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Title	Embedding Alexander quandles into groups
Author(s)	Akita, Toshiyuki
Citation	Journal of knot theory and its ramifications, 32(2) https://doi.org/10.1142/S0218216523500116
Issue Date	2023-03-04
Doc URL	http://hdl.handle.net/2115/91254
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Type	article (author version)
File Information	embedding_Alex_final.pdf

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EMBEDDING ALEXANDER QUANDLES INTO GROUPS

TOSHIYUKI AKITA

ABSTRACT. For any twisted conjugate quandle Q, and in particular any Alexander quandle, there exists a group G such that Q is embedded into the conjugation quandle of G.

1. EMBEDDABLE QUANDLES

A non-empty set Q equipped with a binary operation $Q \times Q \to Q$, $(x,y) \mapsto x * y$ is called a *quandle* if it satisfies the following three axioms:

(1) $x * x = x$ $(x \in Q)$,
(2) $(x * y) * z = (x * z) * (y * z)$ $(x,y,z \in Q)$,
(3) For all $x \in Q$, the map $S_x \colon Q \to Q$ defined by $y \mapsto y * x$ is bijective.

Quandles were introduced independently by Joyce [7] and Matveev [9]. Since then, quandles have been important objects in the study of knots and links, set-theoretical solutions of the Yang-Baxter equation, Hopf algebras and many others. We refer to Nosaka [10] for further details of quandles.

A map $f \colon Q \to Q'$ of quandles is called a *quandle homomorphism* if it satisfies $f(x * y) = f(x) * f(y)$ $(x, y \in Q)$. Given a group G, the set G equipped with a quandle operation $h * g := g^{-1}hg$ is called the *conjugation quandle* of G and is denoted by $\mathrm{Conj}(G)$. A quandle Q is called *embeddable* if there exists a group G and an injective quandle homomorphism $Q \to \mathrm{Conj}(G)$. Not all quandles are embeddable (see the bottom of §2).

In their paper [2], Bardakov-Dey-Singh proposed the question "For which quandles X does there exists a group G such that X embeds in the conjugation quandle $\mathrm{Conj}(G)$?" [2, Question 3.1], and proved that Alexander quandles associated with fixed-point free involutions are embeddable [2, Proposition 3.2]. The following is a list of embeddable quandles of which the author is aware: (1) free quandles and free n-quandles (Joyce [7, Theorem 4.1 and Corollary 10.3]), (2) commutative quandles, latin quandles and simple quandles (Bardakov-Nasybullov [3, §5]), (3) core quandles (Bergman [4, (6.5)]), (4) generalized Alexander quandles associated with fixed-point free automorphisms (Dhanwani-Raundal-Singh [6, Proposition 3.12]), and (5) free c-nilpotent quandles (Darné [5, Proposition 2.18]).

In this short note, we will show that twisted conjugation quandles, which include all Alexander quandles, are embeddable, thereby generalize the aforementioned result of Bardakov-Dey-Singh.

2020 *Mathematics Subject Classification.* Primary 20N02; Secondary 08A05, 57K10.
Key words and phrases. quandle, conjugation quandle, Alexander quandle.

2. Embeddings of twisted conjugation quandles

Let G be an additive abelian group and let $\phi\colon G \to G$ be a group automorphism of G. The *Alexander quandle* $\mathrm{Alex}(G,\phi)$ associated with ϕ is the set G equipped with the quandle operation

$$g * h := \phi(g) + h - \phi(h).$$

Let G be a group and let $\phi\colon G \to G$ be an automorphism of G. The *twisted conjugation quandle* $\mathrm{Conj}(G,\phi)$ associated with ϕ is the the set G equipped with the quandle operation

$$g * h := \phi(h^{-1}g)h.$$

Observe that an Alexander quandle $\mathrm{Alex}(G,\phi)$ is precisely a twisted conjugation quandle $\mathrm{Conj}(G,\phi)$ whose underlying group G is abelian. Twisted conjugation quandles appeared in Andruskiewitsch-Graña [1, §1.3.7] under the name *twisted homogeneous crossed sets*. We prefer the name twisted conjugation quandles because $\mathrm{Conj}(G,\phi) = \mathrm{Conj}(G)$ if ϕ is the identity map. It should be emphasized that $\mathrm{Conj}(G,\phi)$ is different from the *generalized Alexander quandle* associated with (G,ϕ). The latter has the same underlying set G, but with the different quandle operation $g * h := \phi(gh^{-1})h$. Now we prove that $\mathrm{Conj}(G,\phi)$ is embeddable:

Theorem. *Any twisted conjugation quandle is embeddable. In particular, any Alexander quandle is embeddable.*

Proof. Given a twisted conjugation quandle $\mathrm{Conj}(G,\phi)$, we will construct an explicit embedding $\mathrm{Conj}(G,\phi) \to \mathrm{Conj}(H)$. Let $\mathbb{Z}$ be the additive group of integers, and let $H := G \rtimes_\phi \mathbb{Z}$ be the semidirect product of G and $\mathbb{Z}$ associated with ϕ. Namely, H equals to $G \times \mathbb{Z}$ as sets. The group law on H is given by

$$(g,m)\cdot(h,n) := (\phi^n(g)h, m+n).$$

The inverse of $(g,m) \in H$ is

$$(g,m)^{-1} = (\phi^{-m}(g^{-1}), -m).$$

Observe that

$$\begin{aligned}(g,1)*(h,1) &:= (h,1)^{-1}\cdot(g,1)\cdot(h,1) = (\phi^{-1}(h^{-1}),-1)\cdot(g,1)\cdot(h,1)\\ &= (\phi^{-1}(h^{-1}),-1)\cdot(\phi(g)h,2) = (\phi^2(\phi^{-1}(h^{-1}))\phi(g)h,1)\\ &= (\phi(h^{-1})\phi(g)h,1) = (\phi(h^{-1}g)h,1)\end{aligned}$$

holds in $\mathrm{Conj}(H)$, and we conclude that the injective map $G \to H$ defined by $g \mapsto (g,1)$ is an injective quandle homomorphism $\mathrm{Conj}(G,\phi) \to \mathrm{Conj}(H)$, hence verifying the theorem. □

Now let Q be an arbitrary quandle. The *associated group* $\mathrm{As}(Q)$ of Q is the group defined by the presentation

$$\mathrm{As}(Q) := \langle e_x\,(x \in Q) \mid e_y^{-1}e_xe_y = e_{x*y}\,(x,y \in Q)\rangle.$$

A quandle Q is called *injective* if the canonical map $Q \to \mathrm{As}(Q)$ defined by $x \mapsto e_x$ $(x \in Q)$ is injective. The injectivity of finite quandles is important in the study of set-theoretical solutions of the Yang-Baxter equation (see Lebed-Vendramin [8] for instance). According to Joyce [7, Section 6] (see also Dhanwani-Raundal-Singh [6, Theorem 3.8]), a quandle Q is injective if and only if Q is embeddable. As a byproduct of the theorem, we obtain the following corollary:

Corollary. *Any twisted conjugation quandle is injective. In particular, any Alexander quandle is injective.*

Finally, we remark that not all quandles are embeddable. Indeed, there exist quandles which are not injective and hence are not embeddable. See Joyce [7, Section 6] and Bardakov-Nasybullov [3, §4] for examples of such quandles.

Acknowledgement. The author would like to thank the anonymous referee for valuable comments improving this paper. The author was partially supported by JSPS KAKENHI Grant Number 20K03600.

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Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo, 060-0810 Japan

Email address: `akita@math.sci.hokudai.ac.jp`