Various maximum principles for elliptic equations on unbounded domains
(非有界領域上の楕円型方程式に対する種々の最大値原理)

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Contents

Abstract

In this dissertation, we report some topics related to maximum principle. We deal with fully nonlinear second order elliptic partial differential equations on unbounded domains in the n-dimensional Euclidean space. Our argument is based on the viscosity solution theory.

In Chapter 1, we establish two Phragmén–Lindelöf theorems for viscosity subsolutions to fully nonlinear elliptic equations with a dynamical boundary condition. The first result is for an elliptic equation on an epigraph in $\mathbf{R}^n$. Because we assume a good structural condition, which includes wide classes of elliptic equations as well as uniformly elliptic equations, we can benefit from the strong maximum principle. The second result is for an equation that is strictly elliptic in one direction. Because the strong maximum principle does not need to hold for such equations, we adopt a strategy often used to prove the weak maximum principle. Considering such equations on a slab, we can approximate the viscosity subsolution by functions that strictly satisfy the viscosity inequality and obtain a contradiction.

In Chapter 2, we establish the Hadamard three sphere theorem for viscosity supersolutions to fully nonlinear uniformly elliptic equations with a superlinear growth in the gradient. The classical Hadamard property asserts that the circumferential minimum of the supersolution of an elliptic equation has some convexity. We prove this assertion by constructing a radially symmetric solution on the annulus. Moreover, we derive Liouville type theorem by applying the Hadamard theorem. In addition, we apply the argument to singular or degenerate elliptic cases, the ellipticity of which depends on the gradient.

Chapter 1 is essentially based on a paper [1], which is reproduced with permission from Springer Nature. Chapter 2 is based on [2], which is a joint work with Hamamuki.

All sections, formulas, theorems, etc. are cited only in the chapter where they appear.

References

[1] K. Abiko. Phragmén–lindelöf theorems for a weakly elliptic equation with a nonlinear dynamical boundary condition. *Partial Differential Equations and Applications*, 4(3):24, 2023.

[2] K. Abiko and N. Hamamuki. Hadamard and Liouville type theorems for fully nonlinear uniformly elliptic equations with a superlinear growth in the gradient, in preparation.

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Chapter 1

Phragmén–Lindelöf theorems for a weakly elliptic equation with a nonlinear dynamical boundary condition

1.1 Introduction

Background and motivation Maximum principle is one of the most fundamental properties of the solutions of elliptic partial differential equations. On a bounded domain, maximum principle is valid for a wide class of elliptic equations. On an unbounded domain, however, it does not always hold, even for uniformly elliptic equations. For example, $u(x) = x_1 x_n$ is harmonic in the half plane, $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n \mid x_n > 0\}$, and $u = 0$ on the boundary. However, since $u(1, 0, \ldots, 1) = 1$, we see maximum principle does not hold. Under a growth rate assumption on solutions, a similar estimate is valid. We say such assertion as Phragmén–Lindelöf theorem.

In this chapter, we consider an elliptic equation of the form

$$F(x, t, Du(x, t), D^2u(x, t)) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a domain, $T > 0$ is a given constant, and $u = u(x, t)$ is an unknown function. Du and D^2u represent the gradient and the Hessian matrix with respect to x of the solution u, respectively.

Throughout this chapter, we assume that $F : \overline{\Omega} \times (0, T] \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is continuous and degenerate elliptic. That is,

$$F(x, t, p, X) \leq F(x, t, p, Y) \text{ for } X \geq Y \text{ in } \mathbf{S}^n, \text{ and } (x, t, p) \in \overline{\Omega} \times (0, T] \times \mathbf{R}^n. \tag{1.1.2}$$

Here, $\mathbf{S}^n$ is the space of $n \times n$ real-valued symmetric matrices. As we do not assume F to be linear, we depend on the viscosity solution theory (cf. [12]).

As a boundary condition, we consider

$$\partial_t u(x, t) + B(x, Du(x, t)) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.1.3}$$

where B is a given continuous function on $\partial\Omega \times \mathbf{R}^n$, and $\partial_t u$ is the partial derivative with respect to t. The boundary condition that includes a time derivative term is called the dynamical boundary condition. This condition describes various diffusion phenomena such as thermal contact with a perfect conductor and solute diffusion from a stirred liquid or vapor. For studies related to the dynamical boundary condition, see [2, 14–19, 22, 24].

We are interested in the case where B is a nonlinear Neumann-type operator. Precisely, we assume (B1)–(B3).

(B1) $B(x, rp) = rB(x, p)$ for all $(x, p) \in \partial\Omega \times \mathbf{R}^n$ and all $r \geq 0$.

(B2) There exists $L_b > 0$ such that

$$|B(x, p) - B(x, q)| \leq L_b |p - q|$$

for all $(x, p) \in \partial\Omega \times \mathbf{R}^n$.

(B3) There exists $\theta > 0$ such that

$$B(x, p + \tau\nu(x)) - B(x, p) \geq \tau\theta$$

for all $(x, p) \in \partial\Omega \times \mathbf{R}^n$ and $\tau > 0$, where $\nu(x)$ is the unit outer normal vector on $\partial\Omega$.

Typical examples include the Neumann condition ($B(x,p) = \langle \nu(x), p\rangle$) and the oblique condition ($B(x,p) = \langle l(x), p\rangle$, where l is a given vector field with $\langle l, \nu\rangle > 0$). Here, $\langle \cdot, \cdot\rangle$ is the inner product in $\mathbf{R}^n$. We can also deal with the controlled reflection for

$$B(x,p) = \sup_{\alpha\in\mathcal{A}}\{\langle l_\alpha(x), p\rangle - g_\alpha(x)\},$$

with given functions l_α and g_α. For related results, see [4, 5, 22, 25, 26].

As a initial condition, we assume that

$$\limsup_{t\to+0}\ \sup_{|x|=R,\ x\in\partial\Omega} u(x,t) \le 0 \quad \text{for all } R>0. \tag{1.1.4}$$

Elliptic problems with dynamical boundary conditions have been studied extensively. For instance, see [17–19, 22, 28, 38, 39].

We denote the initial boundary value problem defined by equations (1.1.1), (1.1.3), and (1.1.4) as DBP. Our aim is to establish the Phragmén–Lindelöf theorems for viscosity subsolutions of DBP.

The Phragmén–Lindelöf theorem has been extensively studied. Classically, for subharmonic functions in $\Omega \subset \mathbf{R}^n_+$, the theorem states that if $u \le 0$ on $\partial\Omega$ and $u = o(|x|)$ as $|x| \to \infty$, then $u \le 0$ in Ω. Gilbarg proved it for $n = 2$ in [21] and Hopf did so for $n \ge 3$ in [23]. Later, Oddson ($n = 2$) and Miller ($n \ge 3$) dealt with the general linear equations ([29, 32]). For classical arguments, we refer the reader to [21, 23, 29–32, 34, 36].

For nonlinear equations with Dirichlet boundary conditions, Capuzzo Dolcetta and Vitolo proved the Phragmén–Lindelöf theorems via the Alexandrov–Bakelman–Pucci estimate and the weak boundary Harnack inequality for viscosity solutions (see [8]). Later, maximum principles and other estimates including the Phragmén–Lindelöf properties were studied. For related issues, we refer the reader to [7–10, 40]. Furthermore, the Phragmén–Lindelöf theorems have been shown in the framework of L^p-viscosity solutions. Koike and Nakagawa proved this in [27] for elliptic equations, and Tateyama did so in [37] for parabolic equations.

Uniformly elliptic equations Ishige and Nakagawa proved the Phragmén–Lindelöf theorems for fully nonlinear elliptic problems with a linear Neumann-type dynamical boundary condition in [24], which assumed (1.1.5) as a structure condition;

$$\mathcal{P}^-_{\lambda,\Lambda}(X) - L(x)|p'| \le F(x,t,p,X), \tag{1.1.5}$$

for $(x,t,p,X) \in \overline{\Omega}\times(0,T]\times\mathbf{R}^n\times\mathbf{S}^n$. Here, L is a positive continuous function on $\overline{\Omega}$, $p = (p', p_n) \in \mathbf{R}^{n-1}\times\mathbf{R}$, and $\mathcal{P}^-_{\lambda,\Lambda}(X)$ is the *Pucci's minimal operator* defined by

$$\mathcal{P}^-_{\lambda,\Lambda}(X) := \min\{-\operatorname{Tr}(AX) \mid \lambda I \le A \le \Lambda I,\ A \in \mathbf{S}^n\},$$

where $0 < \lambda \le \Lambda$.

A simple calculation confirms that (1.1.5) is satisfied if F is uniformly elliptic and Lipschitz continuous with respect to the $p' \in \mathbf{R}^{n-1}$ variable. Here, we say F is uniformly elliptic if there exists $0 < \lambda \le \Lambda$ such that

$$\mathcal{P}^-_{\lambda,\Lambda}(X-Y) \le F(x,t,p,X) - F(x,t,p,Y) \le \mathcal{P}^+_{\lambda,\Lambda}(X-Y)$$

for all $(x,t,p) \in \overline{\Omega}\times(0,T]\times\mathbf{R}^n$ and $X, Y \in \mathbf{S}^n$. Here, $\mathcal{P}^+_{\lambda,\Lambda}(X) := -\mathcal{P}^-_{\lambda,\Lambda}(-X)$ is the *Pucci's maximal operator*. We note that (1.1.5) does not necessarily imply the uniform ellipticity (see [8]).

Here, we present a result from [24].

Proposition 1 ([24, Theorem 5]). *Let $u \in C(\overline{\mathbf{R}^n_+}\times(0,T])$ be a viscosity subsolution of*

$$\begin{cases} F(x,t,Du,D^2u) = 0 & \text{in } \mathbf{R}^n_+\times(0,T], \\ \partial_t u + \partial_\nu u = 0 & \text{on } \partial\mathbf{R}^n_+\times(0,T], \end{cases}$$

where $F \in C(\overline{\mathbf{R}^n_+}\times(0,T]\times\mathbf{R}^n\times\mathbf{S}^n)$ satisfies (1.1.2) *and* (1.1.5)*. If u satisfies* (1.1.4) *and*

$$\liminf_{R\to\infty}\ \sup_{|x|=R,t\in(0,T)} \frac{u(x,t)}{1+x_n} \le 0, \tag{1.1.6}$$

then $u \le 0$ in $\overline{\mathbf{R}^n_+}\times(0,T]$.

We extend Proposition 1 to an assertion for nonlinear boundary problems. That is one of our main results in this chapter.

Theorem 1. *Assume that Ω is an epigraph in $\mathbf{R}^n_+$ such that*

$$\Omega = \{x \in \mathbf{R}^n \mid x_n > \rho(x')\}$$

for a nonnegative function $\rho \in C^2(\mathbf{R}^{n-1})$, F satisfies (1.1.2) *and* (1.1.5)*, and B satisfies* (B1)–(B3)*. Let $u \in C(\overline{\Omega}\times(0,T])$ be a viscosity subsolution of* DBP *and satisfy* (1.1.6)*, then $u \le 0$ in $\overline{\Omega}\times(0,T]$.*

We prove this assertion by a similar argument in [24]. After transforming the subsolution appropriately, it is attributed to an argument on a bounded domain, and the strong maximum principle is used to derive the contradiction.

Although we do not require F to be uniformly elliptic in Theorem 1, we still obtain the benefit of ellipticity by the structure condition (1.1.5) because the Pucci's minimal operator is uniformly elliptic. In Section 1.2, we review the strong maximum principle and the Hopf's boundary point lemma, as they play a significant role in our proof.

Directionally elliptic equations Next, we consider the case in which the strong maximum principle may not hold. Thus, we cannot expect significant ellipticity in the structure. In this case, we deal with a slab such that $\Omega = \{x \in \mathbf{R}^n \mid 0 < x_n < 1\}$.

Instead of the structure condition (1.1.5), we assume the following properties.

(F1) $F(x,t,0,O) = 0$ for $(x,t) \in \overline{\Omega} \times (0,T]$.

(F2) There exists a positive, continuous and bounded function $L(x,t)$ such that

$$|F(x,t,p,X) - F(x,t,q,X)| \le L(x,t)|p-q|$$

for all $(x,t,X) \in \overline{\Omega} \times (0,T] \times \mathbf{S}^n$ and $p,\ q \in \mathbf{R}^n$.

(F3) There exists a positive function $\gamma(x,t)$ such that $\liminf_{|x|\to\infty} \inf_{t\in(0,T]} \gamma(x,t) > 0$ and

$$F(x,t,p,X+\tau e_n \otimes e_n) - F(x,t,p,X) \le -\gamma(x,t)\tau$$

for all $(x,t,p,X) \in \overline{\Omega} \times (0,T] \times \mathbf{R}^n \times \mathbf{S}^n$ and $\tau > 0$. Here, we mean $\xi \otimes \eta := (\xi_i \eta_j)_{ij}$ for $\xi,\ \eta \in \mathbf{R}^n$ and $e_n := (0,\cdots,0,1) \in \mathbf{R}^n$.

(F4) For any sequence $\{(x_\varepsilon,t_\varepsilon)\} \subset \overline{\Omega} \times (0,T]$ such that $|x_\varepsilon| \to \infty$ as $\varepsilon \to +0$,

$$\liminf_{\varepsilon\to+0} F\left(x_\varepsilon, t_\varepsilon, 0, \frac{\varepsilon}{|x_\varepsilon|} I'\right) = 0.$$

Here, $I' = I - e_n \otimes e_n$.

(F3) refers to the directional ellipticity, which is a strictly monotone property in the e_n direction. With a small calculation, we can see that all uniformly elliptic functions are directionally elliptic in any direction. However, the directional ellipticity does not imply the uniform ellipticity. Furthermore, it does not ensure the strong maximum principle. (F4) applies a growth condition in the unbounded direction. For Dirichlet problems, the Phragmén–Lindelöf property is already known (see [7, 9, 10]). These assumptions are equivalent to those made in [9].

Example 1 ([9, Example 1.1, 1.3])**.** Consider the linear case

$$F(p,X) = -\operatorname{Tr}(AX) + \langle b, p\rangle,$$

where $A = (a_{ij})_{i,j} \in \mathbf{S}^n$ is positive-semidefinite, $b \in \mathbf{R}^n$. In this case, F satisfies (F3) if $a_{nn} > 0$ and the other properties are clearly satisfied. When F depends on (x,t), (F3) is fulfilled if $a_{nn} = a_{nn}(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times (0,T]$. (F4) is equivalent to $\limsup_{\varepsilon\to+0} \frac{\varepsilon}{|x_\varepsilon|} \operatorname{Tr}(I'A(x_\varepsilon,t_\varepsilon)) = 0$. This is fulfilled if $a_{ij}(x,t) = \mathcal{O}(|x|)$ as $|x| \to \infty$.

Furthermore, if F is Isaacs type such as

$$F(x,t,p,X) = \sup_\alpha \inf_\beta \{-\operatorname{Tr}(A^{\alpha\beta}(x,t)X) + \langle b^{\alpha\beta}(x,t), p\rangle\},$$

F satisfies (F1)–(F4) if, for all α and β,

$$A^{\alpha\beta} \ge O,\ a_{nn}^{\alpha\beta} > 0,\ a_{ij}^{\alpha\beta}(x,t) = \mathcal{O}(|x|)\ (|x| \to \infty),\ \text{and } b^{\alpha\beta} \text{ is bounded.}$$

Under these hypotheses, we establish the second main result of this chapter. We provide proof in Section 1.4.

Theorem 2. *Assume F satisfies* (1.1.2) *and* (F1)–(F4)*, and B satisfies* (B1)–(B3)*. Let $u \in C(\overline{\Omega} \times (0,T])$ be a viscosity subsolution of* DBP*. If u satisfies*

$$\liminf_{R\to\infty} \sup_{|x|=R, t\in(0,T]} \frac{u(x,t)}{1+|x|} \le 0, \tag{1.1.7}$$

then $u \le 0$ in $\overline{\Omega} \times (0,T]$.

Our proof is a modified version of the arguments in [9, 10], which discuss the Dirichlet problem.

The rest of this chapter is organized as follows. In Section 1.2, we prepare the definition of viscosity solutions of DBP and review some standard facts. In Section 1.3 and Section 1.4, we prove the Phragmén–Lindelöf theorems for DBP.

1.2 Preliminaries and basic results

We define a viscosity subsolution for the initial boundary value problem DBP.

Definition 1. Let $u \in C(\overline{\Omega} \times (0,T))$ and $\varphi \in C^2(\overline{\Omega} \times (0,T))$.

We say that u is a viscosity subsolution of DBP if u satisfies (1.1.4) and

(1)

$$F(\hat{x}, \hat{t}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) \leq 0 \tag{1.2.1}$$

holds whenever $u - \varphi$ attains its local maximum value at $(\hat{x}, \hat{t}) \in \Omega \times (0,T)$.

(2)

$$\min\{F(\hat{x}, \hat{t}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})), \partial_t\varphi(\hat{x}, \hat{t}) + B(\hat{x}, D\varphi(\hat{x}, \hat{t}))\} \leq 0 \tag{1.2.2}$$

holds whenever $u - \varphi$ attains its local maximum value at $(\hat{x}, \hat{t}) \in \partial\Omega \times (0,T)$.

Remark 1. We give two comments on treatment of $t = T$.

(i) In the definition above, $\hat{t} = T$ can be allowed. Precisely, if $u \in C(\overline{\Omega} \times (0,T])$ is a viscosity subsolution of DBP and $u - \varphi$ attains its local maximum value at $(\hat{x}, T) \in \overline{\Omega} \times (0,T]$, then the inequality (1.2.1) (or (1.2.2)) holds with $\hat{t} = T$. In fact, letting $\varphi_\varepsilon(x,t) = \varphi(x,t) + \frac{\varepsilon}{T-t}$ for each $\varepsilon > 0$, we see that $u - \varphi_\varepsilon$ attains its local maximum at some $(x_\varepsilon, t_\varepsilon) \in \overline{\Omega} \times (0,T)$ and that $(x_\varepsilon, t_\varepsilon) \to (\hat{x}, T)$ as $\varepsilon \to 0$. See [11, Lemma 5.7] for the detailed argument. In this chapter, we prove Theorems 1 and 2 by using this fact.

(ii) In [1], we proved Phragmén–Lindelöf type results for subsolutions $u \in C(\overline{\Omega} \times (0,T))$ under an assumption that u does not diverge as $t \to T$. However, this assumption can be removed by applying Theorem 1 or 2. In fact, for any $T' \in (0,T)$, we have $u \in C(\overline{\Omega} \times (0,T'])$. Thus Theorem 1 or 2 implies that $\sup_{\overline{\Omega} \times (0,T']} u \leq 0$. Since $T' \in (0,T)$ is arbitrary, we find that $\sup_{\overline{\Omega} \times (0,T)} u \leq 0$.

We require the strong maximum principle for our proof of Theorem 1. For related studies, refer to [3, 6, 13, 20, 33].

Proposition 2 ([13, Theorem 2.1])**.** *Let $\Omega \subset \mathbf{R}^n$ be a domain and $u \in USC(\overline{\Omega} \times [0,T])$ be a viscosity subsolution of*

$$G(x, t, u, \partial_t u, Du, D^2_{xt}u) = 0 \quad \text{in } \Omega \times (0,T),$$

where $G : \overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^{n+1} \to \mathbf{R}$ is a locally bounded and lower semicontinuous function. In addition, assume that G is proper in the sense of [12] *and satisfies the following properties:*

(G1) There exists $\rho_0 > 0$ such that for some $\gamma_0 \geq 0$,

$$G(x, t, 0, s, p, I - \gamma(s,p) \otimes (s,p)) > 0$$

for all $0 < |(s,p)| < \rho_0$ with $p \in \mathbf{R}^n \setminus \{0\}$, $\gamma > \gamma_0$, and $(x,t) \in \Omega \times (0,T)$.

(G2) For all $\eta > 0$, there exists a function $\varphi : (0,1) \to (0,+\infty)$, $\varepsilon_\eta > 0$ and $\gamma_0 \geq 0$ such that for all $\varepsilon \in (0, \varepsilon_n]$ and $\gamma > \gamma_0$ the following condition holds uniformly:

$$G(x, t, \varepsilon r, \varepsilon s, \varepsilon p, \varepsilon(I - \gamma(p,s) \otimes (p,s))) \geq \varphi(\varepsilon) G(x, t, r, s, p, I - \gamma(p,s) \otimes (p,s))$$

for all $(x,t) \in B_{(x_0,t_0)}(\eta)$ with the given $(x_0, t_0) \in \Omega \times (0,T)$, $r \in [-1, 0]$, $0 < |p| \leq \eta$, $|s| \leq \eta$. Here, $B_{(x_0,t_0)}(\eta)$ is a ball in $\mathbf{R}^{n+1}$ with its center at (x_0, t_0) and with a radius of η.

Then, the strong maximum principle is valid. Namely, if u attains a non-negative maximum at $(x_0, t_0) \in \Omega \times (0,T)$, u is constant in $\overline{\Omega} \times [0, t_0]$.

Although Da Lio [13] proved this assertion with a similar argument in the classical parabolic case, G is not necessarily parabolic. Particularly, elliptic equations on $\Omega \times (0,T)$ are also included.

In the classical argument for linear elliptic problems with the mixed boundary condition (see [34] for example), the Hopf's boundary point lemma plays an important role in the proof. Thus, we require the interior sphere condition. Namely, for all $x \in \partial\Omega$, there exists a ball $B \subset \Omega$ satisfying $x \in \partial B$. For $x \in \partial\Omega$, we set the radius $R(x)$ and the center $c(x)$.

Proposition 3 (Hopf's boundary point lemma)**.** *Assume the same hypotheses as in Proposition 5. In addition, assume that $\partial\Omega$ satisfies the interior sphere condition. Assume that $u(x_*, t_*) = M \geq 0$ for $(x_*, t_*) \in \partial\Omega \times (0,T]$, and $u(x,t) < M$ for all $(x,t) \in B_{R(x_*)}(c(x_*)) \times (t_* - R(x_*), t_*)$. Then,*

$$\liminf_{s \to +0} \frac{u(x_* + sw, t_* + s\tau) - u(x_*, t_*)}{s} < 0,$$

where $(w, \tau) \in \mathbf{R}^{n+1}$ satisfies $(x_* + sw, t_* + s\tau) \in B_{R(x_*)}(c(x_*)) \times (t_* - R(x_*), t_*)$.

We can prove this assertion by the standard argument (cf. [3, 6, 13]).

Proof. Set $B_* := B_{R(x_*)}(c(x_*))$ and $R_* := R(x_*)$. Define an auxiliary function,

$$h(x,t) := -e^{-\alpha|(x,t)-(c(x_*),t_*)|^2} + e^{-\alpha R_*^2} \tag{1.2.3}$$

with $\alpha > 0$. With a small calculation, we observe that

$$\begin{cases} h(x,t) < 0 & (|(x,t)-(c(x_*),t_*)| < R_* \text{ and } t \le t_*), \\ h(x,t) = 0 & (|(x,t)-(c(x_*),t_*)| = R_* \text{ and } t \le t_*), \\ h(x,t) > 0 & (|(x,t)-(c(x_*),t_*)| > R_* \text{ and } t \le t_*). \end{cases} \tag{1.2.4}$$

Then, using (G1) and (G2), we get

$$\begin{aligned} & G(x_*, t_*, h(x_*,t_*), \partial_t h(x_*,t_*), Dh(x_*,t_*), D^2_{xt}h(x_*,t_*)) \\ &= G(x_*, t_*, 0, 0, 2\alpha e^{-\alpha R_*^2} R_* \nu(x_*), 2\alpha e^{-\alpha R_*^2} I - 4\alpha^2 e^{-\alpha R_*^2} R_*^2 (\nu(x_*),0)\otimes(\nu(x_*),0)) \\ &\ge \varphi(2\alpha e^{-\alpha R_*^2}) G(x_*, t_*, 0, 0, \nu(x_*), I - 2\alpha R_*^2 (\nu(x_*),0)\otimes(\nu(x_*),0)) \\ &> 0 \end{aligned}$$

for sufficiently large $\alpha > 0$. By the lower semicontinuity of G, we can find a small $r > 0$ such that

$$G(x,t,h(x,t),\partial_t h(x,t), Dh(x,t), D^2_{xt}h(x,t)) \ge C > 0 \ ((x,t) \in D_*)$$

where

$$D_* := \{(x,t) \in \overline{\Omega} \times (0,T] \mid |(x,t)-(c(x_*),t_*)| < R_*, |(x,t)-(x_*,t_*)| < r, t < t_*\}.$$

Clearly, we find that εh is also strict supersolution of (1.3.2) in a neighborhood of (x_*, t_*).

Now, for small $\varepsilon > 0$, we prove that

$$u(x,t) \le \varepsilon h(x,t) + M \ (\text{for all } (x,t) \in \overline{D_*})$$

by contradiction.

Suppose that $\max_{\overline{D_*}}(u - \varepsilon h - M) = (u - \varepsilon h - M)(\hat{z}) > 0$ where $\hat{z} = (\hat{x}, \hat{t}) \in \overline{D}_*$.

Consider the case $|\hat{z} - (x_*, t_*)| = r$ and $|\hat{z} - (c(x_*), t_*)| < R_*$. Since $\inf_{|z-(x_*,t_*)|=r} |\hat{z} - z| > 0$, there exists a constant $C > 0$ such that $u - M > C > 0$. Also considering (1.2.4), we obtain $u - \varepsilon h - M \le 0$ for sufficiently small $\varepsilon > 0$, thus this case is unsuitable.

Therefore $\hat{z} \in D_*$ and thus we can regard $\varepsilon h + M$ as a test function. So the following inequality has to hold:

$$G(\hat{z}, \varepsilon h(\hat{z}), \varepsilon\partial_t h(\hat{z}), \varepsilon Dh(\hat{z}), \varepsilon D^2_{xt}h(\hat{z})) \le G(\hat{z}, u(\hat{z}), \varepsilon\partial_t h(\hat{z}), \varepsilon Dh(\hat{z}), \varepsilon D^2_{xt}h(\hat{z})) \le 0.$$

However, this contradicts the fact that εh is a strict supersolution.

Thus, we find $u(x_* + sw, t_* + s\tau) \le \varepsilon h(x_* + sw, t_* + s\tau) + M$. Then,

$$\begin{aligned} \frac{u(x_* + sw, t_* + s\tau) - u(x_*, t_*)}{s} &\le \varepsilon \frac{h(x_* + sw, t_* + s\tau) - h(x_*, t_*)}{s} \\ &\to \varepsilon \langle Dh(x_*, t_*), w\rangle + \varepsilon\tau\partial_t h(x_*, t_*) \ (s \to +0) \\ &< 0. \end{aligned}$$

Then, we reach the conclusion. □

Remark 2. If $\rho \in C^2(\mathbf{R}^{n-1})$, the epigraph satisfies the interior sphere condition.

Lastly, we prepare a lemma for proof of Theorem 2. In this case, recall that $\Omega = \{x \in \mathbf{R}^n \mid 0 < x_n < 1\}$. We set $\partial_0\Omega := \{x \in \mathbf{R}^n \mid x_n = 0\}$ and $\partial_1\Omega := \{x \in \mathbf{R}^n \mid x_n = 1\}$. Lemma 1 is the standard technique for proof of the weak maximum principle and comparison results. See [12] for instance.

Lemma 1. *Assume that F satisfies* (F2) *and* (F3). *Let $u \in C(\overline{\Omega} \times (0,T])$ be a viscosity subsolution of* DBP. *Then, there exists $u_\varepsilon \in C(\overline{\Omega} \times (0,T))$ such that $u_\varepsilon \to u$ as $\varepsilon \to +0$ locally uniformlly, $u_\varepsilon(x,t) \le u(x,t)$ for all $(x,t) \in \overline{\Omega} \times (0,T)$, and it is a viscosity subsolution of* (1.2.5)*:*

$$\begin{cases} F(x,t,Du_\varepsilon, D^2u_\varepsilon) + \varepsilon C = 0 & \text{in } \Omega \times (0,T), \\ \partial_t u_\varepsilon + B(x, Du_\varepsilon) + \varepsilon C = 0 & \text{on } \partial\Omega \times (0,T), \\ \limsup\limits_{t\to+0} \sup\limits_{|x|=R, x\in\partial\Omega} u_\varepsilon(x,t) \le 0 & \text{for all } R > 0, \end{cases} \tag{1.2.5}$$

where $C = C(x,t) > 0$ is a positive continuous function such that $\liminf_{|x|\to\infty} \inf_{t\in(0,T]} C(x,t) > 0$.

Proof. Define

$$u_\varepsilon(x,t) := u(x,t) - \varepsilon(e^\alpha - e^{\alpha x_n}) - \frac{\delta}{T-t} \tag{1.2.6}$$

for ε, δ, $\alpha > 0$. Then, in $\Omega \times (0,T)$, we obtain the following estimation in the viscosity sense:

$$\begin{aligned} 0 &\geq F(x,t,Du_\varepsilon - \varepsilon\alpha e^{\alpha x_n} e_n, D^2 u_\varepsilon - \varepsilon\alpha^2 e^{\alpha x_n} e_n \otimes e_n) \\ &\geq F(x,t,Du_\varepsilon, D^2 u_\varepsilon) + \varepsilon\gamma\left(\alpha^2 \frac{L}{\gamma} - \alpha e^{-\alpha}\right). \end{aligned}$$

Thus $F(x,t,Du_\varepsilon, D^2u_\varepsilon) + C \leq 0$ for sufficiently large $\alpha > 0$.

On $\partial_0\Omega \times (0,T)$,

$$\begin{aligned} 0 &\geq \partial_t u_\varepsilon + \frac{\delta}{(T-t)^2} + B(x, Du_\varepsilon - \varepsilon\alpha\nu(x)) \\ &\geq \partial_t u_\varepsilon + \frac{\delta}{(T-t)^2} + B(x, Du_\varepsilon) - \varepsilon\alpha L_b \\ &\geq \alpha e L_b \varepsilon + \partial_t u_\varepsilon + B(x, Du_\varepsilon) \end{aligned}$$

for $\delta = 2T^2\alpha L_b \varepsilon e$. We also calculate on $\partial_1\Omega \times (0,T)$ by a similar line. ☐

1.3 Proof of Theorem 1

Proof of Theorem 1. Let

$$v(x,t) := \frac{u(x,t)}{e^{L_b t}(1+x_n)}.$$

Then, (1.1.6) implies $\liminf_{R\to\infty} \sup_{|x|=R, t\in(0,T]} v(x,t) \leq 0$. Particularly, for all $\varepsilon > 0$, there exists $\{R_k\}$ such that $R_k \to \infty$ as $k \to \infty$ and

$$\sup_{|x|=R_k, t\in(0,T]} v(x,t) \leq \varepsilon$$

for large k.

Set

$$\Omega_k := \{x \in \Omega \mid |x| < R_k\},\ \Gamma_k := \partial\Omega \cap \partial\Omega_k,\ \Gamma_k' := \Omega \cap \partial\Omega_k.$$

We argue by contradiction. Suppose that there exists k such that $\sup_{|x|\leq R_k, t\in(0,T]} v(x,t) > \varepsilon$. Let $M_k := \sup_{|x|\leq R_k, t\in(0,T]} v(x,t)$. There exists $(x_*, t_*) \in \overline{\Omega_k} \times (0,T]$ which satisfies $v(x_*, t_*) = M_k$.

By using (1.1.5), (B1), and (B2), we find that v is a viscosity subsolution of

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(D^2 v) - \tilde{L}|Dv| = 0 & \text{in } \Omega_k \times (0,T] \\ \partial_t v + B(x,Dv) + L_b v - L_b|v| = 0 & \text{on } \Gamma_k \times (0,T], \\ v = \varepsilon & \text{on } \Gamma_k' \times (0,T]. \end{cases} \tag{1.3.1}$$

Here, $\tilde{L} := \frac{2n\Lambda}{1+x_n} + L$.

Therefore, we can apply the strong maximum principle to v. Considering the initial condition, we obtain $(x_*, t_*) \in \Gamma_k \times (0,T]$ and $v < M_k$ in $\Omega_k \times (0,T]$.

Define

$$\varphi(x,t) := \delta h(x,t) + M_k,$$

where h is the auxiliary function defined by (1.2.3), and $\delta > 0$ is sufficiently small. Then, by the same argument used in the proof of Proposition 3, we observe that $\varphi \geq v$ in $\{(x,t) \mid |(x,t) - (c(x_*), t_*)| \leq R(x_*)\}$. Additionally, because $\varphi > M_k$ holds in the outside of the ball, there exists a neighborhood of (x_*, t_*) such that $\varphi \geq v$ holds. Therefore, we can apply φ as a test function.

We know that $Dh(x_*, t_*) = 2\alpha e^{-\alpha R(x_*)^2} R(x_*)\nu(x_*)$ and $\partial_t h(x_*, t_*) = 0$. Thus, we find $\mathcal{P}^-_{\lambda,\Lambda}(D^2\varphi(x_*, t_*)) - \tilde{L}(x_*)|D\varphi(x_*, t_*)| > 0$. Furthermore, we obtain

$$\begin{aligned} &\partial_t\varphi(x_*, t_*) + B(x_*, D\varphi(x_*, t_*)) + L_b v(x_*, t_*) - L_b|v(x_*, t_*)| \\ &= \delta\partial_t h(x_*, t_*) + \delta B(x_*, Dh(x_*, t_*)) \\ &= 2\delta\alpha e^{-\alpha R(x_*)^2} R(x_*) B(x_*, \nu(x_*)) \\ &\geq 2\delta\alpha e^{-\alpha R(x_*)^2} R(x_*)\theta \\ &> 0, \end{aligned}$$

and this is a contradiction. Therefore, we obtain $M_k \leq \varepsilon$ for all k. Considering $k \to \infty$ and $\varepsilon \to +0$, we reach the conclusion. □

Proof of Proposition 1. Apply Theorem 1 with $\rho \equiv 0$ and $B(x,p) = -p_n$. Then the conclusion immediately follows. □

Remark 3. We can apply the same argument as above to the parabolic equations $\partial_t u + F(x,t,Du,D^2u) = 0$ because the strong maximum principle similarly holds.

Remark 4. The growth condition (1.1.6) is essential. Consider the case in which $|D\rho| < C$ for some constant $C > 0$, $F(x,t,p,X) = -\operatorname{Tr}(X)$, and $B(x,p) = \langle \nu(x), p\rangle$. Then, $u(x,t) = \frac{t}{\sqrt{C^2+1}} + x_n$ is a subsolution of DBP and $u > 0$ in $\Omega \times (0,T]$.

In consideration of this observation, some alternative ideas would be as follows. Instead of (1.1.6), we consider the following growth condition of the form

$$\liminf_{R\to\infty} \sup_{|x|=R, t\in(0,T]} \frac{u(x,t)}{1+g(x,t)} \leq 0.$$

If $g \in C^2(\overline{\Omega} \times [0,T])$ is nonnegative and

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(D^2 g) - L|D'g| \geq 0 & \text{in } \Omega \times (0,T], \\ \partial_t g + B(x, Dg) \geq 0 & \text{on } \partial\Omega \times (0,T], \\ g(x,0) \geq 0 & \text{in } \overline{\Omega}, \end{cases} \tag{1.3.2}$$

we can apply the same argument used above. However, if g is just continuous, and not expected to be differentiable, our argument does not work. If g satisfies (1.3.2) in the viscosity sense, a method using the comparison theorem can be employed. This approach is treated in [35].

Example 2. Now we consider the minimal surface equation, in which

$$F(p,X) = -\operatorname{Tr}\left\{\left(I - \frac{p\otimes p}{1+|p|^2}\right)X\right\}. \tag{1.3.3}$$

As argued in [20, Remark 2.7], F is uniformly elliptic if there exists $M > 0$ such that $|p| \leq M$ with the ellipticity constants $\lambda = \frac{1}{1+M^2}$ and $\Lambda = 1$. In addition, $p \mapsto F(p,X)$ is Lipschitz continuous if $|p|$ and $\|X\|$ are bounded. Namely, the benefits of a good structure can only be localized. In addition, without some modification of the argument or additional assumptions, it seems unlikely that equation corresponding to (1.3.1) can be expected to have a good structure. In order to apply Theorem 1, we need to make some modifications.

1.4 Proof of Theorem 2

Proof of Theorem 2. It suffices that we prove Theorem 2 for subsolutions of (1.2.5).

Suppose that there exists $(x_*,t_*) \in \overline{\Omega} \times [0,T)$ such that $u(x_*,t_*) =: M > 0$. (1.1.7) implies that for all $\varepsilon > 0$, there exists a subsequence $\{R_\varepsilon\}$ such that $R_\varepsilon \to \infty$ as $\varepsilon \to +0$, and

$$\sup_{|x|=R_\varepsilon, t\in(0,T)} \frac{u(x,t)}{\psi(x)} \leq \varepsilon, \tag{1.4.1}$$

where $\psi(x',x_n) := \sqrt{1+|x'|^2}$.

Let $u^\varepsilon := u - \varepsilon\psi$ and $M_\varepsilon := \sup_{\overline{\Omega}\times(0,T)} u^\varepsilon$. Then we find $M_\varepsilon > \frac{M}{2}$ for sufficiently small $\varepsilon > 0$. Because we know $u^\varepsilon(x,t) \leq 0$ for all $|x| \geq R_\varepsilon$ and $t \in (0,T)$ form (1.4.1), we obtain

$$M_\varepsilon = \sup_{|x|\leq R_\varepsilon, t\in(0,T)} u^\varepsilon(x,t).$$

Considering the initial condition and (1.2.6), we can find $(x_\varepsilon,t_\varepsilon) \in \overline{\Omega_\varepsilon} \times (0,T)$ such that $M_\varepsilon = u^\varepsilon(x_\varepsilon,t_\varepsilon)$. Here, $\Omega_\varepsilon := \{x \in \Omega \mid |x| < R_\varepsilon\}$.

First, consider the case $x_\varepsilon \in \partial\Omega$. Because ψ is smooth, we can regard ψ as a test function. We thus know that

$$0 \geq C + B(x_\varepsilon, \varepsilon D\psi(x_\varepsilon)) \geq C - \varepsilon L_b.$$

Thus, this is a contradiction for small $\varepsilon > 0$.

Next, consider the case $x_\varepsilon \in \Omega$. We know that

$$F(x_\varepsilon, t_\varepsilon, \varepsilon D\psi(x_\varepsilon), \varepsilon D^2\psi(x_\varepsilon)) + C \leq 0.$$

By directional calculation, we have

$$D\psi(x) = \frac{1}{\psi(x)}x', \quad D^2\psi(x) = \frac{1}{\psi(x)}I' - \frac{1}{\psi(x)^3}x' \otimes x'.$$

Here, we identify $x' \in \mathbf{R}^{n-1}$ with $(x', 0) \in \mathbf{R}^n$.

Take $\varepsilon \to +0$. Because $\{t_\varepsilon\}$ is bounded, there exist a subsequence and $\hat{t}$ such that $t_\varepsilon \to \hat{t}$. The initial condition implies $\hat{t} > 0$. By examining (1.2.6), we observe $u(x,t) \to -\infty$ as $t \to T$. Thus, $\hat{t} < T$ because $M_\varepsilon > 0$.

If $\{x_\varepsilon\}$ is bounded, there exists a subsequence such that $(x_\varepsilon, t_\varepsilon) \to (\hat{x}, \hat{t})$ for $(\hat{x}, \hat{t}) \in \overline{\Omega} \times (0,T)$.

Then,

$$0 \geq C + \liminf_{\varepsilon \to +0} F(x_\varepsilon, t_\varepsilon, \varepsilon D\psi(x_\varepsilon), \varepsilon D^2\psi(x_\varepsilon)) > F(\hat{x}, \hat{t}, 0, O) = 0.$$

If $\{x_\varepsilon\}$ is unbounded, there exist $x_\varepsilon \in \Omega$ with $|x'_\varepsilon| \to \infty$. Thus,

$$0 \geq C + F\left(x_\varepsilon, t_\varepsilon, 0, \frac{\varepsilon}{\psi(x_\varepsilon)}I'\right) - \varepsilon L(x_\varepsilon, t_\varepsilon),$$

and taking $\varepsilon \to +0$, we find a contradiction by (F4). □

Remark 5. The aforementioned argument is still valid for domains with a curved boundary. Precisely, we can also obtain the same estimation for the case $\Omega = \{x \in \mathbf{R}^n \mid \rho_0(x') < x_n < \rho_1(x')\}$, where $\rho_0,\ \rho_1 \in C^1(\mathbf{R}^{n-1})$ are given bounded functions, which satisfy $\rho_0 < \rho_1$ and $|D'\rho_0|$, $|D'\rho_1|$ are bounded in $\mathbf{R}^{n-1}$.

Remark 6. Instead of (B2), we can prove this assertion when we assume the following condition (B4):

(B4) For all $x_\varepsilon \in \partial_0\Omega$ with $|x_\varepsilon| \to \infty$ as $\varepsilon \to +0$,

$$\liminf_{\varepsilon \to +0} \varepsilon B\left(x_\varepsilon, \frac{x'_\varepsilon}{\psi(x_\varepsilon)}\right) > 0.$$

First, return to Lemma 1. By the same definition of u_ε, the argument in $\Omega \times (0,T)$ and $\partial_1\Omega \times (0,T)$ are still valid. Therefore, in the proof of Theorem 2, the same argument still works if $x_\varepsilon \in \Omega$ or $x_\varepsilon \in \partial_1\Omega$.

On $\partial_0\Omega \times (0,T)$, we obtain

$$0 \geq \partial_t u_\varepsilon + \frac{\delta}{T^2} + B(x, Du_\varepsilon - \varepsilon\alpha\nu).$$

Consider the case in which $x_\varepsilon \in \partial_0\Omega \times (0,T)$. If the maximum value M_ε is achieved at some interior point, we already know that it is a contradiction. Thus, the maximum value is achieved only on the boundary. Applying the same argument for the Hopf's boundary point lemma, we can find that there exists a smooth function $\varphi(x,t)$ such that $u^\varepsilon - \varphi$ attains its local maximum value at $(x_\varepsilon, t_\varepsilon)$, and $D\varphi(x_\varepsilon, t_\varepsilon) = -Ae_n = A\nu(x_\varepsilon)$ for some $A > 0$.

Therefore, we have

$$0 \geq \frac{\delta}{T^2} + A'\theta + \varepsilon B\left(x_\varepsilon, \frac{x'_\varepsilon}{\psi(x_\varepsilon)}\right),$$

where $A' < A$ is a positive constant. Taking $\varepsilon \to +0$, we have a contradiction.

Remark 7. The minimal surface equation (1.3.3) is not directional elliptic with any fixed directions because the direction depends on the gradient. Thus we can not apply Theorem 2 directlly.

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Chapter 2

Hadamard and Liouville type theorems for fully nonlinear uniformly elliptic equations with a superlinear growth in the gradient

2.1 Introduction

In this chapter we consider a fully nonlinear partial differential equation

$$F(x, u(x), Du(x), D^2u(x)) = 0. \tag{2.1.1}$$

Here u is a real-valued unknown function defined in a subset of $\mathbf{R}^n$, where $n \geq 2$ is a given natural number. Also, $Du = (u_{x_i})_{i=1}^n$ and $D^2u = (u_{x_ix_j})_{i,j=1}^n$ denote the gradient and the Hesse matrix of u, respectively. Our goal of this chapter is to establish Hadamard and Liouville type theorems.

Assumptions. We list assumptions on $F : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$. Throughout this chapter we assume the following (F1)–(F3).

(F1) $F : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is a continuous function, where $\mathbf{S}^n$ denotes the set defined by $n \times n$ real symmetric matrices.

(F2) There exist positive constants $\lambda \leq \Lambda$ such that

$$F(x, s, p, X) \leq F(x, s, p, O) + \mathcal{P}^+_{\lambda,\Lambda}(X)$$

for all $(x, s, p, X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$.

(F3) There exist constants $k \in (1, 2]$, $\alpha \geq 1$, a continuous function $\sigma : [0, \infty) \to [0, \infty)$, and a continuous fnction $h : \mathbf{R}^n \to (-\infty, 0]$ such that

$$F(x, s, p, O) \leq \sigma(|x|)|p|^k + h(x)s^\alpha$$

for all $(x, s, p) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$.

Here $\mathcal{P}^+_{\lambda,\Lambda} : \mathbf{S}^n \to \mathbf{R}$ is the *Pucci's maximal operator* defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := \sup\{-\operatorname{Tr}(AX) \mid A \in \mathbf{S}^n, \lambda I \leq A \leq \Lambda I\},$$

where $I \in \mathbf{S}^n$ is the identity matrix. We will later use the following representation

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = -\lambda \sum_{e_i \geq 0} e_i - \Lambda \sum_{e_i < 0} e_i, \tag{2.1.2}$$

where $e_1, \ldots, e_n$ are the eigenvalues of $X \in \mathbf{S}^n$. See [5, Chap. 2] for more details.

We note that, by (F2) and (F3), any nonnegative viscosity supersolution u of (2.1.1) is also a supersolution of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u(x)) + \sigma(|x|)|Du|^k + h(x)\{u(x)\}^\alpha = 0 \tag{2.1.3}$$

and

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u(x)) + \sigma(|x|)|Du|^k = 0. \tag{2.1.4}$$

Background. In partial differential equations theory, Liouville type property is a highly interesting subject. There are some ideas to prove it. For solutions of suitable elliptic equations, one of the most popular ways to derive it is to employ Harnack's inequality. Amendola, Rossi, and Vitolo obtained the Liouville property for continuous viscosity solutions of (2.1.4) in [2] via Harnack's inequality (see [5, 12] for basic arguments). For general arguments related to the Liouville property, see [8, 12, 17, 18] and references therein.

A method using the Hadamard three spheres theorem is also known and we adopt this approach. One advantage of this approach is that it is also valid for supersolutions.

Now we explain the Hadamard three spheres theorem. For $r_1 > 0$ and $u : \mathbf{R}^n \to \mathbf{R}$ we define a function $m : [r_1, \infty) \to \mathbf{R}$ as

$$m(r) := \min_{r_1 \leq |x| \leq r} u(x). \tag{2.1.5}$$

If u is a supersolution of some elliptic equation, it is known that m is a concave function of a nonnegative function ψ. Precisely, there exists a function $\psi : \mathbf{R} \to [0, \infty)$ such that

$$m(r) \geq \frac{\psi(r) - \psi(r_1)}{\psi(r_2) - \psi(r_1)} m(r_2) + \frac{\psi(r_2) - \psi(r)}{\psi(r_2) - \psi(r_1)} m(r_1) \quad (r \in [r_1, r_2]), \tag{2.1.6}$$

where $0 < r_1 < r_2$. Such assertion is called the Hadamard three spheres theorem. If u is a superharmonic function, ψ is a fundamental solution of Laplace's equation, that is, $\psi(r) = \log r$ $(n = 2)$ or r^{2-n} $(n \geq 3)$. See [17, Chap. 2, Sec. 12] for arguments for linear equations.

Cutrì and Leoni obtained (2.1.6) for viscosity supersolutions of a nonlinear uniformly elliptic equation of the form $F(x, D^2u) + h(x)u^p = 0$ in [10]. They characterized ψ as a fundamental solution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2u) = 0$ in $\mathbf{R}^n \setminus \{0\}$. After that, in [6], Capuzzo Dolcetta and Cutrì extended this result to viscosity supersolutions of equations including the first order derivative term. In other words, they dealt with (2.1.1) under the assumption (F3) with $k = 1$. In their Hadamard type result, ψ was defined by

$$\psi(r) = \int_{r_1}^{r} s^{-\frac{\Lambda(n-1)}{\lambda}} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s} \sigma(\tau)\, d\tau\right) ds.$$

Key idea for the Hadamard property is to find a radially symmetric solution $\Phi = \Phi(x)$ of Dirichlet problem

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi) + \sigma(|x|)|D\Phi| = 0 & \text{in } \Omega, \\ \Phi(x) = m(r_1) & \text{for } |x| = r_1, \\ \Phi(x) = m(r_2) & \text{for } |x| = r_2, \end{cases}$$

where $m(r)$ is defined by (2.1.5) and $\Omega := \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\}$. Indeed, the right hand side of (2.1.6) with $r = |x|$ is a solution of above boundary value problem.

Liouville type result can be obtained by the Hadamard property. Indeed, if the supersolution u is nonnegative and $\psi(r_1) = 0$, (2.1.6) implies

$$m(r) \geq m(r_1)\left(1 - \frac{\psi(r)}{\psi(r_2)}\right).$$

In addition, if $\lim_{r_2 \to \infty} \psi(r_2) = \infty$, we obtain $m(r) \geq m(r_1)$ for $r_1 \leq r$, and we immediately reach the conclusion by the strong minimum principle. [10, Theorem 3.2] and [6, Theorem 4.1] are established by this argument. Capuzzo Dolcetta and Cutrì also proved the Liouville theorem even if $\lim_{r_2 \to \infty} \psi(r_2) < \infty$ under some additional assumptions ([6, Theorem 4.2]). For arguments related to the Hadamard property and its applications, we refer the reader to [1, 4, 6, 7, 10, 11, 13].

We apply this approach to elliptic equations with a superlinear growth in the gradient term. As we noted before, the Liouville property for viscosity solutions was proved via Harnack's inequality in [2] but we also deal with supersolutions. In addition, for the case where $\sigma \leq 0$, the Liouville property was established in [8] however the case where $\sigma \geq 0$ was not solved. Thus our results are different from previous studies in these aspects.

We remark on two points. First, we require $k \leq 2$ for ensuring the strong minimum principle. Secondly, because the equation is nonlinear, we can only obtain a radically symmetric solution including a parameter implicitly. Therefore we have to analyze closely how the solution behaves with respect to the parameter in order to obtain the Liouville property.

The rest of this chapter is organized as follows. In Section 2.2, we recall the definition of viscosity solutions and review some related facts. In Section 2.3 we find a radially symmetric solution on an annulus and prove the Hadamard three spheres theorem. In Section 2.4 we prove Liouville type theorem and analyze the sufficient condition. We also consider an equation which is degenerate elliptic or has a singularity on its ellipticity in Section 2.5.

2.2 Preliminaries

We define a notion of viscosity solutions and state some results on maximum principles. For $D \subset \mathbf{R}^n$ let us set $\mathrm{USC}(D) := \{u : D \to \mathbf{R} \mid u \text{ is upper semicontinuous in } D\}$ and $\mathrm{LSC}(D) := \{u : D \to \mathbf{R} \mid u \text{ is lower semicontinuous in } D\}$.

Definition 2 (Viscosity solutions, [9])**.** Let $D \subset \mathbf{R}^n$ be a domain.

(1) We say $u \in \mathrm{LSC}(D)$ (resp. $u \in \mathrm{USC}(D)$) is a viscosity supersolution (resp. viscosity subsolution of (2.1.4)) in D if

$$F(z, u(z), D\phi(z), D^2\phi(z)) \geq 0 \text{ (resp. } \leq 0)$$

holds for any $z \in D$ and $\phi \in C^2(D)$ such that $u - \phi$ attains a local minimum (resp. local maximum) at z.

(2) If $u \in C(D)$ is both a viscosity supersolution and a viscosity subsolution of (2.1.1) in D, we say that u is a viscosity solution of (2.1.1) in D.

We next state the comparison principle and the strong minimum principle for (2.1.4).

Proposition 4 (Comparison principle)**.** *Let $D \subset \mathbf{R}^n$ be a bounded domain. Let $u \in \mathrm{LSC}(\overline{D})$ and $v \in C^2(D) \cap C(\overline{D})$ be respectively a viscosity supersolution of* (2.1.4) *and a classical subsolution of* (2.1.4) *in D. If $v \leq u$ on ∂D, then $v \leq u$ in D.*

We omit the proof because it is based on the classical argument. See e.g. [16].

Proposition 5 (Strong minimum principle)**.** *Let $D \subset \mathbf{R}^n$ be a domain and let $u \in \mathrm{LSC}(D)$ be a viscosity supersolution of* (2.1.4)*. If u achieves a nonpositive minimum in D, then u is constant.*

We omit the proof here because it is the same argument in [3], which is based on the Hopf's boundary point lemma. See also [2, 15].

2.3 Radially symmetric solutions

Let $0 < r_1 < r_2$, and $c_1 > c_2 > 0$. In this section we consider

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi(x)) + \sigma(|x|)|D\Phi(x)|^k = 0 \quad \text{in } \Omega = \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\} \tag{2.3.1}$$

under a boundary condition

$$\Phi(x) = c_1 \quad \text{on } \partial B_{r_1}, \tag{2.3.2}$$

and

$$\Phi(x) = c_2 \quad \text{on } \partial B_{r_2}. \tag{2.3.3}$$

We seek a radically symmetric classical solution $\Phi \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2.3.1)–(2.3.3). For this purpose, we let $\phi \in C^2((r_1, r_2)) \cap C([r_1, r_2])$ and assume that Φ is of the form

$$\Phi(x) = \phi(|x|).$$

Then we have

$$\begin{aligned} D\Phi(x) &= \phi'(|x|)\frac{x}{|x|}, \\ D^2\Phi(x) &= \phi''(|x|)\frac{x \otimes x}{|x|^2} + \phi'(|x|)\frac{|x|I_n - \frac{x\otimes x}{|x|}}{|x|^2} \\ &= \frac{\phi'(|x|)}{|x|}I_n + \left\{\frac{\phi''(|x|)}{|x|^2} - \frac{\phi'(|x|)}{|x|^3}\right\}(x \otimes x). \end{aligned}$$

Here $x \otimes x = (x_i x_j)^n_{ij}$ for $x = (x_i)^n_{i=1} \in \mathbf{R}^n$. From the above representation of $D^2\Phi(x)$, it follows that its eigen values are $\phi''(|x|)$ which is simple and $\frac{\phi'(|x|)}{|x|}$ with multiplicity $(n-1)$; $\mathbf{R}x$ and $(\mathbf{R}x)^\perp$ are respectively the eigenspaces associated with $\phi''(|x|)$ and $\frac{\phi'(|x|)}{|x|}$. See [10, Lemma 3.1].

Hereafter we assume that

$$\phi' \leq 0, \quad \phi'' \geq 0 \quad \text{in } (r_1, r_2). \tag{2.3.4}$$

Then

$$|D\Phi(x)|^k = |\phi'(|x|)|^k = \{-\phi'(|x|)\}^k.$$

Also, taking the signs of the eigenvalues of $D^2\Phi(x)$ into account, we see by (2.1.2) that

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi(x)) = -\lambda\phi''(|x|) - \Lambda(n-1)\frac{\phi'(|x|)}{|x|}.$$

Plugging these into (2.3.1) and letting $|x| = r$, we are led to the ordinary differential equation

$$-\lambda\phi''(r) - \Lambda(n-1)\frac{\phi'(r)}{r} + \sigma(r)\{-\phi'(r)\}^k = 0 \quad (r \in (r_1, r_2)). \tag{2.3.5}$$

By (2.3.2) and (2.3.3) the boundary condition for ϕ is

$$\phi(r_1) = c_1, \tag{2.3.6}$$
$$\phi(r_2) = c_2. \tag{2.3.7}$$

In order to solve (2.3.5)–(2.3.7), we put $y(r) := -\phi'(r)$. Then (2.3.4) implies

$$y \geq 0, \quad y' \leq 0 \quad \text{in } (r_1, r_2),$$

and (2.3.1) becomes

$$\lambda y'(r) + \Lambda(n-1)\frac{y(r)}{r} + \sigma(r)\{y(r)\}^k = 0 \quad (r \in (r_1, r_2)).$$

This is a Bernoulli equation, and as is well known, this equation reduces to a linear equation by the transformation $z(r) := \{y(r)\}^{1-k}$. In fact, if $y > 0$ in (r_1, r_2), then z is well-defined and solves

$$z'(r) - A(k-1)\frac{z(r)}{r} - \frac{k-1}{\lambda}\sigma(r) = 0 \quad (r \in (r_1, r_2)), \tag{2.3.8}$$

where

$$A := \frac{\Lambda(n-1)}{\lambda}.$$

From (2.3.8) we deduce

$$\begin{aligned} z(r) &= e^{\int_{r_1}^r A(k-1)\frac{1}{s}\,ds}\left(\int_{r_1}^r \frac{k-1}{\lambda}\sigma(s)e^{-\int_{r_1}^s A(k-1)\frac{1}{t}\,dt}\,ds + z(r_1)\right) \\ &= \left(\frac{r}{r_1}\right)\left(\int_{r_1}^r \frac{k-1}{\lambda}\sigma(s)\left(\frac{s}{r_1}\right)^{-A(k-1)}\,ds + z(r_1)\right) \\ &= r^{A(k-1)}\left(\frac{k-1}{\lambda}\int_{r_1}^r \sigma(s)s^{-A(k-1)}\,ds + r_1^{-A(k-1)}z(r_1)\right), \end{aligned}$$

and therefore

$$y(r) = \{z(r)\}^{-\frac{1}{k-1}} = r^{-A}\left(\frac{k-1}{\lambda}\int_{r_1}^r \sigma(s)s^{-A(k-1)}\,ds + \{r_1^A y(r_1)\}^{-(k-1)}\right)^{-\frac{1}{k-1}}.$$

We now regard the term $\{r_1^A y(r_1)\}^{-(k-1)}$ as a parameter $\theta > 0$ and set

$$y(r,\theta) = r^{-A}\left(\frac{k-1}{\lambda}\int_{r_1}^r \sigma(s)s^{-A(k-1)}\,ds + \theta\right)^{-\frac{1}{k-1}}. \tag{2.3.9}$$

Remark 8. Because σ is nonnegative, we have $y(r,\theta) > 0$ for $(r,\theta) \in (r_1, r_2) \times (0,\infty)$. Moreover, $y(r,\cdot)$ and $y(\cdot,\theta)$ are decreasing.

Recall that we put $y(r) = -\phi'(r)$. Thus the function

$$\phi(r,\theta) := c_1 - \int_{r_1}^r y(t,\theta)\,dt \tag{2.3.10}$$

is a solution of (2.3.1) and satisfies (2.3.6) for every $\theta > 0$.

It remains to consider the other boundary condition (2.3.7), $\phi(r_2,\theta) = c_2$. To achieve this, we want to select a suitable $\theta > 0$ so that $\phi(r_2,\theta) = c_2$. Let us define $Y : (0,\infty) \to \mathbf{R}$ by

$$Y(\theta) := \int_{r_1}^{r_2} y(t,\theta)\,dt. \tag{2.3.11}$$

Then $\phi(r_2,\theta) = c_2$ if and only if $Y(\theta) = c_1 - c_2$.

We investigate properties of Y.

Proposition 6. *Let $0 < r_1 < r_2$ and Y be the function defined by* (2.3.11). *Then*

(1) *Y is continuous and decreasing in $(0, \infty)$.*

(2) *For any $c_0 > 0$ there exists a unique $\theta_0 = \theta_0(c_0; r_1, r_2)$ such that $Y(\theta_0) = c_0$.*

Proof. (1) This is obvious by the definition of Y.

(2) As Y is continuous and decreasing, it suffices to prove that

$$\lim_{\theta \to +0} Y(\theta) = \infty, \tag{2.3.12}$$

$$\lim_{\theta \to \infty} Y(\theta) = 0. \tag{2.3.13}$$

The limit (2.3.12) is immediately derived. Indeed, since $y(r, \theta) \le r_1^{-A} \theta^{-\frac{1}{k-1}}$, we have

$$0 \le Y(\theta) \le r_1^{-A} \theta^{-\frac{1}{k-1}} (r_2 - r_1) \to 0 \ (\theta \to 0),$$

which shows (2.3.12).

Let us next prove (2.3.13). We set $K := \frac{k-1}{\lambda} (\max_{[r_1 r_2]} \sigma) r_1^{-A(k-1)} > 0$. Then

$$\frac{k-1}{\lambda} \int_{r_1}^{r} r_2^{-A} (K(t - r_1) + \theta)^{-\frac{1}{k-1}} \, dt = r_2^{-A} K^{-\frac{1}{k-1}} \int_{r_1}^{r_2} \left(t - r_1 + \frac{\theta}{K} \right)^{-\frac{1}{k-1}} \, dt.$$

When $k = 2$, we have

$$\begin{aligned} r_2^A K Y(\theta) &\ge \int_{r_1}^{r_2} \left(t - r_1 + \frac{\theta}{K} \right)^{-1} \, dt \\ &= \left[-\frac{k-1}{2-k} \left(t - r_1 + \frac{\theta}{K} \right)^{-\frac{2-k}{k-1}} \right]_{r_1}^{r_2} \\ &= \log \left(\frac{K(r_2 - r_1)}{\theta} + 1 \right) \\ &\to \infty \ (\theta \to +0). \end{aligned}$$

Thus (2.3.13) holds. Assume next that $1 < k < 2$. Then

$$\begin{aligned} r_2^A K^{\frac{1}{k-1}} Y(\theta) &\ge \int_{r_1}^{r_2} \left(t - r_1 + \frac{\theta}{K} \right)^{-\frac{1}{k-1}} \, dt \\ &= \left[-\frac{k-1}{2-k} \left(t - r_1 + \frac{\theta}{K} \right)^{-\frac{2-k}{k-1}} \right]_{r_1}^{r_2} \\ &= -\frac{k-1}{2-k} \left\{ \left(r_2 - r_1 + \frac{\theta}{K} \right)^{-\frac{2-k}{k-1}} - \left(\frac{\theta}{K} \right)^{-\frac{2-k}{k-1}} \right\} \\ &\to \infty \ (\theta \to +0), \end{aligned}$$

which implies (2.3.13).

□

As a result of Proposition 6, we see that the function $\phi(\cdot, \theta_0)$ with $\theta_0 = \theta_0(c_1 - c_2; r_1, r_2)$ is a solution of (2.3.5)–(2.3.7). Let us summarize the above argument.

Proposition 7 (Radially symmetric solution)**.** *Let $0 < r_1 < r_2$ and define $\Omega = \{x \in \mathbf{R}^n \mid r_1 < x < r_2\}$. Let c_1, $c_2 \in \mathbf{R}$ be constants such that $c_1 > c_2$. Furthermore, let $\theta_0 = \theta_0(c_1 - c_2; r_1, r_2)$ be the constant given in Proposition 6 and let ϕ_{θ_0} be the function given by* (2.3.10). *Then $\phi(\cdot, \theta_0)$ is a solution of* (2.3.5)–(2.3.7). *Moreover, $\Phi(x) = \phi(|x|, \theta_0(m(r_1) - m(r_2); r_1, r_2))$ is a solution of* (2.3.1)–(2.3.3).

We are now in a position to prove the Hadamard theorem for (2.1.1).

Theorem 3 (Hadamard three spheres theorem)**.** *Let $0 < r_1 < r_2$ and define $\Omega = \{x \in \mathbf{R}^n \mid r_1 < x < r_2\}$. Let $u \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity supersolution of* (2.1.1) *in Ω. Define m by* (2.1.5). *Moreover, assume that $m(r_1) > m(r_2)$ and let $\theta_0 := \theta_0(c_1 - c_2; r_1, r_2)$ be the constant given in Proposition 6. Define $\phi(\cdot, \theta_0)$ by* (2.3.10). *Then*

$$m(r) \ge \phi(r, \theta_0) \quad \text{for all } r \in [r_1, r_2]. \tag{2.3.14}$$

Proof. We recall that u is a viscosity supersolution of (2.1.4) in Ω. Also, $\phi(|\cdot|, \theta_0)$ is a classical solution of (2.1.4) in Ω (Proposition 7). Moreover, on the boundary ∂B_{r_1}, we have $u(\cdot) \geq m(r_1) = \phi(r_1, \theta_0) = \phi(|\cdot|, \theta_0)$. In the same way, we see that $u(\cdot) \geq \phi(|\cdot|, \theta_0)$ on ∂B_{r_2}. Therefore, the comparison principle (Proposition 4) implies that $u(\cdot) \geq \phi(|\cdot|, \theta_0)$ in $\overline{\Omega}$.

Fix $r \in [r_1, r_2]$. Then, for any $x \in \overline{\Omega}$ with $r_1 \leq |x| \leq r$, we have $u(x) \geq \phi(|x|, \theta_0) \geq \phi(r, \theta_0)$, where we used the fact that $\phi(\cdot, \theta_0)$ is decreasing. Taking the minimum with respect to x, we obtain (2.3.14). □

Remark 9. In the case where $m(r_1) = m(r_2)$ in Theorem 3, we have

$$m(r) = m(r_1) \quad \text{for all } r \in [r_1, r_2]$$

since m is nonincreasing.

2.4 The Liouville theorem

As applications of the Hadamard theorem (Theorem 3), let us derive the Liouville theorem.

2.4.1 The Liouville theorem

Our Liouville theorem is derived under the following condition:

$$\begin{cases} \displaystyle\lim_{r_2 \to \infty} \int_{r_1}^{r} y(t, \theta_0)\, dt = 0 \quad \text{for all } r \geq r_1 > 0 \text{ and } c_1 > c_2 \geq 0, \\ \text{where } \theta_0 = \theta_0(c_1 - c_2; r_1, r_2) \text{ is the constant given in Proposition 6 (2).} \end{cases} \tag{2.4.1}$$

Sufficient conditions for (2.4.1) will be discussed later.

Theorem 4 (Liouville theorem). *Let $u \in \mathrm{LSC}(\mathbf{R}^n)$ be a nonnegative viscosity supersolution of* (2.1.1) *in $\mathbf{R}^n$. Assume* (2.4.1)*. Then u is constant in $\mathbf{R}^n$. Moreover, if there exists some $x_0 \in \mathbf{R}^n$ such that $h(x_0) < 0$, then $u \equiv 0$ in $\mathbf{R}^n$.*

The proof is almost the same as that [6, Theorem 4.1], but we need to pay additional attention to the relation between $m(r_1)$ and $m(r_2)$ since a radial solution of (2.3.5)–(2.3.7) was constructed only when $c_1 > c_2$.

Proof.

1. We recall that u is a viscosity supersolution of (2.1.4) in $\mathbf{R}^n$. In order to prove that u is constant in $\mathbf{R}^n$, it suffices to show that

$$\min_{\overline{B_r}} u = u(0) \quad \text{for all } r > 0. \tag{2.4.2}$$

In fact, if this is true, the strong minimum principle implies that $u \equiv u(0)$ in B_r for every $r > 0$. Therefore $u \equiv u(0)$ in $\mathbf{R}^n$.

2. To prove (2.4.2), we fix $r > 0$ and take any $z \in \overline{B_r} \setminus \{0\}$. We also take $r_1 \in (0, |z|)$ and define m by (2.1.5). Let us prove that

$$m(r_1) = m(r). \tag{2.4.3}$$

Since m is nonincreasing, we always have $m(r_1) \geq m(r)$. Suppose that $m(r_1) > m(r)$. Then, for any $r_2 > r_1$, we have $m(r_1) > m(r) \geq m(r_2)$. Thus the Hadamard theorem (Theorem 3) implies that

$$m(r) \geq \phi_{\theta_0}(r) = m(r_1) - \int_{r_1}^{r} y(t, \theta_0)\, dt.$$

Here $\theta_0 = \theta_0(m(r_1) - m(r_2); r_1, r_2)$. By (2.4.1), sending $r_2 \to \infty$ gives $m(r) \geq m(r_1)$. This is a contradiction.

Since $r_1 < |z| \leq r$ and (2.4.3) hold, we have

$$u(z) \geq \min_{\overline{B_R} \setminus B_{r_1}} u = m(r) = m(r_1) = \min_{\partial B_{r_1}} u.$$

As u is lower semicontinuous, sending $\liminf_{r_1 \to +0}$ yields $u(z) \geq u(0)$. We therefore conclude (2.4.2).

3. Assume that $h(x_0) < 0$ for some $x_0 \in \mathbf{R}^n$. We let $u(x) \equiv C \geq 0$ for $x \in \mathbf{R}^n$. Since u is a classical supersolution of (2.1.3), we have

$$h(x)C^\alpha \geq 0 \quad \text{for all } x \in \mathbf{R}^n.$$

In particular, $h(x_0)C^\alpha \geq 0$. Since $h(x_0) < 0$, we see that C must be 0. □

Next, we consider the case where (2.4.1) does not hold. Precisely, we assume that there exists $r_1 > 0$, $c_0 > 0$, and $L > 0$ such that $\lim_{r_2\to\infty} \int_{r_1}^{r} y(t, \theta_0(c_0; r_1, r_2))\ dt = L < +\infty$.

In this case, the Liouville property does not necessarily hold. We show a counter example.

Fix $r_2 > r_1$ and let $\varphi(r) := L - \int_{r_1}^{r} y(t, \theta_0(c_0; r_1, r_2))\ dt$. Then we immediately find that φ is a classical supersolution of

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi(|x|)) + \sigma(|x|)|D\varphi(|x|)|^k = 0 & x \in \Omega, \\ \varphi(r_1) = L, \quad \varphi(r_2) = L - c_0. \end{cases}$$

Let

$$u(x) := \begin{cases} L & |x| \leq r_1, \\ \varphi(|x|) & r_1 \leq |x|. \end{cases}$$

Then we can immediately find that u is a viscosity supersolution of (2.1.4).

This observation implies that the condition (2.4.1) is equivalent to the Liouville property for (2.1.4). Precisely, if the function h appearing in the hypothesis (F3) is identically zero and (2.4.1) holds, there is no nonnegative viscosity supersolution of (2.1.1) except constant functions, and vice versa. However, it is not sure for (2.1.3). In other words, the Liouville property for the case where $h < 0$ and (2.4.1) does not hold still remains open question.

2.4.2 Sufficient conditions for the Liouville property

Let us consider sufficient conditions for (2.4.1). Since $y(\cdot, \theta)$ is decreasing, we know

$$\int_{r_1}^{r} y(t, \theta_0)\ dt \leq (r - r_1) \sup_{(r_1, r)} y(\cdot, \theta_0) = (r - r_1)y(r_1, \theta_0) = (r - r_1)r_1^{-A}\theta_0^{-\frac{1}{k-1}}.$$

Therefore, if

$$\theta_0 \to \infty \quad (r_2 \to \infty) \tag{2.4.4}$$

for arbitrarily fixed $r_1 > 0$ and $c_1 > c_2 > 0$, then (2.4.1) is valid.

We next discuss a sufficient condition for (2.4.4). We aim to explicitly describe sufficient conditions using σ, as in [6]. So we focus on how Y behaves with r_2. Thus we fix $r_1 > 0$ arbitrarily and write $Y(\theta)$ for $Y(r_2, \theta)$, that is,

$$Y(r_2, \theta) := Y(\theta) = \int_{r_1}^{r_2} y(t, \theta)\ dt. \tag{2.4.5}$$

Remark 10. (2.3.9) implies that $Y(\cdot, \theta)$ is increasing and $Y(r, \cdot)$ is decreasing.

In addition let $\theta_0(c_0, r_2)$ denote $\theta_0(c_0; r_1, r_2)$. In other words we define a function $\theta_0 : (0, \infty) \times (r_1, \infty) \to (0, \infty)$ by

$$Y(r_2, \theta_0(c_0, r_2)) = c_0, \tag{2.4.6}$$

where $r_1 > 0$ is arbitrarily fixed. Here Proposition 6 (2) ensures that θ_0 is well-defined. Using the monotonicity of y and Y, one can easily show the following Lemma 2. We omit the proof here.

Lemma 2. *Fix $r_1 > 0$ arbitrarily and let $(c_0, r_2) \mapsto \theta_0(c_0, r_2)$ be the function defined by (2.4.6). Then the following properties hold.*

(1) *For all $c_0 > 0$, $r_2 \mapsto \theta_0(c_0, r_2)$ is increasing.*

(2) *For all $r_2 > r_1$, $c \mapsto \theta_0(c_0, r_2)$ is decreasing.*

We are now ready to obtain the sufficient condition we seek.

Proposition 8. *Fix $r_1 > 0$ arbitrarily. Let $Y(r_2, \theta)$ be the function defined by (2.4.5). If $\lim_{r_2\to\infty} Y(r_2, \theta) = \infty$ for all $\theta > 0$, then $\lim_{r_2\to\infty} \theta_0(c, r_2) = \infty$ for all $c > 0$.*

Proof. We prove by contradiction. By Lemma 2 (1), it is sufficient to suppose that there exists $c > 0$ and $\hat{\theta} < \infty$ such that $\theta_0(c, r_2) \leq \hat{\theta}$ for all r_2.

The assumption implies that there exists $r_c > 0$ such that $Y(r_2, \hat{\theta}) > c$ for all $r_2 > r_c$. Fix $r_2 > r_c$. Since $Y(r_2, \cdot)$ is decreasing, we know $\theta_0(c, r_2) > \hat{\theta}$, and this is contradiction. □

Next, we consider the inverse of the previous discussion. Precisely, we consider the situation under the assumption where there exist $\theta' > 0$ and $c' = c'(\theta')$ such that $\lim_{r_2\to\infty} Y(r_2, \theta') = c'$. In this case, we find $\lim_{r_2\to\infty} \theta(c', r_2) < \infty$.

Proposition 9. *Fix $r_1 > 0$ arbitrary. Let $Y(r_2, \theta)$ be the function defined by (2.4.5). Assume that there exists $\theta' > 0$ and $c' = c'(\theta')$ such that $\lim_{r_2\to\infty} Y(r_2, \theta') = c'$. Then $\lim_{r_2\to\infty} \theta_0(c, r_2) < +\infty$ for all $c > 0$.*

Proof. Since $Y(\cdot, \theta)$ is decreasing, we see

$$c' = \sup_{r_2 \geq r_1} Y(r_2, \theta') \geq Y(r_2, \theta') \quad \text{for all } r_2 \geq r_1.$$

Let $c \geq c'$. Because $Y(r_2, \theta_0(c, r_2)) = c$, we have

$$c = Y(r_2, \theta_0(c, r_2)) \geq c' \geq Y(r_2, \theta') \quad \text{for all } r_2 \geq r_1.$$

Therefore we obtain $\theta(c, r_2) \leq \theta'$.

Next, let $0 < c < c'$. Again the monotonicity of $Y(\cdot, \theta)$ implies that $\tilde{Y}(\theta) := \sup_{r_2 \geq r_1} Y(r_2, \theta)$ is well-defined for $\theta \geq \theta'$. The triangle inequality implies that $\tilde{Y}$ is decreasing. Furthermore, because of (2.3.13), $\tilde{Y}$ also vanishes at infinity. Therefore we obtain

$$c > \tilde{Y}(\theta) = \sup_{r_2 \geq r_1} Y(r_2, \theta)$$

for sufficiently large $\theta > \theta'$, and thus $\theta_0(c, r_2)$ is bounded. □

According to Proposition 9, if the limit $\lim_{r_2\to\infty} Y(r_2, \theta)$ is not infinity, we have to check (2.4.1) directly in order to derive the Liouville property for viscosity suersolutions of (2.1.1). Observing the limit $r_2 \to \infty$, since $\theta_0(c, r_2)$ converges to some positive number θ', we find $y(r, \theta_0(c, r_2)) \to y(r, \theta')$ and then (2.4.1) is equivalent to

$$\int_{r_1}^{r} y(t, \theta')\, dt = 0 \quad \text{for all } r \geq r_1 > 0.$$

However, this is impossible since y is positive.

Let us summarize the above argument.

Theorem 5. (i)–(iii) *are all equivalent.*

(i) (2.4.1)*, that is,* $\lim_{r_2\to\infty} Y(r, \theta_0(c_0, r_2)) = 0$ *for all $r > r_1 > 0$ and $c_0 > 0$, where Y is the function defined by* (2.4.5) *for each $r_1 > 0$.*

(ii) $\lim_{r_2\to\infty} \theta_0(c_0, r_2) = \infty$ *for all $r_1 > 0$ and $c_0 > 0$, where θ_0 is the function defined by* (2.4.6) *for each $r_1 > 0$.*

(iii) $\lim_{r_2\to\infty} Y(r_2, \theta) = \infty$ *for all $\theta > 0$ and $r_1 > 0$, where Y is the function defined by* (2.4.5) *for each $r_1 > 0$.*

2.4.3 Observation in the case $\sigma(r) = r^p$

Since θ_0 is defined only implicitly, it may be easier to check (i) or (iii) than (ii) in order to see if there exists non-constant supersolution for (2.1.4).

Now we especially deal with the equation (2.1.4) where $\sigma(r) = r^p$ with $p \in \mathbf{R}$ and for $r \geq 1$. We aim to characterize the power p which qualifies for the Liouville property. As we argued before, an observation of the limit $\lim_{r_2\to\infty} Y(r_2, \theta)$ will give us the answer. We note that $y(r, \theta)$ can be reformulated as

$$y(r, \theta) = r^{-A} \left(\frac{k-1}{\lambda} \int_{r_1}^{r} s^{-A(k-1)+p}\, ds + \theta \right)^{-\frac{1}{k-1}},$$

where $A = \frac{\Lambda(n-1)}{\lambda}$ and $1 < k \leq 2$.

Theorem 6. *Fix $r_1 > 0$ arbitrarily. Let $\sigma(r) = r^p$ and $Y(r_2, \theta)$ is defined by (2.4.5). If $p \leq k - 2$, $\lambda = \Lambda$ and $n = 2$, then $\lim_{r_2\to\infty} Y(r_2, \theta) = +\infty$ for all $\theta > 0$. Otherwise the limit $\lim_{r_2\to\infty} Y(r_2, \theta)$ converges.*

Proof.

Case 1: For $p = A(k-1) - 1$. We obtain

$$y(r,\theta) = r^{-A} \left(\frac{k-1}{\lambda} \int_{r_1}^{r} s^{-1} \, ds + \theta \right)^{-\frac{1}{k-1}}$$

and thus

$$Y(r_2,\theta) = \int_{r_1}^{r_2} r^{-A} \left(\frac{k-1}{\lambda} (\log r - \log r_1) + \theta \right)^{-\frac{1}{k-1}} \, dr.$$

Therefore $\lim_{r_2\to\infty} Y(r,\theta) = \infty$ if and only if $A = 1$ and $\frac{1}{k-1} = 1$, that is $\lambda = \Lambda$, $n = 2$ and $k = 2$, and thus $p = 0$.

Case 2: For $p > A(k-1) - 1$. Let $\beta := -A(k-1) + p + 1$. Then we have

$$y(r,\theta) = r^{-A} \left(\frac{k-1}{\lambda\alpha} r^{\beta} - \frac{k-1}{\lambda\alpha} r_1^{\beta} + \theta \right)^{-\frac{1}{k-1}}$$

and then

$$Y(r_2,\theta) = C \int_{r_1}^{r_2} r^{-A} (r^{\beta} + C')^{-\frac{1}{k-1}} \, dr,$$

where C, C' are constants. Since $\beta > 0$, $\lim_{r_2\to\infty} Y(r,\theta) = \infty$ if and only if $A + \frac{\beta}{k-1} \leq 1$. Recalling that $\beta = -A(k-1) + p + 1$, we obtain a condition

$$A(k-1) - 1 < p \leq k - 2. \tag{2.4.7}$$

However, because

$$A(k-1) - 1 = \frac{\Lambda}{\lambda}(n-1) - 1 \geq (n-1)(k-1) - 1,$$

there is no real number p satisfying (2.4.7).

Case 3: For $p < A(k-1) - 1$. Similarly we obtain

$$Y(r_2,\theta) = C \int_{r_1}^{r_2} r^{-A} (r^{\beta} + C')^{-\frac{1}{k-1}} \, dr.$$

Since $\beta < 0$, we easily find $\lim_{r_2\to\infty} Y(r,\theta) = \infty$ if and only if $A \leq 1$, that is $\lambda = \Lambda$ and $n = 2$. Thus we obtain a condition

$$p < A(k-1) - 1, \;\; \lambda = \Lambda, \;\; \text{and } n = 2,$$

that is $p < k - 2$. □

Eventually, since the key point is whether $\int_{r_1}^{\infty} y(r,\theta) \, dr$ converges or not, a similar characterization can be made for the case $\sigma(r) = \mathcal{O}(r^p)$ $(r \to \infty)$.

Corollary 1. *Fix $r_1 > 0$ arbitrarily and let $Y(r_2,\theta)$ be the function defined by (2.4.5). Assume that there exists $C > 0$ and $r_0 > 0$ such that $\sigma(r) \leq Cr^{k-2}$. If $\lambda = \Lambda$ and $n = 2$, then $\lim_{r_2\to\infty} Y(r_2,\theta) = +\infty$ for all $\theta > 0$.*

Proof. (2.3.9) implies that

$$y(r,\theta) \geq r^{-A} \left(\frac{k-1}{\lambda} \int_{r_1}^{r} s^{-A(k-1)+k-2} \, ds + \theta \right)^{-\frac{1}{k-1}}$$

for $r > 0$ under $\sigma(r) \leq r^{k-2}$. By the same argument in Theorem 6, if $\lambda = \Lambda$ and $n = 2$, then the integral

$$\int_{r_1}^{\infty} r^{-A} \left(\frac{k-1}{\lambda} \int_{r_1}^{r} s^{-A(k-1)+k-2} \, ds + \theta \right)^{-\frac{1}{k-1}} \, dr$$

tends to ∞, and thus $\lim_{r_2\to\infty} Y(r_2,\theta) = \int_{r_1}^{\infty} y(r,\theta) \, dr = \infty$. □

2.5 Application for degenerate elliptic case

In this section we consider an elliptic equation of the form

$$G(Du, D^2u) + \sigma(|x|)|Du|^k = 0, \tag{2.5.1}$$

where $G : (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is a continuous function with $G(p, O) = 0$ for all $p \in \mathbf{R}^n \setminus \{0\}$, $\sigma : [0, \infty) \to [0, \infty)$ is a continuous function, and $k \in \mathbf{R}$. In addition we assume that there exist positive constants $\lambda \leq \Lambda$ and a constant $q > -1$ such that

$$G(p, X) \leq |p|^q \mathcal{P}^+_{\lambda,\Lambda}(X) \tag{2.5.2}$$

for all $p \in \mathbf{R}^n \setminus \{0\}$ and $X \in \mathbf{S}^n$. (2.5.2) typically includes the $(q+2)$-Laplacian ($-G(p, X) = |p|^q \operatorname{Tr}(X) + q|p|^{q-2}\langle Xp, p\rangle$) and the mean curvature ($-G(p, X) = \operatorname{Tr}(X) - \frac{\langle Xp,p\rangle}{|p|^2}$). Other examples are listed in [4].

We note that (2.5.2) includes functions which have singularity at $p = 0$ therefore we use a slightly weakened definition of viscosity solutions treated in [3].

Definition 3. Let $D \subset \mathbf{R}^n$ be a domain.

(1) We say $u \in \mathrm{LSC}(D)$ (resp. $u \in \mathrm{USC}(D)$) is a viscosity supersolution (resp. viscosity subsolution) of (2.5.1) in D if

$$G(D\phi(z), D^2\phi(z)) + \sigma(|z|)|D\phi(z)|^k \geq 0 \text{ (resp. } \leq 0)$$

holds for any $z \in D$ and $\phi \in C^2(D)$ such that $D\phi(z) \neq 0$ and $u - \phi$ attains a local minimum (resp. local maximum) at z.

(2) If $u \in C(D)$ is both a viscosity supersolution and a viscosity subsolution of (2.5.1) in D, we say that u is a viscosity solution of (2.5.1) in D.

We note that any nonnegative viscosity supersolutions of (2.5.1) are also supersolution of

$$|Du|^q \mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du|^k = 0, \tag{2.5.3}$$

and thus we especially consider viscosity supersolutions of (2.5.3).

We aim to establish the Hadamard and the Liopuville property for viscosity supersolutions of (2.5.3). For this purpose, we prepare the comparison principle and the strong minimum principle.

Proposition 10 (Comparison principle)**.** *Let $D \subset \mathbf{R}^n$ be a bounded domain. Let $u \in \mathrm{LSC}(\overline{D})$ be a viscosity supersolution of* (2.5.3)*. Let $v \in C^2(D) \cap C(\overline{D})$ be a classical subsolution of* (2.1.4) *and $Dv \neq 0$ in D. If $v \leq u$ on ∂D, then $v \leq u$ in D.*

Although we should pay attention for the singularity, the proof is almost same for the uniformly elliptic case by the assumption $Dv \neq 0$. We omit the proof here and refer the reader to [4, 14].

Proposition 11 (Strong minimum principle)**.** *Let $D \subset \mathbf{R}^n$ be a domain, $1 \leq k - q \leq 2$, and $u \in \mathrm{LSC}(D)$ be a viscosity supersolution of* (2.5.3)*. If u achieves a nonpositive minimum in D, then u is constant.*

It is sufficient to confirm that the sufficient condition listed in [3] is achieved. We omit the proof here.

As in argued in Section 2.3 we aim to solve the following problem

$$|D\Phi_q(x)|^q \mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi_q(x)) + \sigma(|x|)|D\Phi_q(x)|^k = 0 \quad \text{in } \Omega = \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\} \tag{2.5.4}$$

under the boundary condition

$$\Phi_q(x) = c_1 \quad \text{on } \partial B_{r_1},$$
$$\Phi_q(x) = c_2 \quad \text{on } \partial B_{r_2}.$$

Recalling the argument in Section 2.3, in the case $1 < k - q \leq 2$ and $c_1 > c_2 > 0$, we already solved

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi(x)) + \sigma(|x|)|D\Phi(x)|^{k-q} = 0 \quad \text{in } \Omega \tag{2.5.5}$$

with the same boundary condition above. The solution is

$$\Phi(x) = c_1 - \int_{r_1}^{|x|} y_q(t, \theta_1)\, dt, \tag{2.5.6}$$

where

$$y_q(r, \theta_1) := r^{-A} \left(\frac{k-q-1}{\lambda} \int_{r_1}^{r} \sigma(s) s^{-A(k-q-1)}\, ds + \theta_1 \right)^{-\frac{1}{k-q-1}}. \tag{2.5.7}$$

Here $A = \frac{\Lambda(n-1)}{\lambda}$ and $\theta_1 = \theta_1(c_1 - c_2; r_1, r_2)$ is a parameter such that $\Phi(x) = c_2$ for $x \in \partial B_{r_2}$. We can find such parameter θ_1 by a similar argument in the proof of Proposition 6.

Comparing (2.5.5) with (2.5.4), we see Φ is also solution of (2.5.4), that is, $\Phi_q = \Phi$.

Remark 11. Since $D\Phi \neq 0$, Φ_q is also a solution of (2.5.5).

Therefore we can establish the Hadamard theorem and the Liouville type property.

Theorem 7. *Let $0 < r_1 < r_2$, $1 < k - q \leq 2$, and define $\Omega = \{x \in \mathbf{R}^n \mid r_1 < x < r_2\}$. Let $u \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity supersolution of* (2.5.1) *in Ω. Define m by* (2.1.5)*. Moreover, assume that $m(r_1) > m(r_2)$ and let $\theta_1 = \theta_1(m(r_1) - m(r_2); r_1, r_2)$ be the constant appearing in* (2.5.6)*. Then*

$$m(r) \geq m(r_1) - r^{-A}\left(\frac{k-q-1}{\lambda}\int_{r_1}^{r} \sigma(s)s^{-A(k-q-1)}\, ds + \theta_1\right)^{-\frac{1}{k-q-1}} \quad \textit{for all } r \in [r_1, r_2].$$

Theorem 8. *Let $u \in \mathrm{LSC}(\mathbf{R}^n)$ be a nonnegative viscosity supersolution of* (2.5.1) *in $\mathbf{R}^n$. Assume $1 < k - q \leq 2$ and*

$$\lim_{r_2 \to \infty} \int_{r_1}^{r} y_q(t, \theta_1)\, dt = 0 \quad \textit{for all } r_1 > 0 \textit{ and } c_1 > c_2 \geq 0,$$

where y_q is the function defined by (2.5.7) *and $\theta_1 = \theta_1(m(r_1) - m(r_2); r_1, r_2)$ be the constant appearing in* (2.5.6)*. Then u is constant in $\mathbf{R}^n$.*

These proofs are exactly the same as those in Theorem 3 and Theorem 4, except that k has been changed to $k - q$, so we shall omit them.

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