博士学位論文

Asymptotic analysis of mean curvature flow equations via games
（ゲームを用いた平均曲率流方程式の漸近解析）

Kuniyasu Misu

Department of Mathematics, Graduate school of Science, Hokkaido University

2024, March

Abstract

In part I we consider the asymptotic behavior of solutions to an obstacle problem for the mean curvature flow equation by using a game-theoretic approximation, to which we extend that of Kohn and Serfaty [37]. The paper [37] gives a deterministic two-person zero-sum game whose value functions approximate the solution to the level set mean curvature flow equation without obstacle functions. We prove that moving curves governed by the mean curvature flow converge in time to the boundary of the convex hull of obstacles under some assumptions on the initial curves and obstacles. Convexity of the initial set, as well as smoothness of the initial curves and obstacles, are not needed. In these proofs, we utilize properties of the game trajectories given by very elementary game strategies and consider reachability of each player. Also, when the equation has a driving force term, we present several examples of the asymptotic behavior, including a problem dealt in [22].

In part II we study the initial value problem for a fully nonlinear degenerate parabolic equation with discontinuous source terms, to which a usual type of comparison principles do not apply. Examples include singular equations appearing in surface evolution problems such as the level set mean curvature flow equation with a driving force term and a discontinuous source term. By a suitable scaling, we establish weak comparison principles for a viscosity sub- and supersolution to the equation. We also present uniqueness and existence results of possibly discontinuous viscosity solutions.

In part III we consider the asymptotic shape of solutions to the level set mean curvature flow equation with a negative driving force and a discontinuous source term. This is a model equation of crystal growth phenomenon called a two-dimensional nucleation. A typical source term in our mind is a characteristic function of a set Ω. It turns out that, if Ω satisfies some weak convexity condition, then the asymptotic shape of the solution is given by the unique solution of the corresponding stationary problem with the Dirichlet boundary condition. We also apply the game-theoretic interpretation established in [49]. By using the game, we construct a solution with non-trivial growth speed when Ω consists of two disks touching each other. We also give another non-uniqueness result by using the game, which is a counter-example to a weak comparison principle in [33] when the source term does not satisfy the assumption of the weak comparison principle.

Each part I, II, and III of this doctoral thesis corresponds to the reference [49, 33, 31] respectively. Since all the parts are independent, there are some common definitions and similar arguments in them.

Lastly we note the sign of the driving force term. Through the thesis, we consider both the competitive situation and the cooperative situation. Namely the mean curvature term and the driving force term are competitive or cooperative when considering a closed hypersurface ∂A of a bounded convex set A. We denote the driving force by $\nu \in \mathbb{R}$. The competitive situation corresponds to $\nu > 0$ in part I and II and to $\nu < 0$ in part III.

Acknowledgements

The author would like to express his sincere gratitude to the following individuals for their invaluable contributions and support throughout the completion of this paper:

1. Professor Nao Hamamuki: giving the author helpful comments and careful reading of all parts
2. Professor Hiroyoshi Mitake: giving the author helpful comments and careful reading of part III
3. Professor Qing Liu: conducting a fruitful discussion with the author
4. Professor Keisuke Takasao: providing the impetus for part I and informing the author of several related studies
5. Professor Etsuro Yokoyama: conducting a great discussion on the model in part III and informing the author of some references on crystal growth

Also, the author was supported by JST SPRING, Grant Number JPMJSP2119.

Contents

Part I

A game-theoretic approach to the asymptotic behavior of solutions to an obstacle problem for the mean curvature flow equation

1 Introduction

Obstacle problem for the mean curvature flow equation. We consider the following obstacle problem for the mean curvature flow equation:

$$\begin{cases} V = -\kappa \text{ on } \partial D_t, \\ O_- \subset D_t, \end{cases} \tag{1.1}$$

where $\{D_t\}_{t>0}$ is the unknown family of open sets in $\mathbb{R}^d$, V is the velocity of a point in ∂D_t in the direction of its outward normal vector, κ is the mean curvature of ∂D_t at the point and O_- is a fixed open set in $\mathbb{R}^d$. Our main goal is to investigate the asymptotic behavior of solutions to (1.1) by the level set equation and its game-theoretic approximation. We mainly deal with the case $d = 2$ in this manuscript.

The mean curvature flow equation has been attracting much attention. In the early stages, the smoothness of the initial surface was naturally assumed and the surface evolution was considered as long as singularities do not occur. In paticular when $d = 2$, the mean curvature flow equation is often called *the curve shortening problem* and the curve evolution was analysed in e.g. [15, 26].

The level set method for surface evolution equations was first rigorously analyzed in [8, 13]. The basic idea of this method is to represent moving surfaces as level sets of auxiliary functions and to rewrite surface evolution equations by level set equations, whose unknown functions are the auxiliary functions. A great advantage is the point that viscosity solutions of level set equations follow the long time behavior of the moving surfaces even after topological change of surfaces. The level set method is applied to various surface evolution equations including the mean curvature flow equation. See also [17] in detail.

Recently obstacle problems for the mean curvature flow equation have been considered in [25, 35, 48]. Obstacle problems are problems that have regions called obstacles which the solutions cannot exceed.

According to the unpublished paper [arXiv:1409.7657v3] by Mercier (We denote this paper by [Mercier] hereafter.), the level set method is still valid for (1.1). The corresponding level set equation to (1.1) is the following:

$$\begin{cases} u_t(x,t) + F(Du(x), D^2u(x)) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \\ \Psi_-(x) \le u(x,t) & \text{in } \mathbb{R}^d \times (0,\infty), \end{cases} \tag{1.2}$$

where $\Psi_- \in Lip(\mathbb{R}^d)$ is a given obstacle function that satisfies $O_- = \{x \in \mathbb{R}^d \mid \Psi_-(x) > 0\}$. The function $u_0 \in C(\mathbb{R}^d)$ is an initial datum and F is given by

$$F(Du, D^2u) = -|Du|\text{div}\left(\frac{Du}{|Du|}\right).$$

Namely F is the level-set mean curvature flow operator defined as

$$F(p,X) = -\text{Tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right)X\right), \quad p \in \mathbb{R}^d \setminus \{0\},\ X \in \mathbb{S}^d,$$

where $p \otimes p = (p_i p_j)_{i,j=1}^d$ for a vector $p = (p_1, \dots, p_d) \in \mathbb{R}^d$ and $\mathbb{S}^d$ is the set of $d \times d$ real symmetric matrices. For a comparison principle to (1.2), see also [35].

Throughout this paper we follow the 0 level set of the solution u. Together with it, we assume on the initial data u_0 as follows:

$$\text{For some } a < 0 \text{ and } R > 0,\ u_0 = a \text{ in } B^c(0,R). \tag{1.3}$$

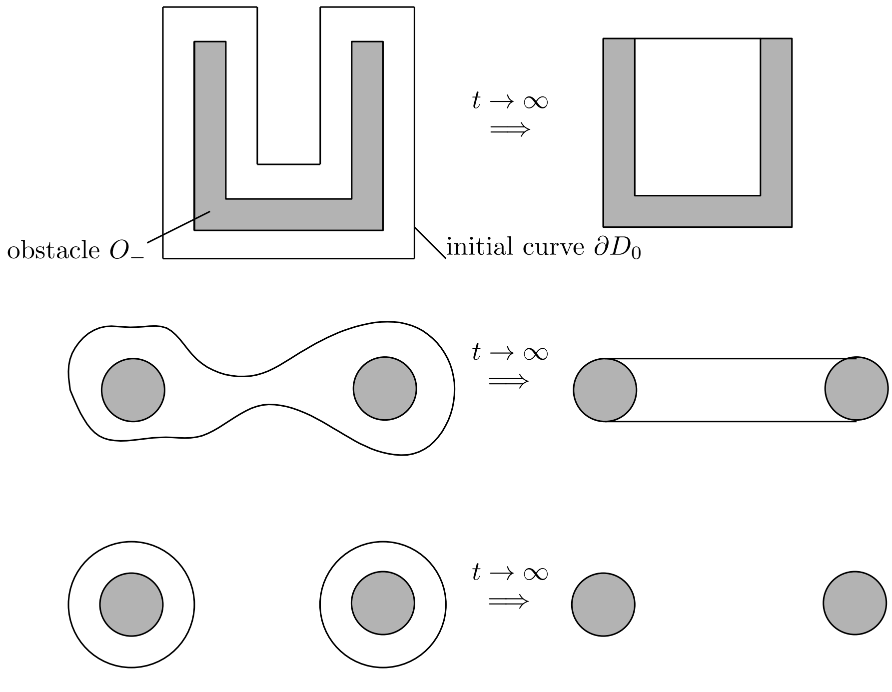


Figure 1: Conjecutures on the asymptotic shapes

Intuitive observation. For the solution to (1.1) with $d = 2$ and without obstacles, it is known that D_t becomes convex at some time, the moving curve ∂D_t converges to a single point and then vanishes, provided ∂D_0 is a smooth closed curve ([15, 26]). On the other hand, for our problem (1.1), it is obvious that the solution does not converge to any single point. Also it is not clear whether D_t becomes convex at some time. However it is natural to expect that in many cases D_t converges to the convex hull of O_-, which we hereafter denote by $Co(O_-)$, as $t \to \infty$ because of the curve shortening property of the solution and the smoothing effect of the curvature flow as we draw some examples in Figure 1. As shown in Figure 1, even for the same obstacles, different initial curves may converge to different limits. Thus we shall assume at least that one connected component of D_0 contains the whole O_- and expect that the asymptotic shape is $Co(O_-)$ under this assumption. Our main theorem (Theorem 3.2) is intended to justify this expectation as much as possible.

Game interpretation. Our first result is the extension of [37] to problems including (1.2). First, let us briefly explain the game rule for (1.2) with $d = 2$ and without obstacle function by following [37, Section 1.6]. The game is a deterministic two-person zero-sum game. For convenience, we name the first player Paul and the second player Carol. Let $\epsilon > 0$. Also, let $x_0 = x \in \mathbb{R}^2$ be the initial position of this game and $t > 0$ be the terminal time. At the i-th round of this game, Paul chooses directions $v_i \in \mathbb{R}^2$ with $|v_i| = 1$ and Carol chooses a number $b_i = \pm 1$ after Paul's choice. Then the game position that we henceforth regard as Paul's position conveniently moves from x_{i-1} to the next place x_i depending on their choice as follows:

$$x_i = x_{i-1} + \sqrt{2}\epsilon b_i v_i \tag{1.4}$$

After the N-th round, where $N \sim t\epsilon^{-2}$, the game ends and Carol pays the terminal cost $u_0(x_N)$ to Paul. Paul's goal is maximizing the cost while Carol's goal is minimizing it. The value function $u^\epsilon(x, t)$ is defined as the cost optimized by both the players, that is,

$$u^\epsilon(x, t) = \max_{v_1} \min_{b_1} \ldots \max_{v_N} \min_{b_N} u_0(x_N).$$

This value function approximates the viscosity solution u of (1.2) with $d = 2$ and without obstacle function. In fact the convergence $u^\epsilon \to u$ is shown in [37].

In order to handle (1.2) that has the obstacle function Ψ_-, we modify the game rule as follows. At each i-th round, we suppose that Paul has the right to quit the game. If Paul quits the game, the game cost is given by $\Psi_-(x_i)$. By doing this modification, the value function u^ϵ is restricted to $\Psi_- \leq u^\epsilon$. Such an interpretation of parameters of PDEs is well understood for first order equations; see [2]. The cost $\Psi_-(x_i)$ is called *stopping cost* and an optimal control problem with stopping cost is called *optimal stopping time problem*. For second order equations, see e.g. [46, 9], which deal with the optimal stopping time problem corresponding to the obstacle problem for the infinity Laplacian equation and p-Laplacian equation respectively.

The value function $u^\epsilon(x,t)$ satisfies the following *Dynamic Programming Principle*:

$$u^\epsilon(x,t) = \max\{\Psi_-(x), \max_{|v|=1} \min_{b=\pm 1} u^\epsilon(x+\sqrt{2}\epsilon bv, t-\epsilon^2)\}$$

for $t > 0$. This is a key equation in the proof of the convergence result.

The paper [37] also mention the game interpretation for higher dimensional case. Based on this, we can generalize our game to the case $d \geq 3$. In the game, Paul chooses $d-1$ orthogonal unit vectors $v_i^j (j = 1, 2, \cdots, d-1)$, Carol chooses $d-1$ values $b_i^j \in \{\pm 1\} (j = 1, 2, \cdots, d-1)$, and the state equation is $x_i = x_{i-1} + \sqrt{2}\epsilon \sum_{j=1}^{d-1} b_i^j v_i^j$ instead of (A.1).

The precise statement of the convergence of the value functions is described by the half relaxed limits of the value functions, which are defined as follows:

$$\overline{u}(x,t) := \overline{\lim_{\substack{(y,s)\to(x,t)\\ \epsilon\searrow 0}}} u^\epsilon(y,s), \ \underline{u}(x,t) := \underline{\lim_{\substack{(y,s)\to(x,t)\\ \epsilon\searrow 0}}} u^\epsilon(y,s).$$

As a consequence of Proposition A.3, we present the convergence result for (1.2) at the moment. We describe a game interpretation and the same type of convergence result for more general PDEs than (1.2) in Appendix A.

Proposition 1.1. *The functions $\overline{u}$ and $\underline{u}$ are respectively viscosity sub- and supersolution of* (1.2). *Moreover $\overline{u}(x,0) = \underline{u}(x,0) = u_0(x)$ for $x \in \mathbb{R}^d$.*

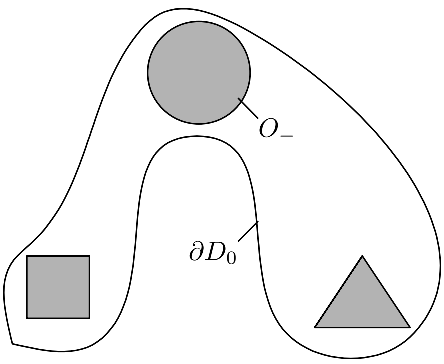


Figure 2: Example of D_0 and O_-

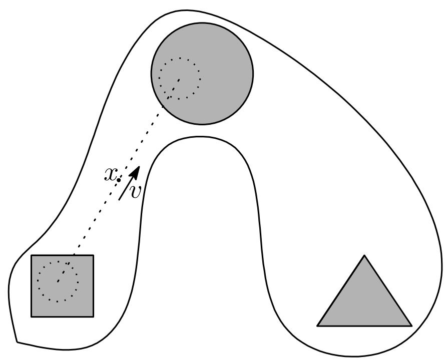


Figure 3: Strategy

Asymptotic behavior. We study the asymptotic behavior of solutions to (1.1). To explain an outline of the proof of the main theorem (Theorem 3.2), at the moment, we identify u^ϵ with u and consider a specific figure (Figure 2). Since we consider 0 level set of solutions to (1.2), our concern is whether the game cost is positive or negative. Thus, from Paul's point of view, the victory condition is that the game cost becomes positive. Namely, from Carol's point of view, the victory condition is that the game cost becomes negative. There is no need to give optimal strategies. Hereafter, even if a strategy taken by the players is not optimal, we often use present tense such as "Paul takes some strategy when he is in some domain". To show that the asymptotic shape is $Co(O_-)$, we have to prove that Paul wins if he starts from $Co(O_-)$ and Carol wins if Paul starts from $\overline{Co(O_-)}^c$. (We avoid the argument on the boundary of $Co(O_-)$.) Furthermore, by the rule of the game explained above, we see that the victory condition of Paul is whether he reaches O_- at some round or D_0 at the final round.

The easiest situation for Paul is that the initial game position $x \in O_-$. In this case, it suffices for Paul to quit the game at the first round and gain the stopping cost $\Psi_-(x) > 0$. If we take x as shown in Figure 3, a strategy for Paul to win is the following: He keeps taking v parallel to the dotted line segment as in Figure 3 until he reaches the domain inside the dotted circle. Once he gets there, he quits the game and gains the positive stopping cost. Even if he does not get there, he can gain the positive

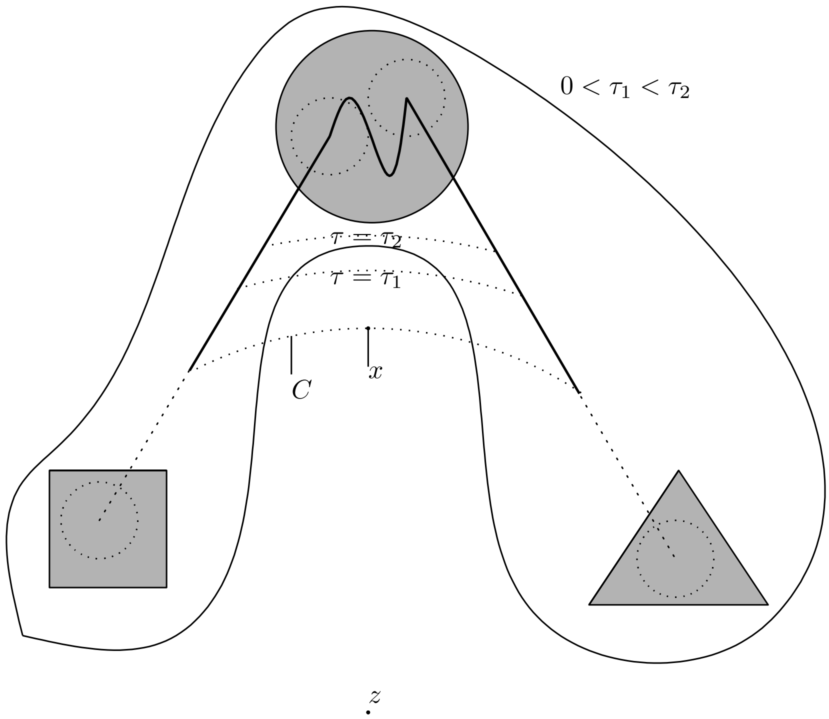


Figure 4: Strategy for $x \in Co(O_-)$

terminal cost at the final round of the game because the dotted line segment in Figure 3 is contained in D_0.

For the other $x \in Co(O_-)$, we consider a strategy for Paul to reach the domain from where we above overview that he could win if he started. To construct it, we prepare a type of strategies of the game called *concentric strategy*, which is also introduced in [37] and [43, Lemma 2.5 2.6]. See Definition 2.5. If Paul takes a concentric strategy, he can choose his favorable point $z \in \mathbb{R}^2$ and can control the distance from z to game positions regardless of Carol's choices as follows:

$$|x_n - z| = \sqrt{|x_0 - z|^2 + 2n\epsilon^2}.$$

In paticular $|x_n - z|$ is monotonically increasing with respect to n and, denoting the game time $n\epsilon^2$ by τ, it goes to infinity as $\tau \to \infty$. Figure 4 shows an example of $x \in Co(O_-)$ and an appropriate concentric strategy. In Figure 4 the center of the arc C is z. Paul's strategy is to choose this z and keep taking the concentric strategy until he reaches a neighborhood of the bold curve in D_0. Since the domain enclosed by the arc C and the bold curve is bounded, he indeed reaches a neighborhood of the bold curve. Therefore he wins if he starts at this initial position x.

In the main theorem we state a condition on D_0 and O_- that we are able to apply above technique. To indicate above bounded domain, we construct an appropriate Jordan closed curve in the proof and Appendix D and then use the Jordan curve theorem.

One also define a concentric strategy of Carol (Definition 2.5) that has similar effect to that of Paul. Namely if Carol chooses a point z and takes the concentric strategy, she can force the distance $|x_n - z|$ to be monotonically increasing with respect to n and go to infinity as $\tau \to \infty$. For $x \in \overline{Co(O_-)}^c$ we can take an open ball B such that $\overline{Co(O_-)} \subset B$ and $x \in B^c$ by the hyperplane separation theorem and the boundedness of $\overline{Co(O_-)}$. If Carol chooses z and takes the concentric strategy, she wins for sufficiently large τ owe to the boundedness of D_0. If we assume a kind of strict convexity on the obstacle O_-, we can take above open ball B for $x \in \overline{Co(O_-)}^c$ whose radius does not depend on x. This means that the moving surface sticks to the obstacle in finite time (Theorem 3.7).

Literature. We give some other related works on the asymptotic behavior of solutions to obstacle problems for the mean curvature flow equation. Spadaro considers (1.1) to characterize the mean-convex hull set in his unpublished paper [arXiv:1112.4288v1]. He considers (1.1) by a variational discrete scheme, which is different from our approach, but is guaranteed to approximate the viscosity solution to (1.2) by

[Mercier]. According to [arXiv:1112.4288v1], the part of the limit hypersurface that does not touch the obstacle is a minimal surface. (This result enhances the plausibility of our expectation.) Compared to our result, his result works in higher dimensional case $d \le 7$, while it needs to assume at least that the initial set D_0 is convex when $d = 2$. For $d \ge 8$, [53, Proposition 4.2] implies that the limit hypersurface may have non empty boundary. [48] proves the convergence of moving surfaces in a situation that both initial surface and obstacles are given as periodic graphs. For problems with driving force, [25] proves the solution $u(x,t)$ to the problem (2.1) with $f = 0$ (We will introduce it later.) converges as $t \to \infty$ to the stationary solution. They also give the result concerning to the shape of the stationary solution. However it is limited to the case where the initial data and the obstacle function are radially symmetric.

Concerning to an approach other than the level set method, Takasao [54] considers an obstacle problem for the mean curvature flow equation in the sense of Brakke's mean curvature flow ([5]). He proves the global existence of the weak solution by using the Allen-Cahn equation with forcing term.

While [35, 54] and this paper consider given obstacle problems, they arise from many different situations. In [22] an obstacle problem naturally appears in the motion of the top and the bottom of the solution of birth and spread type equations though the equations have no obstacle functions. [45] shows that large exponent limit of power mean curvature flow equation is formulated by an obstacle problem involving 1-Laplacian.

Organization. This paper is organized as follows. Section 2 contains definitions, notations and lemmas that are needed to prove our results. In Section 3 we prove the theorems on the asymptotic shape of the solution to (1.1). Section 4 is devoted to compute several examples of asymptotic shapes of solutions to problems with driving force. The convergence of the value functions of the game and some arguments to complement the proof of the main theorem are presented in appendices.

2 Preliminary result

2.1 Definitions and notations

In this subsection we introduce the notion of viscosity solution to the following obstacle problem, which is the most general form in this manuscript. Moreover we remark some known results.

$$\begin{cases} u_t(x,t) - \nu|Du(x,t)| + F(Du(x,t), D^2u(x,t)) = f(x) & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \\ \Psi_-(x) \le u(x,t) \le \Psi_+(x) & \text{in } \mathbb{R}^d \times (0,\infty), \end{cases} \tag{2.1}$$

where $\Psi_+, \Psi_- \in Lip(\mathbb{R}^d)$ are obstacle functions which satisfy $\Psi_- \le \Psi_+$ in $\mathbb{R}^d$. A real number ν is a constant and f is a locally bounded function. This equation without obstacles is a birth and spread type equation introduced in [22]. Though the source term f is not considered in the proof of the main theorem, we take it into consideration in Appendix A to prepare for the forthcoming paper "Asymptotic shape of solutions to the mean curvature flow equation with discontinuous source terms" by Hamamuki and the author. We do so all the more because it is natural extension in light of optimal control theory.

We denote the upper and lower semicontinous envelope of u by u^* and u_* respectively.

Definition 2.1 (Viscosity solution).

1. A function u is a viscosity subsolution of (2.1) if it satisfies the following conditions.
   (a) $\Psi_-(x) \le u^*(x,t) \le \Psi_+(x)$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.
   (b) $u^*(x,0) \le u_0(x)$ for all $x \in \mathbb{R}^d$.
   (c) Whenever $\phi(x,t)$ is smooth, $u^* - \phi$ has a local maximum at $(x_0,t_0) \in \mathbb{R}^d \times (0,\infty)$ and $u^*(x_0,t_0) - \Psi_-(x_0) > 0$, we have
   $$\phi_t(x_0,t_0) - \nu|D\phi(x_0,t_0)| + F_*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \le f^*(x_0).$$
2. A function u is a viscosity supersolution of (2.1) if it satisfies the following conditions.
   (a) $\Psi_-(x) \le u_*(x,t) \le \Psi_+(x)$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.
   (b) $u_*(x,0) \ge u_0(x)$ for all $x \in \mathbb{R}^d$.

(c) Whenever $\phi(x,t)$ is smooth, $u_* - \phi$ has a local minimum at $(x_0,t_0) \in \mathbb{R}^d \times (0,\infty)$ and $u_*(x_0,t_0) < \Psi_+(x_0)$, we have

$$\phi_t(x_0,t_0) - \nu|D\phi(x_0,t_0)| + F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \geq f_*(x_0).$$

3. A function u is a viscosity solution of (2.1) if it is a viscosity subsolution and a viscosity supersolution of (2.1).

We now give the definition of solutions of the following surface evolution equation:

$$\begin{cases} V = -\kappa + \nu, \text{ on } \partial D_t, \\ O_- \subset D_t \subset O_+, \end{cases} \tag{2.2}$$

where $\{D_t\}_{t>0}$ is the unknown family of open sets in $\mathbb{R}^d$. Furthermore O_- and O_+ are fixed open sets in $\mathbb{R}^d$. We also introduce the closed version of (2.2):

$$\begin{cases} V = -\kappa + \nu, \text{ on } \partial E_t \\ C_- \subset E_t \subset C_+, \end{cases} \tag{2.3}$$

where $\{E_t\}_{t>0}$ is the unknown family of closed sets in $\mathbb{R}^d$. C_- and C_+ are fixed closed sets in $\mathbb{R}^d$. The PDE (2.1) with $f = 0$ is the level set equation for these surface evolution equations. Since we only consider bounded initial surfaces in this manuscript, we employ the following class of solutions:

$$\begin{aligned} &K_a(\mathbb{R}^d \times [0,\infty)) := \\ &\{u \in C(\mathbb{R}^d \times [0,\infty)) \mid \forall T > 0\ \exists R > 0 \text{ s.t. } u = a \text{ in } B_R^c(0) \times [0,T]\}. \end{aligned}$$

Definition 2.2. 1. Let D_0, O_- and O_+ be open sets in $\mathbb{R}^d$. A family of open sets $\{D_t\}_{t\geq 0}$ is called an *open evolution* of (2.2) with D_0, O_- and O_+ if there exist $\Psi_-, \Psi_+ \in Lip(\mathbb{R}^d)$, $u_0 \in C(\mathbb{R}^d)$ and a solution $u \in K_a(\mathbb{R}^d \times [0,\infty))$ of (2.1) with Ψ_-, Ψ_+, u_0 and $f = 0$ such that $O_- = \{x \in \mathbb{R}^d \mid \Psi_-(x) > 0\}$, $O_+ = \{x \in \mathbb{R}^d \mid \Psi_+(x) > 0\}$ and $D_t = \{x \in \mathbb{R}^d \mid u(x,t) > 0\}$ for $t \geq 0$.

2. Let E_0, C_- and C_+ be closed sets in $\mathbb{R}^d$. A family of closed sets $\{D_t\}_{t\geq 0}$ is called an *closed evolution* of (2.3) with E_0, C_- and C_+ if there exist $\Psi_-, \Psi_+ \in Lip(\mathbb{R}^d)$, $u_0 \in C(\mathbb{R}^d)$ and a solution $u \in K_a(\mathbb{R}^d \times [0,\infty))$ of (2.1) with Ψ_-, Ψ_+, u_0 and $f = 0$ such that $C_- = \{x \in \mathbb{R}^d \mid \Psi_-(x) \geq 0\}$, $C_+ = \{x \in \mathbb{R}^d \mid \Psi_+(x) \geq 0\}$ and $E_t = \{x \in \mathbb{R}^d \mid u(x,t) \geq 0\}$ for $t \geq 0$.

Remark 2.3. The open evolutions and the closed evolutions uniquely exist ([Mercier]).

Remark 2.4. Our main equation in this manuscript is (1.1), which has an obstacle on one side. We interpret problems that have only O_- as (2.2) with $O_+ = \mathbb{R}^d$ and problems that have only O_+ as (2.2) with $O_- = \emptyset$.

Whenever we consider (2.2) in this manuscript, we simultaniouly consider the solution $\{E_t\}_{t\geq 0}$ to (2.3) with $E_0 = \overline{D_0}$, $C_- = \overline{O_-}$ and $C_+ = \overline{O_+}$. Throughout the manuscript, we denote an open evolution by $\{D_t\}_{t\geq 0}$ and a closed evolution by $\{E_t\}_{t\geq 0}$. As explained above, we assume that D_0 is bounded.

Notations. For a point $z \in \mathbb{R}^d$, we denote the set $\{x \in \mathbb{R}^d \mid |x-z| < r\}$ by $B_r(z)$ or sometimes $B(z,r)$. For a set $S \subset \mathbb{R}^d$, we denote the set $\{x \in \mathbb{R}^d \mid dist(x,S) < r\}$ by $B_r(S)$. We denote by S^{d-1} the set of unit vectors in $\mathbb{R}^d$. The line segment with end points x and y will be denoted by $l_{x,y}$. When two lines l_1 and l_2 are parallel, we will write $l_1 \parallel l_2$. For a set A, we denote the convex hull of A by $Co(A)$. For a family of sets $\{D_t\}_{t\geq 0}$, we define

$$\overline{\lim_{t\to\infty}}\, D_t := \bigcap_{\tau>0} \bigcup_{t>\tau} D_t, \quad \underline{\lim_{t\to\infty}}\, D_t := \bigcup_{\tau>0} \bigcap_{t>\tau} D_t.$$

If $\overline{\lim}_{t\to\infty} D_t = \underline{\lim}_{t\to\infty} D_t$, we will write

$$\lim_{t\to\infty} D_t := \overline{\lim_{t\to\infty}}\, D_t = \underline{\lim_{t\to\infty}}\, D_t.$$

2.2 Basic strategy of the game

We prepare special strategies of both players that we explained in Section 1.

Definition 2.5 (Concentric strategy)**.** Let $\epsilon > 0$ and $z \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$ be the current position of the game.

1. A set of $d-1$ orthogonal unit vectors $v^j \in S^{d-1}(j = 1, 2, \cdots d-1)$ chosen by Paul is called a z *concentric strategy* (by Paul) if $\langle v^j, x - z\rangle = 0$ for all j, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product.
2. Let $\{v^1, v^2, \cdots, v^{d-1}\}$ be a choice by Paul in the same round. A choice $(b^1, b^2, \cdots, b^{d-1}) \in \{\pm 1\}^{d-1}$ by Carol is called a z *concentric strategy* (by Carol) if $\langle b^j v^j, x - z\rangle \geq 0$ for all j.

One can easily understand the behaviors of trajectories given when one player takes above strategies. Let $d_n = |x_n - z|$ for fixed $z \in \mathbb{R}^d$. If Paul takes a z concentric strategy through the game, then we see that the sequence $\{d_n\}$ satisfies

$$R_{n+1} = \sqrt{R_n^2 + 2(d-1)\epsilon^2} \tag{2.4}$$

by the Pythagorean theorem regardless of Carol's choices. The solution $\{R_n\}$ of (2.4) is explicitly obtained by

$$R_n = \sqrt{R_0^2 + 2(d-1)n\epsilon^2}. \tag{2.5}$$

On the other hand, if Carol takes a z concentric strategy through the game, we have

$$\begin{aligned}\left| x + \sum_j (\sqrt{2}\epsilon b^j v^j) - z \right|^2 &= |x-z|^2 + 2(d-1)\epsilon^2 + \sum_j \sqrt{2}\epsilon \langle b^j v^j, x - z\rangle \\ &\geq |x-z|^2 + 2(d-1)\epsilon^2,\end{aligned}$$

which implies $d_{n+1} \geq \sqrt{d_n^2 + 2(d-1)\epsilon^2}$. Therefore we obtain $d_n \geq \sqrt{d_0^2 + 2(d-1)n\epsilon^2}$.

Remark 2.6. When Carol takes a z concentric strategy, she can control the distance $|x_n - z|$. However we notice that she can not control the moves in tangential direction. For instance, let $d = 2$, $z = (0, 0)$ and the current game position $x_n = (0, 1)$. If Paul chooses $v = (1, 0)$ at this point, both $b = 1$ and $b = -1$ are $(0, 0)$ concentric strategies by Carol. One may think that if Carol makes the further decision that for Paul's choice v tangential to the circle centered at the origin passing through x_n, she chooses b so that bv becomes clockwise, then she could control the trajectory of the game to be clockwise. However this is not true. Carol's greedy attempt to move as she pleases in the tangential direction will be thwarted by Paul. Indeed when Carol makes above decision, Paul can move counterclockwise while the distance $|x_n - z|$ meets almost the condition (2.5) by slightly leaning the vector v from tangential one. (e.g. $v = (\cos -\epsilon^2, \sin -\epsilon^2)$ at $x_n = (0, 1)$)

3 Asymptotic behavior of solutions

Throughout this section we consider the main equation (1.1).

In the following lemma, we estimate the asymptotic shape from above by considering Carol's strategies.

Lemma 3.1.

$$\overline{\lim_{t\to\infty}}\, E_t \subset Co(C_-).$$

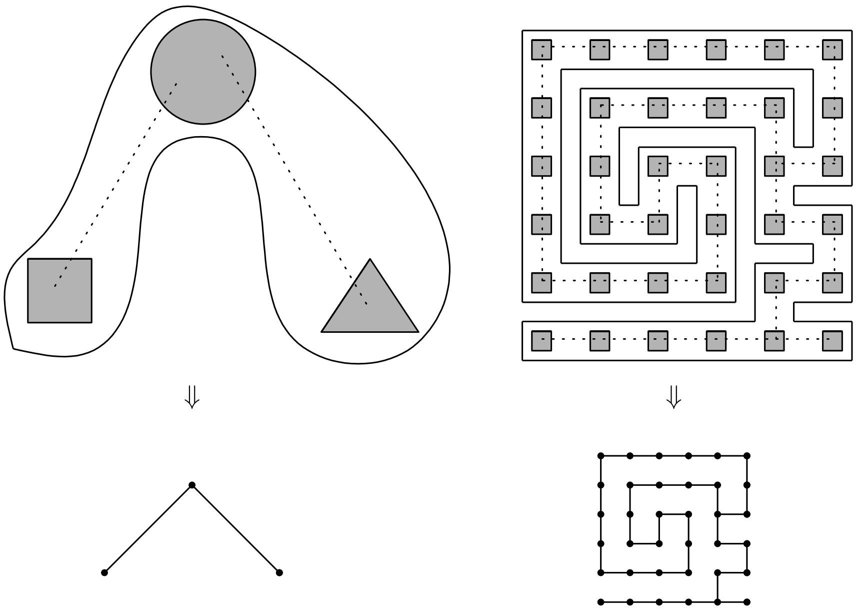
Proof. Let u be the unique solution to (1.2) with u_0 and Ψ_- that are as in Definition 2.2. We notice that the conclusion holds if and only if for $x \in Co(C_-)^c$, there exists $\tau > 0$ such that $u(x, t) < 0$ for $t > \tau$. To prove $u^\epsilon < 0$, it is sufficient to give a Carol's strategy that makes the game cost negative but is not necessarily optimal one. For $x \in Co(C_-)^c$ we can take an open ball B such that $C_- \subset B$ and $x \in \partial B$ by the hyperplane separation theorem and the boundedness of $Co(C_-)$. Let z be the center of B and $r = |z - x|$. If Carol takes a z concentric strategy, then we see that regardless of Paul's choice, the game trajectory $\{x_n\}$ satisfies $|x_n - z| \geq \sqrt{r^2 + 2n(d-1)\epsilon^2}$, where $\sqrt{r^2 + 2n(d-1)\epsilon^2}$ is the solution of (2.4) with $R_0 = r$. Letting $\tau = 2R(R + r)/(d-1)$, we have

$$|x_N - z| \geq r + 2R \tag{3.1}$$

for the last position x_N of the game, where $R > 0$ is a constant taken in (1.3). The inequality (3.1) and $Co(C_-) \subset B$ imply $dist(x_N, Co(C_-)) \geq dist(x_N, B) \geq 2R$. Also $C_- \subset E_0 \subset B_R(0)$ implies $Co(C_-) \subset B_R(0)$. Hence we have $x_N \notin B_R(0)$, which means $u_0(x_N) = a < 0$ by (1.3). If Paul quits the game on the way, the stopping cost is at most

$$\sup_{b \in B^c} \Psi_-(b) < 0.$$

The comparison principle ([35]) for (1.2) and the convergence results in Appendix A imply that $u^\epsilon(x,t)$ converges to $u(x,t)$ locally uniformly in (x,t). See also [2, Chapter V Lemma 1.9] if necessary. We also notice that the uniform boundedness of u^ϵ is satisfied owe to the rule of the game and the boundedness of u_0 and Ψ_-. Since both upper bound of the terminal cost a and that of the stopping cost $\sup_{b\in B^c} \Psi_-(b)$ do not depend on ϵ, we conclude that $u(x,t) < 0$ for $t > \tau$ and $x \in Co(C_-)^c$. □



The graph G is connected. The graph G is connected.

Figure 5: D_0 and O_- that satisfy the assumption of Theorem 3.2 (In this figure the loops of G are ignored.)

For an obstacle O_- and an initial set D_0, we define the graph $G = (V, E)$ as follows:

$$V := \{O \subset \mathbb{R}^2 \mid O \text{ is a connected component of } O_-\},$$

$$E := \{\langle O, P\rangle \mid O, P \in V \text{ and } l_{x,y} \subset D_0 \text{ for some } x \in O \text{ and } y \in P\}.$$

See Appendix C for definitions of terms in graph theory.

Theorem 3.2. *Assume that $d = 2$ and the graph G is connected. Then*

$$Co(O_-) \subset \varliminf_{t\to\infty} D_t \subset \varlimsup_{t\to\infty} D_t \subset \overline{Co(O_-)}$$

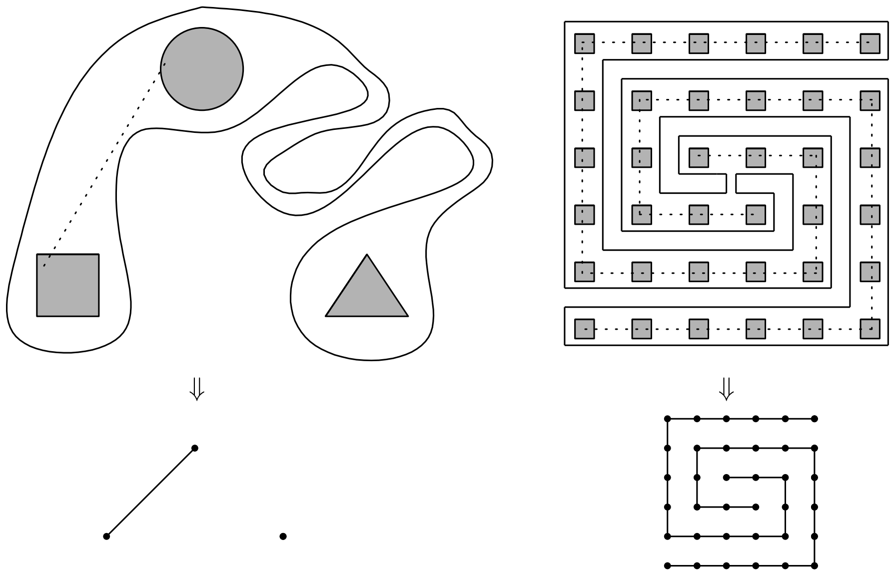
and

$$Co(O_-) \subset \varliminf_{t\to\infty} E_t \subset \varlimsup_{t\to\infty} E_t \subset \overline{Co(O_-)}.$$

Remark 3.3. Figure 5 (resp. Figure 6) shows examples of D_0 and O_- that satisfy (resp. do not satisfy) the assumption of Theorem 3.2.

Proof. For D_0 and E_0, we take u_0 as in Definition 2.2. Indeed it suffices to let

$$u_0(x) = \begin{cases} dist(x, \partial D_0), \ x \in D_0 \\ \max\{a, -dist(x, \partial D_0)\}, \ x \in D_0^c. \end{cases} \tag{3.2}$$



The graph G is not connected. The graph G is not connected.

Figure 6: D_0 and O_- that do not satisfy the assumption of Theorem 3.2

Similarly it suffices to let

$$\Psi_-(x) = \begin{cases} dist(x, \partial O_-), \ x \in O_- \\ \max\{a, -dist(x, \partial O_-)\}, \ x \in O_-^c. \end{cases} \tag{3.3}$$

By Lemma 3.1 and $D_t \subset E_t$, it suffices to prove $Co(O_-) \subset \underline{\lim}_{t\to\infty} D_t$. Namely our goal is to prove that for $x \in Co(O_-)$, there exists $\tau > 0$ such that $u(x,t) > 0$ for $t > \tau$. To prove $u^\epsilon > 0$, it is sufficient to give a Paul's strategy, which makes the game cost positive and is not necessarily optimal one. Let $x \in Co(O_-)$. It would be convenient to introduce the set

$$L := \{z \in l_{x,y} \mid x, y \in O_-, l_{x,y} \subset D_0\}$$

in doing case analysis for $x \in Co(O_-)$.

1) $\boldsymbol{x \in O_-}$. In this case, it suffices for Paul to quit the game at the first round and gain the stopping cost $\Psi_-(x) > 0$. Recall that $u^\epsilon(x,t)$ converges to $u(x,t)$ locally uniformly in (x,t). Thus we obtain $u(x,t) > 0$ for any $t > 0$.

2) $\boldsymbol{x \in L \setminus O_-}$. Let $z, w \in O_-$ satisfy $x \in l_{z,w}$ and $l_{z,w} \subset D_0$. We take $\delta > 0$ to satisfy $\overline{B_\delta(z)} \subset O_-$ and $\overline{B_\delta(w)} \subset O_-$. For the initial position x, Paul's strategy is to keep taking $v = \frac{z-x}{|z-x|}$ until he reaches $B_\delta(z) \cup B_\delta(w)$. If he reaches $B_\delta(z) \cup B_\delta(w)$, then he quits the game. By doing this, Paul gains positive game cost in either case he quits the game or not. See Figure 3. More precisely, Paul gains at least

$$\min\left\{\min_{y\in\overline{B_\delta(z)\cup B_\delta(w)}} \Psi_-(y), \ \min_{y\in l_{z,w}} u_0(y)\right\} > 0$$

regardless of $\epsilon \in (0, \delta/\sqrt{2})$, where ϵ is taken small enough for Paul not to stride over $B_\delta(z)$ or $B_\delta(w)$. Hence, as in the case 1), we obtain $u(x,t) > 0$ for any $t > 0$.

3) $\boldsymbol{x \in Co(O_-) \setminus L}$. Henceforth we give a strategy by Paul that includes a z concentric strategy and makes the game cost positive. To do so, we are going to construct a closed curve that consists of an arc C with its center at z and a path $\hat{\Gamma}$ in L. Since $O_- \subset L \subset Co(O_-)$, we have $Co(O_-) = Co(L)$. By Lemma B.2 in Appendix B, we can take $a, b \in L$ such that $x \in l_{a,b}$. We only show the case $a, b \notin O_-$, since otherwise we would prove it in a simpler manner.

We first explain how to construct a path $\Gamma \subset L$ that contains a and b. We take a specific path rather than just a path. By doing so, we are able to indicate a region that includes final positions of the games to guarantee that u^ϵ is uniformly positive. Since $a \in L$, there is a line segment that is in D_0, contains a, and has endpoints in some connected components A and B of O_- respectively. Similarly there is a line segment

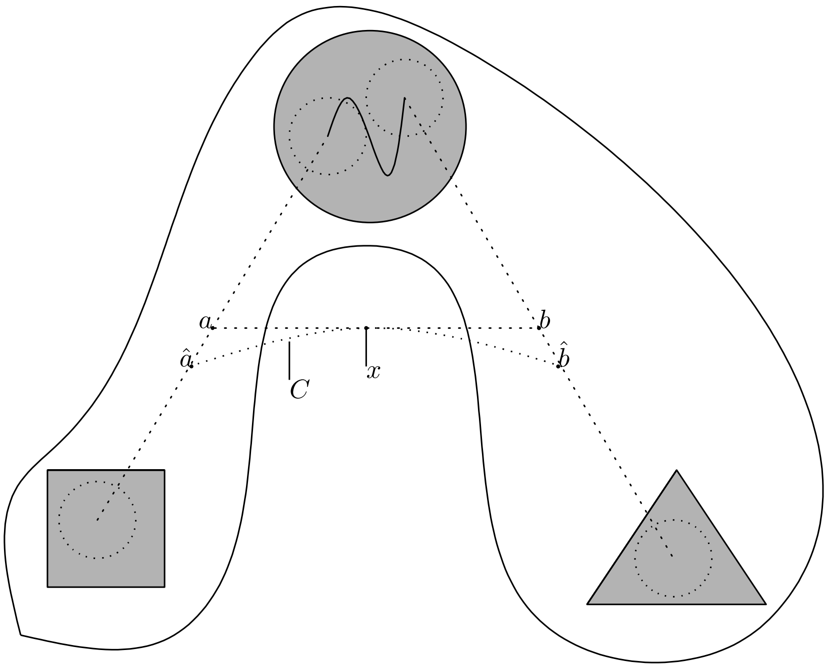


Figure 7: strategy in the case 3)

that is in D_0, contains b, and has endpoints in some connected components C and D of O_- respectively. Notice that $\langle A, B\rangle, \langle C, D\rangle \in E$, recalling E is the set of unordered pairs of the graph G defined above. From Proposition C.2 there is a path $P = (V', E')$ of the graph G such that $A, B, C, D \in V'$ and $\langle A, B\rangle, \langle C, D\rangle \in E'$. Writing $V' = \{O_0, O_1, \cdots, O_n\}$ and $E' = \{\langle O_i, O_{i+1}\rangle \mid i = 0, 1, \cdots, n-1\}$, we see that for some points $y_i, \tilde{y}_i \in O_i$ $(i = 0, 1, \cdots, n-1)$, there are line segments $l_{\tilde{y}_i, y_{i+1}}$ $(i = 0, 1, \cdots, n-1)$ such that $a \in l_{\tilde{y}_0, y_1}$, $b \in l_{\tilde{y}_{n-1}, y_n}$ and $l_{\tilde{y}_i, y_{i+1}} \subset D_0$ $(i = 0, 1, \cdots, n-1)$. Let Γ_i be a polygonal line in O_i with endpoints y_i and $\tilde{y}_i$. Now we define

$$\Gamma := l_{\tilde{y}_0, y_1} \cup \Gamma_1 \cup l_{\tilde{y}_1, y_2} \cup \Gamma_2 \cup l_{\tilde{y}_2, y_3} \cup \cdots \cup l_{\tilde{y}_{n-2}, y_{n-1}} \cup \Gamma_{n-1} \cup l_{\tilde{y}_{n-1}, y_n}.$$

We may assume $l_{a,b} \nparallel l_{\tilde{y}_0, y_1}$ and $l_{a,b} \nparallel l_{\tilde{y}_{n-1}, y_n}$, since otherwise we would retake either a or b as an element of O_-. Without loss of generality we can also assume that Γ and $l_{a,b}$ do not cross each other except at a and b. By Lemma B.1 in Appendix B we take $\delta > 0$ small enough to satisfy $B_{3\delta}(\Gamma) \subset L$, noticing that L is an open set and Γ is a compact set. Let w_0, w_1, w_2 be unit vectors in $\mathbb{R}^2$ such that $w_0 \parallel l_{a,b}$, $w_1 \parallel l_{\tilde{y}_0, y_1}$, $w_2 \parallel l_{\tilde{y}_{n-1}, y_n}$ and $(w_0 \cdot w_1)(w_0 \cdot w_2) \geq 0$. Let $\hat{a} = a + \delta w_1$ and $\hat{b} = b + \delta w_2$. We temporarily define C as the arc passing through $\hat{a}$, $\hat{b}$ and x. Combining Γ and C, we make a closed curve $C \cup \hat{\Gamma}$, where

$$\hat{\Gamma} := l_{\hat{a}, y_1} \cup \Gamma_1 \cup l_{\tilde{y}_1, y_2} \cup \Gamma_2 \cup l_{\tilde{y}_2, y_3} \cup \cdots \cup l_{\tilde{y}_{n-2}, y_{n-1}} \cup \Gamma_{n-1} \cup l_{\tilde{y}_{n-1}, \hat{b}}.$$

For the closed curve $C \cup \hat{\Gamma}$ and x in it, there is a Jordan closed curve $\hat{C}$ such that $x \in \hat{C}$ and $\hat{C} \subset C \cup \hat{\Gamma}$ (Figure 8). See Appendix D in detail. Thus, based on the Jordan curve theorem, we let Ω be the bounded domain that satifies $\partial\Omega = \hat{C}$. If Ω touches the arc C from inside, then we retake the other pair of $(\hat{a}, \hat{b})$ and, together with it, retake C and Ω so that Ω touches the arc C from outside (Figure 9). We notice that the domain enclosed by $l_{\tilde{y}_0, y_1}$, $l_{\tilde{y}_{n-1}, y_n}$ and the two arcs shown in Figure 9 is bounded, and hence, so is the new Ω. We further notice that we can take C so that C and $\hat{\Gamma}$ intersect only at $\hat{a}$ and $\hat{b}$ by taking δ smaller if necessary.

We now give a strategy by Paul for the initial position x. Paul first takes a z concentric strategy, where z is the center of the arc C. If Paul enters $B_\delta(\Gamma_i)$ for some i, then he quits the game at this point. Once he enters $B_\delta(l_{\tilde{y}_i, y_{i+1}})$ for some i, he takes a similar strategy to that in the case 2). To see that it attains positive game cost, we notice two propeties of game trajectories $\{x_n\}$ given when Paul keeps taking a z concentric strategy by some round. One is that $x_n \in B\left(z, \sqrt{|x_0 - z|^2 + 2n\epsilon^2}\right)^c$. The other is that $x_n \in (C \cup \Omega) \setminus N_\delta$ implies $x_{n+1} \in \Omega$ for $\sqrt{2}\epsilon < \delta$, where we denote $(\bigcup_i B_\delta(\Gamma_i)) \cup (\bigcup_i B_\delta(l_{\tilde{y}_i, y_{i+1}}))$

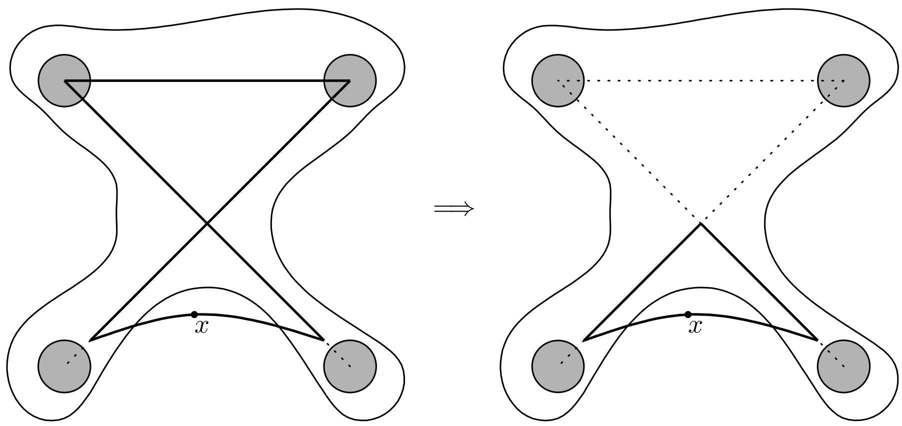


Figure 8: make a Jordan closed curve

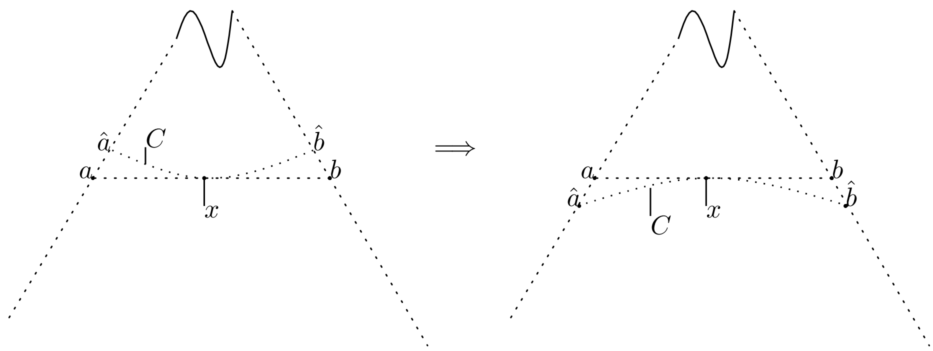


Figure 9: retake C and Ω

by N_δ. Thus it takes at most finite time τ (satisfying $\Omega \subset B(z, \sqrt{|x_0 - z|^2 + 2\tau})$) for Paul to reach N_δ. We see that the game ends at some point in $N_{2\delta}$ and Paul gains at least

$$\min \left\{ \inf_{y \in \bigcup_i B_{2\delta}(\Gamma_i)} \Psi_-(y), \inf_{y \in \bigcup_i B_\delta(l_{\tilde{y}_i, y_{i+1}})} u_0(y) \right\} > 0,$$

regardless of $\epsilon \in (0, \delta/\sqrt{2})$. Therefore we conclude that $u(x,t) > 0$ for any $t > \tau$, noticing that τ may depend on x, but does not depend on ϵ. □

Remark 3.4. In general, fattening of the level set may occur under the assumption of Theorem 3.2. i.e., $\overline{D_t} = E_t$ may fail at some $t > 0$. Theorem 3.2 states that even if the curve evolutions are not unique, they have the same limit.

We give some sufficient conditions to the assumption of Theorem 3.2.

Corollary 3.5. *Assume that $Co(O_-) \subset D_0$. Then the same conclusion as that of Theorem 3.2 holds.*

Corollary 3.6. *Assume that O_- is connected (Figure 10 and 11). Then the same conclusion as that of Theorem 3.2 holds.*

Under the following assumption, the moving surface sticks to the obstacle in finite time:

$$\exists r > 0 \ \forall w \in \partial O_- \ \exists z \in B_r(w), \text{ s.t. } O_- \subset B_{|z-w|}(z). \tag{3.4}$$

In the following theorem, there is no need to asuume $d = 2$.

Theorem 3.7. *Assume* (3.4). *Then*

$$\lim_{t \to \infty} D_t = O_-$$

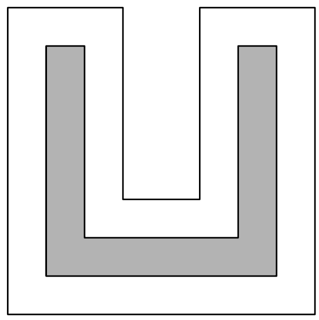


Figure 10: O_- is connected

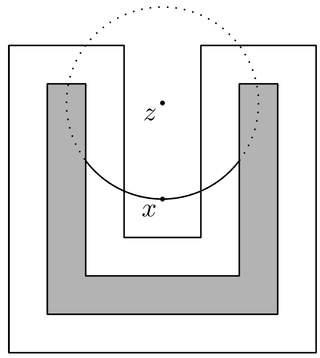


Figure 11: Paul's strategy to achieve positive game cost for the initial position x

and

$$\lim_{t\to\infty} E_t = C_-.$$

Moreover there exists $\tau > 0$ *such that* $D_t = D_\tau = O_-$ *and* $E_t = E_\tau = C_-$ *for* $t \geq \tau$.

Proof. We notice that the condition (3.4) implies that O_- is convex (Proposition B.4). It is now clear that $O_- \subset D_t$ for any $t > 0$ and hence $O_- \subset \bigcap_{t>0} D_t \subset \underline{\lim}_{t\to\infty} D_t$. We prove that there exists $\tau > 0$ such that $\bigcup_{t\geq\tau} E_t \subset \overline{O_-}$. Namely we show that there exists $\tau > 0$ such that $u(x,t) < 0$ for $t \geq \tau$ and $x \in (\overline{O_-})^c$. The difference from Lemma 3.1 is that we now have to take τ independent of x. Indeed, for $x \in (\overline{O_-})^c$, we can take a ball $B_{|z-w|}(z)$ in (3.4) such that $x \notin B_{|z-w|}(z)$ and it suffices for Carol to keep taking a z concentric strategy until the game ends. The value r in (3.1) is now taken independent of x and then so is τ. Therefore we obtain

$$O_- \subset \underline{\lim}_{t\to\infty} D_t \subset \overline{\lim}_{t\to\infty} D_t \subset \bigcup_{t\geq\tau} D_t \subset \bigcup_{t\geq\tau} E_t \subset \overline{O_-}.$$

Since $\bigcup_{t\geq\tau} D_t$ is open, we have $O_- = \bigcup_{t\geq\tau} D_t$, which means $\underline{\lim}_{t\to\infty} D_t = \overline{\lim}_{t\to\infty} D_t$ and moreover $D_t = D_\tau = O_-$ for $t \geq \tau$.

We also have

$$O_- \subset \bigcap_{t>0} D_t \subset \bigcap_{t>0} E_t \subset \underline{\lim}_{t\to\infty} E_t \subset \overline{\lim}_{t\to\infty} E_t \subset \overline{O_-}.$$

Since $\bigcap_{t>0} E_t$ is closed, we similarly have $\overline{O_-} = \bigcap_{t>0} E_t$, which means $\underline{\lim}_{t\to\infty} E_t = \overline{\lim}_{t\to\infty} E_t$ and moreover $E_t = E_0 = \overline{O_-}$ for $t \geq 0$. □

Remark 3.8. The hair-clip solution, which can be regarded as an explicit solution to the curve shortening problem with the Dirichlet condition (See e.g. [47]), implies that the solution of our obstacle problem can be apart from the asymptotic shape at any time. i.e., there are D_0 and O_- such that $Co(O_-) \subsetneq D_t$ for any $t > 0$.

4 With driving force

In this section we consider the following surface evolution equations in the plane that have obstacles on one side:

$$\begin{cases} V = -\kappa + \nu, \text{ on } \partial D_t, \\ O_- \subset D_t, \end{cases} \tag{4.1}$$

or

$$\begin{cases} V = -\kappa + \nu, \text{ on } \partial D_t, \\ D_t \subset O_+, \end{cases} \tag{4.2}$$

where D_t, O_- and O_+ are open sets in $\mathbb{R}^2$ and $\nu > 0$ is a constant. These equations are specific cases of (2.2).

While Theorem 3.2 is invarient with respect to similarity transformation, the behaviors of moving curves governed by (4.1) or (4.2) depend not only on shape of the obstacles and the initial curve but also on size of them. It is easily understood by considering the case ∂D_0 is a circle. For $D_0 = B_R((0,0))$, the circle ∂D_0 shrinks if $R < \nu^{-1}$, and it spreads if $R > \nu^{-1}$ as time goes. So it is difficult to present a concise result as Theorem 3.2. We give several examples of computations of the asymptotic shapes here.

4.1 Basic strategy of the game

Concerning to the game interpretation for (4.1), the difference from the game in Section 1 is that Paul has the right to choose $w_i \in S^1$ at each round i and the control system is

$$x_i = x_{i-1} + \sqrt{2}\epsilon b_i v_i + \nu\epsilon^2 w_i,$$

instead of (A.1). In the game for (4.2), not Paul but Carol has the right to quit the game. If Carol quits the game at round i, then the cost is given by $\Psi_+(x_i)$. See Appendix A in detail. As explained later in the proof of Lemma 3.1, the value functions u^ϵ locally uniformly converge to the solution u of the corresponding level set equation.

We now prepare several types of game strategies and give the properties of the game trajectories when they are used. The first one is similar to the one in Definition 2.5.

Definition 4.1 (Concentric strategy). Let $\nu > 0$, $\epsilon > 0$ and $z \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ be the current position of the game.

1. A choice $(v, w) \in S^1 \times S^1$ by Paul is called a z *concentric strategy* (by Paul) if

$$w = \frac{z - x}{|z - x|} \text{ and } v \perp w.$$

   When $x = z$, any $(v, w) \in S^1 \times S^1$ satisfying $v \perp w$ is called a z *concentric strategy.*

2. Let $(v, w) \in S^1 \times S^1$ be a choice by Paul in the same round. A choice $b \in \{\pm 1\}$ by Carol is called a z *concentric strategy* (by Carol) if

$$\langle bv, x + \nu\epsilon^2 w - z\rangle \geq 0.$$

As in Section 2.2, let $d_n = |x_n - z|$ for fixed $z \in \mathbb{R}^2$. If Paul takes a z concentric strategy through the game, then the sequence $\{d_n\}$ satisfies

$$R_{n+1} = \sqrt{(R_n - \nu\epsilon^2)^2 + 2\epsilon^2}. \tag{4.3}$$

If Carol takes a z concentric strategy through the game, we have $d_n \geq R_n$, where $\{R_n\}$ is the solution to (4.3) with $R_0 = d_0$. In the following lemmas, we give basic propeties of the behaviors of the solutions to (4.3).

Lemma 4.2. *Fix $\nu > 0$. Let $\epsilon > 0$ and $\{R_n\}$ be a sequence satisfying the condition* (4.3)*. Then the following properties hold.*

1. *If $R_0 = \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n = \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n. If $R_0 > \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n > \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n and $\{R_n\}$ is decreasing for $\epsilon \leq \sqrt{2}\nu^{-1}$. If $\nu\epsilon^2 \leq R_0 < \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n < \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n and $\{R_n\}$ is increasing.*

2. $$\lim_{n\to\infty} R_n = \nu^{-1} + \frac{\nu}{2}\epsilon^2.$$

3. $$R_{n+1} - R_n \leq \frac{\epsilon^2}{R_n} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2R_n}. \tag{4.4}$$

   If $R_{n+1} \geq R_n$, then

$$\frac{\epsilon^2}{R_{n+1}} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2R_{n+1}} \leq R_{n+1} - R_n. \tag{4.5}$$

Proof. 1. We first notice that $R_{n+1} \leq R_n$ is equivalent to $R_n \geq \frac{\nu}{2}\epsilon^2 + \nu^{-1}$. Also $R_{n+1} \geq R_n$ is equivalent to $R_n \leq \frac{\nu}{2}\epsilon^2 + \nu^{-1}$. These facts imply the first assertion.

Assume that $R_k > \frac{\nu}{2}\epsilon^2 + \nu^{-1}$ for some k. Since $(R_k - \nu\epsilon^2)^2 > \left(\nu^{-1} - \frac{\nu}{2}\epsilon^2\right)^2$ for $\epsilon \leq \sqrt{2}\nu^{-1}$, we have

$$R_{k+1} = \sqrt{(R_k - \nu\epsilon^2)^2 + 2\epsilon^2} > \sqrt{\left(\nu^{-1} - \frac{\nu}{2}\epsilon^2\right)^2 + 2\epsilon^2} = \frac{\nu}{2}\epsilon^2 + \nu^{-1}.$$

Hence, if $R_0 > \frac{\nu}{2}\epsilon^2 + \nu^{-1}$, we have by induction that $R_n > \frac{\nu}{2}\epsilon^2 + \nu^{-1}$ for all n. The second assertion follows from this.

Assume that $\nu\epsilon^2 \leq R_k < \frac{\nu}{2}\epsilon^2 + \nu^{-1}$. Since $(R_k - \nu\epsilon^2)^2 < \left(\nu^{-1} - \frac{\nu}{2}\epsilon^2\right)^2$ for $\epsilon \in \left(0, \sqrt{2}\nu^{-1}\right)$, we get $R_{k+1} < \frac{\nu}{2}\epsilon^2 + \nu^{-1}$. In the same way, $\nu\epsilon^2 \leq R_0 < \frac{\nu}{2}\epsilon^2 + \nu^{-1}$ gives $R_n < \frac{\nu}{2}\epsilon^2 + \nu^{-1}$. Therefore the third assertion is obtained.

2. Since $\{R_n\}$ is monotone and bounded, it is convergent. Its limit value is given by taking limit for both sides of (4.3) and solving the limit equation.

3. The inequality (4.4) is given by the following computation.

$$R_{n+1} - R_n = \frac{R_{n+1}^2 - R_n^2}{R_{n+1} + R_n} \leq \frac{R_{n+1}^2 - R_n^2}{2R_n} = \frac{\epsilon^2}{R_n} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2R_n}.$$

Similarly (4.5) is obtained by

$$R_{n+1} - R_n = \frac{R_{n+1}^2 - R_n^2}{R_{n+1} + R_n} \geq \frac{R_{n+1}^2 - R_n^2}{2R_{n+1}} \geq \frac{\epsilon^2}{R_{n+1}} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2R_{n+1}}.$$

□

It is sometimes convenient to describe the behavior of the trajectory by an operator. For fixed $\nu > 0$, we define the operator $T_h : \mathbb{R} \to \mathbb{R}$ as

$$T_h(R) := \sqrt{(R - \nu h)^2 + 2h}.$$

We denote n times composition of T_h by T_h^n. The solution $\{R_n\}$ to (4.3) with $R_0 = R$ is described by $R_n = T_{\epsilon^2}^n(R)$.

Lemma 4.3. *1. $T_h(R) < T_h(R')$, provided $\nu h \leq R < R'$.*

2. $T_h(R) > T_{h/2}^2(R)$.

Proof. The proofs are done by direct computation, so are omitted. □

We also prepare a notation that represents the time to pass through the set $B_b(z) \setminus B_a(z)$ when Paul takes a z concentric strategy. For $a, b \geq 0$ and $\epsilon > 0$ satisfying $a \leq b \leq \nu^{-1} + \frac{\nu}{2}\epsilon^2$, we define

$$t_\epsilon(a, b) := \epsilon^2 |\{n \in \mathbb{N} \mid T_{\epsilon^2}^n(a) < b\}|, \tag{4.6}$$

where we denote the set $\{0, 1, 2, \cdots\}$ by $\mathbb{N}$.

Lemma 4.4. *Let $t > 0$ and $0 < R < \nu^{-1}$. Define $N = \lceil t\epsilon^{-2} \rceil$. Then $T_{\epsilon^2}^N(R) < \nu^{-1}$ for sufficiently small $\epsilon > 0$.*

Proof. We prove that

$$t_\epsilon\left(\nu^{-1} - \epsilon, \nu^{-1}\right) \geq C \log_2 \epsilon^{-1}$$

for some constant $C > 0$.

First let us explain the property of $t_\epsilon(a, b)$. Let $0 < a \leq b \leq c \leq \nu^{-1} + \frac{\nu}{2}\epsilon^2$. By the definition of t_ϵ, we see

$$t_\epsilon(a, b) \leq t_\epsilon(a, c).$$

By Lemma 4.3 1, we have

$$t_\epsilon(b, c) \leq t_\epsilon(a, c).$$

Let $n^* = t_\epsilon(a,b)\epsilon^{-2}$ and $d = T_{\epsilon^2}^{n^*-1}(a)$. Then we have

$$t_\epsilon(a,c) = t_\epsilon(a,b) + t_\epsilon(d,c) - \epsilon^2.$$

Since $d < b$, we deduce that

$$t_\epsilon(a,c) \geq t_\epsilon(a,b) + t_\epsilon(b,c) - \epsilon^2. \tag{4.7}$$

We next estimate $t_\epsilon\left(\nu^{-1}-\epsilon, \nu^{-1}\right)$. When $\nu^{-1} - 2^{-m}\epsilon \leq R_n \leq \nu^{-1} - 2^{-m-1}\epsilon$ for $m, n \in \mathbb{N}$, the inequality (4.4) implies

$$R_{n+1} - R_n \leq \frac{\epsilon^2}{\nu^{-1} - 2^{-m}\epsilon} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2\nu^{-1} - 2^{-m+1}\epsilon} = \frac{2^{-m}\nu\epsilon^3 + \nu^2\epsilon^4}{2\nu^{-1} - 2^{-m+1}\epsilon},$$

and in particular, if $\epsilon \leq \nu^{-1}$,

$$R_{n+1} - R_n \leq \frac{2^{-m}\nu\epsilon^3 + \nu^2\epsilon^4}{2\nu^{-1} - 2^{-m+1}\epsilon} \leq \frac{2^{-m}\nu\epsilon^3 + \nu^2\epsilon^4}{\nu^{-1}} = \nu^2\epsilon^3\left(2^{-m} + \nu\epsilon\right).$$

Thus $t_\epsilon\left(\nu^{-1} - 2^{-m}\epsilon, \nu^{-1} - 2^{-m-1}\epsilon\right)$ can be compared to the exit time for sequences with the constant speed of $\nu^2\epsilon^3\left(2^{-m} + \nu\epsilon\right)$ per round. Then we have

$$\begin{aligned} & t_\epsilon\left(\nu^{-1} - 2^{-m}\epsilon, \nu^{-1} - 2^{-m-1}\epsilon\right) \\ & \geq \left(\left(\nu^{-1} - 2^{-m-1}\epsilon\right) - \left(\nu^{-1} - 2^{-m}\epsilon\right)\right)\frac{\epsilon^2}{\nu^2\epsilon^3\left(2^{-m} + \nu\epsilon\right)} = \frac{1}{2\nu^2 + 2^{m+1}\nu^3\epsilon}. \end{aligned}$$

In particular, if $2^m \leq \epsilon^{-1}$, we get

$$\frac{1}{2\nu^2 + 2^{m+1}\nu^3\epsilon} \geq \frac{1}{2\nu^2 + 2\nu^3} = \frac{1}{2\nu^2(\nu+1)}.$$

Let m^* be the minimal integer m satisfying $2^m > \epsilon^{-1}$. Using (4.7), we obtain

$$\begin{aligned} t_\epsilon\left(\nu^{-1} - \epsilon, \nu^{-1}\right) &\geq \sum_{k=0}^{m^*-1}\left(t_\epsilon\left(\nu^{-1} - 2^{-k}\epsilon, \nu^{-1} - 2^{-k-1}\epsilon\right) - \epsilon^2\right) \\ &\geq \sum_{k=0}^{m^*-1}\frac{1}{3\nu^2(\nu+1)} = \frac{\log_2\epsilon^{-1}}{3\nu^2(\nu+1)}, \end{aligned}$$

which is the conclusion. □

Lemma 4.5. *For $\nu > 0$ and $\nu^{-1} \geq \delta > 0$, there exist $\epsilon_0 > 0$ and $M > 0$ such that*

$$t_\epsilon(a, \nu^{-1} - \delta) \leq M$$

for all $0 < \epsilon < \epsilon_0$ and $0 \leq a \leq \nu^{-1} - \delta$.

Proof. Let $n_\epsilon = t_\epsilon(a, \nu^{-1} - \delta)\epsilon^{-2}$. Namely $T_{\epsilon^2}^{n_\epsilon}(a) < \nu^{-1} - \delta$ and $T_{\epsilon^2}^{n_\epsilon+1}(a) \geq \nu^{-1} - \delta$ are satisfied. Since $T_{\epsilon^2}(R) - R$ is monotonically decreasing with respect to R and $T_{\epsilon^2}^{n_\epsilon+1}(a) \leq \nu^{-1} - \delta/2$ for sufficiently small $\epsilon > 0$, we have by (4.5) in Lemma 4.2

$$T_{\epsilon^2}^{n+1}(a) - T_{\epsilon^2}^{n}(a) \geq T_{\epsilon^2}^{n_\epsilon+1}(a) - T_{\epsilon^2}^{n_\epsilon}(a) \geq \frac{\epsilon^2}{T_{\epsilon^2}^{n_\epsilon+1}(a)} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2T_{\epsilon^2}^{n_\epsilon+1}(a)} \geq \frac{\delta\nu^2}{2-\delta\nu}\epsilon^2$$

for $n = 0, 1, \cdots, n_\epsilon$. This means

$$n_\epsilon \leq (\nu^{-1} - \delta - a)\frac{2-\delta\nu}{\delta\nu^2\epsilon^2}$$

and hence

$$t_\epsilon(a, \nu^{-1} - \delta) \leq (\nu^{-1} - \delta - a)\frac{2-\delta\nu}{\delta\nu^2\epsilon^2}\epsilon^2 \leq (\nu^{-1} - \delta)\frac{2-\delta\nu}{\delta\nu^2},$$

which is the conclusion. □

To study the asymptotic behavior of solutions, the following types of strategies of the game are also useful.

Definition 4.6 (Push by moving circle)**.** For a sequence $\{z_n\} \subset \mathbb{R}^2$ such that $|z_{n+1} - z_n| \leq C$ for some constant $C > 0$, a strategy for Paul (resp. for Carol) to keep taking a z_n concentric strategy at each round n is called a *push by moving circle strategy* by Paul (resp. by Carol).

Lemma 4.7 (Properties of push by moving circle strategies)**.** *Let $\delta > 0$. Let $\epsilon > 0$ be sufficiently small. i.e., we assume that $0 < \epsilon < \epsilon_0(\nu, \delta)$ for some $\epsilon_0(\nu, \delta) > 0$.*

1. *If either Paul or Carol takes a push by moving circle strategy with $C = \frac{\delta}{2}\nu^2\epsilon^2$ and $x_0 \in \overline{B(z_0, \nu^{-1} - \delta)}^c$, then $x_n \in \overline{B(z_n, \nu^{-1} - \delta)}^c$ for all round n.*
2. *If Paul takes a push by moving circle strategy with $C = \frac{\nu^2\delta}{2(1+\nu\delta)}\epsilon^2$ and $x_0 \in \overline{B(z_0, \nu^{-1} + \delta)}$. then $x_n \in \overline{B(z_n, \nu^{-1} + \delta)}$ for all round n.*

Proof. 1. We notice that $|x_{n+1} - z_n| \geq T_{\epsilon^2}(|x_n - z_n|)$ whichever player takes the push by moving circle strategy. We prove that $x_n \in \overline{B(z_n, \nu^{-1} - \delta)}^c$ implies $x_{n+1} \in \overline{B(z_{n+1}, \nu^{-1} - \delta)}^c$. If $|x_n - z_n| \geq \nu^{-1}$, we see from Lemma 4.2 that $|x_{n+1} - z_n| \geq \nu^{-1}$ and hence $|x_{n+1} - z_{n+1}| \geq \nu^{-1} - \frac{\delta}{2}\nu^2\epsilon^2 \geq \nu^{-1} - \delta$ for sufficiently small ϵ. Thus we asuume $|x_n - z_n| < \nu^{-1}$ hereafter.

It suffices to show $|x_{n+1} - z_n| > \nu^{-1} - \delta + \frac{\delta}{2}\nu^2\epsilon^2$. Indeed this inequality is obtained by

$$|x_{n+1} - z_n| \geq T_{\epsilon^2}(|x_n - z_n|) > T_{\epsilon^2}(\nu^{-1} - \delta) \geq \nu^{-1} - \delta + \frac{\delta}{2}\nu^2\epsilon^2.$$

The second inequality is derived from the monotonicity property of T_{ϵ^2} stated in Lemma 4.3. The third inequality is computed by (4.5) in Lemma 4.2. We notice that $T_{\epsilon^2}(\nu^{-1} - \delta) \leq \nu^{-1} - \delta/2$ for sufficiently small ϵ. Letting $R_n = \nu^{-1} - \delta$ and hence $R_{n+1} = T_{\epsilon^2}(\nu^{-1} - \delta)$, we indeed have

$$R_{n+1} - R_n \geq \frac{\epsilon^2}{R_{n+1}} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2R_{n+1}} \geq \frac{\epsilon^2}{\nu^{-1} - \delta/2} - \nu\epsilon^2 + \frac{\nu^2\epsilon^4}{2\nu^{-1} - \delta} \geq \frac{\delta}{2}\nu^2\epsilon^2.$$

Thus the proof is complete.

2. We prove that $x_n \in \overline{B(z_n, \nu^{-1} + \delta)}$ implies $x_{n+1} \in \overline{B(z_{n+1}, \nu^{-1} + \delta)}$. If $|x_n - z_n| < \nu\epsilon^2$, it is clear that $|x_{n+1} - z_{n+1}| \leq \nu^{-1} + \delta$ for sufficiently small ϵ. If not, we have

$$|x_{n+1} - z_{n+1}| \leq T_{\epsilon^2}(\nu^{-1} + \delta) + \frac{\nu^2\delta}{2(1 + \nu\delta)}\epsilon^2.$$

Letting $R_n = \nu^{-1} + \delta$ and hence $R_{n+1} = T_{\epsilon^2}(\nu^{-1} + \delta)$, we have from (4.4) in Lemma 4.2

$$R_n - R_{n+1} \geq -\frac{\epsilon^2}{R_n} + \nu\epsilon^2 - \frac{\nu^2\epsilon^4}{2R_n} = \frac{\nu^2\delta}{1 + \nu\delta}\epsilon^2 + o(\epsilon^2).$$

Therefore we obtain

$$|x_{n+1} - z_{n+1}| \leq \nu^{-1} + \delta$$

for sufficiently small ϵ.

□

We set

$$\mathcal{L}_\nu := \{\gamma([0,1]) \mid \gamma \in C^2([0,1]; \mathbb{R}^2),\ \gamma \text{ is regular and } |\kappa_{\gamma(t)}| \leq \nu \text{ for all } t \in (0,1)\},$$

where we denote the curvature of $\gamma([0,1])$ at $\gamma(t)$ by $\kappa_{\gamma(t)}$.

Definition 4.8 (Γ tube strategy)**.** Let $\nu > 0$ and $\Gamma \in \mathcal{L}_\nu$. Let $x \in \mathbb{R}^2$ be the current position of the game. Let $\hat{x}$ satisfy $\min_{y \in \Gamma}|x - y| = |x - \hat{x}|$. A $\hat{x} + \frac{\hat{x} - x}{\nu|\hat{x} - x|}$ concentric strategy is called a Γ *tube strategy* (Figure 12).

Lemma 4.9 (Property of Γ tube strategies)**.** *Let $\Gamma = \gamma([0,1]) \in \mathcal{L}_\nu$. Let $\delta > 0$ and $\epsilon \ll \delta$. If Paul takes a Γ tube strategy at round n and $x_n \in B_\delta(\Gamma) \setminus (B_\delta(\gamma(0)) \cup B_\delta(\gamma(1)))$, then $x_{n+1} \in B_\delta(\Gamma)$.*

Proof. It is clear that the statement holds if $x_n \in B_{\delta/2}(\Gamma)$. Thus, there is no loss of generality to assume that $\hat{x}_n = (0,0)$ and $x_n \in \{(0,q) \mid \delta/2 \leq q < \delta\}$, where $\hat{x}_n$ is a minimizer taken in Definition 4.8. There are a graph $f : \mathbb{R} \to \mathbb{R}$ and $\delta_0 > 0$ such that $\hat{\Gamma} := \{(p,q) \mid q = f(p), -\delta_0 < p < \delta_0\} \subset \Gamma$. Let $C = \partial B_{\nu^{-1}}((0, -\nu^{-1}))$. If Paul takes a Γ tube strategy, we see that $dist(x_{n+1}, C) < dist(x_n, C) < \delta$. Since $f(p) \geq -\nu^{-1} + \sqrt{\nu^{-2} - p^2}$ for $-\delta_0 < p < \delta_0$, we have

$$dist(x_{n+1}, \Gamma) \leq dist(x_{n+1}, \hat{\Gamma}) \leq dist(x_{n+1}, C) < dist(x_n, C) < \delta,$$

which is the conclusion. □

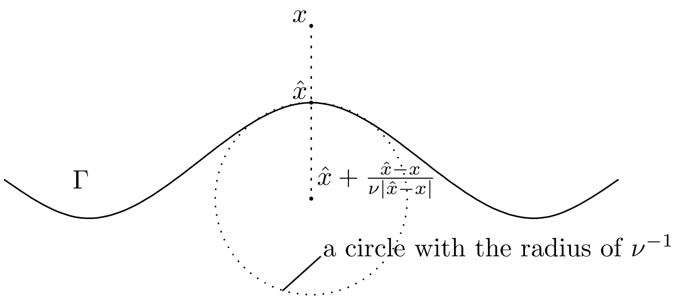


Figure 12: Γ tube strategy

4.2 Examples

We first observe the behavior of the solution to (4.1) by considering two examples of O_-. In both of the problems we assume

(A1) $D_0 \subset B_R(z)$ for some $z \in \mathbb{R}^2$ and $R < \nu^{-1}$.

Set

$$A := \bigcap \left\{ \overline{B_{\nu^{-1}}(z)} \mid O_- \subset \overline{B_{\nu^{-1}}(z)},\ z \in \mathbb{R}^2 \right\}$$

for $O_- \subset \mathbb{R}^2$. Under the assumption (A1), this A is a major candidate of the asymptotic shape. Indeed we can show at least that the asymptotic shape is bounded by A from above for general O_- satisfying (A1).

Lemma 4.10. *Let $O \subset \mathbb{R}^d$ and $r > 0$. Then the set $\{y \in \mathbb{R}^d \mid O \subset \overline{B_r(y)}\}$ is convex.*

Proof. Let $y_1, y_2 \in \mathbb{R}^d$ satisfy $O \subset \overline{B_r(y_1)}$ and $O \subset \overline{B_r(y_2)}$. Then, for $x \in O$, we have $|y_i - x| \le r$ $(i = 1, 2)$. This implies that

$$\left| \frac{y_1 + y_2}{2} - x \right| = \frac{|y_1 - x + y_2 - x|}{2} \le r$$

for all $x \in O$ and the lemma follows. $\square$

We hereafter take u_0 and Ψ_- as (3.2) and (3.3) respectively. We set $O_{\delta_1} := \{x \in \mathbb{R}^2 \mid \Psi_-(x) > -\delta_1\}$ and $C_{\delta_1,\delta_2} := \{z \in \mathbb{R}^2 \mid O_{\delta_1} \subset \overline{B(z, \nu^{-1} - \delta_2)}\}$ for $\delta_1, \delta_2 > 0$.

Lemma 4.11. *Assume* (A1). *For $x \in A^c$, there exist $\delta_1, \delta_2 > 0$ and $\hat{z} \in \mathbb{R}^2$ such that $\hat{z} \in C_{\delta_1,\delta_2}$ and $x \in \overline{B_{\nu^{-1}}(\hat{z})}^c$.*

Proof. **Step 1.** We first show that $B_{\delta_1+\delta_2}(z) \subset C_{0,0}$ implies $z \in C_{\delta_1,\delta_2}$. Indeed $B_{\delta_1+\delta_2}(z) \subset C_{0,0}$ implies

$$O_- \subset \bigcap \{\overline{B_{\nu^{-1}}(\tilde{z})} \mid \tilde{z} \in B_{\delta_1+\delta_2}(z)\} = \overline{B(z, \nu^{-1} - (\delta_1 + \delta_2))}.$$

By taking δ_1 neighborhood of both sides, we have

$$O_{\delta_1} \subset \overline{B(z, \nu^{-1} - \delta_2)},$$

which means $z \in C_{\delta_1,\delta_2}$.

Step 2. Let $C^x_{0,0} := \{z \in \mathbb{R} \mid O_- \subset \overline{B_{\nu^{-1}}(z)}, x \in \overline{B_{\nu^{-1}}(z)}^c\}$. We next show $(C^x_{0,0})^{int} \neq \emptyset$. By the assumption (A1), we see that $\overline{B(z, \nu^{-1} - R)} \subset C_{0,0}$ for some $z \in \mathbb{R}^2$. We also notice that $C^x_{0,0} = C_{0,0} \setminus \overline{B_{\nu^{-1}}(x)}$. If $\overline{B(z, \nu^{-1} - R)} \setminus \overline{B_{\nu^{-1}}(x)} \neq \emptyset$, then it has the interior and so does $C^x_{0,0}$. Since $x \in A^c$, we have $C^x_{0,0} \neq \emptyset$ and let $\tilde{z} \in C^x_{0,0}$. Since $C_{0,0}$ is convex (Lemma 4.10), we have $Co\left(\{\tilde{z}\} \cup \overline{B(z, \nu^{-1} - R)}\right) \subset C_{0,0}$. Hence, even if $\overline{B(z, \nu^{-1} - R)} \subset \overline{B_{\nu^{-1}}(x)}$, the set $Co\left(\{\tilde{z}\} \cup \overline{B(z, \nu^{-1} - R)}\right) \setminus \overline{B_{\nu^{-1}}(x)}$ has the interior and so does $C^x_{0,0}$.

Therefore, for small $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\hat{z} \in C_{\delta_1,\delta_2}$ and $x \in \overline{B_{\nu^{-1}}(\hat{z})}^c$. $\square$

Lemma 4.12. *Assume* (A1). *Then*

$$\overline{\lim_{t\to\infty}} E_t \subset A.$$

Proof. We give appropriate strategies of Carol as in the proof of Lemma 3.1. For $x \in A^c$, there exist $\delta_1, \delta_2 > 0$ and $z_0 \in \mathbb{R}^2$ such that $R < \nu^{-1} - \delta_2$, $z_0 \in C_{\delta_1,\delta_2}$ and $x \notin \overline{B(z_0, \nu^{-1} - \delta_2)}$ (Lemma 4.11). We define the sequence $\{z_n\}$ by

$$z_n := z_0 + \min\left\{\frac{\delta_2}{2} n\nu^2\epsilon^2, |z - z_0|\right\} \frac{z - z_0}{|z - z_0|},$$

where z is a point taken in the assumption (A1). Carol's strategy is to take a push by moving circle strategy with this $\{z_n\}$.

If she does so, then $z_n \in C_{\delta_1,\delta_2}$ for all n because $z_0, z \in C_{\delta_1,\delta_2}$ and C_{δ_1,δ_2} is convex (Lemma 4.10). By 1 in Lemma 4.7 we see that if Paul quits the game at round i, the game position x_i is not in O_{δ_1}. Thus the stopping cost is at most $-\delta_1$ regardless of ϵ. If Paul does not quit the game, the last position x_N of the game is not in $\overline{B(z, \nu^{-1} - \delta_2)}$ for sufficiently large t. Thus the terminal cost is at most $R - \nu^{-1} + \delta_2 < 0$ for large t regardless of ϵ. Therefore we conclude that for $x \in A^c$, there exists $\tau > 0$ such that $u(x,t) < 0$ for $t > \tau$. □

The first example of O_- is the following:

$$O_- := B_{R'}((0,0)) \setminus \{(p,q) \mid q \geq |p|\},$$

where $R' < \nu^{-1}$ (Figure 13). Notice that
$A^{int} = B_{R'}((0,0)) \setminus \left\{(p,q) \mid q \geq \sqrt{\nu^{-2} - p^2} + R' - \sqrt{\nu^{-2} - R'^2}\right\}$ for this obstacle, where we denote the interior of A by A^{int}.

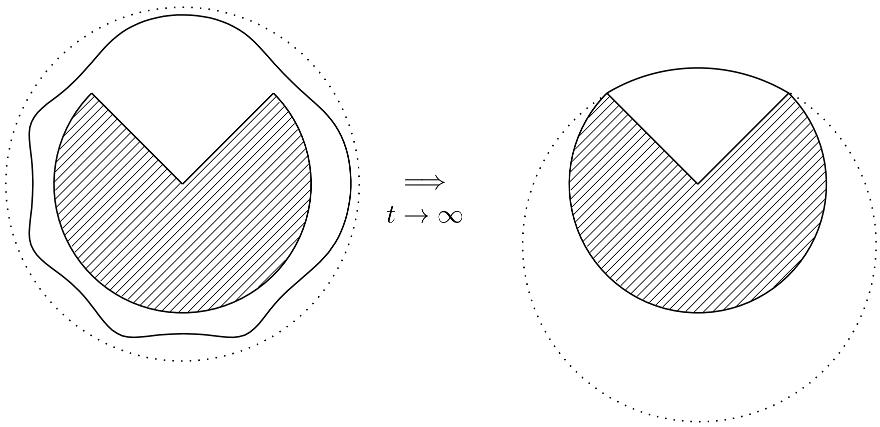


Figure 13: Small Pac-Man

Proposition 4.13 (Small Pac-Man)**.** *Assume* (A1)*. Then*

$$A^{int} \subset \varliminf_{t\to\infty} D_t \subset \varlimsup_{t\to\infty} D_t \subset A$$

and

$$A^{int} \subset \varliminf_{t\to\infty} E_t \subset \varlimsup_{t\to\infty} E_t \subset A.$$

Proof. By Lemma 4.12, it suffices to show $A^{int} \subset \varliminf_{t\to\infty} D_t$. As in the proof of Theorem 3.2, we do case analysis for the initial game position $x \in A^{int}$ and give an appropriate strategy of Paul.

1) $\boldsymbol{x \in O_-}$**.** In this case, it suffices for Paul to quit the game at the first round and gain the stopping cost $\Psi_-(x) > 0$ as in the proof of Theorem 3.2.

2) $\boldsymbol{x \in A^{int} \setminus O_-}$**.** We set $O^{\delta_1} := \{x \in \mathbb{R}^2 \mid \Psi_-(x) > \delta_1\}$. We see that for $x \in A^{int} \setminus O_-$, there exist $\delta_1, \delta_2 > 0$ and $z_0 \in \{(0,q) \mid q \leq x\cdot(0,1)\}$ such that $x \in \overline{B(z_0, \nu^{-1} + \delta_2)}$ and $O^{\delta_1} \cap \partial B(z_0, \nu^{-1} + \delta_2) \cap \{(p,q) \mid q > 0\} \neq \emptyset$ (Figure 14). Define the sequence $\{z_n\}$ so that

$$z_{n+1} = z_n + \left(0, -\frac{\nu^2\delta_2}{2(1+\nu\delta_2)}\epsilon^2\right).$$

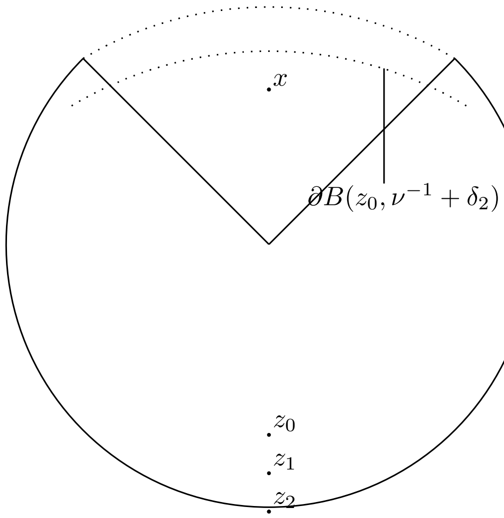


Figure 14: A strategy of Paul

Paul's strategy is to keep taking a push by moving circle strategy with this $\{z_n\}$ until he reaches O^{δ_1}, where he quits the game.

By doing this strategy, Paul actually gains positive game cost. Set

$$\Omega_n = B_{\delta_1}(\{(p,q) \mid q \geq |p|\}) \cap B(z_n, \nu^{-1} + \delta_2)$$

for $n = 0, 1, \cdots$. By 2 in Lemma 4.7 we see that if $x_n \in \Omega_n$, then $x_{n+1} \in \Omega_{n+1}$ or Paul quits the game. If $n\epsilon^2$ is sufficiently large, then $\Omega_n = \emptyset$. That means Paul definitely quits the game in finite time and the stopping cost is at least

$$\inf_{y \in O^{\delta_1}} \Psi_-(y) = \delta_1 > 0$$

regardless of ϵ. Hence, together with the case 1), it turns out that for $x \in A^{int}$, there exists $\tau > 0$ such that $u(x,t) > 0$ for $t > \tau$. $\square$

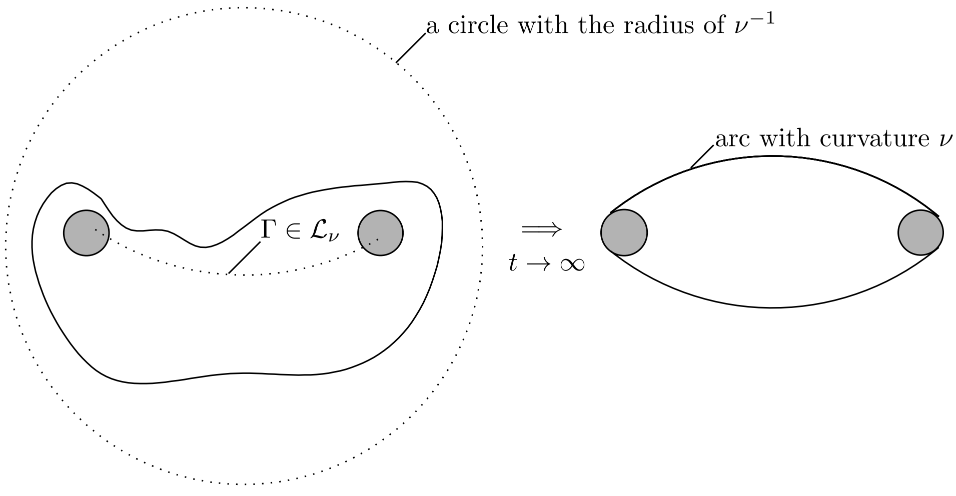


Figure 15: Two small balls

We next consider (4.1) with $O_- = B_r((d,0)) \cup B_r((-d,0))$. In this problem we further assume

(A2) There exists a function $f : [a,b] \to \mathbb{R}$ such that $\Gamma := \{(p,q) \mid q = f(p), a \leq p \leq b\} \in \mathcal{L}_\nu$, $(a, f(a)) \in B_r((-d,0))$, $(b, f(b)) \in B_r((d,0))$ and $\Gamma \subset D_0$.

Proposition 4.14 (Two small balls). *Assume* (A1) *and* (A2). *Then*

$$A^{int} \subset \varliminf_{t\to\infty} D_t \subset \varlimsup_{t\to\infty} D_t \subset A$$

and

$$A^{int} \subset \varliminf_{t\to\infty} E_t \subset \varlimsup_{t\to\infty} E_t \subset A.$$

Proof. We take $\delta > 0$ small enough to satisfy $B_{2\delta}(\Gamma) \subset D_0$ and $B_{2\delta}((a, f(a))) \cup B_{2\delta}((b, f(b))) \subset O_-$.

1) $\boldsymbol{x \in O_-}$**.** The proof for this case is the same as before.

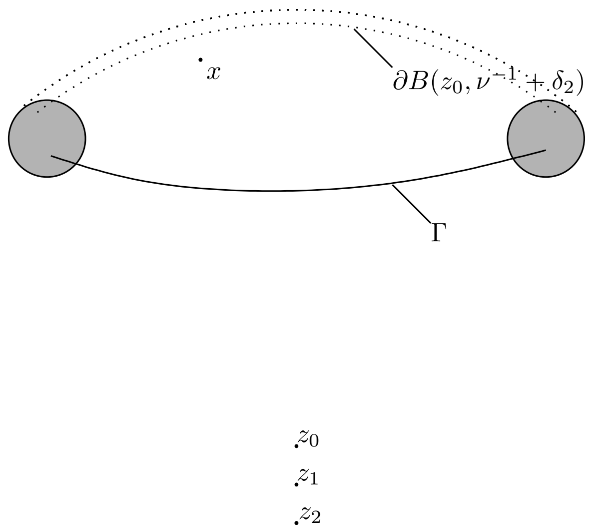


Figure 16: A strategy of Paul

2) $\boldsymbol{x \in B_\delta(\Gamma) \setminus O_-}$**.** Paul keeps taking a Γ tube strategy until he reaches $B_\delta((a, f(a))) \cup B_\delta((b, f(b)))$. Once he reaches $B_\delta((a, f(a))) \cup B_\delta((b, f(b)))$, he quits the game.

By his doing this strategy, the game trajectory $\{x_n\}$ is restricted to $B_\delta(\Gamma)$ as shown in Lemma 4.9. Thus, whether he quits the game or not, he gains at least $\delta > 0$.

3) $\boldsymbol{x \in A^{int} \setminus (B_\delta(\Gamma) \cup O_-)}$**.** We extend f as follows:

$$\tilde{f}(p) := \begin{cases} \max\{f(p), \sqrt{(r-\delta_1)^2 - (p+d)^2}\}, & -d - r + \delta_1 \le p \le -d + r - \delta_1 \\ \max\{f(p), \sqrt{(r-\delta_1)^2 - (p-d)^2}\}, & d - r + \delta_1 \le p \le d + r - \delta_1 \\ f(p), & -d + r - \delta_1 < p < d - r + \delta_1. \end{cases}$$

Without loss of generality we can assume $x \in \{(p,q) \mid q \ge \tilde{f}(p), -d - r + \delta_1 \le p \le d + r - \delta_1\}$. We take $\delta_1, \delta_2 > 0$ and $\{z_n\}$ as in the case 2) in the proof of Proposition 4.13. See also Figure 16.

Paul's strategy is to keep taking a push by moving circle strategy with this $\{z_n\}$ until he reaches $O^{\delta_1} \cup B_\delta(\Gamma)$. Once he reaches O^{δ_1}, he quits the game. Once he reaches $B_\delta(\Gamma)$, he takes the same strategy as in the case 2).

By adopting this strategy, Paul actually gains positive game cost. Set

$$\Omega_n = \{(p,q) \mid q \ge \tilde{f}(p), -d - r + \delta_1 \le p \le d + r - \delta_1\} \cap B(z_n, \nu^{-1} + \delta_2) \setminus B_\delta(\Gamma)$$

for $n = 0, 1, \cdots$. By 2 in Lemma 4.7 we see that if $x_n \in \Omega_n$, then $x_{n+1} \in \Omega_{n+1}$ or Paul reaches $O^{\delta_1} \cup B_\delta(\Gamma)$ (Figure 16). If $n\epsilon^2$ is sufficiently large, then $\Omega_n = \emptyset$. That means Paul definitely reaches $O^{\delta_1} \cup B_\delta(\Gamma)$ in finite time. Since the stopping cost is at least $\delta_1 > 0$ and the terminal cost is at least $\delta > 0$, it turns out that for $x \in A^{int}$, there exists $\tau > 0$ such that $u(x,t) > 0$ for $t > \tau$. □

Remark 4.15. We expect that the same conclusion holds for more general O_-. As an analogue of Theorem 3.2, we define the graph $G = (V, E)$ as follows:

$$V := \{O \subset \mathbb{R}^2 \mid O \text{ is a connected component of } O_-\},$$

$$E := \{\langle O, P\rangle \mid \text{There exist } x \in O,\ y \in P \text{ and } \Gamma_{x,y} \in \mathscr{L}_\nu \text{ such that } \Gamma_{x,y} \subset D_0\},$$

where we denote a curve $\Gamma = \gamma([0,1]) \in \mathscr{L}_\nu$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$ by $\Gamma_{x,y}$. It seems that if O_- satisfies (A1) and the graph G is connected, then the same conclusion holds.

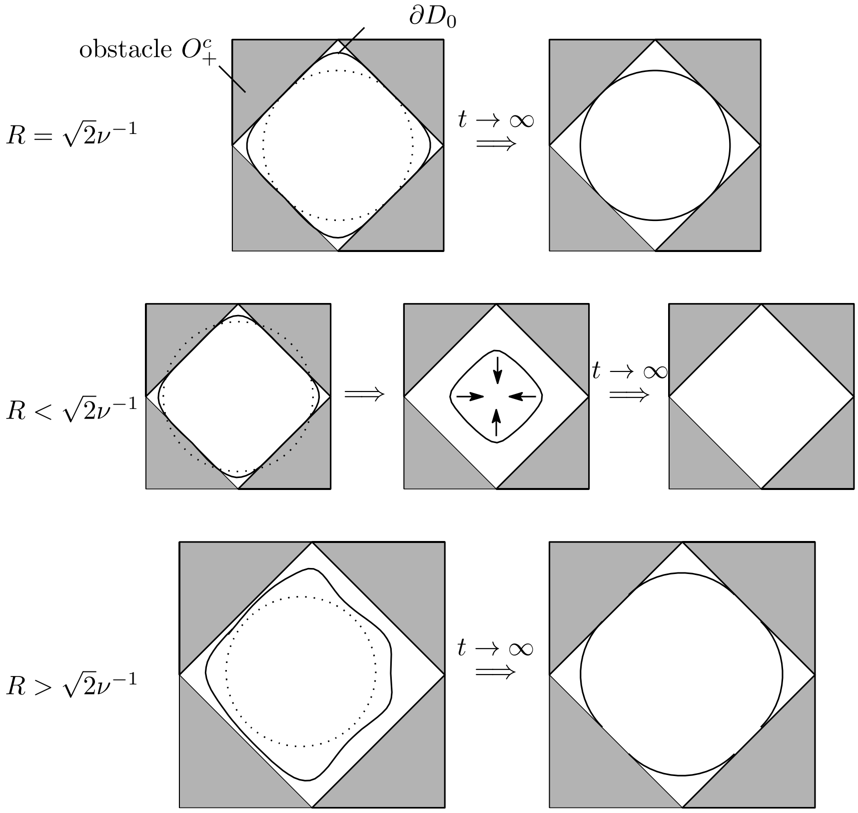


Figure 17: Square boxes

Finally we give an example of computation of the asymptotic shape of solutions to (4.2). We can deal with the problem that remains in [22, Section 6], where $O_+ = \mathbb{R}^2 \setminus \{(p,q) \in \mathbb{R}^2 \mid |p| + |q| \le R\}$ with $R = \sqrt{2}\nu^{-1}$. We also give a game theoritic proof for the case $R \neq \sqrt{2}\nu^{-1}$, which is considered in a different way in [22, Section 6].

We set

$$A := \bigcup \left\{ B_{\nu^{-1}}(z) \mid B_{\nu^{-1}}(z) \subset O_+, \ z \in \mathbb{R}^2 \right\}.$$

We notice that $A = B_{\nu^{-1}}((0,0))$ if $R = \sqrt{2}\nu^{-1}$ and $A = \emptyset$ if $R < \sqrt{2}\nu^{-1}$. Figure 17 shows the result of the asymptotic shapes. In Figure 17, dotted circles are circles with the radius of ν^{-1}.

Proposition 4.16. *Assume either of the following:*

1. $R = \sqrt{2}\nu^{-1}$ *and* $B_{\nu^{-1}}((0,0)) \subset D_0$,
2. $R < \sqrt{2}\nu^{-1}$,
3. $R > \sqrt{2}\nu^{-1}$ *and* $B(\hat{z}, \nu^{-1} + \delta) \subset D_0$ *for some* $\hat{z} \in \mathbb{R}^2$ *and* $\delta > 0$.

Then

$$A \subset \varliminf_{t\to\infty} D_t \subset \varlimsup_{t\to\infty} D_t \subset \overline{A}$$

and

$$A \subset \varliminf_{t\to\infty} E_t \subset \varlimsup_{t\to\infty} E_t \subset \overline{A}.$$

Proof. It suffices to take u_0 as (3.2). Similarly it suffices to let

$$\Psi_+(x) = \begin{cases} dist(x, \partial O_+), & x \in O_+ \\ \max\{a, -dist(x, \partial O_+)\}, & x \in O_+^c. \end{cases}$$

1. **1)** $\boldsymbol{x \in B_{\nu^{-1}}((0,0))}$**.** Paul's strategy is to keep taking a (0,0) concentric strategy. We denote by $V^\epsilon(x,t)$ the total cost when Paul takes this strategy with the game variables (x,t,ϵ) and Carol does not quit the game on the way. Notice that $V^\epsilon(x,t)$ does not depend on Carol's choices $\{b_n\}$ because u_0 is radially symmetric. Since $u_0(x_n)$ is monotonically decreasing with respect to n and $u_0(x_n) \le \Psi_+(x_n)$, Carol's optimal strategy for the strategy of Paul is not to quit the game on the way. Thus the inequality $u^\epsilon(x,t) \ge V^\epsilon(x,t)$ is satisfied. Fix $x \in B_{\nu^{-1}}((0,0))$ and $t > 0$. By Lemma 4.4 we have $V^\epsilon(x,t) > 0$ for sufficiently small $\epsilon > 0$. Lemma 4.3 2 implies that for any subsequence $\{\epsilon_n\}$ satisfying $\epsilon_n = 2^{-n}\epsilon_0$, $V^{\epsilon_n}(x,t)$ is monotonically increasing with respect to n. Therefore we obtain $u(x,t) > 0$.

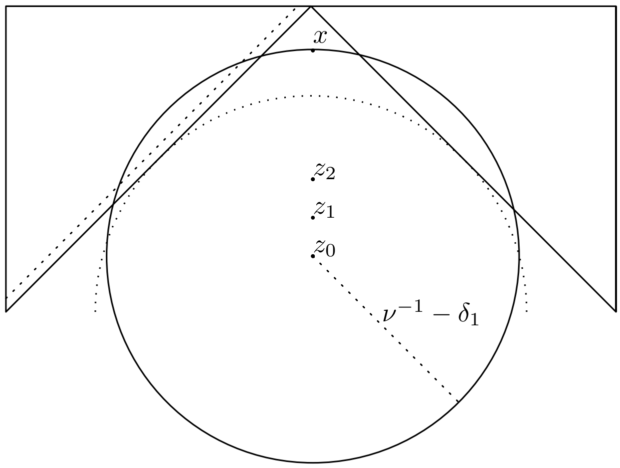


Figure 18: Carol's strategy

   2) $\boldsymbol{x \in \overline{O_+}^c}$**.** Carol's strategy is to quit the game at the first round.

   3) $\boldsymbol{x = (p,q) \in \overline{A}^c \setminus \overline{O_+}^c}$**.** We may assume $q \ge |p|$. We can take $z_0 \in \mathbb{R}^2$, $\delta_1 > 0$ and $\delta_2 > 0$ so that $|x - z_0| \ge \nu^{-1} - \delta_1$, $z_0 \in \{(0,y) \mid y > 0\}$ and $\partial B(z_0, \nu^{-1} - \delta_1) \cap \overline{B_{\delta_2}(O_+)}^c \ne \emptyset$. Carol's strategy is to take a push by moving circle strategy with $\{z_n\} \subset \{(0,q) \mid q > 0\}$ satisfying $z_{n+1} = z_n + (0, \frac{\delta_1}{2}\nu^2\epsilon^2)$ for all n until Paul reaches $\overline{B_{\delta_2}(O_+)}^c$. Once he reaches there, she quits the game. See Figure 18.

   By doing above strategy, Carol actually pays nagative game cost. To show it, set

$$\Omega_n = \{(p,q) \mid q \ge |p|\} \cap \left(\overline{B_{\delta_2}(O_+)} \setminus B(z_n, \nu^{-1} - \delta_1)\right)$$

   for $n = 0, 1, \cdots$. We see that if $x_n \in \Omega_n$, then $x_{n+1} \in \Omega_{n+1}$ or Carol quits the game at round $n+1$. If $n\epsilon^2$ is sufficiently large, then $\Omega_n = \emptyset$. That means Carol definitely quits the game in finite time and the stopping cost is at most

$$\sup_{y \in \overline{B_{\delta_2}(O_+)}} \Psi_+(y) = -\delta_2 < 0$$

   regardless of ϵ. Therefore we obtain $u(x,t) < 0$ for $x \in \overline{B_{\nu^{-1}}((0,0))}^c$ and sufficiently large $t > 0$.

2. We take $\delta_1 > 0$ to satisfy $B((0,0), \nu^{-1} - 2\delta_1) \cap \overline{O_+}^c \ne \emptyset$.

   1) $\boldsymbol{x \in \overline{O_+}^c}$**.** Carol's strategy is to quit the game at the first round.

   2) $\boldsymbol{x \in \overline{O_+} \setminus B((0,0), \nu^{-1} - \delta_1)}$**.** Carol's strategy is similar to that in the case 3) in 1.

   3) $\boldsymbol{x \in \overline{O_+} \cap B((0,0), \nu^{-1} - \delta_1)}$**.** Carol keeps taking a $(0,0)$ concentric strategy until Paul is forced to reach $B((0,0), \nu^{-1} - \delta_1)^c \cap B_{\delta_1}(O_+)^c$. By Lemma 4.5 it takes at most finite time for Paul to reach there. Once Paul reaches $B_{\delta_1}(O_+)^c$, Carol quits the game. Once Paul reaches $\overline{O_+} \setminus B((0,0), \nu^{-1} - \delta_1)$, Carol's strategy is as in the case 2).

   Therefore, for $x \in \mathbb{R}^2$, we obtain $u(x,t) < 0$ for sufficiently large $t > 0$.

3. **1)** $\boldsymbol{x \in \overline{O_+}^c}$**.** Carol's strategy is to quit the game at the first round.

   2) $\boldsymbol{x \in \overline{O_+} \setminus A}$**.** Carol's strategy is similar to that in the case 3) in 1.

3) $\boldsymbol{x \in A}$. We set $O^{\delta_1} := \{x \in \mathbb{R}^2 \mid \Psi_+(x) > \delta_1\}$ and $C_{\delta_1,\delta_2} := \{z \in \mathbb{R}^2 \mid B(z, \nu^{-1} + \delta_2) \subset O^{\delta_1}\}$ for $\delta_1, \delta_2 > 0$. For $x \in A$, there exist $\delta_1, \delta_2 > 0$ such that

$$x \in \bigcup \{B(z, \nu^{-1} + \delta_2) \mid z \in C_{\delta_1,\delta_2}\}$$

and $C_{0,\delta} \subset C_{\delta_1,\delta_2}$. Let $z_0 \in \mathbb{R}^2$ be a point satisfying $x \in B(z_0, \nu^{-1} + \delta_2)$ and $z_0 \in C_{\delta_1,\delta_2}$. We define the sequence $\{z_n\}$ by

$$z_n := z_0 + \min\left\{\frac{\delta_2}{2} n\nu^2\epsilon^2, |\hat{z} - z_0|\right\} \frac{\hat{z} - z_0}{|\hat{z} - z_0|}.$$

Since C_{δ_1,δ_2} is now convex, we have $l_{z_0,\hat{z}} \subset C_{\delta_1,\delta_2}$.

Paul's strategy is to take a push by moving circle strategy with this $\{z_n\}$. Indeed if Carol quits the game on the way, the stopping cost is at least $\delta_1 > 0$ because the game trajectory $\{x_n\}$ is contained in O^{δ_1}. If Carol does not quit the game, the terminal cost is at least $\delta - \delta_1$, which is positive. That is because $x_N \in B(\hat{z}, \nu^{-1} + \delta_1) \subset B(\hat{z}, \nu^{-1} + \delta) \subset D_0$ for any last position x_N of the game.

□

A Game interpretation and convergence of value functions

In this appendix we give a game whose value functions converge to the viscosity solution to (2.1). We introduce the rule of the game corresponding to (2.1) with $\nu \geq 0$ and $d = 2$ and give the proof of the convergence result for the case. We also remark on the other $\nu \in \mathbb{R}$ and d.

The game is almost the same as explained in Section 1. We define the total number N of rounds by $N = \lceil t\epsilon^{-2} \rceil$, where $\lceil r \rceil$ stands for the minimum integer that is no less than r. The actions of both players in each round i $(i = 1, 2, \cdots, N)$ are modified as follows:

1. Paul decides whether to quit the game.
2. Carol decides whether to quit the game.
3. Paul chooses $v_i, w_i \in S^1$. (S^1 is the set of unit vectors in $\mathbb{R}^2$.)
4. Carol chooses $b_i \in \{\pm 1\}$ after Paul's choice.
5. Determine the next states as follows.

$$x_i = x_{i-1} + \sqrt{2}\epsilon b_i v_i + \nu\epsilon^2 w_i. \tag{A.1}$$

The total cost is also modified as follows. If Paul quits the game at round i, the total cost is given by $\Psi_-(x_i)$. If Carol quits the game at round i, it is given by $\Psi_+(x_i)$. If both players go throughout N rounds of the game, it is given by $u_0(x_N) + \sum_{i=0}^{N-1} \epsilon^2 f(x_i)$. The value function $u^\epsilon(x, t)$ is defined inductively based on the following *Dynamic Programming Principle* and the initial condition:

$$\begin{aligned} &u^\epsilon(x,t) = \\ &\max\{\Psi_-(x), \min\{\Psi_+(x), \sup_{v,w\in S^1} \min_{b=\pm 1} [u^\epsilon(x + \sqrt{2}\epsilon b v + \nu w\epsilon^2, t - \epsilon^2) + \epsilon^2 f(x)]\}\} \end{aligned} \tag{A.2}$$

for $t > 0$.

$$u^\epsilon(x,t) = u_0(x) \tag{A.3}$$

for $t \leq 0$.

These value functions mean the total cost optimized by both players.

Remark A.1. As explained in Section 1, we can generalize our game to the case $d \geq 3$. In the game corresponding to (2.1) with $\nu \geq 0$, Paul chooses a unit vector w_i and $d - 1$ orthogonal unit vectors $v_i^j (j = 1, 2, \cdots d - 1)$. Carol chooses $d - 1$ values $b_i^j \in \{\pm 1\} (j = 1, 2, \cdots d - 1)$. The state equation is $x_i = x_{i-1} + \nu w_i \epsilon^2 + \sqrt{2}\epsilon \sum_{j=1}^{d-1} b_i^j v_i^j$ instead of (A.1).

Remark A.2. The Dynamic Programming Principle corresponding to (2.1) with $\nu < 0$ is given by

$$u^\epsilon(x,t) = \max\{\Psi_-(x), \min\{\Psi_+(x), \sup_{v\in S^1} \inf_{\substack{w\in S^1\\ b=\pm 1}} [u^\epsilon(x+\sqrt{2}\epsilon b v + \nu w\epsilon^2, t-\epsilon^2) + \epsilon^2 f(x)]\}\}.$$

Namely, not Paul but Carol has the right to choose $w_i \in S^1$.

For these value functions, the same type of result as Proposition 1.1 holds.

Proposition A.3. *the functions* $\overline{u}$ *and* $\underline{u}$ *are respectively viscosity sub- and supersolution of* (2.1). *Moreover* $\overline{u}(x,0) = \underline{u}(x,0) = u_0(x)$ *for* $x \in \mathbb{R}^d$.

Remark A.4. As explained before, (2.1) with $f = 0$ is a level set equation. By choosing Ψ_+ so that $\Psi_+ > \|u_0\|_\infty$ for $O_+ = \mathbb{R}^d$, we can ignore Carol's stopping cost Ψ_+ when we consider obstacle problems that have an obstacle on one side such as (1.1). Similarly, by choosing Ψ_- so that $\Psi_- < -\|u_0\|_\infty$ for $O_- = \emptyset$, we can ignore Paul's stopping cost Ψ_-.

We especially show the proof of Proposition A.3 with $d = 2$ and $\nu \geq 0$ because the other case is similar. Our proof is based directly on the game as in [37], whereas those in [28, 38] are based on the properties of the operator whose fixed point is the solution of the Dynamic Programming Principle. Also since the proof in [37] is local argument, roughly speaking, all we have to do is to do the local argument in $\{(x,t) \mid \Psi_-(x) < \overline{u}(x,t)\}$ or in $\{(x,t) \mid \Psi_+(x) > \underline{u}(x,t)\}$. However we need to care about the point that $\{(x,t) \mid \Psi_-(x) < \overline{u}(x,t)\}$ or $\{(x,t) \mid \Psi_+(x) > \underline{u}(x,t)\}$ may not be open.

The proof consists of three steps. We show that the limits of the value functions satisfy the conditions (a) and (c) in Definition 2.1 in the first two propositions, (Proposition A.5 and A.7) and they satisfy the initial condition (b) in the last one. (Proposition A.11) We mention that the initial data u_0 is assumed to be just continuous, not to be Lipschitz continuous as in [28] or bounded uniformly continuous as in [38]. Regarding the last proposition, the idea of the proof is similar to that of [20, Proposition 3.1] though the situation is different.

To visualize choices of players of the game, we give another description of the level-set mean curvature flow operator F:

$$F(Du, D^2u) = -\left\langle D^2u\frac{D^\perp u}{|Du|}, \frac{D^\perp u}{|Du|}\right\rangle$$

for $Du \neq 0$. Here we denote by $D^\perp u$ a vector field satisfying $Du \cdot D^\perp u = 0$ and $|Du| = |D^\perp u|$ in $\mathbb{R}^2$.

Proposition A.5. *The function* $\underline{u}$ *is a viscosity supersolution of* (2.1) *in* $\mathbb{R}^2 \times (0,\infty)$.

Proof. As for Definition 2.1-2(a), we directly have $\Psi_-(x) \leq u^\epsilon(x,t) \leq \Psi_+(x)$ by the Dynamic Programing Principle (A.2). Thus we obtain $\Psi_-(x) \leq \underline{u}(x,t) \leq \Psi_+(x)$ since Ψ_+ and Ψ_- are continuous. To prove the viscosity inequality, we argue by contradiction. For a smooth function $\phi : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$, a positive constant $\theta_0 > 0$ and $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, we consider the following condition (C):

$$\partial_t\phi(x,t) - \nu|D\phi(x,t)| - \left\langle D^2\phi(x,t)\frac{D^\perp\phi(x,t)}{|D\phi(x,t)|}, \frac{D^\perp\phi(x,t)}{|D\phi(x,t)|}\right\rangle - f_*(x) \leq -\theta_0 < 0 \tag{A.4}$$

if $D\phi(x,t) \neq 0$, and

$$\partial_t\phi(x,t) - \nu|D\phi(x,t)| - \inf_{|\zeta|=1} \langle D^2\phi(x,t)\zeta, \zeta\rangle - f_*(x) \leq -\theta_0 < 0 \tag{A.5}$$

if $D\phi(x,t) = 0$.

We assume that there exist a smooth function ϕ and $(\hat{x},\hat{t})$ such that $(\hat{x},\hat{t})$ is a strict local minimum of $\underline{u} - \phi$, $\underline{u} < \Psi_+$ at $(\hat{x},\hat{t})$ and the condition (C) is satisfied at $(\hat{x},\hat{t})$ with ϕ and some $\theta_0 > 0$. Then we can take a δ neighborhood of $(\hat{x},\hat{t})$ where $\underline{u} - \phi$ attains its unique minimum at $(\hat{x},\hat{t})$ and the condition (C) holds, retaking smaller $\theta_0 > 0$ if necessary. For technical reasons, we take such δ neighborhood as $N_\delta((\hat{x},\hat{t})) := \{(x,t) \in \mathbb{R}^2 \times [0,\infty) \mid |x-\hat{x}| + |t-\hat{t}| < \delta\}$ and $\delta > 0$ small enough to satisfy $\delta \leq \frac{a}{2\max\{L,M\}}$, where $a := \Psi_+(\hat{x}) - \underline{u}(\hat{x},\hat{t})$, L is the Lipschitz constant of Ψ_+ and $M = \sup_{y\in B_1(\hat{x})}|f(y)|$.

From the definition of $\underline{u}$, there are sequence $\{\epsilon_n\}$, $\{x^0_{\epsilon_n}\}$, and $\{t^0_{\epsilon_n}\}$ satisfying

$$\epsilon_n \searrow 0, \quad (x^0_{\epsilon_n}, t^0_{\epsilon_n}) \to (\hat{x},\hat{t}), \quad u^{\epsilon_n}(x^0_{\epsilon_n}, t^0_{\epsilon_n}) \to \underline{u}(\hat{x},\hat{t}).$$

We may substitute ϵ for ϵ_n hereafter. We take ϵ small enough to satisfy $(x_\epsilon^0, t_\epsilon^0) \in N_{\delta/2}((\hat{x}, \hat{t}))$ and $\Psi_+(x_\epsilon^0) - u^\epsilon(x_\epsilon^0, t_\epsilon^0) \geq a/2$. For any ϵ, we construct the sequence $\{X_k\}$ satisfying

$$X_0 = (x_\epsilon^0, t_\epsilon^0),$$

$$X_{k+1} = X_k + (\sqrt{2}\epsilon b_k \eta_k^\perp + \nu \eta_k \epsilon^2, -\epsilon^2), \tag{A.6}$$

where $\eta_k = \frac{D\phi(X_k)}{|D\phi(X_k)|}$ if $D\phi(X_k) \neq 0$, and is an arbitrary unit vector if $D\phi(X_k) = 0$. We will determine b_k later. Let x_k and t_k denote the spatial and time component of X_k respectively hereafter. Let k_ϵ be the maximal k satisfying

$$X_j \in N_{\delta/2}((\hat{x}, \hat{t})) \text{ for any } j = 0, 1, \dots, k-1.$$

Indeed such k_ϵ exists because of the definition of the sequence $\{X_k\}$. We prove by induction that $\Psi_+(x_k) > u^\epsilon(X_k)$ and

$$u^\epsilon(X_0) - u^\epsilon(X_k) \geq \epsilon^2 \sum_{m=0}^{k-1} f(x_m) \tag{A.7}$$

for all $k < k_\epsilon$. These inequalities hold for $k = 0$. We assume that these hold for some $k < k_\epsilon$. Then the Dynamic Programming Principle (A.2) and $\Psi_+(x_k) > u^\epsilon(X_k)$ imply

$$u^\epsilon(X_k) = \max\{\Psi_-(x_k), \sup_{v,w \in S^1} \min_{b=\pm 1} u^\epsilon(x_k + \sqrt{2}\epsilon b v + \nu w \epsilon^2, t_k - \epsilon^2) + \epsilon^2 f(x_k)\}.$$

Thus we have

$$u^\epsilon(X_k) \geq \min_{b=\pm 1} u^\epsilon(x_k + \sqrt{2}\epsilon b \eta_0^\perp + \nu \eta_0 \epsilon^2, t_k - \epsilon^2) + \epsilon^2 f(x_k). \tag{A.8}$$

We determine b_k in $(A.6)$ as a minimizer b in (A.8). We then get

$$u^\epsilon(X_k) - u^\epsilon(X_{k+1}) \geq \epsilon^2 f(x_k).$$

Adding (A.7), we have

$$u^\epsilon(X_0) - u^\epsilon(X_{k+1}) \geq \epsilon^2 \sum_{m=0}^{k} f(x_m)$$

and consequently

$$u^\epsilon(X_0) + M(k+1)\epsilon^2 \geq u^\epsilon(X_{k+1}). \tag{A.9}$$

If $k+1 < k_\epsilon$, we see from the definition of k_ϵ that $X_{k+1} \in N_{\delta/2}((\hat{x}, \hat{t}))$ and thus

$$|x_{k+1} - x_0| + |t_{k+1} - t_0| \leq |x_{k+1} - \hat{x}| + |x_0 - \hat{x}| + |t_{k+1} - \hat{t}| + |t_0 - \hat{t}| \leq \delta. \tag{A.10}$$

From the Lipschitz continuity of Ψ_+, we have

$$|\Psi_+(x_{k+1}) - \Psi_+(x_0)| \leq L|x_{k+1} - x_0| \leq L(\delta - |t_{k+1} - t_0|) = L(\delta - (k+1)\epsilon^2). \tag{A.11}$$

Combining (A.9) and (A.11), we obtain

$$\begin{aligned}
\Psi_+(x_{k+1}) - u^\epsilon(X_{k+1}) &\geq \Psi_+(x_0) - L(\delta - (k+1)\epsilon^2) - u^\epsilon(X_0) - M(k+1)\epsilon^2 \\
&\geq \Psi_+(x_0) - \max\{L, M\}(\delta - (k+1)\epsilon^2) - u^\epsilon(X_0) - \max\{L, M\}(k+1)\epsilon^2 \\
&\geq \Psi_+(x_0) - u^\epsilon(X_0) - \max\{L, M\}\delta \\
&> \frac{a}{2} - \frac{a}{2} = 0
\end{aligned}$$

and conclude the induction.

Next we take the continuous path that affinely interpolates among $\{X_k\}$, i.e., $X(s) = X_k + \left(s\epsilon^{-2} - k\right)(X_{k+1} - X_k)$ for $k\epsilon^2 \leq s \leq (k+1)\epsilon^2$, and we write $X(s) = (x(s), t_\epsilon^0 - s)$. Using Taylor's theorem for $\phi(X(t))$ at $t = k\epsilon^2$, we get

$$\phi(X_{k+1}) - \phi(X_k) = \epsilon^2\{-\partial_t \phi(X_k) + \nu|D\phi(X_k)| + \langle D^2\phi(X_k)\eta_k^\perp, \eta_k^\perp\rangle\} + \Psi_k(\epsilon), \tag{A.12}$$

where $\Psi_k(\epsilon) = o(\epsilon^2)$. Moreover, from the assumption (A.4), we have

$$\phi(X_{k+1}) - \phi(X_k) \geq \epsilon^2(\theta_0 - f_*(x_k)) + \Psi_k(\epsilon).$$

This inequality is also obtained in the case $D\phi(X_k) = 0$, using (A.12) and (A.5). Summing up both sides, we have

$$\phi(X_k) - \phi(X_0) \geq k\epsilon^2\theta_0 - \epsilon^2 \sum_{m=0}^{k-1} f_*(x_m) + \sum_{m=0}^{k-1} \Psi_m(\epsilon). \tag{A.13}$$

Provided $k < k_\epsilon$, we have

$$|\Psi_k(\epsilon)| \leq C\epsilon^3,$$

where C depends on ϕ in δ neighborhood around $(\hat{x}, \hat{t})$, and does not depend on k. This estimation is derived from the Taylor expansion (A.12). Hence (A.13) becomes

$$\phi(X_k) - \phi(X_0) \geq k\epsilon^2\theta_0 - \epsilon^2 \sum_{m=0}^{k-1} f_*(x_m) + kC\epsilon^3, \qquad k \leq k_\epsilon.$$

Adding this relation to (A.7), we have

$$\begin{aligned} u^\epsilon(X_0) - \phi(X_0) &\geq u^\epsilon(X_k) - \phi(X_k) + \epsilon^2 \sum_{m=0}^{k-1} (f(x_m) - f_*(x_m)) + k\epsilon^2\theta_0 + kC\epsilon^3 \\ &\geq u^\epsilon(X_k) - \phi(X_k) + k\epsilon^2\theta_0 + kC\epsilon^3. \end{aligned}$$

For sufficiently small ϵ, we get

$$-\frac{k}{2}\epsilon^2\theta_0 \geq u^\epsilon(X_k) - u^\epsilon(X_0) - \phi(X_k) + \phi(X_0), \qquad k \leq k_\epsilon. \tag{A.14}$$

By the definition of k_ϵ, we see that $Y_\epsilon \in \overline{N_{3\delta/4}((\hat{x}, \hat{t})) \setminus N_{\delta/2}((\hat{x}, \hat{t}))}$, where we substitute Y_ϵ for X_{k_ϵ}. So there is a subsequence $\{Y_{\epsilon_n}\}_n$ such that

$$\lim_{n\to\infty} Y_{\epsilon_n} = (x', t'),$$

where $(x', t') \in B_\delta((\hat{x}, \hat{t}))$ and $(x', t') \neq (\hat{x}, \hat{t})$. From (A.14), we have

$$u^{\epsilon_n}(x^0_{\epsilon_n}, t^0_{\epsilon_n}) - \phi(x^0_{\epsilon_n}, t^0_{\epsilon_n}) \geq u^{\epsilon_n}(Y_{\epsilon_n}) - \phi(Y_{\epsilon_n}).$$

Letting n go to ∞, we obtain

$$\underline{u}(\hat{x}, \hat{t}) - \phi(\hat{x}, \hat{t}) \geq \underline{u}(x', t') - \phi(x', t').$$

This is a contradiction since $\underline{u} - \phi$ attains its unique minimum at $(\hat{x}, \hat{t})$. □

The following lemma can be found in [37, Lemma 2.3].

Lemma A.6. *Let ϕ be a C^3 function on a compact subset K of $\mathbb{R}^2$. Let $x \in K$ and $\epsilon \in [0, \infty)$. If $D\phi(x) \neq 0$, there exists a constant C_1 (depending only on the C^2 norm of ϕ) with the following two properties for all unit vectors $v \in \mathbb{R}^2$.*

1. *If $\sqrt{2}|\langle D\phi(x), v\rangle| \geq C_1\epsilon$,*

$$\sqrt{2}\epsilon|\langle D\phi, v\rangle| + \epsilon^2\langle D^2\phi v, v\rangle \geq \epsilon^2 \left\langle D^2\phi \frac{D^\perp\phi}{|D\phi|}, \frac{D^\perp\phi}{|D\phi|} \right\rangle$$

   at x.

2. *If $\sqrt{2}|\langle D\phi(x), v\rangle| \leq C_1\epsilon$, there exists a constant C_2 (depending only on the C^2 norm of ϕ) such that*

$$\sqrt{2}\epsilon|\langle D\phi, v\rangle| + \epsilon^2\langle D^2\phi v, v\rangle \geq \epsilon^2 \left\langle D^2\phi \frac{D^\perp\phi}{|D\phi|}, \frac{D^\perp\phi}{|D\phi|} \right\rangle - \frac{C_2\epsilon^3}{|D\phi|}$$

   at x.

Proposition A.7. *The function* $\overline{u}$ *is a viscosity subsolution of* (2.1) *in* $\mathbb{R}^2 \times (0,\infty)$.

Proof. As in Proposition A.5, we obtain $\Psi_-(x) \le \overline{u}(x,t) \le \Psi_+(x)$ and argue by contradiction to prove the viscosity inequality. We prepare the following conditions:

$$\partial_t\phi(x,t) - \nu|D\phi(x,t)| - \left\langle D^2\phi(x,t)\frac{D^\perp\phi(x,t)}{|D\phi(x,t)|}, \frac{D^\perp\phi(x,t)}{|D\phi(x,t)|}\right\rangle - f^*(x) \ge \theta_0 > 0, \tag{A.15}$$

$$\partial_t\phi(x,t) - \nu|D\phi(x,t)| - \sup_{|\zeta|=1}\left\langle D^2\phi(x,t)\zeta,\zeta\right\rangle - f^*(x) \ge \theta_0 > 0. \tag{A.16}$$

We assume that there exist a smooth function ϕ and $(\hat{x},\hat{t})$ such that $(\hat{x},\hat{t})$ is a strict local maximum of $\overline{u}-\phi$, $\overline{u} > \Psi_-$ at $(\hat{x},\hat{t})$ and, for some $\theta_0 > 0$, (A.15) is satisfied at $(\hat{x},\hat{t})$ provided $D\phi(\hat{x},\hat{t}) \ne 0$, and (A.16) is satisfied at $(\hat{x},\hat{t})$ provided $D\phi(\hat{x},\hat{t}) = 0$. If $D\phi(\hat{x},\hat{t}) \ne 0$, we take a δ neighborhood of $(\hat{x},\hat{t})$ where $\overline{u}-\phi$ attains its unique maximum at $(\hat{x},\hat{t})$, $|D\phi| > \theta_1$ for some $\theta_1 > 0$, and (A.15) holds, retaking smaller $\theta_0 > 0$ if necessary. If $D\phi(\hat{x},\hat{t}) = 0$, we take a δ neighborhood of $(\hat{x},\hat{t})$ where $\overline{u}-\phi$ attains its unique maximum at $(\hat{x},\hat{t})$ and (A.16) holds, retaking smaller $\theta_0 > 0$ if necessary. We take $\delta > 0$ small enough to satisfy $\delta \le \frac{b}{3\max\{L,M\}}$, where $b := \overline{u}(\hat{x},\hat{t}) - \Psi_-(\hat{x})$. From the definition of $\overline{u}$, there are some sequences $\{\epsilon_n\}$, $\{x^0_{\epsilon_n}\}$, and $\{t^0_{\epsilon_n}\}$ satisfying

$$\epsilon_n \searrow 0, \quad (x^0_{\epsilon_n}, t^0_{\epsilon_n}) \to (\hat{x},\hat{t}), \quad u^{\epsilon_n}(x^0_{\epsilon_n}, t^0_{\epsilon_n}) \to \overline{u}(\hat{x},\hat{t}).$$

We may substitute ϵ for ϵ_n hereafter. We take ϵ small enough to satisfy $(x^0_\epsilon, t^0_\epsilon) \in N_{\delta/2}((\hat{x},\hat{t}))$ and $u^\epsilon(x^0_\epsilon, t^0_\epsilon) - \Psi_-(x^0_\epsilon) \ge b/2$.

We construct the sequence $\{X_k\}$ and the functions $\Psi_k : S^1 \times S^1 \to \mathbb{R}$ inductively as follows. We first let

$$X_0 := (x^0_\epsilon, t^0_\epsilon),$$

and

$$\Psi_0(v,w) := \min_{b=\pm1} u^\epsilon(x^0_\epsilon + \sqrt{2}\epsilon bv + \nu\epsilon^2 w, t^0_\epsilon - \epsilon^2) + \epsilon^2 f(x^0_\epsilon).$$

Then let (v_0,w_0) satisfying

$$\Psi_0(v_0,w_0) \ge \sup_{(v,w)\in S^1\times S^1} \Psi_0(v,w) - \epsilon^3,$$

and we determine

$$X_1 = X_0 + (\sqrt{2}\epsilon b_0 v_0 + \nu\epsilon^2 w_0, -\epsilon^2),$$

where we will decide b_0 later. For any $k \in \mathbb{N}$, we similarly define

$$\Psi_k(v,w) := \min_{b=\pm1} u^\epsilon(x_k + \sqrt{2}\epsilon bv + \nu\epsilon^2 w, t_k - \epsilon^2) + \epsilon^2 f(x_k),$$

and

$$X_{k+1} := X_k + (\sqrt{2}\epsilon b_k v_k + \nu\epsilon^2 w_k, -\epsilon^2), \tag{A.17}$$

where $(v_k,w_k) \in S^1 \times S^1$ satisfies

$$\Psi_k(v_k,w_k) \ge \sup_{(v,w)\in S^1\times S^1} \Psi_k(v,w) - \epsilon^3.$$

Define k_ϵ as in the proof of Proposition A.5. We prove by induction that $u^\epsilon(X_k) > \Psi_-(x_k)$ and

$$u^\epsilon(X_0) - u^\epsilon(X_k) \le \epsilon^2 \sum_{m=0}^{k-1}[f(x_m)] + k\epsilon^3 \tag{A.18}$$

for all $k < k_\epsilon$. These inequalities hold for $k = 0$. We assume that these hold for some $k < k_\epsilon$. Then the Dynamic Programming Principle (A.2) and $u^\epsilon(X_k) > \Psi_-(x_k)$ imply

$$u^\epsilon(X_k) = \min\{\Psi_+(x_k), \sup_{v,w\in S^1}\min_{b=\pm1} u^\epsilon(x_k + \sqrt{2}\epsilon bv + \nu w\epsilon^2, t_k - \epsilon^2) + \epsilon^2 f(x_k)\}.$$

Thus we have

$$\begin{aligned} u^\epsilon(X_k) &\le \sup_{v,w\in S^1} \Psi_k(v,w) \\ &\le \Psi_k(v_k,w_k) + \epsilon^3 \\ &= u^\epsilon(x_k + \sqrt{2}\epsilon b_k v_k + \nu w_k\epsilon^2, t_k - \epsilon^2) + \epsilon^2 f(x_k) + \epsilon^3 \end{aligned}$$

and hence

$$u^\epsilon(X_k) - u^\epsilon(X_{k+1}) \le \epsilon^2 f(x_k) + \epsilon^3, \tag{A.19}$$

which means (A.18) holds for $k+1$. From the Lipschitz continuity of Ψ_- and (A.10), we have

$$-\Psi_-(x_{k+1}) \ge -\Psi_-(x_0) - L(\delta - (k+1)\epsilon^2) \tag{A.20}$$

provided $k+1 < k_\epsilon$. Combining (A.19) and (A.20), we obtain

$$\begin{aligned}
&u^\epsilon(X_{k+1}) - \Psi_-(x_{k+1}) \\
&\ge u^\epsilon(X_0) - \Psi_-(x_0) - M(k+1)\epsilon^2 - L(\delta - (k+1)\epsilon^2) - (k+1)\epsilon^3 \\
&\ge u^\epsilon(X_0) - \Psi_-(x_0) - \max\{M, L\}\delta - (k+1)\epsilon^3.
\end{aligned}$$

We notice that $(k+1)\epsilon^3 \le L\delta\epsilon$ from (A.11). Therefore $u^\epsilon(X_{k+1}) > \Psi_-(x_{k+1})$ holds for sufficiently small ϵ and we conclude the induction.

Next we take the continuous path $X(s)$ and use the Taylor's theorem in the same way as in Proposition A.5. Then we have

$$\begin{aligned}
&\phi(X_{k+1}) - \phi(X_k) \\
&= \sqrt{2}\epsilon b_k \langle D\phi(X_k), v_k\rangle + \epsilon^2\{-\partial_t\phi(X_k) + \nu\langle D\phi(X_k), w_k\rangle + \langle D^2\phi(X_k)v_k, v_k\rangle\} + \Phi_k(\epsilon),
\end{aligned}$$

where $\Phi_k(\epsilon) = o(\epsilon^2)$. By taking b_k in (A.17) properly, we get

$$\begin{aligned}
&\phi(X_{k+1}) - \phi(X_k) \\
&\le -\sqrt{2}\epsilon|\langle D\phi(X_k), v_k\rangle| + \epsilon^2\{-\partial_t\phi(X_k) + \nu|D\phi(X_k)| + \langle D^2\phi(X_k)v_k, v_k\rangle\} + \Phi_k(\epsilon).
\end{aligned} \tag{A.21}$$

We first consider the case $D\phi(\hat{x}, \hat{t}) \ne 0$. If $k < k_\epsilon$, we just consider ϕ in $N_\delta((\hat{x}, \hat{t}))$. We now use the assumption (A.15) and Lemma A.6 replacing ϕ by $-\phi$ to get

$$\begin{aligned}
&\phi(X_{k+1}) - \phi(X_k) \\
&\le \epsilon^2\left\{-\partial_t\phi(X_k) + \nu|D\phi(X_k)| + \left\langle D^2\phi(X_k)\frac{D^\perp\phi(X_k)}{|D\phi(X_k)|}, \frac{D^\perp\phi(X_k)}{|D\phi(X_k)|}\right\rangle\right\} \\
&+ \frac{C_2\epsilon^3}{|D\phi(X_k)|} + \Phi_k(\epsilon) \\
&\le \epsilon^2\left(-\frac{\theta_0}{2} - f^*(x_k)\right),
\end{aligned} \tag{A.22}$$

for sufficiently small ϵ. This inequality is also obtained in the case $D\phi(\hat{x}, \hat{t}) = 0$, using (A.21) and the assumption (A.16). From (A.19) and (A.22), we have

$$u^\epsilon(X_0) - u^\epsilon(X_k) - \phi(X_0) + \phi(X_k) \le -\frac{k}{4}\epsilon^2\theta_0, \qquad k \le k_\epsilon. \tag{A.23}$$

By the same argument as in Proposition A.5, we obtain

$$\overline{u}(\hat{x}, \hat{t}) - \phi(\hat{x}, \hat{t}) \le \overline{u}(x', t') - \phi(x', t'),$$

where $(x', t') \ne (\hat{x}, \hat{t})$. This is a contradiction since $\overline{u} - \phi$ attains its unique maximum at $(\hat{x}, \hat{t})$. □

The proof of Proposition A.3 is completed by checking that $\overline{u}$ and $\underline{u}$ satisfy the initial condition. To prove the last proposition, we need additional property of the solution to (4.3) and strategies of the game.

Lemma A.8. *Let $\delta \in (0, \nu^{-1}]$. For sufficiently small $\epsilon > 0$, we have*

$$\frac{r_2 - \delta}{\delta^{-1} - \frac{\nu}{2}} \le t_\epsilon(r_1, r_2),$$

for any r_1 and r_2 satisfying $0 \le r_1 \le \delta \le r_2 \le \nu^{-1}$.

Proof. Let $\delta \le r \le \nu^{-1}$. We take $\epsilon > 0$ small enough to satisfy $T_{\epsilon^2}(r) \ge r$. Concretely we assume $\nu\epsilon^2 \le \delta$. Then the inequality (4.4) implies

$$T_{\epsilon^2}(r) - r \le \left(\delta^{-1} - \nu\right)\epsilon^2 + \frac{\nu^2\epsilon^4}{2\delta}.$$

Hence we obtain

$$t_\epsilon(r_1, r_2) \ge t_\epsilon(\delta, r_2) \ge \frac{r_2 - \delta}{\left(\delta^{-1} - \nu\right)\epsilon^2 + \frac{\nu^2\epsilon^4}{2\delta}}\epsilon^2 \ge \frac{r_2 - \delta}{\delta^{-1} - \nu + \frac{\nu}{2}} = \frac{r_2 - \delta}{\delta^{-1} - \frac{\nu}{2}}.$$

□

Definition A.9 (Reversed concentric strategy)**.** Let $\nu > 0$, $\epsilon > 0$ and $z \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ be the current position of the game. Let $(v, w) \in S^1 \times S^1$ be a choice by Paul in the same round. A choice $b \in \{\pm 1\}$ by Carol is called a *reversed* z *concentric strategy* if

$$\langle bv, x + \nu\epsilon^2 w - z\rangle \le 0.$$

If Carol takes reversed z concentric strategy through the game, we get $|x_n - z| \le P_n$, where P_n satisfies

$$P_{n+1} = \sqrt{(P_n + \nu\epsilon^2)^2 + 2\epsilon^2} \tag{A.24}$$

with $P_0 = |x_0 - z|$. We define $t_\epsilon(a, b)$ in the same way as (4.6), replacing the operator T_h as follows:

$$T_h(R) := \sqrt{(R + \nu h)^2 + 2h}.$$

Lemma A.10. *Let $\delta > 0$. For sufficiently small $\epsilon > 0$, we have*

$$\frac{r_2 - \delta}{\delta^{-1} + \nu + 1} \le t_\epsilon(r_1, r_2)$$

for any r_1 and r_2 satisfying $0 \le r_1 \le \delta \le r_2$.

Proof. The proof is similar to that of Lemma A.8, so is omitted. □

Proposition A.11. *Let u_0 be a continuous function. Then $\overline{u}(x, 0) = \underline{u}(x, 0) = u_0(x)$ for all $x \in \mathbb{R}^2$.*

Proof. Let $x \in \mathbb{R}^2$. For the initial position $y \in \mathbb{R}^2$, the terminal time $s > 0$ and the step size $\epsilon > 0$, we define $V^-(y, s, \epsilon)$ as the minimum total cost when Paul takes a x concentric strategy through the game. Similarly we define $V^+(y, s, \epsilon)$ as the supremum total cost when Carol takes reversed x concentric strategy through the game. It is clear by the property of the value functions that

$$V^-(y, s, \epsilon) \le u^\epsilon(y, s) \le V^+(y, s, \epsilon).$$

It is sufficient to show

$$\varliminf_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} V^-(y, s, \epsilon) \ge u_0(x) \tag{A.25}$$

and

$$\varlimsup_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} V^+(y, s, \epsilon) \le u_0(x). \tag{A.26}$$

We first analyze V^-. We denote by $V^-_{quit}(y, s, \epsilon)$ (resp. $V^-_{end}(y, s, \epsilon)$) the minimum total cost when Paul takes x concentric strategy through the game and Carol quits (resp. does not quit) the game on the way. Then we write

$$V^-(y, s, \epsilon) = \min\{V^-_{end}(y, s, \epsilon), V^-_{quit}(y, s, \epsilon)\}.$$

Furthermore we analyze V^-_{end}. We denote by $V^-_{run}(y, s, \epsilon)$ (resp. $V^-_{ter}(y, s, \epsilon)$) the minimum running cost (resp. terminal cost) in the same situation as $V^-_{end}(y, s, \epsilon)$. Obviously we have

$$V^-_{end}(y, s, \epsilon) \ge V^-_{run}(y, s, \epsilon) + V^-_{ter}(y, s, \epsilon).$$

Since Paul takes a x concentric strategy, he can stay in $B(x, 2\nu^{-1})$ by Lemma 4.2. So the running cost is at most $M := \sup_{z\in B(x,2\nu^{-1})}|f(z)|$, and at least $-M$ per round. Hence we have

$$|V^-_{run}(y, s, \epsilon)| \le \epsilon^2 NM = \epsilon^2 \lceil s\epsilon^{-2} \rceil M \le \epsilon^2 (s\epsilon^{-2} + 1) M = (s + \epsilon^2)M. \tag{A.27}$$

We denote by $Ter(y, s, \epsilon)$ a terminal point x_N in the situation of $V^-_{end}(y, s, \epsilon)$. Since u_0 is continuous, what we have to prove about the terminal cost is

$$\lim_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} Ter(y, s, \epsilon) = x$$

for any choices of Carol. Let $\{(y_n, s_n, \epsilon_n)\} \subset \mathbb{R}^2 \times (0, \infty) \times (0, \infty)$ be any sequence satisfying

$$\epsilon_n \searrow 0, \quad y_n \to x, \quad s_n \to 0.$$

Let $\delta > 0$. Then we shall show that $|Ter(y_n, s_n, \epsilon_n) - x| < \delta$ for sufficiently large n. Indeed, from Lemma A.8, there exists $\tilde{\epsilon} > 0$ such that

$$\frac{\delta/3}{3\delta^{-1} - \nu + 1} \le t_\epsilon(r, 2\delta/3) \tag{A.28}$$

for all $r \in [0, \delta/3)$ and all $\epsilon \in (0, \tilde{\epsilon})$. We take n large enough so that

$$|y_n - x| < \delta/3,\ |s_n| < \frac{\delta/3}{3\delta^{-1} - \nu + 1},\ \epsilon_n < \tilde{\epsilon}.$$

Hence, together with (A.27), we obtain

$$\lim_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} V^-_{run}(y, s, \epsilon) + V^-_{ter}(y, s, \epsilon) = u_0(x). \tag{A.29}$$

If Carol quits the game on the way, the game positions $\{x_n\}$ are in $B(x, |Ter(y, s, \epsilon) - x|)$. Thus we have

$$V^-_{quit}(y, s, \epsilon) \ge \inf\{\Psi_+(z) \mid z \in B(x, |Ter(y, s, \epsilon) - x|)\}$$

and hence

$$\begin{aligned}\varliminf_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} V^-_{quit}(y, s, \epsilon) &\ge \lim_{\substack{(y,s)\to(x,0)\\ \epsilon\searrow 0}} \inf\{\Psi_+(z) \mid z \in B(x, |Ter(y, s, \epsilon) - x|)\}\\ &= \Psi_+(x) \ge u_0(x).\end{aligned}$$

Together with (A.29), we obtain (A.25).

We next estimate V^+. We denote by $V^+_{quit}(y, s, \epsilon)$ (resp. $V^+_{end}(y, s, \epsilon)$) the supremum total cost when Carol takes a reversed x concentric strategy through the game and Paul quits (resp. does not quit) the game on the way. Then we write

$$V^+(y, s, \epsilon) = \max\{V^+_{end}(y, s, \epsilon), V^+_{quit}(y, s, \epsilon)\}.$$

We further denote by $V^+_{run}(y, s, \epsilon)$ (resp. $V^+_{ter}(y, s, \epsilon)$) the supremum running cost (resp. terminal cost) in the same situation as $V^+_{end}(y, s, \epsilon)$. Obviously we have

$$V^+_{end}(y, s, \epsilon) \le V^+_{run}(y, s, \epsilon) + V^+_{ter}(y, s, \epsilon).$$

From Lemma A.10, Paul is forced to stay in a compact set for sufficiently small s. Thus as in (A.27), we have

$$|V^+_{run}(y, s, \epsilon)| \le \epsilon^2 NM \le (s + \epsilon^2)M.$$

The values V^+_{ter} and V^+_{quit} are also estimated in the same way as V^-_{ter} and V^-_{quit} respectively. The only difference is

$$\frac{\delta/3}{3\delta^{-1} + \nu + 1} \le t_\epsilon(r, 2\delta/3)$$

instead of (A.28). Thus (A.26) is also obtained. □

B Set theory

The following are supplementary propositions related to general topology and convex sets.

Lemma B.1. *Let $A \subset \mathbb{R}^d$ be an open set. Let $K \subset A$ be a compact set. Then, for sufficiently small $\delta > 0$,*

$$B_\delta(K) \subset A.$$

Proof. We can assume without loss of generality that A is bounded. We define $f(x) := \sup\{\delta > 0 \mid B_\delta(x) \subset A\}$ for $x \in A$ and check that it is a lower semicontinuous function. Let $\epsilon > 0$. For $x, y \in A$ satisfying $|x - y| < \epsilon$, it is clear that $B_{f(x)-\epsilon}(y) \in A$ and then $f(y) \geq f(x) - \epsilon$.

Since f is lower semicontinuous, it has a minimizer $\hat{x}$ in K by the extreme value theorem. Letting $\delta = f(\hat{x})$, we obtain the conclusion. □

Lemma B.2. *Let $A \subset \mathbb{R}^2$ be a connected open set. Then $Co(A) = \{x \in l_{a,b} \mid a, b \in A\}$.*

Proof. It is clear that $Co(A) \supset \{x \in l_{a,b} \mid a, b \in A\}$. By Carathéodory's theorem, we have

$$Co(A) = \left\{ \sum_{i=1}^{3} \lambda_i x_i;\ x_i \in A, \lambda_i \in [0,1], \sum_{i=1}^{3} \lambda_i = 1 \right\}.$$

Fix any element $x \in Co(A)$. Then we can write $x = \sum_{i=1}^{3} \lambda_i x_i$ for some $x_i \in A$ and $\lambda_i \in [0,1]$. It suffices to consider the case x_1, x_2, x_3 are different and $\lambda_i \in (0,1)$. We can assume $x = (0,0)$, $x_1 = (0,1)$, $x_2 \in \{(p,q) \in \mathbb{R}^2 \mid p < 0\}$, and $x_3 \in \{(p,q) \in \mathbb{R}^2 \mid p > 0\}$. Since A is a a connected open set, it is also path-connected. Thus there is a continuous path $\Gamma \subset A$ that connects x_2 and x_3. By the intermediate value theorem, the path Γ crosses y-axis. If Γ crosses $x_4 \in \{(0,q) \in \mathbb{R}^2 \mid q < 0\}$, then $x_1, x_4 \in A$ and $x \in l_{x_1,x_4}$. Otherwise let l be the line satisfying $x \in l$ and $l_{x_2,x_3} \parallel l$. The path Γ crosses points $x_4 \in l \cap \{(p,q) \in \mathbb{R}^2 \mid p < 0\}$ and $x_5 \in l \cap \{(p,q) \in \mathbb{R}^2 \mid p > 0\}$. Thus we have $x_4, x_5 \in A$ and $x \in l_{x_4,x_5}$. Therefore we conclude that $x \in l_{a,b}$ for some $a, b \in A$. □

Proposition B.3. *If $A \subset \mathbb{R}^d$ is open, then $Co(A)$ is open.*

Proof. Fix $x \in Co(A)$. By Carathéodory's theorem, we have $x = \sum_{i=1}^{d+1} \lambda_i x_i$ for some $x_i \in A$ and $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^{d+1} \lambda_i = 1$. Since A is open, we see that $\cup_{i=1}^{d+1} B_{r_0}(x_i) \subset A$ for some $r_0 > 0$. Therefore, for any unit vector $v \in S^1$, we have $x + rv \in Co(A)$ for $0 \leq r < r_0$ since $x + rv = \sum_{i=1}^{d+1} \lambda_i (x_i + rv)$. □

Proposition B.4. *Let $O_- \subset \mathbb{R}^d$. If O_- satisfies* (3.4)*, then $\overline{O_-}$ is strictly convex.*

Proof. Assume that $\overline{O_-}$ is not strictly convex, i.e., there exist $x, y \in \overline{O_-}$ such that $\lambda x + (1-\lambda) y \in (O_-^{int})^c$ for some $\lambda \in (0,1)$.

1) $\boldsymbol{\lambda x + (1-\lambda) y =: z \in \partial O_-}$ for some $\boldsymbol{\lambda \in (0,1)}$. By (3.4) we can take an open ball B that satisfies $z \in \partial B$ and $\overline{O_-} \subset \overline{B}$. Since $x \in \overline{B}$ and $z \in \partial B$, we have $y \in (\overline{B})^c$, which is a contradiction.

2) $\boldsymbol{\lambda x + (1-\lambda) y \in (\overline{O_-})^c}$ for any $\boldsymbol{\lambda \in (0,1)}$. Let $z = \frac{x+y}{2}$. Let $\delta > 0$ satisfy $B_\delta(z) \subset \left(\overline{O_-}\right)^c$. Since $x \in \partial O_-$, there exists $w \in O_-$ such that $w \in B_{2\delta}(x)$. Since $\frac{w+y}{2} \in \left(\overline{O_-}\right)^c$, we see that $\lambda w + (1-\lambda) y \in \partial O_-$ for some $\lambda \in (0,1)$ and hence deduce a contradiction. □

C Graph theory

We present the notion of graph and some related notions in the graph theory.

Definition C.1. For a non empty set V and a set E of unordered pairs in V, the pair of the sets (V, E) is called a *graph*. A graph $H = (V', E')$ is called a *subgraph* of G if $V' \subset V$ and $E' \subset E$. A subgraph $H = (V', E') \subset G$ is called a *path* of G if V' is a finite set $\{x_0, x_1, \cdots, x_n\}$(duplication is permitted.) and $E' = \{\langle x_i, x_{i+1} \rangle \mid i = 0, 1, \cdots, n-1\}$, where we denote unordered pairs by $\langle , \rangle$. A graph $G = (V, E)$ is *connected if for any $v_1, v_2 \in V$, there is a path of G whose endpoints are v_1 and v_2.*

To precisely indicate the path of graph introduced in the proof of Theorem 3.2 , we present the following proposition, though the assertion seems to be obvious.

Proposition C.2. *Let $G = (V, E)$ be a connected graph. Let $\langle a, b \rangle, \langle c, d \rangle \in E$. Then there is a path $P = (V', E')$ such that $a, b, c, d \in V'$ and $\langle a, b \rangle, \langle c, d \rangle \in E'$.*

Proof. If $a = c$, $(\{b, a, d\}, \{\langle b, a\rangle, \langle a, d\rangle\})$ is a required path. Hereafter we consider the case neither $a = c$, $b = c$, $a = d$ nor $b = d$. Let $P_0 = (V_0, E_0)$ be a path with endpoints a and c.

If $b, d \notin V_0$, define $V_2 := V_0 \cup \{b, d\}$ and $E_2 := E_0 \cup \{\langle b, a\rangle, \langle c, d\rangle\}$.

If $b \notin V_0$ and $d \in V_0$, define $V_1 := V_0 \cup \{b\}$ and $E_1 := E_0 \cup \{\langle b, a\rangle\}$. Writing

$$\begin{aligned} &V_1 = \{x_0, x_1, \cdots, x_n\}, \\ &E_1 = \{\langle x_i, x_{i+1}\rangle \mid i = 0, 1, \cdots, n-1\}, \end{aligned} \tag{C.1}$$

we see that $x_0 = b$, $x_1 = a$, $x_n = c$ and $x_j = d$ for some $j \in \{2, 3, \cdots, n\}$. Then we further define $V_2 := \{x_0, x_1, \cdots, x_j, x_n\}$ and $E_2 := \{\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \cdots \langle x_{j-1}, x_j\rangle, \langle x_j, x_n\rangle\}$.

We finally consider the case $b, d \in V_0$. When we follow the path P_0 from a to c, there are two cases: whether we find b earlier than d or not. In the former case, writing V_0 and E_0 as (C.1), we see $x_j = b$, $x_m = d$ for some $0 \leq j < m \leq n$. We then define $V_2 := \{x_0, x_j, x_{j+1}, \cdots, x_m, x_n\}$ and $E_2 := \{\langle x_0, x_j\rangle, \langle x_j, x_{j+1}\rangle, \cdots \langle x_{m-1}, x_m\rangle, \langle x_m, x_n\rangle\}$. In the latter case, we see $x_j = d$, $x_m = b$ for some $0 \leq j < m \leq n$. We define $V_2 := \{x_l, x_0, x_1, \cdots, x_j, x_n\}$ and $E_2 := \{\langle x_l, x_0\rangle, \langle x_0, x_1\rangle, \cdots \langle x_{j-1}, x_j\rangle, \langle x_j, x_n\rangle\}$.

In any case, $P_2 := (V_2, E_2)$ is a required path. □

D Curve theory

To complement the proof of Thorem 3.2, we will mathematically describe the construction of a Jordan curve $\hat{C}$ that is included in a given closed curve C and includes a given point $x \in C$. We begin with a general property of connected sets. In what follows we especially notice that two points in an open and connected subset of $\mathbb{R}^d$ can be connected by a polygonal line.

Definition D.1 (polygonal line connected)**.** A path is called a *polygonal line* if it consists of finite line segments. Let $A \subset \mathbb{R}^d$. The set A is called *polygonal line connected* if for any two points $x, y \in A$, there exists a polygonal line in A that connects x and y.

Proposition D.2. *Let $A \subset \mathbb{R}^d$ be an open set. Then the following statements are equivalent.*

1. *A is connected.*
2. *A is path-connected.*
3. *A is polygonal line connected.*

Proof. Without loss of generality, we can assume $A \neq \emptyset$ since otherwise all the statements are obviously true.

3⇒2.

This is clear because a polygonal line is a path.

2⇒1.

Fix $x \in A$. Since a path is a connected set, all elements $y \in A$ are in the connected component including x. Therefore A is connected.

1⇒3.

Fix $x \in A$. We define
$O := \{y \in A \mid$ there exists a polygonal line in A that connects x and $y\}$. We first show that O is an open set. Let $y \in O$. Since $y \in A$, there is an open ball $B_\delta(y)$ such that $B_\delta(y) \subset A$. For any $z \in B_\delta(y)$, we can make a polygonal line in A that connects x and z, combining the line segment between y and z with a polygonal line between x and y. Therefore we have $B_\delta(y) \subset O$, which means O is an open set.

We show that $A \setminus O$ is also an open set. Let $y \in A \setminus O$. As before, there is an open ball $B_\delta(y)$ such that $B_\delta(y) \subset A$. If a point z in $B_\delta(y)$ is in O, we can make a polygonal line in A that connects x and y. This is a contradiction. Hence we have $B_\delta(y) \subset A \setminus O$.

Since A is connected, O must be $\emptyset$ or A. Since $x \in O$, we have $O = A$ and conclude that A is polygonal line connected. □

We state the condition on components of the closed curve C. In what follows we call a map $f|_A$ injective at $t \in A$ if $s \in A$ and $f(s) = f(t)$ imply $s = t$. Also we call a map $f|_A$ injective in $B(\subset A)$ if $s \in A$, $t \in B$ and $f(s) = f(t)$ imply $s = t$. We set a class $\mathscr{C}$ of curves in $\mathbb{R}^2$. We make the assumption on $\mathscr{C}$:

(A1) There exists a map $\mathscr{C} \ni C \mapsto \gamma_C \in C([0, 1]; \mathbb{R}^2)$ such that

1. $\gamma_C([0,1]) = C$ and the set $\{t \in (0,1) \mid \gamma_C \text{ is not injective at } t\}$ is at most finite.
2. For any $C, D \in \mathscr{C}$, $\{t \in (0,1) \mid \gamma_C(t) \notin D\}$ is at most a finite union of open intervals.

We now state the assumption on the closed curve C:

(A2) For some $C_1, C_2, \cdots, C_N \in \mathscr{C}$, $C = \cup_{i=1}^N C_i$, $\gamma_{C_i}(1) = \gamma_{C_{i+1}}(0)$ for $i \in \{1, 2, \cdots, N-1\}$ and $\gamma_{C_N}(1) = \gamma_{C_1}(0)$.

Remark D.3. The set of line segments and arcs in $\mathbb{R}^2$ satisfies (A1). Hence the closed curve in the proof of Theorem 3.2 satisfies (A2).

Set $\gamma : [0, N] \to \mathbb{R}^2$ as

$$\gamma(t) := \gamma_{C_i}(t - [t]),$$

if $i - 1 \leq t \leq i$. Here we denote by $[t]$ the maximal integer that is no more than t. For a point $x \in C$, there is no loss of generality to assume $\gamma(0) = x$. The reason is the following:

Proposition D.4. *Let $\mathscr{C}$ satisfy* (A1)*. Let $C' \in \mathscr{C}$. Then $\mathscr{C}' := \mathscr{C} \cup \{C'_1, C'_2\}$ also satisfies* (A1)*, where we define*

$$C'_1 := \gamma_{C'}([0, a]) \text{ and } C'_2 := \gamma_{C'}([a, 1])$$

for $a \in (0, 1)$.

Proof. Let $\gamma_{C'_1}(t) = \gamma_{C'}(t/a)$ and $\gamma_{C'_2}(t) = \gamma_{C'}(a + (1-a)t)$. The proof is done by checking the assumption (A1) directly. □

We assume that $\gamma|_{[0,N)}$ is injective in a neighborhood of 0. i.e.,

(A3) There exists $\delta > 0$ such that $\gamma|_{[0,N)}$ is injective in $[0, \delta)$.

Remark D.5. The closed curve $C \cup \hat{\Gamma}$ in the proof of Theorem 3.2 satisfies (A3) because $B_{3\delta}(\hat{\Gamma}) \subset L$ and $x \notin L$.

We inductively define

$$t_1 := 0,$$
$$s_i := \sup\{\tau \mid \gamma|_{(t_i,N]} \text{ is injective in } (t_i, \tau)\},$$
$$t_{i+1} := \sup\{\tau \mid \gamma(s_i) = \gamma(\tau)\}$$

for $i = 1, 2, \cdots$.

Proposition D.6. *For some $m \in \mathbb{N}$, $s_m = N$ or $t_m = N$.*

Proof. We first prove $t_j < s_j$ for all j. The assumption that $\gamma|_{[0,N)}$ is injective in a neighborhood of 0 implies $t_1 < s_1$. If $s_1 < N$, then fix $j \in \{2, 3, \cdots\}$. Let $i = [t_j] + 1$. Set

$$A_i := \bigcap_{i+1 \leq k \leq N} \{t \in (i-1, i) \mid \gamma(t) \notin C_k\}$$

for $1 \leq i \leq N-1$ and $A_N := (N-1, N)$. We also set

$$B_i := \{t \in (i-1, i) \mid \gamma|_{(i-1,i)} \text{ is not injective at } t\}.$$

From the assumption (A1), A_i is a finite union of open intervals and B_i is at most finite. By the definition of t_j we see that if $i - 1 < t_j < i$, then $t_j \in A_i$. Hence, if $i - 1 < t_j < i$, we have $(t_j, t_j + \delta) \subset A_i \cap B_i^c$ for some $\delta > 0$. Also this assertion holds for $t_j = i - 1$. To show it, we prove by contradiction that $\inf A_i = i - 1$. We assume that $\inf A_i > i - 1$. Then we have $\inf\{t \in (i-1, i) \mid \gamma(t) \notin C_k\} > i - 1$ for $i + 1 \leq k \leq N$. From the continuity of γ, we obtain $\gamma(t_j) \in C_k$, which is a contradiction with the definition of t_j.

Now it turns out that there exists $l \geq 0$ such that $i - 1 \leq t_j < i$ implies $i \leq s_{j+l}$ or $i \leq t_{j+l+1}$. Indeed, if $s_j < i$, then $s_j \in \partial A_i \cup B_i$. If $s_j < i$ and $s_j \in \partial A_i$, then $i \leq t_{j+1}$. If $s_j < i$ and $s_j \in B_i$, then $t_{j+1} \in B_i$. Thus the proof is complete. □

If $s_m = N$ or $t_{m+1} = N$, then let $T = \sum_{i=1}^{m}(s_i - t_i)$. Define $\hat{\gamma} : [0,T] \to \mathbb{R}^2$ as follows:

$$\hat{\gamma}(t) := \gamma(t)$$

if $t \leq s_1$ and

$$\hat{\gamma}(t) := \gamma\left(t + t_{k+1} - \sum_{i=1}^{k}(s_i - t_i)\right)$$

if $\sum_{i=1}^{k}(s_i - t_i) \leq t \leq \sum_{i=1}^{k+1}(s_i - t_i)$.

Proposition D.7. *$\hat{C} := \hat{\gamma}([0,T])$ is a Jordan closed curve.*

Proof. This assertion is obvious from the definitions of t_i and s_i. □

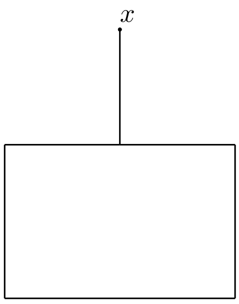


Figure 19: An example of C and x that we avoid

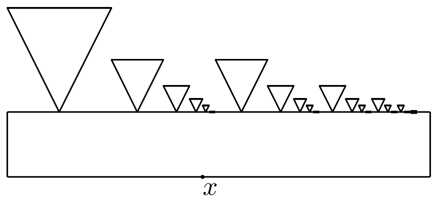


Figure 20: An example of C that includes an infinite number of loops

Remark D.8. By assuming (A3), we can avoid a closed curve C and a point x in it such as Figure 19. We also avoid a closed curve C that includes an infinite number of loops such as Figure 20 by the assumption (A2).

Part II

Weak comparison principles for fully nonlinear degenerate parabolic equations with discontinuous source terms

1 Introduction

Equation and purpose. We study a fully nonlinear parabolic partial differential equation of the form

$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f(x) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{1.1}$$

under the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Here $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown function and u_t, $\nabla u = (u_{x_i})_{i=1}^n$ and $\nabla^2 u = (u_{x_i x_j})_{i,j=1}^n$ stand for the time derivative, the spatial gradient and the Hessian matrix of u, respectively. Moreover, we assume the following conditions throughout this paper:

- $F : (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R}$ is a continuous function, where $\mathbb{S}^n$ denotes the set of $n \times n$ real symmetric matrices with the usual ordering,
- $H : \mathbb{R}^n \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function called a Hamiltonian,
- $f : \mathbb{R}^n \to \mathbb{R}$ is a locally bounded, possibly discontinuous source term,
- $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a continuous initial datum such that $\lim_{|x|\to\infty} u_0(x) = 0$.

Further assumptions on F, H and f will be given later. We note that $F = F(p,X)$ is allowed to be singular at $p = 0$.

The goal of this paper is to establish new comparison principles for a viscosity sub- and supersolution of (1.1)–(1.2). A difficulty lies in the discontinuity of the source term f, and due to this, classical comparison results do not apply to (1.1). Under a condition on discontinuity of f, we prove a weak version of comparison principles for (1.1). Moreover, we derive uniqueness of solutions to (1.1)–(1.2) in suitable classes. Our results guarantee uniqueness of solutions which are possibly discontinuous. We also investigate existence of solutions.

Typical equation and physical background. Our assumptions on F are very mild; indeed, we only need (2.1)–(2.3) in Section 2. Examples of F include typical second order operators such as the Laplacian $-\Delta u(x,t)$, the Pucci extremal operators $\mathcal{P}^{\pm}(\nabla^2 u(x,t))$ which are fully nonlinear ([7]) and Bellman–Isaacs operators arising in stochastic control problems ([14]).

Among many other second order equations, a typical one in our mind is the level-set mean curvature flow equation with a driving force term and a source term. Namely,

$$u_t(x,t) - \nu(x,t)|\nabla u(x,t)| - \Delta_1 u(x,t) = f(x) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.3}$$

where $|\cdot|$ stands for the standard Euclidean norm. In this case, the function F in (1.1) is given by

$$F(\nabla u(x,t), \nabla^2 u(x,t)) = -\Delta_1 u(x,t) := -\frac{1}{n-1}|\nabla u(x,t)| \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

or equivalently

$$F(p,X) = -\frac{1}{n-1}\operatorname{tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right) \quad ((p,X) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n). \tag{1.4}$$

Here $p \otimes p = (p_i p_j)_{i,j=1}^n$ for a vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$. Also, the Hamiltonian is

$$H(x,t,p) = -\nu(x,t)|p|, \tag{1.5}$$

where a continuous function $\nu : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ stands for the driving force. A typical source term f is a characteristic function

$$f(x) = c\chi_\Omega(x) \quad (c > 0,\, \Omega \subset \mathbb{R}^n), \tag{1.6}$$

where $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \notin \Omega$.

The equation of the form (1.3) appears in a crystal growth phenomenon called *two dimensional nucleation* ([6, 50, 52]). In this phenomenon, crystals grow by catching molecules on some area of the crystal surface. Let us briefly explain the derivation of the equation describing this growth. Let $u(x, t)$ be the height of the crystal surface at a position $x \in \mathbb{R}^n$ and a time $t \in [0, \infty)$. See Figure 21. We assume that the crystal growth in the horizontal direction and the vertical direction are respectively governed by the following laws (A) and (B):

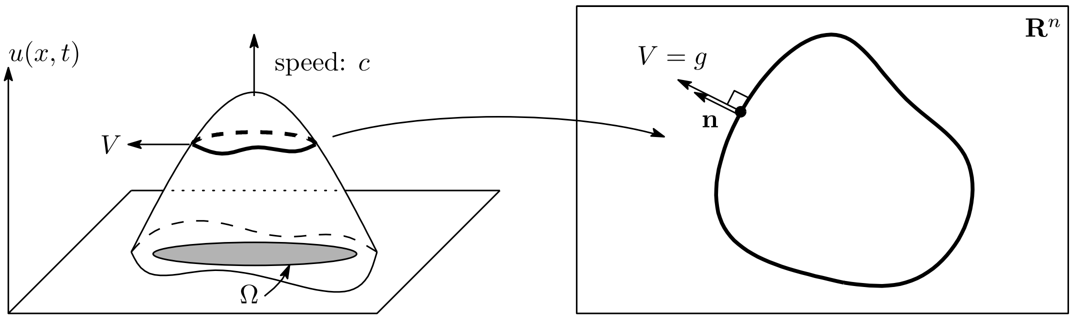


Figure 21: Growth laws in the case of (1.6).

(A) For each $l \in \mathbb{R}$, the level-set $\Gamma_l(t) = \{x \in \mathbb{R}^n \mid u(x, t) = l\}$ of u evolves in the horizontal direction according to a surface evolution equation of the form

$$V = g(x, t, \ltimes, \nabla \ltimes) \quad \text{on } \Gamma_l(t). \tag{1.7}$$

(B) The height u changes at a rate of $f(x)$ due to nucleation.

Here g is a given function, $\ltimes = \ltimes(x, t) \in \mathbb{R}^n$ is the unit normal vector to $\Gamma_l(t)$ at x from $\{x \in \mathbb{R}^n \mid u(x, t) > l\}$ to $\{x \in \mathbb{R}^n \mid u(x, t) < l\}$, $V = V(x, t)$ is the normal velocity of $\Gamma_l(t)$ at x in the direction of $\ltimes$, and $-\nabla \ltimes$ is the second fundamental form in the direction of $\ltimes$. If $\nabla u(x, t) \neq 0$ and u is smooth near (x, t), we have the following representation:

$$V = \frac{u_t(x, t)}{|\nabla u(x, t)|}, \quad \ltimes = -\frac{\nabla u(x, t)}{|\nabla u(x, t)|}, \quad \nabla \ltimes = -\frac{1}{|\nabla u(x, t)|} Q_{\nabla u(x,t)}(\nabla^2 u(x, t)),$$

where

$$Q_p(X) = R_p X R_p \quad \left(R_p := I - \frac{p \otimes p}{|p|^2}\right).$$

See [17, Chapter 1] for derivations of these representations. Substituting these for (1.7), we obtain

$$u_t(x, t) + G(x, t, \nabla u(x, t), \nabla^2 u(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1.8}$$

with

$$G(x, t, p, X) = -|p| g \left(x, t, -\frac{p}{|p|}, -\frac{1}{|p|} Q_p(X)\right), \tag{1.9}$$

which is a possibly singular function at $p = 0$. The equation (1.8) is often called a *level-set equation.* See [8, 13, 17] for rigorous analysis of such level-set equations. We turn to the condition (B). Since the growth speed in the vertical direction is given by $u_t(x, t)$, the condition requires that

$$u_t(x, t) = f(x) \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.10}$$

As both (A) and (B) occur in the two dimensional nucleation, the equation describing this phenomenon is the mixed one of (1.8) and (1.10), that is,

$$u_t(x, t) + G(x, t, \nabla u(x, t), \nabla^2 u(x, t)) = f(x) \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.11}$$

We thus get (1.1) provided that the second order term of G is separable.

Trotter–Kato product formula is another tool to derive (1.11); see [22] for the details.

One of typical surface evolution equations is the mean curvature flow equation with a driving force, which is of the form

$$V = \kappa + \nu \quad \text{on } \Gamma_l(t).$$

Here $\kappa = -(\mathrm{div}_{\Gamma_l(t)} \ltimes)(x)/(n-1)$ is the mean curvature of $\Gamma_l(t)$ at x, and $\nu = \nu(x,t)$ is a driving force. The equation (1.11) in this case is given by (1.3).

In this paper we often focus on (1.3) with a constant driving force $\nu \in \mathbb{R}$, that is,

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = f(x) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.12}$$

where the Hamiltonian is

$$H(x,t,p) = H(p) = -\nu|p|. \tag{1.13}$$

Results. A usual comparison principle for viscosity solutions to initial value problems asserts that, if u and v are respectively are a viscosity subsolution and a viscosity supersolution and if $u^*(\cdot,0) \leqq v_*(\cdot,0)$ in $\mathbb{R}^n$, then

$$u^* \leqq v_* \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.14}$$

Here the asterisks $*$ stand for the semicontinuous envelopes; see Section 2 for the definitions. This comparison result guarantees that viscosity solutions are unique and that the unique solution is continuous. Also, this type of comparison principle is established under a suitable continuity of equations ([10]).

When the equation is discontinuous, it is possible that viscosity solutions are not unique and that discontinuous solutions exist. See Section 6 and [18, 22]. Thus, we cannot expect the inequality (1.14) as a conclusion of our comparison principle for (1.1). For this reason, we establish a weaker version of the comparison principle. We prove

$$(u^*)_* \leqq v_*, \quad u^* \leqq (v_*)^* \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.15}$$

Since $(u^*)_* \leqq u^*$ and $v_* \leqq (v_*)^*$, these estimates are actually weaker than (1.14). In this sense, we call our result (1.15) a *weak comparison principle.*

We establish two kinds of weak comparison principles in Section 3 under different assumptions. In both the proofs, we change the scale of either a subsolution u or a supersolution v. Such an idea can be found in [37] for elliptic equations. For the first comparison principle (Theorem 3.1), we assume that either u or v is Lipschitz continuous with respect to the space variable x; see (2.7). This Lipschitz regularity guarantees that derivatives of a test function are bounded uniformly in parameters, and it enables us to extract a subsequence of the derivatives and take a limit for viscosity inequalities.

The condition (2.7) requires Lipschitz continuity in x which is locally uniform in $t \in (0,\infty)$. In particular, the initial time is excluded in the condition (2.7). Thanks to this, there is a chance to build a unique solution even if the initial datum is not Lipschitz continuous, provided that the Lipschitz regularizing effect occurs for (1.1). See Remark 5.2 (3) for more comments on this.

Our second comparison principle (Theorem 3.2) does not need the Lipschitz regularity of one of the solutions. Instead, we assume that the Hamiltonian H satisfies an additional condition (2.8), which covers the case (1.12) with a nonpositive driving force $\nu \leqq 0$.

In Section 4 we derive some uniqueness results of solutions. The results are obtained as a consequence of the weak comparison principles. Among other things, we prove that semicontinuous solutions are unique. We also discuss existence of solutions in Section 5 via approximation of the source term f by continuous ones. For (1.12) with a negative driving force $\nu < 0$, with the aid of Perron's method, we prove that solutions have compact supports which are uniform in $t \geqq 0$.

Literature overview. In [18] Hamilton-Jacobi equations with a discontinuous source term

$$u_t(x,t) + H(x, \nabla u(x,t)) = f(x) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{1.16}$$

are studied. This is the case where $F = 0$ in (1.1). Introducing a new notion of solutions, the authors of [18] prove uniqueness and existence of solutions when H is coercive. The large time behavior of the solution is investigated in [27].

The level-set mean curvature flow equation (1.12) is studied in [22] when the driving force ν is a positive constant. The asymptotic speed of the maximal solution is investigated. We remark that, in [22, Proposition 2.1] a comparison result named a "weak comparison principle" is presented, but its assertion is different from ours. It asserts that (1.14) holds if v is a viscosity supersolution of (1.12) with $g(x)$ on the right-hand side such that $f^* \leqq g_*$ in $\mathbb{R}^n$. Our comparison results differ from [22, Proposition 2.1] in that ours apply to equations with the common source term.

When the source term f is continuous, some further results are obtained in [23, 21]. The asymptotic shape is studied in [23] for a radially symmetric source term. In [21] the asymptotic speed is investigated for equations with a general degenerate elliptic operator F.

Our second weak comparison principle (Theorem 3.2) is applied in the forthcoming paper [31], where we consider (1.12) with a negative driving force $\nu < 0$. The asymptotic shape of solutions is investigated. In [31] we also provide a game-theoretic interpretation for the equation and apply the result to study the asymptotic speed of solutions.

In [16] some weak versions of comparison principles are established for first order equations

$$u_t(x,t) + H(u(x,t), \nabla u(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

whose solutions may develop jump discontinuity (shock). We also refer the reader to the paper [4], which studies connection between nonempty interior condition on evolving surfaces and uniqueness of semicontinuous solutions to geometric equations without source terms. See also [3] for uniqueness results of semicontinuous solutions to first order Hamilton–Jacobi equations whose Hamiltonian is convex.

Organization. This paper is organized as follows: Section 2 is devoted to preliminaries. In Section 3 we establish two kinds of weak comparison principles as main results of this paper. In Sections 4 and 5 we study uniqueness and existence of solutions, respectively. Some examples are given in Section 6.

A part of the results in this paper is announced in [32].

2 Preliminaries

2.1 Assumptions

Let us denote by $B_r(x)$ the open ball centered at x with a radius $r > 0$. We first recall a notion of semicontinuous envelopes. For a subset $K \subset \mathbb{R}^N$ and a function $h : K \to \mathbb{R}$, we define the *upper semicontinuous envelope* $h^* : \overline{K} \to \mathbb{R} \cup \{\infty\}$ and the *lower semicontinuous envelope* $h_* : \overline{K} \to \mathbb{R} \cup \{-\infty\}$ by

$$h^*(x) = \lim_{r\to+0} \sup\{h(y) \mid y \in B_r(x) \cap K\}, \quad h_*(x) = \lim_{r\to+0} \inf\{h(y) \mid y \in B_r(x) \cap K\}$$

for $x \in \overline{K}$.

We assume the following conditions on F:

$$F(p,X) \leqq F(p,Y) \text{ for all } p \in \mathbb{R}^n \setminus \{0\} \text{ and } X, Y \in \mathbb{S}^n \text{ such that } X \geqq Y, \tag{2.1}$$

$$-\infty < F_*(0,O) = F^*(0,O) < \infty, \tag{2.2}$$

$$F(rp,X) = F(p,X) \text{ for all } (p,X) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \text{ and } r > 0. \tag{2.3}$$

Remark 2.1. It is not difficult to see that the level-set mean curvature flow operator (1.4) satisfies all of (2.1)–(2.3) above.

A key assumption on the source term f is

$$\begin{cases} \text{For any discontinuous point } x \in \mathbb{R}^n \text{ of } f \text{ and any sequence } \{x_\lambda\}_{\lambda>1} \subset \mathbb{R}^n \\ \text{such that } \lim\limits_{\lambda\to1+0} x_\lambda = x, \text{ we have } \limsup\limits_{\lambda\to1+0}\{f^*(\lambda x_\lambda) - f_*(x_\lambda)\} \leqq 0. \end{cases} \tag{2.4}$$

Remark 2.2. It is clear that (2.4) is fulfilled if

$$f^*(\lambda x) \leqq f_*(x) \text{ for all } x \in \mathbb{R}^n \text{ and } \lambda > 1. \tag{2.5}$$

When f is a characteristic function (1.6), the condition (2.5) holds if and only if Ω is star-shaped with respect to the origin, that is,

$$\overline{\Omega} \subset \lambda\Omega^\circ \text{ for all } \lambda > 1, \tag{2.6}$$

where Ω° is the interior of Ω and $\lambda\Omega^\circ = \{\lambda x \mid x \in \Omega^\circ\}$.

We present two weak comparison principles in Section 3. For the first comparison principle, we assume that either a subsolution u or a supersolution v is Lipschitz continuous with respect to the space variable. More precisely, the following condition is imposed on $w = u$ or $w = v$:

$$\begin{cases} \text{For every } \gamma > 0 \text{ and } T > \gamma \text{ there is } L > 0 \text{ such that} \\ |w(x,t) - w(y,t)| \leqq L|x - y| \text{ for all } x, y \in \mathbb{R}^n \text{ and } t \in [\gamma, T]. \end{cases} \tag{2.7}$$

We note that the Lipschitz continuity is not required at the initial time $t = 0$.

Our second comparison principle does not need (2.7); instead we assume that the Hamiltonian H satisfies

$$H \text{ is independent of } (x,t), \text{ and } H(p) \leqq H(\lambda p) \text{ for all } p \in \mathbb{R}^n \text{ and } \lambda > 1. \tag{2.8}$$

Remark 2.3. When H is of the form (1.13) with a nonpositive $\nu \leqq 0$, the condition (2.8) is fulfilled. More generally, if

$$H(x,t,p) = H(p) = -\nu|p|^a \quad (\nu \leqq 0,\ a \geqq 0 \text{ are constants}),$$

then H satisfies (2.8). Indeed, for $p \in \mathbb{R}^n$ and $\lambda > 1$, we have

$$H(\lambda p) - H(p) = -\nu\lambda^a|p|^a + \nu|p|^a = -\nu(\lambda^a - 1)|p|^a \geqq 0.$$

2.2 Viscosity solution

We next introduce a notion of viscosity solutions. The reader is referred to [10, 17] for the basic theory of viscosity solutions. Let $C^{2,1}(\mathbb{R}^n \times (0,\infty))$ denote the set of functions $\phi = \phi(x,t)$ that are of class C^2 in x and C^1 in t.

Definition 2.4 (Viscosity solution). (1) Let $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$. We say that u is a *viscosity subsolution* (resp. a *viscosity supersolution*) *of* (1.1) if the following (i)–(ii) hold:

(i) $u^* < \infty$ (resp. $u_* > -\infty$) in $\mathbb{R}^n \times (0,\infty)$,

(ii) Whenever $u^* - \phi$ (resp. $u_* - \phi$) attains a local maximum (resp. a local minimum) at $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ for $\phi \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$, we have

$$\phi_t(x_0,t_0) + H(x_0,t_0,\nabla\phi(x_0,t_0)) + F_*(\nabla\phi(x_0,t_0), \nabla^2\phi(x_0,t_0)) \leqq f^*(x_0)$$
$$(\text{resp. } \phi_t(x_0,t_0) + H(x_0,t_0,\nabla\phi(x_0,t_0)) + F^*(\nabla\phi(x_0,t_0), \nabla^2\phi(x_0,t_0)) \geqq f_*(x_0)).$$

(2) Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$. We say that u is a *viscosity subsolution* (resp. a *viscosity supersolution*) *of* (1.1)–(1.2) if u is a viscosity subsolution of (1.1) and $u^*(\cdot,0) \leqq u_0$ (resp. $u_*(\cdot,0) \geqq u_0$) in $\mathbb{R}^n$.

(3) A function u is called a *viscosity solution* if u is both a viscosity subsolution and a viscosity supersolution.

Remark 2.5. When u is a viscosity solution of (1.1)–(1.2), it is continuous on $\mathbb{R}^n \times \{0\}$.

Remark 2.6. The definition of viscosity solutions can be rephrased by using parabolic semijets $\mathcal{P}^{2,\pm}u(x,t)$ and the extended ones $\overline{\mathcal{P}}^{2,\pm}u(x,t)$; for their definitions, see [10, 17]. In fact, the condition (ii) in Definition 2.4 can be replaced by the following one:

(ii)′ Whenever $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $(p,\tau,X) \in \mathcal{P}^{2,+}u^*(x_0,t_0)$ (resp. $(p,\tau,X) \in \mathcal{P}^{2,-}u_*(x_0,t_0)$), we have

$$\tau + H(x_0,t_0,p) + F_*(p,X) \leqq f^*(x_0) \quad (\text{resp. } \tau + H(x_0,t_0,p) + F^*(p,X) \geqq f_*(x_0)).$$

Moreover, one may replace "$\mathcal{P}^{2,\pm}u(x,t)$" in (ii)′ by "$\overline{\mathcal{P}}^{2,\pm}u(x,t)$".

In our comparison principles we assume decay conditions on a subsolution u and a supersolution v as follows:

$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } u(x,t) \leqq \delta \text{ in } B_R(0)^c \times [0,T], \tag{2.9}$$
$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } v(x,t) \geqq -\delta \text{ in } B_R(0)^c \times [0,T]. \tag{2.10}$$

For later use we prepare notations of classes of viscosity sub- and supersolutions.

Definition 2.7.

$$\mathrm{SUB} := \{u \mid u \text{ is a viscosity subsolution of (1.1)–(1.2) and satisfies (2.9)}\},$$
$$\mathrm{SUP} := \{u \mid u \text{ is a viscosity supersolution of (1.1)–(1.2) and satisfies (2.10)}\}$$

and $\mathrm{SOL} := \mathrm{SUB} \cap \mathrm{SUP}$. Moreover,

$$\mathrm{SUB_{Lip}} := \{u \in \mathrm{SUB} \mid u \text{ is continuous in } \mathbb{R}^n \times [0,\infty) \text{ and satisfies (2.7)}\},$$
$$\mathrm{SUP_{Lip}} := \{u \in \mathrm{SUP} \mid u \text{ is continuous in } \mathbb{R}^n \times [0,\infty) \text{ and satisfies (2.7)}\}$$

and $\mathrm{SOL_{Lip}} := \mathrm{SUB_{Lip}} \cap \mathrm{SUP_{Lip}}$.

3 Weak comparison principles

We establish two kinds of weak comparison principles for a viscosity subsolution and a viscosity supersolution of (1.1).

3.1 Comparison under Lipschitz continuity of solutions

We prove a weak comparison principle under the assumption that either a subsolution or a supersolution satisfies the Lipschitz condition (2.7).

Theorem 3.1 (Weak comparison principle 1)**.** *Assume* (2.1)–(2.4)*. Let* $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be a viscosity subsolution of* (1.1) *satisfying* (2.9)*, and let* $v : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be a viscosity supersolution of* (1.1) *satisfying* (2.10)*. Assume that either* u *or* v *satisfies* (2.7)*. If* $u^*(\cdot,0) \leqq v_*(\cdot,0)$ *in* $\mathbb{R}^n$*, then* $(u^*)_* \leqq v_*$ *and* $u^* \leqq (v_*)^*$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. For simplicity let us write u and v for u^* and v_*, respectively. We only give the proof of $u_* \leqq v$ in $\mathbb{R}^n \times (0,\infty)$ since the other one is derived in a parallel way.

1. Let $\lambda > 1$. We rescale the subsolution u by

$$u_\lambda(x,t) = \frac{1}{\lambda^2} u(\lambda x, \lambda^2 t).$$

A direct calculation shows that u_λ is a viscosity subsolution of

$$u_t(x,t) + H(\lambda x, \lambda^2 t, \lambda \nabla u(x,t)) + F(\lambda \nabla u(x,t), \nabla^2 u(x,t)) = f(\lambda x) \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

and, by (2.3), it is also a viscosity subsolution of

$$u_t(x,t) + H(\lambda x, \lambda^2 t, \lambda \nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f(\lambda x) \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{3.1}$$

Note that $u_* \leqq \liminf_{\lambda\to 1+0} u_\lambda$ in $\mathbb{R}^n \times [0,\infty)$. Thus, in order to derive $u_* \leqq v$ in $\mathbb{R}^n \times (0,\infty)$, it suffices to prove that

$$\liminf_{\lambda\to 1+0} u_\lambda \leqq v \quad \text{in } \mathbb{R}^n \times (0,T) \tag{3.2}$$

for every $T > 0$. Suppose by contradiction that

$$\theta := \sup_{\mathbb{R}^n \times (0,T)} \left(\liminf_{\lambda\to 1+0} u_\lambda - v \right) > 0.$$

We choose a point $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ such that

$$\liminf_{\lambda\to 1+0} u_\lambda(x_0,t_0) - v(x_0,t_0) \geqq \frac{4\theta}{5},$$

and then there exists some $\lambda_0 > 1$ such that

$$u_\lambda(x_0,t_0) - v(x_0,t_0) \geqq \frac{3\theta}{5} \quad \text{for all } \lambda \in (1,\lambda_0). \tag{3.3}$$

We fix $\lambda \in (1,\lambda_0)$ and define a function $\Psi_\lambda : \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,T] =: \mathcal{O} \to \mathbb{R} \cup \{-\infty\}$ by

$$\Psi_\lambda(x,t,y,s) := u_\lambda(x,t) - v(y,s) - \phi(x,t,y,s)$$

with

$$\phi(x,t,y,s) = \frac{|x-y|^4}{\varepsilon} + \frac{|t-s|^2}{\varepsilon} + \frac{\sigma}{T-t}.$$

Here $\varepsilon \in (0,1]$ and

$$\sigma = \frac{\theta(T-t_0)}{5} > 0. \tag{3.4}$$

We interpret $\sigma/(T-t) = \infty$ when $t = T$. Note that σ is independent of λ. From (3.3) and (3.4) it follows that

$$\Psi_\lambda(x_0,t_0,x_0,t_0) \geqq \frac{2\theta}{5}. \tag{3.5}$$

2. For later use we prepare some constants.

- Since u and v satisfy (2.9) and (2.10) respectively, there exists a constant $R > 0$ independent of $\lambda \in (1,\lambda_0)$ such that

$$u_\lambda \leqq \frac{\theta}{5}, \quad v \geqq -\frac{\theta}{5} \quad \text{in } B_R(0)^c \times [0,T]. \tag{3.6}$$

- By (3.6) and the semicontinuity of u and v, there exists a constant $M > 0$ independent of $\lambda \in (1,\lambda_0)$ such that

$$u_\lambda \leqq M, \quad v \geqq -M \quad \text{in } \mathbb{R}^n \times [0,T]. \tag{3.7}$$

- Hereafter we take $\varepsilon \in (0,1]$ so small that $2\varepsilon M \leqq R^4$.

3. Define

$$K := \left\{ (x,t,y,s) \in \mathcal{O} \;\middle|\; \Psi_\lambda(x,t,y,s) \geqq \frac{2\theta}{5} \right\}, \tag{3.8}$$

which is nonempty by (3.5) and is closed by the upper semicontinuity of Ψ_λ. Let us prove that

$$K \subset B_{2R}(0) \times [0,T] \times B_{2R}(0) \times [0,T] =: K_*.$$

Since K_* is bounded, the above inclusion guarantees that K is a nonempty compact set. Take $(x,t,y,s) \in K$ arbitrarily. Then, by (3.7)

$$\frac{|x-y|^4}{\varepsilon} + \frac{|t-s|^2}{\varepsilon} + \frac{\sigma}{T-t} \leqq u_\lambda(x,t) - v(y,s) - \frac{2\theta}{5} \leqq 2M - \frac{2\theta}{5} < 2M,$$

and especially

$$|x-y|^4 \leqq 2\varepsilon M, \quad |t-s|^2 \leqq 2\varepsilon M. \tag{3.9}$$

The former inequality implies that $|x-y| \leqq R$ due to the smallness of ε. Suppose that $(x,t,y,s) \notin K_*$, i.e., $x \notin B_{2R}(0)$ or $y \notin B_{2R}(0)$. Since $|x-y| \leqq R$, we then have $x \notin B_R(0)$ and $y \notin B_R(0)$. It follows from (3.6) that

$$\Psi_\lambda(x,t,y,s) \leqq \frac{\theta}{5} + \frac{\theta}{5} - \frac{|x-y|^4}{\varepsilon} - \frac{|t-s|^2}{\varepsilon} - \frac{\sigma}{T-t} < \frac{2\theta}{5},$$

which is contrary to the assumption $(x,t,y,s) \in K$. We thus conclude that $K \subset K_*$.

4. Since K is a nonempty compact set, Ψ_λ attains a maximum over $\mathcal{O}$ at some $Z_{\lambda,\varepsilon} = (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}, y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) \in K \subset K_*$. In particular, letting $\phi_1(x,t) := \phi(x,t,y_{\lambda,\varepsilon},s_{\lambda,\varepsilon})$ and $\phi_2(y,s) := -\phi(x_{\lambda,\varepsilon},t_{\lambda,\varepsilon},y,s)$, we have

$$\begin{cases} u_\lambda - \phi_1 \text{ attains a maximum at } (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}), \\ v - \phi_2 \text{ attains a minimum at } (y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}). \end{cases} \tag{3.10}$$

Moreover, the definition of K_* implies that the family of maximizers $\{Z_{\lambda,\varepsilon}\}$ is bounded uniformly in λ and ε. Thus, for every $\lambda \in (1,\lambda_0)$, we may assume that there exists some $(\bar{x}_\lambda, \bar{t}_\lambda, \bar{y}_\lambda, \bar{s}_\lambda) \in \overline{K_*}$ such that

$$\lim_{\varepsilon\to+0} Z_{\lambda,\varepsilon} = \lim_{\varepsilon\to+0} (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}, y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = (\bar{x}_\lambda, \bar{t}_\lambda, \bar{y}_\lambda, \bar{s}_\lambda).$$

By (3.9) we have $\bar{x}_\lambda = \bar{y}_\lambda \in \overline{B_{2R}(0)}$ and $\bar{t}_\lambda = \bar{s}_\lambda \in [0,T]$. Since $\{\bar{x}_\lambda\}$ and $\{\bar{t}_\lambda\}$ are bounded uniformly in λ, we may again assume that they are convergent. Namely,

$$\lim_{\lambda\to1+0} (\bar{x}_\lambda, \bar{t}_\lambda) = (\bar{x}, \bar{t})$$

for some $(\bar{x}, \bar{t}) \in \overline{B_{2R}(0)} \times [0, T]$.

Let us show that $\bar{t} \in (0, T)$. Set

$$\Theta_\lambda := \sup_{(x,t)\in\mathbb{R}^n\times(0,T)} \Psi_\lambda(x,t,x,t) = \sup_{(x,t)\in\mathbb{R}^n\times(0,T)} \left\{ u_\lambda(x,t) - v(x,t) - \frac{\sigma}{T-t} \right\}.$$

From (3.5) it follows that $\Theta_\lambda \geqq 2\theta/5 > 0$. Moreover, since $\Psi_\lambda(Z_{\lambda,\varepsilon}) \geqq \Psi_\lambda(x,t,x,t)$ for any $(x,t) \in \mathbb{R}^n \times (0,T)$, we deduce that $\Psi_\lambda(Z_{\lambda,\varepsilon}) \geqq \Theta_\lambda$. This inequality implies that

$$u_\lambda(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) - v(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) - \frac{\sigma}{T - t_{\lambda,\varepsilon}} \geqq \frac{|x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}|^4}{\varepsilon} + \frac{|t_{\lambda,\varepsilon} - s_{\lambda,\varepsilon}|^2}{\varepsilon} + \Theta_\lambda \geqq \frac{2\theta}{5}.$$

Taking $\limsup_{\varepsilon\to+0}$, we obtain

$$u_\lambda(\bar{x}_\lambda, \bar{t}_\lambda) - v(\bar{x}_\lambda, \bar{t}_\lambda) - \frac{\sigma}{T - \bar{t}_\lambda} \geqq \frac{2\theta}{5},$$

and then sending $\limsup_{\lambda\to1+0}$ yields

$$u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - \frac{\sigma}{T - \bar{t}} \geqq \frac{2\theta}{5}.$$

By this inequality we see that $\bar{t} \neq T$. Moreover, we have $\bar{t} \neq 0$; otherwise the initial conditions on u and v would imply that

$$u(\bar{x}, 0) - v(\bar{x}, 0) - \frac{\sigma}{T} \leqq 0 - \frac{\sigma}{T} < 0,$$

which is a contradiction. Therefore $\bar{t} \in (0, T)$.

When $\lambda \in (1, \lambda_0)$ is sufficiently close to 1, we have $\bar{t}_\lambda \in (\bar{t}/2, T)$. In addition, for every such λ, we have $t_{\lambda,\varepsilon}, s_{\lambda,\varepsilon} \in (\bar{t}/2, T)$ if $\varepsilon \in (0, 1]$ is sufficiently small.

5. Let us define

$$p_{\lambda,\varepsilon} := \nabla_x \phi(Z_{\lambda,\varepsilon}) = -\nabla_y \phi(Z_{\lambda,\varepsilon}) = \frac{4}{\varepsilon}|x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}|^2 (x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} := \phi_t(Z_{\lambda,\varepsilon}) - \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} = -\phi_s(Z_{\lambda,\varepsilon}) = \frac{2}{\varepsilon}(t_{\lambda,\varepsilon} - s_{\lambda,\varepsilon}).$$

For $\gamma := \bar{t}/2$ we apply the assumption that either u or v satisfies the Lipschitz condition (2.7). In either case, we see by (3.10) that $\{p_{\lambda,\varepsilon}\}$ is bounded uniformly in λ and ε. Thus, for a fixed $\lambda \in (1, \lambda_0)$, extracting a subsequence if necessary, we deduce that

$$\lim_{\varepsilon\to+0} p_{\lambda,\varepsilon} = \bar{p}_\lambda$$

for some $\bar{p}_\lambda \in \mathbb{R}^n$. Furthermore, we may assume that there is $\bar{p} \in \mathbb{R}^n$ such that

$$\lim_{\lambda\to1+0} \bar{p}_\lambda = \bar{p}.$$

Fix $\lambda \in (1, \lambda_0)$, and let us divide the situation into two cases.

Case 1: We study the case where there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset (0, 1]$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $p_{\lambda,\varepsilon_k} \neq 0$ for all $k \in \mathbb{N}$. To simplify notation we omit the index k below. We apply Crandall-Ishii's lemma (see, e.g., [10, Theorems 3.2 and 8.3]) for Ψ_λ at $Z_{\lambda,\varepsilon}$. Then there exist $X_{\lambda,\varepsilon}, Y_{\lambda,\varepsilon} \in \mathbb{S}^n$ such that $X_{\lambda,\varepsilon} + Y_{\lambda,\varepsilon} \leqq O$ and

$$\left(p_{\lambda,\varepsilon}, \tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2}, X_{\lambda,\varepsilon} \right) \in \overline{\mathcal{P}}^{2,+} u_\lambda(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}), \quad (p_{\lambda,\varepsilon}, \tau_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) \in \overline{\mathcal{P}}^{2,-} v(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}).$$

Recall that u_λ is a subsolution of (3.1). By Remark 2.6 we then have

$$\tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, \lambda p_{\lambda,\varepsilon}) + F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) \leqq f^*(\lambda x_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} + H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, p_{\lambda,\varepsilon}) + F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) \geqq f_*(y_{\lambda,\varepsilon}).$$

Here, since $p_{\lambda,\varepsilon} \neq 0$, we have used the fact that $F_*(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) = F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon})$ and $F^*(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) = F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon})$. Also, note that $F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) \geqq F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon})$ by (2.1). Hence, subtracting the two inequalities above, we obtain

$$\frac{\sigma}{T^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, \lambda p_{\lambda,\varepsilon}) - H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, p_{\lambda,\varepsilon}) \leqq f^*(\lambda x_{\lambda,\varepsilon}) - f_*(y_{\lambda,\varepsilon}). \tag{3.11}$$

Sending $\limsup_{\varepsilon\to+0}$ gives

$$\frac{\sigma}{T^2} + H(\lambda \bar{x}_\lambda, \lambda^2 \bar{t}_\lambda, \lambda \bar{p}_\lambda) - H(\bar{x}_\lambda, \bar{t}_\lambda, \bar{p}_\lambda) \leqq f^*(\lambda \bar{x}_\lambda) - f_*(\bar{x}_\lambda). \tag{3.12}$$

Case 2: We next consider the case where $p_{\lambda,\varepsilon} = 0$ for all $\varepsilon > 0$ small enough. Let us recall the fact (3.10). For the functions ϕ_1 and ϕ_2 in (3.10), we have

$$\nabla\phi_1(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) = \nabla\phi_2(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = 0, \quad \nabla^2\phi_1(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) = \nabla^2\phi_2(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = O.$$

Thus, the definition of viscosity solutions imply

$$\tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, 0) + F_*(0, O) \leqq f^*(\lambda x_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} + H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, 0) + F^*(0, O) \geqq f_*(y_{\lambda,\varepsilon}).$$

By (2.2), subtracting these two inequalities and letting ε go to 0, we are again led to (3.12) with $\bar{p}_\lambda = 0$.

Recall that σ does not depend on λ. We take $\limsup_{\lambda\to1+0}$ in (3.12) and apply (2.4) to obtain

$$\frac{\sigma}{T^2} + H(\bar{x}, \bar{t}, \bar{p}) - H(\bar{x}, \bar{t}, \bar{p}) \leqq 0,$$

which is a contradiction since $\sigma/T^2 > 0$. □

3.2 Comparison for special Hamiltonians

We establish the other version of a weak comparison principle which is valid for H satisfying (2.8). In this case, we do not need the Lipschitz continuity (2.7).

Theorem 3.2 (Weak comparison principle 2)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be a viscosity subsolution of* (1.1) *satisfying* (2.9)*, and let* $v : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *be a viscosity supersolution of* (1.1) *satisfying* (2.10)*. If* $u^*(\cdot, 0) \leqq v_*(\cdot, 0)$ *in* $\mathbb{R}^n$*, then* $(u^*)_* \leqq v_*$ *and* $u^* \leqq (v_*)^*$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. We only state the difference from the proof of Theorem 3.1. Due to a lack of the Lipschitz continuity (2.7) of u or v, $\{p_{\lambda,\varepsilon}\}$ may not have a convergent subsequence. However, (3.11) and (2.8) give

$$\frac{\sigma}{T^2} \leqq f^*(\lambda x_{\lambda,\varepsilon}) - f_*(y_{\lambda,\varepsilon}).$$

Sending $\varepsilon \to +0$, we deduce that $\sigma/T^2 \leqq f^*(\lambda\bar{x}_\lambda) - f_*(\bar{x}_\lambda)$. Thus, taking $\limsup_{\lambda\to1+0}$, we reach a contradiction by (2.4). □

4 Uniqueness of solutions

From Theorems 3.1 and 3.2, we derive uniqueness results of solutions to (1.1)–(1.2).

4.1 Uniqueness under Theorem 3.1

As an immediate consequence of Theorem 3.1, we see that that Lipschitz continuous solutions are unique in the following sense:

Theorem 4.1 (Uniqueness of Lipschitz continuous solutions)**.** *Assume* (2.1)–(2.4)*. Let* $u \in \mathrm{SOL_{Lip}}$*. If* $v \in \mathrm{SOL}$*, then* $u = v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. Note that u is continuous in $\mathbb{R}^n \times [0,\infty)$. We apply Theorem 3.1 for $u \in \mathrm{SOL_{Lip}} \subset \mathrm{SUB_{Lip}}$ and $v \in \mathrm{SOL} \subset \mathrm{SUP}$ to obtain $u = (u^*)_* \leqq v_*$ in $\mathbb{R}^n \times [0,\infty)$. Next, changing the role of u and v, we deduce that $v^* \leqq (u_*)^* = u$ in $\mathbb{R}^n \times [0,\infty)$. Combining the two inequalities implies that $u = v$ in $\mathbb{R}^n \times [0,\infty)$. □

We next show that solutions given as an envelope of Lipschitz continuous solutions are unique. For this purpose, we introduce

Definition 4.2 (Envelope solution)**.**

$$\mathrm{SOL}^+ = \{u \in \mathrm{SOL} \mid \text{there exists a family } \mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}} \text{ such that } u = \inf_{w\in\mathcal{G}} w\},$$
$$\mathrm{SOL}^- = \{u \in \mathrm{SOL} \mid \text{there exists a family } \mathcal{G} \subset \mathrm{SUB}_{\mathrm{Lip}} \text{ such that } u = \sup_{w\in\mathcal{G}} w\}.$$

We call $u \in \mathrm{SOL}^+$ an *upper envelope solution* and $u \in \mathrm{SOL}^-$ a *lower envelope solution.*

Remark 4.3. We do not require that Lipschitz constants of $w \in \mathcal{G}$ are uniform in the definitions above, and hence upper- and lower envelope solutions may not satisfy (2.7). However, since $\mathrm{SUP}_{\mathrm{Lip}}$ and $\mathrm{SUB}_{\mathrm{Lip}}$ consist of continuous functions in $\mathbb{R}^n \times [0,\infty)$, we see that every upper envelope solution is upper semicontinuous in $\mathbb{R}^n \times [0,\infty)$ and every lower envelope solution is lower semicontinuous in $\mathbb{R}^n \times [0,\infty)$.

The following comparison result immediately follows from Theorem 3.1.

Corollary 4.4. *Assume* (2.1)–(2.4)*.*

(1) *If* $u \in \mathrm{SUB}$ *and* $v \in \mathrm{SOL}^+$*, then* $u^* \leqq v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

(2) *If* $u \in \mathrm{SOL}^-$ *and* $v \in \mathrm{SUP}$*, then* $u \leqq v_*$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

(3) *If* $u \in \mathrm{SOL}^-$ *and* $v \in \mathrm{SOL}^+$*, then* $u^* \leqq v$ *and* $u \leqq v_*$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. (1) Take $\mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}}$ such that $v = \inf_{w\in\mathcal{G}} w$. For any $w \in \mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}}$, Theorem 3.1 implies that $u^* \leqq (w_*)^* = w$ since w is continuous. Thus, taking the infimum, we conclude that $u^* \leqq \inf_{w\in\mathcal{G}} w = v$ in $\mathbb{R}^n \times [0,\infty)$. The proof of (2) is parallel, and (3) is a consequence of (1) and (2). □

Let us prove that upper envelope solutions and lower envelope solutions are unique.

Theorem 4.5 (Uniqueness of envelope solutions)**.** *Assume* (2.1)–(2.4)*. Let* $u^+ \in \mathrm{SOL}^+$ *and* $u^- \in \mathrm{SOL}^-$*.*

(1) *If* $v \in \mathrm{SOL}$*, then* $u^- \leqq v_* \leqq v \leqq v^* \leqq u^+$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

(2) *If* $v \in \mathrm{SOL}^+$*, then* $u^+ = v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

(3) *If* $v \in \mathrm{SOL}^-$*, then* $u^- = v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. (1) Since $v \in \mathrm{SOL} \subset \mathrm{SUB}$ and $u^+ \in \mathrm{SOL}^+$, Corollary 4.4 (1) implies that $v^* \leqq u^+$ in $\mathbb{R}^n \times [0,\infty)$. Similarly, since $u^- \in \mathrm{SOL}^-$ and $v \in \mathrm{SOL} \subset \mathrm{SUP}$, we deduce from Corollary 4.4 (2) that $u^- \leqq v_*$ in $\mathbb{R}^n \times [0,\infty)$.

(2) We have $u^+ \in \mathrm{SOL}^+ \subset \mathrm{SOL} \subset \mathrm{SUB}$ and $v \in \mathrm{SOL}^+$. Thus, noting that u^+ is upper semicontinuous (Remark 4.3), we see that $u^+ = (u^+)^* \leqq v$ in $\mathbb{R}^n \times [0,\infty)$ by Corollary 4.4 (1). The same argument shows that $v \leqq u^+$ in $\mathbb{R}^n \times [0,\infty)$. One can prove (3) in a similar manner. □

Remark 4.6. Let $u^+ \in \mathrm{SOL}^+$ and $u^- \in \mathrm{SOL}^-$. Theorem 4.5 (1) asserts that u^+ and u^- are respectively a *maximal solution* and a *minimal solution.* In this sense, envelope solutions can characterize maximal solutions and minimal solutions. In [22, Section 2] uniqueness and existence of maximal solutions are established for the equation (1.12) with a positive $\nu > 0$.

4.2 Uniqueness under Theorem 3.2

We begin with a result similar to Theorem 4.1 as a consequence of Theorem 3.2. We omit the proof since it is almost the same as before.

Theorem 4.7 (Uniqueness of continuous solutions)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u \in \mathrm{SOL} \cap C(\mathbb{R}^n \times [0,\infty))$*. If* $v \in \mathrm{SOL}$*, then* $u = v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Theorem 3.2 also guarantees that semicontinuous solutions are unique.

Theorem 4.8 (Uniqueness of semicontinuous solutions)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u \in \mathrm{SOL}$*.*

(1) $(u_*)^* = u^*$ *and* $(u^*)_* = u_*$ *in* $\mathbb{R}^n \times [0,\infty)$*. In particular,* $u^*, u_* \in \mathrm{SOL}$*.*

(2) *If* $v \in \mathrm{SOL}$*, then* $u^* = v^*$ *and* $u_* = v_*$ *in* $\mathbb{R}^n \times [0,\infty)$*. In particular, if both* $u, v \in \mathrm{SOL}$ *are upper semicontinuous or lower semicontinuous in* $\mathbb{R}^n \times [0,\infty)$*, then* $u = v$ *in* $\mathbb{R}^n \times [0,\infty)$*.*

Proof. Let $v \in \mathrm{SOL}$. Then Theorem 3.2 yields

$$(u^*)_* \leqq v_*, \quad u^* \leqq (v_*)^*, \quad (v^*)_* \leqq u_*, \quad v^* \leqq (u_*)^* \quad \text{in } \mathbb{R}^n \times [0,\infty). \tag{4.1}$$

By the first and third inequalities, we have

$$(u^*)_* \leqq v_* \leqq (v^*)_* \leqq u_* \leqq (u^*)_* \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

and hence $(u^*)_* = u_* = v_*$ in $\mathbb{R}^n \times [0,\infty)$. Similarly, the second and fourth inequalities in (4.1) imply that $(u_*)^* = u^* = v^*$ in $\mathbb{R}^n \times [0,\infty)$. The former assertions of (1) and (2) are thus proved. The proofs of the latter ones are immediate. □

Remark 4.9. Let $u \in \mathrm{SOL}$. Then the unique upper semicontinuous solution and the unique lower semicontinuous solution are given by u^* and u_*, respectively. Moreover, as a consequence of Theorem 4.8, we see that u^* is a maximal solution and u_* is a minimal solution. Therefore they give another characterization of the maximal and minimal solution (Remark 4.6).

Let us recall that every upper envelope solution $u^+ \in \mathrm{SOL}^+$ is upper semicontinuous (Remark 4.3). Accordingly, we have $u^* = u^+$ since upper semicontinuous solutions are unique. Similaly, if $u^- \in \mathrm{SOL}^-$, then $u_* = u^-$.

5 Existence of solutions

We turn to the issue of existence of solutions.

5.1 Envelope solutions

We discuss construction of upper- and lower envelope solutions, which are unique under the assumptions in Theorem 4.5. To do this, we approximate the source term f by continuous functions f^ε and f_ε such that $f_\varepsilon \leqq f \leqq f^\varepsilon$, and we solve

$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f^\varepsilon(x) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{5.1}$$

$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f_\varepsilon(x) \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{5.2}$$

We define $\mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$ as the set of viscosity solutions u of (5.1)–(1.2) satisfying (2.7), (2.9) and (2.10). In a similar manner, we define $(\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$ by replacing "(5.1)" by "(5.2)" above.

Proposition 5.1. *Assume* (2.1)–(2.4)*. Let* $\{f^\varepsilon\}_{\varepsilon>0}, \{f_\varepsilon\}_{\varepsilon>0} \subset C(\mathbb{R}^n)$ *be sequences such that*

$$f_\varepsilon \leqq f_\delta \leqq f_* \leqq f^* \leqq f^\delta \leqq f^\varepsilon \quad \text{in } \mathbb{R}^n \text{ for } 0 < \delta < \varepsilon, \tag{5.3}$$

$$f^* = \inf_{\varepsilon>0} f^\varepsilon, \;\; f_* = \sup_{\varepsilon>0} f_\varepsilon \quad \text{in } \mathbb{R}^n. \tag{5.4}$$

Assume that $u^\varepsilon \in \mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$*,* $u_\varepsilon \in (\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$ *for* $\varepsilon > 0$ *and that*

$$u_\varepsilon \leqq u_\delta \leqq u^\delta \leqq u^\varepsilon \quad \text{in } \mathbb{R}^n \times [0,\infty) \text{ for } 0 < \delta < \varepsilon. \tag{5.5}$$

Define $u^+ := \inf_{\varepsilon>0} u^\varepsilon$ *and* $u^- := \sup_{\varepsilon>0} u_\varepsilon$*. Then*

(1) $u^+ \in \mathrm{SOL}^+$ *and* $u^- \in \mathrm{SOL}^-$*.*

(2) *If either* u^+ *or* u^- *satisfies* (2.7)*, then* $u^+ = u^-$ *in* $\mathbb{R}^n \times [0,\infty)$ *and* $u^\pm = v$ *in* $\mathbb{R}^n \times [0,\infty)$ *for any* $v \in \mathrm{SOL}$*.*

Proof. (1) We first note that (5.3) implies that $u^\varepsilon \in \mathrm{SUP}_{\mathrm{Lip}}$ and $u_\varepsilon \in \mathrm{SUB}_{\mathrm{Lip}}$ for $\varepsilon > 0$.

Next, by (5.5) we have $u_\varepsilon \leqq u^- \leqq u^+ \leqq u^\varepsilon$ for every $\varepsilon > 0$. This shows that $u^\pm$ satisfy the initial condition (1.2) and the decay conditions (2.9) and (2.10).

Let us prove that u^+ is a viscosity solution of (1.1). To do this, we apply stability results for viscosity solutions ([10, Sections 4 and 6]).

- Since u^ε is a viscosity supersolution of (1.1), the infimum $u^+ = \inf_{\varepsilon>0} u^\varepsilon$ is also a viscosity supersolution of (1.1).
- We next apply stability under the relaxed half limits. From the monotonicity (5.5) and (5.4) it follows that

$$\limsup_{\varepsilon\to+0}{}^* u^\varepsilon = \inf_{\varepsilon>0} u^\varepsilon = u^+, \quad \limsup_{\varepsilon\to+0}{}^* f^\varepsilon = \inf_{\varepsilon>0} f^\varepsilon = f^*.$$

  Since u^ε is a viscosity subsolution of (5.1), the limit u^+ is a viscosity subsolution of (1.1).

We thus conclude that $u^+ \in \mathrm{SOL}^+$. One can prove that $u^- \in \mathrm{SOL}^-$ in the same way.

(2) This follows from Theorem 4.1. □

A technique similar to the above proof can be found in [18, Proposition 3.7] and [22, Theorem 2.2].

Remark 5.2. We comment on the assumptions in Proposition 5.1.

(1) When the usual comparison principle (in the sense of (1.14)) holds for (5.1) and (5.2), it implies the monotonicity (5.5) of solutions.

(2) When the initial datum u_0 is Lipschitz continuous or more regular, there is a chance that the unique solutions of (5.1)–(1.2) and (5.2)–(1.2) preserve the Lipschitz continuity, i.e., $u^\varepsilon \in \mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$ and $u_\varepsilon \in (\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$. See, e.g., [39] for linear and quasi-linear equations, [40, Lemma 7.28] for viscous Hamilton–Jacobi equations and [1, Theorems 8.1 and 8.2] for first order equations. For the equation (1.12) with $\nu > 0$, Lipschitz continuity of solutions is shown in [21, Section 4]. See also [36], [17, Chapter 3.5] and [24, Section 5] for related results.

(3) Let us recall that (2.7) does not require the Lipschitz regularity at the initial time. This implies that, even though the initial datum u_0 is not Lipschitz continuous, Proposition 5.1 can be applied if Lipschitz regularizing effect occurs for (5.1) and (5.2). Here, by Lipschitz regularizing, we mean that the solution $u(x,t)$ immediately gets Lipschitz regularity in x after the initial time. Such Lipschitz regularizing effect occurs for some uniformly parabolic equations and Hamilton–Jacobi equations. See, e.g., [11, 51, 12] for second order equations and [41, 42] for first order equations.

5.2 Semicontinuous solutions

To build semicontinuous solutions, whose uniqueness are guaranteed in Theorem 4.8, we only have to find some solution $u \in \mathrm{SOL}$. Indeed, the semicontinuous envelopes u^* and u_* then give the unique upper semicontinuous solution and the unique lower semicontinuous solution, respectively.

To find a viscosity solution $u \in \mathrm{SOL}$, *Perron's method* ([10, Section 4], [17, Chapter 2.4]) is a well-known and powerful tool; the method can give a solution without approximating f by continuous ones. For Perron's method we need so-called barrier functions. Namely, we need $h^- \in \mathrm{SUB}$ and $h^+ \in \mathrm{SUP}$ such that

$$\begin{cases} (h^-)^* \leqq (h^+)_* \text{ in } \mathbb{R}^n \times [0,\infty), \\ h^\pm(\cdot,0) = u_0 \text{ in } \mathbb{R}^n,\ h^\pm \text{ are continuous on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{5.6}$$

If we define

$$u_P(x,t) := \sup\{w(x,t) \mid w \in \mathrm{SUB} \text{ and } (h^-)^* \leqq w \leqq (h^+)_* \text{ in } \mathbb{R}^n \times [0,\infty)\}, \tag{5.7}$$

then $u_P \in \mathrm{SOL}$.

The remaining problem is the existence of barrier functions. In this paper we do not pursue this issue too far; the reader is refereed to [17, Chapter 4.3] and so on.

We state a simple sufficient condition for existence of the barriers for (1.1).

Proposition 5.3. *Assume that $u_0 \in C^2(\mathbb{R}^n)$ and that both ∇u_0 and $\nabla^2 u_0$ are bounded in $\mathbb{R}^n$. Assume that F is locally bounded in $\mathbb{R}^n \times \mathbb{S}^n$, H is bounded in $\mathbb{R}^n \times [0,\infty) \times B_R(0)$ for every $R > 0$ and that f is bounded in $\mathbb{R}^n$. For $M > 0$ we define*

$$h^-(x,t) = -Mt + u_0(x), \quad h^+(x,t) = Mt + u_0(x).$$

If $M > 0$ is large enough, then $h^- \in \mathrm{SUB}$, $h^+ \in \mathrm{SUP}$ and $h^\pm$ satisfy (5.6).

Proof. We define

$$M := \sup_{(x,t)\in\mathbb{R}^n\times[0,\infty)} |H(x,t,\nabla u_0(x)) + F(\nabla u_0(x), \nabla^2 u_0(x)) - f(x)|,$$

which is finite by assumptions. It then easily seen that h^- and h^+ are respectively a classical subsolution and a classical supersolution of (1.1). Furthermore, h^- satisfies (2.9) and h^+ satisfies (2.10) since $\lim_{|x|\to\infty} u_0(x) = 0$. We thus have $h^- \in \mathrm{SUB}$ and $h^+ \in \mathrm{SUP}$. The condition (5.6) is obvious. □

Remark 5.4. Let $\nu \in \mathbb{R}$ and consider a geometric equation

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{5.8}$$

Then, there exist barrier functions $h_0^- \in \mathrm{SUB}$ and $h_0^+ \in \mathrm{SUP}$ satisfying (5.6). See [17, Chapter 4.3]. Furthermore, if the support

$$\operatorname{supp} u_0 = \overline{\{x \in \mathbb{R}^n \mid u_0(x) \neq 0\}}$$

of the initial datum u_0 is bounded, then $h_0^\pm$ can be chosen so that $\operatorname{supp} h_0^\pm(\cdot,t)$ are bounded for every $t \geqq 0$.

Modifying $h_0^\pm$, one easily obtains barrier functions for (1.12) provided that f is bounded in $\mathbb{R}^n$. In fact, if

$$m_1 := \inf_{\mathbb{R}^n} f > -\infty, \quad m_2 := \sup_{\mathbb{R}^n} f < \infty,$$

then it is easily seen that the functions

$$h^-(x,t) = \min\{m_1,\, 0\}t + h_0^-(x,t), \quad h^+(x,t) = \max\{m_2,\, 0\}t + h_0^+(x,t)$$

are barriers for (1.12).

We next restrict ourselves to (1.12) with a negative driving force $\nu < 0$ and build barrier functions $h^\pm$ whose supports $\operatorname{supp} h^\pm(\cdot,t)$ are bounded uniformly in $t \geqq 0$. As a consequence, we see that the support $\operatorname{supp} u(\cdot,t)$ of any solution $u \in \mathrm{SOL}$ is also bounded uniformly in $t \geqq 0$.

For this purpose, we prepare solutions to the elliptic problem associated to

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = c\chi_{B_R(0)}(x) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{5.9}$$

where $c, R > 0$. We solve the elliptic problem in $B_R(0)$, so that discontinuity of the source term disappears.

Example 5.5. Let $c, R > 0$ and $\nu < 0$. We consider

$$-\nu|\nabla U(x)| - \Delta_1 U(x) = c \quad \text{in } B_R(0) \tag{5.10}$$

under the Dirichlet boundary condition:

$$U(x) = 0 \quad \text{on } \partial B_R(0). \tag{5.11}$$

Here $U : \overline{B_R(0)} \to \mathbb{R}$ is unknown. We now suppose that there is a smooth solution $U(x)$ and that it is radially symmetric $U(x) = \psi(|x|)$. By direct calculations we have

$$\nabla U(x) = \psi'(|x|)\frac{x}{|x|}, \quad \nabla^2 U(x) = \psi''(|x|)\frac{x\otimes x}{|x|^2} + \psi'(|x|)\frac{1}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right) \quad (x \neq 0).$$

Substituting these for (5.10), we find that

$$-\nu|\psi'(r)| - \frac{1}{r}\psi'(r) = c \quad \text{in } (0,R). \tag{5.12}$$

Also, by (5.11) we have

$$\psi(R) = 0.$$

We now assume that $\psi' \leqq 0$. Then, the equation (5.12) gives $\psi'(r) = cr/(\nu r - 1)$ in $(0,R)$, and thus

$$\psi(r) = \int_R^r \frac{cs}{\nu s - 1}\, ds = -\frac{c}{\nu}(R - r) + \frac{c}{\nu^2}\log\frac{-\nu r + 1}{-\nu R + 1} \quad (0 \leqq r \leqq R).$$

Therefore, we conclude that

$$U_{R,c}(x) := U(x) = \psi(|x|) = -\frac{c}{\nu}(R - |x|) + \frac{c}{\nu^2}\log\frac{-\nu|x|+1}{-\nu R+1} \quad (|x| \leqq R). \tag{5.13}$$

See Figure 22 for the graph. One can check that the function U above is a viscosity solution of (5.10)–(5.11). In fact, $U \in C^2(\overline{B_R(0)})$ and U solves (5.10) in the classical sense in $B_R(0) \setminus \{0\}$. At the origin $x = 0$, we have

$$\nabla U(0) = 0, \quad \nabla^2 U(0) = -cI.$$

These facts show that $F_*(0, -cI) = F^*(0, -cI) = -c$, where F is the operator given by (1.4). This implies that U solves (5.10) in the viscosity sense at $x = 0$.

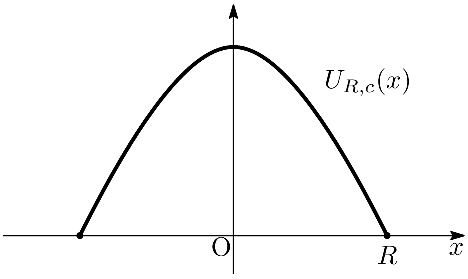


Figure 22: The graph of $U_{R,c}(x)$.

In the following proposition we use notations SOL etc. for the problem (1.12)–(1.2).

Proposition 5.6. *Let $\nu < 0$ and consider* (1.12)*. Assume that both f and u_0 are nonnegative and their supports* $\operatorname{supp} f$ *and* $\operatorname{supp} u_0$ *are bounded in $\mathbb{R}^n$. Let $h_0^\pm$ are barrier functions for* (5.8) *given in Remark 5.4 such that* $\operatorname{supp} h_0^\pm(\cdot, t)$ *are bounded for every $t \geqq 0$. For $c, R > 0$ we define*

$$\tilde{U}_{R,c}(x) := \begin{cases} U_{R,c}(x) & \text{if } |x| \leqq R, \\ 0 & \text{if } |x| > R, \end{cases}$$

where $U_{R,c}$ is the function defined in (5.13)*. We further define*

$$h^-(x,t) = \sup\{(h_0^-)^*(x,t),\, 0\}, \quad h^+(x,t) = \inf\{(h_0^+)_*(x,t),\, \tilde{U}_{R,c}(x)\}.$$

If c, R are chosen so that $f \leqq c\chi_{\overline{B_R(0)}}$ in $\mathbb{R}^n$ and $u_0 \leqq \tilde{U}_{R,c}$ in $\mathbb{R}^n$, then $h^- \in$ SUB, $h^+ \in$ SUP, $h^\pm$ satisfy (5.6) *and* $\operatorname{supp} h^\pm(\cdot, t) \subset \overline{B_R(0)}$ *for all $t \geqq 0$. In particular,* $\operatorname{supp} u(\cdot, t) \subset \overline{B_R(0)}$ *for all $u \in$ SOL and $t \geqq 0$.*

Proof. **1.** We prove that $\tilde{U}_{R,c}$ is a viscosity supersolution of (1.12). First, recall that $U_{R,c}$ is a solution to (5.10). Since $f \leqq c$ in $B_R(0)$ by assumption, we see that $\tilde{U}_{R,c}$ is a viscosity supersolution of (1.12) in $B_R(0) \times (0, \infty)$. Next, it is easily seen that a constant function 0 is a supersolution of (1.12) in $\overline{B_R(0)}^c \times (0, \infty)$.

It remains to prove the viscosity property of $\tilde{U}_{R,c}$ on $\partial B_R(0) \times (0, \infty)$. Assume that $\tilde{U}_{R,c} - \phi$ attains a local minimum at $(x_0, t_0) \in \partial B_R(0) \times (0, \infty)$ for $\phi \in C^{2,1}(\mathbb{R}^n \times (0, \infty))$. Since $\tilde{U}_{R,c}$ is independent of t, we have $\phi_t(x_0, t_0) = 0$. Also, by [22, Lemma A.1 (i)] we see that there is $s \leqq 0$ such that

$$F^*(\nabla\phi(x_0, t_0), \nabla^2\phi(x_0, t_0)) \geqq -\frac{s}{R},$$

where F is the operator defined in (1.4). Accordingly,

$$\phi_t(x_0, t_0) - \nu|\nabla\phi(x_0, t_0)| + F^*(\nabla\phi(x_0, t_0), \nabla^2\phi(x_0, t_0)) \geqq 0 = c\chi_{B_R(0)}(x_0) = f(x_0).$$

We thus conclude that $\tilde{U}_{R,c}$ is a supersolution of (1.12).

2. By the previous step and the stability result for viscosity solutions ([10, Lemma 4.2]), we see that $h^+ \in$ SUP. Similarly, we have $h^- \in$ SUB since the constant 0 is a subsolution of (1.12). Moreover, since $h_0^\pm$ satisfy (5.6) and $0 \leqq u_0 \leqq \tilde{U}_{R,c}$ in $\mathbb{R}^n$, we deduce from Theorem 3.2 that $(h^-)^* \leqq (h^+)_*$ in $\mathbb{R}^n \times [0,\infty)$. The remaining conditions in (5.6) also hold since $(h_0^-)^* \leqq (h^-)^* \leqq (h^+)_* \leqq (h_0^+)_*$ in $\mathbb{R}^n \times [0,\infty)$ and $h_0^\pm$ satisfy (5.6).

We now have

$$0 \leqq h^- \leqq h^+ \leqq \tilde{U}_{R,c} \quad \text{in } \mathbb{R}^n \times [0,\infty)$$

and $\operatorname{supp} \tilde{U}_{R,c} = \overline{B_R(0)}$. This shows that $\operatorname{supp} h^\pm(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$.

3. Let $u_P \in \mathrm{SOL}$ be the solution given by (5.7). By definition we have $\operatorname{supp} u_P(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$. Take any $u \in \mathrm{SOL}$. Then Theorem 4.8 guarantees that $(u_P)_* \leqq u \leqq (u_P)^*$ in $\mathbb{R}^n \times [0,\infty)$, which implies that $\operatorname{supp} u(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$. The proof is complete. □

6 Examples

Let us give some examples of solutions to illustrate our results. Throughout this section we let $\nu, c > 0$. We consider the level-set mean curvature flow equation

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = c\chi_\Omega(x) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{6.1}$$

or the Hamilton–Jacobi equation

$$u_t(x,t) - \nu|\nabla u(x,t)| = c\chi_\Omega(x) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{6.2}$$

with the source term $f = c\chi_\Omega$ given as (1.6). We solve them under the initial condition

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^n. \tag{6.3}$$

We study solutions for several Ω. Solutions of (6.1) and (6.2) are respectively investigated in [22] and [18] in the case of $\nu = 1$. We utilize the results, but we present them for a general $\nu > 0$.

6.1 Discontinuous solutions

Example 6.1. Let

$$\Omega = \overline{B_R(0)} \quad (R > 0),$$

and let us study (6.1) with this Ω. Clearly, (2.6) is fulfilled and hence Theorem 3.1 is applicable to (6.1). In [22, Section 4] the maximal solution of (6.1)–(6.3) is investigated for $\nu = 1$. For a general $\nu > 0$ the unique maximal solution u_R of (6.1)–(6.3) is given as follows:

- If $R < \nu^{-1}$, then $u_R(x,t) = \min\{ct,\, \Psi_R(|x|)\}$. Here
$$\Psi_R(r) = \begin{cases} \dfrac{c}{\nu}\left(r - R + \dfrac{1}{\nu}\log\dfrac{-\nu r + 1}{-\nu R + 1}\right) & \text{if } 0 \leqq r \leqq R, \\ 0 & \text{if } r > R. \end{cases}$$
- If $R = \nu^{-1}$, then $u_{\nu^{-1}}(x,t) = ct\chi_{\overline{B_{\nu^{-1}}(0)}}(x)$. This is a discontinuous solution.
- If $R > \nu^{-1}$, then
$$u_R(x,t) = \begin{cases} ct & \text{if } |x| \leqq R, \\ \max\{ct + \Phi_R(|x|),\, 0\} & \text{if } |x| > R. \end{cases}$$
Here
$$\Phi_R(r) = \frac{c}{\nu}\left(R - r + \nu^{-1}\log\frac{-\nu R + 1}{-\nu r + 1}\right).$$

We discuss in what sense u_R is the unique solution. Let $R \neq \nu^{-1}$. Then it is easily seen that u_R satisfies the Lipschitz continuity (2.7), i.e., $u_R \in \mathrm{SOL}_{\mathrm{Lip}}$, where we use notations $\mathrm{SOL}_{\mathrm{Lip}}$ etc. for the problem (6.1)–(6.3). Thus, u_R is the unique solution in the sense of Theorem 4.1. We next let $R = \nu^{-1}$. Observe that $u_R \in \mathrm{SUP}_{\mathrm{Lip}}$ if $R > \nu^{-1}$, $u_R \in \mathrm{SUB}_{\mathrm{Lip}}$ if $0 < R < \nu^{-1}$, and that

$$u_{\nu^{-1}} = \inf_{R > \nu^{-1}} u_R, \quad (u_{\nu^{-1}})_* = ct\chi_{B_{\nu^{-1}}(0)} = \sup_{0 < R < \nu^{-1}} u_R.$$

We therefore have $u_{\nu^{-1}} \in \mathrm{SOL}^+$ and $(u_{\nu^{-1}})_* \in \mathrm{SOL}^-$. From Theorem 4.5 it follows that $u_{\nu^{-1}}$ and $(u_{\nu^{-1}})_*$ are respectively the unique upper envelope solution and the unique lower envelope solution of (6.1)–(6.3).

6.2 Counter-examples

We give counter-examples to our weak comparison principles when f does not satisfy (2.4).

Example 6.2. Let $n = 2$. Let $\Omega \subset \mathbb{R}^2$ be two touching disks given by

$$\Omega = \overline{B_{\nu^{-1}}((-\nu^{-1}, 0))} \cup \overline{B_{\nu^{-1}}((\nu^{-1}, 0))}.$$

We study (6.1) with this Ω.

We check that (2.4) does not hold for $f = c\chi_\Omega$. Note that $f^* = c\chi_\Omega$ and $f_* = c\chi_{\Omega^\circ}$. By this we see that f is discontinuous at the origin $x = 0 \in \mathbb{R}^2$. Then, letting $x_\lambda = 0$ for all $\lambda > 1$, we have $\lim_{\lambda\to 1+0} x_\lambda = x = 0$ and

$$\limsup_{\lambda\to 1+0}\{f^*(\lambda x_\lambda) - f_*(x_\lambda)\} = \limsup_{\lambda\to 1+0}\{f^*(0) - f_*(0)\} = \limsup_{\lambda\to 1+0}(c - 0) = c > 0.$$

Thus, (2.4) does not hold.

Solutions to (6.1)–(6.3) is studied in [22, Appendix B], where the authors prove that there are at least two different viscosity solutions $u, v \in \mathrm{SOL}$. One solution u is a trivial one given as

$$u(x, t) = ct\chi_\Omega(x).$$

On the other hand, it is shown in [22] that the maximal solution v satisfies

$$\liminf_{t\to\infty} \frac{v(x, t)}{t} \geqq \alpha \quad \text{locally uniformly in } x \in \mathbb{R}^2$$

for some $\alpha > 0$. This implies that neither $(v^*)_* \leqq u_*$ nor $v^* \leqq (u_*)^*$ holds in $\mathbb{R}^n \times (0, \infty)$.

In [31] we study the behavior of the maximal solution v in more detail by applying the game theoretic interpretation of (6.1) ([37]).

Example 6.3. We study the first order equation (6.2). Let

$$\Omega = \{0\}.$$

Since $f^*(0) = c$ and $f_*(0) = 0$, for the same reason as the previous example, the condition (2.4) is not satisfied at $x = 0$. For $\alpha \in \mathbb{R}$ let us set

$$u^\alpha(x, t) = \max\left\{\alpha\left(t - \frac{|x|}{\nu}\right), 0\right\},$$

which belongs to $\mathrm{SOL}_{\mathrm{Lip}}$. Then, as in [18, Examples 2.3, 5.5 and 5.6], u^α is a viscosity solution of (6.2)–(6.3) for every $\alpha \in [0, c]$. In other words, there are infinitely many Lipschitz continuous solutions, and thus Theorem 4.1 fails.

Part III
Asymptotic shape of solutions to the mean curvature flow equation with discontinuous source terms

1 Introduction

Equation We consider the initial value problem for the level-set forced mean curvature flow equation of the form

$$\begin{cases} u_t + \nu|Du| + F(Du, D^2u) = f(x) & \text{in } \mathbb{R}^d \times (0,\infty), & (1.1a) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, & (1.1b) \end{cases}$$

where $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is the unknown function and u_t, $Du = (u_{x_i})_{i=1}^n$ and $D^2u = (u_{x_ix_j})_{i,j=1}^n$ stand for the time derivative, the spatial gradient and Hessian matrix of u respectively. Also $\nu \geq 0$ is a constant, $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded, possibly discontinuous function and $u_0 \in BC(\mathbb{R}^d)$ is a bounded continuous initial datum. The function F is given by

$$F(Du, D^2u) = -\frac{1}{d-1}|Du|\mathrm{div}\left(\frac{Du}{|Du|}\right).$$

Namely F is the level-set mean curvature flow operator defined as

$$F(p,X) = -\frac{1}{d-1}\mathrm{Tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right), \quad p \in \mathbb{R}^d \setminus \{0\},\ X \in \mathbb{S}^d, \tag{1.2}$$

where $p \otimes p = (p_ip_j)_{i,j=1}^n$ for a vector $p = (p_1, \dots, p_n) \in \mathbb{R}^d$ and $\mathbb{S}^d$ is the set of $d \times d$ real symmetric matrices.

In [33] we establish a uniqueness and existence of viscosity solutions to fully nonlinear degenerate parabolic PDEs including (1.1). Based on [33], the goal of this paper is to investigate the asymptotic behavior (large time behavior) of the solution u, which is of the form

$$u(x,t) \sim at + \phi(x) \quad \text{as } t \to \infty. \tag{1.3}$$

Here we call $a \in \mathbb{R}$ the *asymptotic speed* and $\phi(x)$ the *asymptotic shape* of u.

A typical f in our mind is a characteristic function, i.e., we are especially interested in

$$\begin{cases} u_t + \nu|Du| + F(Du, D^2u) = c\chi_\Omega(x) & \text{in } \mathbb{R}^d \times (0,\infty), & (1.4a) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, & (1.4b) \end{cases}$$

where $c > 0$ and χ_Ω is a function such that $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \notin \Omega$ for a given subset $\Omega \subset \mathbb{R}^d$.

Physical background and motivation The equation (1.1) appears in a crystal growth phenomenon. Let $u(x,t)$ represent the height of the crystal at location x and at time t. Assume that the crystal grows at speed $f(x)$ in the vertical direction. The crystal simultaneously grows in the horizontal direction; the horizontal growth speed V of each level set is given by the surface evolution equation

$$V = \kappa - \nu, \tag{1.5}$$

where κ is the mean curvature in the direction of the outer normal vector. In particular, on the convex part of the level set, the speed V is negative, so that the crystal shrinks.

Let us explain the physical background briefly. We consider a perfectly flat surface of a crystal immersed in a supersaturated liquid. In this situation, solute molecules adsorb on the surface. At some time, a large number of molecules present at some site. Then a stable grouping form. This grouping grows and expands across the surface by adding new molecules. Such a phenomenon is called *two-dimensional nucleation* [50]. The case the crystal spread at finite velocity in horizontal direction is especially called

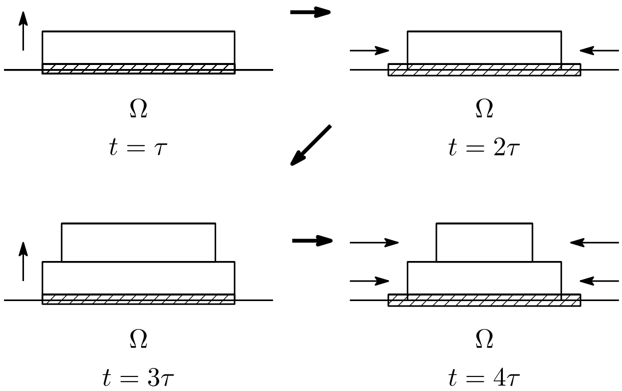


Figure 23: Birth and spread model.

birth and spread model, which includes our problem. The birth and spread model can be heuristically observed by Trotter-Kato product formula. Trotter-Kato product formula is an approximation, in which vertical growth and horizontal growth occur alternately per time $\tau > 0$. See Figure 23 and [22, 21] in detail.

The equation of the type (1.1) or (1.4) was first rigorously analyzed in [22] and has been continued in [23, 21] for a negative ν, where the asymptotic speed of the maximal viscosity solution is studied. In the case $\nu < 0$, if Ω is a ball for instance, the asymptotic speed of the maximal solution depends on a radius of the ball, while in our case with $\nu \geq 0$, the speed is 0 for any radius. (In fact, the speed is 0 for more general bounded sets Ω.) This is one of differences between the two cases $\nu < 0$ and $\nu \geq 0$, and thus our interest lies in the asymptotic shape of solutions rather than the asymptotic speed.

We employ the theory of viscosity solutions to solve (1.1). A notion of viscosity solutions originated from the optimal control theory. It is known that the value function of an optimal control problem or a differential game is characterized as the unique viscosity solution of the corresponding first order equation. See [2]. In this context Kohn and Serfaty [37] present a deterministic game whose value functions u^ϵ ($\epsilon > 0$) approximate the unique viscosity solution u of the level set mean curvature flow equation. We modify their game so that the limit of the value functions gives a solution of our problem (1.1) and use the representation to study the asymptotic behavior of the solution. Such a game-theoretic approach is also valid for the model studied in [22]. Its corresponding game is made by reversing goals of two players in the game for our model.

Results We investigate the asymptotic behavior of solutions to (1.4) under a geometrical assumption on Ω that makes our situation simple. We study the case where the normal velocity V is negative on $\partial\Omega$, so that the solution is 0 on $\partial\Omega$ and grows only in Ω as in Figure 23. Moreover, in this case the solution seems to become stable in a finite time. Since V is now given by (1.5), the condition $V < 0$ on $\partial\Omega$ is written as

$$\kappa_{\partial\Omega}(x) < \nu^{-1} \quad \text{for all } x \in \partial\Omega, \tag{1.6}$$

where $\kappa_{\partial\Omega}(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$. Our geometrical condition on Ω is a generalized version of (1.6). It allows concave parts of $\partial\Omega$. Especially a convex Ω always satisfies this assumption.

Due to the above considerations, it is reasonable to consider the stationary problem:

$$\begin{cases} \nu|DU(x)| + F(DU(x), D^2U(x)) = c & \text{in } \Omega, \quad (1.7a) \\ U(x) = 0 & \text{on } \partial\Omega \quad (1.7b) \end{cases}$$

for the unknown function $U : \overline{\Omega} \to \mathbb{R}$. It is then expected that the solution of the Dirichlet problem (1.7) gives the asymptotic shape of solutions to (1.4). We prove that this expectation is in fact true. Our assumption on Ω plays a crucial role to construct a viscosity solution of (1.7). More precisely the assumption guarantees the existence of barrier functions, a subsolution and a supersolution satisfying the boundary condition of (1.7) for all $x \in \partial\Omega$. Such barriers are needed to apply Perron's method. We also establish a comparison principle for (1.7) with a general positive function $g \in C(\Omega)$ on the right-hand side of (1.7a). The proof is similar to that for eikonal equations [34] since the left-hand side of the equation (1.7a) is homogeneous. This comparison result shows that a unique viscosity solution is continuous.

Using the unique solution U of (1.7), we prove that the function

$$u(x,t) = \begin{cases} \min\{U(x),\, ct\}, & x \in \Omega, \\ 0, & x \in \Omega \end{cases} \tag{1.8}$$

is a viscosity solution of (1.1) with $u_0 = 0$. Since this u is continuous, our uniqueness result ([33, Theorem 4.7]) implies that u is a unique viscosity solution of (1.4). In particular this representation implies that $u(x,t) = U(x)$ for t large enough. Thus u becomes the asymptotic shape U in a finite time. We furthermore prove that, even if u_0 is allowed to be nonnegative in Ω, the solution v of (1.4) satisfies $v(x,t) = U(x)$ for t large enough. This is shown by applying the weak comparison principle. We estimate v by the solution u in (1.8) from below and by the solution w of (1.4a) with $w(x,0) = (\sup u_0)\chi_\Omega(x)$ from above. Since both u and w have the same asymptotic shape U, it turns out that v also has it.

We turn to the game-theoretic approach. First let us briefly explain the game rule for (1.1) with $f \equiv 0$ by following [37, Example 2]. Let $d = 2$. There are two players, Paul and Carol. Let $\epsilon > 0$. Also let $x_0 = x \in \mathbb{R}^2$ be the initial position of this game and $t > 0$ be the terminal time. At the i-th round of this game, Paul chooses directions $v_i, w_i \in \mathbb{R}^2$ with $|v_i| = |w_i| = 1$ and Carol chooses a number $b_i = \pm 1$. Then the game position, that we henceforth call *Paul's position* conveniently, moves from x_{i-1} to the next place x_i depending on their choice. After the N-th round, where $N \sim t\epsilon^{-2}$, the game ends and Paul pays the terminal cost $u_0(x_N)$ to Carol. Paul's goal is minimizing the cost while Carol's goal is maximizing it. The value function $u^\epsilon(x,t)$ is defined as the cost optimized by both the players, that is,

$$u^\epsilon(x,t) = \inf_{v_1,w_1} \max_{b_1} \ldots \inf_{v_N,w_N} \max_{b_N} u_0(x_N).$$

This value function approximates the viscosity solution u of (1.1) with $f \equiv 0$. In fact the convergence $u^\epsilon \to u$ is shown in [37].

In order to handle (1.1) with a discontinuous source f, we modify the game rule as follows. At each i-th round, we suppose that Paul has to pay the running cost $\epsilon^2 f(x_i)$ depending on the current place. Namely the resulting value function is

$$u^\epsilon(x,t) = \inf_{v_1,w_1} \max_{b_1} \ldots \inf_{v_N,w_N} \max_{b_N} \left[u_0(x_N) + \sum_{i=0}^{N-1} \epsilon^2 f(x_i) \right].$$

Such an interpretation of source terms (inhomogeneous terms) of PDEs as the running cost is well understood for first order equations; see [2]. We refer the reader to [49] for the proof of the convergence. We also point out that, if the goals of two players are reversed, then the value function approximates solutions of (1.1) with $\nu < 0$.

Applying the game above, we consider the problem given in [22, Appendix B], where $\Omega = \overline{B}((-1,0),1) \cup \overline{B}((1,0),1)$(Figure 28) in (1.4) with $\nu = -1$ and $d = 2$. In this setting, we construct a viscosity solution by the game, which is different from the explicit solution $c\chi_\Omega(x)$. Moreover we improve the result of the asymptotic speed of a nontrivial solution in [22]. They prove that the asymptotic speed $\lim_{t\to\infty} u(x,t)/t$ is strictly greater than 0 at any point in $\mathbb{R}^2$. We improve this result to that the asymptotic speed equals to c. Roughly speaking on our proof, whatever decision Carol makes, Paul is able to enter the set Ω after several rounds and remain there until the game ends. Since the running cost has a positive value c only in Ω, this gives an lower estimate of the value function. As a result, this estimate and Perron's method imply that there is a viscosity solution of (1.4) whose asymptotic speed is c. We also point out that there are at least two viscosity solutions for the problem (1.4) with $\nu > 0$ and $\Omega = B((0,0),R_0) \setminus (\overline{B((1,0),1) \cup B((-1,0),1)})$ (Figure 36). This statement is proved in a similar way as the former problem.

Literature overview. A rigorous treatment of the level-set mean curvature flow equation:

$$u_t(x,t) + F(Du(x,t), D^2u(x,t)) = 0 \tag{1.9}$$

was first established by [8] and [13] independently. They proved a uniqueness and existence of viscosity solutions. For the equation (1.1) with a discontinuous source term, the asymptotic speed of the maximal viscosity solution is studied in [22] when $\nu < 0$. When the source term f is continuous, some further results are obtained in [21, 23]. The asymptotic shape is studied in [23] for a radially symmetric source term. In [21] the asymptotic speed is investigated when F is a general degenerate elliptic operator.

Concerning nonlinear PDEs with discontinuous source terms, the paper [18] studies Hamilton-Jacobi equations:

$$u_t(x,t) + H(x, Du(x,t)) = f(x)$$

with a discontinuous f. In [18] a notion of envelope solutions is introduced and the unique existence of envelope solutions is established. When H is a so-called Bellman type operator, a representation formula of the solution is derived by considering the corresponding optimal control problem. The large time behavior of the envelope solution is studied in [27].

As we have already explained, a game theoretic representation of the solution to (1.9) was first established in [37]. The representation is also used to investigate geometric properties of solutions. For instance, a game theoretic proof for fattening phenomenon is given in [43]. Also, by studying game strategies, it is shown in [44] that the solution of (1.9) preserves convexity. As an extension of [37], the paper [30] gives a game theoretic representation of the solution to (1.9) with nonlinear dynamic boundary conditions. For general elliptic and parabolic equations, a game is given by [38]. Furthermore, for fully nonlinear parabolic equations with dynamic boundary conditions, the paper [29] gives a game.

Organization This paper is organized as follows. Section 2 contains the definition of viscosity solutions to (1.1) and a basic property of them. Section 2 also remarks the existence of solutions to (1.1). Section 3 deals with the stationary problem (1.7). Based on the results in Section 2 and 3, we find the asymptotic behavior of solutions to (1.1) in Section 4. In Section 5 we give a game interpretation to (1.1), state the convergence of the value functions, and introduce the notions of strategy of the game. In Section 6 we investigate asymptotic speed of solutions for domain of touching balls by applying the properties of game trajectories. By using similar technique, we also give an example of Ω where uniqueness of solutions fails even in the meaning of semicontinuous envelopes.

Throughout this paper, we basically assume that $\nu \geq 0$ and F is given by (1.2). Meanwhile Section 6.2 specifically deals with the case where ν is negative. Also some of our results are valid for a negative ν and more general F with suitable structure conditions. We will mention such possible generalization as remarks.

2 Preliminary results

2.1 Definition of viscosity solutions

We define a notion of viscosity solutions and give a basic property of them. We state them only for parabolic equations (1.1); those for elliptic equations are given in a similar way, so are omitted. The reader is referred to [10, 17] for notions and results presented in this subsection.

We first introduce a notion of semicontinuous envelopes.

Definition 2.1 (Semicontinuous envelopes)**.** Let $K \subset \mathbb{R}^d$ be a subset. For $h : K \to \mathbb{R}$, we define the *semicontinuous envelopes* $h^* : \overline{K} \to \mathbb{R} \cup \{\infty\}$ and $h_* : \overline{K} \to \mathbb{R} \cup \{-\infty\}$ by

$$h^*(x) = \lim_{\epsilon \searrow 0} \sup\{h(y) \mid y \in B(x,\epsilon) \cap K\},$$

$$h_*(x) = \lim_{\epsilon \searrow 0} \inf\{h(y) \mid y \in B(x,\epsilon) \cap K\} \quad (x \in \overline{K}),$$

where $B(x,\epsilon) := \{y \in \mathbb{R}^d \mid |x-y| < \epsilon\}$.

Though the level-set mean curvature flow operator (1.2) is not defined at $p = 0$, the semicontinuous envelopes $F_*(p,X)$ and $F^*(p,X)$ are defined for all $(p,X) \in \mathbb{R}^d \times \mathbb{S}^d$. These extensions are used in the definitions of viscosity solutions.

Definition 2.2 (Viscosity solution)**.** 1. A function $u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ is called a *viscosity subsolution* (resp. *viscosity supersolution*) *of* (1.1a) if it satisfies

(a) $u^* < \infty$ (resp. $u_* > -\infty$) in $\mathbb{R}^d \times (0,\infty)$.

(b) Whenever $\phi(x,t)$ is smooth and $u^* - \phi$ has a local maximum (resp. local minimum) at $(x_0,t_0) \in \mathbb{R}^d \times (0,\infty)$, we have

$$\phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F_*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \leq f^*(x_0).$$
$$(\text{resp. } \phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \geq f_*(x_0).)$$

2. A function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is called a *viscosity subsolution* (resp. *viscosity supersolution*) *of the initial value problem* (1.1) if it is a viscosity subsolution of (1.1a) and $u^*(\cdot, 0) \leq u_0$ (resp. $u_*(\cdot, 0) \geq u_0$) in $\mathbb{R}^d$.

3. A function u is called a *viscosity solution* if it is a viscosity subsolution and a viscosity supersolution.

Remark 2.3. Due to the ellipticity of F, it is easily seen that a classical subsolution (resp. supersolution) is also a viscosity subsolution (resp. supersolution).

The term "viscosity" is often omitted, and we just say "solution" (or "subsolution", "supersolution") in this paper. The following is a stability property of viscosity solutions ([17, Lemma 2.4.1]).

Proposition 2.4 (Stability). *Let $\mathcal{T}$ be a family of subsolutions (resp. supersolutions) of* (1.1a). *Define*

$$u(x,t) = \sup\{v(x,t) \mid v \in \mathcal{T}\} \quad (\text{resp. } u(x,t) = \inf\{v(x,t) \mid v \in \mathcal{T}\}).$$

If $u^* < \infty$ (resp. $u_* > -\infty$) in $\mathbb{R}^d \times (0, \infty)$, then u is a subsolution (resp. supersolution) of (1.1a).

2.2 Weak comparison principle and uniqueness

In this subsection we present several results on the uniqueness of solutions for readers convenience.

The following comparison result applies to a sub- and supersolution of (1.1a) with different source terms. The proof is almost the same as that of [22, Proposition 2.1], so is omitted.

Proposition 2.5 (Comparison principle). *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a bounded function such that $f^* \leq g_*$ in $\mathbb{R}^d$. Let u be a subsolution of* (1.1), *and v be a supersolution of* (1.1) *with g instead of f on the right-hand side of* (1.1a). *Assume that, for every $T > 0$, there exists some $R > 0$ such that $\sup_{B(0,R)^c \times [0,T]}(u^* - v_*) \leq 0$. Then $u^* \leq v_*$ in $\mathbb{R}^d \times [0, \infty)$.*

In [33] we give comparison principles with the same source term. A key assumption on the source term f is

$$\begin{cases} \text{For any discontinuous point } x \in \mathbb{R}^d \text{ of } f \text{ and any sequence } \{x_\lambda\}_{\lambda>1} \subset \mathbb{R}^d \\ \text{such that } \lim\limits_{\lambda \to 1+0} x_\lambda = x, \text{ we have } \limsup\limits_{\lambda \to 1+0} \{f^*(\lambda x_\lambda) - f_*(x_\lambda)\} \leqq 0. \end{cases} \tag{2.1}$$

A sufficient condition for (2.1) is the following:

$$f^*(\lambda x) \leqq f_*(x) \text{ for all } x \in \mathbb{R}^d \text{ and } \lambda > 1. \tag{2.2}$$

When f is a characteristic function $c\chi_\Omega$ $(c > 0)$ as in (1.4), the condition (2.2) holds if and only if Ω is star-shaped with respect to the origin, that is,

$$\overline{\Omega} \subset \lambda \Omega^\circ \text{ for all } \lambda > 1, \tag{2.3}$$

where Ω° is the interior of Ω and $\lambda\Omega^\circ = \{\lambda x \mid x \in \Omega^\circ\}$.

We also assume decay conditions on a subsolution u and a supersolution v as follows:

$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } u(x,t) \leqq \delta \text{ in } B_R(0)^c \times [0,T], \tag{2.4}$$

$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } v(x,t) \geqq -\delta \text{ in } B_R(0)^c \times [0,T]. \tag{2.5}$$

Based on these assumptions, it is natural to define a class of viscosity solutions as follows:

$$\begin{aligned} &\text{SUB} := \{u \mid u \text{ is a viscosity subsolution of (1.1a)–(1.1b) and satisfies (2.4)}\} \\ &\text{SUP} := \{u \mid u \text{ is a viscosity supersolution of (1.1a)–(1.1b) and satisfies (2.5)}\} \\ &\text{SOL} := \text{SUB} \cap \text{SUP} \end{aligned}$$

Theorem 2.6 (Comparison principle [33]). *Assume* (2.1). *Let $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be a viscosity subsolution of* (1.1a) *satisfying* (2.4), *and let $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be a viscosity supersolution of* (1.1a) *satisfying* (2.5). *If $u^*(\cdot, 0) \leqq v_*(\cdot, 0)$ in $\mathbb{R}^d$, then $(u^*)_* \leqq v_*$ and $u^* \leqq (v_*)^*$ in $\mathbb{R}^d \times [0, \infty)$.*

As a consequence of Theorem 2.6, we give the following uniqueness results.

Corollary 2.7 (Uniqueness of continuous solutions)**.** *Assume* (2.1)*. Let* $u \in \mathrm{SOL} \cap C(\mathbb{R}^n \times [0,\infty))$*. If* $v \in \mathrm{SOL}$*, then* $u = v$ *in* $\mathbb{R}^d \times [0,\infty)$*.*

Corollary 2.8 (Uniqueness of semicontinuous solutions)**.** *Assume* (2.1)*. Let* $u \in \mathrm{SOL}$*.*

(1) $(u_*)^* = u^*$ *and* $(u^*)_* = u_*$ *in* $\mathbb{R}^d \times [0,\infty)$*. In particular,* $u^*, u_* \in \mathrm{SOL}$*.*

(2) *If* $v \in \mathrm{SOL}$*, then* $u^* = v^*$ *and* $u_* = v_*$ *in* $\mathbb{R}^d \times [0,\infty)$*. In particular, if both* $u, v \in \mathrm{SOL}$ *are upper semicontinuous or lower semicontinuous in* $\mathbb{R}^d \times [0,\infty)$*, then* $u = v$ *in* $\mathbb{R}^d \times [0,\infty)$*.*

Remark 2.9. In comparison principles here, the level-set mean curvature flow operator (1.2) can be generalized to F satisfying

$$F \in C((\mathbb{R}^d \setminus \{0\}) \times \mathbb{S}^d), \tag{2.6}$$

$$F(p, X) \le F(p, Y) \text{ for all } p \in \mathbb{R}^d \setminus \{0\} \text{ and } X, Y \in \mathbb{S}^d \text{ such that } X \ge Y, \tag{2.7}$$

$$-\infty < F_*(0, O) = F^*(0, O) < \infty, \tag{2.8}$$

$$F(rp, X) = F(p, X) \text{ for all } (p, X) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{S}^d \text{ and } r > 0.$$

Remark 2.10. In this manuscript, we give two examples of Ω such that there is a discontinuous viscosity solution and hence the usual comparison principle fails. One is Proposition 4.8 in Section 4. The other is in Section 6.3.

2.3 Sign of the driving force

It is worth emphasizing that (1.1) is convertible to the following problem:

$$\begin{cases} v_t - \nu|Dv| + F(Dv, D^2 v) = c - f(x) & \text{in } \mathbb{R}^d \times (0,\infty), \\ v(x, 0) = -u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{2.9}$$

Proposition 2.11. *The following are equivalent for functions* u *and* v *that satisfy* $v(x,t) = ct - u(x,t)$*.*

1. *u is a subsolution (resp. supersolution) to* (1.1)*.*
2. *v is a supersolution (resp. subsolution) to* (2.9)*.*

Proof. Assume that u and v are smooth functions that satisfy $v(x,t) = ct - u(x,t)$. We only show that if u is a subsolution to (1.1), then v is a supersolution to (2.9) because the proofs of the other statements are similar. If u is a subsolution to (1.1), we have by a direct computation that

$$\begin{aligned} f(x) \ge u_t + \nu|Du| + F(Du, D^2 u) &\ge c - v_t + \nu|-Dv| + F(-Dv, -D^2 v) \\ &\ge c - v_t + \nu|Dv| - F(Dv, D^2 v), \end{aligned}$$

because $F(-p, -X) = -F(p, X)$. Hence we obtain $v_t - \nu|Dv| + F(Dv, D^2 v) \ge c - f(x)$. When v is not smooth, we carry out the same computation for test functions. □

2.4 Examples

We give some examples of solutions.

Example 2.12. Let us study (1.4) with $\Omega = B(0, R)$ for $R > 0$. For this purpose, we first consider the elliptic problem (1.7) with $\Omega = B(0, R)$. In this case, solutions to (1.7) are unique as is shown in Corollary 3.2 for a more general Ω, and the unique solution $U = U_{R,c}$ of (1.7) is explicitly obtained as follows. Though the construction of the unique solution $U = U_{R,c}$ of (1.7) is explained in [33, Example 5.5], we will repeat it here.

Let us suppose that $U_{R,c}$ is radially symmetric. Namely $U_{R,c}(x) = \psi(|x|)$ for some $\psi = \psi(r)$. Computing the derivatives of $U_{R,c}(x) = \psi(|x|)$, we have

$$DU_{R,c}(x) = \psi_r(|x|)\frac{x}{|x|}, \quad D^2 U_{R,c}(x) = \psi_{rr}(|x|)\frac{x \otimes x}{|x|^2} + \psi_r(|x|)\frac{1}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right).$$

Substituting these for (1.7), we get

$$\begin{cases} \nu|\psi_r(r)| - \dfrac{1}{r}\psi_r(r) = c, & 0 < r < R, \\ \psi(R) = 0. \end{cases} \tag{2.10}$$

We now assume that $\psi_r \leq 0$. Then the equation of (2.10) gives $\psi_r(r) = \frac{cr}{-\nu r - 1}$, and thus

$$\psi(r) = \int_R^r \frac{cs}{-\nu s - 1}\, ds = \frac{c}{\nu}(R - r) + \frac{c}{\nu^2}\log\frac{\nu r + 1}{\nu R + 1}. \tag{2.11}$$

Therefore we have

$$U_{R,c}(x) = \frac{c}{\nu}(R - |x|) + \frac{c}{\nu^2}\log\frac{\nu|x| + 1}{\nu R + 1}. \tag{2.12}$$

This function $U_{R,c}$ is a solution of (1.7). Indeed following the above discussion in reverse, we see that (2.12) is a solution in $\Omega \setminus \{0\}$. Moreover (2.12) is twice differentiable at $x = 0$ and

$$DU_{R,c}(0) = 0, \quad D^2U_{R,c}(0) = -cI.$$

Thus the viscosity inequalities hold at $x = 0$, since

$$F^*(0, -cI) = F_*(0, -cI) = c.$$

Consequently (2.12) is the unique viscosity solution of (1.7).

Let us come back to the parabolic problem (1.4). Define $u_{R,c}(x,t)$ by

$$u_{R,c}(x,t) := \begin{cases} \min\{U_{R,c}(x),\, ct\}, & x \in B(0,R), \\ 0, & x \in B(0,R)^c. \end{cases} \tag{2.13}$$

In Theorem 4.6 we prove that this $u_{R,c}$ is a solution of (1.4) with $u_0 \equiv 0$. In particular we have

$$\lim_{t\to\infty} \frac{u_{R,c}(x,t)}{t} = 0 \quad \text{uniformly in } x \in \mathbb{R}^d, \tag{2.14}$$

that is, the asymptotic speed is 0 for any $R > 0$.

More generally, for $h \geq 0$ the following $u^h_{R,c}$ is a supersolution of (1.4).

$$u^h_{R,h}(x,t) := \begin{cases} U_{R,c}(x), & x \in B(0,R) \text{ and } U_{R,c}(x) \leq ct, \\ h + ct, & x \in B(0,R) \text{ and } U_{R,c}(x) > ct, \\ 0, & x \in B(0,R)^c. \end{cases} \tag{2.15}$$

This fact is proved in the proof of Theorem 4.7. Note that $u^h_{R,h}(\cdot, 0) = h\chi_{B(0,R)}$ in $\mathbb{R}^d$.

When Ω is a ball and $\nu < 0$, explicit solutions are obtained in [22, Section 4]. Converting their results to our case $\nu > 0$ by Proposition 2.11, we obtain the following results for the case where Ω is the complement of a ball.

Proposition 2.13. *When $\Omega = B(0,R)^c$ for $R > 0$, the following assertions hold for the equation* (1.4) *with $u_0 = 0$ and $\nu > 0$.*

(1) When $R < \nu^{-1}$, the function

$$u(x,t) = \begin{cases} \max\{ct - \Psi_R(|x|), 0\} & \text{if } |x| \leqq R, \\ ct & \text{if } |x| > R. \end{cases}$$

is a solution, where

$$\Psi_R(r) = \frac{c}{\nu}\left(r - R + \frac{1}{\nu}\log\frac{-\nu r + 1}{-\nu R + 1}\right).$$

(2) When $R = \nu^{-1}$, the function $u(x,t) = ct\chi_\Omega(x)$ is a solution.

(3) When $R > \nu^{-1}$, the function

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \leqq R, \\ \min\{\Psi_R(|x|),\, ct\} & \text{if } |x| > R. \end{cases}$$

is a solution.

Unlike (2.14) for the case where Ω is a ball, the asymptotic speed $\lim_{t\to\infty} \frac{u(x,t)}{t}$ of solutions u in Proposition 2.13 depends on the radius R. Indeed it is c in (1) and 0 in (3) at any point $x \in \mathbb{R}^d$. In the critical case (2), the speed is c in $B(0,R)^c$ and 0 in $B(0,R)$.

2.5 Existence result

Perron's method ([17, Theorem 2.4.3]) is a basic tool to prove the existence of solutions.

Proposition 2.14 (Perron's method)**.** *Let h^- and h^+ be a sub- and supersolution of* (1.1) *such that $h^- \le h^+$ in $\mathbb{R}^d \times [0,\infty)$. Define*

$$u(x,t) := \\ \sup\{w(x,t) \mid w \text{ is a subsolution of (1.1) such that } h_- \le w \le h_+ \text{ in } \mathbb{R}^d \times [0,\infty)\}.$$

Then u is a solution of (1.1a)*. Furthermore, if $h^\pm$ satisfies*

$$h^\pm(\cdot,0) = u_0 \text{ in } \mathbb{R}^d, \quad h^\pm \text{ are continuous on } \mathbb{R}^d \times \{0\}, \tag{2.16}$$

then u is a solution of (1.1)*.*

Constructing the barrier functions $h_\pm$ in this proposition, we prove the existence of solutions to (1.1).

Theorem 2.15 (Existence)**.** *Assume that u_0 is uniformly continuous in $\mathbb{R}^d$. Then there exists a solution u of* (1.1)*. Moreover, if we further assume that both f and u_0 are nonnegative and their supports are bounded in $\mathbb{R}^d$, then, there exists $R > 0$ such that $\operatorname{supp} u(\cdot,t) \subset \overline{B_R(0)}$ for all $u \in \mathrm{SOL}$ and $t \geqq 0$.*

Proof. It is well-known that there are an upper semicontinuous subsolution v^- and a lower semicontinuous supersolution v^+ of (1.1) with $f \equiv 0$, which is the standard level-set mean curvature flow equation, such that $v^- \le u_0 \le v^+$ in $\mathbb{R}^d \times [0,\infty)$ and that (2.16) holds for $v^\pm$. See, e.g., [19, Section 4.2]. Let us set

$$c_1 = \inf_{\mathbb{R}^d} f, \quad c_2 = \sup_{\mathbb{R}^d} f, \tag{2.17}$$

and define

$$V^-(x,t) := v^-(x,t) + \min\{c_1,\, 0\}t, \quad V^+(x,t) := v^+(x,t) + \max\{c_2,\, 0\}t.$$

Then V^- and V^+ are respectively a subsolution and a supersolution of (1.1) such that

$$V^- \le u_0 \le V^+ \text{ in } \mathbb{R}^d \times [0,\infty), \tag{2.18}$$

and moreover they satisfy (2.16). Thus, adopting $V^\pm$ as barrier functions in Proposition 2.14, we deduce the existence of a viscosity solution of (1.1).

The latter part of this theorem is proved in [33, Proposition 5.6], so its proof is omitted.

□

3 Stationary problem

We study uniqueness and existence of solutions to the stationary problem (1.7), where $\Omega \subsetneq \mathbb{R}^d$ is an open set. The boundary condition of (1.7) is interpreted not in the viscosity sense but in the classical sense. Namely we say that U is a subsolution (resp. supersolution) of (1.7) if it is a subsolution (resp. supersolution) of (1.7a) and $U^*(x) \le 0$ (resp. $U_*(x) \ge 0$) for all $x \in \partial\Omega$.

3.1 Comparison principle

We give the comparison result for (1.7a) with a general positive function $g(x)$ instead of $c > 0$ on the right-hand side. Moreover F can also be generalized. In fact it suffices to assume (2.6), (2.7), (2.8) and homogeneity. Namely

$$F(tp, tX) = tF(p, X) \text{ for all } (p, X) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{S}^d \text{ and } t > 0. \tag{3.1}$$

Also the constant ν can be negative. In the proof of Proposition 3.1, we employ the technique introduced in [34] for eikonal equations.

Proposition 3.1 (Comparison principle)**.** *Assume that Ω is bounded. Let U and V be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of* (1.7a) *with a positive function $g \in C(\Omega)$ on the right-hand side. If $U \le V$ on $\partial\Omega$, then $U \le V$ in $\overline{\Omega}$.*

Proof. Suppose by contradiction that $\sup_{\overline{\Omega}}(U - V) = 2\theta > 0$. Let $W = \lambda U$ for $\lambda \in (0, 1)$. It then follows from (3.1) that W is a subsolution of

$$\nu|DW| + F(DW, D^2W) = \lambda g(x) \quad \text{in } \Omega.$$

We also have

$$W(x) - V(x) = U(x) - V(x) - (1-\lambda)U(x) \ge U(x) - V(x) - (1-\lambda)\sup_{\overline{\Omega}} U \quad (x \in \overline{\Omega}).$$

Thus letting λ be close to 1, we get $\sup_{\overline{\Omega}}(W - V) \ge \theta > 0$.

For $\epsilon > 0$ we set

$$\Psi(x, y) = W(x) - V(y) - \phi(x, y), \quad \phi(x, y) = \frac{|x-y|^4}{\epsilon}.$$

Let $(x_\epsilon, y_\epsilon) \in \overline{\Omega} \times \overline{\Omega}$ be a maximizer of Ψ. Since $\overline{\Omega}$ is bounded, we may assume that $\lim_{\epsilon \searrow 0}(x_\epsilon, y_\epsilon) = (\hat{x}, \hat{x})$ for some $\hat{x} \in \overline{\Omega}$. Then the standard argument shows that

$$\lim_{\epsilon \searrow 0}(W(x_\epsilon), V(y_\epsilon)) = (W(\hat{x}), V(\hat{x})), \quad (W - V)(\hat{x}) \ge \theta.$$

If $\hat{x} \in \partial\Omega$, the boundary condition implies

$$0 < \theta \le (W - V)(\hat{x}) = (\lambda U - V)(\hat{x}) \le (\lambda - 1)V(\hat{x}) \le (\lambda - 1)\inf_{\partial\Omega} V.$$

This is a contradiction for λ sufficiently close to 1. Accordingly we have $\hat{x} \in \Omega$ and $(x_\epsilon, y_\epsilon) \in \Omega \times \Omega$ for sufficiently small ϵ.

Let $p_\epsilon = D_x\phi(x_\epsilon, y_\epsilon)$. Assume first that there is a sub sequence $\{p_{\epsilon_n}\}$ such that $p_{\epsilon_n} \ne 0$ for all n. We write $\{p_\epsilon\}$ for $\{p_{\epsilon_n}\}$. By Crandall-Ishii's lemma [10, Theorems 3.2], there exist $X, Y \in \mathbb{S}^d$ satisfying $X + Y \le O$ and

$$(p_\epsilon, X) \in \overline{J}^{2,+}W(x_\epsilon), \quad (p_\epsilon, -Y) \in \overline{J}^{2,-}V(y_\epsilon),$$

where $\overline{J}^{2,\pm}$ denotes the extended semijets. (See [10].) Since W and V are respectively a subsolution and a supersolution, we have

$$\nu|p_\epsilon| + F(p_\epsilon, X) \le \lambda g(x_\epsilon), \quad \nu|p_\epsilon| + F(p_\epsilon, -Y) \ge g(y_\epsilon).$$

Subtracting these inequalities yields

$$F(p_\epsilon, X) - F(p_\epsilon, -Y) \le \lambda g(x_\epsilon) - g(y_\epsilon).$$

By (2.7) the left-hand side is nonnegative. Therefore, letting $\epsilon \to 0$, we obtain $0 \le \lambda g(\hat{x}) - g(\hat{x})$. This is a contradiction since $g(\hat{x}) > 0$.

If $p_\epsilon = 0$ for sufficiently small $\epsilon > 0$, we are again led to a contradiction more directly. We apply the test function $V(y_\epsilon) - \phi(\cdot, y_\epsilon)$(resp. $W(x_\epsilon) - \phi(x_\epsilon, \cdot)$) that touches W at $x = x_\epsilon$ from above. (resp. V at $y = y_\epsilon$ from below.) Since $p_\epsilon = D_x\phi(x_\epsilon, y_\epsilon) = D_y\phi(x_\epsilon, y_\epsilon) = 0$ and $D_x^2\phi(x_\epsilon, y_\epsilon) = D_y^2\phi(x_\epsilon, y_\epsilon) = O$, we indeed deduce a contradiction by using (2.8). □

From Proposition 3.1 we deduce

Corollary 3.2 (Uniqueness)**.** *Assume that Ω is bounded. Then there exists at most one viscosity solution of* (1.7)*. Moreover, if the solution exists, it is continuous in $\overline{\Omega}$.*

3.2 Geometrical discussion on domains

Though solutions are unique for any bounded set Ω, the existence of solutions depends on a geometrical character of Ω. To prove the existence of solutions, we assume (3.2) given below. In this subsection we discuss a sufficient condition and a necessary condition for (3.2). We also give some examples of Ω satisfying or not satisfying (3.2). The proof of the existence result under (3.2) will be given in the next subsection.

For $\rho > 0$ let us define

$$C_\rho := \{x \in \mathbb{R}^d \mid B(x,\rho) \subset \Omega^c\} = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \overline{\Omega}) \geq \rho\}$$

and

$$d(z,\rho) := \operatorname{dist}(z, C_\rho) - \rho.$$

By definition C_ρ is a closed set. We first prepare

Lemma 3.3. *Let $z \in \partial\Omega$. Then $d(z,\cdot)$ is nonnegative and nondecreasing in $(0,\infty)$.*

Proof. Fix $\rho > 0$ and let us prove that $d(z,\rho) \geq 0$. Take $x_0 \in C_\rho$ such that $\operatorname{dist}(z, C_\rho) = |z - x_0|$. The fact $x_0 \in C_\rho$ implies that $\operatorname{dist}(x_0, \overline{\Omega}) \geq \rho$, and we also have $\operatorname{dist}(x_0, \overline{\Omega}) \leq |x_0 - z|$ since $z \in \partial\Omega$. Collecting the relations above, we deduce that $\operatorname{dist}(z, C_\rho) \geq \rho$, that is, $d(z,\rho) \geq 0$.

We next let $0 < \rho_1 < \rho_2$ to check that $d(z,\cdot)$ is nondecreasing. Let $x_0 \in C_{\rho_2}$ be a point satisfying $\operatorname{dist}(z, C_{\rho_2}) = |z - x_0|$. Note that we have $|z - x_0| \geq \rho_2$ since $d(z,\rho_2) \geq 0$. Now let us set $v = (\rho_2 - \rho_1)\frac{z - x_0}{|z - x_0|}$. Then $|v| = \rho_2 - \rho_1$, and thus $x_0 + v$ lies on the line segment joining x_0 and z. This implies that

$$|z - (x_0 + v)| = |z - x_0| - (\rho_2 - \rho_1).$$

We also have $x_0 + v \in C_{\rho_1}$ because $B(x_0 + v, \rho_1) \subset B(x_0, \rho_2) \subset \Omega^c$. Therefore

$$\begin{aligned}\operatorname{dist}(z, C_{\rho_1}) = \min_{y \in C_{\rho_1}} |z - y| \leq |z - (x_0 + v)| &= |z - x_0| - (\rho_2 - \rho_1)\\ &= \operatorname{dist}(z, C_{\rho_2}) - (\rho_2 - \rho_1).\end{aligned}$$

This shows that $d(z,\cdot)$ is nondecreasing. ☐

For $\nu > 0$, the assumption on Ω for our existence result is

$$\inf_{\rho > \nu^{-1}} \frac{1}{\nu - \rho^{-1}} d(z,\rho) = 0 \quad \text{for all } z \in \partial\Omega. \tag{3.2}$$

Let us provide a sufficient condition and a necessary condition for (3.2). We consider *exterior sphere conditions* on Ω.

Proposition 3.4 (Sufficient condition for (3.2))**.** *Ω satisfies* (3.2) *if*

$$\forall z \in \partial\Omega,\ \exists z_0 \in \mathbb{R}^d,\ s.t.\ |z - z_0| > \nu^{-1},\ B(z_0, |z - z_0|) \subset \Omega^c. \tag{3.3}$$

Proof. Fix $z \in \partial\Omega$ and take $z_0 \in \mathbb{R}^d$ in (3.3). Let us set $\rho := |z - z_0| > \nu^{-1}$. Since $B(z_0,\rho) \subset \Omega^c$, we have $z_0 \in C_\rho$. Then

$$0 \leq d(z,\rho) = \operatorname{dist}(z, C_\rho) - \rho \leq |z - z_0| - \rho = 0.$$

Hence $d(z,\rho) = 0$, and this together with the nonnegativity of $d(z,\cdot)$ yield (3.2). ☐

The assumption (3.2) is a weak convexity condition because the condition (3.3) includes the case where Ω is convex. Moreover (3.3) permits a concave part of $\partial\Omega$ on which the outward mean curvature is strictly less than ν.

Proposition 3.5 (Necessary condition for (3.2))**.** *If Ω satisfies* (3.2)*, then*

$$\forall z \in \partial\Omega,\ \exists z_0 \in \mathbb{R}^d,\ s.t.\ |z - z_0| \geq \nu^{-1},\ B(z_0, |z - z_0|) \subset \Omega^c. \tag{3.4}$$

Proof. Let $z \in \partial\Omega$. Take $z_0 \in C_{\nu^{-1}}$ such that $\operatorname{dist}(z, C_{\nu^{-1}}) = |z - z_0|$. If we prove that $|z - z_0| = \nu^{-1}$, then (3.4) holds. Now, from the monotonicity of $d(z,\cdot)$ and (3.2), it follows that $\lim_{\rho \searrow \nu^{-1}} d(z,\rho) = 0$. Since $d(z,\cdot)$ is nonnegative, we get $d(z,\nu^{-1}) = 0$, i.e., $|z - z_0| = \nu^{-1}$. ☐

If Ω satisfies (3.4) but does not satisfy (3.3), that is, if there is $z \in \partial\Omega$ at which the radius of the exterior sphere must be ν^{-1}, then the situation is delicate. We give some examples of such Ω.

Let $d = 2$ and

$$\Omega_1 = ((-L, L) \times (-L, L)) \cap \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y < \frac{\nu}{2}x^2\right\}$$

for large $L > 0$. See Figure 24. It is easily seen that $(0, 0)$ is the unique point on $\partial\Omega_1$ that does not satisfy (3.3).

Proposition 3.6. Ω_1 *satisfies* (3.2).

Proof. It suffices to check (3.2) for $z = (0, 0)$. Let $0 < x_0 < 2$. We will find the exterior circle such that it touches Ω_1 at $\left(x_0, \frac{\nu}{2}x_0^2\right)$ and the center is on y axis. Since the equation of the normal line to $y = \frac{\nu}{2}x^2$ at $\left(x_0, \frac{\nu}{2}x_0^2\right)$ is $y = -\frac{1}{\nu x_0}x + \nu^{-1} + \frac{\nu}{2}x_0^2$, the center of the exterior circle is $\left(0, \nu^{-1} + \frac{\nu}{2}x_0^2\right)$ and its radius is $\rho := \sqrt{x_0^2 + (\nu^{-1})^2}$. This implies that $\left(0, \nu^{-1} + \frac{\nu}{2}x_0^2\right) \in C_\rho$ and

$$d((0, 0), \rho) \leq \nu^{-1} + \frac{\nu}{2}x_0^2 - \rho = \nu^{-1} + \frac{\nu}{2}\left(\rho^2 - \nu^{-2}\right) - \rho.$$

Therefore we have

$$0 \leq \frac{1}{\nu - \rho^{-1}} d((0, 0), \rho) \leq \frac{\rho}{2}\left(\rho - \nu^{-1}\right).$$

Letting $x_0 \searrow 0$, i.e., $\rho \searrow \nu^{-1}$, we obtain (3.2). □

The more points on $\partial\Omega$ fail the condition (3.3), the more likely Ω fails (3.2). To see this, let us set

$$\Omega_2 = ((-L, L) \times (-L, 0)) \cap \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 > \nu^{-2}\right\}$$

for large $L > 0$. See Figure 25. The above Ω_2 does not satisfy (3.3) at any $(x, y) \in \partial B\left((0, 0), \nu^{-1}\right)$ $(y < 0)$.

Proposition 3.7. Ω_2 *does not satisfy* (3.2).

Proof. We show that (3.2) fails at $z = \left(0, -\nu^{-1}\right)$. Let $\rho > \nu^{-1}$. By geometric observations we notice that $\left(0, \sqrt{\rho^2 - \nu^{-2}}\right) \in C_\rho$ and

$$\operatorname{dist}\left(\left(0, -\nu^{-1}\right), C_\rho\right) = \left|\left(0, -\nu^{-1}\right) - \left(0, \sqrt{\rho^2 - \nu^{-2}}\right)\right|.$$

Thus we have

$$\begin{aligned}\frac{1}{\nu - \rho^{-1}} d\left(\left(0, \nu^{-1}\right), \rho\right) &= \frac{1}{\nu - \rho^{-1}}\left(\sqrt{\rho^2 - \nu^{-2}} + \nu^{-1} - \rho\right) \\ &= \frac{\rho}{\nu}\left(\sqrt{\frac{\rho + \nu^{-1}}{\rho - \nu^{-1}}} - 1\right).\end{aligned}$$

This implies that

$$\inf_{\rho > \nu^{-1}} \frac{1}{\nu - \rho^{-1}} d\left(\left(0, -\nu^{-1}\right), \rho\right) = \nu^{-2} > 0,$$

which shows that the condition (3.2) fails. □

Define $\Omega_3 \subset \mathbb{R}^d$ by

$$\Omega_3 = B(0, R_0) \setminus \overline{B\left(0, \nu^{-1}\right)}, \quad R_0 > \nu^{-1}. \tag{3.5}$$

See Figure 26. It is easily seen that Ω_3 does not satisfy (3.2) at any $z \in \partial B\left(0, \nu^{-1}\right)$. Furthermore it turns out in Proposition 3.10 that there is no solution to (1.7).

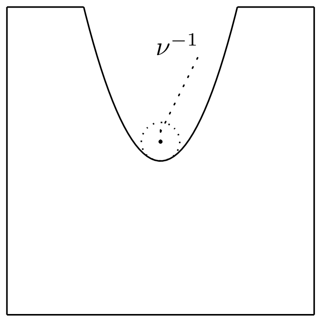


Figure 24: Shape of Ω_1

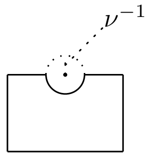


Figure 25: Shape of Ω_2

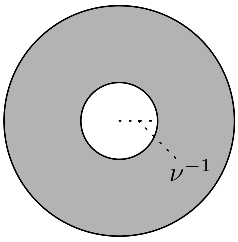


Figure 26: Shape of Ω_3

3.3 Existence result

We construct a unique solution of (1.7) under (3.2).

Theorem 3.8 (Unique existence)**.** *Let $\nu > 0$. Assume that Ω is bounded and satisfies* (3.2)*. Then, there exists a unique viscosity solution $U \in C(\overline{\Omega})$ of* (1.7)*. Moreover $U > 0$ in Ω.*

Proof. The uniqueness and continuity of solutions have already been established in Corollary 3.2.

We prove the existence of solutions by Perron's method, i.e., we construct a subsolution W_1 and a supersolution W_2 of (1.7) such that $W_1 \le W_2$ in $\overline{\Omega}$ and

$$W_1 = W_2 = 0 \text{ on } \partial\Omega, \quad W_1 \text{ and } W_2 \text{ are continuous on } \partial\Omega. \tag{3.6}$$

It is clear that $W_1 \equiv 0$ is a subsolution of (1.7) that satisfies (3.6). In order to construct W_2, we set

$$\mathscr{L} := \left\{(z_0, \rho) \in \mathbb{R}^d \times \left(\nu^{-1}, \infty\right) \,\middle|\, z_0 \in C_\rho\right\}$$

and for any $(z_0, \rho) \in \mathscr{L}$

$$l_{z_0,\rho}(x) := \frac{c}{\nu - \rho^{-1}}(|x - z_0| - \rho).$$

Then we define W_2 by

$$W_2(x) := \inf_{(z_0,\rho)\in\mathscr{L}} l_{z_0,\rho}(x), \qquad x \in \overline{\Omega}.$$

Since $l_{z_0,\rho}$ is continuous, we see that W_2 is upper semicontinuous in $\overline{\Omega}$. Moreover it is easily seen that $l_{z_0,\rho} \ge 0$ in $\overline{\Omega}$ for any $(z_0, \rho) \in \mathscr{L}$. Thus $W_2 \ge 0 = W_1$ in $\overline{\Omega}$.

We show that W_2 is a supersolution of (1.7). By stability (Proposition 2.4) it suffices to show that each $l_{z_0,\rho}$ is a supersolution. Let $(z_0, \rho) \in \mathscr{L}$ and $x \in \Omega$. By direct calculation, we have

$$Dl_{z_0,\rho}(x) = L\frac{x - z_0}{|x - z_0|}, \quad F(Dl_{z_0,\rho}(x), D^2 l_{z_0,\rho}(x)) = \frac{-L}{|x - z_0|} \quad \left(L := \frac{c}{\nu - \rho^{-1}}\right).$$

Thus, noting that $|x - z_0| \ge \rho$, we find

$$\nu|Dl_{z_0,\rho}(x)| + F(Dl_{z_0,\rho}(x), D^2 l_{z_0,\rho}(x)) = L\left(\nu - 1/|x - z_0|\right) \ge L\left(\nu - \rho^{-1}\right) = c.$$

Accordingly $l_{z_0,\rho}$ is a supersolution of (1.7).

Next let us check that W_2 satisfies the conditions in (3.6). Fix $z \in \partial\Omega$. For any $\rho > \nu^{-1}$ we let $z_\rho \in C_\rho$ be a point satisfying $\mathrm{dist}(z, C_\rho) = |z - z_\rho|$. Then

$$l_{z_\rho,\rho}(z) = \frac{c}{\nu - \rho^{-1}}(|z - z_\rho| - \rho) = \frac{c}{\nu - \rho^{-1}} d(z, \rho).$$

By (3.2) we obtain

$$0 \le (W_2)_*(z) \le (W_2)^*(z) = W_2(z) \le \inf_{\rho > \nu^{-1}} l_{z_\rho,\rho}(z) = 0,$$

which implies that W_2 satisfies (3.6). By Proposition 2.14 we obtain a viscosity solution U of (1.7) such that $0 = W_1 \le U \le W_2$ in $\overline{\Omega}$.

We finally prove that $U > 0$ in Ω. If there were some $x_0 \in \Omega$ such that $U(x_0) = 0$, then $U - \phi$ would attain a minimum at x_0 for $\phi \equiv 0$. However

$$\nu|D\phi(x_0)| + F^*(D\phi(x_0), D^2\phi(x_0)) = \nu|0| + F^*(0, O) = 0 < c,$$

which contradicts the fact that U is a supersolution. □

Remark 3.9. When $\nu = 0$, the existence result is obtained by [13, Theorem 7.4], where Ω is assumed to be bounded, open and strictly convex. Also the boundary $\partial\Omega$ is assumed to be smooth and connected. The paper [13] proves the existence by using approximate equations. On the other hand, our technique based on Perron's method works if we make the following assumption:

$$\forall z \in \partial\Omega,\ \exists z_0 \in \mathbb{R}^d,\ \text{s.t. } \Omega \subset B(z_0, |z - z_0|). \tag{3.7}$$

As in the proof of Theorem 3.8, we set the following supersolutions:

$$l_{z_0,\rho}(x) := -\rho c(|x - z_0| - \rho),$$

where $\rho = |z - z_0|$. Under the assumption (3.7), we can take such a supersolution $l_{z_0,\rho}$ for $z \in \partial\Omega$ that $l_{z_0,\rho} \geq 0$ in $\overline{\Omega}$ and $l_{z_0,\rho}(z) = 0$. Thus we obtain $W_2(x) := \inf_{z\in\partial\Omega} l_{z_0,\rho}(x)$ satisfying (3.6).

3.4 Non-existence result

We prove that, when Ω is given by (3.5), the problem (1.7) does not admit a solution.

Proposition 3.10. *Let Ω be the set in* (3.5). *Then there exists no solution to* (1.7).

Proof. Suppose by contradiction that there is a solution U to (1.7). In order to estimate U, we set $\Omega_d = B(0, R_0) \setminus \overline{B(0, \nu^{-1} + d)}$ for $d \in \left(0, R_0 - \nu^{-1}\right)$ and consider

$$\begin{cases} \nu|DU| + F(DU, D^2U) = c & \text{in } \Omega_d, \\ U = 0 & \text{on } \partial\Omega_d. \end{cases} \tag{3.8}$$

Since Ω_d satisfies (3.3), there exists a unique solution U_d to (3.8). Let us give an explicit form of the solution U_d. For this purpose, we first let $\psi_1(r)$ be the function in (2.11) with $R = R_0$. Then $\psi_1(|x|)$ is a solution to (1.7) with $\Omega = B(0, R_0)$. Next set

$$\psi_2^d(r) := \int_{\nu^{-1}+d}^{r} \frac{cs}{\nu s - 1}\, ds = \frac{c}{\nu}\left(r - \frac{1}{\nu} - d\right) + \frac{c}{\nu^2}\log\left(\frac{r - \nu^{-1}}{d}\right) \tag{3.9}$$

for $d \in \left(0, R_0 - \nu^{-1}\right)$ and $r \in (\nu^{-1}, \infty)$. Then $\psi_2^d(r)$ is a solution to

$$\begin{cases} \nu|\psi_r| - \frac{\psi_r}{r} = c, & \nu^{-1} + d < r \\ \psi\left(\nu^{-1} + d\right) = 0, \end{cases}$$

and solves (1.7) with $\Omega = B\left(0, \nu^{-1} + d\right)^c$. Let us define $\psi^d(r) := \min\{\psi_1(r), \psi_2^d(r)\}$.

We now claim that $\psi^d(|x|)$ is a solution to (3.8). It is clear that the boundary condition is fulfilled. To check the equation, we only have to consider the points where $\psi^d(|x|)$ is not smooth. Namely we test $\psi^d(|x|)$ at x_0 with $|x_0| = r_0$, where $r_0 \in \left(\nu^{-1} + d, R_0\right)$ is the value satisfying $\psi_1(r_0) = \psi_2(r_0)$. Since $\psi_1'(r_0) < 0$ and $(\psi_2^d)'(r_0) > 0$, there is no test function that touches $\psi^d(|x|)$ from below. Assume that $\psi^d(|x|) - \phi(x)$ has a maximum at x_0. Then we have by [22, Lemma A.1] that

$$|D\phi(x_0)| = s \text{ and } F_*(D\phi(x_0), D^2\phi(x_0)) \leq -s/r_0$$

for some $s \in [a, b]$ with $a := \psi_1'(r_0)$ and $b := (\psi_2^d)'(r_0)$. Thus we have

$$\nu|D\phi(x_0)| + F_*(D\phi(x_0), D^2\phi(x_0)) \leq \nu|s| - \frac{s}{r_0}.$$

It is clear that

$$\sup_{s\in[a,b]}\left[\nu|s| - \frac{s}{r_0}\right] = c$$

since $\nu|a| - \frac{a}{r_0} = c$, $\nu|b| - \frac{b}{r_0} = c$ and $\nu - \frac{1}{r_0} > 0$. Hence we conclude that $\psi^d(|x|)$ is the viscosity solution to (3.8), that is, $U_d(x) = \psi^d(|x|)$.

We get back to (1.7). Let us define $\tilde{U}_d$ by

$$\tilde{U}_d(x) = \begin{cases} U_d(x), & \nu^{-1} + d < |x|, \\ 0, & \nu^{-1} \le |x| \le \nu^{-1} + d. \end{cases}$$

This is a subsolution to (1.7) for any $d \in (0, R_0 - \nu^{-1})$. Indeed U_d is a subsolution in $\Omega_d \subset \Omega$ and 0 is a subsolution in $\Omega \setminus \Omega_d$. Concerning a point x_0 with $|x_0| = \nu^{-1} + d$, there is no test function that touches $\tilde{U}_d$ at x_0 from above since $(\psi_2^d)'(d) > 0$.

Therefore the comparison principle (Proposition 3.1) implies that

$$\sup_{d \in (0, R_0 - \nu^{-1})} \tilde{U}_d \le U \quad \text{in } \Omega.$$

Now, by (3.9) we see that $\sup_{d \in (0, R_0 - \nu^{-1})} \psi_2(r) = \infty$ for $r > \nu^{-1}$. Thus $\sup_{d \in (0, R_0 - \nu^{-1})} \tilde{U}_d(x) = \psi_1(|x|)$ for $|x| > \nu^{-1}$. This gives $\psi_1(|x|) \le U(x)$ for $x \in \Omega$, and by the continuity of U on $\partial\Omega$ we have $\psi_1(|x|) \le U(x)$ for $|x| = \nu^{-1}$. However this is a contradiction since $\psi_1(\nu^{-1}) > 0$ and $U(x) = 0$ for $|x| = \nu^{-1}$. □

4 Asymptotic behavior

We investigate the asymptotic behavior (1.3) of solutions u to (1.1) or (1.4). We first study the asymptotic speed a and prove that it is 0 if the support of the source term f is bounded. Moreover, with further assumptions, it turns out that solutions of (1.4) becomes stable in a finite time. The asymptotic shape $\phi(x)$ is given by the unique solution $U(x)$ of the elliptic problem (1.7).

4.1 Asymptotic speed

We first give an upper and lower bound of the asymptotic speed of solutions.

Proposition 4.1 (Upper/lower bound of the asymptotic speed). *Let c_1, c_2 be the constants in* (2.17). *Let u be a subsolution (resp. supersolution) of* (1.1) *that is bounded from above (resp. from below) in $\mathbb{R}^d \times [0, T]$ for every $T > 0$. Then $u^* \le c_2 t + \|u_0\|$ (resp. $u_* \ge c_1 t - \|u_0\|$) in $\mathbb{R}^d \times [0, \infty)$.*

In particular, if u is a solution of (1.1) *that is bounded in $\mathbb{R}^d \times [0, T]$ for every $T > 0$, then*

$$c_1 \le \liminf_{t \to \infty} \frac{u(x,t)}{t} \le \limsup_{t \to \infty} \frac{u(x,t)}{t} \le c_2 \quad \text{for all } x \in \mathbb{R}^d. \tag{4.1}$$

Proof. (4.1) is an immediate consequence of the first part of the theorem.

Let us prove that $u^* \le c_2 t + \|u_0\|$ in $\mathbb{R}^d \times [0, \infty)$ for a subsolution u. We omit the proof for a supersolution since it is parallel. Take $\delta, T > 0$ arbitrarily. It suffices to show that $u^* \le (c_2 + \delta)t + \|u_0\|$ in $\mathbb{R}^d \times [0, T)$. Suppose by contradiction that there is some $(x_0, t_0) \in \mathbb{R}^d \times [0, T)$ satisfying $M := u^*(x_0, t_0) - (c_2 + \delta)t_0 - \|u_0\| > 0$. Let us define $\Phi(x,t) := u^*(x,t) - \phi(x,t)$ with

$$\phi(x,t) := (c_2 + \delta)t + \|u_0\| + \epsilon\langle x \rangle + \frac{\sigma}{T - t},$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$ and $\epsilon, \sigma \in (0,1)$ are small constants such that $\epsilon\langle x_0 \rangle + \frac{\sigma}{T - t_0} < M$. We then have

$$\Phi(x_0, t_0) = M - \epsilon\langle x_0 \rangle - \frac{\sigma}{T - t_0} > 0.$$

Let us define

$$K(\epsilon) := \sup_{x \in \mathbb{R}^d} |\nu\epsilon D\langle x \rangle| + |F_*(\epsilon D\langle x \rangle, \epsilon D^2\langle x \rangle)|.$$

Since $F_*(0, O) = 0$ and $D\langle\cdot\rangle$, $D^2\langle\cdot\rangle$ are bounded in $\mathbb{R}^d$, we see that $K(\epsilon) \to 0$ as $\epsilon \to 0$. Hereafter we choose ϵ small so that $K(\epsilon) < \delta$.

By the boundedness of u^* from above, Φ attains a maximum over $\mathbb{R}^d \times [0,T)$ at some $(\hat{x},\hat{t}) \in \mathbb{R}^d \times [0,T)$. Then $\hat{t} \neq 0$. Indeed, if $\hat{t}=0$, we would have

$$\Phi(\hat{x},\hat{t}) = u^*(\hat{x},0) - \phi(\hat{x},0) \le u_0(\hat{x}) - \|u_0\| - \epsilon\langle\hat{x}\rangle - \frac{\sigma}{T-\hat{t}} \le -\epsilon\langle\hat{x}\rangle - \frac{\sigma}{T-\hat{t}} < 0.$$

This is a contradiction because $\Phi(\hat{x},\hat{t}) \ge \Phi(x_0,t_0) > 0$.

Since u is a subsolution of (1.1a), we have

$$I := \phi_t(\hat{x},\hat{t}) + \nu|D\phi(\hat{x},\hat{t})| + F_*(D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \le f^*(\hat{x}) \le c_2.$$

However, by the definition of ϕ, we have the estimate

$$I = c_2 + \delta + \frac{\sigma}{(T-\hat{t})^2} + \nu\epsilon|D\langle\hat{x}\rangle| + F_*(\epsilon D\langle\hat{x}\rangle, \epsilon D^2\langle\hat{x}\rangle) > c_2 + \delta - K(\varepsilon) > c_2,$$

which is a contradiction. □

Remark 4.2. This theorem holds for a more general F if it satisfies (2.6) and (2.8) with $F_*(0,O) = F^*(0,O) = 0$. Also, since we did not use the fact that ν is nonnegative in the proof, the theorem holds for any $\nu \in \mathbb{R}$.

Theorem 4.3 (Asymptotic speed 0)**.** *Assume that* $\operatorname{supp} f$ *is bounded in* $\mathbb{R}^d$*. Let* u *be a solution of* (1.1) *satisfying the following: there exist* $m, \rho : (0,\infty) \to (0,\infty)$ *such that* $m(T)/T \to 0$ *as* $T \to \infty$ *and* $|u(x,t)| \le m(T)$ *in* $B(0,\rho(T))^c \times [0,T]$ *for every* $T > 0$*. Then*

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = 0 \quad \textit{uniformly in } x \in \mathbb{R}^d. \tag{4.2}$$

Proof. Set $c = c_2 - c_1 + 1 > 0$ with the constants c_1, c_2 given in (2.17). Choose $R > 0$ such that $\operatorname{supp} f \subset B(0,R)$. Then $f^* \le c\chi_{B(0,R)}$ in $\mathbb{R}^d$. Let us fix $T > 0$ and define

$$W_T(x,t) := u_{R,c}(x,t) + \max\{m(2T),\ \|u_0\|\},$$

where $u_{R,c}$ is the solution (2.13) of (1.4a) with $\Omega = B(0,R)$. By the definition above, we see that W_T is a supersolution of (1.4) with $\Omega = B(0,R)$ and that $\sup_{B(0,\rho(2T))^c\times[0,2T]}(u^* - W_T) \le 0$. Comparing u^* and W_T over $\mathbb{R}^d \times [0,2T)$ as in Proposition 2.5, we obtain $u^* \le W_T$ in $\mathbb{R}^d \times [0,2T)$. In the same manner, we deduce that $-W_T \le u_*$ in $\mathbb{R}^d \times [0,2T)$. Thus we have

$$\frac{|u(x,T)|}{T} \le \frac{W_T(x,T)}{T} \le \frac{u_{R,c}(x,T)}{T} + \max\left\{\frac{2m(2T)}{2T},\ \frac{\|u_0\|}{T}\right\},$$

which yields (4.2). □

Remark 4.4. When $\nu < 0$, the asymptotic speed of solutions may not be 0 as we have already seen in Proposition 2.13.

4.2 Asymptotic shape

We focus on the problem (1.4) with an open set $\Omega \subsetneq \mathbb{R}^d$. We first investigate the asymptotic shape when $u_0 = 0$. In this case, the solution is given by an explicit form. Also the solution will be used to study the case of more general initial data.

Before the theorems, we prepare the following lemma, which is proved in the same way as [22, Lemma A.1].

Lemma 4.5. *Let* $R > 0$ *and* $W : \mathbb{R}^d \to \mathbb{R}$ *be a nonnegative function such that* $W = 0$ *in* $\overline{B(0,R)}$*. Let* $x_0 \in \partial B(0,R)$ *and* $\phi \in C^2(\mathbb{R}^d)$*. If* $W - \phi$ *has a minimum at* x_0*, then there exists* $s \ge 0$ *such that*

$$D\phi(x_0) = s\frac{x_0}{R} \ \textit{and}\ F^*(D\phi(x_0), D^2\phi(x_0)) \ge -\frac{s}{R}.$$

Theorem 4.6 (Asymptotic shape with $u_0 = 0$)**.** *Assume that* Ω *is bounded and satisfies* (3.2)*. Let* $U \in C(\overline{\Omega})$ *be the unique solution of* (1.7)*. Define* $u \in C(\mathbb{R}^d \times [0,\infty))$ *by*

$$u(x,t) := \begin{cases} \min\{U(x),\, ct\}, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases} \tag{4.3}$$

Then u *is a solution of* (1.4) *with* $u_0 = 0$*. In particular the solution* u *satisfies*

$$u(x,t) = U(x), \quad x \in \mathbb{R}^d,\ t \geq t_* := \frac{1}{c} \max_{\overline{\Omega}} U. \tag{4.4}$$

If Ω *further satisfies* (2.3)*, then* u *is a unique solution in* SOL*.*

Proof. It is clear that $u(\cdot,0) = 0$ in $\mathbb{R}^d$. We discuss the viscosity solution property of u by dividing $\mathbb{R}^d \times (0,\infty)$ into the following three sets:

$$D_1 := (\overline{\Omega})^c \times (0,\infty), \quad D_2 := \partial\Omega \times (0,\infty), \quad D_3 := \Omega \times (0,\infty).$$

1) D_1**.** We have $u \equiv 0$ in D_1, and thus u is a classical sub- and supersolution of (1.4) in D_1. Hence it is a viscosity solution.

2) D_2**.** Let $(x_0,t_0) \in D_2$. Assume that $u - \phi$ attains its maximum at (x_0,t_0), where ϕ is a smooth function. Then, since $u(x_0,t_0) = 0$ and $u \geq 0$, we see that ϕ attains a minimum at (x_0,t_0). It thus follows that $\phi_t(x_0,t_0) = 0$, $D\phi(x_0,t_0) = 0$ and $D^2\phi(x_0,t_0) \geq O$. From these we deduce

$$\phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F_*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \leq F_*(0,O) = 0 \leq c.$$

This is the inequality to be checked for a viscosity subsolution.

Assume next that $u - \phi$ attains a minimum at (x_0,t_0), where ϕ is a smooth function. Since $u(x_0,\cdot) = 0$, we see that $\phi(x_0,\cdot)$ attains a maximum at t_0. Thus $\phi_t(x_0,t_0) = 0$. To compute the spatial derivatives, let us recall that (3.4) holds. This implies that there is an exterior ball $B = B(z_0,\rho) \subset \Omega^c$ with the radius $\rho \geq \nu^{-1}$ that touches $\partial\Omega$ at x_0. By the definition of u we have $u(\cdot,t_0) \geq 0$ in $\mathbb{R}^d$ and $u(\cdot,t_0) = 0$ in B. Thus Lemma 4.5 shows that, for some $s \geq 0$

$$|D\phi(x_0,t_0)| = s, \quad F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \geq -s/\rho.$$

Then we get

$$\phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \geq \nu s - \frac{s}{\rho} \geq 0.$$

This is the desired inequality.

3) D_3**.** Since U solves (1.7a), we notice that $v(x,t) = U(x) - ct$ is a solution of

$$v_t + \nu|Dv| + F(Dv, D^2v) = 0 \quad \text{in } D_3 = \Omega \times (0,\infty). \tag{4.5}$$

Let us recall the invariance property ([22, Theorem 4.2.1]) for a geometric equation (4.5). This asserts that, if h is a subsolution (resp. supersolution) of (4.5) and if $\theta : \mathbb{R} \to \mathbb{R}$ is an upper semicontinuous (resp. lower semicontinuous) and nondecreasing function, then $\theta \circ v$ is also a subsolution (resp. supersolution) of (4.5).

Now let $\theta(r) = \min\{r, 0\}$. Then $(\theta \circ v)(x,t) = \min\{U(x) - ct, 0\}$ is a solution of (4.5), and hence, $(\theta \circ v)(x,t) + ct = u(x,t)$ is a solution of (1.4) in D_3.

(4.4) is an immediate consequence of the definition of u. Also, since u is a continuous solution, the last assertion of the theorem follows from Corollary 2.7. □

We prove that the asymptotic shape is U even if the initial datum u_0 is allowed to be positive in Ω.

Theorem 4.7 (Asymptotic shape)**.** *Assume that* Ω *is bounded and satisfies* (2.3) *and* (3.2)*. Assume that* $u_0 \geq 0$ *in* Ω *and* $u_0 = 0$ *in* Ω^c*. Let* $U \in C(\overline{\Omega})$ *be the unique solution of* (1.7)*. Then* (4.4) *holds for any solution* $u \in$ SOL *of* (1.4)*.*

Proof. Let $u \in \mathrm{SOL}$ be a solution of (1.4). Denote by $\hat{u}$ the solution of (1.4) given in (4.3). Since $\hat{u}(\cdot, 0) = 0 \le u_0$ in $\mathbb{R}^d$ and $\hat{u} \in \mathrm{SUB}$, Theorem 2.6 yields

$$\hat{u} = \hat{u}_* \le u_* \quad \text{in } \mathbb{R}^d \times [0, \infty). \tag{4.6}$$

We next construct a supersolution of (1.4) whose asymptotic shape is $U(x)$. Let $h \ge \|u_0\|$ and we define a function $w(x,t)$ by

$$w(x,t) := \begin{cases} U(x), & x \in \Omega \text{ and } U(x) \le ct, \\ h + ct, & x \in \Omega \text{ and } U(x) > ct, \\ 0, & x \in \Omega^c. \end{cases} \tag{4.7}$$

Then we have $w(\cdot, 0) = h\chi_\Omega \ge u_0$ in $\mathbb{R}^d$ and $w \in \mathrm{SUP}$. We show that w is a supersolution of (1.4a). It is proved in the same way as Theorem 4.6 that w is a supersolution in $\Omega^c \times (0, \infty)$. In $\Omega \times (0, \infty)$, we consider $v(x,t) = U(x) - ct$. Set

$$\theta(r) := \begin{cases} h, & r > 0, \\ r, & r \le 0, \end{cases}$$

which is a nondecreasing and lower semicontinuous function. Since v is a solution of (4.5), the invariance property implies that $\theta \circ v$ is a supersolution of (4.5). Actually $\theta \circ v$ is also a subsolution of (4.5) since $(\theta \circ v)^* = \theta^* \circ v$ and θ^* is upper semicontinuous. Therefore $(\theta \circ v)(x,t) + ct = w(x,t)$ is a solution of (1.4) in $\Omega \times (0, \infty)$. From Theorem 2.6 we deduce

$$u^* \le w^* \quad \text{in } \mathbb{R}^d \times [0, \infty). \tag{4.8}$$

Since $w^*(\cdot, t) = \hat{u}(\cdot, t)$ for $t \ge t_*$, we conclude the proof by (4.6) and (4.8). □

Last part of this section is devoted to construct an example of discontinuous solutions since we use similar test function argument to the proof of Theorem 4.6. We consider Ω given by (3.5). Recall that, for this Ω, there is no solution to (1.7) as proved in Proposition 3.10. To avoid difficulty, we consider a viscosity solution only in a short time.

Proposition 4.8. *Assume that $u_0 = 0$ and Ω is given by* (3.5). *Let $U : \Omega \to \mathbb{R}$ be the function given by* (2.12)*, and define u as in* (4.3). *Then u is a viscosity solution of* (1.4) *in $\mathbb{R}^d \times [0, T)$, where $T < \frac{1}{c} \sup_{x \in \Omega} U(x)$.*

Proof. It suffices to show that u is a subsolution on $\partial B(0, \nu^{-1}) \times (0, \infty)$. In fact the same proof as Theorem 4.6 works for the proof of the other assertions.

Assume that $u^* - \phi$ attains its maximum at $(x_0, t_0) \in \partial B\left(0, \nu^{-1}\right) \times (0, \infty)$. If $ct_0 < U^*(x_0)$, we have

$$\phi_t(x_0, t_0) = c, \quad |D\phi(x_0, t_0)| = s, \quad F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \le -\nu s$$

for some $s \in [0, \infty)$. Hence we obtain by the analogue of [22, Lemma A.1] that

$$\phi_t(x_0, t_0) + \nu |D\phi(x_0, t_0)| + F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \le c.$$

□

Remark 4.9. If $ct_0 > U^*(x_0)$, we have

$$\phi_t(x_0, t_0) = 0, \quad D\phi(x_0, t_0) = s\nu x_0, \quad F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \le -\nu s$$

for some $s \in [-\frac{c}{2\nu}, \infty)$. Note that $\psi'(|x_0|) = \psi'(\nu^{-1}) = -\frac{c}{2\nu}$, where $\psi(|x|) = U(x)$. Hence we obtain

$$\begin{aligned} \phi_t(x_0, t_0) + \nu |D\phi(x_0, t_0)| + F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0)) &\le \nu |s\nu x_0| - \nu s \\ &= \nu |s| - \nu s \le 2\nu \frac{c}{2\nu} = c. \end{aligned}$$

The remaining case $ct_0 = U^*(x_0)$ is non trivial because we only have $0 \le \phi_t(x_0, t_0) \le c$.

5 Game interpretation

In this section we describe a game whose value functions converge to a viscosity solution to our equation (1.1) with $\nu \in \mathbb{R}$. Also we introduce notions of strategy of the game to clarify the statements on the game.

5.1 The rule of the game

We introduce the rule of the game corresponding to (1.1) with $\nu \geq 0$ and $d = 2$. The game is a deterministic two-person zero-sum game. For convenience, we name the first player Paul and the second player Carol. Paul's goal is to minimize the total cost of the game and Carol's goal is to maximize it. Let $x \in \mathbb{R}^2$, $t > 0$, and $\epsilon > 0$ be the initial setting of the game. We call the initial setting (x, t, ϵ) the "variables of the game" henceforth. The game consists of $N = \lceil t\epsilon^{-2} \rceil$ rounds, where $\lceil r \rceil$ stands for the minimum integer that is no less than r. In each round i $(i = 1, 2, \cdots, N)$, the following procedures are carried out.

1. Paul chooses $v_i, w_i \in S^1$. (S^1 is the set of unit vectors in $\mathbb{R}^2$.)
2. Carol chooses $b_i = 1$ or $b_i = -1$ after Paul's choice.
3. Determine the next states as follows.

$$x_i = x_{i-1} + \sqrt{2}\epsilon b_i v_i + \nu\epsilon^2 w_i. \tag{5.1}$$

Here $x_i \in \mathbb{R}^2$ denotes the game position at the end of round i. The initial position is $x_0 = x$. For convenience, we often regard the game position as Paul's position.

The total cost is given by $u_0(x_N) + \sum_{i=0}^{N-1} \epsilon^2 f(x_i)$. We define the value function $u^\epsilon(x, t)$ as the total cost given when Paul starts at x with terminal time t and both players take optimal choices, i.e.

$$u^\epsilon(x,t) = \inf_{v_1,w_1 \in S^1} \max_{b_1=\pm 1} \inf_{v_2,w_2 \in S^1} \max_{b_2=\pm 1} \cdots \inf_{v_N,w_N \in S^1} \max_{b_N=\pm 1} \left[u_0(x_N) + \sum_{i=0}^{N-1} \epsilon^2 f(x_i) \right] \tag{5.2}$$

for $t > 0$, and

$$u^\epsilon(x,t) = u_0(x)$$

for $t \leq 0$.

The value function $u^\epsilon(x, t)$ satisfies the *Dynamic Programming Principle*

$$u^\epsilon(x,t) = \inf_{v,w \in S^1} \max_{b=\pm 1} [u^\epsilon(x + \sqrt{2}\epsilon b v + \nu w \epsilon^2, t - \epsilon^2) + \epsilon^2 f(x)], \qquad t > 0.$$

We consider the following half relaxed limits of the value functions.

$$\overline{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t) \\ \epsilon \searrow 0}} u^\epsilon(y,s), \quad \underline{u}(x,t) = \liminf_{\substack{(y,s)\to(x,t) \\ \epsilon \searrow 0}} u^\epsilon(y,s). \tag{5.3}$$

Proposition 5.1. *Assume that u_0 is continuous. Then $\overline{u}$ and $\underline{u}$ are viscosity sub- and supersolution of (1.1) respectively.*

Remark 5.2. The paper [37] deals mainly with the game interpretation to (1.1) without driving force and without source terms. They give the proof of the convergence of the value functions in this case. They also mention the game interpretation to (1.1) with driving force and higher dimensional case. Generalizing our game interpretation to the case $d \geq 3$, Paul chooses $w_i \in S^{d-1}$ and $d-1$ orthogonal unit vectors $v_i^j \in S^{d-1} (j = 1, 2, \cdots d-1)$, Carol chooses $d-1$ values $b_i^j \in \{\pm 1\} (j = 1, 2, \cdots d-1)$ and the control system becomes $x_i = x_{i-1} + \sqrt{2}\epsilon \sum_{j=1}^{d-1} b_i^j v_i^j + \nu\epsilon^2 w_i$.

Remark 5.3. If the source term f is continuous, we see $\overline{u} = \underline{u}$ from the standard comparison principle. This reduces our convergence result to

$$\lim_{\epsilon \searrow 0} u^\epsilon(x,t) = u(x,t),$$

where u is the unique viscosity solution. Moreover it is well known that this convergence is locally uniform one. Even if only the weak comparison principle, which is stated in Theorem 2.6, holds, then any solution $w \in SOL$ is characterized by $\underline{u} \leq w \leq \overline{u}$, provided that $\overline{u}$ and $\underline{u}$ respectively satisfy (2.4) and (2.5).

5.2 Notions of strategy

In our game we see from (5.2) that each player chooses many values with complex dependency. Strategy is a notion of which each player chooses one mathematical object through the game and a notion of mathematical objects we are going to construct later. In this subsection we verify that so called 'feedback strategy' is such a notion of discrete games including our game.

For simplicity we denote Paul's choice (v_i, w_i) at each round i by a_i. From (5.2) the dependency of choices of Paul is the following:

$$a_1, a_2(a_1, b_1), a_3(a_1, b_1, a_2, b_2), \cdots .$$

Since a_1 has no dependency (except for the variables (x, t, ϵ) of the game), the dependency is reduced to

$$a_1, a_2(b_1), a_3(b_1, a_2, b_2), \cdots .$$

Since a_2 depends on b_1, we have

$$a_1, a_2(b_1), a_3(b_1, b_2), \cdots$$

and so on. Similarly the dependency of choices of Carol is the following:

$$b_1(a_1), b_2(a_1, a_2), b_3(a_1, a_2, a_3), \cdots .$$

To describe such a situation, we first introduce the following maps, suggesting the relation between differential games and discrete games.

Definition 5.4 (Non-anticipating map)**.** Let A and B be sets. Let $\mathscr{A} = A^N$ and $\mathscr{B} = B^N$. We denote the i-th component of $\alpha \in \mathscr{A}$ by $\alpha(i)$ or α_i.

1. A map $\eta : \mathscr{A} \to \mathscr{B}$ is called *non-anticipating map* if for all $j \in \{1, 2, \cdots, N\}$ and all $\alpha, \tilde{\alpha} \in \mathscr{A}$, $\alpha(i) = \tilde{\alpha}(i)$ for all i with $1 \leq i \leq j$ implies $\eta[\alpha](i) = \eta[\tilde{\alpha}](i)$ for all i with $1 \leq i \leq j$.
2. A map $\eta : \mathscr{A} \to \mathscr{B}$ is called *slightly delayed non-anticipating map* if it satisfies the following conditions for all $\alpha, \tilde{\alpha} \in \mathscr{A}$.
   (a) For all $j \in \{1, 2, \cdots, N-1\}$, $\alpha(i) = \tilde{\alpha}(i)$ for all i with $1 \leq i \leq j$ implies $\eta[\alpha](i+1) = \eta[\tilde{\alpha}](i+1)$ for all i with $0 \leq i \leq j$.
   (b) $\eta[\alpha](1) = \eta[\tilde{\alpha}](1)$.

Remark 5.5. If $\eta : \mathscr{A} \to \mathscr{B}$ is a slightly delayed non-anticipating map, then it is a non-anticipating map.

As for our game, we fix variables of the game (x, t, ϵ). Let $\mathscr{A}$ and $\mathscr{B}$ be the sets of admissible controls of Paul and Carol respectively, i.e.

$$\mathscr{A} = \left\{ \{(v_n, w_n)\}_{n=1}^{\lceil t\epsilon^{-2} \rceil} \;\middle|\; v_n, w_n \in S^1 \right\}, \quad \mathscr{B} = \left\{ \{b_n\}_{n=1}^{\lceil t\epsilon^{-2} \rceil} \;\middle|\; b_n \in \{\pm 1\} \right\}.$$

When Paul and Carol take controls $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$ respectively, we denote the value of the game by $J^\epsilon(x, t, \alpha, \beta)$. We call a map $\eta : \mathscr{A} \to \mathscr{B}$ *non-anticipating strategy* of Carol if it is a non-anticipating map. Then the value function u^ϵ is written as

$$u^\epsilon(x, t) = \sup_{\eta \in \mathscr{H}} \inf_{\alpha \in \mathscr{A}} J^\epsilon(x, t, \alpha, \eta[\alpha]),$$

where $\mathscr{H}$ is the set of non-anticipating strategies of Carol. The proof of this fact is similar to that of [2, Chapter 8 Theorem 3.18], so is omitted. On the other hand, we call a map $\zeta : \mathscr{B} \to \mathscr{A}$ *non-anticipating strategy* of Paul if it is a slightly delayed non-anticipating map. Then the value function is also expressed as

$$u^\epsilon(x, t) = \inf_{\zeta \in \mathscr{Z}} \sup_{\beta \in \mathscr{B}} J^\epsilon(x, t, \zeta[\beta], \beta),$$

where $\mathscr{Z}$ is the set of non-anticipating strategies of Paul. For a fixed non-anticipating strategy ζ of Paul, we let

$$V_\zeta(x, t, \epsilon) := \sup_{\beta \in \mathscr{B}} J^\epsilon(x, t, \zeta[\beta], \beta). \tag{5.4}$$

For a fixed non-anticipating strategy η of Carol, we let

$$V^\eta(x, t, \epsilon) := \inf_{\alpha \in \mathscr{A}} J^\epsilon(x, t, \alpha, \eta[\alpha]).$$

Then $V^\eta(x,t,\epsilon) \le u^\epsilon(x,t) \le V_\zeta(x,t,\epsilon)$ obviously holds.

Now we introduce the notion of feedback strategy as a special class of non-anticipating strategy. In the notion of non-anticipating strategy, one player knows the current and past choices of the control made by the other player. In the notion of feedback strategy, one player knows just the current position and its past positions. So the dependency of choices of Paul is the following.

$$a_1(x_0), a_2(x_0,x_1), a_3(x_0,x_1,x_2), \cdots .$$

If Paul decides one profile with this information pattern, the corresponding non-anticipating strategy is automatically given. Indeed, for the variables (t,ϵ) of the game, a_1 is determined by x_0. Since Paul has already prepared the function $(x_0,x_1) \mapsto a_2$ and x_1 is determined by (x_0,a_1,b_1), only the value of b_1 is needed to determine a_2. Similarly the values of b_1 and b_2 are needed to determine a_3. In this meaning, the set of feedback strategies is naturally embedded in the set of non-anticipating strategies. See [2, Chapter 8 Lemma 3.5] for the corresponding statement of differential games. Later in this subsection, we concisely define feedback strategy and prove the discrete version of [2, Chapter 8 Lemma 3.5].

Definition 5.6 (Feedback strategy)**.** Let $(x,t,\epsilon) \in \mathbb{R}^2 \times (0,\infty) \times (0,\infty)$. For the game introduced in Section 5.1, we call a slightly delayed non-anticipating map $\mathscr{X} \to \mathscr{A}$ a *feedback strategy* of Paul, where

$$\mathscr{X} := \left\{ \{x_i\}_{i=1}^{\lceil t\epsilon^{-2} \rceil} \,\middle|\, \{x_i\}_{i=1}^{\lceil t\epsilon^{-2} \rceil} \text{ satisfies (5.1) with } x_0 = x \text{ for some } \alpha \in \mathscr{A} \text{ and } \beta \in \mathscr{B} \right\}.$$

We denote the set of feedback strategies of Paul by $\mathscr{F}$.

For simplicity we temporarily denote (5.1) by

$$x_i = x_{i-1} + g(a_i, b_i).$$

Proposition 5.7. *1. For $\zeta \in \mathscr{F}$ and $\beta = \{b_i\}_i \in \mathscr{B}$, there is a unique solution $\{x_i\}_i = \chi(\zeta,\beta) \in \mathscr{X}$ to the following equation.*

$$\begin{cases} x_i = x_{i-1} + g(\zeta[\chi(\zeta,\beta)]_i, b_i), \ i = 1,2,\cdots \\ x_0 = x. \end{cases} \tag{5.5}$$

2. For $\zeta \in \mathscr{F}$, $\chi(\zeta,\cdot) : \mathscr{B} \to \mathscr{X}$ is a non-anticipating map.

3. $\zeta[\chi(\zeta,\cdot)] : \mathscr{B} \to \mathscr{A}$ is a non-anticipating strategy of Paul.

Proof. 1. We prove by induction with respect to j that

$$\text{the equation (5.5) restricted to } 1 \le i \le j \text{ has a unique solution.} \tag{5.6}$$

Since $\zeta[\chi]_1$ is determined regardless of $\chi = \{x_i\}_i \in \mathscr{X}$, the condition (5.6) holds for $j = 1$. Assume that (5.6) holds for j. Let

$$\tilde{\mathscr{X}} := \left\{ \chi = \{x_i\} \in \mathscr{X} \,\middle|\, \begin{array}{l} \{x_i\}_{i=0}^{j} \text{ is the unique solution} \\ \text{to the equation (5.5) restricted to } 1 \le i \le j \end{array} \right\}.$$

Since $\zeta[\chi]_{j+1}$ is determined regardless of $\chi \in \tilde{\mathscr{X}}$, the position x_{j+1} is uniquely determined. Therefore the condition (5.6) holds for $j+1$ and we obtain the conclusion.

2. Let $\beta = \{b_i\}_i \in \mathscr{B}$ and $\tilde{\beta} = \{\tilde{b}_i\}_i \in \mathscr{B}$. We denote $\chi(\zeta,\beta) \in \mathscr{X}$ by $\chi = \{x_i\}_i$ and $\chi(\zeta,\tilde{\beta}) \in \mathscr{X}$ by $\tilde{\chi} = \{\tilde{x}_i\}_i$. First the relation $b_1 = \tilde{b}_1$ implies $x_1 = \tilde{x}_1$ since $\zeta[\chi]_1 = \zeta[\tilde{\chi}]_1$. We assume that if $b_i = \tilde{b}_i$ for all $i \in \{1,2,\cdots,k\}$, then $x_i = \tilde{x}_i$ for all $i \in \{1,2,\cdots,k\}$. If $b_i = \tilde{b}_i$ for all $i \in \{1,2,\cdots,k+1\}$, we have $x_i = \tilde{x}_i$ for all $i = 1,2,\cdots,k$ and $\zeta[\chi]_{k+1} = \zeta[\tilde{\chi}]_{k+1}$. Therefore we obtain $x_{k+1} = \tilde{x}_{k+1}$ and conclude by induction that $\chi(\zeta,\cdot)$ is a non-anticipating map.

3. Since $\zeta[\chi(\zeta,\cdot)] : \mathscr{B} \to \mathscr{A}$ is the composition of the slightly delayed non-anticipating map $\zeta \in \mathscr{F}$ and the non-anticipating map $\chi(\zeta,\cdot)$, it is a slightly delayed non-anticipating map.

□

5.3 The inverse game

As a natural modification of the game, we consider the "inverse game". By this we mean the game with the same rules but the opposite goals. Its value function is given by

$$v^\epsilon(x,t) = \sup_{v_1,w_1} \min_{b_1} \sup_{v_2,w_2} \min_{b_2} \ldots \sup_{v_N,w_N} \min_{b_N} \left[v_0(x_N) + \sum_{i=0}^{N-1} \epsilon^2 g(x_i) \right].$$

We set the limit continuum as (5.3).

$$\overline{v}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ \epsilon\searrow 0}} v^\epsilon(y,s), \quad \underline{v}(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ \epsilon\searrow 0}} v^\epsilon(y,s).$$

Actually v is a viscosity solution of

$$\begin{cases} v_t - \nu|Dv| + F(Dv, D^2v) = g(x) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{5.7}$$

where $\nu > 0$.

The difference from (1.1) is the sign of coefficient of $|Dv|$.

Proposition 5.8. *Assume that v_0 is continuous. Then $\overline{v}$ and $\underline{v}$ are viscosity sub- and supersolution of* (5.7) *respectively.*

Proof. Let $u^\epsilon(y,s) := -v^\epsilon(y,s)$. From the definition of $v^\epsilon(y,s)$, we have

$$u^\epsilon(x,t) = \inf_{v_1,w_1} \max_{b_1} \inf_{v_2,w_2} \max_{b_2} \ldots \inf_{v_N,w_N} \max_{b_N} \left[-v_0(x_N) - \sum_{i=0}^{N-1} \epsilon^2 g(x_i) \right].$$

This $u^\epsilon(y,s)$ is the value function defined in Section 5 with $u_0 = -v_0$ and $f = -g$. So $\overline{u} = -\underline{v}$ and $\underline{u} = -\overline{v}$ are respectively viscosity sub- and supersolution of

$$\begin{cases} u_t + \nu|Du| + F(Du, D^2u) = -g(x) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = -v_0(x) & \text{in } \mathbb{R}^2. \end{cases}$$

Hence changing this equation to that on v, we obtain (5.7). □

6 Application

In this section we give applications of the game interpretation introduced in Section 5.

6.1 Basic strategy of the game

In this subsection, we prepare special non-anticipating strategies of each player, which are easy to understand geometrically and are components of strategies we require in the proofs later. The strategies of Paul below are especially *feedback memoryless strategies*, which are feedback strategies without dependence of past trajectories. In other words, Paul selects his choices in such strategies by solely referring to his current position. Thus we can write such a strategy as a map $\mathbb{R}^2 \to S^1 \times S^1$.

Definition 6.1 (Concentric strategy)**.** Let $\nu \in \mathbb{R}$, $\epsilon > 0$ and $z \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ be the current position of the game.

1. A choice $(v,w) \in S^1 \times S^1$ by Paul is called a z *concentric strategy* (by Paul) if

   $$w = H(\nu)\frac{z-x}{|z-x|}, \qquad v \perp w,$$

   where we define

   $$H(\nu) := \begin{cases} 1, & \nu \geq 0 \\ -1, & \nu < 0. \end{cases}$$

   When $x = z$, any $(v,w) \in S^1 \times S^1$ satisfying $v \perp w$ is called a z *concentric strategy.*

2. Let $(v, w) \in S^1 \times S^1$ be a choice by Paul in the same round. A choice $b \in \{\pm 1\}$ by Carol is called a z *concentric strategy* (by Carol) if

$$\langle bv, x + \nu\epsilon^2 w - z\rangle \geq 0.$$

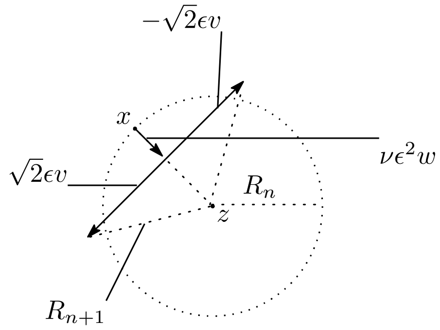


Figure 27: z concentric strategy

We investigate the behaviors of trajectories given when one player takes above strategies. Later in this subsection, we assume $\nu > 0$. We can apply the follwing argument to the case $\nu < 0$ by replacing ν by $|\nu|$. A z concentric strategy has the ability to control the distance from z to game positions. Let $d_n = |x_n - z|$ for fixed $z \in \mathbb{R}^2$. If Paul takes a z concentric strategy through the game, then we see that regardless of Carol's choices, the sequence $\{d_n\}$ satisfies

$$R_{n+1} = \sqrt{(R_n - \nu\epsilon^2)^2 + 2\epsilon^2} \tag{6.1}$$

by the Pythagorean theorem. See Figure 27.

Also a z concentric strategy of Carol is a maximizer of $\Phi(b) := |x - z + \sqrt{2}\epsilon bv + \nu\epsilon^2 w|$. If Carol takes z concentric strategy through the game, we have $d_n \geq R_n$, where R_n satisfies (6.1) with $R_0 = d_0$. Both players can control the distance from z to game positions by taking z concentric strategy. We notice that not only Paul but also Carol can not control moves in the direction perpendicular to $x_n - z$ in this strategy.

In what follows, we state the properties of the sequences satisfying (6.1).

Remark 6.2. When $\nu = 0$, the solution of the equation (6.1) is explicitly obtained by $R_n = \sqrt{R_0^2 + 2n\epsilon^2}$.

Lemma 6.3 ([49]). *Fix $\nu > 0$. Let $\epsilon > 0$ and $\{R_n\}$ be a sequence satisfying the condition* (6.1). *Then the following properties hold.*

1. *If $R_0 = \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n = \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n. If $R_0 > \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n > \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n and $\{R_n\}$ is decreasing for $\epsilon \leq \frac{\sqrt{2}}{\nu}$. If $\nu\epsilon^2 \leq R_0 < \nu^{-1} + \frac{\nu}{2}\epsilon^2$, then $R_n < \nu^{-1} + \frac{\nu}{2}\epsilon^2$ for all n and $\{R_n\}$ is increasing.*

2. $$\lim_{n\to\infty} R_n = \nu^{-1} + \frac{\nu}{2}\epsilon^2.$$

It is sometimes convenient to describe the behavior of the trajectory by an operator. For fixed $\nu > 0$, we define the operator $T_h : \mathbb{R} \to \mathbb{R}$ as

$$T_h(R) := \sqrt{(R - \nu h)^2 + 2h}.$$

We denote n times composition of T_h by T_h^n. The solution $\{R_n\}$ to (6.1) with $R_0 = R$ is described by $R_n = T_{\epsilon^2}^n(R)$.

Lemma 6.4. *$T_h(R) < T_h(R')$, provided $\nu h \leq R < R'$.*

Proof. The proof is done by direct computation, so is omitted. □

To see the behavior of sequences satisfying (6.1), we prepare a notation $t_\epsilon(a,b)$. For $a,b \geq 0$ and $\epsilon > 0$ satisfying $a \leq b \leq \nu^{-1} + \frac{\nu}{2}\epsilon^2$, we define

$$t_\epsilon(a,b) := \epsilon^2 |\{n \in \mathbb{N} \mid T^n_{\epsilon^2}(a) < b\}|,$$

where we denote the set $\{0, 1, 2, \cdots\}$ by $\mathbb{N}$. For a, b satisfying $\nu^{-1} + \frac{\nu}{2}\epsilon^2 \leq b \leq a$, we define

$$t_\epsilon(a,b) := \epsilon^2 |\{n \in \mathbb{N} \mid b < T^n_{\epsilon^2}(a)\}|.$$

In a radially symmetric setting of the equation (1.1), $(0,0)$ concentric strategies are optimal strategies. Let $u_0(x) = \phi(|x|)$ and $f(x) = \psi(|x|)$ where ϕ and ψ are nondecreasing. Then$(0,0)$ concentric strategies by Paul and by Carol are respectively optimal strategies. Above $t_\epsilon(a,b)$ means time to exit from the set $B((0,0),b) \setminus B((0,0),a)$ (or $B((0,0),a) \setminus B((0,0),b)$) when Paul starts at x_0 with $|x_0| = a$ and takes a $(0,0)$ concentric strategy.

Now let us consider the case $u_0 = 0$ and $f(x) = \chi_\Omega(x)$, where $\Omega = B((0,0),R)^c$. When $R < \nu^{-1}$, Lemma 6.3 implies that Paul cannot exit from Ω and pays running cost through the game if he starts in Ω and Carol takes a $(0,0)$ concentric strategy. In other words, the value functions converge to 1 in Ω as $\epsilon \searrow 0$. When $R > \nu^{-1}$, Paul can stay in $B((0,0),R)$ if he starts there. This means the value functions converge to 0 in $B((0,0),R)$ as $\epsilon \searrow 0$. The critical case $R = \nu^{-1}$ is difficult because the limit of the sequence $\{R_n\}$ is a little larger than the critical radius ν^{-1}. However the following Lemma 6.5.1 implies that Paul can stay in $B((0,0),R)$ by taking sufficiently small ϵ that depends on terminal time t of the game. It is worth emphasizing that the limit function of value functions are discontinuous on $\partial B((0,0),R)$ because Paul cannot exit from Ω. As for assertions on solutions for these settings, see Proposition 2.13. Similar things hold for the inverse game explained in Section 5.3 by replacing $\Omega = B((0,0),R)^c$ with $\Omega = B((0,0),R)$.

We use Lemma 6.5 in the proof of Lemma 6.8 in the next subsection.

Lemma 6.5. *Let $\nu > 0$. Then the following hold.*

1. $$t_\epsilon\left(\nu^{-1} - \epsilon, \nu^{-1}\right) \geq C \log_2 \epsilon^{-1}$$

   for some constant $C > 0$.

   Especially, if $a < \nu^{-1}$, then

   $$\lim_{\epsilon \searrow 0} t_\epsilon\left(a, \nu^{-1}\right) = \infty.$$

2. *Let $\delta > 0$ and $R > \nu^{-1} + \delta$. Then there exist $A > 0$ and $\epsilon_0 > 0$ such that $t_\epsilon\left(R, \nu^{-1} + \delta\right) \leq AR$ for all $\epsilon \in (0, \epsilon_0)$.*

Proof. 1. The proof is in [49, Lemma 4.4], so is omitted.

2. Let n^* be the maximal n that satisfies $\nu^{-1} + \delta < T^n_{\epsilon^2}(R)$. Then we get by Lemma 6.4

$$\sqrt{(\nu^{-1} + \delta - \nu\epsilon^2)^2 + 2\epsilon^2} < T^{n^*+1}_{\epsilon^2}(R) \leq \nu^{-1} + \delta. \tag{6.2}$$

We take ϵ small enough to satisfy $\nu^{-1} + \frac{\delta}{2} \leq \sqrt{(\nu^{-1} + \delta - \nu\epsilon^2)^2 + 2\epsilon^2}$, $\nu^{-1} + \delta \leq \sqrt{(\nu^{-1} + 2\delta - \nu\epsilon^2)^2 + 2\epsilon^2}$, and $\epsilon \leq \sqrt{\delta\nu^{-1}}$. The condition $\epsilon \leq \sqrt{\delta\nu^{-1}}$ guarantees $T^{n+1}_{\epsilon^2}(R) \geq T^n_{\epsilon^2}(R)$.

From (6.2) we have

$$\nu^{-1} + \frac{\delta}{2} < T^{n^*+1}_{\epsilon^2}(R) \leq \nu^{-1} + \delta.$$

We also have

$$\nu^{-1} + \delta \leq T^{n^*}_{\epsilon^2}(R) \leq \nu^{-1} + 2\delta.$$

This is because $\nu^{-1}+2\delta < T_{\epsilon^2}^{n^*}(R)$ implies $\nu^{-1}+\delta < T_{\epsilon^2}^{n^*+1}(R)$. We notice that $|T_{\epsilon^2}^{n+1}(R)-T_{\epsilon^2}^{n}(R)|$ is decreasing with respect to n. Then we have

$$\begin{aligned} t_\epsilon(R,\nu^{-1}+\delta) &\le t_\epsilon(R,T_{\epsilon^2}^{n^*+1}(R)) \le \left\lceil \frac{R-T_{\epsilon^2}^{n^*+1}(R)}{|T_{\epsilon^2}^{n^*+1}(R)-T_{\epsilon^2}^{n^*}(R)|}\right\rceil \epsilon^2 \\ &\le \left(\frac{R-\nu^{-1}-\frac{\delta}{2}}{|T_{\epsilon^2}^{n^*+1}(R)-T_{\epsilon^2}^{n^*}(R)|}+1\right)\epsilon^2 \\ &= \frac{T_{\epsilon^2}^{n^*}(R)+T_{\epsilon^2}^{n^*+1}(R)}{|2(1-\nu T_{\epsilon^2}^{n^*}(R))\epsilon^2+\nu^2\epsilon^4|}\left(R-\nu^{-1}-\frac{\delta}{2}\right)\epsilon^2+\epsilon^2. \qquad (6.3) \end{aligned}$$

The first inequality is derived from $T_{\epsilon^2}^{n^*+1}(R) \le \nu^{-1}+\delta$. The second inequality is an estimation by the sequence with the constant speed $|T_{\epsilon^2}^{n^*+1}(R)-T_{\epsilon^2}^{n^*}(R)|$. Moreover we get

$$\begin{aligned} \frac{T_{\epsilon^2}^{n^*}(R)+T_{\epsilon^2}^{n^*+1}(R)}{|2(1-\nu T_{\epsilon^2}^{n^*}(R))\epsilon^2+\nu^2\epsilon^4|} &\le \frac{2\nu^{-1}+3\delta}{2(\nu T_{\epsilon^2}^{n^*}(R)-1)\epsilon^2+\nu^2\epsilon^4} \\ &\le \frac{2\nu^{-1}+3\delta}{2\nu\delta\epsilon^2+\nu^2\epsilon^4} \le \frac{2\nu^{-1}+3\delta}{\nu\delta}. \end{aligned}$$

Together with (6.3) we obtain

$$t_\epsilon(R,\nu^{-1}+\delta) \le \frac{2\nu^{-1}+3\delta}{\nu\delta}\left(R-\nu^{-1}-\frac{\delta}{2}\right)\epsilon^2+\epsilon^2 \le \left(\frac{2}{\nu^2\delta}+\frac{3}{\nu}\right)R.$$

□

Remark 6.6. In light of Theorem 4.6, if we assume Ω is star shaped and satisfies exterior sphere condition (3.3), the support of $u(\cdot,t)$ is included in $\overline{\Omega}$. This fact is also implied by the game. Indeed, let $x \in \overline{\Omega}^c$ and z be the center of an exterior sphere that includes x. Then Paul can avoid any cost by taking z concentric strategy. If we permit exterior spheres with critical radius ν^{-1}, the limit of value functions may be discontinuous as explained before. Also it is the same for solutions. See Proposition 4.8 for example.

6.2 Asymptotic speed of solutions for domain of touching balls ($\nu < 0$)

Let $\nu < 0$, $d = 2$, $u_0 = 0$ and $f = c\chi_\Omega$ ($c > 0$) and

$$\Omega = \overline{B((-|\nu|^{-1},0),|\nu|^{-1})} \cup \overline{B((|\nu|^{-1},0),|\nu|^{-1})}.$$

To see the connection with the previous study, we write the problem with negative ν in this subsection. We can convert the problem to that with positive ν by Proposition 2.11. Also, without loss of generality, we assume $\nu = -1$ and

$$\Omega = \overline{B((-1,0),1)} \cup \overline{B((1,0),1)}$$

later in this subsection.

Let us first consider this problem heuristically in line with the idea of Trotta-Kato product formula. At the first time step, a thin crystal in the shape of Ω is stacked as Figure 23. Then the crystal grows horizontally. However one may ask for which curve enclosing Ω one should consider the horizontal growth. There are at least two possible interpretations. One interpretation is that two circles enclose Ω as Figure 29. The other is that one curve encloses Ω as Figure 30. If we adopt the former interpretation, there is no horizontal growth, and hence the crystal will have seemed to be stacked vertically, keeping in the shape of Ω. If we adopt the latter interpretation, both curvature and driving force at the origin make the crystal spread.

Actually the paper [22] gives an explicit viscosity solution $u(x,t) = c\chi_\Omega(x)t$, and proves that there exists another solution whose asymptotic speed is strictly greater than 0 regardless of $x \in \mathbb{R}^2$. We improve this result to that the asymptotic speed is c.

In our proof, we construct a feedback strategy of Paul for each variables of the game (y,s,ϵ). Actually it is sufficient to consider variables with sufficiently small ϵ since we are concerned about the half relaxed limits $\overline{u}$ and $\underline{u}$. Precisely we call a function $\epsilon_0 : \mathbb{R}^2 \times (0,\infty) \to (0,\infty)$ locally positive if for any compact subset K of $\mathbb{R}^2 \times (0,\infty)$,

$$\inf_{(x,t)\in K} \epsilon_0(x,t) > 0.$$

Let $\epsilon_0 : \mathbb{R}^2 \times (0,\infty) \to (0,\infty)$ be a locally positive function. Then the set of variables we have to consider is

$$A_{\epsilon_0} := \{(x,t,\epsilon) \in \mathbb{R}^2 \times (0,\infty) \times (0,\infty) \mid \epsilon < \epsilon_0(x,t)\}.$$

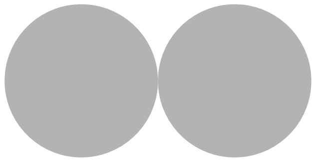


Figure 28: Shape of domain

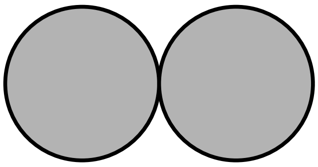


Figure 29: Two circles enclosing Ω

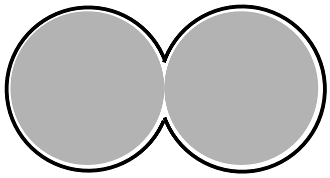


Figure 30: One curve enclosing Ω

The following lemma guarantees that Paul can enter the convex hull of Ω from everywhere when he takes the strategy stated in Lemma 6.8.

Lemma 6.7. *Let $r \in (0,1]$, $C_1 = \{(x,y) \in \mathbb{R}^2 \mid (x-r)^2 + y^2 = r^2\}$, and $C_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y+a)^2 = (1+a+b)^2\}$. Then the following statements are equivalent.*

1. $$\frac{3-\sqrt{3}}{2} < r \leq 1.$$

2. *There exist $a > 0$ and $b > 0$ such that C_1 and C_2 intersect at different two points and $a + r > 1$.*

Proof. Let $d(\theta)$ be the distance between $(0,-a)$ and $(r(1+\sin\theta), r\cos\theta)$. Then we see

$$\sqrt{a^2+r^2} - r \leq d(\theta) \leq \sqrt{a^2+r^2} + r.$$

Thus C_1 and C_2 intersect at different two points if and only if

$$\begin{aligned}\sqrt{a^2+r^2} - r &< 1 + a + b \\ &< \sqrt{a^2+r^2} + r.\end{aligned} \tag{6.4}$$

Here (6.4) is true for any $a > 0$, $b > 0$ and $r > 0$. Thus the condition 2 is transformed equivalently by elementary computation as follows:

$$1 + a + b < \sqrt{a^2+r^2} + r \text{ and } a + r > 1 \text{ for some } a > 0 \text{ and } b > 0.$$

$$\iff 1 + a < \sqrt{a^2+r^2} + r \text{ and } a + r > 1 \text{ for some } a > 0.$$

$$\iff 2a(1-r) < 2r - 1 \text{ and } 1 - r < a \text{ for some } a > 0.$$

$$\iff 2(1-r)^2 < 2r - 1.$$

$$\iff \frac{3-\sqrt{3}}{2} < r \leq 1.$$

□

Lemma 6.8. *There are a locally positive function ϵ_0 and a locally bounded function $\psi : \mathbb{R}^2 \to [0, \infty)$ such that for all $(y, s, \epsilon) \in A_{\epsilon_0}$,*

$$c\epsilon^2 \lceil s\epsilon^{-2} \rceil - \psi(y) \leq V_\zeta(y, s, \epsilon) \tag{6.5}$$

holds for some feedback strategy $\zeta \in \mathscr{A}$.

Proof. Note that the corresponding game is now 'inverse game' introduced in Section 5.3, because ν is negative. So V_ζ is defined as

$$V_\zeta(y, s, \epsilon) := \inf_{\beta \in \mathscr{B}} J^\epsilon(y, s, \zeta[\beta], \beta)$$

instead of (5.4). We denote by x_n the position at round n in the game with variables (y, s, ϵ). We are going to construct feedback strategies $\zeta \in \mathscr{A}$ for each (y, s, ϵ) that are not necessarily optimal ones. The strategies are combinations of the following six strategies:

Strategy I : $(0, 0)$ concentric strategy,
Strategy II : $(1, 0)$ concentric strategy,
Strategy III : $(-1, 0)$ concentric strategy,
Strategy IV : $(0, -1/2)$ concentric strategy,
Strategy V : $(0, 1/2)$ concentric strategy,
Strategy VI : a strategy where Paul takes $v = (1, 0)$ and $w = (0, -1)$ if he is in $\mathbb{R} \times [0, \infty)$ and takes $v = (1, 0)$ and $w = (0, 1)$ if he is in $\mathbb{R} \times (-\infty, 0)$.

From the definition of V_ζ, we see $V_\zeta(y, s, \epsilon) = c\epsilon^2 \lceil s\epsilon^{-2} \rceil - cn^*(y, s, \epsilon)\epsilon^2$, where we denote by n^* the maximal number of rounds such that $x_n \in \Omega^c$. Thus, to prove (6.5), we show that for any initial position $x_0 = y$, Paul can enter Ω in at most finite time and remain there until the game ends. We do case analysis for the initial positions $x_0 = y$.

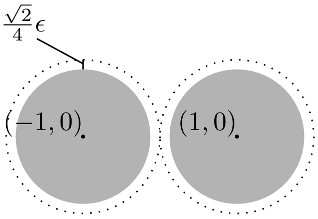


Figure 31: Shape of D_1

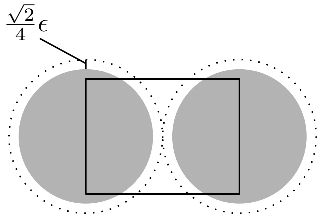


Figure 32: Shape of D_2

Strategy $\boldsymbol{\zeta}$ We prepare the following notations of sets in $\mathbb{R}^2$.

$$\begin{aligned}
D_1 &:= \overline{B\left((-1,0), 1 - \frac{\sqrt{2}}{4}\epsilon\right)} \cup \overline{B\left((1,0), 1 - \frac{\sqrt{2}}{4}\epsilon\right)} \text{ (Figure 31)}, \\
D_2 &:= ([-1,1] \times [-3/4, 3/4]) \setminus D_1, \\
D_3 &:= \left(\overline{B\left((0,0), 1 + \frac{\sqrt{5}-1}{4}\right)} \cap (\mathbb{R} \times [0, \infty))\right) \setminus (D_1 \cup D_2), \\
D_4 &:= \left(\overline{B\left((0,0), 1 + \frac{\sqrt{5}-1}{4}\right)} \cap (\mathbb{R} \times (-\infty, 0))\right) \setminus (D_1 \cup D_2), \\
D_5 &:= \mathbb{R}^2 \setminus \left(\overline{B\left((0,0), 1 + \frac{\sqrt{5}-1}{4}\right)} \cup D_1 \cup D_2\right).
\end{aligned}$$

1) $\boldsymbol{x_0 \in D_1}$.

If $x_0 \in \overline{B\left((1,0), 1 - \frac{\sqrt{2}}{4}\epsilon\right)}$, Paul keeps taking Strategy II until the game ends. By the properties of concentric strategies stated in Lemma 6.3 and Lemma 6.5.1, it turns out that Paul keeps staying in Ω for $\epsilon \leq \min\{2^{-2s/C}, D\}$, where we let $C > 0$ be a constant in Lemma 6.5.1 and $D > 0$ be a small constant taken in Lemma 6.3. See also Remark 6.6. If $x_0 \in \overline{B\left((-1,0), 1-\frac{\sqrt{2}}{4}\epsilon\right)}$, Paul keeps taking Strategy III until the game ends. Similarly he keeps staying in Ω by doing it.

2) $x_0 \in D_2$.

In this case, Paul keeps taking Strategy VI until he reaches D_1. Indeed he reaches D_1. By his doing this strategy, the distance between the current position x_n and x-axis decreases ϵ^2 per round. Thus, even if Carol prevent Paul from entering D_1 as long as possible, Paul can enter D_1 by way of the following set D_6.

$$D_6 := \left([-1,1] \times \left[-\epsilon^2/2, \epsilon^2/2\right]\right) \setminus D_1$$

Since $D_6 \pm \sqrt{2}\epsilon(1,0) \pm \epsilon^2(0,1) \subset D_1$, Paul can enter D_1 from D_3 by just one round. See Figure 33. Hence it takes at most $\lceil \frac{3}{4\epsilon^2} \rceil + 1$ round for Paul to enter D_1 from any point in D_2.

3) $x_0 \in D_3$.

In this case, Paul keeps taking Strategy IV until he reaches D_1 or D_2. Indeed Lemma 6.7 guarantees that he actually reaches D_1 or D_2. This elementary lemma implies that if $r = 1$, there exist $a > 0$ and $b > 0$ such that C_1 and C_2 intersect at different two points (Figure 34). The values $a = 1/2$ and $b = \frac{\sqrt{5}-1}{4}$ are such a and b for example. If Carol prevent Paul from entering D_1, Paul keeps staying in the upper half space because circles with its center at $(0, -1/2)$ passing through x_n are divided by D_1. Paul can enter D_2 because the sequence $|x_n - (0, -1/2)|$ is monotonically decreasing with respect to n (Lemma 6.3 1) and gets less than $5/4$ in finite time (Lemma 6.5.2). Once Paul reaches D_1, he changes his strategy to that in the case 1). Once he reaches D_2, he changes his strategy to that in the case 2).

4) $x_0 \in D_4$.

Since this case and case 3) are symmetrical with respect to x-axis, Paul first takes Strategy V instead of Strategy IV. After he reaches D_1 or D_2, he takes the same strategies as in 3).

5) $x_0 \in D_5$.

In this case, Paul keeps taking Strategy I until he reaches D_1 or $\overline{B\left((0,0), 1+\frac{\sqrt{5}-1}{4}\right)}$. Even if Carol prevents him from entering D_1, he can enter $\overline{B\left((0,0), 1+\frac{\sqrt{5}-1}{4}\right)}$ in at most $A|x_0|$ seconds for some constant $A > 0$ by Lemma 6.5.2. If Paul reaches D_1, he takes the same strategies as in the case 1). If he reaches $\overline{B((0,0), 1+\frac{\sqrt{5}-1}{4})}$, he takes the same strategies as in the case 3) or 4).

Estimation of the value V_ζ In $\overline{B\left((0,0), 1+\frac{\sqrt{5}-1}{4}\right)}^c$, the time Paul is in Ω^c is at most $A|x_0|$ from the case 5). In $\overline{B\left((0,0), 1+\frac{\sqrt{5}-1}{4}\right)}$, the time he is in Ω^c is at most some finite time B from the cases 2), 3) and 4). To achieve such behavior of the game trajectories, we let $\epsilon_0 = \min\{2^{-2s/C}, D\}$ as in 1). We take a smaller constant $D > 0$ for the game trajectories in 3) or 4) not to cross over D_1 if necessary. Therefore, letting $\epsilon_0 = \min\{2^{-2s/C}, D\}$ and $\psi(y) = cA|y| + cB$, we obtain (6.5).

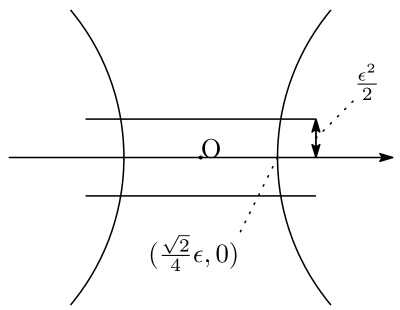


Figure 33: Shape of D_3

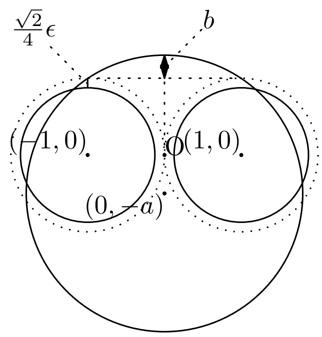


Figure 34: Strategy to enter $D_1 \cup D_2$

□

Theorem 6.9. *There exists a viscosity solution u satisfying*

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = c \quad \textit{locally uniformly for } x \in \mathbb{R}^2.$$

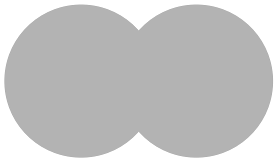


Figure 35: Shape of domain

Proof. From Lemma 6.8 and the rule of the game, we have

$$c\epsilon^2 \lceil s\epsilon^{-2} \rceil - \psi(y) \le V_\zeta(y, s, \epsilon) \le u^\epsilon(y, s) \le c\epsilon^2 \lceil s\epsilon^{-2} \rceil$$

for all $(y, s, \epsilon) \in A_{\epsilon_0}$. Letting $\epsilon \searrow 0$, $y \to x$, and $s \to t$, we get

$$ct - \psi_*(x) \le \overline{u}(x, t) \le ct.$$

Since $\overline{u}(x,t)$ and ct are viscosity sub- and supersolution respectively, and both functions are continuous at $t = 0$, we can use Perron's method. i.e., there exists a viscosity solution w such that

$$\overline{u}(x, t) \le w(x, t) \le ct.$$

Then we obtain

$$c - \frac{\psi_*(x)}{t} \le \frac{w(x, t)}{t} \le c.$$

This implies the statement of this theorem. ☐

When $\Omega = \overline{B}((-r, 0), 1) \cup \overline{B}((r, 0), 1)$ with $0 < r < 1$ (Figure 35), the following Lemma 6.10 plays the same role as Lemma 6.7. Hence, in this case, we see that there is a solution whose asymptotic speed is c.

Lemma 6.10. *Let $r \in (0, 1)$, $C_1 = \{(x, y) \in \mathbb{R}^2 \mid (x - r)^2 + y^2 = 1\}$, and $C_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y + a)^2 = (1 + a + b)^2\}$. Then there exist $a > 0$ and $b > 0$ such that C_1 and C_2 intersect at different two points.*

Remark 6.11. When $\Omega = \Omega_r := \overline{B}((-r, 0), r) \cup \overline{B}((r, 0), r)$, Lemma 6.7 implies that Paul can enter the convex hull of Ω even if $\frac{3-\sqrt{3}}{2} < r < 1$. However the case $\Omega = \Omega_r$ $(r < 1)$ is complicated. That is because Paul may be forced to exit from Ω. Actually, if $\Omega = \Omega_r$ with $\frac{3-\sqrt{3}}{2} < r < 1$, there exists a viscosity solution u satisfying

$$\lim_{t\to\infty} \frac{u(x, t)}{t} = \alpha \quad \text{locally uniformly for } x \in \mathbb{R}^2,$$

where $\alpha > 0$. We omit the proof, but roughly speaking, we construct a strategy ζ of Paul such that

$$\alpha s \le V_\zeta(y, s, \epsilon)$$

for some $\alpha > 0$.

6.3 Non uniqueness result $(\nu > 0)$

Converting the above problem to $\nu > 0$ in light of Proposition 2.11, we can construct a counter-example to a weak comparison principle without the assumption (2.1).

Let $\nu = 1$, $d = 2$, $u_0 = 0$, and
$\Omega = B((0, 0), R_0) \setminus \overline{(B((1, 0), 1) \cup B((-1, 0), 1))}$ (Figure 36). In this setting, the source term $f(x) = c\chi_\Omega(x)$ does not satisfy the assumption (2.1) and we can construct two solutions. Let

$$w_1(x, t) := \min\{U(x)\chi_\Omega(x), ct\},$$

where U is the function (2.12). To reduce difficulty, we just check that w_1 is a viscosity solution at least in a short time. We notice that the weak comparison principle (Theorem 2.6) holds for solutions defined in $\mathbb{R}^d \times [0, T)$ as can be seen from the proof [33, Theorem 3.1] under the assumption (2.1). Accordingly the uniqueness of solutions (Corollary 2.8 (2)) holds for solutions defined in $\mathbb{R}^d \times [0, T)$.

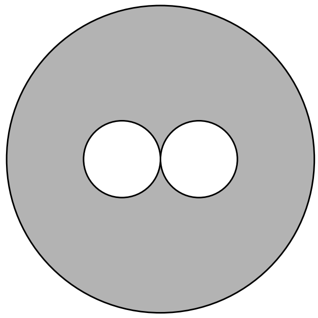


Figure 36: A domain that is not star shaped

Proposition 6.12. *Under the settings above, the function w_1 is a solution to* (1.4) *in* $\mathbb{R}^2 \times [0,T)$ *for some* $T > 0$.

Proof. We take $T > 0$ so small that the set $\{x \in B((0,0),R_0) \mid U(x) = cT\}$ does not touch $D := \partial(B((1,0),1) \cup B((-1,0),1))$. Let $x_0 \in \mathbb{R}^2$. Except for the case $x_0 \in D$, we can check that the viscosity inequalities hold at x_0 in the same way as the proof of Theorem 4.6. Also the case $x_0 \in D$ is similarly proved as [22, Proposition B.1]. □

On the other hand, we can construct a solution w_2 that satisfies $0 \leq w_2 \leq \underline{u}$ by Perron's method, where $\underline{u}$ is the lower limit of the value functions of the game. The following proposition states that w_1 and w_2 are different solutions even in the meaning of semicontinuous envelopes. (Corollary 2.8 (2))

Proposition 6.13. *For some non-empty open set* $\mathscr{O} \subset (\mathbb{R}^2 \times [0,T])$, $\overline{u} < w_1$ *in* $\mathscr{O}$.

Proof. We compare w_1 and $\overline{u}$ in the following open set.

$$\mathscr{O} = \{(x,y,t) \mid |x| < 1, |y| < t/2, (x,y) \in \Omega\}.$$

Since T is small, it is clear that $w_1(x,t) = ct$ in $\mathscr{O}$. On the other hand, by taking a strategy of Paul explained in the case 2) in Lemma 6.8, it takes at most $\lceil \frac{t}{2\epsilon^2} \rceil + 1$ rounds for Paul to enter $B((1,0),1) \cup B((-1,0),1)$. Once Paul enters $B((1,0),1) \cup B((-1,0),1)$, he can stay there by taking a strategy explained in the case 1) in Lemma 6.8. Thus we see that $\overline{u}(x,t) \leq \frac{c}{2}t$ in $\mathscr{O}$. Hence we conclude that $\overline{u} < w_1$ in $\mathscr{O}$. □

Remark 6.14. [22] suggests a notion of solutions to (1.1), *maximal solution*, which is defined as follows.

$$v(x,t) := \sup_{u \in S} u(x,t),$$

where S is the set of the viscosity solutions u satisfying the following.

$$u(x,t) = 0 \text{ for all } x \in B(0,R_T)^c,\ t \in [0,T] \text{ and some } R_T > 0.$$

In the same way, we can define *minimal solution* $w(x,t) := \inf_{u \in S} u(x,t)$. Based on these terms, the result in Section 6.2 ($\nu < 0$) means that the solution given by the game is not the minimal solution. In light of Proposition 2.11, it is natural that the game solution is not the maximal solution for the above problem ($\nu > 0$).

References

[1] Y. Achdou, G. Barles, H. Ishii, and G. L. Litvinov. *Hamilton–Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Mathematics*. Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2011.

[2] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhäuser Boston Inc., Boston, MA, 1997.

[3] G. Barles. Discontinuous viscosity solutions of first-order Hamilton–Jacobi equations: a guided visit. *Nonlinear Anal.*, 20:1123–1134, 1993.

[4] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31:439–469, 1993.

[5] K. A. Brakke. *The motion of a surface by its mean curvature.* Princeton University Press, Princeton, N.J., 1978.

[6] W. K. Burton, N. Cabrera, and F. C. Frank. The growth of crystals and the equilibrium structure of their surfaces. *Philos. Trans. Roy. Soc. London. Ser. A.*, 243:299–358, 1951.

[7] L. A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*, volume 43 of *Amer. Math. Soc. Colloq. Publ.* American Mathematical Society, Providence, RI, 1995.

[8] Y. G. Chen, Y. Giga, and S. I. Goto. Uniqueness and existence of viscosity solutions of generalized mean-curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[9] L. Codenotti, M. Lewicka, and J. Manfredi. Discrete approximations to the double-obstacle problem and optimal stopping of tug-of-war games. *Trans. Amer. Math. Soc.*, 369:7387–7403, 2017.

[10] M. G. Crandall, H. Ishii, and P. L. Lions. Users guide to viscosity solutions of 2nd-order partial-differential equations. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[11] M. G. Crandall, P. L. Lions, and P. E. Souganidis. Maximal solutions and universal bounds for some partial differential equations of evolution. *Arch. Rational Mech. Anal.*, 105:163–190, 1989.

[12] N. Dirr and V. D. Nguyen. Some new results on Lipschitz regularization for parabolic equations. *J. Evol. Equ.*, 19:1149–1166, 2019.

[13] L. C. Evans and J. Spruck. Motion of level sets by mean-curvature.1. *J. Differential Geom.*, 33:635–681, 1991.

[14] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability.* Springer, New York, second edition, 2006.

[15] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Differntial Geom.*, 26:69–96, 1986.

[16] Y. Giga. Viscosity solutions with shocks. *Commun. Pure Appl. Math.*, 55:431–480, 2002.

[17] Y. Giga. *Surface evolution equations. A level set approach.* Birkhäuser Verlag, Basel, 2006.

[18] Y. Giga and N. Hamamuki. Hamilton-Jacobi equations with discontinuous source terms. *Comm. Partial Differential Equations*, 38:199–243, 2013.

[19] Y. Giga and N. Hamamuki. On a dynamic boundary condition for singular degenerate parabolic equations in a half space. *NoDEA Nonlinear Differential Equations Appl.*, 25(6):Paper No. 51, 39, 2018.

[20] Y. Giga and Q. Liu. A billiard-based game interpretation of the neumann problem for the curve shortening equation. *Adv. Differential Equations*, 14(3-4), 2009.

[21] Y. Giga, H. Mitake, T. Ohtsuka, and H. V. Tran. Existence of asymptotic speed of solutions to birth-and-spread type nonlinear partial differential equations. *Indiana Univ. Math. J.*, 70:121–156, 2021.

[22] Y. Giga, H. Mitake, and H. V. Tran. On asymptotic speed of solutions to level-set mean curvature flow equations with driving and source terms. *SIAM J. Math. Anal.*, 48:3515–3546, 2016.

[23] Y. Giga, H. Mitake, and H. V. Tran. Remarks on large time behavior of level-set mean curvature flow equations with driving and source terms. *Discrete Contin. Dyn. Syst. Ser. B*, 25:3983–3999, 2020.

[24] Y. Giga and N. Požár. Viscosity solutions for the crystalline mean curvature flow with a nonuniform driving force term. *Partial Differ. Equ. Appl.*, 1:Paper No. 39, 26, 2020.

[25] Y. Giga, H. V. Tran, and L. Zhang. On obstacle problem for mean curvature flow with driving force. *Geometric flows*, 4:9–29, 2019.

[26] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.*, 26:285–314, 1987.

[27] N. Hamamuki. On large time behavior of Hamilton-Jacobi equations with discontinuous source terms. nonlinear analysis in interdisciplinary sciences modellings, theory and simulations. *GAKUTO Internat. Ser. Math. Sci. Appl.*, 36:83–112, 2013.

[28] N. Hamamuki and Q. Liu. A deterministic game interpretation for fully nonlinear parabolic equations with dynamic boundary conditions. *ESAIM Control Optim. Calc. Var.*, 26-13, 2020.

[29] N. Hamamuki and Q. Liu. A deterministic game interpretation for fully nonlinear parabolic equations with dynamic boundary conditions. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 13, 42, 2020.

[30] N. Hamamuki and Q. Liu. A game-theoretic approach to dynamic boundary problems for level-set curvature flow equations and applications. *Partial Differ. Equ. Appl.*, 2:Paper No. 30, 27, 2021.

[31] N. Hamamuki and K. Misu. Asymptotic shape of solutions to the mean curvature flow equation with discontinuous source terms. in preparation.

[32] N. Hamamuki and K. Misu. A weak comparison principle and asymptotic behavior of viscosity solutions to the mean curvature flow equation with discontinuous source terms (geometric aspects of solutions to partial differential equations). *RIMS Kôkyûroku (Japanese)*, 2212, 2022.

[33] N. Hamamuki and K. Misu. Weak comparison principles for fully nonlinear degenerate parabolic equations with discontinuous source terms. *Minimax Theory Appl.*, 8(1):37–60, 2023.

[34] H. Ishii. A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type. *Proc. Amer. Math. Soc.*, 100:247–251, 1987.

[35] K. Ishii, H. Kamata, and S. Koike. Remarks on viscosity solutions for mean curvature flow with obstacles. *Springer Proc. Math. Stat.*, 215:83–103, 2017.

[36] B. Kawohl and N. Kutev. Comparison principle and Lipschitz regulality for viscosity solutions of some classes of nonlinear partial differencial equations. *Funkcialaj Ekvacioj*, 43:241–253, 2000.

[37] R. Kohn and S. Serfaty. A deterministic-control-based approach to motion by curvature. *Commun. Pure Appl. Math.*, 59:344–407, 2006.

[38] R. Kohn and S. Serfaty. A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. *Comm. Pure Appl. Math.*, 10:1298–1350, 2010.

[39] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[40] N. Q. Le, H. Mitake, and H. V. Tran. *Dynamical and geometric aspects of Hamilton-Jacobi and linearized Monge-Ampère equations—VIASM 2016.*

[41] P. L. Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[42] P. L. Lions. Regularizing effects for first-order Hamilton–Jacobi equations. *Applicable Anal.*, 20:283–307, 1985.

[43] Q. Liu. Fattening and comparison principle for level set equations of mean curvature type. *SIAM J. Control Optim.*, 49:2518–2541, 2011.

[44] Q. Liu, A. Schikorra, and X. Zhou. A game-theoretic proof of convexity-preserving properties for motion by curvature. *Indiana Univ. Math. J.*, 65:171–197, 2016.

[45] Q. Liu and N. Yamada. An obstacle problem arising in large exponent limit of power mean curvature flow equation. *Trans. Amer. Math. Soc.*, 372:2103–2141, 2019.

[46] J. J. Manfredi, J. D. Rossi, and S. J. Somersille. An obstacle problem for tug-of-war games. *Commun. Pure Appl. Math.*, 14:217–228, 2015.

[47] N. R. McDonald. The fundamental solutions of the curve shortening problem via the Schwarz function. *Complex Anal. Synerg.*, 8(1):Paper No. 5, 6, 2022.

[48] G. Mercier and M. Novaga. Mean curvature flow with obstacles: Existence, uniqueness and regularity of solutions. *Interfaces and Free Boudaries*, 17:399–426, 2015.

[49] K. Misu. A game-theoretic approach to the asymptotic behavior of solutions to an obstacle problem for the mean curvature flow equation. *Hokkaido University Preprint Series in Mathematics*, 1149:1–40, 2023.

[50] M. Ohara and R. C. Reid. *Modeling Crystal Growth Rates from Solution.* Prentice-Hall, Englewood Cliffs, NJ, 1973.

[51] A. Porretta and E. Piriora. Global lipschitz regularizing effects for linear and nonlinear parabolic equations. *J. Math. Pures Appl.*, 100:633–686, 2013.

[52] G. Sazaki, S. Zepeda, S. Nakatsubo, E. Yokoyama, and Y. Furukawa. Elementary steps at the surface of ice crystals visualized by advanced optical microscopy. *Proc. Nat. Acad. Sci. USA.*, 107:19702–19707, 2010.

[53] E. Spadaro. Mean-convex sets and minimal barriers. *Matematiche(Catania)*, 1:353–375, 2020.

[54] K. Takasao. On obstacle problem for Brakke's mean curvature flow. *SIAM J. Math. Anal.*, 53(6):6355–6369, 2021.