2K\eta d. \tag{2.6}$$

Meanwhile, letting $\xi := (\frac{p}{|p|}, \frac{q}{|q|}) \in \mathbb{R}^{2d}$ and applying ξ to (2.4) yield

$$-\frac{2K\eta}{\varepsilon^2} \le \operatorname{tr}\left(\frac{p \otimes p}{|p|^2}X\right) - \operatorname{tr}\left(\frac{q \otimes q}{|q|^2}Y\right) \le \frac{2K\eta}{\varepsilon^2}\left(1 - \frac{\langle p, q\rangle}{|p||q|}\right) + 2K\eta. \tag{2.7}$$

Combining (2.6) and (2.7) together with the Schwarz inequality gives

$$-2K\left(1 + \frac{d+2}{\varepsilon^2}\right)\eta \le F(q,Y) - F(p,X) \le 2K\left(d + \frac{1}{\varepsilon^2}\right)\eta.$$

Thus, we define $\omega_{R,K}(s) := 2K\left(d + \frac{1}{\varepsilon^2}\right)s$ for each $s \in [0,\infty)$ and observe that $\lim_{s\downarrow 0}\omega_{R,K}(s) = 0$ and

$$F(q,Y) - F(p,X) \le \omega_{R,K}(\eta) \le \omega_{R,K}\left(\eta + |x-y|(1+|p| \vee |q|) + \frac{|x-y|^2}{\varepsilon^2}\right).$$

Hence, F satisfies (F2). For (F3), it is known that $F_*(\mathbf{0}, O) = F^*(\mathbf{0}, O) = 0$ by [23, Lemma 1.6.16]. Let us check that B satisfies (B1). We compute

$$\begin{aligned} B(u, p + \lambda\nu_\Omega(x)) - B(u, p) &= \langle p + \lambda\nu_\Omega(x), \nu_\Omega(x)\rangle + \beta(x)|p + \lambda\nu_\Omega(x)| - \langle p, \nu_\Omega(x)\rangle - \beta(x)|p| \\ &= \lambda + \beta(x)(|p + \lambda\nu_\Omega(x)| - |p|) \geq \lambda - \|\beta\|_\infty \, ||p + \lambda\nu_\Omega(x)| - |p|| \\ &\geq \lambda - \|\beta\|_\infty |p + \lambda\nu_\Omega(x) - p| = (1 - \|\beta\|_\infty)\lambda. \end{aligned}$$

Thus, letting $C_R := 1 - \|\beta\|_\infty > 0$, we have (B1). For (B2), we compute

$$\begin{aligned} |B(x, p) - B(y, q)| &= |\langle p, \nu_\Omega(x)\rangle + \beta(x)|p| - \langle q, \nu_\Omega(y)\rangle - \beta(y)|q|| \\ &\leq |\langle p, \nu_\Omega(x)\rangle - \langle q, \nu_\Omega(y)\rangle| + |\beta(x)|p| - \beta(y)|q||. \end{aligned} \tag{2.8}$$

The first term of the right-hand side of (2.8) can be estimated as follows.

$$\begin{aligned} &|\langle p, \nu_\Omega(x)\rangle - \langle q, \nu_\Omega(y)\rangle| = |\langle p - q, \nu_\Omega(x)\rangle + \langle q, \nu_\Omega(x) - \nu_\Omega(y)\rangle| \\ &\leq |p - q| + |q|\|\nabla_{\partial\Omega}\nu_\Omega\|_\infty|x - y| \leq |p - q| + \|\nabla_{\partial\Omega}\nu_\Omega\|_\infty(|p| + |q|)|x - y|. \end{aligned} \tag{2.9}$$

Whereas, for the second term of the right-hand side of (2.8), we compute

$$\begin{aligned} &|\beta(x)|p| - \beta(y)|q|| \leq |\beta(x) - \beta(y)||p| + |\beta(y)|p| - \beta(y)|q|| \\ &\leq \|\nabla_{\partial\Omega}\beta\|_\infty|p||x - y| + \|\beta\|_\infty||p| - |q|| \leq \|\nabla_{\partial\Omega}\beta\|_\infty(|p| + |q|)|x - y| + |p - q|. \end{aligned} \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) and setting $C := \max\{2, \|\nabla_{\partial\Omega}\beta\|_\infty, \|\nabla_{\partial\Omega}\nu_\Omega\|_\infty\} > 0$, we derive the desired inequality. $\square$

3 Capillary Chambolle type scheme

In this section, we will explore some properties of a minimizer of the energy functional $E_h^\beta(u)$ defined by

$$E_h^\beta(u) := C_\beta(u) + \frac{1}{2h}\int_\Omega (u - g)^2 \, \mathrm{d}\mathcal{L}^d, \tag{3.1}$$

where $g \in L^2(\Omega)$ is a given data.

The following lemma asserts the monotonicity of the minimizer of $E_h^\beta(u)$ with respect to the data g:

Lemma 3.1. *Let w_f denote the unique minimizer of (3.1). Let $f, g \in L^2(\Omega)$ and suppose that $f \leq g$ holds a.e. in Ω. Then, $w_f \leq w_g$ holds a.e. in Ω.*

Proof. Though the proof is quite similar to that of [9], we include it for the reader's convenience. Our present purpose is to show that the set $\{w_f > w_g\}$ has zero d-dimensional Lebesgue measure. Since w_f and w_g are respectively minimizers of (3.1) for the data f and g, we have

$$C_\beta(w_f) + \frac{1}{2h}\int_\Omega (w_f - f)^2 \, \mathrm{d}\mathcal{L}^d \quad \leq \quad C_\beta(w_f \wedge w_g) + \frac{1}{2h}\int_\Omega (w_f \wedge w_g - f)^2 \, \mathrm{d}\mathcal{L}^d, \tag{3.2}$$

$$C_\beta(w_g) + \frac{1}{2h}\int_\Omega (w_g - g)^2 \, \mathrm{d}\mathcal{L}^d \quad \leq \quad C_\beta(w_f \vee w_g) + \frac{1}{2h}\int_\Omega (w_f \vee w_g - g)^2 \, \mathrm{d}\mathcal{L}^d. \tag{3.3}$$

We now recall a well-known inequality:

$$\int_\Omega |\nabla(u \wedge v)| + \int_\Omega |\nabla(u \vee v)| \leq \int_\Omega |\nabla u| + \int_\Omega |\nabla v|. \tag{3.4}$$

Moreover, it is easily observed that

$$\int_{\partial\Omega} \beta\gamma(u \wedge v) d\mathcal{H}^{d-1} + \int_{\partial\Omega} \beta\gamma(u \vee v) d\mathcal{H}^{d-1} = \int_{\partial\Omega} \beta\gamma u d\mathcal{H}^{d-1} + \int_{\partial\Omega} \beta\gamma v d\mathcal{H}^{d-1}. \tag{3.5}$$

Thus, we obtain

$$\begin{aligned} \int_\Omega (w_f - f)^2 \,\mathrm{d}\mathcal{L}^d &\leq \int_\Omega (w_f \wedge w_g - f)^2 \,\mathrm{d}\mathcal{L}^d, \\ \int_\Omega (w_g - g)^2 \,\mathrm{d}\mathcal{L}^d &\leq \int_\Omega (w_f \vee w_g - g)^2 \,\mathrm{d}\mathcal{L}^d. \end{aligned} \tag{3.6}$$

Splitting Ω into $\{w_f \leq w_g\}$ and $\{w_f > w_g\}$ and summing up (3.6) yield

$$\begin{aligned} \int_{\{w_f > w_g\}} (w_f - f)^2 \,\mathrm{d}\mathcal{L}^d &\leq \int_{\{w_f > w_g\}} (w_g - f)^2 \,\mathrm{d}\mathcal{L}^d, \\ \int_{\{w_f > w_g\}} (w_g - g)^2 \,\mathrm{d}\mathcal{L}^d &\leq \int_{\{w_f > w_g\}} (w_f - g)^2 \,\mathrm{d}\mathcal{L}^d. \end{aligned}$$

Adding two inequalities yield

$$\int_{\{w_f > w_g\}} (w_f - w_g)(g - f) \,\mathrm{d}\mathcal{L}^d \leq 0. \tag{3.7}$$

Since $f \leq g$ a.e. in Ω, the integral domain $\{w_f > w_g\}$ should have zero d dimensional measure. $\square$

We next prove that the map $L^2(\Omega) \ni g \mapsto w_g^h \in L^2(\Omega)$ is Lipschitz continuous:

Proposition 3.1. *For every $h > 0$ and $f, g \in L^2(\Omega)$, it holds that*

$$\|w_f^h - w_g^h\|_2 \leq \|f - g\|_2.$$

Proof. Set $p_f := -(w_f^h - f)/h$ and $p_g := -(w_g^h - g)/h$. Then, we see that $p_f \in \partial C_\beta(w_f^h)$ and $p_g \in \partial C_\beta(w_g^h)$. Hence, we get

$$\begin{aligned} C_\beta(w_g^h) &\geq \int_\Omega p_f (w_g^h - w_f^h) \,\mathrm{d}\mathcal{L}^d + C_\beta(w_f^h), \\ C_\beta(w_f^h) &\geq \int_\Omega p_g (w_f^h - w_g^h) \,\mathrm{d}\mathcal{L}^d + C_\beta(w_g^h). \end{aligned} \tag{3.8}$$

Summing up both sides of (3.8) gives

$$0 \leq \int_\Omega (p_f - p_g)(w_f^h - w_g^h) \,\mathrm{d}\mathcal{L}^d = \int_\Omega \left(\frac{f - g - w_f^h + w_g^h}{h} \right) (w_f^h - w_g^h) \,\mathrm{d}\mathcal{L}^d. \tag{3.9}$$

Thus, we obtain from Cauchy-Schwarz' inequality that

$$\|w_f^h - w_g^h\|_2^2 \leq \|f - g\|_2 \|w_f^h - w_g^h\|_2.$$

The proof is now complete. $\square$

In the sequel, we investigate conditions on the domain Ω and on the contact angle function β to ensure that a solution w_g^h to (3.10) is equi-continuous with respect to $h > 0$:

$$\left\{ \begin{array}{rcl} w + h \operatorname{div} \nabla\phi(\nabla w) & = & g \text{ in } \Omega, \\ B(\cdot, \nabla w) & = & 0 \text{ on } \partial\Omega, \end{array} \right. \tag{3.10}$$

where $\phi(p) := |p|$, and $B(x,p) := \langle p, \nu_\Omega(x)\rangle + \beta(x)|p|$ for $x \in \partial\Omega$ and $p \in \mathbb{R}^d$. Precisely speaking, we obtain the following result:

Theorem 3.1. *Suppose that Ω is a convex domain and $\beta \in C^1(\partial\Omega)$ with $\|\beta\|_\infty < 1$. Suppose that w is a solution to (3.10). Assume that $\nabla_{\partial\Omega}\beta(x)$ is orthogonal to the kernel of the Weingarten map $\nabla_{\partial\Omega}\nu_\Omega(x)$ at each $x \in \partial\Omega$. Assume that $|\nabla_{\partial\Omega}\beta(x)| \leq k(x)$ where $k(x)$ is the minimal nonnegative principal (inward) curvature of $\partial\Omega$ at $x \in \partial\Omega$. Then, it holds that*

$$\|\nabla w\|_\infty \leq \|\nabla g\|_\infty.$$

In other words, if the gradient of g is bounded, then w_g^h is equi-continuous with respect to $h > 0$.

Remark 3.1. *Our assumption on β implies that β must be a constant function if $\partial\Omega$ is flat. Precisely speaking, if $\partial\Omega$ is flat in the direction of x_i-axis, then β must be independent of x_i.*

Remark 3.2. *We note that the function w_g^h turns out to be equi-continuous with respect to g in the case when g is the signed geodesic distance function.*

To prove Theorem 3.1, we need several lemmas.

Lemma 3.2. *Let L be a degenerate elliptic differential operator of the form:*

$$L := \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{\ell=1}^{d} b_\ell \frac{\partial}{\partial x_\ell}$$

with a non-positive definite symmetric matrix $(a_{ij})_{1\leq i,j\leq d}$ and b_ℓ, where $(a_{ij})^{1/2}$ is Lipschitz and b_ℓ's are uniformly continuous in $\overline{\Omega}$ where Ω is a domain in $\mathbb{R}^d$. Assume that $\partial\Omega$ is uniformly C^1. Let ξ be a bounded C^1 vector field on $\partial\Omega$ such that $\inf_{\partial\Omega}\langle \xi, \nu_\Omega\rangle > 0$ where ν_Ω is the exterior unit normal vector field of $\partial\Omega$. If $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\left\{ \begin{array}{ll} v + Lv \leq \lambda & \text{in } \Omega, \\ \langle \xi, \nabla v\rangle \leq 0 & \text{on } \partial\Omega \end{array} \right. \tag{3.11}$$

for some constant λ, then it holds that $v \leq \lambda$ in Ω.

Proof. Since $v \equiv \lambda$ is a solution, the assertion follows from a classical comparison principle for linear equations (see e.g. [35]). □

Remark 3.3. *The comparison principle for the oblique derivative boundary problem is well known even for a viscosity solution as stated in Theorem 2.1. See e.g. [3, 30].*

Lemma 3.3. *Assume that $\partial\Omega$ is uniformly C^2 so that ν_Ω is uniformly C^1. Assume that $w \in C^3(\Omega)$ is C^2 up to the boundary. Assume that w satisfies $B(\cdot, \nabla w(\cdot)) = 0$ on $\partial\Omega$. Assume that β is C^1, $\|\beta\|_\infty < 1$ and $\|\nabla_{\partial\Omega}\beta\|_\infty < \infty$. If B satisfies*

$$\sum_{i=1}^{d} w_i(x) \frac{\partial B}{\partial x_i}(x, \nabla w(x)) \geq 0 \quad \text{for} \quad x \in \partial\Omega,$$

then for $u := \frac{1}{2}|\nabla w|^2$, the C^1 vector field $\xi = \nabla_p B(\cdot, \nabla w(\cdot))$ satisfies

$$\langle \xi, \nabla u \rangle \leq 0 \quad \text{on} \quad \partial\Omega$$

and

$$\langle \xi, \nu_\Omega \rangle \geq 1 - \|\beta\|_\infty > 0 \quad \text{on} \quad \partial\Omega.$$

Proof. First differentiate $B(x, \nabla w(x)) = 0$ in x_i and multiply $w_i(x)$ to get

$$\sum_{\ell=1}^{d} w_i(x) \frac{\partial B}{\partial p_\ell}(x, \nabla w(x)) w_{i\ell}(x) + w_i(x) \frac{\partial B}{\partial x_i}(x, \nabla w(x)) = 0. \tag{3.12}$$

We sum up (3.12) from $i = 1$ to d to get

$$\sum_{\ell=1}^{d} \frac{\partial B}{\partial p_\ell}(x, \nabla w(x)) \frac{\partial u}{\partial x_\ell}(x) + \sum_{i=1}^{d} w_i(x) \frac{\partial B}{\partial x_i}(x, \nabla w(x)) = 0.$$

By our assumption, we now obtain

$$\langle \xi(x), \nabla u(x) \rangle = \nabla_p B(x, \nabla w(x)) \cdot \nabla u(x) = \sum_{\ell=1}^{d} \frac{\partial B}{\partial p_\ell}(x, \nabla w(x)) \frac{\partial u}{\partial x_\ell}(x) \leq 0.$$

Meanwhile, a direct calculation shows

$$\frac{\partial B}{\partial p_\ell}(x, p) = \nu_\ell(x) + \beta(x) \frac{p_\ell}{|p|},$$

where ν_ℓ denotes the ℓ-th element of ν_Ω. We deduce from the Schwarz inequality that

$$\langle \xi(x), \nu_\Omega(x) \rangle = \sum_{\ell=1}^{d} \left(\nu_\ell(x)^2 + \beta(x) \frac{w_\ell(x)\nu_\ell(x)}{|\nabla w(x)|} \right) = 1 + \beta(x) \frac{\langle \nabla w(x), \nu_\Omega(x) \rangle}{|\nabla w(x)|} \geq 1 - \|\beta\|_\infty.$$

We now complete the proof. $\square$

Lemma 3.4. *Assume that $\partial\Omega$ is uniformly C^2. Assume that $\nabla_{\partial\Omega}\beta(x)$ is orthogonal to the kernel of the inward Weingarten map $\nabla_{\partial\Omega}\nu_\Omega(x)$ for each $x \in \partial\Omega$. If $|\nabla_{\partial\Omega}\beta(x)|$ is bounded by the minimal nonnegative principal (inward) curvature $\kappa(x)$ of $\partial\Omega$ at each $x \in \partial\Omega$, then*

$$\sum_{i=1}^{d} w_i(x) \frac{\partial B}{\partial x_i}(x, \nabla w(x)) \geq 0$$

is fulfilled for every $x \in \partial\Omega$.

Proof. Since

$$\frac{\partial B}{\partial x_i}(x,\nabla w(x)) = \left\langle \frac{\partial \nu_\Omega}{\partial x_i}(x), \nabla w(x) \right\rangle + \frac{\partial \beta}{\partial x_i}(x)|\nabla w(x)|,$$

we see that

$$\sum_{i=1}^{d} w_i(x)\frac{\partial B}{\partial x_i}(x,\nabla w(x)) = \sum_{\ell,i=1}^{d} \frac{\partial \nu_\ell}{\partial x_i}(x)w_\ell(x)w_i(x) + \sum_{i=1}^{d} w_i(x)\frac{\partial \beta}{\partial x_i}(x)|\nabla w(x)|.$$

We extend β as constant to the ν_Ω-direction. Since $\nabla_{\partial\Omega}\beta$ is orthogonal to the kernel of the Weingarten map $\nabla_{\partial\Omega}\nu_\Omega$, we proceed

$$\begin{aligned}\sum_{\ell,i=1}^{d} \frac{\partial \nu_\ell}{\partial x_i}(x)w_\ell(x)w_i(x) + \sum_{i=1}^{d} w_i(x)\frac{\partial \beta}{\partial x_i}(x)|\nabla w(x)| &\geq \kappa(x)|\nabla w(x)|^2 - |\nabla w(x)|^2|\nabla_{\partial\Omega}\beta(x)| \\ &= (\kappa(x) - |\nabla_{\partial\Omega}\beta(x)|)|\nabla w(x)|^2 \geq 0.\end{aligned}$$

Our assumption guarantees that the right-hand side is positive. The proof is now complete. □

We are now in the position to prove Proposition 3.1.

Proof of Proposition 3.1. We define $u := \frac{1}{2}|\nabla w|^2$ and argue by Bernstein's method (see e.g. [28, Chapter 15]). We differentiate the first equation of (3.10) in the direction x_k ($1 \leq k \leq d$) and multiply it by $w_k(x)$ to get

$$w_k(x)^2 - h\sum_{i=1}^{d} w_k(x)\partial_i\left(\sum_{j=1}^{d}\phi_{ij}(\nabla w(x))w_{jk}(x)\right) = w_k(x)g_k(x). \tag{3.13}$$

Meanwhile, we calculate

$$\begin{aligned}&\sum_{i=1}^{d} w_k(x)\partial_i\left(\sum_{j=1}^{d}\phi_{ij}(\nabla w(x))w_{jk}(x)\right) \qquad (3.14)\\ &=\sum_{i=1}^{d}\partial_i\left(\sum_{j=1}^{d} w_k(x)\phi_{ij}(\nabla w(x))w_{jk}(x)\right) - \sum_{i,j=1}^{d} w_{ik}(x)\phi_{ij}(\nabla w(x))w_{jk}(x)\\ &\leq \sum_{i=1}^{d}\partial_i\left(\sum_{j=1}^{d}\phi_{ij}(\nabla w(x))w_k(x)w_{jk}(x)\right) = \sum_{i=1}^{d}\partial_i\left\{\sum_{j=1}^{d}\phi_{ij}(\nabla w(x))\partial_j\left(\frac{w_k(x)^2}{2}\right)\right\}.\end{aligned}$$

Here, the inequality in (3.14) follows from the positive definiteness of $(\phi_{ij}(\nabla w(x)))_{1\leq i,j\leq d}$. Summing up (3.13) over $1 \leq k \leq d$ and taking (3.14) into account, we obtain

$$
\begin{aligned}
&2u(x) - h\sum_{i=1}^{d} \partial_i \left(\sum_{j=1}^{d} \phi_{ij}(\nabla w(x)) u_j(x) \right) \\
&\leq \sum_{k=1}^{d} \left\{ w_k(x)^2 - h\sum_{i=1}^{d} w_k(x)\partial_i \left(\sum_{j=1}^{d} \phi_{ij}(\nabla w(x)) w_{jk}(x) \right) \right\} \\
&= \sum_{k=1}^{d} w_k(x) g_k(x) \leq \frac{1}{2}|\nabla w(x)|^2 + \frac{1}{2}|\nabla g(x)|^2 \leq u(x) + \frac{1}{2}\|\nabla g\|_\infty^2.
\end{aligned}
$$

Hence, u satisfies the first equation of (3.11) in the case where $\lambda := \frac{1}{2}\|\nabla g\|_\infty^2$ and $L(u) = -\operatorname{div}(A(x)\nabla u)$ with $A(x) := (\phi_{ij}(\nabla w(x)))_{1\leq i,j\leq d}$. We have already seen that $\langle \xi, \nabla u\rangle \leq 0$ with $\xi := \nabla_p B(\cdot, \nabla w(\cdot))$ in Lemma 3.3 and Lemma 3.4. Therefore, we conclude from Lemma 3.2 that $\|\nabla w\|_\infty \leq \|\nabla g\|_\infty$. □

Remark 3.4. *To guarantee that w is C^2 up to the boundary, we need a regularity assumption on Ω which is slightly more than C^2, say $C^{2,\alpha}$. The $C^{2,1}$ assumption is sufficient to guarantee C^2 regularity for w (see e.g., [28, §6.4]).*

We are now in the position to define a set operator $T_h : \mathcal{P}(\overline{\Omega}) \to \mathcal{P}(\overline{\Omega})$ where $\mathcal{P}(\overline{\Omega})$ denotes the set of all subsets in $\overline{\Omega}$. For each $E \subset \overline{\Omega}$, we set

$$
T_h(E) := \{w_E^h \leq 0\},
$$

where w_E^h is the unique minimizer of $E_h^\beta(u)$. Then, we have two important properties of the set operator T_h as follows.

Lemma 3.5 (Monotonicity of T_h)**.** *Suppose that $E \subset F \subset \overline{\Omega}$. Then, it holds that $T_h(E) \subset T_h(F)$.*

Proof. Since $d_{\Omega,E} \geq d_{\Omega,F}$ in $\overline{\Omega}$, we deduce from Lemma 3.1 that $w_E^h \geq w_F^h$ in $\overline{\Omega}$. This readily yields $T_h(E) \subset T_h(F)$. □

Remark 3.5. *It is crucial to use the geodesic signed distance function $d_{\Omega,E}$ as an initial data g for the variational problem* (3.1)*. Thanks to the monotonicity of $d_{\Omega,E}$ with respect to E (see e.g. [19, Lemma 1]), we do not need to assume the convexity of Ω to obtain the monotonicity of $T_h(E)$.*

We are now in the position to establish the continuity of the scheme T_h:

Lemma 3.6 (Continuity of T_h)**.** *Let $\{E_n\}_n$ be a non-increasing sequence of relatively closed subsets in $\overline{\Omega}$. Then, it holds that*

$$
T_h\left(\bigcap_{n=1}^{\infty} E_n\right) = \bigcap_{n=1}^{\infty} T_h(E_n).
$$

Proof. Let $E := \bigcap_{n=1}^\infty E_n$. Since $\bigcap_{n=1}^\infty T_h(E_n) \supset T_h(E)$ is obvious, we prove the converse inclusion. Suppose that $x \notin T_h(E)$. Then, we have $w_E^h(x) > 0$. Since $d_{\Omega,E_n} \to d_{\Omega,E}$ pointwise as $n \to \infty$ and Ω is bounded, this convergence is uniform in $\overline{\Omega}$. Hence, we deduce from

Proposition 3.1 that $w^h_{E_n} \to w^h_E$ a.e. in Ω as $n \to \infty$. Since $|\nabla d_{\Omega,E_n}|$ is uniformly bounded with respect to $n \in \mathbb{N}$, we see that $\{w^h_{E_n}\}_n$ is equi-continuous by Theorem 3.1 and Remark 3.2. Moreover, the maximum principle guarantees that $w^h_{E_n}$ is uniformly bounded with respect to $n \in \mathbb{N}$. Hence, we can extract a subsequence $\{w^h_{E_{n_k}}\}_k$ by the Ascoli–Arzelà theorem such that $w^h_{E_{n_k}} \to w$ uniformly in $\overline{\Omega}$ for some $w \in UC(\overline{\Omega})$. This w should correspond to w^h_E. Letting $k \to \infty$, we derive $w^h_{E_{n_k}}(x) \to w^h_E(x)$ which implies $w^h_{E_{n_k}}(x) > 0$ holds for sufficiently large $k \in \mathbb{N}$. This leads to $x \notin \bigcap_{n=1}^{\infty} T_h(E_n)$. $\square$

As a conclusion of this section, we characterize the unique minimizer of (3.1) as the projection to the data function onto a closed convex set in $L^2(\Omega)$:

Proposition 3.2. *Let $g \in L^2(\Omega)$. Then, there exists $\overline{z} \in \mathbf{X}_2(\Omega)$ such that*

$$\overline{z} = \operatorname{argmin} \left\{ \|\operatorname{div} z - g\|_2 \;\middle|\; \begin{array}{c} z \in \mathbf{X}_2(\Omega),\ \|z\|_\infty \le 1, \\ [z \cdot \nu] = -\beta\ \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \end{array} \right\}. \tag{3.15}$$

Proof. We take a minimizing sequence $\{z_i\}_i \subset \mathbf{X}_2(\Omega)$ of (3.15). Since $\{z_i\}_i$ is bounded in $L^\infty(\Omega;\mathbb{R}^d)$, up to a subsequence, there exists $\overline{z} \in L^\infty(\Omega;\mathbb{R}^d)$ such that

$$z_i \rightharpoonup \overline{z} \quad \text{weakly-}* \quad \text{in} \quad L^\infty(\Omega;\mathbb{R}^d).$$

Meanwhile, since $\{\operatorname{div} z_i\}_i$ is bounded in $L^2(\Omega)$, there exists $z_{div} \in L^2(\Omega)$ such that

$$\operatorname{div} z_i \rightharpoonup z_{div} \quad \text{weakly} \quad \text{in} \quad L^2(\Omega)$$

by taking a further subsequence. Then, we have $\operatorname{div} \overline{z} = z_{div}$ in $\mathcal{D}'(\Omega)$. Indeed, for any $\varphi \in C_0^\infty(\Omega)$, we deduce

$$\begin{aligned} \int_\Omega z_{div} \varphi \, \mathrm{d}\mathcal{L}^d &= \lim_{i\to\infty} \int_\Omega (\operatorname{div} z_i)\, \varphi \, \mathrm{d}\mathcal{L}^d \\ &= \lim_{i\to\infty} \int_\Omega -z_i \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^d = -\int_\Omega \overline{z} \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^d. \end{aligned} \tag{3.16}$$

Here, the second equality is deduced from [2, Proposition C.4]. Thus, we see $\overline{z} \in \mathbf{X}_2(\Omega)$. Moreover, the lower semi-continuity of $\{z_i\}_i$ in the topology of weakly-$*$ $L^\infty(\Omega;\mathbb{R}^d)$ yields

$$\|\overline{z}\|_\infty \le \liminf_{i\to\infty} \|z_i\|_\infty \le 1.$$

For any $\varphi \in C^\infty(\overline{\Omega})$, we obtain

$$\begin{aligned} \int_\Omega z_{div}\, \varphi \, \mathrm{d}\mathcal{L}^d &= \lim_{i\to\infty} \int_\Omega (\operatorname{div} z_i)\, \varphi \, \mathrm{d}\mathcal{L}^d \\ &= \lim_{i\to\infty} \left(\int_{\partial\Omega} [z_i \cdot \nu]\, \varphi \, d\mathcal{H}^{d-1} - \int_\Omega z_i \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^d \right) \\ &= \lim_{i\to\infty} \left(\int_{\partial\Omega} -\beta\, \varphi \, d\mathcal{H}^{d-1} - \int_\Omega z_i \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^d \right) \\ &= \int_{\partial\Omega} -\beta\, \varphi \, d\mathcal{H}^{d-1} - \int_\Omega \overline{z} \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^d. \end{aligned} \tag{3.17}$$

The left-hand side of (3.17) is also deformed as follows:

$$\int_\Omega z_{div}\,\varphi\;\mathrm{d}\mathcal{L}^d = \int_\Omega (\operatorname{div}\overline{z})\,\varphi\;\mathrm{d}\mathcal{L}^d = \int_{\partial\Omega} [\overline{z}\cdot\nu]\,\varphi\,d\mathcal{H}^{d-1} - \int_\Omega \overline{z}\cdot\nabla\varphi\;\mathrm{d}\mathcal{L}^d. \tag{3.18}$$

Combining (3.17) and (3.18) gives

$$\int_{\partial\Omega} [\overline{z}\cdot\nu]\,\varphi\,d\mathcal{H}^{d-1} = \int_{\partial\Omega} -\beta\,\varphi\,d\mathcal{H}^{d-1}.$$

Since $\varphi \in C^\infty(\overline{\Omega})$ is arbitrary, we see that $[\overline{z}\cdot\nu] = -\beta$ holds $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Therefore, we conclude that $\overline{z}$ is a minimizer of (3.15). □

Proposition 3.3. *Let $g \in L^2(\Omega)$. Suppose that w is a solution of*

$$\frac{u-g}{h} + \partial C_\beta(u) \ni 0. \tag{3.19}$$

Then, $w = g - \pi_{hK_\beta}(g)$ holds where $K_\beta = \partial C_\beta(0)$ and π_{hK_β} is the orthogonal projection to the set hK_β in $L^2(\Omega)$.

Proof. The variational problem (3.19) is equivalent to $w \in \partial C_\beta^*((g-w)/h)$ due to Proposition 2.1. Setting $\overline{w} := (g-w)/h$, the problem is rewritten as

$$0 \in h\overline{w} - g + \partial C_\beta^*(\overline{w}). \tag{3.20}$$

This implies that

$$\overline{w} = \mathrm{argmin}_{\overline{u}\in L^2(\Omega)} \left\{ \frac{\|h\overline{u}-g\|_2^2}{2} + C_\beta^*(\overline{u}) \right\}.$$

Due to Remark 2.1, the above problem is reduced to

$$\overline{w} = \mathrm{argmin}_{\overline{u}\in K_\beta} \|h\overline{u}-g\|_2. \tag{3.21}$$

The formula (3.21) implies that

$$\overline{w} = \mathrm{argmin}_{\overline{u}\in K_\beta} \|h\overline{u}-g\|_2 = \frac{1}{h}\,\mathrm{argmin}_{u\in hK_\beta} \|u-g\|_2 = \frac{1}{h}\pi_{hK_\beta}(g).$$

Recalling $\overline{w} = (g-w)/h$, we conclude that $w = g - \pi_{hK_\beta}(g)$.

□

Remark 3.6. *Similar arguments as in Proposition 3.2 and Proposition 3.3 can be found in [10, §3]. Therein, Chambolle considered the dual problem ((3.20) in our case) instead of the original one ((3.19) in our case) to establish a gradient descent algorithm to compute a time discrete solution to the mean curvature flow.*

4 Convergence of the proposed scheme

In this section, we shall show convergence of the approximate scheme S_h to (1.4). For this purpose, it is crucial to confirm that S_h fulfills the conditions from (1.9) to (1.12).

We now define an approximate scheme S_h for (1.4) with the aid of the capillary Chambolle type scheme T_h as follows:

$$S_h u_0(x) := \sup\{\lambda \in \mathbb{R} \mid x \in T_h(\{u_0 \geq \lambda\})\}. \tag{4.1}$$

In terms of S_h, we define an approximate solution u^h to (1.4) by

$$u^h(t,x) := S_h^{\lfloor \frac{t}{h} \rfloor} u_0(x).$$

We now state a main result of this section as follows:

Theorem 4.1. *Suppose that Ω is convex. Assume that there exist constants $\underline{\beta} < 0$ and $\overline{\beta} > 0$ such that $-1 < \underline{\beta} \leq \beta \leq \overline{\beta} < 1$ on $\partial\Omega$. Moreover, assume that β satisfies the hypotheses of Theorem 3.1. Then, the approximate scheme S_h defined by* (4.1) *satisfies the conditions from* (1.9) *to* (1.12)*.*

Once Theorem 4.1 is established, the statement of Theorem 1.1 immediately follows from the result by Barles and Souganidis:

Theorem 4.2 ([4], Theorem 2.1)**.** *Suppose that $F : (\mathbb{R}^d \backslash \{\mathbf{0}\}) \times \mathbb{S}^d \to \mathbb{R}$ is degenerate elliptic, geometric, continuous, and satisfies $-\infty < F_*(\mathbf{0}, O) = F^*(\mathbf{0}, O) < \infty$. Assume that the approximate scheme S_h fulfills the conditions from* (1.9) *to* (1.12)*. Then, u^h uniformly converges to the unique viscosity solution of* (1.4)*.*

We begin with confirmation that S_h is monotone and translation invariant:

Proposition 4.1 (Monotonicity and translation invariance of S_h)**.** *The function operator S_h satisfies the criteria* (1.9) *and* (1.10)*.*

Proof. Suppose that $u \leq v$ in $\overline{\Omega}$. Then, we see that $\{u \geq \lambda\} \subset \{v \geq \lambda\}$ for every $\lambda \in \mathbb{R}$. Since T_h is monotone from Lemma 3.5, we have $T_h(\{u \geq \lambda\}) \subset T_h(\{v \geq \lambda\})$. Thus, it follows from the definition of S_h that $S_h u \leq S_h v$ in $\overline{\Omega}$. The formula $S_h(u + c) = S_h u + c$ is straightforward by the definition of S_h. □

The following lemma states a relationship between S_h and T_h which will be crucial for our study.

Lemma 4.1. *For every $\lambda \in \mathbb{R}^2$ and $u_0 \in UC(\overline{\Omega})$, $S_h u_0(x) \geq \lambda$ holds if and only if $x \in T_h(\{u_0 \geq \lambda\})$.*

Proof. The assertion is straightforward by Lemma 3.1, Lemma 3.6 and [20, Lemma 4.3]. □

We need a soliton-like rigorous solution to the mean curvature flow with the constant contact angle condition to capture a general flow. A candidate for such a solution is a translative soliton:

Definition 4.1 (Translative soliton)**.** *A function $f:\overline{\Omega}\to\mathbb{R}$ is called a translative soliton if there exist a constant $k\in(-1,1)$ and a function $\Phi_k(x')$ on some subset $\tilde{\Omega}\subset\mathbb{R}^{d-1}$ such that $f(x',x_d)=\Phi_k(x')-x_d$ holds and f solves the following partial differential equation:*

$$\begin{cases} w-h\operatorname{div}\nabla\phi(\nabla w)=d_{\Omega,F} & \text{in } \Omega,\\ \langle\nabla w,\nu_\Omega\rangle+k|\nabla w|=0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $F:=\{(x',x_d)\mid x_d\le\Phi_k(x')+(\arctan\alpha)h\}$ with $\alpha:=\arccos k$.

Remark 4.1. *A translative soliton is often called either a translator or a translating soliton in the literature, and it originally means a rigorous solution to the mean curvature flow with a contact angle condition which can be represented as a graph over an ambient space. We note that the problem* (4.2) *is a discrete variant of level-set equations for the mean curvature flow. The translative soliton which we treat here corresponds to a bowl soliton which is restricted to a cylindrical domain. For a summary of existing works for translators, see e.g. [29, §4].*

To prove Theorem 4.1, we need an assumption and several lemmas:

Lemma 4.2. *For every point $z\in\Omega$ and a vector $\mathbf{v}$, there exists a translative soliton which evolves in the direction either $\mathbf{v}$ or $-\mathbf{v}$ and includes z in its level set.*

Proof. This is straightforward from the result by Zhou [36, Collorary 4.2]. □

If Ω is a smooth bounded domain, then the above lemma can be proved by approximating Ω with a cylindrical domain. We can compute an exact form of $\Phi_\beta(x')$ if $d=2$ and Ω is a cylindrical domain as follows:

Lemma 4.3. *For each $\alpha>0$, we define a function $u_\alpha:[-1,1]\times[0,T]\to\mathbb{R}$ by*

$$u_\alpha(x,t):=\frac{1}{\arctan\alpha}\log|\cos((-\arctan\alpha)x)|+(-\arctan\alpha)t. \tag{4.3}$$

Let $\Omega_b:=[-1,1]\times[-b,b]$ for some large $b>0$. Namely, Ω_b is supposed to be a long cylinder. Set $E_t:=\{(x,y)\in\Omega_b \mid y\le u_\alpha(x,t)\}$ for each $t\ge0$. Then, it holds that $T_h(E_t)=E_{t+h}$ for every $t\ge0$ whenever $-(t+h)/\arctan\alpha\ge-b$ and $\alpha=-\beta/\sqrt{1-\beta^2}$. In other words, the capillary Chambolle type scheme yields the translative soliton.

Proof. Let ν be the unit normal vector field of ∂E_t and suppose that ν is extended to whole Ω by $\nu(x,y):=\nu(x,u_\alpha(x,t))$ for all $y\in[-b,b]$. Then, we define $w:=d_{\Omega,E_t}-h\operatorname{div}\nu=d_{\Omega,E_t}-h\kappa$. We deduce from the assumptions that ν satisfies the conditions on $z\in L^\infty(\Omega;\mathbb{R}^2)$ in [19, Theorem 2]. Therefore, w is a unique minimizer of $E_h^\beta(u)$ with the data E_t. We easily observe that the function u_α defined by (4.3) is an exact solution to (1.1) with $\theta\equiv\arccos\beta$. This implies that evolving E_t in the normal direction by its curvature is equivalent to translate it downward (parallel to the y-axis) at the speed $\arctan\alpha$. Hence, the resulting $T_h(E_t)$ is nothing but E_{t+h}. □

Let us prove a key result to show the consistency of the scheme S_h:

Proposition 4.2. *Let $\varphi\in C^2(\overline{\Omega})$ and $h>0$. Assume that there exist constants $\overline{\beta}>0$ and $\underline{\beta}<0$ such that $-1<\underline{\beta}\le\beta\le\overline{\beta}<1$. For each $\mu\in\mathbb{R}$, we set $E_\mu^\varphi:=\{x\in\overline{\Omega}\mid\varphi(x)\ge\mu\}$. Assume that $\nabla\varphi(z)\neq\mathbf{0}$ for some $z\in\overline{\Omega}$. If either $z\in\Omega$ or $z\in\partial\Omega$ and $\langle\nabla\varphi(z),\nu_\Omega(z)\rangle+\beta(z)|\nabla\varphi(z)|>0$ (resp, $\langle\nabla\varphi(z),\nu_\Omega(z)\rangle+\beta(z)|\nabla\varphi(z)|<0$), then, up to a modification of φ in a neighborhood of z, the problem* (3.10) *with $g:=d_{\Omega,E^\varphi_{\varphi(z)}}$ has a viscosity supersolution $\overline{w}$ (resp, subsolution $\underline{w}$) satisfying the following condition:*

- *For every $\varepsilon > 0$, there exist $\delta > 0$, $h_0 > 0$, $r > 0$ and $C > 0$ such that*

$$\overline{w} \leq d_{\Omega, E^{\varphi}_{\lambda}} - h\kappa_{E^{\varphi}_{\lambda}} + 3\varepsilon h \quad \text{in} \quad U_{\delta,r} \tag{4.4}$$

$$(\text{resp}, \quad \underline{w} \geq d_{\Omega, E^{\varphi}_{\lambda}} - h\kappa_{E^{\varphi}_{\lambda}} - 3\varepsilon h \quad \text{in} \quad U_{\delta,r}) \tag{4.5}$$

for any $h \in (0, h_0)$ and for any $\lambda \in \mathbb{R}$ with $|\varphi(z) - \lambda| \leq C\sqrt{h}$, where

$$U_{\delta,r} := \{x \in \Omega \mid x \in B(z,\delta) \text{ and } |d_{\Omega,E^{\varphi}_{\lambda}}(x)| < r\}.$$

- *In particular, it holds that*

$$\left| w^h_{E^{\varphi}_{\lambda}} - d_{\Omega,E^{\varphi}_{\lambda}} + h\kappa_{E^{\varphi}_{\lambda}} \right| \leq \varepsilon h \quad \text{in} \quad U_{\delta,r}.$$

Proof. The proof is a modification of [20, Proposition 5.2]. First, we treat the case where $z \in \Omega$. We define $s_{\mu,\beta}(x) := \Phi_\beta(x') - x_d + \mu$ for each $x \in \overline{\Omega}$. Since $\nabla\varphi(z) \neq \mathbf{0}$, there exists a $\delta > 0$ for which $\{\varphi = \mu\} \cap B(z, 3\delta)$ is a smooth hypersurface. We introduce a smooth cutoff function $\eta : \Omega \to [0,1]$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if} \quad x \in B(z,\delta), \\ 0 & \text{if} \quad x \in \Omega \backslash \overline{B(z,2\delta)}. \end{cases}$$

Then, we replace φ with $(1-\eta)s_{\mu,\beta} + \eta\varphi$. We still write it as φ for simplicity. We observe that $s_{\mu+h,\underline{\beta}}$ is a classical supersolution to (3.10) with $g := d_{\Omega,E^{\varphi}_{\mu}}$. Indeed, we have

$$s_{\mu+h,\underline{\beta}} - h \operatorname{div} \nabla\phi(\nabla s_{\mu+h,\underline{\beta}}) = d_{\Omega,E^{s_{\mu+h,\underline{\beta}}}_{\mu}} \geq d_{\Omega,E^{\varphi}_{\mu}} \quad \text{in} \quad \Omega.$$

Here, we note that $E^{s_{\mu+h,\underline{\beta}}}_{\mu} \subset E^{\varphi}_{\mu}$ hence $d_{\Omega,E^{s_{\mu+h,\underline{\beta}}}_{\mu}} \geq d_{\Omega,E^{\varphi}_{\mu}}$. Moreover, we derive

$$\left\langle \nabla s_{\mu+h,\underline{\beta}}, \nu_\Omega \right\rangle + \beta |\nabla s_{\mu+h,\underline{\beta}}| \geq \left\langle \nabla s_{\mu+h,\underline{\beta}}, \nu_\Omega \right\rangle + \underline{\beta} |\nabla s_{\mu+h,\underline{\beta}}| = 0 \quad \text{on} \quad \partial\Omega.$$

We shall construct a viscosity supersolution to (3.10) in a neighborhood of z. To this end, we introduce a smooth cutoff function $\tilde{\eta} : \Omega \to [0,1]$ satisfying:

$$\tilde{\eta}(x) = \begin{cases} 1 & \text{if} \quad |d_{\Omega,E^{\varphi}_{\mu}}(x)| \leq r, \\ 0 & \text{if} \quad |d_{\Omega,E^{\varphi}_{\mu}}(x)| \geq 2r, \end{cases} \qquad \|\nabla\tilde{\eta}\|_\infty + \|\nabla^2\tilde{\eta}\|_\infty \leq L,$$

where $L > 0$ is independent of ε, h, and r. Moreover, suppose that $\tau(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Then, we define

$$\tilde{w} := d_{\Omega,E^{\varphi}_{\mu}} - h\tilde{\eta}\kappa_{\tau(\varepsilon)} + (1-\tilde{\eta})h + 2\varepsilon h.$$

Here, we have used the notation that $\kappa_\tau := \rho_\tau * \kappa_{E^{\varphi}_{\mu}}$ with the standard mollifying kernel ρ_τ. Take $\tau(\varepsilon)$ so small that $\|\kappa_{\tau(\varepsilon)} - \kappa_{E^{\varphi}_{\mu}}\|_{C(E^{\varphi}_{\mu})} < \varepsilon$. Then, as discussed in [20, Proposition 5.2], the function $\tilde{w}$ turns out to be a classical supersolution of (3.10) with $g := d_{\Omega,E^{\varphi}_{\mu}}$ in $U_{\delta,r}$. In terms of $s_{\mu+h,\underline{\beta}}$ and $\tilde{w}$, we define

$$\overline{w} := \begin{cases} \min\{\tilde{w}, s_{\mu+h,\underline{\beta}}\} & \text{in} \quad \overline{U_{\delta,r}}, \\ s_{\mu+h,\underline{\beta}} & \text{in} \quad \overline{\Omega} \backslash U_{\delta,r}. \end{cases}$$

Since viscosity supersolutions are closed under taking minimum, we see that $\overline{w}$ is also a viscosity supersolution of (3.10) with $g := d_{\Omega,E_\mu^\varphi}$. Thus, we deduce that

$$\overline{w} \le \tilde{w} = d_{\Omega,E_\mu^\varphi} - h\kappa_{\tau(\varepsilon)} + 2\varepsilon h \le d_{\Omega,E_\mu^\varphi} - h\kappa_{E_\mu^\varphi} + 3\varepsilon h \ \text{ in } \ U_{\delta,r}.$$

Here, we should take $\delta > 0$ so small that $B(z,\delta) \subset \{|d_{E_\mu^\varphi}| < r\}$ if necessary. Consequently, we derive the desired $\overline{w}$. The comparison principle for viscosity solutions implies $w^h_{E_\mu^\varphi} \le \overline{w}$ and hence

$$w^h_{E_\mu^\varphi} \le \tilde{w} = d_{\Omega,E_\mu^\varphi} - h\kappa_{\tau(\varepsilon)} + 2\varepsilon h \le d_{\Omega,E_\mu^\varphi} - h\kappa_{E_\mu^\varphi} + 3\varepsilon h \ \text{ in } \ U_{\delta,r}.$$

If $z \in \partial\Omega$, φ is already a supersolution of the second equality of (3.10). Moreover, we see that the hypersurface $\{\varphi = \mu\} \cap B(z, 3\delta)$ intersects $\partial\Omega$ with the angle larger than $\arccos\beta$. Thus, we can find a translative soliton whose level set is included in E_μ^φ in a neighborhood of z. Hence, the whole argument for $z \in \Omega$ will work. A desired viscosity subsolution can be obtained in the same manner. We conclude the proof. $\square$

Remark 4.2. *Let us mention difference from the researches [20, 21] regarding construction of a sub- and supersolution which approximate a solution of* (3.10) *in the case where* $\Omega = \mathbb{R}^d$. *Note that the boundary condition in* (3.10) *vanishes due to* $\partial\Omega$ *is an empty set. Therein, the hypersurface* $\{\varphi = \mu\}$ *was approximated with the help of an open bounded set* $V_1 \subset \mathbb{R}^d$ *whose boundary is tangent to* $\{\varphi = \mu\}$. *Then, the function* d_{Ω,V_1} *was bounded by rigorous solutions of* (3.10) *with* $\Omega := \mathbb{R}^d$, $g := d_{\Omega,B}$ *and a ball* B. *This solution was explicitly computed in [8, §B]. However, this result is not available in our case due to the boundary condition. We are forced to modify a test function* φ *to cope the boundary condition. But, we should notice that the definition of viscosity solutions only uses local information of* φ. *Thus, this modification does not affect the following discussion.*

The following lemma establishes a kind of monotonicity of the scheme S_h with respect to the contact angle function β:

Lemma 4.4. *Suppose that* $\beta_1 : \partial\Omega \to [-1, 1]$ *and* $\beta_2 : \partial\Omega \to [-1, 1]$ *satisfy* $\beta_1 \le \beta_2$ *on* $\partial\Omega$. *Let* $S_{h,b}$ *be the associated function operator which is induced from the solution to* (3.10) *with* $\beta := b$ *and* $g = d_{\Omega,E}$ *for each function* $b : \partial\Omega \to [-1, 1]$. *Then, it holds that*

$$S_{h,\beta_2}\varphi \le S_{h,\beta_1}\varphi \qquad in \qquad \overline{\Omega} \tag{4.6}$$

for any function $\varphi \in C(\overline{\Omega})$.

Proof. For each $b : \partial\Omega \to [-1, 1]$, let $T_{h,b}(E) := \{w^h_{E,b} \le 0\}$ where $w^h_{E,b}$ the unique solution to (3.10) with $\beta := b$ and $g = d_{\Omega,E}$. Then, we observe that w^h_{E,β_1} is a viscosity subsolution of (3.10) with $\beta := \beta_2$. Hence, we deduce from the comparison principle that $w^h_{E,\beta_1} \le w^h_{E,\beta_2}$ in $\overline{\Omega}$. This estimate implies that $T_{h,\beta_2}(E) \subset T_{h,\beta_1}(E)$. Therefore, by the definition of S_h, it follows that $S_{h,\beta_2}\varphi \le S_{h,\beta_1}\varphi$. $\square$

We are now in the position to prove the consistency of the scheme S_h.

Theorem 4.3 (Consistency of S_h)**.** *Let $\varphi \in C^2(\overline{\Omega})$. Suppose that Ω and β satisfy the criteria of Proposition 4.2 and Theorem 3.1. Then, it holds that*

$$\limsup_{h\to 0}{}^{*} \frac{S_h\varphi(z) - \varphi(z)}{h} \leq -F_*(\nabla\varphi(z), \nabla^2\varphi(z)) \tag{4.7}$$

$$(resp, \quad \operatorname{liminf}_{*h\to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} \geq -F^*(\nabla\varphi(z), \nabla^2\varphi(z))) \tag{4.8}$$

whenever $z \in \overline{\Omega}$ satisfies one of the following conditions:

- *$z \in \Omega$ and either $\nabla\varphi(z) \neq 0$ or $\nabla\varphi(z) = 0$ and $\nabla^2\varphi(z) = O$.*
- *$z \in \partial\Omega$, $\nabla\varphi(z) \neq 0$ and $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| > 0$ (resp, $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| < 0$).*

Proof. Set $\mu := \varphi(z)$. Fix any $\varphi \in C^2(\overline{\Omega})$ and any $\varepsilon > 0$.

[**Case** $z \in \Omega$ **and** $\nabla\varphi(z) \neq \mathbf{0}$]
Then, we deduce from Lemma 4.2 that there exist a smooth function $\tilde{\varphi}$ and a positive constant δ such that the estimate

$$|w^h_{E^{\tilde{\varphi}}_\lambda} - d_{\Omega, E^{\tilde{\varphi}}_\lambda} + h\kappa_{E^{\tilde{\varphi}}_\lambda}| \leq \varepsilon h \quad \text{in} \quad U_{\delta,r} \tag{4.9}$$

holds for sufficiently small $h > 0$ and $r > 0$ and $\lambda \in \mathbb{R}$ with $|\mu - \lambda| \leq C\sqrt{h}$ with $C := \sqrt{2}|\nabla\varphi(z)|$ and $\tilde{\varphi} = \varphi$ in $U_{\delta,r}$. For simplicity, we still write $\tilde{\varphi}$ as φ. We now define

$$\lambda_h^\pm := \varphi(z_h^\pm) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) + \varepsilon\}h,$$

where

$$z_h^\pm := z \pm \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\sqrt{2h}.$$

Then, we shall show that $S_h\varphi(z_h^\pm) \leq \lambda_h^\pm$ for sufficiently small $h > 0$. We note that this statement is equivalent to $z_h^\pm \notin T_h(E^\varphi_{\lambda_h^\pm})$ by Lemma 4.1. First, we prove that $S_h\varphi(z_h^-) \leq \lambda_h^-$. We use (4.9) with $\mu := \lambda_h^-$ to derive

$$w^h_{E^\varphi_{\lambda_h^-}}(z_h^-) \geq d_{\Omega, E^\varphi_{\lambda_h^-}}(z_h^-) - h\kappa_{E^\varphi_{\lambda_h^-}}(z_h^-) - \varepsilon h \geq d_{\Omega, E^\varphi_{\lambda_h^-}}(z_h^-) - (K + \varepsilon)h, \tag{4.10}$$

where $K := \sup_{0 \leq h \leq 1} \|\kappa_{E^\varphi_{\lambda_h^-}}\|_{C(\overline{U_{\delta,r}})}$. Since $d_{\Omega, E^\varphi_{\lambda_h^-}}$ is smooth, we have

$$d_{\Omega, E^\varphi_{\lambda_h^-}}(z_h^-) = d_{\Omega, E^\varphi_{\lambda_h^-}}(z) - \left\langle \nabla d_{\Omega, E^\varphi_{\lambda_h^-}}(\widetilde{z_h^-}), \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} \right\rangle \sqrt{2h}, \tag{4.11}$$

where $\widetilde{z_h^-} = z - \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\widetilde{h}$ for some $\widetilde{h} \in (0, \sqrt{2h})$. We deduce from the geometry (see Figure 1) that

$$d_{\Omega, E^\varphi_{\lambda_h^-}}(z) \to 0 \qquad \text{and} \qquad \left\langle \nabla d_{\Omega, E^\varphi_{\lambda_h^-}}(\widetilde{z_h^-}), \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} \right\rangle \to -1 \tag{4.12}$$

as $h \to 0$. Here, we have recalled that $|\nabla d_{\Omega, E^\varphi_{\lambda_h^-}}| = 1$ to derive the second convergence of (4.12).

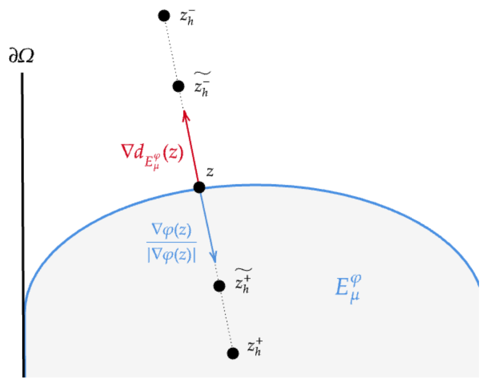

Figure 1: The location of important points associated with $z \in \Omega$.

Combining (4.10), (4.11) and (4.12), we conclude that $w^h_{E^\varphi_{\lambda_h^-}}(z_h^-) > 0$ for sufficiently small $h > 0$. Thus, we obtain that $z_h^- \notin T_h(E^\varphi_{\lambda_h^-})$.

Second, we show that $S_h\varphi(z_h^+) \leq \lambda_h^+$. Comparing the super level sets E^φ_μ and $E_\mu^{s_{\mu+\frac{\underline{C}h}{2},\underline{\beta}}}$ (see Figure 2) and applying Lemma 4.4, we compute

$$S_h\varphi \leq S_h\left(s_{\mu+\frac{\underline{C}h}{2},\underline{\beta}}\right) \leq S_{h,\underline{\beta}}\left(s_{\mu+\frac{\underline{C}h}{2},\underline{\beta}}\right) = s_{\mu-\frac{\underline{C}h}{2},\underline{\beta}} \quad \text{in} \quad \overline{\Omega}, \tag{4.13}$$

where

$$\underline{C} := \arctan\left(\frac{-\underline{\beta}}{\sqrt{1-\underline{\beta}^2}}\right).$$

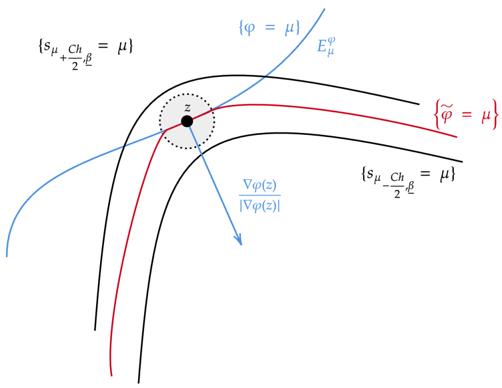

Figure 2: The boundaries of the super level sets.

Evaluating (4.13) at z_h^+ yields

$$S_h\varphi(z_h^+) \leq s_{\mu-\frac{\underline{C}h}{2},\underline{\beta}}(z_h^+) = \mu + |\nabla\varphi(z)|\frac{\underline{C}h}{2} \leq \mu + |\nabla\varphi(z)|\sqrt{2h} + h\Delta\varphi(z) \tag{4.14}$$

for sufficiently small $h > 0$. Here, to derive the last inequality of (4.14), we note that for every $C_1 \in \mathbb{R}$ and $C_2 > 0$, $C_1 h < C_2\sqrt{h}$ holds for sufficiently small $h > 0$ (we may set

$C_1 := |\nabla\varphi(z)|\frac{C}{2} - \Delta\varphi(z)$ and $C_2 := \sqrt{2}|\nabla\varphi(z)|$). Meanwhile, the Taylor expansion gives

$$\varphi(z_h^+) = \mu + |\nabla\varphi(z)|\sqrt{2h} + \left\langle \nabla^2\varphi(\widetilde{z_h^+})\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} \right\rangle h, \tag{4.15}$$

where $\widetilde{z_h^+} = z + \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\widetilde{h}$ for some $\widetilde{h} \in (0, \sqrt{2h})$. Combining (4.14) and (4.15), we derive

$$S_h\varphi(z_h^+) \le \varphi(z_h^+) + \left\{ \Delta\varphi(z) - \left\langle \nabla^2\varphi(\widetilde{z_h^+})\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} \right\rangle \right\} h.$$

Noting that

$$-F(\nabla\varphi(z), \nabla^2\varphi(z)) = \Delta\varphi(z) - \left\langle \nabla^2\varphi(z)\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} \right\rangle$$

and that $\nabla^2\varphi$ is continuous, we can take $h > 0$ so small that

$$S_h\varphi(z_h^+) \le \varphi(z_h^+) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) + \varepsilon\}h.$$

The similar argument yields

$$S_h\varphi(z_h^\pm) \ge \varphi(z_h^\pm) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) - \varepsilon\}h$$

for sufficiently small $h > 0$. We complete the proof for this case.

[**Case** $z \in \partial\Omega$ **and** $\nabla\varphi(z) \neq \mathbf{0}$]
In this case, we have to consider a sequence $\{z_h\}_h$ on $\partial\Omega$ which converges to z. To this end, for each $z \in \partial\Omega$, we set

$$\tau_\Omega(z) := \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|} - \Phi(z)\nu_\Omega(z) \qquad \text{with} \qquad \Phi(z) := \left\langle \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \nu_\Omega(z) \right\rangle.$$

Note that $\tau_\Omega(z)$ is nothing but the projection of the vector $\nu(z)$ onto $\partial\Omega$. Then, we define

$$z_h^\pm := z \pm \tau_\Omega(z)\sqrt{2h}.$$

First, suppose that $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| > 0$. Then, we have

$$\Phi(z) > -\beta(z) \ge -\overline{\beta}.$$

Then, geometrically speaking, it holds that either the graph of φ is bounded by $s_{\mu,-\overline{\beta}}$ from above or that it is bounded from $s_{\mu,\overline{\beta}}$ below. See Figure 3 to grasp the situation.

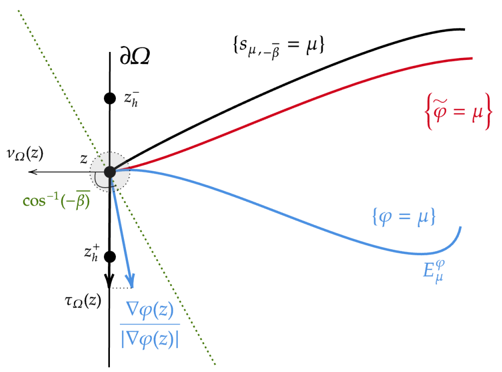

Figure 3: The graphs of a translating soliton and the level set of φ.

We assume that the former statement is valid. Then, comparing the level sets of φ and $s_{\mu,-\overline{\beta}}$, we deduce that

$$s_{\mu,-\overline{\beta}} \geq \varphi \qquad \text{in} \qquad \overline{\Omega}. \tag{4.16}$$

Applying S_h both sides of (4.16) and the monotonicity of S_h together with Lemma 4.4 yield

$$s_{\mu-\overline{C}h,-\overline{\beta}} = S_{h,-\overline{\beta}}(s_{\mu,-\overline{\beta}}) \geq S_h(s_{\mu,-\overline{\beta}}) \geq S_h\varphi, \tag{4.17}$$

where

$$\overline{C} := \arctan\left(\frac{\overline{\beta}}{\sqrt{1-\overline{\beta}^2}}\right).$$

We evaluate the equation (4.17) at z_h^+ to obtain

$$\begin{aligned}
S_h\varphi(z_h^+) &\leq s_{\mu-\overline{C}h-\overline{\beta}}(z_h^+) = \varphi(z) + \overline{C}h\,\langle \tau_\Omega(z), \nabla\varphi(z)\rangle \\
&\leq \mu + \langle \tau_\Omega(z), \nabla\varphi(z)\rangle\sqrt{2h} + h\Delta\varphi(z) \\
&\quad + 2h\Phi(z)\left\langle \nabla^2\varphi(z)\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \nu_\Omega(z)\right\rangle - h\Phi(z)^2\left\langle \nabla^2\varphi(z)\nu_\Omega(z), \nu_\Omega(z)\right\rangle,
\end{aligned} \tag{4.18}$$

where $h > 0$ is taken small enough, and we have used that $\langle \tau_\Omega(z), \nabla\varphi(z)\rangle > 0$. Meanwhile, the Taylor expansion shows

$$\varphi(z_h^+) = \mu + \langle \tau_\Omega(z), \nabla\varphi(z)\rangle\sqrt{2h} + \left\langle \nabla^2\varphi(\widetilde{z_h^+})\tau_\Omega(z), \tau_\Omega(z)\right\rangle h \tag{4.19}$$

and

$$\begin{aligned}
\left\langle \nabla^2\varphi(\widetilde{z_h^+})\tau_\Omega(z), \tau_\Omega(z)\right\rangle &= \left\langle \nabla^2\varphi(\widetilde{z_h^+})\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\right\rangle \\
&\quad + 2\Phi(z)\left\langle \nabla^2\varphi(\widetilde{z_h^+})\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \nu_\Omega(z)\right\rangle \\
&\quad - \Phi(z)^2\left\langle \nabla^2\varphi(\widetilde{z_h^+})\nu_\Omega(z), \nu_\Omega(z)\right\rangle,
\end{aligned} \tag{4.20}$$

where $\widetilde{z_h^+} = z + \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\widetilde{h}$ with $\widetilde{h} \in (0, \sqrt{2h})$. Since $\nabla^2\varphi(z)$ is continuous, we deduce from (4.18), (4.19) and (4.20) that

$$\begin{aligned}
S_h\varphi(z_h^+) &\leq \varphi(z_h^+) + \left\{\Delta\varphi(z) - \left\langle \nabla^2\varphi(\widetilde{z_h^+})\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}\right\rangle\right\} h \\
&\quad + 2\Phi(z)\left\langle \left\{\nabla^2\varphi(\widetilde{z_h^+}) - \nabla^2\varphi(z)\right\}\frac{\nabla\varphi(z)}{|\nabla\varphi(z)|}, \nu_\Omega(z)\right\rangle h \\
&\quad + \Phi(z)^2\left\langle \left\{\nabla^2\varphi(\widetilde{z_h^+}) - \nabla^2\varphi(z)\right\}\nu_\Omega(z), \nu_\Omega(z)\right\rangle h \\
&\leq \varphi(z_h^+) + \left\{-F(\nabla\varphi(z), \nabla^2\varphi(z)) + \varepsilon\right\}
\end{aligned}$$

for sufficiently small $h > 0$. Let us estimate $S_h\varphi$ at z_h^-. Fix any $\varepsilon > 0$, we set $\lambda_h^- := \varphi(z_h^-) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) + \varepsilon\}h$. Then, it suffices to prove that $w^h_{E^\varphi_{\lambda_h^-}}(z_h^-) > 0$ for $h > 0$

small enough. We deduce from Proposition 4.2 that

$$\begin{aligned} w^h_{E^\varphi_{\lambda_h^-}}(z_h^-) &\geq d_{\Omega,E^\varphi_{\lambda_h^-}}(z_h^-) - h\kappa_{E^\varphi_{\lambda_h^-}}(z_h^-) - \varepsilon h \\ &\geq d_{\Omega,E^\varphi_{\lambda_h^-}}(z) - \left\langle \tau_\Omega(z), \nabla d_{\Omega,E^\varphi_{\lambda_h^-}}(z) \right\rangle \sqrt{2h} - h(K+\varepsilon). \end{aligned} \tag{4.21}$$

Here, we note that the coefficient of $\sqrt{2h}$ always positive by geometry. Letting $h > 0$ small enough, we see that the left-hand side of (4.21) is positive, which means

$$S_h\varphi(z_h^-) \leq \lambda_h^- = \varphi(z_h^-) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) + \varepsilon\}h.$$

In the case where the graph of φ is bounded by $s_{\mu,\overline{\beta}}$ from below, the previous arguments still work, replacing $-\overline{\beta}$ and $-\overline{C}$ with $\overline{\beta}$ and $\overline{C}$ respectively. Therefore, we conclude that the estimate (4.7) is valid whenever $z \in \partial\Omega$.

Though, the estimate (4.8) can be deduced by the similar argument, we shall show it for completeness. Suppose that $\langle \nabla\varphi(z), \nu_\Omega(z)\rangle + \beta(z)|\nabla\varphi(z)| < 0$. Then, we see that

$$\Phi(z) < -\beta(z) \leq -\underline{\beta}. \tag{4.22}$$

As discussed before, it holds that either the graph of φ is bounded by $s_{\mu,\underline{\beta}}$ from above or that it is bounded by $s_{\mu,-\underline{\beta}}$ from below. We only deal with the former case. For z_h^+, we deduce from Proposition 4.2 and the Taylor expansion that

$$\begin{aligned} w^h_{E^\varphi_{\lambda_h^+}} &\leq d_{\Omega,E^\varphi_{\lambda_h^+}}(z_h^+) + h\kappa_{E^\varphi_{\lambda_h^+}} + \varepsilon h \\ &\leq d_{\Omega,E^\varphi_{\lambda_h^+}}(z) + \left\langle \tau_\Omega(z), \nabla d_{\Omega,E^\varphi_{\lambda_h^+}}(\widetilde{z_h^+}) \right\rangle \sqrt{2h} + h(K+\varepsilon), \end{aligned} \tag{4.23}$$

where $\widetilde{z_h^+} = z + \tau_\Omega(z)\widetilde{h}$ for some $\widetilde{h} \in (0, \sqrt{2h})$. Note that the coefficient of $\sqrt{2h}$ in (4.23) is always negative for sufficiently small $h > 0$. Thus, we see that $z_h^+ \in T_h(E^\varphi_{\lambda_h^+})$. In other words, we obtain

$$S_h\varphi(z_h^+) \geq \lambda_h^+ = \varphi(z_h^+) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) - \varepsilon\}h.$$

We deduce from geometry that

$$s_{\mu,\underline{\beta}} \leq \varphi \qquad \text{in} \qquad \overline{\Omega}. \tag{4.24}$$

Applying S_h to both sides of (4.24) and the monotonicity of S_h show

$$s_{\mu-\underline{C}h,\underline{\beta}} = S_h(s_{\mu,\underline{\beta}}) \leq S_h\varphi. \tag{4.25}$$

Evaluating (4.25) at z_h^-, we derive

$$\mu - \langle \tau_\Omega(z), \nabla\varphi(z)\rangle \sqrt{2h} \leq \mu - \underline{C}h \langle \tau_\Omega(z), \nabla\varphi(z)\rangle = s_{\mu-\underline{C}h,\underline{\beta}}(z_h^-) \leq S_h\varphi(z_h^-) \tag{4.26}$$

for sufficiently small $h > 0$. Here, we have used $\langle \tau_\Omega(z), \nabla\varphi(z)\rangle > 0$ by geometry. Hence, we again apply the Taylor expansion to the left-hand side of (4.26) to obtain

$$S_h\varphi(z_h^-) \geq \varphi(z_h^-) - \left\langle \nabla^2\varphi(\widetilde{z_h^-})\tau_\Omega(z), \tau_\Omega(z) \right\rangle h,$$

where $\widetilde{z_h^-} = z - \tau_\Omega(z)\widetilde{h}$ for some $\widetilde{h} \in (0, \sqrt{2h})$. In the same argument in the previous case, we deduce that

$$S_h\varphi(z_h^-) \geq \varphi(z_h^-) + \{-F(\nabla\varphi(z), \nabla^2\varphi(z)) - \varepsilon\}h$$

for sufficiently small $h > 0$.

[**Case** $\nabla\varphi(z) = \mathbf{0}$ **and** $\nabla^2\varphi(z) = O$]
In this case, we note that $F^*(\mathbf{0}, O) = F_*(\mathbf{0}, O) = 0$ (see e.g. [23, Lemma 1.6.16]). Thus, our aim is to prove that

$$\operatorname{liminf}_{*h\to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} = \operatorname*{limsup^*}_{h\to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} = 0. \tag{4.27}$$

Fix any $\varepsilon > 0$ and take $h > 0$ so small that $h^2 < \varepsilon h$. We may assume that φ equals a constant $\mu \in \mathbb{R}$ in $B(z, \varepsilon h)$ by taking $h > 0$ much smaller if necessary. For each $\boldsymbol{v} \in \mathbb{R}^d$ with $|\boldsymbol{v}| = 1$, we define

$$z_h^\pm := z \pm h^2\boldsymbol{v}.$$

By Lemma 4.2, we can choose a translative soliton which moves to the direction of $\boldsymbol{v}$. Then, we easily observe (see Figure 4) that

$$s_{\mu-\varepsilon h-\overline{C}h,\overline{\beta}} \leq \varphi \leq s_{\mu+\varepsilon h+\underline{C}h,\underline{\beta}} \qquad \text{in} \qquad B(z, \varepsilon h). \tag{4.28}$$

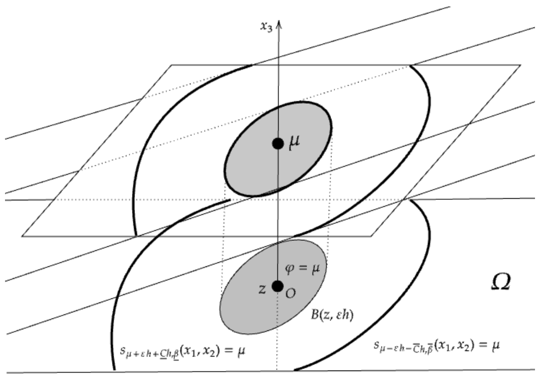

Figure 4: Level sets of $s_{\mu-\varepsilon h-\overline{C}h,\overline{\beta}}$, $s_{\mu+\varepsilon h+\underline{C}h,\underline{\beta}}$ and φ.

The estimate (4.28) holds in $\overline{\Omega}$ under modification of φ outside $B(z, \varepsilon h)$. Hence, applying S_h to (4.28) yields

$$s_{\mu-\varepsilon h,\overline{\beta}} = S_{h,\overline{\beta}}(s_{\mu-\varepsilon h-\overline{C}h,\overline{\beta}}) \leq S_h(s_{\mu-\varepsilon h-\overline{C}h,\overline{\beta}}) \leq S_h\varphi, \tag{4.29}$$

$$S_h\varphi \leq S_h(s_{\mu+\varepsilon h+\underline{C}h,\underline{\beta}}) \leq S_{h,\underline{\beta}}(s_{\mu+\varepsilon h+\underline{C}h,\underline{\beta}}) = s_{\mu+\varepsilon h,\underline{\beta}}. \tag{4.30}$$

We now evaluate (4.29) and (4.30) at z_h^+ to get

$$\varphi(z_h^+) - \varepsilon h = \mu - \varepsilon h \leq s_{\mu-\varepsilon h,\overline{\beta}}(z_h^+) \leq S_h\varphi(z_h^+),$$
$$S_h\varphi(z_h^+) \leq s_{\mu+\varepsilon h,\underline{\beta}}(z_h^+) \leq \mu + \varepsilon h = \varphi(z_h^+) + \varepsilon h.$$

Here, we note that $\partial_{x_d} s_{\mu,k} = -1$ for every $\mu \in \mathbb{R}$ and $k \in (-1, 1)$ and that $z_h^+ \in B(z, \varepsilon h)$. The same argument works even if z_h^+ is replaced with z_h^-. Since the vector $\boldsymbol{v}$ can be chosen arbitrarily, we obtain the equality (4.27). $\square$

Remark 4.3. *We note that the convexity of Ω is used only to derive the equi-continuity of w_E^h with respect to $h > 0$; it yields the continuity of T_h and hence Lemma 4.1 follows. Therefore, we might not need the convexity of Ω to establish the consistency of S_h.*

5 Conclusion

In this paper, we have confirmed that the capillary Chambolle type scheme, which was proposed in [19], is convergent under several assumptions on the domain Ω and the contact angle function β. To this end, it is crucial to derive a generator of the function operator S_h due to Barles and Souganidis [4]. For this, we have established the equi-continuity of the minimizers w_E^h which leads to an important relation between S_h and T_h and have shown that the translative soliton can be mapped to another translative soliton by S_h and T_h. In the course of acquisition of the generator, we frequently use the comparison principle of viscosity solutions and compare the hypersurface Γ_t induced from a test function φ with translative solitons which bound Γ_t from above and from below, respectively. Finally, let us give a concluding remark. When we show that the scheme is convergent, we have assumed that $\|\beta\|_\infty$ is less than 1. In other words, the hypersurface Γ_t must not be tangent to $\partial\Omega$. However, we expect that $\|\beta\|_\infty$ might be allowed to equal 1 by approximation of equations for β with $\|\beta\|_\infty < 1$.

6 Acknowledgements

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