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Free Vibration of Annular Plates Constrained byTranslational and/or Rotational Springs on Outer and Inner Edges

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Abstract

This paper presents comprehensive lists of natural frequencies for vibration of thin isotropic annular plates with outer and inner edges elastically constrained by translational and rotational springs. A method is extended from a previously presented Ritz approach to include effects of the springs. In numerical examples, sixteen models are considered to cover all general cases of locating translational and/or rotational springs on outer and/or inner edges in addition to classical boundary conditions of free and simple supported edges. After convergence study and comparison test in specific cases are made to validate solution accuracy, frequency parameters of the sixteen models are summarized and frequency variations with increasing spring stiffness are illustrated.

Keywords: Annular plate, vibration, natural frequency, elastic springs, Ritz method

1. Introduction

An annular plate is defined as a flat plate consisting of an outer circular boundary and inner concentric circular boundary, and is often found as structural component or as a model of whole structure in the fields of architectural, mechanical and ocean engineering. The applications in the areas are usually exposed to dynamic environment, and there is a long research history in studying the natural frequencies and mode shapes [1].

For thin annular plates, it was known that the free vibration problem has an exact solution in terms of the Bessel functions J_n and Y_n of the first and second kinds, respectively, and the modified Bessel functions I_n and K_n of the first and second kinds, respectively. The first complete numerical sets of natural frequencies were obtained in 1965 with using these Bessel functions by Vogel and Skinner [2] for nine possible combinations of classical boundary conditions (i.e., free (F), simply supported (S) or clamped (C) edges) along outer and inner edges. Their results are presented in the three significant figures, and are not so with good accuracy due to the subroutines for the special functions in those days. Later in the 1970's and 1980's, there found a number of technical papers [3-8] on the topic, most of them consider polar-orthotropy.

In the 2010's, new analysis techniques are introduced, for example, by Hamiltonian approach [9] and convolution and differential quadrature methods [10]. Other techniques are also used in references [11,12]. More recently, the author presented numerical results in five significant figures for the nine combinations [13].

For practical model of annular plates elastically constrained at two edges, there exist more combinations in edge conditions. Avalos and Laura [14,15] presented analysis in using two-term approximation, and Raju and Rao [16,17] used the finite element method only to give the lowest frequency. Kim and Dickinson [18] covered broader parameters, but still five sets of boundary conditions. The objective of the present paper is therefore to cover all the frequency results for sixteen combinations of translational spring constraint (TS) and rotational spring constraint (RS) in addition to the classical three edge constraints (F, S and C).

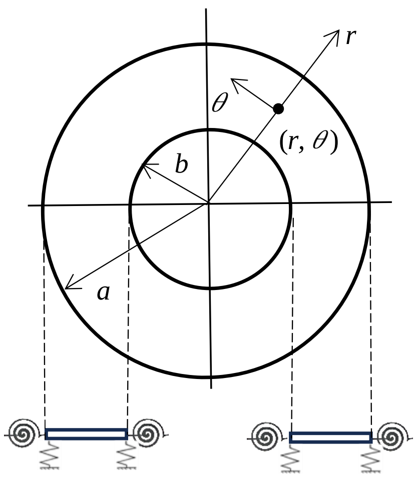

Figure 1. Annular plate with translational and rotational springs and co-ordinate

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2. Method of Analysis

Free vibration of a thin annular plate is considered in polar coordinates, as shown in Fig.1. Outer and inner radii are given by *a* and *b*, respectively, and the uniform thickness is by *h*. The plate can be constrained by translational and/or rotational springs along outer and inner edges.

The maximum strain energy of the plate under bending is expressed by

$$U_p = \frac{1}{2}\int_b^a \int_0^{2\pi} \left\{ D_r \left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\nu_\theta D_r \left(\frac{\partial^2 w}{\partial r^2}\right) \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 + 4D_k \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2 \right\} r\,d\theta\,dr \quad (1)$$

where $w(r,\theta)$ is an amplitude (maximum deflected shape) of plate vibrating in radian frequency ω. A set of the bending stiffness is defined by

$$D_r = \frac{E_r h^3}{12(1-\nu_r\nu_\theta)},\ D_\theta = \frac{E_\theta h^3}{12(1-\nu_r\nu_\theta)},\ D_k = \frac{Gh^3}{12} \quad (2)$$

for polar-orthotropic material, where E_r and E_θ are Young's moduli in *r* and θ direction, respectively, *G* is a shear modulus, ν_r and ν_θ are major and minor Poisson's ratios ($\nu_\theta D_r = \nu_r D_\theta$). For isotropic material, they reduce to

$$D_r = D_\theta = D = \frac{Eh^3}{12(1-\nu^2)},\ H = D_r\nu_\theta + 2D_k = D \quad (3)$$

The expressions of strain energy in the edge springs are

$$U_{ts} = \frac{1}{2}\int_0^{2\pi} k_{ta} \{w(a,\theta)\}^2 a\,d\theta + \frac{1}{2}\int_0^{2\pi} k_{tb} \{w(b,\theta)\}^2 b\,d\theta \quad (4)$$

for translational springs at $r=a$ and $r=b$, respectively, with translational spring stiffness k_{ta} and k_{tb}, and

$$U_{rs} = \frac{1}{2}\int_0^{2\pi} k_{ra} \left\{\frac{\partial w(a,\theta)}{\partial r}\right\}^2 a\,d\theta + \frac{1}{2}\int_0^{2\pi} k_{rb} \left\{\frac{\partial w(b,\theta)}{\partial r}\right\}^2 b\,d\theta \quad (5)$$

for rotational springs with rotational spring stiffness k_{ra} and k_{rb}. The maximum kinetic energy is given by

$$T = \frac{1}{2}\rho h\omega^2 \int_b^a \int_0^{2\pi} w^2 r\,d\theta\,dr \quad (6)$$

where ρ is the mass per unit volume.

Next, the amplitude is approximated by a finite series

$$w(r,\theta) \doteq w_{mn}(\eta,\theta) = \sum_{m=0}^{M-1} A_m Y_m(\eta)\cos n\theta,\ \left(\eta = \frac{r}{a}\right) \quad (7)$$

where A_m are undetermined coefficients, $Y_m(\eta)$ is a function in the radial direction to satisfy the kinematical boundary condition at both inner and outer edges, and *n* is an integer to indicate the number of nodal diameters.

This equation is substituted into

$$\frac{\partial}{\partial A_{\bar{m}}}\left[U_p(w_{mn}) + U_{ts}(w_{mn}) + U_{rs}(w_{mn}) - T(w_{mn})\right] = 0 \quad (\bar{m} = 0,1,..,M-1) \quad (8)$$

The resulting frequency equation is given by

$$\sum_{m=0}^{M-1}\Big\{ f_1^{(22)} + \nu_\theta\left[f_0^{(21)} + f_0^{(12)} - n^2\left(f_{-1}^{(20)} + f_{-1}^{(02)}\right)\right] + \left(\frac{D_\theta}{D_r}\right)\left[f_{-1}^{(11)} + n^4 f_{-3}^{(00)} - n^2\left(f_{-2}^{(10)} + f_{-2}^{(01)}\right)\right] + 2n^2\left[\frac{H}{D_r} - \nu_\theta\right]\left(f_{-3}^{(00)} + f_{-1}^{(11)} - f_{-2}^{(01)} - f_{-2}^{(10)}\right) + k_{ta}^* Y_m(1)Y_{\bar{m}}(1) + k_{tb}^* Y_m(\alpha)Y_{\bar{m}}(\alpha) + k_{ra}^* \frac{dY_m(1)}{d\eta}\frac{dY_{\bar{m}}(1)}{d\eta} + k_{rb}^* \frac{dY_m(\alpha)}{d\eta}\frac{dY_{\bar{m}}(\alpha)}{d\eta} - \Omega^2 f_1^{(00)}\Big\}_{m\bar{m}} A_m = 0 \quad (9)$$

for $\bar{m} = 0,1,..,(M-1)$ and a specific integer *n*. The frequency parameter is defined by

$$\Omega = \omega a^2 \left(\frac{\rho h}{D}\right)^{1/2} \quad (10)$$

and an aspect ratio is $\alpha = b/a$. Function *f* in equation (9) is defined by

$$f_{t,m\bar{m}}^{(pq)} = \int_\alpha^1 \eta^t \left(\frac{d^{(p)} Y_m(\eta)}{d\eta^{(p)}}\right)\left(\frac{d^{(q)} Y_{\bar{m}}(\eta)}{d\eta^{(q)}}\right) d\eta \quad (11)$$

A set of the non-dimensional spring stiffness is given by

$$k_{ta}^* = \frac{k_{ta}a^3}{D} \text{ and } k_{tb}^* = \frac{k_{tb}a^3}{D} \quad (12)$$

for translational springs at outer ($r=a$) and inner ($r=b$) edge, respectively, and similarly by

$$k_{ra}^* = \frac{k_{ra}a}{D} \text{ and } k_{rb}^* = \frac{k_{rb}a}{D} \quad (13)$$

for rotational spring at outer and inner edge, respectively.

In the present Ritz method, an idea of using the boundary index [19,20] is introduced to deal with any combination of classical boundary conditions, i.e., free, simply supported and clamped edges. For this purpose, a function $Y_m(\eta)$ is introduced in the form

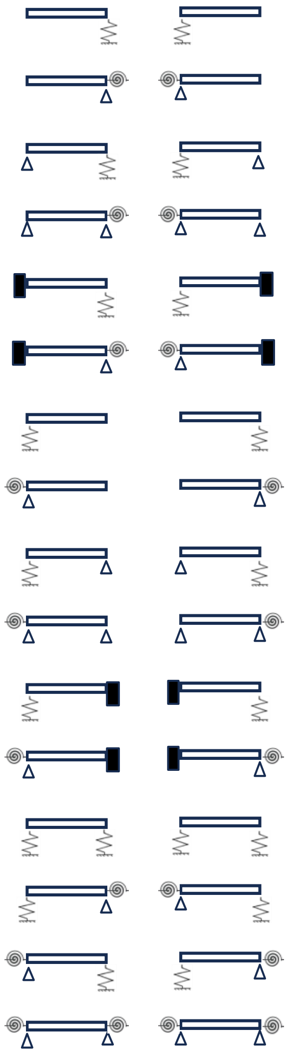
$$Y_m(\eta) = \eta^m (\eta-\alpha)^{Bi} (\eta-1)^{Bo} \quad (14)$$

where *Bi* (abbreviation of Boundary index at inner edge) takes 0, 1 and 2 to satisfy kinematical condition for F, S and C, respectively, along inner circular boundary ($r=b$) and similarly, *Bo* (Boundary index at outer edge) does 0, 1 and 2 to satisfy the condition for F, S and C, respectively, along outer circular boundary ($r=a$). Therefore, one can choose any of the nine possible sets of classical boundary conditions, and include the effects of translational and rotational springs additionally.

Table 1. Numerical examples of annular plates with translational and/or rotational springs.

examples	BC	Limiting case $k^*=0$	$k^*=\infty$
Ex.1	F-TS	F-F	F-S
Ex.2	F-RS	F-S	F-C
Ex.3	S-TS	S-F	S-S
Ex.4	S-RS	S-S	S-C
Ex.5	C-TS	C-F	C-S
Ex.6	C-RS	C-S	C-C
Ex.7	TS-F	F-F	S-F
Ex.8	RS-F	S-F	C-F
Ex.9	TS-S	F-S	S-S
Ex.10	RS-S	S-S	C-S
Ex.11	TS-C	F-C	S-C
Ex.12	RS-C	S-C	C-C
Ex.13	TS-TS	F-F	S-S
Ex.14	TS-RS	F-S	S-C
Ex.15	RS-TS	S-F	C-S
Ex.16	RS-RS	S-S	C-C

Cross-section of plate



3. Numerical results

Numerical examples are listed in Table 1. When one considers three classical boundary conditions (i.e., free (F), simply supported (S), clamped (C)) at each edge, there are nine different combinations (three by three) and their results are already summarized by using Ritz method [13].

In this study, an intermediate condition between free edge and simply supported edge is considered for plates constrained only by translational spring (denoted by TS (Translational Spring)), and this condition can become the free edge in the limit of $k_{ta}=k_{tb}=0$ and simply supported edge in $k_{ta}=k_{tb}=\infty$. Similarly, between simply supported edge and clamped edge, intermediate edge condition is obtained by using rotational spring (denoted by RS (Rotational Spring)) on simply supported edge. In the limit of $k_{ra}=k_{rb}=\infty$, the edge becomes fully clamped. Such limiting cases are listed for each of the sixteen examples in the table.

Table 2 Convergence of frequency parameters Ω (b/a =0.3, ν=0.3).

	M	(n,s) (0,0)	(1,0)	(2,0)	(3,0)	(4,0)
Ex.2	6	4.656	4.490	6.542	12.73	21.90
(k^*=3.333)	7	4.656	4.490	6.542	12.73	21.90
	8	4.656	4.490	6.452	12.73	21.90
	9	4.656	4.490	6.542	12.73	21.90
Ref.[18]		4.656	4.490	6.542	12.73	21.90
Ex.2	6	6.607	6.492	7.903	13.25	22.06
(k^*=333.3)	7	6.606	6.492	7.901	13.25	22.06
	8	6.606	6.492	7.900	13.25	22.06
	9	6.606	6.492	7.900	13.25	22.06
Ref.[18]		6.606	6.492	7.900	13.25	22.06
Ex.8	6	6.108	13.84	25.11	39.77	38.37*
(k^*=1)	7	6.108	13.84	25.11	39.77	38.37*
	8	6.108	13.84	25.11	39.77	38.37*
	9	6.108	13.84	25.11	39.77	38.37*
Ref.[18]		6.108	13.84	25.11	39.77	38.37*

*: (0,1) mode.

Totally, this table summarizes the sixteen examples among twenty five combinations of “five cases (F,TS,S,RS and C) at outer edge” times “five cases (F,TS,S,RS and C) at inner edge”. Since the nine cases of “(F,S,C) at outer edge times (F,S,C) at inner edge” was already published [13], the twenty five combinations deducted by nine end up with the sixteen examples as illustrated in Table.1.

For values of spring stiffness used in numerical examples, the same values are used for translational and rotational springs, when these different types of springs appear in one example, as

$$k^* = k^*_{ta} = k^*_{tb} = k^*_{ra} = k^*_{rb} \quad \text{(e.g., } k^* = 1, 10, 10^2, 10^4\text{)} \qquad (15)$$

for simplicity. But of course, one can calculate frequencies of plates with independent stiffness values of each spring.

Table 2 presents convergence study of Ex.2 and Ex.8 with respect to the number of series terms M in equation (7), and the number of terms is increased from M=6 to 9. It is clearly seen that the fast convergence is seen in the four significant figures. For clarity, the identical values are underlined and are obtained already for M=7. In numerical tables hereafter, M=8 is employed.

The frequency parameters are given basically in increasing order with (n,s) (n: number of nodal diameters, s: number of nodal circles)). Depending on aspect ratio b/a and boundary conditions, the order may change, typically the (0,0) and (1,0) modes, as seen in Ex.2 and Ex.8 in the table. These examples are chosen in the convergence study, because these natural frequencies are available for comparison in [18], and the excellent agreement is found between the present converged values and those cited in the table.

Table 3 Frequency parameters Ω for **(Ex.1)** Annular plate with F outer edge and TS inner edge (ν =0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-F)	RBM	RBM	5.304	12.44	21.84
	1	0.446	0.063	5.304	12.44	21.84
	10	1.326	0.199	5.304	12.44	21.84
	100	2.783	0.612	5.305	12.44	21.84
	10000	3.443	2.278	5.365	12.44	21.84
	∞(F-S)	3.450	2.439	5.429	12.44	21.84
0.3	0(F-F)	RBM	RBM	4.906	12.27	21.78
	1	0.793	0.328	4.909	12.27	21.78
	10	2.095	0.998	4.934	12.27	21.78
	100	3.188	2.367	5.136	12.29	21.79
	10000	3.420	3.357	6.034	12.57	21.85
	∞(F-S)	3.422	3.374	6.080	12.61	21.88
0.5	0(F-F)	RBM	RBM	4.271	11.43	21.07
	1	1.119	0.723	4.307	11.43	21.07
	10	2.804	2.098	4.595	11.51	21.09
	100	3.920	4.081	6.058	12.05	21.29
	10000	4.119	4.852	7.946	13.96	22.68
	∞(F-S)	4.120	4.860	7.985	14.04	22.79

Table 4 Frequency parameters Ω for **(Ex.2)** Annular plate with F outer edge and RS inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-S)	3.450	2.439	5.429	12.44	21.84
	1	3.527	2.521	5.435	12.44	21.84
	10	3.858	2.913	5.480	12.44	21.84
	100	4.171	3.369	5.583	12.45	21.84
	10000	4.236	3.479	5.624	12.45	21.84
	∞(F-C)	4.238	3.479	5.624	12.45	21.84
0.3	0(F-S)	3.422	3.374	6.080	12.61	21.88
	1	3.937	3.827	6.248	12.65	21.88
	10	5.505	5.322	7.007	12.87	21.94
	100	6.487	6.359	7.781	13.18	22.04
	10000	6.658	6.550	7.956	13.27	22.07
	∞(F-C)	6.660	6.552	7.957	13.28	22.07
0.5	0(F-S)	4.120	4.860	7.985	14.04	22.79
	1	5.652	6.166	8.753	14.41	22.97
	10	9.841	10.09	11.71	16.18	23.94
	100	12.53	12.78	14.19	18.11	25.24
	10000	13.02	13.28	14.70	18.56	25.59
	∞(F-C)	13.02	13.29	14.70	18.56	25.60

Table 5 Frequency parameters Ω for **(Ex.3)** Annular plate with S outer edge and TS inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-F)	4.854	13.87	25.40	39.94	56.84
	1	4.933	13.87	25.40	39.94	56.84
	10	5.574	13.89	25.40	39.94	56.84
	100	8.979	14.03	25.40	39.94	56.84
	10000	14.36	16.36	25.66	39.94	56.84
	∞(S-S)	14.49	16.78	25.94	39.98	56.84
0.3	0(S-F)	4.664	12.82	24.12	38.78	56.25
	1	4.939	12.87	24.13	38.78	56.25
	10	6.867	13.37	24.23	38.80	56.25
	100	14.06	16.61	25.09	38.99	56.29
	10000	20.97	23.15	29.98	41.55	57.26
	∞(S-S)	21.08	23.32	30.27	41.91	57.55
0.5	0(S-F)	5.077	11.61	22.36	35.64	52.03
	1	5.540	11.80	22.44	35.67	52.05
	10	8.606	13.38	23.14	36.01	52.22
	100	20.68	22.55	28.47	38.91	53.76
	10000	39.63	41.33	46.46	55.08	67.23
	∞(S-S)	40.04	41.80	47.09	55.96	68.38

Table 6 Frequency parameters Ω for **(Ex.4)** Annular plate with S outer edge and RS inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-S)	14.49	16.78	25.94	39.98	56.84
	1	14.73	16.93	25.96	39.98	56.84
	10	15.97	17.79	26.14	39.99	56.84
	100	17.43	19.04	26.55	40.04	56.85
	10000	17.79	19.40	26.72	40.06	56.85
	∞(S-C)	17.79	19.40	26.72	40.06	56.85
0.3	0(S-S)	21.08	23.32	30.27	41.91	57.55
	1	21.89	24.00	30.69	42.10	57.62
	10	25.53	27.19	32.83	43.21	58.08
	100	29.17	30.61	35.54	44.94	58.96
	10000	29.97	31.39	36.24	45.45	59.27
	∞(S-C)	29.98	31.40	36.24	45.46	59.27
0.5	0(S-S)	40.04	41.80	47.09	55.96	68.38
	1	41.53	43.20	48.28	56.89	69.06
	10	48.88	50.24	54.49	61.96	72.94
	100	57.64	58.83	62.53	69.12	78.99
	10000	59.80	60.96	64.61	71.08	80.78
	∞(S-C)	59.82	60.99	64.63	71.11	80.80

Table 7 Frequency parameters Ω for **(Ex.5)** Annular plate with C outer edge and TS inner edge (ν=0.3).

b/a	*k**	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(C-F)	10.16	21.20	34.54	50.99	69.66
	1	10.22	21.20	34.54	50.99	69.66
	10	10.74	21.22	34.54	50.99	69.66
	100	14.26	21.41	34.54	50.99	69.66
	10000	22.48	24.66	34.96	51.01	69.66
	∞(C-S)	22.70	25.28	35.41	51.07	69.67
0.3	0(C-F)	11.42	19.54	32.59	49.07	68.58
	1	11.60	19.61	32.61	49.07	68.58
	10	13.01	20.19	32.76	49.11	68.59
	100	20.99	24.41	34.07	49.45	68.67
	10000	33.53	35.58	42.21	53.97	70.52
	∞(C-S)	33.77	35.91	42.73	54.61	71.06
0.5	0(C-F)	17.72	22.02	32.12	45.81	63.02
	1	17.91	22.16	32.20	45.86	63.04
	10	19.56	23.44	32.96	46.29	63.28
	100	30.63	32.77	39.20	50.06	65.48
	10000	63.05	64.47	68.87	76.54	87.79
	∞(C-S)	63.97	65.49	70.14	78.18	89.86

Table 8 Frequency parameters Ω for **(Ex.6)** Annular plate with C outer edge and RS inner edge (ν=0.3).

b/a	*k**	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(C-S)	22.70	25.28	35.41	51.07	69.67
	1	23.02	25.49	35.45	51.07	69.67
	10	24.66	26.64	35.72	51.09	69.67
	100	26.75	28.40	36.35	51.17	69.67
	10000	27.28	28.91	36.62	51.22	69.68
	∞(C-C)	27.28	28.92	36.62	51.22	69.68
0.3	0(C-S)	33.77	35.91	42.73	54.61	71.06
	1	34.69	36.72	43.28	54.90	71.19
	10	39.16	40.75	46.19	56.56	71.95
	100	44.17	45.49	50.09	59.22	73.42
	10000	45.33	46.63	51.13	60.02	73.94
	∞(C-C)	45.35	46.64	51.14	60.03	73.95
0.5	0(C-S)	63.97	65.49	70.14	78.18	89.86
	1	65.61	67.06	71.54	79.35	90.76
	10	74.33	75.52	79.25	85.95	96.09
	100	86.10	87.10	90.25	95.99	104.8
	10000	89.21	90.19	93.29	98.89	107.5
	∞(C-C)	89.25	90.23	93.32	98.93	107.6

Table 9 Frequency parameters Ω for **(Ex.7)** Annular plate with TS outer edge and F inner edge (ν=0.3).

b/a	*k**	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-F)	RBM	RBM	5.304	12.44	21.84
	1	1.379	1.988	5.783	12.69	22.00
	10	3.438	5.953	8.893	14.75	23.44
	100	4.661	12.07	18.97	25.68	33.41
	10000	4.852	13.85	25.32	39.74	56.40
	∞(S-F)	4.854	13.87	25.40	39.94	56.84
0.3	0(F-F)	RBM	RBM	4.906	12.27	21.78
	1	1.432	1.995	5.410	12.52	21.95
	10	3.453	5.928	8.589	14.58	23.39
	100	4.508	11.44	18.38	25.37	33.32
	10000	4.663	12.80	24.05	38.59	55.83
	∞(S-F)	4.664	12.82	24.12	38.78	56.25
0.5	0(F-F)	RBM	RBM	4.271	11.43	21.07
	1	1.575	2.047	4.842	11.69	21.23
	10	3.775	5.963	8.220	13.79	22.66
	100	4.908	10.60	17.59	24.25	32.30
	10000	5.075	11.60	22.31	35.49	51.70
	∞(S-F)	5.077	11.61	22.36	35.64	52.03

Table 10 Frequency parameters Ω for **(Ex.8)** Annular plate with RS outer edge and F inner edge (ν=0.3).

b/a	*k**	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-F)	4.854	13.87	25.40	39.94	56.84
	1	5.998	14.94	26.44	40.98	57.88
	10	8.700	18.50	30.57	45.61	62.89
	100	9.964	20.80	33.90	50.05	68.40
	10000	10.16	21.19	34.53	50.98	69.65
	∞(C-F)	10.16	21.20	34.54	50.99	69.66
0.3	0(S-F)	4.664	12.82	24.12	38.78	56.25
	1	6.108	13.84	25.11	39.76	57.26
	10	9.504	17.14	28.99	44.13	62.11
	100	11.16	19.19	32.02	48.22	67.38
	10000	11.42	19.54	32.59	49.06	68.57
	∞(C-F)	11.42	19.54	32.59	49.07	68.58
0.5	0(S-F)	5.077	11.61	22.36	35.64	52.03
	1	7.590	13.11	23.47	36.61	52.95
	10	13.66	18.06	27.86	40.91	57.30
	100	17.12	21.41	31.42	44.96	61.96
	10000	17.71	22.01	32.11	45.80	63.01
	∞(C-F)	17.72	22.02	32.12	45.81	63.02

Table 11 Frequency parameters Ω for **(Ex.9)** Annular plate with TS outer edge and S inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-S)	3.450	2.439	5.429	12.44	21.84
	1	4.006	3.216	5.902	12.70	22.00
	10	7.000	6.810	8.998	14.76	23.44
	100	12.76	14.11	19.21	25.69	33.41
	10000	14.47	16.75	25.86	39.78	56.40
	∞(S-S)	14.49	16.78	25.94	39.98	56.84
0.3	0(F-S)	3.422	3.374	6.080	12.61	21.88
	1	4.067	4.057	6.533	12.87	22.04
	10	7.563	7.721	9.602	14.94	23.49
	100	16.47	17.55	20.80	26.09	33.49
	10000	21.03	23.25	30.15	41.67	57.08
	∞(S-S)	21.08	23.32	30.27	41.91	57.55
0.5	0(F-S)	4.120	4.860	7.985	14.04	22.79
	1	4.846	5.501	8.409	14.30	22.96
	10	8.979	9.405	11.48	16.43	24.45
	100	22.94	23.45	25.24	28.89	35.08
	10000	39.76	41.48	46.68	55.38	67.52
	∞(S-S)	40.04	41.80	47.09	55.96	68.38

Table 12 Frequency parameters Ω for **(Ex.10)** Annular plate with RS outer edge and S inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-S)	14.49	16.78	25.94	39.98	56.84
	1	15.67	17.92	27.00	41.02	57.88
	10	19.65	21.97	31.27	45.67	62.89
	100	22.25	24.78	34.74	50.12	68.40
	10000	22.70	25.28	35.40	51.06	69.65
	∞(C-S)	22.70	25.28	35.41	51.07	69.67
0.3	0(S-S)	21.08	23.32	30.27	41.91	57.55
	1	22.58	24.75	31.55	43.05	58.62
	10	28.34	30.39	36.93	48.27	63.86
	100	32.89	35.00	41.75	53.48	69.71
	10000	33.76	35.90	42.72	54.60	71.05
	∞(C-S)	33.77	35.91	42.73	54.61	71.06
0.5	0(S-S)	40.04	41.80	47.09	55.96	68.38
	1	42.11	43.81	48.95	57.62	69.87
	10	51.65	53.18	57.87	65.94	77.59
	100	61.69	63.19	67.80	75.79	87.37
	10000	63.95	65.46	70.11	78.16	89.83
	∞(C-S)	63.97	65.49	70.14	78.18	89.86

Table 13 Frequency parameters Ω for **(Ex.11)** Annular plate with TS outer edge and C inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-C)	4.238	3.479	5.624	12.45	21.84
	1	4.754	4.107	6.089	12.71	22.00
	10	7.798	7.524	9.161	14.77	23.44
	100	15.03	15.79	19.55	25.71	33.41
	10000	17.76	19.36	26.64	39.86	56.41
	∞(S-C)	17.79	19.40	26.72	40.06	56.85
0.3	0(F-C)	6.660	6.552	7.957	13.28	22.07
	1	7.095	6.998	8.343	13.53	22.24
	10	10.10	10.07	11.16	15.59	23.69
	100	21.15	21.53	23.17	27.08	33.80
	10000	29.87	31.28	36.06	45.16	58.76
	∞(S-C)	29.98	31.40	36.24	45.46	59.27
0.5	0(F-C)	13.02	13.29	14.70	18.56	25.60
	1	13.34	13.60	14.99	18.79	25.77
	10	15.87	16.09	17.30	20.71	27.24
	100	29.92	30.11	30.98	33.36	38.20
	10000	59.19	60.32	63.85	70.11	79.49
	∞(S-C)	59.82	60.99	64.63	71.11	80.80

Table 14 Frequency parameters Ω for **(Ex.12)** Annular plate with RS outer edge and C inner edge (ν=0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-C)	17.79	19.40	26.72	40.06	56.85
	1	19.05	20.61	27.82	41.11	57.88
	10	23.55	25.06	32.25	45.79	62.90
	100	26.71	28.32	35.91	50.27	68.41
	10000	27.27	28.91	36.62	51.21	69.67
	∞(C-C)	27.28	28.92	36.62	51.22	69.68
0.3	0(S-C)	29.98	31.40	36.24	45.46	59.27
	1	31.57	32.95	37.66	46.72	60.41
	10	38.24	39.50	43.90	52.58	66.03
	100	44.14	45.42	49.87	58.68	72.43
	10000	45.33	46.63	51.12	60.02	73.93
	∞(C-C)	45.35	46.64	51.14	60.03	73.95
0.5	0(S-C)	59.82	60.99	64.63	71.11	80.80
	1	62.01	63.14	66.69	73.01	82.53
	10	72.97	73.98	77.16	82.93	91.78
	100	86.05	87.02	90.09	95.66	104.3
	10000	89.22	90.19	93.29	98.89	107.5
	∞(C-C)	89.25	90.23	93.32	98.93	107.6

Table 15 Frequency parameters Ω for **(Ex.13)** Annular plate with TS outer and inner edges (ν =0.3).

b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-F)	RBM	RBM	5.304	12.44	21.84
	1	1.460	1.989	5.783	12.69	22.00
	10	3.993	5.959	8.893	14.75	23.44
	100	8.508	12.19	18.98	25.68	33.41
	10000	14.35	16.34	25.59	39.75	56.40
	∞(S-S)	14.49	16.78	25.94	39.98	56.84
0.3	0(F-F)	RBM	RBM	4.906	12.27	21.78
	1	1.674	2.024	5.413	12.52	21.95
	10	4.989	6.106	8.612	14.58	23.39
	100	12.70	14.16	18.83	25.43	34.32
	10000	20.91	23.08	29.86	41.32	56.80
	∞(S-S)	21.08	23.32	30.27	41.91	57.55
0.5	0(F-F)	RBM	RBM	4.271	11.43	21.07
	1	1.993	2.183	4.874	11.70	21.24
	10	6.171	6.714	8.464	13.87	22.69
	100	17.76	18.44	20.75	25.36	32.68
	10000	39.36	41.03	46.07	54.53	66.42
	∞(S-S)	40.04	41.80	47.09	55.96	68.38

Table 16 Frequency parameters Ω for **(Ex.14)** Annular plate with TS outer edge and RS inner edge (ν=0.3).

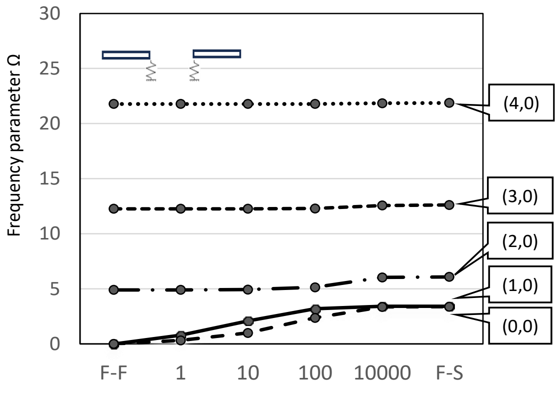
b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(F-S)	3.450	2.439	5.429	12.44	21.84
	1	4.077	3.282	5.908	12.70	22.00
	10	7.403	7.113	9.041	14.76	23.44
	100	14.81	15.58	19.78	25.70	33.41
	10000	17.76	19.36	26.65	39.87	56.41
	∞(S-C)	17.79	19.40	26.72	40.06	56.85
0.3	0(F-S)	3.442	3.374	6.080	12.61	21.88
	1	4.524	4.452	6.693	12.91	22.05
	10	9.093	9.057	10.35	15.19	23.55
	100	20.82	21.22	22.93	26.94	33.75
	10000	29.86	31.27	36.05	45.16	58.76
	∞(S-C)	29.98	31.40	36.24	45.46	59.27
0.5	0(F-S)	4.120	4.860	7.985	14.04	22.79
	1	6.216	6.691	9.148	14.67	23.14
	10	13.04	13.25	14.57	18.43	25.59
	100	29.39	29.58	30.47	32.89	37.80
	10000	59.17	60.30	63.82	70.09	79.47
	∞(S-C)	59.82	60.99	64.63	71.11	80.80

Table 17 Frequency parameters Ω for **(Ex.15)** Annular plate with RS outer and TS inner edges (ν =0.3).

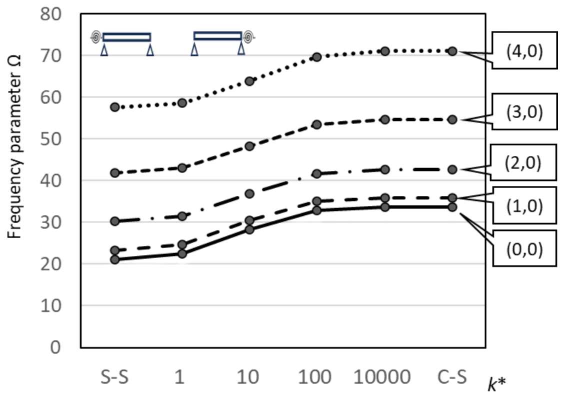
b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-F)	4.854	13.87	25.40	39.94	56.84
	1	6.067	14.94	26.44	40.98	57.88
	10	9.282	18.52	30.57	45.61	62.89
	100	14.03	21.00	33.90	50.05	68.40
	10000	22.47	24.66	34.95	51.00	69.65
	∞(C-S)	22.70	25.28	35.41	51.07	69.67
0.3	0(S-F)	4.664	12.82	24.12	38.78	56.25
	1	6.335	13.90	25.12	39.77	57.26
	10	11.13	17.74	29.13	44.16	62.12
	100	20.64	23.97	33.45	48.58	67.46
	10000	33.52	35.58	42.20	53.96	70.51
	∞(C-S)	33.77	35.91	42.73	54.61	71.06
0.5	0(S-F)	5.077	11.61	22.36	35.64	52.03
	1	7.921	13.29	23.55	36.65	52.97
	10	15.69	19.53	28.66	41.33	57.52
	100	30.00	32.09	38.40	49.11	64.34
	10000	63.02	64.45	68.85	76.52	87.77
	∞(C-S)	63.97	65.49	70.14	78.18	89.86

Table 18 Frequency parameters Ω for **(Ex.16)** Annular plate with RS outer and inner edge (ν=0.3).

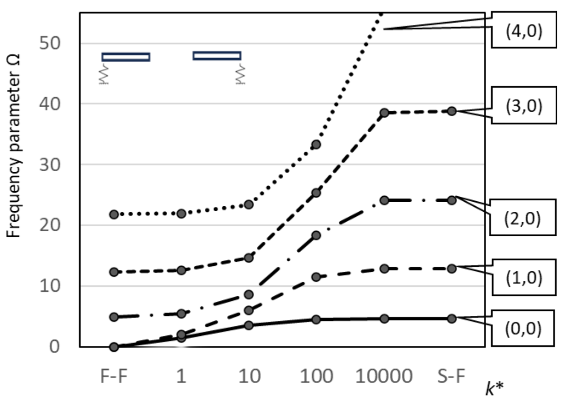
b/a	k*	(n,s)				
		(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
0.1	0(S-S)	14.49	16.78	25.94	39.98	56.84
	1	15.92	18.08	27.03	41.02	57.88
	10	21.35	23.14	31.52	45.69	62.89
	100	26.19	27.82	35.65	50.23	68.41
	10000	27.27	28.91	36.62	51.21	69.67
	∞(C-C)	27.28	28.92	36.62	51.22	69.68
0.3	0(S-S)	21.08	23.32	30.27	41.91	57.55
	1	23.39	25.43	31.97	43.25	58.69
	10	33.11	34.62	39.86	49.84	64.53
	100	43.01	44.31	48.87	57.89	71.93
	10000	45.32	46.62	51.11	60.01	73.92
	∞(C-C)	45.35	46.64	51.14	60.03	73.95
0.5	0(S-S)	40.04	41.80	47.09	55.96	68.38
	1	43.58	45.20	50.14	58.56	70.56
	10	60.81	62.00	65.77	72.53	82.73
	100	83.05	84.04	87.17	92.88	101.7
	10000	89.18	90.16	93.25	98.86	107.5
	∞(C-C)	89.25	90.23	93.32	98.93	107.6



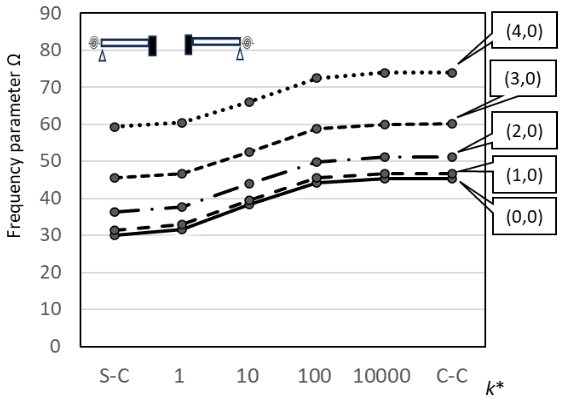
Ex.1 F-TS



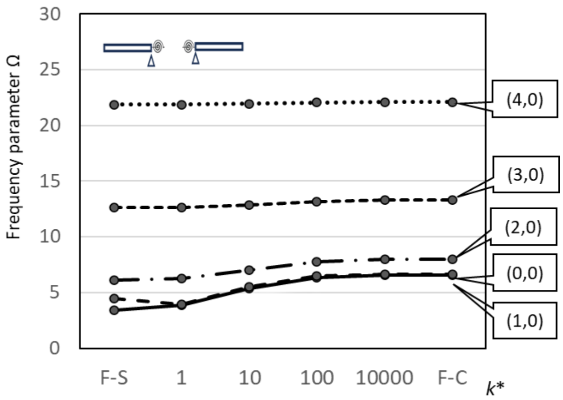
Ex.10 RS-S



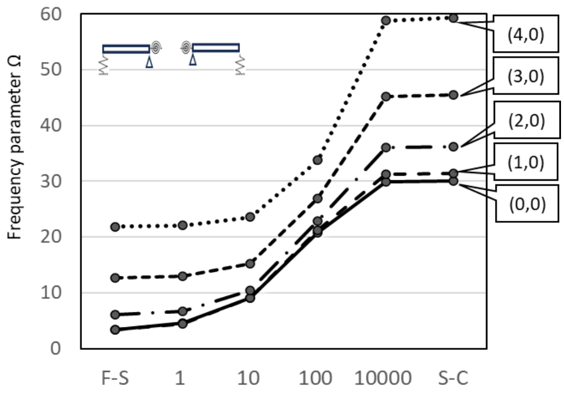
Ex.7 TS-F



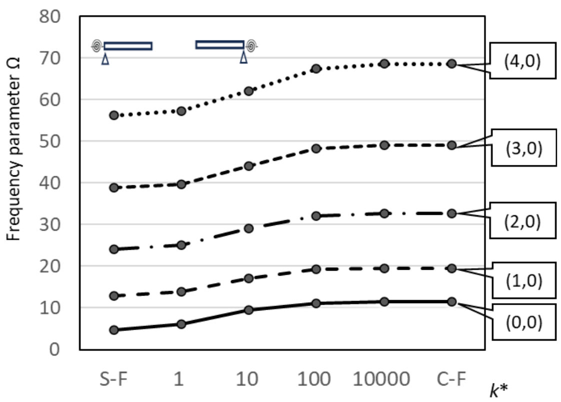
Ex.12 RS-C



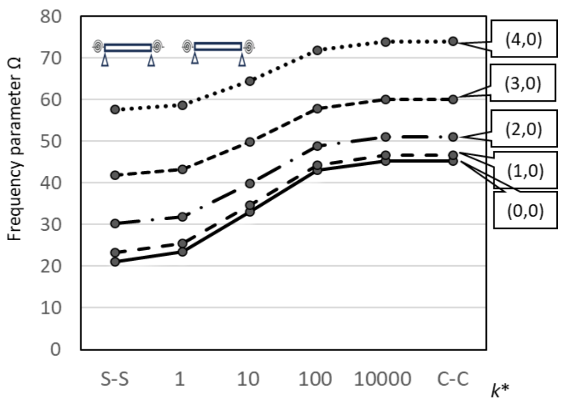
Ex.2 F-RS



Ex.14 TS-RS



Ex.8 RS-F



Ex.16 RS-RS

Figure 2. Variations of frequency parameter Ω with stiffness k^* for annular plates (b/a =0.3, ν=0.3).

Tables 3-18 summarize lists of frequency parameter Ω for modes (0,0), (1,0), (2,0), (3,0) and (4,0) of Ex.1-16, respectively. In each table, results are given in the four significant figures for three aspect ratios b/a=0.1, 0.3 and 0.5 and the representative spring stiffness k*= 0, 1, 10, 100, 10000 and infinity. Basically the lowest five frequencies are shown, but as mentioned above, the sequence of mode changes depending on the aspect ratio and spring stiffness. Therefore, (n,s) is given (n: number of nodal diameters, s: number of nodal circles) in stead of using mode sequence number (e.g., 1st mode, second mode, etc.). For the limiting cases of k*=0 and infinity, a pair of the (non-elastic) boundary condition (F, S or C) is written in the parenthesis.

In all the tables, as the stiffness increases, the frequencies are monotonically increase, and for k*=10000 the frequencies almost coincide with those of the non-elastic boundary condition. In other words, such large stiffness can be used to model the limiting case.

Some results in the tables are plotted for b/a=0.3 in Fig.2 to see the monotonical increase of frequencies as the plate edges are more stiffened. Eight cases out of 16 examples are chosen for b/a=0.3. When one compares the first two sets of Ex.1 (F-TS) and Ex.7 (TS-F), the translational stiffening effect along the outer edge is significantly larger than along the inner edge, because the length of outer edge periphery is larger than the inner periphery by $(2\pi a)/(2\pi b)=1/(b/a)=1/0.3=3.333$. In another comparison between Ex.2 (F-RS) and Ex.8 (RS-F), however, the stiffening effect on the outer edge by rotational spring is not so strong as translational springs. In right column, four cases of Ex.10, Ex.12, Ex.14 and Ex.16 are given. Particularly, two springs are located both on the outer and inner edges, and considerable increase with the stiffness is observed in Ex.14 and Ex.16.

4. Conclusions

Comprehensive freqeuncy data is summarized for the free vibration of thin isotropic annular plates on outer and/or inner edges elastically constrained by translational and rotational springs. General cases of sixteen different combinations are studied, and thorough results may serve for design purpose in structural engineering.

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