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EMBEDDING ALEXANDER QUANDLES INTO GROUPS

TOSHIYUKI AKITA

ABSTRACT. For any twisted conjugate quandle Q , and in particular any Alexander quandle, there exists a group G such that Q is embedded into the conjugation quandle of G .

1. EMBEDDABLE QUANDLES

A non-empty set Q equipped with a binary operation $Q \times Q \rightarrow Q$, $(x, y) \mapsto x * y$ is called a *quandle* if it satisfies the following three axioms:

- (1) $x * x = x$ ($x \in Q$),
- (2) $(x * y) * z = (x * z) * (y * z)$ ($x, y, z \in Q$),
- (3) For all $x \in Q$, the map $S_x: Q \rightarrow Q$ defined by $y \mapsto y * x$ is bijective.

Quandles were introduced independently by Joyce [7] and Matveev [9]. Since then, quandles have been important objects in the study of knots and links, set-theoretical solutions of the Yang-Baxter equation, Hopf algebras and many others. We refer to Nosaka [10] for further details of quandles.

A map $f: Q \rightarrow Q'$ of quandles is called a *quandle homomorphism* if it satisfies $f(x * y) = f(x) * f(y)$ ($x, y \in Q$). Given a group G , the set G equipped with a quandle operation $h * g := g^{-1}hg$ is called the *conjugation quandle* of G and is denoted by $\text{Conj}(G)$. A quandle Q is called *embeddable* if there exists a group G and an injective quandle homomorphism $Q \rightarrow \text{Conj}(G)$. Not all quandles are embeddable (see the bottom of §2).

In their paper [2], Bardakov-Dey-Singh proposed the question “For which quandles X does there exist a group G such that X embeds in the conjugation quandle $\text{Conj}(G)$?” [2, Question 3.1], and proved that Alexander quandles associated with fixed-point free involutions are embeddable [2, Proposition 3.2]. The following is a list of embeddable quandles of which the author is aware: (1) free quandles and free n -quandles (Joyce [7, Theorem 4.1 and Corollary 10.3]), (2) commutative quandles, latin quandles and simple quandles (Bardakov-Nasybullov [3, §5]), (3) core quandles (Bergman [4, (6.5)]), (4) generalized Alexander quandles associated with fixed-point free automorphisms (Dhanwani-Raundal-Singh [6, Proposition 3.12]), and (5) free c -nilpotent quandles (Darné [5, Proposition 2.18]).

In this short note, we will show that twisted conjugation quandles, which include all Alexander quandles, are embeddable, thereby generalize the aforementioned result of Bardakov-Dey-Singh.

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2. EMBEDDINGS OF TWISTED CONJUGATION QUANDLES

Let G be an additive abelian group and let $\phi: G \rightarrow G$ be a group automorphism of G . The *Alexander quandle* $\text{Alex}(G, \phi)$ associated with ϕ is the set G equipped with the quandle operation

$$g * h := \phi(g) + h - \phi(h).$$

Let G be a group and let $\phi: G \rightarrow G$ be an automorphism of G . The *twisted conjugation quandle* $\text{Conj}(G, \phi)$ associated with ϕ is the set G equipped with the quandle operation

$$g * h := \phi(h^{-1}g)h.$$

Observe that an Alexander quandle $\text{Alex}(G, \phi)$ is precisely a twisted conjugation quandle $\text{Conj}(G, \phi)$ whose underlying group G is abelian. Twisted conjugation quandles appeared in Andruskiewitsch-Graña [1, §1.3.7] under the name *twisted homogeneous crossed sets*. We prefer the name twisted conjugation quandles because $\text{Conj}(G, \phi) = \text{Conj}(G)$ if ϕ is the identity map. It should be emphasized that $\text{Conj}(G, \phi)$ is different from the *generalized Alexander quandle* associated with (G, ϕ) . The latter has the same underlying set G , but with the different quandle operation $g * h := \phi(gh^{-1})h$. Now we prove that $\text{Conj}(G, \phi)$ is embeddable:

Theorem. *Any twisted conjugation quandle is embeddable. In particular, any Alexander quandle is embeddable.*

Proof. Given a twisted conjugation quandle $\text{Conj}(G, \phi)$, we will construct an explicit embedding $\text{Conj}(G, \phi) \rightarrow \text{Conj}(H)$. Let \mathbb{Z} be the additive group of integers, and let $H := G \rtimes_{\phi} \mathbb{Z}$ be the semidirect product of G and \mathbb{Z} associated with ϕ . Namely, H equals to $G \times \mathbb{Z}$ as sets. The group law on H is given by

$$(g, m) \cdot (h, n) := (\phi^n(g)h, m + n).$$

The inverse of $(g, m) \in H$ is

$$(g, m)^{-1} = (\phi^{-m}(g^{-1}), -m).$$

Observe that

$$\begin{aligned} (g, 1) * (h, 1) &:= (h, 1)^{-1} \cdot (g, 1) \cdot (h, 1) = (\phi^{-1}(h^{-1}), -1) \cdot (g, 1) \cdot (h, 1) \\ &= (\phi^{-1}(h^{-1}), -1) \cdot (\phi(g)h, 2) = (\phi^2(\phi^{-1}(h^{-1}))\phi(g)h, 1) \\ &= (\phi(h^{-1})\phi(g)h, 1) = (\phi(h^{-1}g)h, 1) \end{aligned}$$

holds in $\text{Conj}(H)$, and we conclude that the injective map $G \rightarrow H$ defined by $g \mapsto (g, 1)$ is an injective quandle homomorphism $\text{Conj}(G, \phi) \rightarrow \text{Conj}(H)$, hence verifying the theorem. \square

Now let Q be an arbitrary quandle. The *associated group* $\text{As}(Q)$ of Q is the group defined by the presentation

$$\text{As}(Q) := \langle e_x (x \in Q) \mid e_y^{-1} e_x e_y = e_{x*y} (x, y \in Q) \rangle.$$

A quandle Q is called *injective* if the canonical map $Q \rightarrow \text{As}(Q)$ defined by $x \mapsto e_x$ ($x \in Q$) is injective. The injectivity of finite quandles is important in the study of

set-theoretical solutions of the Yang-Baxter equation (see Lebed-Vendramin [8] for instance). According to Joyce [7, Section 6] (see also Dhanwani-Raundal-Singh [6, Theorem 3.8]), a quandle Q is injective if and only if Q is embeddable. As a byproduct of the theorem, we obtain the following corollary:

Corollary. *Any twisted conjugation quandle is injective. In particular, any Alexander quandle is injective.*

Finally, we remark that not all quandles are embeddable. Indeed, there exist quandles which are not injective and hence are not embeddable. See Joyce [7, Section 6] and Bardakov-Nasybullov [3, §4] for examples of such quandles.

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REFERENCES

- [1] Nicolás Andruskiewitsch and Matías Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), no. 2, 177–243, DOI 10.1016/S0001-8708(02)00071-3. MR1994219
- [2] Valeriy G. Bardakov, Pinka Dey, and Mahender Singh, *Automorphism groups of quandles arising from groups*, Monatsh. Math. **184** (2017), no. 4, 519–530, DOI 10.1007/s00605-016-0994-x. MR3718201
- [3] Valeriy Bardakov and Timur Nasybullov, *Embeddings of quandles into groups*, J. Algebra Appl. **19** (2020), no. 7, 2050136, 20, DOI 10.1142/S0219498820501364. MR4129183
- [4] George M. Bergman, *On core quandles of groups*, Comm. Algebra **49** (2021), no. 6, 2516–2537, DOI 10.1080/00927872.2021.1874400. MR4255023
- [5] Jacques Darné, *Nilpotent quandles*, preprint (2022), available at <https://arxiv.org/abs/2205.02480>.
- [6] Neeraj K. Dhanwani, Hitesh Raundal, and Mahender Singh, *Dehn quandles of groups and orientable surfaces*, preprint (2021), available at <https://arxiv.org/abs/2106.00290>.
- [7] David Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65, DOI 10.1016/0022-4049(82)90077-9. MR638121
- [8] Victoria Lebed and Leandro Vendramin, *On structure groups of set-theoretic solutions to the Yang-Baxter equation*, Proc. Edinb. Math. Soc. (2) **62** (2019), no. 3, 683–717, DOI 10.1017/s0013091518000548. MR3974961
- [9] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(161)** (1982), no. 1, 78–88, 160 (Russian). MR672410
- [10] Takefumi Nosaka, *Quandles and topological pairs: Symmetry, knots, and cohomology*, SpringerBriefs in Mathematics, Springer, Singapore, 2017. MR3729413

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