博士学位論文

Geometry of timelike minimal surfaces in the three-dimensional Heisenberg group
（3 次元ハイゼンベルグ群における時間的極小曲面の幾何）

清原　悠貴

北海道大学大学院理学院
数学専攻

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1. Introduction

Left invariant Lorentzian metrics on the 3 dimensional Heisenberg group Nil_3 are classified isometrically by S. Rahmani [46], according to whether the center of the Lie algebra $\mathfrak{nil}_3$ is spacelike, timelike, or null. The non-flat two of them can be represented similarly to the well-known left-invariant Riemannian metric on Nil_3 and have the 4 dimensional isometry groups. Then spacelike or timelike surfaces in Nil_3 with a non-flat left-invariant Lorentzian metric can be expected to have properties similar to surfaces in the Riemannian Heisenberg group. In this thesis, we will introduce the results of papers [26] (reproduced with permission from Springer Nature) and [27] about timelike minimal surfaces in a Lorentzian Heisenberg group, containing a Weierstrass-type representation of non-vertical timelike minimal surfaces and the characterization of timelike minimal surfaces that have no counterparts in definite cases.

In 3 dimensional homogeneous spaces with Riemannian metrics, specifically the model spaces of 3 dimensional Thurston geometry, the surface theory has been developed in recent years. In particular, the discovery of a quadratic differential, which is called the generalized Hopf differential or Abresch-Rosenberg differential by U. Abresch and H. Rosenberg [2] accelerated the research of constant mean curvature surfaces in the 3 dimensional homogeneous spaces. As is well known in classical surface theory, the Hopf differential is very useful in studying constant mean curvature surfaces in space forms since the Hopf differential becomes holomorphic if and only if the mean curvature of a surface is constant. The Abresch-Rosenberg differential in homogeneous 3-spaces is an analogy to the Hopf differential because in various classes of 3 dimensional homogeneous spaces, it becomes holomorphic if a surface has a constant mean curvature.

On the other hand, D. A. Berdinskiĭ and I. A. Taĭmanov developed the integral representations of surfaces in 3 dimensional homogeneous spaces [4, 5]. In classical surface theory in Euclidean space, the Kenmotsu-Weierstrass representation is well known as the integral representation. Berdinskiĭ and Taĭmanov generalized the Kenmotsu formula by using the generating spinors and the non-linear Dirac equations for surfaces.

In 3 dimensional Heisenberg group Nil_3, J. Inoguchi and B. Daniel showed independently that the normal Gauss map, which is naturally defined from the unit normal vector field for surfaces by using the stereographic projections, defines a harmonic map into the hyperbolic plane if and only if the surface is non-vertical and minimal. On the other hand, J. F. Dorfmeister, J. Inoguchi, and S.-P. Kobayashi investigated minimal surfaces and established the Weierstrass-type representation via loop group decompositions. The Weierstrass data of non-vertical minimal surfaces is a holomorphic 1-form, called the normalized potential, which is obtained through the loop group decomposition, the so-called Iwasawa decomposition. In that study, they applied the Sym-Bobenko formula, known as an immersion formula of constant mean curvature surfaces in space forms. Moreover, they gave the characterization of the minimality of surfaces in Nil_3 and the harmonicity of the normal Gauss map by using a family of flat connections on a trivial principal bundle. The heart of the above study is combining the Abresch-Rosenberg differential and the non-linear Dirac equation with generating spinors. By doing so an integrable system that is equivalent to the non-linear Dirac equation can be obtained, and then the tools in integrable systems become available.

Therefore it is a natural problem to develop the spacelike and timelike surface theory with generating spinors and to establish a representation formula of spacelike and timelike surfaces of mean curvature 0 in Nil_3 with Lorentzian metrics. The non-flat left-invariant Lorentzian metrics are given in the following form:

$$g_\pm = \mp dx_1^2 + dx_2^2 \pm \left(dx_3 + \frac{1}{2}(x_2 dx_1 - x_1 dx_2)\right)^2 .$$

The spacelike surface theory with respect to the metric g_- is studied by D. Brander and S.-P. Kobayashi [9]. They showed that the generic singularities of spacelike maximal surfaces are cuspidal edge, swallow-tail, and cuspidal cross cap. Moreover they constructed these surfaces via the loop group method. On the other hand, S.-P. Kobayashi and the author investigated the timelike minimal surfaces with respect to the metric g_+ in [26].

In Section 4, we will introduce the research of Weierstrass-type representation of non-vertical timelike minimal surfaces which is constructed in [26]. Because of using a para-complex coordinate system, the existence of generating spinors for timelike surfaces is guaranteed by the conformality of timelike surfaces. Therefore the non-linear Dirac equation of the form

$$\left\{\begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0\end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U}\end{pmatrix}\right\}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}.$$

can be obtained as similar to the Riemannian case. Defining the Abresch-Rosenberg differential for timelike surfaces (see Definition 4.2.1) and combining it with the above equations, we can show the following theorem:

Theorem 4.2.3 ([26])**.** Let f be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into Nil_3 for which the Dirac potential $\mathcal{U}$ satisfies (4.1.3). Then the generating spinors $\widetilde{\psi} = (\psi_1, \psi_2)$ satisfies the system of equations:

$$\widetilde{\psi}_z = \widetilde{\psi}\widetilde{U}, \quad \widetilde{\psi}_{\bar{z}} = \widetilde{\psi}\widetilde{V},$$

where

$$\widetilde{U} = \begin{pmatrix} \frac{1}{2}\partial w + \frac{1}{2}\tilde{\epsilon} e^{-w/2} e^{u/2}\partial H & -\tilde{\epsilon} e^{w/2} \\ Q\tilde{\epsilon} e^{-w/2} & 0 \end{pmatrix},$$

$$\widetilde{V} = \begin{pmatrix} 0 & -\overline{Q}\tilde{\epsilon} e^{-w/2} \\ \tilde{\epsilon} e^{w/2} & \frac{1}{2}\overline{\partial} w + \frac{1}{2}\tilde{\epsilon} e^{-w/2} e^{u/2}\overline{\partial} H \end{pmatrix}.$$

Here, Q is the coefficient of the Abresch-Rosenberg differential and $\tilde{\epsilon} \in \{\pm 1, \pm i'\}$ is the number determined by (4.1.4). Conversely, every solution $\widetilde{\psi}$ to the above equation with (4.1.4) is a solution of the non-linear Dirac equation (4.1.1) with (4.1.2).

One of the main results in Section 4 is the characterization of the minimality of timelike surfaces. The left translated unit normal vector field of a non-vertical timelike surface takes values in the half part of the de-Sitter sphere

$$\widetilde{\mathbb{S}}^2_1 = \left\{ x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathfrak{nil}_3 \;\middle|\; -(x_1)^2 + (x_2)^2 + (x_3)^2 = 1 \right\}.$$

Thus the normal Gauss map for a timelike surface is defined as a map into the de-Sitter sphere in the Minkowski 3-space $\mathbb{L}^3_{(+,-,+)}$ by combining two stereographic projections of de-Sitter spheres. Similarly to the Riemannian case, the following theorem will be proved. Here the family of connections α^μ is parameterized by a hyperbola $\mathbb{S}^1_1 = \{\mu \mid \mu\bar{\mu} = 1, \ \mathrm{Re}\,\mu > 0\}$ and derived from the equation in Theorem 4.2.3.

Theorem 4.4.1 ([26])**.** Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into Nil_3 satisfying (4.1.3). Then the following conditions are mutually equivalent:

(1) f is minimal.
(2) The Dirac potential has purely imaginary values.
(3) $d + \alpha^\mu$ defines a family of flat connections on $\mathbb{D} \times \mathrm{SU}'_{1,1}$.
(4) The normal Gauss map $f^{-1}N$ is a non-conformal Lorentz harmonic map into the de-Sitter sphere $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$.

Because of the harmonicity of the normal Gauss map, timelike minimal surfaces in the Lorentzian Heisenberg group (Nil_3, g_+) induce timelike surfaces of constant mean curvature in the Minkowski 3-space. It has been known that in Minkowski 3-space, timelike surfaces of constant mean curvature are characterized by Lorentz harmonic maps into the de-Sitter sphere. J. F. Dorfmeister, J. Inoguchi, and M. Toda [17] gave the Sym-Bobenko type representation of timelike constant mean curvature surfaces in the Minkowski 3-space, that is, an immersion formula was given from the one parameter family of moving frames, the so-called extended frame (see Definition 4.4.4 and Remark 4.4.5), of a non-conformal Lorentz harmonic map into the de-Sitter sphere. In Section 4, we will also show the following theorem which gives the reformulation of Sym-Bobenko type representation in [17] in terms of para-complex coordinate systems through the identification (4.3.1) of the Minkowski 3-space and the Lie algebra $\mathfrak{su}'_{1,1}$ as a vector space.

Theorem 4.5.2 ([26])**.** Let F^μ be an extended frame of some Lorentz harmonic map φ from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into the de-Sitter sphere $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$. Assume that the coefficient function a of $(1,2)$-entry of $\alpha_{\mathfrak{m}}{}'$ satisfies $a\bar{a} < 0$ on $\mathbb{D}$. Define two maps $f_{\mathbb{L}^3_{(+,-,+)}}$ and $N_{\mathbb{L}^3_{(+,-,+)}}$ by

$$f_{\mathbb{L}^3_{(+,-,+)}} = -i'\mu \left(\frac{\partial}{\partial \mu} F^\mu \right) (F^\mu)^{-1} - \frac{1}{2} \mathrm{Ad}(F^\mu) \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}, \quad N_{\mathbb{L}^3_{(+,-,+)}} = \frac{1}{2} \mathrm{Ad}(F^\mu) \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}.$$

Then by the identification (4.3.1), $f_{\mathbb{L}^3_{(+,-,+)}}$ describes an associated family of timelike surfaces of constant mean curvature $1/2$ in $\mathbb{L}^3_{(+,-,+)}$ with the first fundamental form $I = -16a\bar{a}dzd\bar{z}$ and $N_{\mathbb{L}^3_{(+,-,+)}}$ is the spacelike unit normal vector field of $f_{\mathbb{L}^3_{(+,-,+)}}$ for each $\mu \in \mathbb{S}^1_1$.

Moreover by derivating the surface $f_{\mathbb{L}^3_{(+,-,+)}}$ with respect to the spectral parameter μ and using the linear isomorphism $\Xi : \mathfrak{su}'_{1,1} \to \mathfrak{nil}_3$ defined in (4.5.4), we give an immersion formula of timelike minimal surfaces in Nil_3 complying with the characterization in Theorem 4.4.1, that is, the timelike minimal surfaces which we will give have the normal Gauss map $N_{\mathbb{L}^3_{(+,-,+)}}$:

Theorem 4.5.4 ([26])**.** Let F^μ be an extended frame of some harmonic map from a simply connected domain $\mathbb{D}$ into $\mathbb{H}^2$, and $f_{\mathbb{L}^3_{(+,-,+)}}$ the associated family of constant mean curvature $1/2$ surfaces defined in Theorem 4.5.2. Moreover define a map $f^\mu : \mathbb{D} \to \mathrm{Nil}_3$ by

$$f^\mu = \exp \circ \Xi \circ \hat{f}^\mu$$

where the map $\hat{f^\mu} : \mathbb{D} \to \mathfrak{su}'_{1,1}$ is a $\mathfrak{su}'_{1,1}$-valued map defined by

$$\hat{f^\mu} = \left(f_{\mathbb{L}^3_{(+,-,+)}}\right)^o - \frac{i'}{2}\mu\left(\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}\right)^d$$

Here the superscripts "o" and "d" denote the off-diagonal part and diagonal part, respectively. Then the map f^μ describes a family of timelike minimal surfaces in Nil_3. Moreover the normal Gauss map of f^μ is $N_{\mathbb{L}^3_{(+,-,+)}}$.

Extended frames of Lorentz harmonic maps into the de-Sitter sphere are constructed in [17] through the loop group method. The Weierstrass data of extended frames are two 1-forms since null coordinate systems are adopted in [17]. On the other hand, the Weierstrass-type representation of extended frames can be understood in a unified way in this thesis since para-complex coordinate systems are utilized. More precisely, the Weierstrass data are just a 2 by 2 matrix-valued para-holomorphic 1-form, which is called a normalized potential (see Definition 4.6.4), and then an extended frame of non-conformal Lorentz harmonic map into the de-Sitter sphere in the Minkowski 3-space can be derived as an Iwasawa decomposed factor of the solution to a para-holomorphic differential equation as follows:

Theorem 4.6.5 ([26])**.** Let ξ be a normalized potential of a timelike minimal surface f in Nil_3 defined in (4.6.6), and F_- be the solution of

$$\partial F_- = F_- \xi$$

with the initial condition $F_-(z = 0) = \mathrm{id}$. Moreover let $F^\mu \in \Lambda' \mathrm{SU}'_{1,1\sigma}$ and $F_+ \in \Lambda'^+ \mathrm{SL}_2\mathbb{C}'_\sigma$ be the decomposed factors of F_- with respect to the Iwasawa decomposition, that is $F_- = F^\mu F_+$. Then $F^\mu k$ forms an extended frame of f up to the change of coordinate systems for some $k \in \mathrm{U}'_1$.

The normalized potential for a timelike minimal surface in Nil_3 is determined by the Abresch-Rosenberg differential Qdz^2 and the support function h for the surface. Therefore we can consider timelike minimal surfaces with the Abresch-Rosenberg differential Qdz^2 which satisfies the condition $Q\overline{Q} = 0$. The class satisfying the above condition includes timelike minimal surfaces with $Q = 0$, called horizontal umbrellas. However, the condition $Q\overline{Q} = 0$ does not imply $Q = 0$ as a by-product of using the para-complex structure. It can be seen that such a surface except for $Q = 0$ has no counterparts in definite cases. In Example 4.7.4, although we will show an example of such a surface, the normalized potential can not computed explicitly.

As is well known, the word "null scrolls" is originally defined as ruled surfaces over null curves with null director curves in the Minkowski 3-space. On geometry in the Minkowski 3-space, a point p of a surface is said to be quasi-umbilic if the Hopf differential Adz^2 satisfies $A(p) \neq 0$ and $A(p)\overline{A(p)} = 0$. J. Clelland [13] showed that totally quasi-umbilical timelike surfaces are characterized as null scrolls. On the other hand, Z. M. Sipus, L. P. Gajčić, and I. Protrka [49] gave the reparametrization of a non-degenerate null scroll as a B-scroll, that is a ruled surface with a ruling corresponding to the binormal vectors of a base curve, with no additional assumptions on parameters and curvatures. Other studies of reparametrization of a null scroll as a B-scroll are presented by A. Fujioka and J. Inoguchi [21] and H. Liu [36]. The former authors started from a conformal parametrization of a timelike ruled surface, and the latter considered null scrolls which are parametrized by a distinguished parameter with constant curvatures.

Section 5 treats the class of timelike minimal surfaces defined by the multiplication of a null curve in Nil_3 and the composition of the exponential map $\exp : \mathfrak{nil}_3 \to \mathrm{Nil}_3$ and a curve in the light cone in the Lie algebra $\mathfrak{nil}_3$. Since the definition uses only the structure of Lie groups, such a class can be considered in general Lie groups, and then they can be seen as an analogy of ruled surfaces. We will call these surfaces null scrolls (see Definition 5.2.1). The one of main results in Section 5 is to give an analogy of the results in the Minkowski 3-space, that is, we will give a characterization of minimal null scrolls in Nil_3 as a surface with the Abresch-Rosenberg differential Qdz^2 satisfying the condition $Q\overline{Q} = 0$.

Theorem 5.3.10 ([27])**.** If a null scroll f is minimal, then the Abresch-Rosenberg differential Qdz^2 of f satisfies $Q\overline{Q} = 0$. Conversely, every timelike minimal surface with $Q\overline{Q} = 0$ is a null scroll.

In particular, we construct a frame along null curves in Nil_3 by utilizing the curve theory in the Minkowski 3-space in the proof of the above theorem. Then we can obtain minimal null scrolls in Nil_3 from an arbitrary function.

Corollary 5.3.7 ([27])**.** For an arbitrary real valued function k_1, there exists a minimal null scroll such that the base curve has the first curvature k_1 with respect to some null frame.

Furthermore, we can see another construction of minimal null scrolls by calculating the minimality conditions. Because of the simplicity of minimality conditions, we should consider only three cases (see Theorem 5.2.4). At the last of Section 5 we will show the following theorem:

Theorem 5.4.13 ([27])**.** If a minimal null scroll is not a vertical plane then there exist a null curve γ in Nil_3 and a curve $\widetilde{B}$ in the light cone in $\mathfrak{nil}_3$ which satisfy

$$\gamma^{-1}\frac{d\gamma}{ds} = -4(2g_+(B'', B'')B + B'') \tag{1.0.1}$$

and which define the map $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ describing the original minimal null scroll.

This theorem means non-vertical minimal null scrolls can be reconstructed as B-scroll type minimal surfaces which are timelike minimal surfaces in Nil_3 inducing B-scrolls.

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We devote this section to explaining the basic knowledge and notations we use in the later sections. Let us recall the surface theory in Lie groups developed by D. A. Berdinskiĭ and I. A. Taĭmanov [5] in terms of the spinor representations.

2.1. **Structure equations and spinor representations of surfaces in Lie groups.** The following theorem, which is a result of studies by A. Korn [30] and L. Lichtenstein [32], is fundamental.

Theorem 2.1.1. *Every 2 dimensional orientable Riemannian manifold (M, g) has a structure of Riemann surface, such that each coordinate (x, y) is an isothermal coordinate system for g.*

Let a coordinate system (x, y) of (M, g) be isothermal. Then the first fundamental form I can be given in the form

$$I = e^u dz d\bar{z}.$$

We call the complex coordinate system $z = x + iy$ a *conformal coordinate system* and the coefficient e^u the *conformal factor* of a surface with respect to z. The notations ∂ and $\overline{\partial}$ denote the partial differentiations with respect to the coordinates $z = x+iy$ and $\bar{z} = x-iy$:

$$\partial := \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \overline{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Because we consider surfaces only locally, surfaces immersed in a Riemannian manifold $\widetilde{M}$ can be considered as conformal immersions from a simply connected domain in $\mathbb{C}$ into $\widetilde{M}$ and assume that a conformal coordinate system z is defined on the domain.

Let $f : \mathbb{D} \to G$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}$ into a Lie group G equipped with a left-invariant Riemannian metric g and e^u be the conformal factor with respect to the conformal coordinate system z defined on $\mathbb{D}$.

First we expand the derivative $\Phi := f^{-1}\partial f$ in terms of an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the Lie algebra $\mathfrak{g}$ of G:

$$\Phi = \sum_{j=1}^{n} \phi^j e_j.$$

Here the notation f^{-1} denotes the left translation of vectors from the complexified tangent space at each point to the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Then the conformality $g(\Phi, \Phi) = 0$ and $g(\Phi, \overline{\Phi}) > 0$ of f can be represented as

$$\sum_{j=1}^{n} (\phi^j)^2 = 0, \quad \sum_{j=1}^{n} |\phi^j|^2 = \frac{1}{2}e^u > 0. \tag{2.1.1}$$

Moreover f defines a $\mathfrak{g}$-valued 1-form

$$\alpha := \Phi dz + \overline{\Phi} d\bar{z}.$$

Then α satisfies the Maurer-Cartan equation:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Thus we have the equation

$$\overline{\partial}\Phi - \partial\overline{\Phi} - [\Phi, \overline{\Phi}] = 0. \tag{2.1.2}$$

The following proposition is a basic result of the differential geometry of Lie groups.

Proposition 2.1.2 ([16])**.** *Let $G \subset \mathrm{GL}_m\mathbb{R}$ be a linear Lie group of dimension n equipped with a left-invariant Riemannian metric and $\mathfrak{g}$ be the Lie algebra of G. Moreover let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ and $\Phi = \sum_{j=1}^n \phi^j e_j$ be a non-zero function defined on a simply connected domain $\mathbb{D} \subset \mathbb{C}$, which takes values in $\mathfrak{g}^{\mathbb{C}}$. If the equations*

$$\sum_{j=1}^{n} (\phi^j)^2 = 0, \tag{2.1.3}$$

$$\overline{\partial}\Phi - \partial\overline{\Phi} - [\Phi, \overline{\Phi}] = 0 \tag{2.1.4}$$

hold, then, for any initial condition in G at some base point in $\mathbb{D}$, there exists a conformal immersion $f : \mathbb{D} \to G$ which satisfies $f^{-1}\partial f = \Phi$, and it is unique up to translations.

Secondly, let us consider the condition with respect to the mean curvature of surfaces. Typically, a smooth map $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ from a Riemannian n-manifold M into a Riemannian $\widetilde{n}$-manifold $\widetilde{M}$ defines a section $\nabla d\varphi$ of $TM^* \otimes TM^* \otimes \varphi^{-1}T\widetilde{M}$:

$$\nabla d\varphi(X, Y) := \nabla^{\widetilde{M}}_X d\varphi(Y) - d\varphi(\nabla^M_X Y).$$

Here, ∇^M denotes the Levi-Civita connections of g and $\nabla^{\widetilde{M}}$ denotes the induced connection on the pull-backed bundle $\varphi^{-1}T\widetilde{M}$. In particular, when the map φ is an isometric immersion the section $\nabla d\varphi$ is exactly the second fundamental form of φ. The tension field $\tau(\varphi)$ of φ is defined as the trace of $\nabla d\varphi$:

$$\tau(\varphi) := \mathrm{Trace}_g \nabla d\varphi = \sum_{j=1}^{n} \nabla d\varphi(e_j, e_j),$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame on M with respect to g. The definition of the tension field is independent of the choice of an orthonormal frame on M. As is well known, the mean curvature field $\boldsymbol{H}$ is related to the tension field:

$$\tau(\varphi) = n\boldsymbol{H}. \tag{2.1.5}$$

In particular, we now consider surfaces immersed in a Lie group equipped with a left-invariant Riemannian metric. Then by left translating (2.1.5) to the Lie algebra, we have

$$\overline{\partial}\Phi + \partial\overline{\Phi} + \nabla_{\Phi}\overline{\Phi} + \nabla_{\overline{\Phi}}\Phi = e^u f^{-1}\boldsymbol{H} \tag{2.1.6}$$

where ∇ is the Levi-Civita connection for the left-invariant Riemannian metric on the ambient Lie group.

Theorem 2.1.3 ([16])**.** *Let $G \subset \mathrm{GL}_m\mathbb{R}$ be a linear Lie group of dimension n equipped with a left-invariant Riemannian metric and $\mathfrak{g}$ be the Lie algebra of G. Moreover let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ and $\Phi = \sum_{j=1}^n \phi^j e_j$ be a non-zero function defined on a simply connected domain $\mathbb{D} \subset \mathbb{C}$, which takes values in $\mathfrak{g}^{\mathbb{C}}$. Assume that Φ satisfies* (2.1.3), (2.1.4), *and*

$$\overline{\partial}\Phi + \partial\overline{\Phi} + \{\Phi, \overline{\Phi}\} = 0. \tag{2.1.7}$$

Here $\{\cdot,\cdot\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the bilinear symmetric map defined by

$$\{X, Y\} := \nabla_X Y + \nabla_Y X.$$

The notation ∇ denotes the Levi-Civita connection of the left-invariant Riemannian metric of G. Then there exists a conformal minimal immersion $f : \mathbb{D} \to G$ which satisfies $f^{-1}\partial f = \Phi$, and it is unique up to translations.

Proof. Since Φ satisfies the conditions (2.1.3) and (2.1.4) there exists a conformal immersion $f : \mathbb{D} \to G$ satisfying $f^{-1}\partial f = \Phi$ up to translations by Proposition 2.1.2. The above discussion shows that the condition (2.1.7) means the mean curvature field for f vanishes. $\square$

From now on, we consider immersed surfaces in a Lie group G of dimension 3. Because of the conformality (2.1.1) of surfaces, ϕ^j satisfy

$$(\phi^3)^2 = (-\phi^1 - i\phi^2)(\phi^1 - i\phi^2).$$

By taking complex functions ψ_1 and ψ_2 as

$$(\psi_1)^2 := \frac{-\phi^1 - i\phi^2}{2}, \quad (\overline{\psi_2})^2 := \frac{\phi^1 - i\phi^2}{2},$$

ϕ^j can be rephrased as

$$\phi^1 = (\overline{\psi_2})^2 - (\psi_1)^2, \quad \phi^2 = i\left((\overline{\psi_2})^2 + (\psi_1)^2\right), \quad \phi^3 = 2\psi_1\overline{\psi_2}. \tag{2.1.8}$$

Definition 2.1.4 ([5])**.** The pair (ψ_1, ψ_2) of complex functions defined by (2.1.8) is said to be the *generating spinors* of a surface immersed in a Lie group of dimension 3.

Remark 2.1.5. Since the derivative $f^{-1}\partial f = \sum_{j=1}^{3} \phi^j e_j$ can be written as (2.1.8), the conformal factor e^u is computed as

$$e^{u/2} = 2\left(|\psi_1|^2 + |\psi_2|^2\right).$$

2.2. **Structure equations and spinor representations of timelike surfaces in Lie groups.** The spinor representation of surfaces can be extended to timelike surfaces in Lie groups with a left-invariant Lorentzian metric. To explain this fact, we introduce the Lorentz surface and the para-complex structures.

Lorentz surfaces are orientable manifolds of dimension 2 with a *Lorentz conformal structure*, which is an equivalence class of a Lorentzian metric on the manifold. The global studies of Lorentz conformal structure were started by Kulkarni [31], and a lot of results on Lorentz surfaces have been reported by many geometers, such as Weinstein (T. K. Milnor) [29, 33, 34, 39, 40, 41, 50, 51]. The fundamental properties of Lorentz surfaces are summarized in [57]. Just as a Riemann surface has an isothermal coordinate system, a Lorentz surface has a special coordinate system, called a *null coordinate system.*

Theorem 2.2.1 ([57])**.** *Let (M, g) be a Lorentz surface. Then there exists a coordinate system (x, y) such that the metric g can be represented as*

$$g = e^u dxdy$$

for some function u.

In this thesis to investigate timelike surfaces, we use the coordinate system that is rewritten from a null coordinate system, so-called *para-complex coordinate system.* Para-complex number $\mathbb{C}'$ is a real algebra spanned by 1 and the para-complex unit i' which satisfy the conditions:

$$i'^2 = 1, \quad 1 \cdot i' = i' \cdot 1 = i'.$$

We call the elements of $\mathbb{C}'$ also para-complex numbers. Similarly to complex number $\mathbb{C}$, every $z \in \mathbb{C}'$ can be decomposed uniquely into the real part and the imaginary part, that is,

$$z = x + i'y.$$

Moreover, the para-complex conjugate is also defined as well as the complex conjugate:

$$\bar{z} = x - i'y.$$

However, the para-complex number is not a field. In fact, there exist numbers that do not have inverse elements. This fact is explained in the following proposition.

Proposition 2.2.2. *For a para-complex number $z = x + i'y \in \mathbb{C}'$, following statements hold.*

(1) *There exists a root $w \in \mathbb{C}'$, that is, $z = w^2$ if and only if*

$$x + y \geq 0, \quad x - y \geq 0.$$

(2) *There exists a para-complex number w such that $z = e^w$ if and only if*

$$x + y > 0, \quad x - y > 0.$$

Here the exponential is defined as an infinite series

$$e^w = \sum_{k=0}^{\infty} \frac{w^k}{k!} = e^{\operatorname{Re} w} \left(\cosh(\operatorname{Im} w) + i' \sinh(\operatorname{Im} w) \right).$$

(3) *There exists an inverse number w, that is, $z \cdot w = 1$ if and only if*

$$x^2 - y^2 \neq 0.$$

Proof. By representing w^2 and e^w using the real part and the imaginary part of w, and comparing them with z, we can obtain the statements (1) and (2). If z has a inverse number $w = u + i'v$, we have a system

$$xu + yv = 1, \quad yu + xv = 0.$$

Then removing u or v, we obtain $x^2 - y^2 \neq 0$. Conversely, if $x^2 - y^2 \neq 0$ holds, multiplying $\epsilon \in \{\pm 1, \pm i'\}$ appropriately results in case of $x + y > 0$ and $x - y > 0$. When $x + y > 0$ and $x - y > 0$, by (2), z can be represented into $e^{\widetilde{w}}$ for some para-complex number $\widetilde{w}$. The proof is completed by taking $w = e^{-\widetilde{w}}$. $\square$

The following proposition explains a fundamental property of the square root of para-complex numbers.

Proposition 2.2.3. *If a product xy of two para-complex numbers $x, y \in \mathbb{C}'$ has the square root, then there exists $\epsilon \in \{\pm 1, \pm i'\}$ such that ϵx and ϵy have the square roots.*

Proof. By the assumption,

$$\mathrm{Re}(xy) \pm \mathrm{Im}(xy) \geq 0$$

holds, and a simple computation shows that it is equivalent to

$$(\mathrm{Re}(x) \pm \mathrm{Im}(x))(\mathrm{Re}(y) \pm \mathrm{Im}(y)) \geq 0.$$

Then the claim follows. $\square$

Let (x, y) be a null coordinate on a Lorentz surface (M, g). Then the Lorentzian metric g on M can be written in the form

$$g = e^u dz d\bar{z}$$

by putting the para-complex coordinate z as $z = \ell x + \bar{\ell} y$ with $\ell = (1 + i')/2$. The para-complex coordinate system z is called a conformal coordinate system and the coefficient e^u the conformal factor as in the Riemannian case. In this thesis, we use the notations ∂ and $\overline{\partial}$ as the partial differentiations with respect to the coordinates z and $\bar{z}$, defined formally by

$$\partial = \frac{\partial}{\partial z} = \ell \frac{\partial}{\partial x} + \bar{\ell} \frac{\partial}{\partial y}, \quad \overline{\partial} = \frac{\partial}{\partial \bar{z}} = \bar{\ell} \frac{\partial}{\partial x} + \ell \frac{\partial}{\partial y}.$$

Because we consider timelike surfaces only locally, a timelike surface immersed in a Lorentzian manifold $\widetilde{M}$ can be considered as a conformal immersion from a simply connected domain $\mathbb{D}$ in $\mathbb{C}'$ into $\widetilde{M}$.

Let $f : \mathbb{D} \to G$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into a Lie group G equipped with a left-invariant Lorentzian metric g and e^u be the conformal factor with respect to the conformal coordinate system z defined on $\mathbb{D}$.

Expanding the derivative $\Phi := f^{-1}\partial f$ in terms of an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the Lie algebra $\mathfrak{g}$ of G with a timelike vector e_1:

$$\Phi = \sum_{j=1}^{n} \phi^j e_j,$$

we can represent the conformality of f as

$$\sum_{j=1}^{n} \epsilon_j (\phi^j)^2 = 0, \quad \sum_{j=1}^{n} \epsilon_j \phi^j \overline{\phi^j} = \frac{1}{2} e^u > 0 \tag{2.2.1}$$

where $\epsilon_1 = -1$ and $\epsilon_j = 1$ for $j \neq 1$. Moreover, f defines a $\mathfrak{g}$-valued 1-form

$$\alpha := \Phi dz + \overline{\Phi} d\bar{z}.$$

Then the 1-form α fulfills the Maurer-Cartan equation:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0,$$

that is equivalent to

$$\overline{\partial}\Phi - \partial\overline{\Phi} - [\Phi, \overline{\Phi}] = 0. \tag{2.2.2}$$

Therefore the analogue of Proposition 2.1.2 can be proved.

Proposition 2.2.4. *Let $G \subset \mathrm{GL}_m\mathbb{R}$ be a linear Lie group of dimension n equipped with a left-invariant Lorentzian metric and $\mathfrak{g}$ be the Lie algebra of G. Moreover let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with a timelike vector e_1 and $\Phi = \sum_{j=1}^{n} \phi^j e_j$ be a non-zero function defined on a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ which takes values in the para-complexification $\mathfrak{g}^{\mathbb{C}'} = \mathfrak{g} \otimes \mathbb{C}'$ of $\mathfrak{g}$. If the equations* (2.2.1) *and* (2.2.2) *hold, then there exists a conformal immersion $f : \mathbb{D} \to G$ that satisfies $f^{-1}\partial f = \Phi$ for any initial condition in G at some base point in $\mathbb{D}$, and it is unique up to translations.*

Proof. Since Φ satisfies the Maurer-Cartan equation, there exists a unique map $f : \mathbb{D} \to G^{\mathbb{C}'}$ such that $f^{-1}\partial f = \Phi$ up to translations. A straightforward computation shows $f\overline{f}^{-1}$ is constant. Therefore the initial condition in G induces the G-valued map f. By the assumption, obviously, f is a conformal immersion. □

Next we consider the condition in terms of the mean curvature. For a smooth map $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ from a Lorentzian n-manifold into a Lorentzian $\widetilde{n}$-manifold, the tension field $\tau(\varphi)$ can be defined similarly to the Riemannian case:

$$\tau(\varphi) := \mathrm{Trace}_g \nabla d\varphi = \sum_{j=1}^{n} \epsilon_j \left(\nabla^{\widetilde{M}}_{e_j} d\varphi(e_j) - d\varphi(\nabla^{M}_{e_j} e_j) \right),$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame on M with a timelike vector field e_1 with respect to g and $\epsilon_1 = -1$ and $\epsilon_j = 1$ for $j \neq 1$. In particular, since we now consider timelike surfaces immersed in a Lie group equipped with a left-invariant Lorentzian metric, the computation same with in the Riemannian case derives the following condition:

$$\overline{\partial}\Phi + \partial\overline{\Phi} + \nabla_{\Phi}\overline{\Phi} + \nabla_{\overline{\Phi}}\Phi = e^u f^{-1}\boldsymbol{H} \tag{2.2.3}$$

where ∇ denotes the Levi-Civita connection of the Lorentzian metric on the ambient Lie group. Therefore we can prove the following theorem in the same way to Theorem 2.1.3.

Theorem 2.2.5. *Let $G \subset \mathrm{GL}_m\mathbb{R}$ be a linear Lie group of dimension n equipped with a left-invariant Lorentzian metric and $\mathfrak{g}$ be the Lie algebra of G. Moreover let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ with a timelike vector e_1 and $\Phi = \sum_{j=1}^{n} \phi^j e_j$ be a non-zero function defined on a simply connected domain $\mathbb{D} \subset \mathbb{C}'$, which takes values in $\mathfrak{g}^{\mathbb{C}'}$. Assume that Φ satisfies* (2.2.1) *and* (2.2.2)*, and*

$$\overline{\partial}\Phi + \partial\overline{\Phi} + \{\Phi, \overline{\Phi}\} = 0.$$

Here $\{\cdot, \cdot\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the bilinear symmetric map defined by

$$\{X, Y\} := \nabla_X Y + \nabla_Y X.$$

The notation ∇ denotes the Levi-Civita connection of the left-invariant Lorentzian metric of G. Then there exists a conformal minimal immersion $f : \mathbb{D} \to G$ which satisfies $f^{-1}\partial f = \Phi$, and it is unique up to translations.

From now on we consider timelike surfaces immersed in a Lie group of dimension 3 with a left-invariant Lorentzian metric.

Proposition 2.2.6. *Let $f : \mathbb{D} \to G$ be a conformal immersion from a simply connected domain of $\mathbb{C}'$ into a 3-dimensional Lie group G with a left-invariant Lorentzian metric and* *expand the derivative $f^{-1}\partial f$ in terms of an orthonormal basis $\{e_1, e_2, e_3\}$ with a timelike vector e_1:*

$$\Phi = \sum_{j=1}^{3} \phi^j e_j.$$

Then there exist a para-complex number $\epsilon \in \{\pm i'\}$ and a pair (ψ_1, ψ_2) of para-complex functions ψ_1 and ψ_2 such that

$$\phi_1 = \epsilon\left((\overline{\psi_2})^2 + (\psi_1)^2\right), \quad \phi_2 = \epsilon i'\left((\overline{\psi_2})^2 - (\psi_1)^2\right), \quad \phi_3 = 2\psi_1\overline{\psi_2}. \tag{2.2.4}$$

Proof. Since the first equation of the conformality condition (2.2.1) can be rephrased as

$$(\phi^3)^2 = (\phi^1 + i'\phi^2)(\phi^1 - i'\phi^2),$$

and Proposition 2.2.3 shows that there exists a para-complex number $\epsilon \in \{\pm 1, \pm i'\}$ such that $\epsilon(\phi^1 + i'\phi^2)$ and $\epsilon(\phi^1 - i'\phi^2)$ have the square roots. Thus $\phi^1 + i'\phi^2$ and $\phi^1 - i'\phi^2$ can be represented as

$$\phi^1 + i'\phi^2 = 2\epsilon(\overline{\psi_2})^2, \quad \phi^1 - i'\phi^2 = 2\epsilon(\psi_1)^2$$

for some para-complex functions ψ_1 and ψ_2. Then ϕ^3 can be rephrased as $\phi^3 = 2\psi_1\overline{\psi_2}$, and we obtain the representation in the form of (2.2.4). To complete the proof, we have to show ϵ takes values in $\{\pm i'\}$. A direct computation shows

$$-\phi^1\overline{\phi^1} + \phi^2\overline{\phi^2} + \phi^3\overline{\phi^3} = -2\epsilon\bar{\epsilon}\left(\psi_1\overline{\psi_1} - \epsilon\bar{\epsilon}\psi_2\overline{\psi_2}\right)^2. \tag{2.2.5}$$

Since we assume that the left hand side of (2.2.5) takes positive values on $\mathbb{D}$, ϵ must be i' or $-i'$. □

Remark 2.2.7. The unit normal vector field $N = -2i'\partial f \times \overline{\partial} f/2|g(\partial f, \overline{\partial} f)|^{1/2}$ and the conformal factor e^u for f can be rephrased in terms of (ψ_1, ψ_2) as follows:

$$f^{-1}N = \frac{1}{\psi_2\overline{\psi_2} + \psi_1\overline{\psi_1}}\left(-2\epsilon i' \operatorname{Im}(\psi_1\overline{\psi_2})e_1 + 2\epsilon i' \operatorname{Re}(\psi_1\overline{\psi_2})e_2 - (\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1})e_3\right),$$

$$e^u = 4(\psi_2\overline{\psi_2} + \psi_1\overline{\psi_1})^2.$$

Without loss of generality, we can take $\psi_2\overline{\psi_2}+\psi_1\overline{\psi_1}$ as positive value, if necessary by replacing (ψ_1, ψ_2) into $(-i'\psi_1, i'\psi_2)$. Then we can assume that

$$e^{u/2} = 2(\psi_2\overline{\psi_2} + \psi_1\overline{\psi_1}) \tag{2.2.6}$$

holds. Such a pair (ψ_1, ψ_2) for f is unique up to sign.

Definition 2.2.8. The pair (ψ_1, ψ_2) satisfying (2.2.4) and (2.2.6) is called the *generating spinors* for conformal immersion $f : \mathbb{D} \to \mathrm{Nil}_3$.

2.3. 3 **dimensional Heisenberg group** Nil_3**.** The 3 dimensional Heisenberg Lie algebra $\mathfrak{nil}_3$ is a real algebra spanned by the matrices:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the usual commutator of matrices. The 3 dimensional Heisenberg group Nil_3 is a simply connected linear Lie group corresponding to $\mathfrak{nil}_3$. It consists of 3×3 upper-triangular matrices which have the diagonal components 1. Since the exponential map $\exp : \mathfrak{nil}_3 \to \mathrm{Nil}_3$ gives

$$\exp\left(\sum_{j=1}^{3} x_j e_j\right) = \begin{pmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain the exponential coordinate (x_1, x_2, x_3) of Nil_3, and then Nil_3 can be naturally considered as $(\mathbb{R}^3(x_1, x_2, x_3), \cdot)$ where the notation $\cdot$ denotes the group multiplication:

$$(x_1, x_2, x_3) \cdot (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left(x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, x_3 + \tilde{x}_3 + \frac{1}{2}(x_1\tilde{x}_2 - x_2\tilde{x}_1)\right).$$

By the natural identification of $\mathfrak{nil}_3$ and the space of left-invariant tangent vector fields (or the tangent space at the origin) of Nil_3, we obtain

$$e_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad e_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad e_3 = \frac{\partial}{\partial x_3}.$$

2.4. **Left-invariant Riemannian and Lorentzian metrics on** Nil_3**.** For 3 dimensional Lie groups G with a left-invariant Riemannian metric, Milnor showed that the Lie bracket $[\cdot,\cdot]$ of the Lie algebra $\mathfrak{g}$ can be represented by a uniquely defined linear transformation $L : \mathfrak{g} \to \mathfrak{g}$ and the cross product $\times$ determined by the metric:

$$[u, v] = L(u \times v),$$

and that G is unimodular if and only if L is self adjoint (see section 4 in [38]). Furthermore, for 3 dimensional unimodular Lie groups, Milnor showed a basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of $\mathfrak{g}$ can be chosen so that they are orthonormal with respect to the metric and satisfies

$$[\tilde{e}_1, \tilde{e}_2] = \lambda_3\tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = \lambda_1\tilde{e}_1, \quad [\tilde{e}_3, \tilde{e}_1] = \lambda_2\tilde{e}_2.$$

In particular, in the case of 3 dimensional Heisenberg group, we can choose the orthonormal basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ and λ_j as $\tilde{e}_j = e_j$ and $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ without loss of generality. Therefore the left-invariant Riemannian metric g_R is given by

$$g_R = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + \omega \otimes \omega, \quad \omega = dx_3 + \frac{1}{2}(x_2dx_1 - x_1dx_2).$$

Remark 2.4.1. The 1-form ω is a contact form of Nil_3, and then the pair (Nil_3, ω) is a contact manifold. The Reeb vector field is e_3 and the contact structure is the plane field generated by e_1 and e_2.

On the other hand, left-invariant Lorentzian metrics on Nil_3 are not unique. Using a similar consideration, S. Rahmani [46] classified Lorentzian Heisenberg groups. Let g be a left-invariant Lorentzian metric on Nil_3 and $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ an orthonormal basis of $\mathfrak{nil}_3$ with the timelike vector $\tilde{e}_1$:

$$-g(\tilde{e}_1, \tilde{e}_1) = g(\tilde{e}_2, \tilde{e}_2) = g(\tilde{e}_3, \tilde{e}_3) = 1.$$

There are three structures of Lie algebras in terms of the direction of the center:

(1) $[\tilde{e}_1, \tilde{e}_2] = \lambda\tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = [\tilde{e}_3, \tilde{e}_1] = 0,$
(2) $[\tilde{e}_3, \tilde{e}_2] = \lambda\tilde{e}_1, \quad [\tilde{e}_1, \tilde{e}_2] = [\tilde{e}_3, \tilde{e}_1] = 0,$
(3) $[\tilde{e}_1, \tilde{e}_3] = [\tilde{e}_2, \tilde{e}_3] = \tilde{e}_1 - \tilde{e}_2, \quad [\tilde{e}_1, \tilde{e}_2] = 0.$

The center is generated in (1) by spacelike vector $\tilde{e}_3$, in (2) by timelike vector $\tilde{e}_1$, and in (3) by null vector $\tilde{e}_1 - \tilde{e}_2$. Since the center of $\mathfrak{nil}_3$ is spanned by e_3, we have $\lambda\tilde{e}_3 = e_3$, $\lambda\tilde{e}_1 = e_3$, and $\tilde{e}_1 - \tilde{e}_2 = e_3$, respectively. Let denote the left-invariant Lorentzian metrics corresponding to them as g_+, g_-, and g_0, respectively. In particular, the metrics $g_\pm$ with $\lambda = 1$ is given by

$$g_\pm = \mp dx_1 \otimes dx_1 + dx_2 \otimes dx_2 \pm \omega \otimes \omega.$$

The orthonormal basis can be taken as

(1) $\tilde{e}_1 = e_1, \quad \tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_3,$
(2) $\tilde{e}_1 = e_3, \quad \tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_1.$

The isometry groups with respect to g_R, g_+, and g_- have dimension 4, though only with respect to g_0 has dimension 6. The volume element with respect to the metric g_R, g_+ or g_- is given by $\pm dx_1 \wedge dx_2 \wedge dx_3$. We then orientate Nil_3 for the volume form to be $dx_1 \wedge dx_2 \wedge dx_3$. Moreover, we define the vector product $\times$ as

$$g(X \times Y, Z) = dx_1 \wedge dx_2 \wedge dx_3(X, Y, Z), \quad X, Y, Z \in \mathfrak{nil}_3$$

where $g = g_R$, g_+ or g_-.

Let ∇ denote the Levi-Civita connection for the left invariant metric $g = g_R$, g_+, or g_-, and then the connection ∇ is defined as

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_3 = -\epsilon_- \frac{1}{2} e_2, \\ \nabla_{e_2} e_1 = -\frac{1}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \epsilon_+ \epsilon_- \frac{1}{2} e_1, \\ \nabla_{e_3} e_1 = -\epsilon_- \frac{1}{2} e_2, & \nabla_{e_3} e_2 = \epsilon_+ \epsilon_- \frac{1}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{array}$$

where $\epsilon_\pm$ are signature defined by

$$\epsilon_+ = \begin{cases} 1 & if \quad g \neq g_+ \\ -1 & if \quad g = g_+ \end{cases}, \quad \epsilon_- = \begin{cases} 1 & if \quad g \neq g_- \\ -1 & if \quad g = g_- \end{cases}.$$

3. Surface theory in Riemannian Heisenberg group

In this section, we introduce the Weierstrass-type representation of non-vertical minimal surfaces in Nil_3 studied by J. F. Dorfmeister, J. Inoguchi, and S.-P. Kobayashi [16]. They used a Weierstrass-type representation of harmonic maps into symmetric spaces via loop group decompositions [18]. This method is constructed by J. F. Dorfmeister, F. Pedit, and H. Wu, and called the DPW method by taking the initials of the authors.

3.1. **Non-linear Dirac equation for surfaces.**

D. A. Berdinskiĭ and A. I. Taĭmanov investigated the integral representation of surfaces in 3 dimensional homogeneous spaces, in particular the model spaces of Thurston geometry, by using the generating spinors. For surfaces immersed in such a space, the system of the structure equations (2.1.2) and (2.1.6) are rewritten in the form of the non-linear Dirac equation [55]:

$$\left\{\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.1.1}$$

The complex functions $\mathcal{U}$ and $\mathcal{V}$ are called the *Dirac potentials* for surfaces. The potentials for surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$, Nil_3, $\mathrm{SL}_2\mathbb{R}$, and Sol are given explicitly in [5, 53, 54]. I would like to refer to [47, 56] for the Thurston's geometrization conjecture for manifolds of 3 dimensional. Moreover, because the conformality of immersion guarantees the existence of the generating spinors, spacelike surfaces immersed in a pseudo-Riemannian manifold also have generating spinors.

Theorem 3.1.1 ([55]). *For surfaces immersed in* Nil_3*, the Dirac potentials are*

$$\mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h. \tag{3.1.2}$$

Here the function h is defined by

$$h = 2(|\psi_1|^2 - |\psi_2|^2),$$

and called the support function of surfaces.

Remark 3.1.2. Let us consider the natural projection π from Nil_3 to the plane $\mathbb{R}^2$:

$$\pi : \mathrm{Nil}_3 \ni (x_1, x_2, x_3) \mapsto (x_1, x_2) \in \mathbb{R}^2.$$

From the definition of h and the spinor representation of the unit normal vector field N:

$$N = \frac{1}{|\psi_1|^2 + |\psi_2|^2}\left(2\,\mathrm{Re}(\psi_1\psi_2)e_1 + 2\,\mathrm{Im}(\psi_1\psi_2)e_2 + (|\psi_1|^2 - |\psi_2|^2)e_3\right),$$

we have

$$h = e^{u/2} g_R(N, e_3).$$

Hence the vector e_3 is tangent to a surface when the support function vanishes. This means that a surface that has a vanishing support function is the inverse image of a plane curve by π. Such a surface is called a *Hopf cylinder*. The mean curvature of Hopf cylinders is half of the curvature of corresponding plane curves. Therefore the minimal Hopf cylinders are the plane, called *vertical planes*.

3.2. **Lax type representation for surfaces in** Nil_3**.** The main targets of this section are minimal surfaces in Nil_3 other than vertical planes. Thus we assume that surfaces have the support functions vanishing nowhere if necessary limiting its domain $\mathbb{D}$ and defining $\mathbb{D}$ anew as the limited domain. Such a surface is called *non-vertical*. Then a new complex function w can be defined on $\mathbb{D}$ by

$$e^{w/2} = \mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h \tag{3.2.1}$$

since the Dirac potential vanishes nowhere on the domain. Berdinskiĭ [4] showed that the non-linear Dirac equation (3.1.1) with (3.1.2) is equivalent to another system of partial differential equations for the generating spinors (ψ_1, ψ_2) of a surface in Nil_3.

Definition 3.2.1 ([2, 20, 55])**.** For a conformal immersion $f : \mathbb{D} \to \mathrm{Nil}_3$, define a complex-valued function Q by

$$Q = \frac{2H+i}{4} g_R(\nabla_\partial \partial f, N) + \frac{(\phi^3)^2}{4}. \tag{3.2.2}$$

Here N is the unit normal vector field, H is the mean curvature, and ϕ^3 is the e_3-component of ∂f. The quadratic differential Qdz^2 is well-defined, and then we call it the *Abresch-Rosenberg differential* for f.

Remark 3.2.2. U. Abresch named the non-vertical minimal surfaces of which the Abresch-Rosenberg differential vanishes the *horizontal umbrellas*. The minimal surfaces with the Abresch-Rosenberg differential vanishing anywhere are vertical planes or horizontal umbrellas.

Theorem 3.2.3 ([4])**.** *The non-linear Dirac equation* (3.1.1) *with* (3.1.2) *is equivalent to the following system of partial differential equations*:

$$\partial(\psi_1, \psi_2) = (\psi_1, \psi_2)\widetilde{U}, \quad \overline{\partial}(\psi_1, \psi_2) = (\psi_1, \psi_2)\widetilde{V} \tag{3.2.3}$$

where the 2×2 *matrices* $\widetilde{U}$ *and* $\widetilde{V}$ *are given by*

$$\widetilde{U} = \begin{pmatrix} \frac{1}{2}\partial w + \frac{1}{2}e^{-w/2}e^{u/2}\partial H & -e^{w/2} \\ Qe^{-w/2} & 0 \end{pmatrix}, \quad \widetilde{V} = \begin{pmatrix} 0 & -\overline{Q}e^{-w/2} \\ e^{w/2} & \frac{1}{2}\overline{\partial} w + \frac{1}{2}e^{-w/2}e^{u/2}\overline{\partial} H \end{pmatrix}.$$

Here Q *is the function given by* (3.2.2)*. Then* Qdz^2 *becomes the Abresch-Rosenberg differential for a surface which can be obtained from the Dirac equation* (3.1.1) *with* (3.1.2)*.*

The compatibility condition for the system of partial differential equations (3.2.3), that is $\overline{\partial}\widetilde{U} - \partial\widetilde{V} - [\widetilde{U}, \widetilde{V}] = 0$, gives four equations:

$$\begin{aligned} &\frac{1}{2}\partial\overline{\partial} w + e^w - |Q|^2 e^{-w} + \frac{1}{2}\left(\partial\overline{\partial} H + \partial H \overline{\partial}\frac{-w+u}{2}\right) e^{-w/2}e^{u/2} = 0, \\ &\frac{1}{2}\partial\overline{\partial} w + e^w - |Q|^2 e^{-w} + \frac{1}{2}\left(\partial\overline{\partial} H + \overline{\partial} H \partial\frac{-w+u}{2}\right) e^{-w/2}e^{u/2} = 0, \end{aligned} \tag{3.2.4}$$

$$\begin{aligned} \partial\overline{Q} &= -\frac{1}{2}\overline{Q}\partial H e^{-w/2}e^{u/2} - \frac{1}{2}\overline{\partial} H e^{w/2}e^{u/2}, \\ \overline{\partial} Q &= -\frac{1}{2}Q\overline{\partial} H e^{-w/2}e^{u/2} - \frac{1}{2}\partial H e^{w/2}e^{u/2}. \end{aligned} \tag{3.2.5}$$

From (3.2.5), we immediately obtain the following result.

Theorem 3.2.4 ([2])**.** *For constant mean curvature surfaces in* Nil_3*, the Abresch-Rosenberg differential is holomorphic.*

Remark 3.2.5. In general, the converse statement does not hold, that is, a surface with a holomorphic Abresch-Rosenberg differential does not always have a constant mean curvature. A surface with a holomorphic Abresch-Rosenberg differential is a constant mean curvature surface or a Hopf cylinder (see [16, Appendix A]).

For non-vertical minimal surfaces in Nil_3, the compatibility conditions (3.2.4) and (3.2.5) are more simple as follows:

$$\frac{1}{2}\partial\bar{\partial}w + e^w - |Q|^2 e^{-w} = 0,$$
$$\bar{\partial}Q = 0.$$

In particular, the function w is determined by the support function h by (3.2.1). Thus minimal surfaces in Nil_3 with some initial condition is determined by the support function and the Abresch-Rosenberg differential. Moreover, these equations coincide with the Gauss-Codazzi equations for a spacelike constant mean curvature $1/2$ surface in Minkowski 3-space which has the first fundamental form $h^2 dzd\bar{z}$ and the Hopf differential $4Qdz^2$. Therefore we immediately obtain the following theorem.

Theorem 3.2.6. *For a non-vertical minimal surface in* Nil_3 *with the support function* h *and the Abresch-Rosenberg differential* Qdz^2*, there exists a spacelike constant mean curvature* $1/2$ *surface in Minkowski* 3*-space which has the conformal factor* h^2 *and the Hopf differential* $4Qdz^2$*.*

This correspondence will be explained in subsection 3.5.

3.3. **Normal Gauss map of surfaces in** Nil_3**.**

To obtain a characterization of minimal surfaces in Nil_3, we will introduce the normal Gauss map of surfaces in Nil_3. It is naturally defined from the unit normal vector field and becomes a harmonic map into the hyperbolic plane if the surface has the mean curvature 0.

As is well known, the hyperbolic plane is a Riemannian symmetric space. Let us recall a Riemannian symmetric space representation of the hyperbolic plane. Let $\mathbb{L}^3_{(+,+,-)}$ denote the Minkowski 3-space $(\mathbb{R}^3, \langle , \rangle)$ where $\langle , \rangle$ is an indefinite inner product:

$$\langle (x_1, x_2, x_3), (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\rangle = x_1\tilde{x}_1 + x_2\tilde{x}_2 - x_3\tilde{x}_3.$$

The hyperbolic plane $\mathbb{H}^2$ is a spacelike surface embedded in $\mathbb{L}^3_{(+,+,-)}$ defined by

$$\mathbb{H}^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{L}^3_{(+,+,-)} \middle| (x_1)^2 + (x_2)^2 - (x_3)^2 = -1,\ x_3 > 0 \right\}.$$

Minkowski 3-space $\mathbb{L}^3_{(+,+,-)}$ can be identified as an indefinite scalar product space with the special unitary Lie algebra $\mathfrak{su}_{1,1}$ of index $(1,1)$ defined by

$$\mathfrak{su}_{1,1} = \left\{ \begin{pmatrix} ia & \bar{b} \\ b & -ia \end{pmatrix} \in \mathfrak{sl}_2\mathbb{C} \middle| a \in \mathbb{R}, b \in \mathbb{C} \right\}$$

which is equipped with the following Lorentz scalar product $\langle , \rangle_{m1}$:

$$\langle X, Y\rangle_{m1} = 2\,\mathrm{Trace}(XY), \quad X, Y \in \mathfrak{su}_{1,1}.$$

The correspondence between $\mathfrak{su}_{1,1}$ and $\mathbb{L}^3_{(+,+,-)}$ is given by

$$\mathfrak{su}_{1,1} \ni \frac{1}{2}\begin{pmatrix} ir & -p-iq \\ -p+iq & -ir \end{pmatrix} \longleftrightarrow (p,q,r) \in \mathbb{L}^3_{(+,+,-)}. \tag{3.3.1}$$

Then the hyperbolic plane $\mathbb{H}^2$ can be realized in $\mathfrak{su}_{1,1}$ as

$$\mathbb{H}^2 = \left\{ X \in \mathfrak{su}_{1,1} \middle| \langle X, X\rangle_{m1} = -1,\ \left\langle X, \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\rangle_{m1} < 0 \right\}.$$

The special unitary Lie group $\mathrm{SU}_{1,1}$ of index $(1,1)$ corresponding to the Lie algebra $\mathfrak{su}_{1,1}$ is given by

$$\mathrm{SU}_{1,1} = \left\{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{SL}_2\mathbb{C} \middle| a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\},$$

and acts transitively and isometrically on $\mathbb{H}^2$:

$$\mathrm{SU}_{1,1} \times \mathbb{H}^2 \ni (F, X) \mapsto \mathrm{Ad}(F)X \in \mathbb{H}^2.$$

The isotropy subgroup with respect to this action at $\begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$ is the subgroup U_1 of $\mathrm{SU}_{1,1}$ consisting of diagonal matrices:

$$\mathrm{U}_1 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{SU}_{1,1} \middle| \theta \in \mathbb{R} \right\}.$$

Therefore the hyperbolic plane $\mathbb{H}^2$ has the homogeneous Riemannian space representation $\mathrm{SU}_{1,1}/\mathrm{U}_1$. Since the pair $(\mathrm{SU}_{1,1}, \mathrm{U}_1)$ defines a Riemannian symmetric pair with the involution

$$\sigma : \mathrm{SU}_{1,1} \ni X \mapsto \mathrm{Ad}\left(\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) X \in \mathrm{SU}_{1,1},$$

$\mathbb{H}^2 = \mathrm{SU}_{1,1}/\mathrm{U}_1$ becomes a Riemannian symmetric space.

Although the left translated unit normal vector field for a surface in Nil_3 takes values in the sphere in $\mathfrak{nil}_3$:

$$\widetilde{\mathbb{S}}^2 = \left\{ x_1e_1 + x_2e_2 + x_3e_3 \in \mathfrak{nil}_3 \middle| (x_1)^2 + (x_2)^2 + (x_3)^2 = 1 \right\},$$

it can be considered as a map into the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{L}^3_{(+,+,-)}$ through the stereographic projections (FIGURE 1). For simplicity, we assume that the support function of a surface takes positive values. This means that the left translated unit normal vector field maps into the upper half of sphere $\widetilde{\mathbb{S}}^2$. The composition of the left translated unit normal vector field $f^{-1}N : \mathbb{D} \to \widetilde{\mathbb{S}}^2 \subset \mathfrak{nil}_3$ and the stereographic projections with base points $(0,0,-1) \in \mathbb{L}^3$ and $-e_3 \in \mathfrak{nil}_3$:

$$\pi_{\mathbb{L}^3} : \mathbb{H}^2 \ni (x_1, x_2, x_3) \mapsto \frac{x_1}{1+x_3} + i\frac{x_2}{1+x_3} \in \mathbb{C},$$

$$\pi_{\mathfrak{nil}_3} : \widetilde{\mathbb{S}}^2 \ni x_1e_1 + x_2e_2 + x_3e_3 \mapsto \frac{x_1}{1+x_3} + i\frac{x_2}{1+x_3} \in \mathbb{C}$$

defines a $\mathbb{H}^2$-valued map

$${\pi_{\mathbb{L}^3}}^{-1} \circ \pi_{\mathfrak{nil}_3} \circ f^{-1}N : \mathbb{D} \to \mathbb{H}^2 \subset \mathbb{L}^3_{(+,+,-)}. \tag{3.3.2}$$

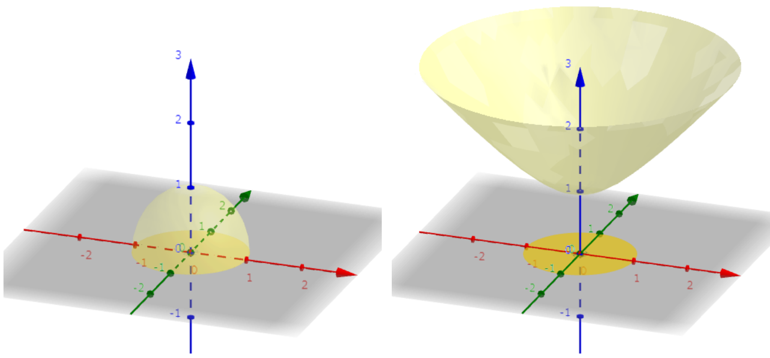

FIGURE 1. Left translated normal vector field takes values in a unit sphere (left). Stereographic projections map the hyperbolic plane (right) and sphere to the same disk.

Since the composition ${\pi_{\mathbb{L}^3}}^{-1} \circ \pi_{\mathfrak{nil}_3} : \widetilde{\mathbb{S}^2} \to \mathbb{H}^2$ is computed as

$$ {\pi_{\mathbb{L}^3}}^{-1} \circ \pi_{\mathfrak{nil}_3}(x_1 e_1 + x_2 e_2 + x_3 e_3) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{1}{x_3} \right), $$

the map (3.3.2) is given explicitly in terms of the generating spinors as

$$ {\pi_{\mathbb{L}^3}}^{-1} \circ \pi_{\mathfrak{nil}_3} \circ f^{-1}N = \frac{1}{|\psi_1|^2 - |\psi_2|^2} \left(2\,\mathrm{Re}(\psi_1 \psi_2), 2\,\mathrm{Im}(\psi_1 \psi_2), |\psi_1|^2 + |\psi_2|^2 \right). \tag{3.3.3} $$

Definition 3.3.1. For a surface in Nil_3, the map ${\pi_{\mathbb{L}^3}}^{-1} \circ \pi_{\mathfrak{nil}_3} \circ f^{-1}N$ defined by (3.3.3) is called the *normal Gauss map* of the surface, and denoted by the same letter $f^{-1}N$ with the left translated unit normal vector field.

Via the identification (3.3.1), the normal Gauss map $f^{-1}N$ of a surface $f : \mathbb{D} \to \mathrm{Nil}_3$ is represented as

$$ f^{-1}N = \frac{1}{2} \mathrm{Ad}(F) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.3.4} $$

where F is a $\mathrm{SU}_{1,1}$-valued map defined by

$$ F = \frac{1}{\sqrt{|\psi_1|^2 - |\psi_2|^2}} \begin{pmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \overline{\psi_1} \end{pmatrix}. \tag{3.3.5} $$

Definition 3.3.2. A $\mathrm{SU}_{1,1}$-valued map F which gives the normal Gauss map by the form (3.3.4) is called a *frame* of the normal Gauss map $f^{-1}N$.

Remark 3.3.3. We would like to note that a frame of the normal Gauss map is not unique. In fact, there is a freedom of $\mathrm{SU}_{1,1}$-valued initial condition F_0 and U_1-valued map k, that is, a frame F and a frame $F_0 F k$ are different in general. Therefore in this paper, we use the particular frame (3.3.5) since the different frames do not always define the same normal Gauss map.

3.4. **Characterization of minimal surfaces in** Nil_3**.** J. Inoguchi [25] and B. Daniel [15] showed that a conformal immersion in Nil_3 is a minimal surface other than a vertical plane if and only if its normal Gauss map is a non-conformal harmonic map into the hyperbolic plane. On the other hand, Dorfmeister, Inoguchi, and Kobayashi characterize the minimality of surfaces with a family of flat connections on a trivial principal bundle in [16].

Let $\widetilde{F}$ be a fundamental system of solutions to the system of partial differential equations (3.2.3). Then we have the matrix differential equations:

$$\partial\widetilde{F} = \widetilde{F}\widetilde{U}, \quad \overline{\partial}\widetilde{F} = \widetilde{F}\widetilde{V}. \tag{3.4.1}$$

Proposition 3.4.1. *Define a* $\mathrm{GL}_2\mathbb{C}$*-valued map* F *by*

$$F := \widetilde{F}\begin{pmatrix} e^{-w/4} & 0 \\ 0 & e^{-w/4} \end{pmatrix}.$$

Then F *satisfies the matrix differential equations*

$$\partial F = FU, \quad \overline{\partial}F = FV \tag{3.4.2}$$

where

$$U = \begin{pmatrix} \frac{1}{4}\partial w + \frac{1}{2}e^{-w/2}e^{u/2}\partial H & -e^{w/2} \\ Qe^{-w/2} & -\frac{1}{4}\partial w \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{4}\overline{\partial}w & -\overline{Q}e^{-w/2} \\ e^{w/2} & \frac{1}{4}\overline{\partial}w + \frac{1}{2}e^{-w/2}e^{u/2}\overline{\partial}H \end{pmatrix}.$$

Proof. The partial derivative of F with respect to z can be computed as

$$\begin{aligned} \partial(\widetilde{F}G) &= (\partial\widetilde{F})G + \widetilde{F}(\partial G) \\ &= \widetilde{F}GG^{-1}\widetilde{U}G + \widetilde{F}GG^{-1}(\partial G) \end{aligned}$$

where $G = \begin{pmatrix} e^{-w/4} & 0 \\ 0 & e^{-w/4} \end{pmatrix}$. Therefore we obtain the first equation of (3.4.2) by putting $U := G^{-1}\widetilde{U}G + G^{-1}(\partial G)$. The second equation can be obtained by computing the partial derivative with respect to $\bar{z}$. □

Remark 3.4.2. It can be checked that the particular frame (3.3.5) of the normal Gauss map for a minimal surface is a solution of the equations (3.4.1) since the Dirac potential takes purely imaginary values. Moreover, the map F defined in Proposition 3.4.1 becomes a frame of the normal Gauss map of minimal surface.

Let us define a family of Maurer-Cartan forms α^λ parametrized by $\lambda \in \mathbb{S}^1 = \{\lambda \in \mathbb{C} | |\lambda| = 1\}$ from the $\mathrm{GL}_2\mathbb{C}$-valued map F defined in Proposition 3.4.1 as

$$\alpha^\lambda = U^\lambda dz + V^\lambda d\bar{z}$$

where

$$U^\lambda = \begin{pmatrix} \frac{1}{4}\partial w + \frac{1}{2}e^{-w/2}e^{u/2}\partial H & -\lambda^{-1}e^{w/2} \\ \lambda^{-1}Qe^{-w/2} & -\frac{1}{4}\partial w \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} -\frac{1}{4}\overline{\partial}w & -\lambda\overline{Q}e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}\overline{\partial}w + \frac{1}{2}e^{-w/2}e^{u/2}\overline{\partial}H \end{pmatrix}.$$

The 1-form α^λ takes values in $\mathfrak{su}_{1,1}$ for each $\lambda \in \mathbb{S}^1$ when the mean curvature is 0. Then minimality of surfaces in Nil_3 is characterized by a family of flat connections on the trivial bundle $\mathbb{D} \times \mathrm{SU}_{1,1}$.

Theorem 3.4.3 ([16])**.** *Let* $f : \mathbb{D} \to \mathrm{Nil}_3$ *be a non-vertical conformal immersion from a simply connected domain into* Nil_3*. The following statements are mutually equivalent*:

(1) *The mean curvature of* f *is* 0.

(2) *The Dirac potential for f has purely imaginary values.*
(3) *$d+\alpha^\lambda$ defines a family of flat connections on the trivial principal bundle $\mathbb{D}\times \mathrm{SU}_{1,1}$.*
(4) *The normal Gauss map of f is a non-conformal harmonic map into the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{L}^3_{(+,+,-)}$.*

Proof. The equivalence between (1) and (2) is trivial from the definition of the Dirac potentials for surfaces in Nil_3.

We first consider the condition for $d+\alpha^\lambda$ to be a family of flat connections on $\mathbb{D}\times\mathrm{SU}_{1,1}$. A straightforward computation shows that the following conditions are the necessary and sufficient conditions:

$$\overline{\partial} U^\lambda - \partial V^\lambda - [U^\lambda, V^\lambda] = 0, \quad \alpha^\lambda \in \mathfrak{su}_{1,1}. \tag{3.4.3}$$

By paying attention to the coefficients of λ^{-1}, λ^0, and λ for each entry, we obtain the following equations

$$\frac{1}{2}e^{u/2}\overline{\partial} H = 0, \quad \overline{\partial} Q + \frac{1}{2} Q e^{-w/2} e^{u/2}\overline{\partial} H = 0, \tag{3.4.4}$$

$$\begin{aligned} &\frac{1}{2}\partial\overline{\partial} w + e^w - |Q|^2 e^{-w} + \frac{1}{2}\left(\partial\overline{\partial} H + \partial H \overline{\partial}\frac{-w+u}{2}\right) e^{-w/2}e^{u/2} = 0,\\ &\frac{1}{2}\partial\overline{\partial} w + e^w - |Q|^2 e^{-w} + \frac{1}{2}\left(\partial\overline{\partial} H + \overline{\partial} H \partial\frac{-w+u}{2}\right) e^{-w/2}e^{u/2} = 0, \end{aligned} \tag{3.4.5}$$

$$\frac{1}{2}e^{u/2}\partial H = 0, \quad \partial \overline{Q} + \frac{1}{2} \overline{Q} e^{-w/2} e^{u/2}\partial H = 0. \tag{3.4.6}$$

Equations (3.4.4), (3.4.5), and (3.4.6) are the coefficient of λ^{-1}, λ^0, and λ, respectively. The equations (3.4.5) coincide the compatibility conditions (3.2.4) for non-linear Dirac equation, and then they always hold.

When the surface $f : \mathbb{D} \to \mathrm{Nil}_3$ is minimal, the conditions (3.4.4) and (3.4.6) are represented as

$$\overline{\partial} Q = 0.$$

Hence from Theorem 3.2.4, the first condition of (3.4.3) holds. Moreover, when $H = 0$ the matrices U^λ and V^λ have a particularly simple form:

$$U^\lambda = \begin{pmatrix} \frac{1}{4}\partial w & -\lambda^{-1} e^{w/2} \\ \lambda^{-1} Q e^{-w/2} & -\frac{1}{4}\partial w \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} -\frac{1}{4}\overline{\partial} w & -\lambda \overline{Q} e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}\overline{\partial} w \end{pmatrix}.$$

Then a direct computation shows α^λ takes values in $\mathfrak{su}_{1,1}$ for each $\lambda \in \mathbb{S}^1$. Hence the statement (3) holds when the statement (1) holds. Conversely, we assume that the condition (3) holds. Since the equations (3.4.4), (3.4.5), and (3.4.6) are satisfied, it can be seen that H is constant. Moreover, since α^λ is valued in $\mathfrak{su}_{1,1}$, comparing $(2,1)$-entry with $(1,2)$-entry derives that H must be 0. Therefore we obtain the equivalence between (1) and (3).

Next, we show the equivalence between (3) and (4). For a map $\varphi : \mathbb{D} \to \mathbb{H}^2$ into the hyperbolic plane $\mathbb{H}^2 = \mathrm{SU}_{1,1}/\mathrm{U}_1$, take a frame F, that is,

$$\varphi = \frac{1}{2}\,\mathrm{Ad}(F)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is known that φ is harmonic if and only if the Maurer-Cartan form $\alpha := F^{-1}dF$ satisfies the condition (see [18, proposition 3.3], [16, Appendix D]):

$$d(*\alpha_1) + [\alpha_0 \wedge *\alpha_1] = 0. \tag{3.4.7}$$

Here the symbol $*$ denotes the Hodge star operator of $\mathbb{D}$ and $\alpha_0 = \alpha'_{\mathfrak{k}}dz + \alpha''_{\mathfrak{k}}d\bar{z}$ and $\alpha_1 = \alpha'_{\mathfrak{m}}dz + \alpha''_{\mathfrak{m}}d\bar{z}$ are given by the decomposition $\alpha = \alpha_0 + \alpha_1$ following the Cartan decomposition $\mathfrak{su}_{1,1} = \mathfrak{k} \oplus \mathfrak{m}$ where

$$\mathfrak{k} = \left\{ \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} \middle| \ r \in \mathbb{R} \right\}, \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & -p-qi \\ -p+qi & 0 \end{pmatrix} \middle| \ p, q \in \mathbb{R} \right\}.$$

By defining a family of 1-forms α^λ with parameter $\lambda \in \mathbb{S}^1$ as

$$\alpha^\lambda = \alpha_0 + \lambda^{-1}\alpha'_{\mathfrak{m}}dz + \lambda\alpha''_{\mathfrak{m}}d\bar{z}, \tag{3.4.8}$$

it can be seen that the harmonicity condition (3.4.7) is equivalent to the condition, so-called *zero-curvature representation*

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$$

for each $\lambda \in \mathbb{S}^1$. The proof of this fact is given by decomposing the left hand side of the zero-curvature representation into λ^{-1}-part, λ^0-part, and λ-part. Since the condition (3) can be rephrased as

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0,$$

we know the map, for a solution F^λ of the system of equations $\partial F^\lambda = F^\lambda U^\lambda$, $\overline{\partial} F^\lambda = F^\lambda V^\lambda$,

$$\frac{1}{2}\operatorname{Ad}(F^\lambda)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

defines a family of harmonic maps into the hyperbolic plane for each $\lambda \in \mathbb{S}^1$. In particular from Remark 3.4.2, $F^\lambda|_{\lambda=1}$ gives the normal Gauss map of f. Moreover, since the $(1,0)$-part of the upper right entry of $\alpha^\lambda|_{\lambda=1}$ is non-degenerate, the normal Gauss map is non-conformal. Therefore the condition (3) derives (4).

Conversely, if we assume that the normal Gauss map (3.3.4) of f is harmonic, then we can construct a minimal surface in Nil_3 by using the representation formula introduced in the next subsection. That minimal surface has the generating spinors (ψ_1, ψ_2) same as the original surface f. Therefore Theorem 2.1.3 shows the minimal surface is the same to f up to translations. Hence f is minimal, that is, the condition (3) holds. □

Definition 3.4.4. For a harmonic map φ into $\mathbb{H}^2 \subset \mathbb{L}^3_{(+,+,-)}$, a $\mathrm{SU}_{1,1}$-valued solution F^λ of the equation $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$, where α^λ is the $\mathfrak{su}_{1,1}$-valued 1-form defined in (3.4.8), is called an *extended frame* of φ.

Remark 3.4.5. For a minimal surface in Nil_3, a $\mathrm{SU}_{1,1}$-valued solution F^λ of the matrix differential equation $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$ with the initial condition $F^\lambda|_{\lambda=1} = F$ is an extended frame of the normal Gauss map of the surface.

3.5. **Sym-Bobenko formula.** As mentioned in Remark 3.2.6, a minimal surface induces a spacelike surface of constant mean curvature $\widetilde{H} = 1/2$. Redefining the generating spinors (ψ_1, ψ_2) of a surface $\tilde{f} : \mathbb{D} \to \mathbb{L}^3_{(+,+,-)}$ as

$$\widetilde{\phi}^1 = (\overline{\psi_2})^2 - (\psi_1)^2, \quad \widetilde{\phi}^2 = i\left((\overline{\psi_2})^2 + (\psi_1)^2\right), \quad \widetilde{\phi}^3 = 2i\psi_1\overline{\psi_2} \tag{3.5.1}$$

where the functions ϕ^j are defined by $\partial\tilde{f} = (\widetilde{\phi}^1, \widetilde{\phi}^2, \widetilde{\phi}^3)$, we can derive the non-linear Dirac equation (3.1.1) with the Dirac potentials $\mathcal{U} = \mathcal{V} = i\widetilde{H}e^{\tilde{u}/2}/2$. Here the function $e^{\tilde{u}}$ denotes the conformal factor of $\tilde{f}$ and is rephrased in terms of the generating spinors as $e^{\tilde{u}} = 4\left(|\psi_1|^2 - |\psi_2|^2\right)^2$. Since the Dirac potentials for minimal surfaces in Nil_3 and surfaces in $\mathbb{L}^3_{(+,+,-)}$ of constant mean curvature $1/2$ coincide, these surfaces can be described by the common generating spinors.

Proposition 3.5.1. *The Gauss map of the induced surface $\tilde{f}$ in $\mathbb{L}^3_{(+,+,-)}$ coincides with the normal Gauss map of the original minimal surface f in Nil_3.*

Proof. The unit normal vector field $\widetilde{N}$ of $\tilde{f}$ is given in terms of the generating spinors as

$$\widetilde{N} = \frac{1}{|\psi_1|^2 - |\psi_2|^2}\left(2\,\mathrm{Re}(\psi_1\psi_2), 2\,\mathrm{Im}(\psi_1\psi_2), |\psi_1|^2 + |\psi_2|^2\right).$$

Then we can see that $\widetilde{N}$ is the normal Gauss map of the minimal surface f by (3.3.3) since f and $\tilde{f}$ have the same generating spinors. □

A. Sym [52] discovered a representation formula of negative constant Gaussian curvature surfaces in the Euclidean space in the sight of integrable systems, and A. I. Bobenko [7] extended it to a representation formula of constant mean curvature surfaces in space forms. These immersion formulas are called *Sym-Bobenko formulas*, and they are investigated in various situations. We now introduce the Sym-Bobenko formula of spacelike surfaces in $\mathbb{L}^3_{(+,+,-)}$ of constant mean curvature $1/2$.

Theorem 3.5.2 ([23])**.** *Let F^λ be an extended frame of some harmonic map φ from a simply connected domain $\mathbb{D} \subset \mathbb{C}$ into $\mathbb{H}^2$. Define the map $f_{\mathbb{L}^3_{(+,+,-)}}$ by*

$$f_{\mathbb{L}^3_{(+,+,-)}} = -i\lambda\left(\frac{\partial F^\lambda}{\partial\lambda}\right)\left(F^\lambda\right)^{-1} - N_{\mathbb{L}^3_{(+,+,-)}}, \quad N_{\mathbb{L}^3_{(+,+,-)}} = \frac{1}{2}\,\mathrm{Ad}\left(F^\lambda\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then $f_{\mathbb{L}^3_{(+,+,-)}}$ describes an associated family of surfaces of constant mean curvature $1/2$ and $N_{\mathbb{L}^3_{(+,+,-)}}$ is the Gauss map of $f_{\mathbb{L}^3_{(+,+,-)}}$ for each $\lambda \in \mathbb{S}^1$.

Proof. After gauging the extended frame, the coefficient a of the upper right entry of $\alpha'_{\mathfrak{m}}$ takes values in purely imaginary. Therefore we assume that a takes purely imaginary values from the beginning. Define a function h by $h = 4ia$, and ψ_1 and ψ_2 by putting

$$F_{11} = \sqrt{2}\psi_1 h^{-1/2}, \quad F_{12} = \sqrt{2}\psi_2 h^{-1/2},$$

respectively. Then a frame $F = F^\lambda|_{\lambda=1}$ of φ is written as

$$F = \sqrt{2}h^{-1/2}\begin{pmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \overline{\psi_1} \end{pmatrix}.$$

Since $\det F = 1$, we have $h = 2\left(|\psi_1|^2 - |\psi_2|^2\right)$. Thus we have

$$F = \frac{1}{\sqrt{|\psi_1|^2 - |\psi_2|^2}} \begin{pmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \overline{\psi_1} \end{pmatrix}.$$

From the continuity of extended frames with respect to the parameter λ, F^λ can be represented in the form of

$$F^\lambda = \frac{1}{\sqrt{|\psi_1(\lambda)|^2 - |\psi_2(\lambda)|^2}} \begin{pmatrix} \psi_1(\lambda) & \psi_2(\lambda) \\ \overline{\psi_2(\lambda)} & \overline{\psi_1(\lambda)} \end{pmatrix}$$

for some complex functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ with $\psi_k(1) = \psi_k$ for $k = 1, 2$. Moreover, $-i\lambda\left(\frac{\partial F^\lambda}{\partial \lambda}\right)\left(F^\lambda\right)^{-1}$ and $N_{\mathbb{L}^3_{(+,+,-)}}$ take velues in $\mathfrak{su}_{1,1}$. Hence $f_{\mathbb{L}^3_{(+,+,-)}}$ is a $\mathfrak{su}_{1,1}$-valued map.

Since F^λ satisfies $\partial F^\lambda = F^\lambda U^\lambda$ and $\overline{\partial} F^\lambda = F^\lambda V^\lambda$ where

$$U^\lambda = \alpha'_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{m}}, \quad V^\lambda = \alpha''_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$$

it is straightforward to be

$$\begin{aligned} \partial f_{\mathbb{L}^3_{(+,+,-)}} &= \partial\left(-i\lambda\left(\frac{\partial F^\lambda}{\partial\lambda}\right)\left(F^\lambda\right)^{-1} - \frac{1}{2}\operatorname{Ad}\left(F^\lambda\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \\ &= -\operatorname{Ad}(F^\lambda)\left(i\lambda\left(\frac{\partial}{\partial\lambda}U^\lambda\right) + \frac{1}{2}\left[U^\lambda, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right]\right) \\ &= 2i\lambda^{-1}\left(-\frac{i}{4}h\right)\operatorname{Ad}(F^\lambda)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\lambda^{-1}\psi_1(\lambda)\overline{\psi_2(\lambda)} & \lambda^{-1}\left(\psi_1(\lambda)\right)^2 \\ -\lambda^{-1}\left(\overline{\psi_2(\lambda)}\right)^2 & \lambda^{-1}\psi_1(\lambda)\overline{\psi_2(\lambda)} \end{pmatrix}. \end{aligned} \tag{3.5.2}$$

Here the notation $[,]$ is the usual bracket of matrices. By the identification (3.3.1), $\partial f_{\mathbb{L}^3_{(+,+,-)}}$ is written as

$$\partial f_{\mathbb{L}^3_{(+,+,-)}} = \left(\phi^1(\lambda), \phi^2(\lambda), i\phi^3(\lambda)\right)$$

with

$$\begin{aligned} \phi^1(\lambda) &:= \lambda^{-1}\left(\left(\overline{\psi_2(\lambda)}\right)^2 - \left(\psi_1(\lambda)\right)^2\right), \\ \phi^2(\lambda) &:= \lambda^{-1} i\left(\left(\overline{\psi_2(\lambda)}\right)^2 + \left(\psi_1(\lambda)\right)^2\right), \\ \phi^3(\lambda) &:= \lambda^{-1} 2\psi_1(\lambda)\overline{\psi_2(\lambda)}. \end{aligned} \tag{3.5.3}$$

A direct computation shows that $f_{\mathbb{L}^3_{(+,+,-)}}$ is a conformal immersion of constant mean curvature 1/2 with the first fundamental form $h(\lambda)^2 dz d\bar{z}$, and $N_{\mathbb{L}^3_{(+,+,-)}}$ is the unit normal vector field of $f_{\mathbb{L}^3_{(+,+,-)}}$ for each $\lambda \in \mathbb{S}^1$.

By the gauge transformation

$$F^\lambda \mapsto \hat{F^\lambda} := F^\lambda \begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}$$

the differential equations $\partial F^\lambda = F^\lambda U^\lambda$ and $\overline{\partial} F^\lambda = F^\lambda V^\lambda$ are rewritten into

$$\partial \hat{F^\lambda} = \hat{F^\lambda}\hat{U^\lambda}, \quad \overline{\partial}\hat{F^\lambda} = \hat{F^\lambda}\hat{V^\lambda} \tag{3.5.4}$$

where

$$\hat{U^\lambda} = \begin{pmatrix} U_{11} & U_{12} \\ \lambda^{-2}U_{21} & U_{22} \end{pmatrix}, \quad \hat{V^\lambda} = \begin{pmatrix} V_{11} & \lambda^2 V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

for $U^\lambda|_{\lambda=1} = (U_{ij})$ and $V^\lambda|_{\lambda=1} = (V_{ij})$. Since $(1,2)$-entry $-U_{12} = ih/4$ of $-\hat{U^\lambda}$ denotes the Dirac potential $ih(\lambda)/4$ for $f_{\mathbb{L}^3_{(+,+,-)}}$, the conformal factor h^2 is independent of λ. Hence the deformation of $f_{\mathbb{L}^3_{(+,+,-)}}$ with respect to λ preserves the metric and the mean curvature. □

Remark 3.5.3. The parametrized frame F^λ obtained from the particular frame (3.3.5) is an extended frame of the Gauss map of a spacelike surface of constant mean curvature $1/2$ by Remark 3.4.5. Therefore by Theorem 3.5.2 an associated family of spacelike surfaces of constant mean curvature $1/2$ can be obtained from a spacelike surface of constant mean curvature $1/2$ [16].

Let us identify the Lie algebra $\mathfrak{nil}_3$ with the Lie algebra $\mathfrak{su}_{1,1}$ as a real vector space. In $\mathfrak{su}_{1,1}$, we choose an orthonormal basis (E_1, E_2, E_3) as

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.5.5}$$

It is easy to check that the vector E_3 is timelike. Then the identification is given by a linear isomorphism $\Xi : \mathfrak{su}_{1,1} \to \mathfrak{nil}_3$, not a Lie algebra isomorphism, which is defined by

$$\Xi(x_1E_1 + x_2E_2 + x_3E_3) = x_1e_1 + x_2e_2 - x_3e_3.$$

We know by the definition of the generating spinors (2.1.8) and (3.5.1) that the induced surface $\tilde{f}$ in $\mathbb{L}^3_{(+,+,-)}$ has the derivative $\partial\tilde{f} = (\phi^1, \phi^2, i\phi^3)$ when the derivative of a minimal surface f in Nil_3 is given by $f^{-1}\partial f = \phi^1e_1 + \phi^2e_2 + \phi^3e_3$. Therefore minimal surfaces in Nil_3 can be constructed by using Theorem 3.5.2.

Theorem 3.5.4 ([16])**.** *Let F^λ be an extended frame of some harmonic map from a simply connected domain $\mathbb{D} \to \mathbb{H}^2$, and $f_{\mathbb{L}^3_{(+,+,-)}}$ the associated family of constant mean curvature $1/2$ surfaces defined by Theorem 3.5.2. Moreover define a map $f^\lambda : \mathbb{D} \to \mathrm{Nil}_3$ by*

$$f^\lambda := \exp \circ \Xi \circ \hat{f^\lambda}$$

where $\hat{f^\lambda} : \mathbb{D} \to \mathfrak{su}_{1,1}$ is a $\mathfrak{su}_{1,1}$-valued map defined by

$$\hat{f^\lambda} := \left(f_{\mathbb{L}^3_{(+,+,-)}}\right)^o - \frac{i}{2}\lambda\left(\frac{\partial}{\partial\lambda}f_{\mathbb{L}^3_{(+,+,-)}}\right)^d.$$

Here the superscripts "o" and "d" denote the off-diagonal part and diagonal part, respectively. Then f^λ describes a family of minimal surfaces in Nil_3*.*

Proof. A direct computation shows $i\lambda\partial(f_{\mathbb{L}^3_{(+,+,-)}})^d/\partial\lambda$ takes values in $\mathfrak{su}_{1,1}$ and then $\hat{f}^\lambda$ also takes values in $\mathfrak{su}_{1,1}$. By using (3.5.2), we have

$$\begin{aligned}\partial\left(\frac{i}{2}\lambda\left(\frac{\partial}{\partial\lambda}f_{\mathbb{L}^3_{(+,+,-)}}\right)\right) &= \frac{i}{2}\lambda\frac{\partial}{\partial\lambda}\left(\lambda^{-1}\frac{1}{2}h\,\mathrm{Ad}(F^\lambda)\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\\ &= -\frac{i}{2}\partial f_{\mathbb{L}^3_{(+,+,-)}}+\frac{1}{2}\left[-f_{\mathbb{L}^3_{(+,+,-)}}-N_{\mathbb{L}^3_{(+,+,-)}},\ \partial f_{\mathbb{L}^3_{(+,+,-)}}\right].\end{aligned}\tag{3.5.6}$$

Then the diagonal part of (3.5.6) is given by

$$\partial\left(\frac{i}{2}\lambda\left(\frac{\partial}{\partial\lambda}f_{\mathbb{L}^3_{(+,+,-)}}\right)\right)^d=-\frac{i}{2}\left(\partial f_{\mathbb{L}^3_{(+,+,-)}}\right)^d-\frac{1}{2}\left[f_{\mathbb{L}^3_{(+,+,-)}},\ \partial f_{\mathbb{L}^3_{(+,+,-)}}\right]^d-\frac{1}{2}\left[N_{\mathbb{L}^3_{(+,+,-)}},\ \partial f_{\mathbb{L}^3_{(+,+,-)}}\right]^d.$$

By using (3.5.2) again and (3.5.3), each term on the right hand side can be computed as

$$\begin{aligned}-\frac{i}{2}\left(\partial f_{\mathbb{L}^3_{(+,+,-)}}\right)^d &= -\frac{i}{2}\begin{pmatrix}-\phi^3(\lambda) & -\phi^1(\lambda)-i\phi^2(\lambda)\\ -\phi^1(\lambda)+i\phi^2(\lambda) & \phi^3(\lambda)\end{pmatrix}^d\\ &= \frac{1}{2}\phi^3(\lambda)E_3,\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\left[f_{\mathbb{L}^3_{(+,+,-)}},\ \partial f_{\mathbb{L}^3_{(+,+,-)}}\right]^d &= -\frac{1}{2}\left[\int\phi^1(\lambda)dzE_1+\int\phi^2(\lambda)dzE_2,\ \phi^1(\lambda)E_1+\phi^2(\lambda)E_2\right]\\ &= \frac{1}{2}\left(\phi^2(\lambda)\int\phi^1(\lambda)dz-\phi^1(\lambda)\int\phi^2(\lambda)dz\right)E_3,\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\left[N_{\mathbb{L}^3_{(+,+,-)}},\partial f_{\mathbb{L}^3_{(+,+,-)}}\right]^d &= -\frac{1}{2}\left[\frac{1}{2}\mathrm{Ad}(F^\lambda)\begin{pmatrix}i&0\\0&-i\end{pmatrix},\ -\lambda^{-1}\frac{1}{2}h\,\mathrm{Ad}(F^\lambda)\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]^d\\ &= -\frac{1}{2}\lambda^{-1}\frac{1}{4}h\,\mathrm{Ad}(F^\lambda)\left[\begin{pmatrix}i&0\\0&-i\end{pmatrix},\ \begin{pmatrix}0&1\\0&0\end{pmatrix}\right]^d\\ &= -\frac{i}{2}\left(\partial f_{\mathbb{L}^3_{(+,+,-)}}\right)^d\\ &= \frac{1}{2}\phi^3(\lambda)E_3.\end{aligned}$$

Thus by combining (3.5.2) and the above results we obtain the derivative of $\hat{f}^\lambda$ with respect to z

$$\partial\hat{f}^\lambda=\phi^1(\lambda)E_1+\phi^2(\lambda)E_2-\left(\phi^3(\lambda)+\frac{1}{2}\left(\phi^2(\lambda)\int\phi^1(\lambda)dz-\phi^1(\lambda)\int\phi^2(\lambda)dz\right)\right)E_3.$$

By the identification (3.5.5) and a left translation, we obtain the derivative of f^λ with respect to z

$$(f^\lambda)^{-1}\partial f^\lambda=\phi^1(\lambda)e_1+\phi^2(\lambda)e_2+\phi^3(\lambda)e_3.$$

Since the conformality of f^λ is derived from

$$g_R\left((f^\lambda)^{-1}\partial f^\lambda,\ (f^\lambda)^{-1}\partial f^\lambda\right)=0,$$

$$g_R\left((f^\lambda)^{-1}\partial f^\lambda,\ (f^\lambda)^{-1}\overline{\partial}f^\lambda\right)=\sum_{j=1}^3|\phi^j(\lambda)|^2=2\left(|\psi_2(\lambda)|^2+|\psi_1(\lambda)|^2\right)^2>0,$$

f^λ describes a family of surfaces in Nil_3. Furthermore the generating spinors of f^λ coincide the ones of $f_{\mathbb{L}^3_{(+,+,-)}}$. Then the Dirac potential $\mathcal{U}$ is given by

$$\mathcal{U} = \frac{i}{4}h(\lambda) = \frac{i}{4}h,$$

This means that f^λ is minimal for each $\lambda \in \mathbb{S}^1$ and the support function of f^λ is h. □

Remark 3.5.5. (1) For a minimal surface $f^\lambda|_{\lambda=1}$ defined in Theorem 3.5.4, denote the Abresch-Rosenberg differential by Qdz^2. Then the Abresch-Rosenberg differential of f^λ is $\lambda^{-2}Qdz^2$. This fact can be confirmed by (3.4.2) with $H = 0$ and (3.5.4).

(2) The Sym-Bobenko formula of minimal surfaces in Nil_3 is written down in [10] in a different way.

3.6. **Weierstrass-type representation via loop group method.** In the previous subsection minimal surfaces are constructed from an extended frame of a harmonic map into $\mathbb{H}^2$. On the other hand, J. F. Dorfmeister, F. Pedit, and H. Wu built a Weierstrass-type construction of harmonic maps into symmetric spaces, so-called DPW method [18]. Therefore we can obtain a non-vertical minimal surface from holomorphic data which is a loop Lie algebra-valued 1-form satisfying some conditions, called a *holomorphic potential*. In particular in this subsection, we will introduce the holomorphic potentials made from non-vertical minimal surfaces, which is called *normalized potentials* [16] and recover an extended frame of the normal Gauss map of minimal surfaces from holomorphic data by using the DPW method.

The key of the DPW method is the decomposition theorems of loop groups, *Birkhoff decomposition* and *Iwasawa decomposition*. First, we consider several *loop groups* which are Lie groups of infinite dimensional.

$$\Lambda\mathrm{SL}_2\mathbb{C}_\sigma = \left\{ g : \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{C} \,\middle|\, g(\lambda) = \sum_{j=-\infty}^{\infty} g_j\lambda^j,\ g(-\lambda) = \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) g(\lambda) \right\}.$$

We assume that the Fourier series of each element of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ is absolutely convergent. Then the topology of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ is determined as a Banach algebra, see [45] for details. Let D^+ and D^- denote the inside of the unit disk and the union of infinite points and the outside of the unit disk, respectively. Then we can consider two loop groups of which the elements can be extended holomorphically to $D^\pm$:

$$\Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma = \{g \in \Lambda\mathrm{SL}_2\mathbb{C}_\sigma | g_j = 0 \text{ for } j < 0\},$$

$$\Lambda^-\mathrm{SL}_2\mathbb{C}_\sigma = \{g \in \Lambda\mathrm{SL}_2\mathbb{C}_\sigma | g_j = 0 \text{ for } j > 0\}.$$

Moreover, we now use the lower subscript $*$ for normalization at $\lambda = 0$ or $\lambda = \infty$ by identity, that is,

$$\Lambda^+_*\mathrm{SL}_2\mathbb{C}_\sigma = \left\{g \in \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma \middle| g_0 = \mathrm{id}\right\},$$

$$\Lambda^-_*\mathrm{SL}_2\mathbb{C}_\sigma = \left\{g \in \Lambda^-\mathrm{SL}_2\mathbb{C}_\sigma \middle| g_0 = \mathrm{id}\right\}.$$

The extended frames of harmonic maps into $\mathbb{H}^2$ can be considered as loops into $\mathrm{SU}_{1,1}$. They are elements of the loop group $\Lambda\mathrm{SU}_{1,1\sigma}$ which is defined as follows:

$$\Lambda\mathrm{SU}_{1,1\sigma} = \left\{ g \in \Lambda\mathrm{SL}_2\mathbb{C}_\sigma \,\middle|\, \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left(\overline{g(1/\overline{\lambda})}^{\top -1}\right) = g(\lambda) \right\}.$$

The Birkhoff and Iwasawa decompositions give fundamental decompositions of the above loop groups.

Theorem 3.6.1 ([45])**.** *The loop group $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ can be decomposed as follows*:

(1) *Birkhoff decomposition*: *The respective multiplication maps*

$$\Lambda_*^-\mathrm{SL}_2\mathbb{C}_\sigma \times \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma \to \Lambda\mathrm{SL}_2\mathbb{C}_\sigma,$$
$$\Lambda_*^+\mathrm{SL}_2\mathbb{C}_\sigma \times \Lambda^-\mathrm{SL}_2\mathbb{C}_\sigma \to \Lambda\mathrm{SL}_2\mathbb{C}_\sigma$$

are diffeomorphisms onto open-dense subsets of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$. These open dense subsets are called the big cells of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$. In particular, the Birkhoff decompositions is unique.

(2) *Iwasawa decomposition*: *The multiplication map*

$$\Lambda\mathrm{SU}_{1,1\sigma} \times \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma \to \Lambda\mathrm{SL}_2\mathbb{C}_\sigma$$

is a diffeomorphism onto an open dense subset of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$. This open-dense subset is also called the big cell. In particular, the Iwasawa decomposition is unique.

From now on we derive a holomorphic potential from the extended frames of normal Gauss maps of minimal surfaces in Nil_3. Let us assume that the extended frame takes values in the big cell of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$.

Theorem 3.6.2 ([16])**.** *Let F^λ be an extended frame of the normal Gauss map of some minimal surface in* Nil_3*, and apply the Birkhoff decomposition in Theorem 3.6.1 to F^λ as $F^\lambda = F_-^\lambda F_+^\lambda$ with $F_-^\lambda \in \Lambda_*^-\mathrm{SL}_2\mathbb{C}_\sigma$ and $F_+^\lambda \in \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma$. Then the Maurer-Cartan form ξ of F_-^λ, that is, $\xi = (F_-^\lambda)^{-1}dF_-^\lambda$, is holomorphic with respect to z. Moreover the Maurer-Cartan form ξ is represented explicitly as follows*:

(3.6.1)
$$\xi = \lambda^{-1}\begin{pmatrix} 0 & -p \\ Qp^{-1} & 0 \end{pmatrix} dz$$

where p is a meromorphic function and Q is the coefficient function of the Abresch-Rosenberg differential.

Definition 3.6.3. We will call the meromorphic 1-form ξ defined in (3.6.1) the *normalized potential* for minimal surface in Nil_3.

Conversely, by using the Iwasawa decomposition, we can recover the extended frame of the normal Gauss map of a minimal surface as the decomposed factor of a primitive loop for the normalized potential.

Theorem 3.6.4. *Let ξ be a normalized potential for a minimal surface f in* Nil_3 *defined in* (3.6.1)*, and F_- be the solution of the partial differential equation*:

(3.6.2)
$$\partial F_- = F_-\xi$$

with the initial condition $F_-(z=0) = \mathrm{id}$. *Moreover let* $F^\lambda \in \Lambda\mathrm{SU}_{1,1\sigma}$ *and* $F_+ \in \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma$ *be the factors of* F_- *in terms of the Iwasawa decomposition, that is,* $F_- = F^\lambda F_+$. *Then* $F^\lambda k$ *forms an extended frame of the normal Gauss map of* f *up to the change of coordinates for some* $k \in \mathrm{U}_1$.

Remark 3.6.5.

(1) The DPW method is constructing an extended frame of harmonic maps by following the process we can see in Theorem 3.6.4.
(2) To obtain minimal surfaces, we can apply the Sym-Bobenko formula in Theorem 3.5.4. Changing the initial condition of the equation (3.6.2) into $F_-(z=0) = A \in \Lambda\mathrm{SU}_{1,1\sigma}$ gives a change of the extended frame, that is, the new extended frame is $\tilde{F^\lambda} = AF^\lambda$. Therefore according to Remark 3.3.3 in general minimal surfaces in Nil_3 with different initial conditions of (3.6.2) are not isometric.

4. Timelike surface theory in Lorentzian Heisenberg group

In this section, we show the theorem which is the timelike surfaces version of Theorem 3.5.4. Moreover, the Weierstrass-type representation of timelike minimal surfaces in (Nil_3, g_+) is obtained by the loop group method, which is constructed similarly to the Riemannian case. From now on, denote the pair (Nil_3, g_+) by Nil_3 simply.

4.1. **Non-linear Dirac equation for timelike surfaces.** By looking at the e_1-terms and e_2-terms of the structure equations (2.2.2) and (2.2.3) we obtain the following theorem.

Theorem 4.1.1. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into the Lorentz Heisenberg group Nil_3 and (ψ_1, ψ_2) be the generating spinors for f. Then the following non-linear Dirac equation holds*:

$$\left\{\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.1.1}$$

Here the para-complex functions $\mathcal{U}$ and $\mathcal{V}$, called the Dirac potentials, are given by

$$\mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i'}{4}h, \quad h = 2(\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}). \tag{4.1.2}$$

By Remark 2.2.7 and the definition of h, we have

$$h = -e^{u/2}g_+(f^{-1}N, e_3).$$

Then h is called the *support function* for a conformal immersion $f : \mathbb{D} \to \mathrm{Nil}_3$. Since the Dirac potential is determined by only the support function for a conformal immersion when the mean curvature vanishes everywhere, we immediately obtain the following lemma by using Proposition 2.2.2.

Lemma 4.1.2. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion with constant mean curvature 0. Then the following statements are equivalent*:

1. *The Dirac potential $\mathcal{U}$ is not invertible at $p \in \mathbb{D}$.*
2. *The support function h is equal to zero at $p \in \mathbb{D}$.*
3. *The vector e_3 is tangent to f at $p \in \mathbb{D}$.*

The equivalence between statements (2) and (3) holds regardless of the values of the mean curvature. Therefore timelike surfaces with h vanishing identically are part of the inverse image of plane curves by the natural projection $\pi : \mathrm{Nil}_3 \to \mathbb{R}^2$ that extracts the first and second components

$$\pi(x_1, x_2, x_3) = (x_1, x_2).$$

These surfaces are called *Hopf cylinders*. The mean curvature of a Hopf cylinder is equal to half of the curvature of the base plane curve.

Definition 4.1.3. A Hopf cylinder in Nil_3 which has a straight line as the base plane curve is called a *vertical plane*.

Remark 4.1.4. Since the straight lines have the constant curvature 0, vertical planes are the only Hopf cylinders that have the constant mean curvature 0.

From now on, we will restrict ourselves to the case that the Dirac potential $\mathcal{U}$ is invertible everywhere, that is,

$$(\mathrm{Re}\mathcal{U})^2 - (\mathrm{Im}\mathcal{U})^2 \neq 0. \tag{4.1.3}$$

Then by Proposition 2.2.2, the Dirac potential can be represented in the exponential formula:

$$\mathcal{U} = \mathcal{V} = \tilde{\epsilon} e^{w/2} \tag{4.1.4}$$

for some para-complex function w and complex number $\tilde{\epsilon} \in \{\pm 1, \pm i'\}$. In particular, if the mean curvature is 0 and the support function h has positive values then $\tilde{\epsilon} = i'$.

4.2. **Lax type representation for timelike surfaces.** In Riemannian case, Berdinskiĭ and Taĭmanov studied surfaces in the 3-dimensional homogeneous manifolds using the Lax type representation which is equivalent to the non-linear Dirac equation for surfaces. In such a way, the generalized Hopf differential, the so-called Abresch-Rosenberg differential, plays an important role instead of the Hopf differential. In Lorentzian case, we can also obtain the Lax type representation for timelike surfaces by using an analogy of the Hopf differential for timelike surfaces.

Definition 4.2.1. Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into Nil_3. Define a para-complex function Q by

$$Q = \frac{1}{4}(2H - i')\tilde{A}, \quad \tilde{A} = A - \frac{(\phi^3)^2}{2H - i'} \tag{4.2.1}$$

where A and ϕ^3 are the Hopf differential

$$A = g_+(\nabla_\partial \partial f, N)$$

and the e_3-component of $f^{-1}f_z$ for f, respectively. Then the quadratic differential Qdz^2 is well-defined and it is called the *Abresch-Rosenberg differential* for timelike surface f.

Definition 4.2.2. A non-vertical timelike minimal surface is called a *horizontal umbrella* if the Abresch-Rosenberg differential vanishes everywhere.

Theorem 4.2.3. *Let f be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into* Nil_3 *for which the Dirac potential $\mathcal{U}$ satisfies* (4.1.3). *Then the generating spinors $\widetilde{\psi} = (\psi_1, \psi_2)$ satisfies the system of equations*:

$$\widetilde{\psi}_z = \widetilde{\psi}\widetilde{U}, \quad \widetilde{\psi}_{\bar{z}} = \widetilde{\psi}\widetilde{V}, \tag{4.2.2}$$

where

$$\widetilde{U} = \begin{pmatrix} \frac{1}{2}\partial w + \frac{1}{2}\tilde{\epsilon} e^{-w/2} e^{u/2}\partial H & -\tilde{\epsilon} e^{w/2} \\ Q\tilde{\epsilon} e^{-w/2} & 0 \end{pmatrix},$$

$$\widetilde{V} = \begin{pmatrix} 0 & -\overline{Q}\tilde{\epsilon} e^{-w/2} \\ \tilde{\epsilon} e^{w/2} & \frac{1}{2}\overline{\partial} w + \frac{1}{2}\tilde{\epsilon} e^{-w/2} e^{u/2}\overline{\partial} H \end{pmatrix}.$$

Here, Q is the coefficient of the Abresch-Rosenberg differential and $\tilde{\epsilon} \in \{\pm 1, \pm i'\}$ is the number determined by (4.1.4). *Conversely, every solution $\widetilde{\psi}$ to the above equation with* (4.1.4) *is a solution of the nonlinear Dirac equation* (4.1.1) *with* (4.1.2).

Proof. By computing the derivative of the Dirac potential $\tilde{\epsilon}e^{w/2}$ with respect to z, we have

$$\frac{1}{2}\tilde{\epsilon}e^{w/2}\partial w = -\frac{1}{2}e^{u/2}\partial H + 2i'H\psi_1\psi_2(\overline{\psi_2})^2 - \frac{2H-i'}{2}\psi_2\partial\overline{\psi_2} - \frac{2H+i'}{2}\overline{\psi_1}\partial\psi_1.$$

Multiplying the equation above by ψ_1 and using the function Q defined in (4.2.1), we derive

$$\partial\psi_1 = \left(\frac{1}{2}\partial w + \frac{1}{2}\tilde{\epsilon}e^{-w/2}e^{u/2}\partial H\right)\psi_1 + Q\tilde{\epsilon}e^{-w/2}\psi_2.$$

The derivative of ψ_2 with respect to z is given by the nonlinear Dirac equation. Thus we obtain the first equation of (4.2.2). We can derive the second equation of (4.2.2) similarly by differentiating the potential with respect to $\bar{z}$.

Conversely, if the vector $\widetilde{\psi} = (\psi_1, \psi_2)$ is a solution of (4.2.2), the terms of $\overline{\partial}\psi_1$ and $\partial\psi_2$ of (4.2.2) are the equations just what we want. $\square$

The compatibility conditions of the system (4.2.2), that is $\overline{\partial}\widetilde{U} - \partial\widetilde{V} - [\widetilde{U}, \widetilde{V}] = 0$, derive the following four equations:

$$\frac{1}{2}\partial\overline{\partial}w + e^w - Q\overline{Q}e^{-w} + \frac{1}{2}\left(\partial\overline{\partial}H + \partial H\overline{\partial}(-w/2 + u/2)\right)\tilde{\epsilon}e^{-w/2}e^{u/2} = 0,$$
$$\frac{1}{2}\partial\overline{\partial}w + e^w - Q\overline{Q}e^{-w} + \frac{1}{2}\left(\partial\overline{\partial}H + \overline{\partial}H\partial(-w/2 + u/2)\right)\tilde{\epsilon}e^{-w/2}e^{u/2} = 0,$$

$$\begin{aligned}\partial\overline{Q}\tilde{\epsilon}e^{-w/2} &= -\frac{1}{2}\overline{Q}\partial He^{-w}e^{u/2} - \frac{1}{2}\overline{\partial}He^{u/2},\\ \overline{\partial}Q\tilde{\epsilon}e^{-w/2} &= -\frac{1}{2}Q\overline{\partial}He^{-w}e^{u/2} - \frac{1}{2}\partial He^{u/2}.\end{aligned}$$

Then we immediately obtain an analogy of Theorem 3.2.4.

Theorem 4.2.4. *For a constant mean curvature timelike surface in* Nil_3 *which has the Dirac potential invertible everywhere, the Abresch-Rosenberg differential is para-holomorphic.*

Remark 4.2.5. (1) To obtain a timelike surface from a solution (w, H, Q) of the compatibility conditions, a solution $\widetilde{\psi}$ must satisfy the additional conditions

$$\tilde{\epsilon}e^{w/2} = -H(\psi_2\overline{\psi_2} + \psi_1\overline{\psi_1}) + \frac{i'}{2}(\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}).$$

(2) When the conformal immersion is minimal the compatibility conditions of the system of differential equations (4.2.2) are more simple:

$$\begin{aligned}\frac{1}{2}\partial\overline{\partial}w + e^w - Q\overline{Q}e^{-w} &= 0,\\ \overline{\partial}Q &= 0.\end{aligned} \tag{4.2.3}$$

These equations coincide with the Gauss-Codazzi equations for a timelike surface in Minkowski 3-space which has the first fundamental form $h^2dzd\bar{z}$, the mean curvature $1/2$, and the Hopf differential $4Qdz^2$. This fact means that a timelike minimal surface in Nil_3 induces a timelike constant mean curvature surface in Minkowski 3-space. The correspondence between these surfaces will be explained later.

Let $\widetilde{F}$ be a fundamental system of solutions to the system (4.2.2):

$$\partial\widetilde{F} = \widetilde{F}\widetilde{U}, \quad \overline{\partial}\widetilde{F} = \widetilde{F}\widetilde{V}. \tag{4.2.4}$$

Then by putting

$$F := \widetilde{F}G, \quad U := G^{-1}\widetilde{U}G + G^{-1}\partial G, \quad V := G^{-1}\widetilde{V}G + G^{-1}\overline{\partial}G$$

where

$$G = \begin{pmatrix} e^{-w/4} & 0 \\ 0 & e^{-w/4} \end{pmatrix},$$

one can see in the same way as Proposition 3.4.1 that the system (4.2.4) is equivalent to the following matrix differential equations

$$\partial F = FU, \quad \overline{\partial}F = FV. \tag{4.2.5}$$

where

$$U = \begin{pmatrix} \frac{1}{4}\partial w + \frac{1}{2}\tilde{\epsilon}e^{-w/2}e^{u/2}\partial H & -\tilde{\epsilon}e^{w/2} \\ Q\tilde{\epsilon}e^{-w/2} & -\frac{1}{4}\partial w \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{4}\overline{\partial}w & -\overline{Q}\tilde{\epsilon}e^{-w/2} \\ \tilde{\epsilon}e^{w/2} & \frac{1}{4}\overline{\partial}w + \frac{1}{2}\tilde{\epsilon}e^{-w/2}e^{u/2}\overline{\partial}H \end{pmatrix}.$$

4.3. **Normal Gauss map of timelike surfaces in** Nil_3**.** Timelike minimal surfaces in Nil_3 can be characterized by the harmonicity of the normal Gauss map, that is naturally defined from the unit normal vector field. Although in the Riemannian case, the normal Gauss map is a map into the hyperbolic plane $\mathbb{H}^2$, the normal Gauss maps of timelike surfaces are maps into the de-Sitter sphere $\mathbb{S}^2_1$ in a Minkowski 3-space $\mathbb{L}^3_{(+,-,+)}$.

The de-Sitter sphere is known as a Lorentzian symmetric space. Let us recall a representation of the de-Sitter sphere as a Lorentzian symmetric space. Let $\mathbb{L}^3_{(+,-,+)}$ denote the Minkowski 3-space $(\mathbb{R}^3, \langle , \rangle)$ where the notation $\langle , \rangle$ is an indefinite inner product:

$$\langle (x_1, x_2, x_3), (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \rangle = x_1\tilde{x}_1 - x_2\tilde{x}_2 + x_3\tilde{x}_3.$$

The de-Sitter sphere $\mathbb{S}^2_1$ is a timelike surface in $\mathbb{L}^3_{(+,-,+)}$ defined by

$$\mathbb{S}^2_1 = \left\{ (x_1, x_2, x_3) \middle| (x_1)^2 - (x_2)^2 + (x_3)^2 = 1 \right\}.$$

On the other hand, let $\mathfrak{su}'_{1,1}$ denote the *special para-unitary Lie algebra of index* $(1,1)$ defined by

$$\mathfrak{su}'_{1,1} = \left\{ \begin{pmatrix} i'a & \bar{b} \\ b & -i'a \end{pmatrix} \middle| \, a \in \mathbb{R}, b \in \mathbb{C}' \right\}.$$

The Lie bracket of $\mathfrak{su}'_{1,1}$ is the usual commutator of matrices. Let us equip $\mathfrak{su}'_{1,1}$ with the following indefinte scalar product $\langle , \rangle_{m2}$:

$$\langle X, Y \rangle_{m2} = 2\,\mathrm{Trace}(XY).$$

Then the Minkowski 3-space $\mathbb{L}^3_{(+,-,+)}$ is identified with $\mathfrak{su}'_{1,1}$ as an indefinite scalar product space by the following correspondence:

$$\mathfrak{su}'_{1,1} \ni \frac{1}{2}\begin{pmatrix} i'r & -p - i'q \\ -p + i'q & -i'r \end{pmatrix} \longleftrightarrow (p, q, r) \in \mathbb{L}^3_{(+,-,+)}, \tag{4.3.1}$$

and the de-Sitter sphere $\mathbb{S}^2_1$ can be realized in $\mathfrak{su}'_{1,1}$ as

$$\mathbb{S}^2_1 = \left\{ X \in \mathfrak{su}'_{1,1} \middle| \langle X, X \rangle_{m2} = 1 \right\}.$$

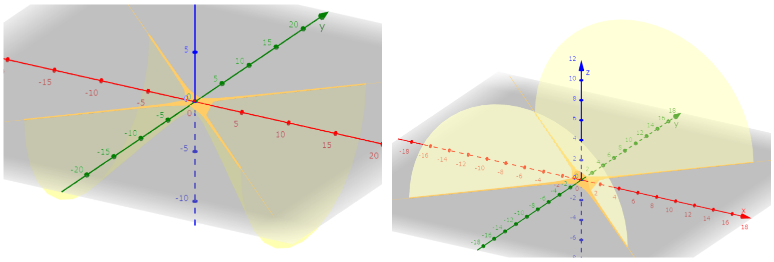

FIGURE 2. Left translated unit normal vector field takes values in the de-Sitter sphere in $\mathfrak{nil}_3$ (left). Stereographic projections map the de-Sitter spheres in $\mathbb{L}^3_{(+,-,+)}$ (right) and $\mathfrak{nil}_3$ to the common subset.

The Lie group $\mathrm{SU}'_{1,1}$ which corresponds to the Lie algebra $\mathfrak{su}'_{1,1}$, that is called the *special para-unitary Lie group of index* $(1,1)$, is given by

$$\mathrm{SU}'_{1,1} = \left\{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C}', a\bar{a} - b\bar{b} = 1 \right\}.$$

Then the Lie group $\mathrm{SU}'_{1,1}$ acts transitively and isomorphically on $\mathbb{S}^2_1$:

$$\mathrm{SU}'_{1,1} \times \mathbb{S}^2_1 \ni (F, X) \mapsto \mathrm{Ad}(F)X \in \mathbb{S}^2_1.$$

The isotropy subgroup with respect to this action at the point $\begin{pmatrix} i'/2 & 0 \\ 0 & -i'/2 \end{pmatrix}$ is the subgroup U'_1 of $\mathrm{SU}'_{1,1}$ consisting of diagonal matrices:

$$\mathrm{U}'_1 = \left\{ \begin{pmatrix} \pm e^{i'\theta} & 0 \\ 0 & \pm e^{-i'\theta} \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\}.$$

The resulting homogeneous space $\mathbb{S}^2_1 = \mathrm{SU}'_{1,1}/\mathrm{U}'_1$ is a Lorentzian symmetric space with involution σ:

$$\sigma : \mathrm{SU}'_{1,1} \ni X \mapsto \mathrm{Ad}\left(\frac{1}{2} \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix} \right) X \in \mathrm{SU}'_{1,1}.$$

Although the left translated unit normal vector field of a timelike surface in Nil_3 takes values in the de-Sitter sphere $\widetilde{\mathbb{S}}^2_1$ in $\mathfrak{nil}_3$:

$$\widetilde{\mathbb{S}}^2_1 = \left\{ x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathfrak{nil}_3 \middle| -(x_1)^2 + (x_2)^2 + (x_3)^2 = 1 \right\},$$

one can regard the map as a map into the de-Sitter sphere $\mathbb{S}^2_1$ in $\mathbb{L}^3_{(+,-,+)}$ via the stereographic projections of $\widetilde{\mathbb{S}}^2_1$ and $\mathbb{S}^2_1$ (FIGURE 2).

For simplicity, we assume that the support function takes positive values. This implies that the left translated unit normal vector field takes values in the lower half of $\widetilde{\mathbb{S}}^2_1$. Then the composition of the left translated unit normal vector field $f^{-1}N : \mathbb{D} \to \widetilde{\mathbb{S}}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$ and the stereographic projections with the base points e_3 and $(0,0,-1) \in \mathbb{L}^3_{(+,-,+)}$:

$$\pi'_{\mathbb{L}^3} : \mathbb{S}^2_1 \ni (x_1, x_2, x_3) \mapsto \frac{x_1}{1 + x_3} + i' \frac{x_2}{1 + x_3} \in \mathbb{C}',$$

$$\pi'_{\mathfrak{nil}_3} : \widetilde{\mathbb{S}}^2_1 \ni x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto \frac{x_1}{1 - x_3} + i' \frac{x_2}{1 - x_3} \in \mathbb{C}'$$

defines a $\mathbb{S}^2_1$-valued map

$$\pi'_{\mathbb{L}^3}{}^{-1} \circ \pi'_{\mathfrak{nil}_3} \circ f^{-1}N : \mathbb{D} \to \mathrm{SU}'_{1,1} \subset \mathbb{L}^3_{(+,-,+)}. \tag{4.3.2}$$

Since the composition $\pi'_{\mathbb{L}^3}{}^{-1} \circ \pi'_{\mathfrak{nil}_3} : \widetilde{\mathbb{S}}^2_1 \to \mathbb{S}^2_1$ is computed as

$$\pi'_{\mathbb{L}^3}{}^{-1} \circ \pi'_{\mathfrak{nil}_3}(x_1e_1 + x_2e_2 + x_3e_3) = \left(-\frac{x_1}{x_3}, -\frac{x_2}{x_3}, -\frac{1}{x_3}\right),$$

the map (4.3.2) is given explicitly in terms of the generating spinors as

$$\pi'_{\mathbb{L}^3}{}^{-1} \circ \pi'_{\mathfrak{nil}_3} \circ f^{-1}N = \frac{1}{\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}}\left(-2\,\mathrm{Im}(\psi_1\psi_2), 2\,\mathrm{Re}(\psi_1\psi_2), \psi_2\overline{\psi_2} + \psi_1\overline{\psi_1}\right). \tag{4.3.3}$$

Definition 4.3.1. For a timelike surface in Nil_3, the map $\pi'_{\mathbb{L}^3}{}^{-1} \circ \pi'_{\mathfrak{nil}_3} \circ f^{-1}N$ defined by (4.3.3) is called the *normal Gauss map* of the timelike surface, and denoted by the same letter $f^{-1}N$ with the left translated unit normal vector field.

Via the identification (4.3.1), the normal Gauss map $f^{-1}N$ of a timelike surface $f : \mathbb{D} \to \mathrm{Nil}_3$ is represented as

$$f^{-1}N = \frac{1}{2}\,\mathrm{Ad}(F)\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix} \tag{4.3.4}$$

where F is a $\mathrm{SU}'_{1,1}$-valued map defined by

$$F = \frac{1}{\sqrt{\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}}}\begin{pmatrix} \overline{\psi_2} & \overline{\psi_1} \\ \psi_1 & \psi_2 \end{pmatrix}. \tag{4.3.5}$$

Definition 4.3.2. A $\mathrm{SU}'_{1,1}$-valued map F which gives the normal Gauss map by the form (4.3.4) is called a *frame* of the normal Gauss map.

Remark 4.3.3.
(1) The particular frame (4.3.5) of the normal Gauss map of a timelike minimal surface is a solution of the system (4.2.5). This can be checked since the pair $(\overline{\psi_2}, \overline{\psi_1})$ is a solution of the system (4.2.2) for a timelike minimal surface with the generating spinors (ψ_1, ψ_2).
(2) A frame of the normal Gauss map is not unique as in the Riemannian case. Let F_0 be an element of $\mathrm{SU}'_{1,1}$, k a U'_1-valued map, and F a frame of the normal Gauss map for some timelike surface. Then F_0Fk defines another frame. Since the arbitrary choice of initial condition does not correspond to a given timelike surface, we use the particular frame in (4.3.5).

4.4. **Characterization of timelike minimal surfaces in** Nil_3**.** The minimality of timelike surfaces in Nil_3 can be characterized by the harmonicity of the normal Gauss map or a family of flat connections on a trivial principal bundle.

We define a family of Maurer-Cartan forms α^μ by

$$\alpha^\mu = U^\mu dz + V^\mu d\bar{z}. \tag{4.4.1}$$

Here the coefficient matrices U^μ and V^μ are defined by parameterizing U and V in (4.2.5) with $\mu \in \mathbb{S}^1_1 = \{\mu \in \mathbb{C}' | \mu\bar{\mu} = 1, \mathrm{Re}\,\mu > 0\}$ as follows:

$$U^\mu = \begin{pmatrix} \frac{1}{4}\partial w + \frac{1}{2}\tilde{\epsilon}e^{-w/2}e^{u/2}\partial H & -\mu^{-1}\tilde{\epsilon}e^{w/2} \\ \mu^{-1}Q\tilde{\epsilon}e^{-w/2} & -\frac{1}{4}\partial w \end{pmatrix}, \quad V^\mu = \begin{pmatrix} -\frac{1}{4}\overline{\partial} w & -\mu\overline{Q}\tilde{\epsilon}e^{-w/2} \\ \mu\tilde{\epsilon}e^{w/2} & \frac{1}{4}\overline{\partial} w + \frac{1}{2}\tilde{\epsilon}e^{-w/2}e^{u/2}\overline{\partial} H \end{pmatrix}.$$

Theorem 4.4.1. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into Nil_3 satisfying* (4.1.3). *Then the following conditions are mutually equivalent*:

(1) *f is minimal.*
(2) *The Dirac potential has purely imaginary values.*
(3) *$d + \alpha^{\mu}$ defines a family of flat connections on $\mathbb{D} \times \mathrm{SU}'_{1,1}$.*
(4) *The normal Gauss map $f^{-1}N$ is a non-conformal Lorentz harmonic map into the de-Sitter sphere $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$.*

Before we prove the Theorem 4.4.1, we discuss the *zero-curvature representation* of Lorentz harmonic maps.

For a map $\varphi : \mathbb{D} \to \mathbb{S}^2_1$ into the de-Sitter sphere $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$, take a frame F, that is,

$$\varphi = \frac{1}{2}\,\mathrm{Ad}(F)\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}.$$

Moreover decompose the Maurer-Cartan form $\alpha = F^{-1}dF$ into $\alpha = \alpha_0 + \alpha_1$ according to the decomposition $\mathfrak{su}'_{1,1} = \mathfrak{k}' + \mathfrak{m}'$ with

$$\mathfrak{k}' = \left\{ \begin{pmatrix} i'r & 0 \\ 0 & -i'r \end{pmatrix} \middle| \; r \in \mathbb{R} \right\}, \quad \mathfrak{m}' = \left\{ \begin{pmatrix} 0 & -p - i'q \\ -p + i'q & 0 \end{pmatrix} \middle| \; p, q \in \mathbb{R} \right\}.$$

Then we obtain the following proposition.

Proposition 4.4.2. *A map $\varphi : \mathbb{D} \to \mathbb{S}^2_1$ is Lorentz harmonic if and only if the following equation holds*:

$$d(*\alpha_1) + [\alpha_0 \wedge *\alpha_1] = 0. \tag{4.4.2}$$

Here the notation $*$ is the Hodge star operator defined by

$$*dz = i'dz, \quad *d\bar{z} = -i'd\bar{z}.$$

Proof. Decompose α_0 and α_1 into the dz-part and the $d\bar{z}$-part:

$$\alpha_0 = {\alpha_{\mathfrak{k}}}' dz + {\alpha_{\mathfrak{k}}}'' d\bar{z}, \quad \alpha_1 = {\alpha_{\mathfrak{m}}}' dz + {\alpha_{\mathfrak{m}}}'' d\bar{z}.$$

Then we put $\alpha' := \alpha'_{\mathfrak{k}} + \alpha'_{\mathfrak{m}}$ and $\alpha'' := \alpha''_{\mathfrak{k}} + \alpha''_{\mathfrak{m}}$, that is, $\alpha = \alpha' dz + \alpha'' d\bar{z}$ denote the decomposition of α into dz-part and $d\bar{z}$-part. A straightforward computation shows

$$\partial\varphi = \frac{1}{2}\,\mathrm{Ad}(F)\left[\alpha', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right].$$

Therefore direct computations show

$$\bar{\partial}\partial\varphi = \frac{1}{2}\,\mathrm{Ad}(F)\left(\left[\alpha'', \left[\alpha', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right] + \left[\bar{\partial}\alpha', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right).$$

By decomposing α' and α'' into $\mathfrak{k}$-part and $\mathfrak{m}$-part one can see that

$$\left[\bar{\partial}\alpha', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right] = \left[\bar{\partial}{\alpha_{\mathfrak{m}}}', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right],$$

$$\left[\alpha'', \left[\alpha', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right] = \left[{\alpha_{\mathfrak{m}}}'', \left[{\alpha_{\mathfrak{m}}}', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right] + \left[{\alpha_{\mathfrak{k}}}'', \left[{\alpha_{\mathfrak{m}}}', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right].$$

The Jacobi identity implies

$$\left[\alpha_{\mathfrak{k}}{}'', \left[\alpha_{\mathfrak{m}}{}', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right] = \left[[\alpha_{\mathfrak{k}}{}'', \alpha_{\mathfrak{m}}{}'], \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right].$$

Therefore we obtain

$$\overline{\partial}\partial\varphi = \frac{1}{2}\operatorname{Ad}(F)\left(\left[\overline{\partial}\alpha_{\mathfrak{m}}{}' + [\alpha_{\mathfrak{k}}{}'', \alpha_{\mathfrak{m}}{}'], \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right] + \left[\alpha_{\mathfrak{m}}{}'', \left[\alpha_{\mathfrak{m}}{}', \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right]\right)$$

Since the first term is in $\mathfrak{m}$ and the second term is in $\mathfrak{k}$, the definition of Lorentz harmonicity, $\overline{\partial}\partial\varphi = \rho\varphi$, implies

$$\overline{\partial}\alpha_{\mathfrak{m}}{}' + [\alpha_{\mathfrak{k}}{}'', \alpha_{\mathfrak{m}}{}'] = 0. \tag{4.4.3}$$

Similar computations for $\partial\overline{\partial}\varphi$ derive

$$\overline{\partial}\alpha_{\mathfrak{m}}{}'' + [\alpha_{\mathfrak{k}}{}', \alpha_{\mathfrak{m}}{}''] = 0.$$

Clearly, this is equivalent to the harmonicity condition (4.4.3). Therefore we have

$$\overline{\partial}\alpha_{\mathfrak{m}}{}'' + \overline{\partial}\alpha_{\mathfrak{m}}{}' + [\alpha_{\mathfrak{k}}{}', \alpha_{\mathfrak{m}}{}''] + [\alpha_{\mathfrak{k}}{}'', \alpha_{\mathfrak{m}}{}'] = 0, \tag{4.4.4}$$

which is equivalent to (4.4.2). Conversely, we assume (4.4.2). Since the map φ satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0,$$

we have the following equations in the $\mathfrak{k}$-part and the $\mathfrak{m}$-part:

$$\begin{aligned} \partial\alpha_{\mathfrak{k}}{}'' - \overline{\partial}\alpha_{\mathfrak{k}}{}' + [\alpha_{\mathfrak{k}}{}', \alpha_{\mathfrak{k}}{}''] + [\alpha_{\mathfrak{m}}{}', \alpha_{\mathfrak{m}}{}''] = 0, \\ -\overline{\partial}\alpha_{\mathfrak{m}}{}' + \partial\alpha_{\mathfrak{m}}{}'' + [\alpha_{\mathfrak{m}}{}', \alpha_{\mathfrak{k}}{}''] + [\alpha_{\mathfrak{k}}{}', \alpha_{\mathfrak{m}}{}''] = 0. \end{aligned} \tag{4.4.5}$$

Subtracting the first equation of (4.4.5) from (4.4.4) derives the harmonicity condition (4.4.3). Therefore the proof completes. □

We define a family of Maurer-Cartan forms α^μ for φ parameterized by $\mu \in \mathbb{S}^1_1$ as follows:

$$\alpha^\mu = \alpha_0 + \mu^{-1}\alpha_{\mathfrak{m}}{}'dz + \mu\alpha''_{\mathfrak{m}}d\bar{z}. \tag{4.4.6}$$

Then we obtain the following theorem. Such a fundamental observation was first due to Pohlmeyer [43]. We can prove it in the same way of [37, Proposition 2.1.4].

Theorem 4.4.3. *A map $\varphi : \mathbb{D} \to \mathbb{S}^2_1$ is Lorentz harmonic if and only if the family α^μ for φ satisfies*

$$d\alpha^\mu + \frac{1}{2}[\alpha^\mu \wedge \alpha^\mu] = 0$$

for every $\mu \in \mathbb{S}^1_1$.

Proof of Theorem 4.4.1. The equivalence between (1) and (2) is trivial from the definition of the Dirac potentials for timelike surfaces in Nil_3.

Define U^μ and V^μ by $\alpha^\mu = U^\mu dz + V^\mu d\bar{z}$. Then the statement (3) holds if and only if

$$\overline{\partial}U^\mu - \partial V^\mu - [U^\mu, V^\mu] = 0 \tag{4.4.7}$$

for all $\mu \in \mathbb{S}^1_1$. The coefficients of μ^{-1}, μ^0, and μ of (4.4.7) are as follows:

$$\frac{1}{2}e^{u/2}\overline{\partial}H = 0, \quad \overline{\partial}Q + \frac{1}{2}Q\tilde{\epsilon}e^{-w/2}e^{u/2}\overline{\partial}H = 0, \tag{4.4.8}$$

$$
(4.4.9)\qquad
\begin{aligned}
&\frac{1}{2}\partial\overline{\partial}w + e^{w} - Q\overline{Q}e^{-w} + \frac{1}{2}(\partial\overline{\partial}H + \overline{\partial}\frac{-w+u}{2}\partial H)\tilde{\epsilon}e^{-w/2}e^{u/2} = 0,\\
&\frac{1}{2}\partial\overline{\partial}w + e^{w} - Q\overline{Q}e^{-w} + \frac{1}{2}(\partial\overline{\partial}H + \partial\frac{-w+u}{2}\overline{\partial}H)\tilde{\epsilon}e^{-w/2}e^{u/2} = 0,
\end{aligned}
$$

$$
(4.4.10)\qquad \partial\overline{Q} + \frac{1}{2}\overline{Q}\tilde{\epsilon}e^{-w/2}e^{u/2}\partial H = 0, \quad \frac{1}{2}e^{u/2}\partial H = 0.
$$

Equations (4.4.8), (4.4.9), and (4.4.10) are the coefficient of μ^{-1}, μ^{0}, and μ, respectively. The equations (4.4.9) appears in the compatibility conditions of (4.2.2) for the original timelike surface, and then they always hold.

When we assume the condition (1), the conditions (4.4.8) and (4.4.10) are rewritten into

$$
\overline{\partial}Q = 0.
$$

Hence it is straightforward that the condition (4.4.7) holds.

Next we assume the condition (3) holds, that is, (4.4.8), (4.4.9), and (4.4.10) are satisfied. Then it is easy to see that H is constant. Furthermore, since α^{μ} takes values in $\mathfrak{su}'_{1,1}$, we can derive that the mean curvature H is 0 by comparing $(2,1)$-entry with $(1,2)$-entry of α^{μ}.

We can see the condition (3) implies (4) by Theorem 4.4.3. In fact, if we assume (3), the particular frame (4.3.5) is a solution of the system (4.2.5) and the Maurer-Cartan equation

$$
d\alpha^{\mu} + \frac{1}{2}[\alpha^{\mu}\wedge\alpha^{\mu}] = 0
$$

holds for every μ. Then the normal Gauss map $\mathrm{Ad}(F^{\mu}|_{\mu=1})\begin{pmatrix} i'/2 & 0 \\ 0 & -i'/2 \end{pmatrix}$ is Lorentz harmonic by Theorem 4.4.3. The non-verticality is obtained since the $(1,2)$-part of U is non-degenerate. Hence (3) implies (4).

To complete the proof, we must prove (4) $\Longrightarrow$ (1). This is proved by applying the results proven in the next section. We can construct a timelike minimal surface from a harmonic map into $\mathbb{S}^2_1$. Assuming that the normal Gauss map (4.3.4) is harmonic, we obtain a timelike minimal surface which has the generating spinor (ψ_1, ψ_2) same as the timelike surface f. Therefore we can see that the timelike minimal surface is f up to a translation by Theorem 2.2.5. Hence f is minimal. $\square$

Definition 4.4.4. For a Lorentz harmonic map φ into $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$, a $\mathrm{SU}'_{1,1}$-valued solution F^{μ} of the equation $(F^{\mu})^{-1}dF^{\mu} = \alpha^{\mu}$, where α^{μ} is the $\mathfrak{su}'_{1,1}$-valued 1-form defined in (4.4.6), is called an *extended frame* of φ.

Remark 4.4.5. Since the normal Gauss map of a timelike minimal surface f is a Lorentz harmonic map into the de-Sitter sphere, we can obtain an extended frame by parameterizing the Maurer-Cartan form of the particular frame (4.3.5) as in (4.4.6). The family α^{μ} is nothing but the one defined in (4.4.1). We call the family of Maurer-Cartan forms α^{μ} the *extended frame* of a timelike minimal surface f.

4.5. **Sym-Bobenko formula of timelike minimal surfaces in** Nil_3**.** In this subsection, we will derive an immersion formula for timelike minimal surfaces in Nil_3 in terms of the extended frame, the so-called *Sym-Bobenko formula.* Unlike the integral representation formula, the so-called *Weierstrass-type representation* [11, 28, 48], the Sym-Bobenko formula will be given by the derivative of the extended frame with respect to the spectral parameter.

A timelike minimal surface in Nil_3 induces a timelike surface of constant mean curvature $\widetilde{H} = 1/2$ in Minkowski 3-space as mentioned in Remark 4.2.5. Let us redefine the generating spinors (ψ_1, ψ_2) of a conformal immersion $\tilde{f} : \mathbb{D} \to \mathbb{L}^3_{(+,-,+)}$ as

$$\widetilde{\phi}^1 = \epsilon\left((\overline{\psi_2})^2 + (\psi_1)^2\right), \quad \widetilde{\phi}^2 = \epsilon i'\left((\overline{\psi_2})^2 - (\psi_1)^2\right), \quad \widetilde{\phi}^3 = 2i'\psi_1\overline{\psi_2} \tag{4.5.1}$$

where the functions $\widetilde{\phi}^j$ are defined by $\partial\tilde{f} = (\widetilde{\phi}^2, \widetilde{\phi}^1, \widetilde{\phi}^3)$. Then the Dirac potentials are changed into $\mathcal{U} = \mathcal{V} = i'\widetilde{H}e^{\tilde{u}/2}/2$, where the function $e^{\tilde{u}}$ is the conformal factor of $\tilde{f}$. The conformal factor is rephrased in terms of the generating spinors as $e^{\tilde{u}} = 4\left(\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}\right)^2$. Since the Dirac potentials for timelike minimal surfaces in Nil_3 and surfaces in $\mathbb{L}^3_{(+,-,+)}$ of constant mean curvature $\widetilde{H} = 1/2$ coincide, the common generating spinors can describe these surfaces.

Proposition 4.5.1. *The Gauss map of the induced surface $\tilde{f}$ in $\mathbb{L}^3_{(+,-,+)}$ coincides with the normal Gauss map of the original timelike minimal surface f in Nil_3.*

Proof. The spinor representation of the normal Gauss map (4.3.3) of the timelike minimal surface f coincides with the Gauss map of $\tilde{f}$ since the generating spinors of f and $\tilde{f}$ are same. □

The Sym-Bobenko formula of timelike surfaces of constant mean curvature in $\mathbb{L}^3_{(+,-,+)}$ is discovered by Inoguchi [24]. In [17, 24], they used a null coordinate system on a Lorentz surface. On the other hand in this thesis, we construct the Sym formula with a para-complex conformal coordinate system. When we construct a Weierstrass-type representation of timelike minimal surfaces in Nil_3 in the next subsection, conformal coordinate systems are useful because only one Weierstrass data can be obtained from surfaces.

Theorem 4.5.2. *Let F^μ be an extended frame of some Lorentz harmonic map φ from a simply connected domain $\mathbb{D} \subset \mathbb{C}'$ into the de-Sitter sphere $\mathbb{S}^2_1 \subset \mathbb{L}^3_{(+,-,+)}$. Assume that the coefficient function a of $(1,2)$-entry of $\alpha_{\mathfrak{m}}{}'$ satisfies $a\bar{a} < 0$ on $\mathbb{D}$. Define two maps $f_{\mathbb{L}^3_{(+,-,+)}}$ and $N_{\mathbb{L}^3_{(+,-,+)}}$ by*

$$f_{\mathbb{L}^3_{(+,-,+)}} = -i'\mu\left(\frac{\partial}{\partial\mu}F^\mu\right)(F^\mu)^{-1} - \frac{1}{2}\operatorname{Ad}(F^\mu)\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}, \quad N_{\mathbb{L}^3_{(+,-,+)}} = \frac{1}{2}\operatorname{Ad}(F^\mu)\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}.$$

Then by the identification (4.3.1), $f_{\mathbb{L}^3_{(+,-,+)}}$ describes an associated family of timelike surfaces of constant mean curvature $1/2$ in $\mathbb{L}^3_{(+,-,+)}$ with the first fundamental form $I = -16a\bar{a}dzd\bar{z}$ and $N_{\mathbb{L}^3_{(+,-,+)}}$ is the spacelike unit normal vector field of $f_{\mathbb{L}^3_{(+,-,+)}}$ for each $\mu \in \mathbb{S}^1_1$.

Proof. After gauging the extended frame, the coefficient a takes purely imaginary values, that is a can be assumed to be purely imaginary from the beginning. For simplicity, we consider only the case of $i'a < 0$. Define h by $h = -4i'a$. Then h has positive values. Moreover for a frame $F^\mu|_{\mu=1} = F = (F_{ij})$ of φ, define ψ_1 and ψ_2 by putting

$$F_{21} = \sqrt{2}\psi_1 h^{-1/2}, \quad F_{22} = \sqrt{2}\psi_2 h^{-1/2},$$

respectively. Then a frame F is written as

$$F = \sqrt{2}h^{-1/2}\begin{pmatrix} \overline{\psi_2} & \overline{\psi_1} \\ \psi_1 & \psi_2 \end{pmatrix}.$$

Since $\det F = 1$, we have $h = 2(\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1})$. Thus we have

$$F = \frac{1}{\sqrt{\psi_2\overline{\psi_2} - \psi_1\overline{\psi_1}}}\begin{pmatrix} \overline{\psi_2} & \overline{\psi_1} \\ \psi_1 & \psi_2 \end{pmatrix}.$$

Because an extended frame is continuous with respect to the spectral parameter μ the frame F^μ can be represented in the form of

$$F = \frac{1}{\sqrt{\psi_2(\mu)\overline{\psi_2(\mu)} - \psi_1(\mu)\overline{\psi_1(\mu)}}}\begin{pmatrix} \overline{\psi_2(\mu)} & \overline{\psi_1(\mu)} \\ \psi_1(\mu) & \psi_2(\mu) \end{pmatrix}$$

for some para-complex functions $\psi_1(\mu)$ and $\psi_2(\mu)$ with $\psi_k(1) = \psi_k$ for $k = 1, 2$. Therefore it is straightforward that the maps $-i'\mu(\partial_\mu F^\mu)(F^\mu)^{-1}$ and $N_{\mathbb{L}^3_{(+,-,+)}}$ take values in $\mathfrak{su}'_{1,1}$, and then the map $f_{\mathbb{L}^3_{(+,-,+)}}$ is $\mathfrak{su}'_{1,1}$-valued.

Since the Maurer-Cartan form α^μ of a frame F^μ is defined by (4.4.6) the frame F^μ satisfies

$$\partial F^\mu = F^\mu U^\mu, \quad \overline{\partial} F^\mu = F^\mu V^\mu \tag{4.5.2}$$

where

$$U^\mu = \alpha_{\mathfrak{k}}{}' + \mu^{-1}\alpha_{\mathfrak{m}}{}', \quad V^\mu = \alpha_{\mathfrak{k}}{}'' + \mu\alpha_{\mathfrak{m}}{}''.$$

By the usual computations, we obtain

$$\begin{aligned}
\partial f_{\mathbb{L}^3} &= \partial\left(-i'\mu\left(\frac{\partial}{\partial\mu}F^\mu\right)(F^\mu)^{-1}) - \frac{1}{2}\operatorname{Ad}(F^\mu)\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right) \\
&= \operatorname{Ad}(F^\mu)\left(-i'\mu\left(\frac{\partial}{\partial\mu}U^\mu\right) - \frac{1}{2}\left[U^\mu, \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}\right]\right) \\
&= 2i'\mu^{-1}\left(-\frac{i'}{4}h\right)\operatorname{Ad}(F^\mu)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \mu^{-1}\begin{pmatrix} \psi_1(\mu)\overline{\psi_2(\mu)} & -(\overline{\psi_2(\mu)})^2 \\ (\psi_1(\mu))^2 & -\psi_1(\mu)\overline{\psi_2(\mu)} \end{pmatrix}.
\end{aligned}$$

Thus by putting $\phi^j(\mu)$ for $j = 1, 2, 3$ as

$$\phi^1(\mu) = \mu^{-1}i'\left((\overline{\psi_2(\mu)})^2 + (\psi_1(\mu))^2\right), \quad \phi^2(\mu) = \mu^{-1}\left((\overline{\psi_2(\mu)})^2 - (\psi_1(\mu))^2\right),$$

and

$$\phi^3(\mu) = \mu^{-1}2\psi_1(\mu)\overline{\psi_2(\mu)},$$

we have

$$\partial f_{\mathbb{L}^3} = \frac{1}{2}\begin{pmatrix} \phi^3(\mu) & -\phi^2(\mu) - i'\phi^1(\mu) \\ -\phi^2(\mu) + i'\phi^1(\mu) & -\phi^3(\mu) \end{pmatrix}.$$

Hence the identification (4.3.1) of $\mathfrak{su}'_{1,1}$ and $\mathbb{L}^3_{(+,-,+)}$ shows

$$\partial f_{\mathbb{L}^3_{(+,-,+)}} = \left(\phi^2(\mu), \phi^1(\mu), i'\phi^3(\mu)\right).$$

It is straightforward that the map $f_{\mathbb{L}^3_{(+,-,+)}}$ is a conformal immersion of constant mean curvature 1/2 with the first fundamental form $4\left(\psi_2(\mu)\overline{\psi_2(\mu)} - \psi_1(\mu)\overline{\psi_1(\mu)}\right)^2 dzd\bar{z}$, and the Gauss map of $f_{\mathbb{L}^3_{(+,-,+)}}$ is $N_{\mathbb{L}^3_{(+,-,+)}}$ for each $\mu \in \mathbb{S}^1_1$.

By considering the gauge transformation

$$F^\mu \mapsto \hat{F}^\mu := F^\mu \begin{pmatrix} \mu^{-1/2} & 0 \\ 0 & \mu^{1/2} \end{pmatrix},$$

it can be shown that the deformation with respect to μ does not change the Dirac potential, that is, $\psi_2(\mu)\overline{\psi_2(\mu)} - \psi_1(\mu)\overline{\psi_1(\mu)}$ is independent of μ. In fact the differential equations (4.5.2) are rewritten into

$$\partial \hat{F}^\mu = \hat{F}^\mu \hat{U}^\mu, \quad \overline{\partial} \hat{F}^\mu = \hat{F}^\mu \hat{V}^\mu \tag{4.5.3}$$

where

$$\hat{U}^\mu = \begin{pmatrix} U_{11} & U_{12} \\ \mu^{-2} U_{21} & U_{22} \end{pmatrix}, \quad \hat{V}^\mu = \begin{pmatrix} V_{11} & \mu^2 V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

for $U^\mu|_{\mu=1} = (U_{ij})$ and $V^\mu|_{\mu=1} = (V_{ij})$. Since $(1,2)$-entry of $-\hat{U}^\mu$ denotes the Dirac potential $i'(\psi_2(\mu)\overline{\psi_2(\mu)} - \psi_1(\mu)\overline{\psi_1(\mu)})/2$, it can be seen the conformal factor of $f_{\mathbb{L}^3_{(+,-,+)}}$ is h^2, and then the deformation with respect to μ preserves the metric and the mean curvature. $\square$

Remark 4.5.3. A Gauss map of a timelike surface in $\mathbb{L}^3_{(+,-,+)}$ of constant mean curvature derives an extended frame by Remark 4.4.5 and Proposition 4.5.1. Therefore an associated family of timelike surfaces in $\mathbb{L}^3_{(+,-,+)}$ of constant mean curvature can be obtained from a timelike surface of constant mean curvature 1/2. In general, timelike constant mean curvature $H \neq 0$ surfaces can be constructed in [17, 24].

To construct timelike minimal surfaces in Nil_3, we identify the Lie algebra $\mathfrak{nil}_3$ with the Lie algebra $\mathfrak{su}'_{1,1}$ as a real vector space. Let $\{E_1, E_2, E_3\}$ denote the basis of $\mathfrak{su}'_{1,1}$ given by

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & -i' \\ i' & 0 \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}.$$

Then the identification is given by a linear isomorphism $\Xi : \mathfrak{su}'_{1,1} \to \mathfrak{nil}_3$, which is not Lie algebra isomorphism, defined by

$$\Xi(x_1 E_1 + x_2 E_2 + x_3 E_3) = x_1 e_1 + x_2 e_2 + x_3 e_3. \tag{4.5.4}$$

When the derivative of a minimal surface f in Nil_3 is given by $f^{-1}\partial f = \phi^1 e_1 + \phi^2 e_2 + \phi^3 e_3$, we know that the induced surface $\tilde{f}$ has the derivative $\partial \tilde{f} = (\phi^2, \phi^1, i'\phi^3)$ by the definition of the generating spinors (2.2.4) and (4.5.1). Therefore timelike minimal surfaces in Nil_3 can be constructed by using Theorem 4.5.2.

Theorem 4.5.4. *Let F^μ be an extended frame of some harmonic map from a simply connected domain $\mathbb{D}$ into $\mathbb{H}^2$, and $f_{\mathbb{L}^3_{(+,-,+)}}$ the associated family of constant mean curvature 1/2 surfaces defined in Theorem 4.5.2. Moreover define a map $f^\mu : \mathbb{D} \to \mathrm{Nil}_3$ by*

$$f^\mu = \exp \circ \Xi \circ \hat{f}^\mu$$

where the map $\hat{f}^\mu : \mathbb{D} \to \mathfrak{su}'_{1,1}$ *is a* $\mathfrak{su}'_{1,1}$*-valued map defined by*

$$\hat{f}^\mu = \left(f_{\mathbb{L}^3_{(+,-,+)}}\right)^o - \frac{i'}{2}\mu\left(\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}\right)^d$$

Here the superscripts "o" and "d" denote the off-diagonal part and diagonal part, respectively. Then the map f^μ *describes a family of timelike minimal surfaces in* Nil_3*. Moreover the normal Gauss map of* f^μ *is* $N_{\mathbb{L}^3_{(+,-,+)}}$.

Proof. Since $f_{\mathbb{L}^3_{(+,-,+)}}$ is $\mathfrak{su}'_{1,1}$-valued, the diagonal entries of $\frac{i'}{2}\mu\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}$ take purely imaginary values and trace of $\frac{i'}{2}\mu\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}$ vanishes. Therefore $\frac{i'}{2}\mu\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}$ takes $\mathfrak{su}'_{1,1}$ values.

We have

$$\begin{aligned}\partial\left(\frac{i'}{2}\mu\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}\right) &= \frac{i'}{2}\mu\frac{\partial}{\partial\mu}\left(-\frac{1}{2}\mu^{-1}h\,\mathrm{Ad}(F^\mu)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\\ &= -\frac{i'}{2}\partial f_{\mathbb{L}^3_{(+,-,+)}} - \frac{1}{2}\left[f_{\mathbb{L}^3_{(+,-,+)}} + N_{\mathbb{L}^3_{(+,-,+)}}, \partial f_{\mathbb{L}^3_{(+,-,+)}}\right]\end{aligned} \tag{4.5.5}$$

Then the diagonal part of (4.5.5) is

$$\partial\left(\frac{i'}{2}\mu\frac{\partial}{\partial\mu} f_{\mathbb{L}^3_{(+,-,+)}}\right)^d = -\frac{i'}{2}\left(\partial f_{\mathbb{L}^3_{(+,-,+)}}\right)^d - \frac{1}{2}\left[f_{\mathbb{L}^3_{(+,-,+)}}, \partial f_{\mathbb{L}^3_{(+,-,+)}}\right]^d - \left[N_{\mathbb{L}^3_{(+,-,+)}}, \partial f_{\mathbb{L}^3_{(+,-,+)}}\right]^d.$$

By a simple computation using

$$\begin{aligned}-\frac{i'}{2}\left(\partial f_{\mathbb{L}^3_{(+,-,+)}}\right) &= -\frac{i'}{4}\begin{pmatrix}\phi^3(\mu) & -\phi^2(\mu) - i'\phi^1(\mu)\\ -\phi^2(\mu) + i'\phi^1(\mu) & -\phi^3(\mu)\end{pmatrix}^d\\ &= -\frac{1}{2}\phi^3(\mu)E_3,\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\left[f_{\mathbb{L}^3_{(+,-,+)}}, \partial f_{\mathbb{L}^3_{(+,-,+)}}\right]^d &= -\frac{1}{2}\left[\int \phi^1(\mu)dzE_1 + \int \phi^2(\mu)dzE_2, \phi^1(\mu)E_1 + \phi^2(\mu)E_2\right]\\ &= -\frac{1}{2}\left(\phi^2(\mu)\int \phi^1(\mu)dz - \phi^1(\mu)\int \phi^2(\mu)dz\right)E_3,\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\left[N_{\mathbb{L}^3_{(+,-,+)}}, \partial f_{\mathbb{L}^3_{(+,-,+)}}\right]^d &= -\frac{1}{2}\left[\frac{1}{2}\mathrm{Ad}(F^\mu)\begin{pmatrix}i' & 0\\ 0 & -i'\end{pmatrix}, -\frac{1}{2}\mu^{-1}h\,\mathrm{Ad}(F^\mu)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]^d\\ &= \frac{i'}{8}\mu^{-1}h\left(\mathrm{Ad}(F^\mu)\left[\begin{pmatrix}i' & 0\\ 0 & -i'\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]\right)^d\\ &= -\frac{i'}{2}(\partial f_{\mathbb{L}^3_{(+,-,+)}})^d\\ &= -\frac{1}{2}\phi^3(\mu)E_3,\end{aligned}$$

we obtain the derivative of $\hat{f}^\mu$

$$\begin{aligned}\partial\hat{f}^\mu &= \phi^1(\mu)E_1 + \phi^2(\mu)E_2 - \partial\left(\frac{i'}{2}\mu\frac{\partial}{\partial\mu}f_{\mathbb{L}^3_{(+,-,+)}}\right)^d\\ &= \phi^1(\mu)E_1 + \phi^2(\mu)E_2 + \left(\phi^3(\mu) + \frac{1}{2}\phi^2(\mu)\int\phi^1(\mu)dz - \frac{1}{2}\phi^1(\mu)\int\phi^2(\mu)dz\right)E_3.\end{aligned}$$

This implies that the derivative of f^μ is computed as

$$f^{\mu-1}\partial f^\mu = \phi^1(\mu)e_1 + \phi^2(\mu)e_2 + \phi^3(\mu)e_3.$$

Therefore the map f^μ describes a family of timelike minimal surfaces in Nil_3 since the generating spinors are the same as ones of $f_{\mathbb{L}^3_{(+,-,+)}}$. Moreover from Proposition 4.5.1, the normal Gauss map of f^μ is given by $N_{\mathbb{L}^3_{(+,-,+)}}$. □

Remark 4.5.5. (1) The support function of f^μ is independent of μ since the conformal factor of the induced surface $f_{\mathbb{L}^3_{(+,-,+)}}$ is independent of μ. On the other hand, the deformation with respect to μ changes the conformal factor of f^μ.

(2) For a timelike minimal surface, the normal Gauss map defines a Lorentz harmonic map into $\mathbb{S}^2_1$. Therefore a family of timelike minimal surfaces that preserves the support function can be obtained by Theorem 4.5.4.

(3) Denote the Abresch-Rosenberg differential by Qdz^2 for a timelike minimal surface $f^\mu|_{\mu=1}$ defined in Theorem 4.5.4. Then the Abresch-Rosenberg differential of f^μ is given by $\mu^{-2}Qdz^2$. This fact can be seen from (4.2.5) and (4.5.3).

4.6. **Weierstrass-type representation via loop group method.** We observed that non-vertical timelike minimal surfaces can be constructed from an extended frame of a Lorentz harmonic map into the de-Sitter sphere $\mathbb{S}^2_1$ in the previous subsection. By adapting the DPW method to the case of Lorentz harmonic maps, it can be expected that we obtain the Weierstrass-type construction of an extended frame of a Lorentz harmonic map. In this subsection, we will give the construction of an extended frame of the normal Gauss map of a timelike minimal surface in Nil_3 with a conformal coordinate system. Without using para-complex coordinate systems, J. F. Dorfmeister, J. Inoguchi, and M. Toda [17] gave the Weierstrass-type representation of an extended frame of timelike surfaces in the Minkowski 3-space of constant mean curvature. Thus we will use their method and the correspondence between para-complex coordinate systems and null coordinate systems.

Let us recall the hyperbola $\mathbb{S}^1_1$ on $\mathbb{C}'$:

$$\mathbb{S}^1_1 = \{\mu\in\mathbb{C}' | \mu\overline{\mu} = 1, \operatorname{Re}\mu > 0\}.$$

An extended frame F^μ of a timelike minimal surface is analytic on $\mathbb{C}'\setminus\{x(1\pm i')|x\in\mathbb{R}\}$ with respect to μ. Therefore, as in the Riemannian case, it is natural to introduce the following loop groups:

$$\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma = \left\{g:\mathbb{S}^1_1\to\mathrm{SL}_2\mathbb{C}'\,\middle|\, g(\mu) = \sum_{j=-\infty}^{\infty}g_j\mu^j,\ g(-\mu) = \mathrm{Ad}\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)g(\mu)\right\},$$

$$\Lambda'^{+}\mathrm{SL}_2\mathbb{C}'_\sigma = \{g\in\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma | g_j = 0 \text{ for } j<0\},$$

$$\Lambda'^{-}\mathrm{SL}_2\mathbb{C}'_\sigma = \{g\in\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma | g_j = 0 \text{ for } j>0\},$$

$$\Lambda'^{+}_{*}\mathrm{SL}_2\mathbb{C}'_\sigma = \left\{ g \in \Lambda'^{+}\mathrm{SL}_2\mathbb{C}'_\sigma \middle| g_0 = \mathrm{id} \right\},$$

$$\Lambda'^{-}_{*}\mathrm{SL}_2\mathbb{C}'_\sigma = \left\{ g \in \Lambda'^{-}\mathrm{SL}_2\mathbb{C}'_\sigma \middle| g_0 = \mathrm{id} \right\},$$

$$\Lambda'\mathrm{SU}'_{1,1\sigma} = \left\{ g \in \Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma \middle| \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left(\overline{g(1/\overline{\mu})}^{\top} \right)^{-1} = g(\mu) \right\}.$$

The following loop group decompositions, *Birkhoff* and *Iwasawa decompositions* are reformations of [17, Theorem 2.5] in terms of the conformal para-complex structure.

Theorem 4.6.1. *The loop group $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$ can be decomposed as follows*:

(1) *Birkhoff decomposition*: *The multiplication maps*

$$\Lambda'^{-}_{*}\mathrm{SL}_2\mathbb{C}'_\sigma \times \Lambda'^{+}\mathrm{SL}_2\mathbb{C}'_\sigma \to \Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma,$$
$$\Lambda'^{+}_{*}\mathrm{SL}_2\mathbb{C}'_\sigma \times \Lambda'^{-}\mathrm{SL}_2\mathbb{C}'_\sigma \to \Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$$

are diffeomorphisms onto the open dense subsets of $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$. These open dense subsets are called the big cells *of $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$.*

(2) *Iwasawa decomposition*: *The multiplication map*

$$\Lambda'\mathrm{SU}'_{1,1\sigma} \times \Lambda'^{+}\mathrm{SL}_2\mathbb{C}'_\sigma \to \Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$$

is a diffeomorphism onto the open dense subset of $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$. This open dense subset also is called the big cell *of $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$.*

Proof. We consider the loop algebras of two real Lie algebras $\mathfrak{sl}_2\mathbb{R}$ and $\mathfrak{su}'_{1,1}$ Let us define several loop algebras as follows:

$$\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma = \left\{ \xi : \mathbb{R}^+ \to \mathfrak{sl}_2\mathbb{R} \middle| \xi(\lambda) = \sum_{j=-\infty}^{\infty} \xi_j \lambda^j, \ \xi(-\lambda) = \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \xi(\lambda) \right\},$$

$$\Lambda^{\pm}\mathfrak{sl}_2\mathbb{R}_\sigma = \left\{ \xi \in \Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \middle| \xi(\lambda) = \sum_{j=0}^{\infty} \xi_{\pm j} \lambda^{\pm j} \right\},$$

$$\Lambda^{\pm}_{*}\mathfrak{sl}_2\mathbb{R}_\sigma = \left\{ \xi \in \Lambda^{\pm}\mathfrak{sl}_2\mathbb{R}_\sigma \middle| \xi_0 = 0 \right\},$$

$$\Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma = \left\{ \tau : \mathbb{S}^1_1 \to \mathfrak{sl}_2\mathbb{C}' \middle| \tau(\mu) = \sum_{j=-\infty}^{\infty} \tau_j \mu^j, \ \tau(-\mu) = \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tau(\mu) \right\},$$

$$\Lambda'^{\pm}\mathfrak{sl}_2\mathbb{C}'_\sigma = \left\{ \tau \in \Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma \middle| \tau(\mu) = \sum_{j=0}^{\infty} \tau_j \mu^j \right\},$$

$$\Lambda'\mathfrak{su}'_{1,1\sigma} = \left\{ \tau \in \Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma | \tau^*(1/\overline{\mu}) = \tau(\mu) \right\}.$$

Since the explicit map

$$\mathfrak{sl}_2\mathbb{R} \ni X \mapsto \ell X + \bar{\ell} X^* \in \mathfrak{su}'_{1,1}, \quad X^* = -\mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \overline{X}^{\top} \tag{4.6.1}$$

defines an isomorphism of real Lie algebras, an isomorphism between $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ and $\Lambda'\mathfrak{su}'_{1,1\sigma}$ is induced as follows: Let $\xi = \sum_{j=-\infty}^{\infty} \xi_j \lambda^j$ be an element of $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ and consider the isomorphism (4.6.1)

$$
\begin{aligned}
\xi \mapsto \xi\ell + \xi^*\bar{\ell} &= \sum_{j=-\infty}^{\infty} (\xi_j \ell + \xi_j^* \bar{\ell})\lambda^j \\
&= \sum_{j=-\infty}^{\infty} (\xi_j \ell + \xi_{-j}^* \bar{\ell})\mu^j \\
&= \xi(\mu)\ell + \xi^*(1/\overline{\mu})\bar{\ell}
\end{aligned}
\tag{4.6.2}
$$

where $\mu \in \mathbb{S}^1_1$ is given by $\lambda = \mu\ell + \mu^{-1}\bar{\ell}$. Since a direct computation shows $\tau(\mu) := \xi\ell + \xi^*\bar{\ell}$ satisfies

$$
(\tau(\mu))^* = \sum_{=-\infty}^{\infty} ({\xi_j}^* \bar{\ell} + \xi_{-j}\ell)\overline{\mu^j}
$$

we have $\tau^*(1/\overline{\mu}) = \tau(\mu)$, and then τ is an element of $\Lambda'\mathfrak{su}'_{1,1\sigma}$. Thus we obtain an isomorphism (4.6.2) between $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ and $\Lambda'\mathfrak{su}'_{1,1\sigma}$.

In general, the para complexification $\mathfrak{g} \otimes \mathbb{C}'$ of a given real Lie algebra $\mathfrak{g}$ gives isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ as a real Lie algebra. In fact, the map

$$
\mathfrak{g} \oplus \mathfrak{g} \ni (X, Y) \mapsto \ell X + \bar{\ell} Y \in \mathfrak{g} \otimes \mathbb{C}' \tag{4.6.3}
$$

is an isomorphism of real Lie algebras. Therefore combining the isomorphism (4.6.2) with (4.6.3) derives an isomorphisms between $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \oplus \Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ and $\Lambda'\mathfrak{su}'_{1,1\sigma} \oplus \Lambda'\mathfrak{su}'_{1,1\sigma} \simeq \Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma$:

$$
\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \oplus \Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \ni (\xi(\lambda), \eta(\lambda)) \mapsto (\xi(\mu)\ell + \xi^*(1/\overline{\mu})\bar{\ell}, \eta(\mu)\ell + \eta^*(1/\overline{\mu})\bar{\ell}) \in \Lambda'\mathfrak{su}'_{1,1\sigma} \oplus \Lambda'\mathfrak{su}'_{1,1\sigma},
$$

$$
\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \oplus \Lambda\mathfrak{sl}_2\mathbb{R}_\sigma \ni (\xi(\lambda), \eta(\lambda)) \mapsto \xi(\mu)\ell + \eta^*(1/\overline{\mu})\bar{\ell} \in \Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma. \tag{4.6.4}
$$

The isomorphisms discussed above induce the isomorphisms of Lie groups $\mathrm{SL}_2\mathbb{R} \simeq \mathrm{SU}'_{1,1}$, $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma \simeq \Lambda'\mathrm{SU}'_{1,1\sigma}$, and $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{R}_\sigma \simeq \Lambda'\mathrm{SU}'_{1,1\sigma} \times \Lambda'\mathrm{SU}'_{1,1\sigma} \simeq \Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$. Moreover the isomorphism (4.6.4) induces the following isomorphisms:

$$
\Lambda^+_*\mathfrak{sl}_2\mathbb{R}_\sigma \oplus \Lambda^-\mathfrak{sl}_2\mathbb{R}_\sigma \simeq \Lambda'^+\mathfrak{sl}_2\mathbb{C}'_\sigma.
$$

Thus we have an isomorphism between loop groups $\Lambda^+_*\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda^-\mathrm{SL}_2\mathbb{R}_\sigma$ and $\Lambda'^+\mathrm{SL}_2\mathbb{C}'_\sigma$.

The Birkhoff and Iwasawa decompositions of $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma$ and $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{R}_\sigma$ were given in Theorem 2.2 and Theorem 2.5 in [17]. Therefore we know that the multiplication maps

$$
\begin{gathered}
\Lambda^-_*\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda^+\mathrm{SL}_2\mathbb{R}_\sigma \to \Lambda\mathrm{SL}_2\mathbb{R}_\sigma, \\
\Lambda^+_*\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda^-\mathrm{SL}_2\mathbb{R}_\sigma \to \Lambda\mathrm{SL}_2\mathbb{R}_\sigma, \\
\Delta(\Lambda\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{R}_\sigma) \times (\Lambda^+_*\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda^-\mathrm{SL}_2\mathbb{R}_\sigma) \to \Lambda\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{R}_\sigma
\end{gathered}
\tag{4.6.5}
$$

are diffeomorphisms onto the open dense subsets of $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma$ and $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{R}_\sigma$, respectively. Hence by combining the isomorphisms of loop groups with the above decompositions, we complete the proof. □

Remark 4.6.2. Since the loop algebra $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ is a Banach Lie algebra [17], we know that the loop algebra $\Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma$ also is a Banach Lie algebra. Moreover, the Lie groups $\Lambda\mathrm{SL}_2\mathbb{R}_\sigma$ and $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$ corresponding to $\Lambda\mathfrak{sl}_2\mathbb{R}_\sigma$ and $\Lambda'\mathfrak{sl}_2\mathbb{C}'_\sigma$ become Banach Lie groups.

From now on we derive para-holomorphic data from an extended frame of normal Gauss maps of timelike minimal surfaces in Nil_3. Let us assume that extended frames take values in the big cell of $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$.

Theorem 4.6.3. *Let F^μ be an extended frame of the normal Gauss map of a timelike minimal surface in Nil_3, and apply the Birkhoff decomposition in Theorem 4.6.1 to F^μ as $F^\mu = F^\mu_- F^\mu_+$ with $F^\mu_- \in \Lambda'^-_*\mathrm{SL}_2\mathbb{C}'_\sigma$ and $F^\mu_+ \in \Lambda'^+\mathrm{SL}_2\mathbb{C}'_\sigma$. Then the Maurer-Cartan form ξ of F^μ_-, that is, $\xi = (F^\mu_-)^{-1}dF^\mu_-$, is para-holomorphic with respect to z. Moreover ξ is represented explicitly as follows*:

$$\xi = \mu^{-1}\begin{pmatrix} 0 & -p \\ Qp^{-1} & 0 \end{pmatrix} dz \tag{4.6.6}$$

where the para-holomorphic function $p(z)$ is defined by using the support function $h(z, \bar z)$ as

$$p(z) = \frac{i' h^2(z,0)}{4h(0,0)}.$$

Proof. The Maurer-Cartan form ξ can be computed as

$$\xi = (F^\mu_-)^{-1}dF^\mu_- = F^\mu_+\alpha^\mu(F^\mu_+)^{-1} - dF^\mu_+(F^\mu_+)^{-1} \tag{4.6.7}$$

where α^μ is the 1-form given in (4.4.1). Since F^μ_- belongs to $\Lambda'^-_*\mathrm{SL}_2\mathbb{C}'_\sigma$ the middle term does not have μ^j for $j \geq 0$. On the other hand, the right hand side does not have μ^j for $j < -1$. Thus the 1-form ξ has only μ^{-1} term. A straightforward computation of the μ^{-1} term of the right hand side shows

$$\xi = \mu^{-1}F_{+0}\begin{pmatrix} 0 & -\frac{i'}{4}h \\ \frac{4i'Q}{h} & 0 \end{pmatrix}(F_{+0})^{-1}dz$$

where the matrix F_{+0} is the constant term of the expansion of F^μ_+ with respect to μ, that is, $F^\mu_+ = F_{+0} + F_{+1}\mu + F_{+2}\mu^2 + \cdots$. Therefore it can be seen $\overline{\partial}F^\mu_- = 0$, and then ξ is para-holomorphic. Hence we obtain

$$\xi = \mu^{-1}F_{+0}\begin{pmatrix} 0 & -\frac{i'}{4}h \\ \frac{4i'Q}{h} & 0 \end{pmatrix}(F_{+0})^{-1}\Big|_{\bar z = 0} dz. \tag{4.6.8}$$

Since F^μ_+ belongs to $\Lambda'\mathrm{SL}_2\mathbb{C}'_\sigma$, we know that F_{+0} is of the form:

$$F_{+0}(z,\bar z) = \begin{pmatrix} a(z,\bar z) & 0 \\ 0 & a^{-1}(z,\bar z) \end{pmatrix}$$

by taking the limit $\mu \to 0$. Hence (4.6.8) is represented as

$$\xi = \mu^{-1}\begin{pmatrix} 0 & -\frac{i'}{4}h(z,0)a^2(z,0) \\ \frac{4i'Q(z)}{h(z,0)}a^{-2}(z,0) & 0 \end{pmatrix} dz.$$

Since F^μ takes values in $\Lambda' \mathrm{SU}'_{1,1\sigma}$, the restriction of the Maurer-Cartan form $\alpha^\mu|_{\bar z=0}$ has only holomorphic part, and then $F^\mu|_{\bar z=0}$ takes values in $\Lambda'^{-}\mathrm{SL}_2\mathbb{C}'_\sigma$. Therefore $F_+|_{\bar z=0} = (F_-)^{-1}F^\mu|_{\bar z=0}$ has only the μ^0-term. By looking at the μ^0-term of (4.6.7) we obtain

$$0 = \left(F_{+0}\alpha'_{\mathfrak{k}}(F_{+0})^{-1} - \partial F_{+0}(F_{+0})^{-1}\right)\Big|_{\bar z=0} dz.$$

Hence the equation $dF_{+0} = F_{+0}\alpha'_{\mathfrak{k}}$ derives $a(z,0) = h^{1/2}(z,0)c$ for some constant c. It is obvious that $c = h^{-1/2}(0,0)$ because $F_{+0}(z=0) = \mathrm{id}$. Hence we complete the proof. □

Definition 4.6.4. We will call the para-holomorphic 1-form ξ in Theorem 4.6.3 the *normalized potential* of timelike minimal surfaces after in the Riemannian case.

Conversely, we can construct an extended frame of timelike minimal surfaces by using the Iwasawa decomposition as the decomposed factor of a primitive loop for a normalized potential.

Theorem 4.6.5. *Let ξ be a normalized potential of a timelike minimal surface f in* Nil_3 *defined in* (4.6.6)*, and F_- be the solution of*

$$\partial F_- = F_-\xi$$

with the initial condition $F_-(z=0) = \mathrm{id}$. Moreover let $F^\mu \in \Lambda'\mathrm{SU}'_{1,1\sigma}$ and $F_+ \in \Lambda'^{+}\mathrm{SL}_2\mathbb{C}'_\sigma$ be the decomposed factors of F_- with respect to the Iwasawa decomposition, that is $F_- = F^\mu F_+$. Then $F^\mu k$ forms an extended frame of f up to the change of coordinates for some $k \in \mathrm{U}'_1$.

Proof. A straightforward computation of the Maurer-Cartan form α^μ of F^μ gives

$$\alpha^\mu = (F_+)^{-1}\xi F_+ - (F_+)^{-1}dF_+.$$

We can see the right hand side of the above equation does not have μ^j for $j < -1$. On the other hand, since the left hand side belongs to the Lie algebra of $\Lambda'\mathrm{SU}'_{1,1\sigma}$, thus α^μ is of the form:

$$\alpha^\mu = \mu^{-1}\alpha_{-1} + \alpha_0 + \mu^1\alpha_1$$

where ${\alpha_j}^* = \alpha_{-j}$. Therefore we can see the Maurer-Cartan form α^μ almost has the form in (4.4.1). In fact, denoting $F_{+0} = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$ and $F_{+1} = \begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}$, straightforward computations show

$$\alpha_{-1} = \begin{pmatrix} 0 & -a^{-2}p \\ a^2Qp^{-1} & 0\end{pmatrix} dz,$$

$$\alpha_0 = \begin{pmatrix} \partial \log a dz + \overline{\partial} \log a d\bar z & 0 \\ 0 & -\partial \log a dz - \overline{\partial}\log a d\bar z\end{pmatrix}.$$

Consequently, a proper choice of $k \in \mathrm{U}'_1$ and the change of coordinates induce the form same to (4.4.1). This completes the proof. □

4.7. **Examples of timelike minimal surfaces.** In [26], S.-P. Kobayashi and the author gave some basic examples of Weierstrass-type representation of timelike minimal surfaces. In this subsection, we introduce them with their para-holomorphic data, that is, normalized potentials. In particular, the *B-scroll type minimal surfaces* given in this subsection will be looked into in the next section.

Example 4.7.1 (Horizontal umbrella). The timelike minimal surfaces given by the following normalized potential have the Abresch-Rosenberg differential which vanishes anywhere.

$$\xi = \mu^{-1}\begin{pmatrix} 0 & -i' \\ 0 & 0 \end{pmatrix} dz.$$

The solution of the equation $dF_- = F_-\xi$ with the initial condition $F_-(z=0) = \mathrm{id}$ is given by

$$F_- = \begin{pmatrix} 1 & -i'\mu^{-1}z \\ 0 & 1 \end{pmatrix}.$$

The $\Lambda'\mathrm{SU}'_{1,1\sigma}$-valued map F^μ derived from the Iwasawa decomposition of F_- is computed as

$$F^\mu = \frac{1}{(1+z\bar{z})^{1/2}}\begin{pmatrix} 1 & -i'\mu^{-1}z \\ i'\mu\bar{z} & 1 \end{pmatrix}.$$

The map F^μ is defined on a simply connected domain $\tilde{\mathbb{D}} = \{z \in \mathbb{C}' | z\bar{z} > -1\}$. By the Sym formulas in Theorem 4.5.2 and Theorem 4.5.4 we obtain a timelike surface $f_{\mathbb{L}^3_{(+,-,+)}}$ in $\mathbb{L}^3_{(+,-,+)}$ of constant mean curvature $1/2$ and a timelike minimal surface f^μ in Nil_3

$$f_{\mathbb{L}^3_{(+,-,+)}} = \left(\frac{2(\mu^{-1}z + \mu\bar{z})}{1+z\bar{z}}, \frac{2i'(\mu^{-1}z - \mu\bar{z})}{1+z\bar{z}}, \frac{3z\bar{z}-1}{1+z\bar{z}}\right),$$

$$f^\mu = \left(\frac{2i'(\mu^{-1}z - \mu\bar{z})}{1+z\bar{z}}, \frac{2(\mu^{-1}z + \mu\bar{z})}{1+z\bar{z}}, 0\right).$$

A straightforward computation shows the first fundamental form I^μ of f^μ is

$$I^\mu = 16\frac{(1-z\bar{z})^2}{(1+z\bar{z})^4} dz d\bar{z}.$$

In general, horizontal umbrellas can be represented as graphs of functions in the form of $F(x_1, x_2) = ax_1 + bx_2 + c$ for $a, b, c \in \mathbb{R}$, and have the non-constant Gaussian curvature K:

$$K = \frac{2(-(a+\frac{1}{2}x_2)^2 + (b-\frac{1}{2}x_1)^2 + 1) + 1}{4(-(a+\frac{1}{2}x_2)^2 + (b-\frac{1}{2}x_1)^2 + 1)^2}.$$

Example 4.7.2 (Hyperbolic paraboloids corresponding to circular cylinders). We take a normalized potential of the form

$$\xi = \mu^{-1}\begin{pmatrix} 0 & -\frac{i'}{4} \\ \frac{i'}{4} & 0 \end{pmatrix} dz.$$

The solution of the equation $dF_- = F_-\xi$ with the initial condition $F_-(z=0) = \mathrm{id}$ is given by

$$F_- = \begin{pmatrix} \cos\frac{\mu^{-1}z}{4} & -i'\sin\frac{\mu^{-1}z}{4} \\ i'\sin\frac{\mu^{-1}}{4} & \cos\frac{\mu^{-1}z}{4} \end{pmatrix}.$$

Then we obtain an extended frame F^μ by applying the Iwasawa decomposition to F_-:

$$F^\mu = \begin{pmatrix} \cos \frac{\mu^{-1}z+\mu\bar z}{4} & -i' \sin \frac{\mu^{-1}z+\mu\bar z}{4} \\ i' \sin \frac{\mu^{-1}z+\mu\bar z}{4} & \cos \frac{\mu^{-1}z+\mu\bar z}{4} \end{pmatrix}.$$

The corresponding surfaces $f_{\mathbb{L}^3_{(+,-,+)}}$ and f^μ in $\mathbb{L}^3_{(+,-,+)}$ and Nil_3, respectively, are computed as

$$f_{\mathbb{L}^3} = \left(\sin \frac{\mu^{-1}z+\mu\bar z}{2}, i'\frac{\mu^{-1}z-\mu\bar z}{2}, -\cos \frac{\mu^{-1}z+\mu\bar z}{2} \right),$$

$$f^\mu = \left(i'\frac{\mu^{-1}z-\mu\bar z}{2}, \sin \frac{\mu^{-1}z+\mu\bar z}{2}, i'\frac{\mu^{-1}z-\mu\bar z}{4} \sin \frac{\mu^{-1}z+\mu\bar z}{2} \right).$$

The surface $f_{\mathbb{L}^3_{(+,-,+)}}$ is known as a circular cylinder. Moreover the surface f^μ describes a part of hyperbolic paraboloid $x_3 = x_1x_2/2$ for each μ. The first fundamental form I^μ of f^μ is

$$I^\mu = \cos^2 \left(\frac{\mu^{-1}z+\mu\bar z}{2} \right) dz d\bar z.$$

Thus f^μ is not defined entirely. The support function h and the Abresch-Rosenberg differential Qdz^2 are given by $h = 1$ and $Q = \mu^{-2}/16$.

Example 4.7.3 (Hyperbolic paraboloids corresponding to hyperbolic cylinders). In [17], we can see the hyperbolic cylinders in $\mathbb{L}^3_{(+,-,+)}$ are obtained from the following normalized potential ξ:

$$\xi = \mu^{-1} \begin{pmatrix} 0 & -\frac{i'}{4} \\ -\frac{i'}{4} & 0 \end{pmatrix} dz.$$

The timelike minimal surface constructed from the same potential describes a hyperbolic paraboloid $x_3 = -x_1x_2/2$ as follows:

The solution F_- of the equation $dF_- = F_-\xi$ with the initial condition $F_-(z=0) = \mathrm{id}$ is given by

$$F_- = \begin{pmatrix} \cosh \frac{\mu^{-1}z}{4} & -i' \sinh \frac{\mu^{-1}z}{4} \\ -i' \sinh \frac{\mu^{-1}z}{4} & \cosh \frac{\mu^{-1}z}{4} \end{pmatrix}.$$

Then an extended frame F^μ is obtained as the factor of Iwasawa decomposition of F_-:

$$F^\mu = \begin{pmatrix} \cosh \frac{-\mu^{-1}z+\mu\bar z}{4} & i' \sinh \frac{-\mu^{-1}z+\mu\bar z}{4} \\ i' \sinh \frac{-\mu^{-1}z+\mu\bar z}{4} & \cosh \frac{-\mu^{-1}z+\mu\bar z}{4} \end{pmatrix}.$$

Therefore the surfaces $f_{\mathbb{L}^3_{(+,-,+)}}$ and f^μ are computed as

$$f_{\mathbb{L}^3} = \left(\frac{\mu^{-1}z+\mu\bar z}{2}, -\sinh i'\frac{-\mu^{-1}z+\mu\bar z}{2}, -\cosh \frac{-\mu^{-1}z+\mu\bar z}{2} \right),$$

$$f^\mu = \left(-\sinh i'\frac{-\mu^{-1}z+\mu\bar z}{2}, \frac{\mu^{-1}z+\mu\bar z}{2}, \frac{\mu^{-1}z+\mu\bar z}{4} \sinh i'\frac{-\mu^{-1}z+\mu\bar z}{2} \right).$$

Since the first fundamental form I^μ of f^μ is $I^\mu = \left(\cosh \frac{i'}{2}(-\mu^{-1}z + \mu\bar{z})\right)^2 dzd\bar{z}$, each surface f^μ can be defined entirely on $\mathbb{C}'$ in contrast to Example 4.7.2. The support function h and the Abresch-Rosenberg differential Qdz^2 are $h = 1$ and $Qdz^2 = -\mu^{-2}/16dz^2$.

Example 4.7.4 (B-scroll type minimal surfaces). We will construct *B-scrolls* in $\mathbb{L}^3_{(+,-,+)}$ which is first introduced by L. Graves [22] and the corresponding surfaces in Nil_3, which we will call *B-scroll type minimal surfaces.*

Let γ be a null curve in $\mathbb{L}^3_{(+,-,+)}$ defined on an open interval. Then by proper reparametrization, there exist a frame (A, B, C) along γ and functions k_j on the interval for $j = 1, 2$ such that

$$\begin{gathered} A = \frac{d\gamma}{ds}, \quad \langle A, B\rangle = \langle C, C\rangle = 1, \\ \langle A, A\rangle = \langle B, B\rangle = \langle A, C\rangle = \langle B, C\rangle = 0, \\ \frac{d}{ds}(A, B, C) = (A, B, C)\begin{pmatrix} 0 & 0 & -k_2 \\ 0 & 0 & -k_1 \\ k_1 & k_2 & 0 \end{pmatrix}. \end{gathered}$$

Such a frame is called *null Frenet frame* of γ. We would like to note that a null Frenet frame is not unique for a null curve in $\mathbb{L}^3_{(+,-,+)}$. The surface $f(s, t)$ defined from a null curve γ with a null Frenet frame (A, B, C) by

$$f(s, t) = \gamma(s) + tB(s)$$

is called a B-scroll. The mean curvature of a B-scroll is given by k_2.

Definition 4.7.5. A non-vertical timelike minimal surface is said to be *B-scroll type* if it induces a B-scroll.

The following normalized potential ξ gives a B-scroll of constant mean curvature 1/2:

$$\xi = \mu^{-1}\begin{pmatrix} 0 & -\frac{i'}{4} \\ -S(z)\bar{\ell} & 0 \end{pmatrix} dz,$$

where $S(z)$ is a para-holomorphic function. The equation

$$dF_- = F_-\xi, \quad F_-(z = 0) = \mathrm{id} \tag{4.7.1}$$

can not be solved explicitly. But we can understand partially as follows: Because of the para-holomorphicity, the function $S(z)$ can be decomposed as

$$S(z) = Q(s)\ell + R(t)\bar{\ell}$$

with $s = x + y$ and $t = x - y$ for the para-complex coordinate system $z = x + i'y$ where Q and R are defined by $Q = \mathrm{Re}\, S + \mathrm{Im}\, S$ and $R = \mathrm{Re}\, S - \mathrm{Im}\, S$. Moreover the normalized potential ξ can be splitted as

$$\xi = \xi^s\ell + \xi^{t*}\bar{\ell}$$

where

$$\xi^s = \lambda^{-1}\begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{pmatrix} ds, \quad \xi^t = \lambda \begin{pmatrix} 0 & -R(t) \\ \frac{1}{4} & 0 \end{pmatrix} dt,$$

$$\xi^{t^*} = -\operatorname{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left(\overline{\xi^t\left(\frac{1}{\bar{\mu}}\right)}\right)^{\top}.$$

Therefore the solution F_- of (4.7.1) is obtained by $F_- = F^s\ell + F^{t^*}\bar{\ell}$ where $F^s, F^t \in \Lambda\mathrm{SL}_2\mathbb{R}_\sigma$ are the solutions of the equations

$$dF^s = F^s\xi^s, \quad dF^t = F^t\xi^t \quad F^s(0) = F^t(0) = \mathrm{id}.$$

Here $F^{t^*} = \operatorname{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\overline{F^t(1/\bar{\mu})}^{\top -1}$. Although the solution F^s is explicitly integrated as

$$F^s = \begin{pmatrix} 1 & -\frac{1}{4}\lambda^{-1}s \\ 0 & 1 \end{pmatrix},$$

we can not compute F^t explicitly. By the Iwasawa decomposition $(F^s, F^t) = (\hat{F}, \hat{F})(\hat{V}_+, \hat{V}_-)$ in the last equation of (4.6.5), we can compute

$$F_- = F^s\ell + F^{t^*}\bar{\ell} = (\hat{F}\ell + \hat{F}^*\bar{\ell})(\hat{V}_+\ell + \hat{V}_-^*\bar{\ell}).$$

Then by setting

$$F^t = \mathrm{id}\sum_{k\geq 1}\lambda^k\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

where $a_{2k+1} = d_{2k+1} = b_{2k} = c_{2k} = 0$ for all $k \geq 1$, we can compute $\hat{F}$.

Proposition 4.7.6. *The map $\hat{F}$ can be computed as follows*:

$$\hat{F} = F^tV, \quad V = \begin{pmatrix} \left(1 + \frac{1}{4}sc_1\right)^{-1} & -\frac{1}{4}\lambda^{-1}s \\ 0 & 1 + \frac{1}{4}sc_1 \end{pmatrix}.$$

Proof. Since $\hat{F} = F^s(\hat{V}_+)^{-1} = F^t(\hat{V}_-)^{-1}$ holds, we have

$$(F^s)^{-1}F^t = (\hat{V}_+)^{-1}\hat{V}_-.$$

This is exactly the Birkhoff decomposition of the left hand side. By multiplying V on $(F^s)^{-1}F^t$ by right, we have

$$\begin{aligned}(F^s)^{-1}F^tV &= \begin{pmatrix} 1 & \frac{1}{4}\lambda^{-1}s \\ 0 & 1 \end{pmatrix}\left(\mathrm{id} + \sum_{k\geq 1}\lambda^k\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}\right)V \\ &= \left\{\begin{pmatrix} 1 + \frac{1}{4}sc_1 & \frac{1}{4}\lambda^{-1}s \\ 0 & 1 \end{pmatrix} + O(\lambda)\right\}\begin{pmatrix} \frac{1}{1+\frac{1}{4}sc_1} & -\frac{1}{4}\lambda^{-1}s \\ 0 & 1 + \frac{1}{4}sc_1 \end{pmatrix} \\ &= \mathrm{id} + O(\lambda).\end{aligned}$$

Therefore $(F^s)^{-1}F^tV$ takes values in $\Lambda_*^+\mathrm{SL}_2\mathbb{R}_\sigma$. Hence the Birkhoff decomposition of $(F^s)^{-1}F^t$ is given by $(F^s)^{-1}F^t = (\mathrm{id} + O(\lambda))V^{-1}$. The uniqueness of the Birkhoff decomposition implies $\hat{V}_- = V^{-1}$. This completes the proof. □

Since the uniqueness of the Iwasawa decomposition implies $F^\mu = \hat{F}\ell + \hat{F}^*\bar{\ell}$, we can obtain the representation of B-scrolls by the Sym-Bobenko formula:

$$f_{\mathbb{L}^3_{(+,-,+)}} = \{\gamma(t) + q(s,t)B(t)\}\,\ell + \{\gamma(t) + q(s,t)B(t)\}^*\,\bar{\ell}$$

where

$$\gamma(t) = -i'\mu(\partial_\mu F^t)(F^t)^{-1} - \frac{i'}{2}\operatorname{Ad}(F^t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$B(t) = -i'\mu\operatorname{Ad}(F^t)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

$$q(s,t) = \frac{s}{2(1+\frac{1}{16}st)}.$$

Furthermore the timelike minimal surface f^μ corresponding to $f_{\mathbb{L}^3_{(+,-,+)}}$ is computed as

$$f^\mu = \exp(\hat{f}^\mu), \quad \hat{f}^\mu = \left\{\hat{\gamma}(t) + q(s,t)\hat{B}(t)\right\}\ell + \left\{\hat{\gamma}(t) + q(s,t)\hat{B}(t)\right\}^*\bar{\ell},$$

where

$$\hat{\gamma}(t) = \gamma(t)^o - \frac{i'}{2}\mu\partial_\mu\gamma(t)^d, \quad \hat{B}(t) = B(t)^o - \frac{i'}{2}\mu\partial_\mu B(t)^d.$$

A direct computation shows that $\exp(\hat{\gamma}(t)\ell + \hat{\gamma}(t)^*\bar{\ell})$ is a null curve in Nil_3 and $\hat{B}(t)\ell + \hat{B}(t)^*\bar{\ell}$ defines a null vector at each point of the curve $\hat{\gamma}$.

By starting from B-scrolls and using the relation between timelike surfaces in $\mathbb{L}^3_{(+,-,+)}$ of constant mean curvature $1/2$ and timelike minimal surfaces in Nil_3 mentioned in subsection 4.5, we can construct B-scroll type minimal surfaces explicitly as follows:

Let γ be a null curve in $\mathbb{L}^3_{(+,-,+)}$ with $\gamma(0) = (0,0,0)$ and (A,B,C) be a null Frenet frame along γ with $k_2 = 1/2$. We consider the B-scroll $\Phi(s,t) = \gamma(s) + tB(s)$. A conformal coordinate system $z = \ell x + \bar{\ell}y$ can be given by

$$s = x, \quad t = \frac{1}{x/8 + 1/y}.$$

Then the derivative $\partial\Phi = (\phi_2, \phi_1, i'\phi_3)$ is computed as

$$\phi_1 = \ell\left(A^1(s) + tB^{1\prime}(s) - \frac{t^2}{8}B^1(s)\right) + \bar{\ell}\frac{t^2}{y^2}B^1,$$

$$\phi_2 = \ell\left(A^2(s) + tB^{2\prime}(s) - \frac{t^2}{8}B^2(s)\right) + \bar{\ell}\frac{t^2}{y^2}B^2,$$

$$\phi_3 = \ell\left(A^3(s) + tB^{3\prime}(s) - \frac{t^2}{8}B^3(s)\right) - \bar{\ell}\frac{t^2}{y^2}B^3.$$

Here $A = (A^2, A^1, A^3)$ and $B = (B^2, B^1, B^3)$.

Lemma 4.7.7.

$$2\operatorname{Re}\left((\phi_1, \phi_2, \phi_3 - \frac{\phi_1}{2}\int_0^z \operatorname{Re}(\phi_2 dz) + \frac{\phi_2}{2}\int_0^z \operatorname{Re}(\phi_1 dz))dz\right) \tag{4.7.2}$$

is a closed form.

Proof. The exterior derivative of (4.7.2) can be computed as

$$(0, 0, 2\partial\overline{\phi_3} + \phi_1\overline{\phi_2} - \phi_2\overline{\phi_1})dz \wedge d\bar{z}.$$

Then the above 2-form vanishes by using the structure equations of Φ. □

Lemma 4.7.7 and the Stokes' theorem imply that the map

$$f(z, \bar{z}) = (f_1, f_2, f_3) = \int_0^z \mathrm{Re}\left((\phi_1, \phi_2, \phi_3 - \frac{\phi_1}{2}\int_0^z \mathrm{Re}(\phi_2 dz) + \frac{\phi_2}{2}\int_0^z \mathrm{Re}(\phi_1 dz))dz\right)$$

is well-defined. f_1 and f_2 are obviously given by

$$f_1 = \int_0^z \mathrm{Re}(\phi_1 dz) = \gamma_1(s) + tB^1(s), \quad f_2 = \int_0^z \mathrm{Re}(\phi_2 dz) = \gamma_2(s) + tB^2(s).$$

Let us represent $f_3 = \int_0^z \mathrm{Re}\left((\phi_3 - \frac{\phi_1}{2}f_2 + \frac{\phi_2}{2}f_1)dz\right)$ in terms of the coordinate system (s, t). By Lemma 4.7.7, the following paths Γ_s and Γ_t can be chosen as the integral path from 0 to $z = \ell s + \bar{\ell}1/(-s/8 + 1/t)$:

$$\Gamma_s : z(\tilde{s}) = \ell\tilde{s}, \qquad 0 \le \tilde{s} \le s,$$

$$\Gamma_t : z(\tilde{t}) = \ell s + \bar{\ell}\frac{1}{-s/8 + 1/\tilde{t}}, \quad 0 \le \tilde{t} \le t.$$

Therefore f_3 can be represented as

$$f_3(z, \bar{z}) = \gamma_3(s) + \int_0^s \left(-\frac{\gamma_2}{2}A^1 + \frac{\gamma_1}{2}A^2\right) ds + t\left(-B^3(s) - \frac{\gamma_2(s)}{2}B^1(s) + \frac{\gamma_1(s)}{2}B^2(s)\right).$$

Defining maps $\widetilde{\gamma}$ and f by

$$\widetilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3 + \int_0^s \left(-\frac{\gamma_2}{2}A^1 + \frac{\gamma_1}{2}A^2\right) ds),$$

$$f = (f_1, f_2, f_3)$$

as a curve and a map into Nil_3, respectively, we can obtain a timelike minimal surface

$$f(z, \bar{z}) = \widetilde{\gamma}(s) \cdot \exp(t(B^1 e_1 + B^2 e_2 - B^3 e_3))$$

which induces the B-scroll Φ.

5. Null scrolls in Lorentzian Heisenberg group

In the previous section, we constructed timelike minimal surfaces in Lorentzian Heisenberg group Nil_3. In this section, we focus on the surfaces that have the Abresch-Rosenberg differential Qdz^2 satisfying $Q\overline{Q} = 0$. When the Abresch-Rosenberg differential satisfies $Q\overline{Q} = 0$, the first equation of the conditions (4.2.3) becomes a hyperbolic-type Liouville equation. The exact solutions of Liouville equations are well-studied in [6, 12, 14, 35, 44]. In the previous section, we called these surfaces B-scroll type minimal surfaces.

5.1. **Non-uniqueness of timelike minimal surfaces for** (w, Q)**.** Timelike surfaces are obtained from the support function and the Abresch-Rosenberg differential, which derive a solution (w, Q) for the compatibility condition (4.2.3) of minimal surfaces and the Weierstrass data. It is remarkable that for a solution (w, Q) of (4.2.3), the uniqueness of the timelike minimal surfaces may not hold.

Theorem 5.1.1. *Let f_1 and f_2 be timelike minimal surfaces, and h_k and $Q_k dz^2$ the support functions and the Abresch-Rosenberg differentials of f_k for $k = 1, 2$. If $h_1 = \pm h_2$ and $Q_1 = Q_2$ hold, then the derivatives ${f_1}^{-1}\partial f_1 = \sum \phi_1^j e_j$ and ${f_2}^{-2}\partial f_2 = \sum \phi_2^j e_j$ are related in*

$$(\phi_2^2, \phi_2^1, i'\phi_2^3) = (\phi_1^2, \phi_1^1, i'\phi_1^3)F_0 \tag{5.1.1}$$

for some matrix element F_0 of the following matrix group:

$$\left\{ X \in M_3\mathbb{R} \;\middle|\; X^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det X = 1 \right\}.$$

Conversely, if f_1 and f_2 satisfy (5.1.1)*, then $h_1 = \pm h_2$ and $Q_1 = Q_2$ hold.*

Proof. Let f_1 and f_2 be timelike minimal surfaces in Nil_3 with the same support function h and the same Abresch-Rosenberg differential Qdz^2. Then the constant mean curvature $1/2$ surfaces $f_{\mathbb{L}^3_{(+,-,+)},1}$ and $f_{\mathbb{L}^3_{(+,-,+)},2}$ induced from f_1 and f_2 have the same metric $h^2 dz d\bar{z}$ and the same Hopf differential $4Qdz^2$ according to the previous section. The fundamental theorem of timelike surfaces in $\mathbb{L}^3$ shows that there exists a orientation preserving isometry $\Psi : \mathbb{L}^3_{(+,-,+)} \to \mathbb{L}^3_{(+,-,+)}$ of $\mathbb{L}^3_{(+,-,+)}$ such that $f_{\mathbb{L}^3_{(+,-,+)},2} = \Psi \circ f_{\mathbb{L}^3_{(+,-,+)},1}$. This implies that

$$(\phi_2^2, \phi_2^1, i'\phi_2^3) = (\phi_1^2, \phi_1^1, i'\phi_1^3)F_0 \tag{5.1.2}$$

holds for some element F_0 of the Lorentz group:

$$\left\{ X \in M_3\mathbb{R} \;\middle|\; X^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det X = 1 \right\}.$$

Conversely we assume that timelike minimal surfaces f_j with h_j, Q_j for $j = 1, 2$ satisfy the condition (5.1.2). Then the induced surfaces $f_{\mathbb{L}^3_{(+,-,+)},j}$ for $j = 1, 2$ in $\mathbb{L}^3$ are same up to orientation preserving isometries. Therefore constant mean curvature $1/2$ surfaces $f_{\mathbb{L}^3_{(+,-,+)},j}$ have the same first fundamental form and the same Hopf differential. The relation mentioned in the previous section between timelike minimal surfaces in Nil_3 and timelike constant mean curvature $1/2$ surfaces in $\mathbb{L}^3$ shows $\tilde{h} = \pm h$ and $\tilde{Q} = Q$. $\square$

5.2. **Null scrolls and their minimality conditions.** In this subsection, we introduce the *null scrolls* in Nil_3 and probe their minimality conditions. They are considered as an generalization of ruled surfaces over null curves with null director curves in the Minkowski space, what is called, null scrolls. Minimal null scrolls in Nil_3 will be characterized with the timelike minimal surfaces with $Q\overline{Q} = 0$ in the next subsection.

Let γ be a null curve in Nil_3 defined on an open interval. We denote the *velocity* of γ by A. Namely, we define the $\mathfrak{nil}_3$-valued vector field A by $A = \sum_{i=1}^{3} A^i e_i = \gamma^{-1}\frac{d\gamma}{ds}$. Here the notation γ^{-1} denotes the left translation from $T_{\gamma(\cdot)}\mathrm{Nil}_3$ to $\mathfrak{nil}_3$. Moreover, for an arbitrary $\mathfrak{nil}_3$-valued vector field $X(s) = \sum_{j=1}^{3} X^j(s)e_j$ defined on an open interval, we denote the derivative (not covariant derivative) of X by X', that is, $X'(s) = \sum_{j=1}^{3} X^{j\prime}(s)e_j$. Here the notation $'$ denotes the derivation with respect to the parameter s. Furthermore in this section, for a surface $f(s,t)$ the derivatives with respect to s and t will be denoted by f_s and f_t, respectively. We will denote the partial differentials with respect to s and t by ∂_s and ∂_t.

Definition 5.2.1 ([27])**.** A timelike surface f into Nil_3 is said to be a *null scroll* if there exist a null curve $\gamma = \gamma(s)$ in Nil_3 and a curve $\widetilde{B} = \widetilde{B}(s)$ which takes values in the light cone in $\mathfrak{nil}_3$ such that f can be represented as

$$f(s,t) = \gamma(s) \cdot \exp(t\widetilde{B}(s)). \tag{5.2.1}$$

Here, $\exp : \mathfrak{nil}_3 \to \mathrm{Nil}_3$ is the exponential map of Lie group Nil_3. Moreover, we call the curve γ a *base curve* and $\widetilde{B}$ a *ruling* of a null scroll f.

Remark 5.2.2.
(1) We use only the structure of a Lie group to define the map of the form (5.2.1). Thus we can define the surfaces of the form (5.2.1) in arbitrary Lie groups. Moreover, they can be considered as a natural generalization of ruled surfaces. In fact, in semi-Euclidean spaces, the exponential map is the identity map and the group structure is given by the usual sum. Then the maps defined in the form (5.2.1) are ruled surfaces. Moreover, in Lie groups endowed with bi-invariant metrics, the exponential maps define geodesics, and then the maps of the form (5.2.1) become ruled surfaces.
(2) By the computations in Example 4.7.4, B-scroll type minimal surface is an example of null scrolls.

Let $f(s,t) = \gamma(s)\cdot\exp(t\widetilde{B}(s))$ be a null scroll and A be the velocity of γ. Moreover expand $\widetilde{B} = \sum_{j=1}^{3} B^j e_j$. The derivatives of f with respect to s and t are computed as

$$\begin{aligned} f^{-1}f_s =& (A^1 + tB^{1\prime})e_1 + (A^2 + tB^{2\prime})e_2 \\ &+ \left(A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + \frac{t^2}{2}(B^2B^{1\prime} - B^1B^{2\prime})\right)e_3, \\ f^{-1}f_t =& B^1e_1 + B^2e_2 + B^3e_3. \end{aligned}$$

Therefore the first fundamental form I of f is given by

$$I = g_{11}ds^2 + 2g_{12}dsdt$$

where the coefficients g_{ij} are computed as

$$
\begin{aligned}
g_{11} =& g_+(f_s, f_s)\\
=& t\left(2g_+(A, \widetilde{B}') + 2(A^1B^2 - A^2B^1)A^3\right)\\
&+ t^2\left(g_+(\widetilde{B}', \widetilde{B}') + 2(A^1B^2 - A^2B^1)B^{3\prime} + (A^1B^2 - A^2B^1)^2 + (B^2B^{1\prime} - B^1B^{2\prime})A^3\right)\\
&+ t^3(B^2B^{1\prime} - B^1B^{2\prime})(B^{3\prime} + A^1B^2 - A^2B^1)\\
&+ t^4\frac{1}{4}(B^2B^{1\prime} - B^1B^{2\prime})^2,\\
g_{12} =& g_+(f_s, f_t)\\
=& g_+(A, \widetilde{B}) + t(A^1B^2 - A^2B^1)B^3 + t^2\frac{1}{2}(B^2B^{1\prime} - B^1B^{2\prime})B^3,\\
g_{22} =& g_+(f_t, f_t) = 0.
\end{aligned}
$$

Hence it is obvious that the first fundamental form of a null scroll degenerates if and only if the coefficient g_{12} of $(1,1)$-part vanishes.

Since g_{22} vanishes, the mean curvature H of a null scroll is given by

$$
H = -\frac{g_{11}h_{22} - 2g_{12}h_{12}}{2{g_{12}}^2}
$$

where the functions h_{ij} are the coefficients of (i,j)-part of the second fundamental form $II = h_{11}ds^2 + 2h_{12}dsdt + h_{22}dt^2$. In particular, we know that a null scroll is minimal if and only if it follows the equation:

$$
g_{11}g_+(\nabla_{\partial_t}f_t, f_s \times f_t) - 2g_{12}g_+(\nabla_{\partial_s}f_t, f_s \times f_t) = 0. \tag{5.2.2}
$$

Straightforward computations show

$$
f^{-1}\nabla_{\partial_t}f_t = -B^2B^3e_1 - B^1B^3e_2,
$$

$$
\begin{aligned}
f^{-1}f_s \times f_t =& \left(\left(A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + \frac{t^2}{2}(B^2B^{1\prime} - B^1B^{2\prime})\right)B^2 - (A^2 + tB^{2\prime})B^3\right)e_1\\
&+ \left(\left(A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + \frac{t^2}{2}(B^2B^{1\prime} - B^1B^{2\prime})\right)B^1 - (A^1 + tB^{1\prime})B^3\right)e_2\\
&+ \left((A^1 + tB^{1\prime})B^2 - (A^2 + tB^{2\prime})B^1\right)e_3,
\end{aligned}
$$

and then we obtain

$$
g_+(\nabla_{\partial_t}f_t, f_s \times f_t) = -g_{12}(B^3)^2.
$$

Thus, the minimality condition (5.2.2) can be represented as

$$
g_{11}(B^3)^2 + 2g_+(\nabla_{\partial_s}f_t, f_s \times f_t) = 0 \tag{5.2.3}
$$

since g_{12} vanishes nowhere. The covariant derivative of f_t with respect to ∂_s is calculated as

$$
\begin{aligned}
&f^{-1}\nabla_{\partial_s} f_t \\
&= \left(B^{1\prime} - \frac{1}{2}\left(A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + \frac{t^2}{2}(B^2B^{1\prime} - B^1B^{2\prime})\right)B^2 - \frac{1}{2}(A^2 + tB^{2\prime})B^3\right)e_1 \\
&\quad + \left(B^{2\prime} - \frac{1}{2}\left(A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + \frac{t^2}{2}(B^2B^{1\prime} - B^1B^{2\prime})\right)B^1 - \frac{1}{2}(A^1 + tB^{1\prime})B^3\right)e_2 \\
&\quad + \left(B^{3\prime} + \frac{1}{2}(A^1 + tB^{1\prime})B^2 - \frac{1}{2}(A^2 + tB^{2\prime})B^1\right)e_3.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
g_+(\nabla_{\partial_s} f_t, f_s \times f_t) =& g_+(A, \widetilde{B} \times \widetilde{B}') + \frac{1}{2} g_+(A, \widetilde{B}) g_+(A, B) \\
&+ t \begin{pmatrix} -(A^1B^2 - A^2B^1)\left(A^3(B^3)^2 + (B^2B^{1\prime} - B^1B^{2\prime})\right) \\ +\frac{1}{2} g_+(A, \widetilde{B})(-B^1B^{1\prime} + B^2B^{2\prime} - B^3B^{3\prime}) \end{pmatrix} \\
&+ t^2 \begin{pmatrix} -\frac{1}{2}(B^2B^{1\prime} - B^1B^{2\prime})\left(A^3(B^3)^2 + (B^2B^{1\prime} - B^1B^{2\prime})\right) \\ -(A^1B^2 - A^2B^1)B^{3\prime}(B^3)^2 \\ -\frac{1}{2}(A^1B^2 - A^2B^1)^2(B^3)^2 \end{pmatrix} \\
&+ t^3 \begin{pmatrix} -\frac{1}{2}(A^1B^2 - A^2B^1)(B^2B^{1\prime} - B^1B^{2\prime})(B^3)^2 \\ -\frac{1}{2}(B^2B^{1\prime} - B^1B^{2\prime})B^{3\prime}(B^3)^2 \end{pmatrix} \\
&+ t^4 \left(-\frac{1}{8}\right)(B^2B^{1\prime} - B^1B^{2\prime})(B^3)^2.
\end{aligned}
$$

Hence the minimality condition of null scrolls (5.2.3) can be converted into the equations of coefficients for t^k, $k = 0, 1, 2, 3, 4$.

Lemma 5.2.3. *For any curve $\widetilde{B} = \sum_{j=1}^3 B^j e_j$ which takes values in the light cone in $\mathfrak{nil}_3$, there uniquely exists a function $\beta : I \to \mathbb{R}$ such that $\widetilde{B} \times \widetilde{B}' = -\beta \widetilde{B}$. Then it follows that*

$$g_+(\widetilde{B}', \widetilde{B}') = \beta^2.$$

Proof. A direct computation shows $\widetilde{B} \times \widetilde{B}'$ takes values in the light cone in $\mathfrak{nil}_3$ and has the product 0 with $\widetilde{B}$ with respect to g_+ at each point. This implies the existence and uniqueness of a function $\beta : I \to \mathbb{R}$ such that $\widetilde{B} \times \widetilde{B}' = -\beta \widetilde{B}$. Since the derivation of $\widetilde{B} \times \widetilde{B}' = -\beta \widetilde{B}$ derives the following equation:

$$-\beta \widetilde{B}' = \beta' \widetilde{B} + \widetilde{B} \times \widetilde{B}''.$$

Therefore we obtain

$$\begin{aligned}\beta^2 g_+(\widetilde{B}', \widetilde{B}') &= g_+(\beta'\widetilde{B} + \widetilde{B} \times \widetilde{B}'', \beta'\widetilde{B} + \widetilde{B} \times \widetilde{B}'')\\ &= g_+(\widetilde{B} \times \widetilde{B}'', \widetilde{B} \times \widetilde{B}'')\\ &= g_+(\widetilde{B}, \widetilde{B}'')^2\\ &= g_+(\widetilde{B}', \widetilde{B}')^2.\end{aligned}$$

Hence the equation $g_+(\widetilde{B}', \widetilde{B}') = \beta^2$ holds. □

Theorem 5.2.4. *Let γ be a null curve in Nil_3, $\widetilde{B} = \sum_{i=1}^{3} B^i e_i$ be a curve taking values in the light cone in $\mathfrak{nil}_3$ and $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ be a null scroll. Then the following statements are equivalent.*

(1) *The null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ has the mean curvature 0.*
(2) *$g_+(A, \widetilde{B}) = 0$ or $g_+(A, B) = 2\beta$ holds.*

Here, the vector field B is given by $B = B^1e_1 + B^2e_2 - B^3e_3$ and the function β is defined for $\widetilde{B}$ in Lemma 5.2.3.

Proof. The coefficient of t^0 for $g_{11}(B^3)^2 + 2g_+(\nabla_{\partial_s} f_t, f_s \times f_t)$ is given by

$$2g_+(A, \widetilde{B} \times \widetilde{B}') + g_+(A, \widetilde{B})g_+(A, B). \tag{5.2.4}$$

By Lemma 5.2.3, (5.2.4) is rewritten into

$$g_+(A, \widetilde{B})(g_+(A, B) - 2\beta),$$

and then the minimality condition (5.2.3) derives

$$g_+(A, \widetilde{B}) = 0 \quad \text{or} \quad g_+(A, B) = 2\beta.$$

It is easy to see that the coefficient functions of t^3 and t^4 for $g_{11}(B^3)^2 + 2g_+(\nabla_{\partial_s} f_t, f_s \times f_t)$ always vanish. Let us prove that the coefficient function of t^2 equals identically to zero. The coefficient of t^2 in $g_{11}(B^3)^2 + 2g_+(\nabla_{\partial_s} f_t, f_s \times f_t)$ is computed as

$$g_+(\widetilde{B}', \widetilde{B}')(B^3)^2 - (B^2B^{1\prime} - B^1B^{2\prime})^2. \tag{5.2.5}$$

It is straightforward that the coefficient (5.2.5) of t^2 vanishes by using Lemma 5.2.3.

Next, we show that the coefficient of t vanishes when $g_+(A, \widetilde{B}) = 0$ or $g_+(A, B) = 2\beta$ holds. When $g_+(A, \widetilde{B}) = 0$, that is, the velocity and the ruling are linearly dependent at each point, we have

$$A^1B^2 - A^2B^1 = 0 \quad \text{and} \quad g_+(A, \widetilde{B}') = 0.$$

Then it can be seen that the coefficient of t in $g_{11}(B^3)^2 + 2g_+(\nabla_{\partial_s} f_t, f_s \times f_t)$,

$$g_+(A, \widetilde{B})(-B^1B^{1\prime} + B^2B^{2\prime} - B^3B^{3\prime}) - 2(A^1B^2 - A^2B^1)(B^2B^{1\prime} - B^1B^{2\prime}) + 2g_+(A, \widetilde{B}')(B^3)^2 \tag{5.2.6}$$

vanishes. Finally, assume that $g_+(A, B) = 2\beta$ holds. If $\beta = 0$, B and B' are linearly dependent at each point. Therefore it is easy to see that the coefficient (5.2.6) of t vanishes.

Considering the case of $\beta \neq 0$, since $A \times \widetilde{B}$ can be represented as

$$A \times \widetilde{B} = \frac{g_+(A, \widetilde{B}')}{\beta}\widetilde{B} - \frac{g_+(A, \widetilde{B})}{\beta}\widetilde{B}',$$

we have

$$A^1B^2 - A^2B^1 = \frac{g_+(A, \widetilde{B}')}{\beta}B^3 - \frac{g_+(A, \widetilde{B})}{\beta}B^{3\prime}.$$

Thus a simple computation shows (5.2.6) vanishes. $\square$

5.3. **Null frames for null curves in** Nil_3**.** By the computations of B-scroll type minimal surfaces in the previous section, the curve theory in the Minkowski space is useful to investigate null scrolls in Nil_3. We give a frame along a null curve in Nil_3, which we call a *null frame*. In Minkowski space, it is known as a null Frenet frame or Cartan frame [3, 8].

Let γ be a null curve in Nil_3 and denote the bundle along γ consisting of the vectors which have product 0 with $\frac{d\gamma}{ds}$ by $T^\perp\gamma$, that is,

$$T^\perp_{\gamma(s)}\gamma = \left\{ v \in T_{\gamma(s)}\mathrm{Nil}_3 \;\middle|\; g_+\left(v, \frac{d\gamma}{ds}(s)\right) = 0 \right\}.$$

It is obvious that the subspace $T^\perp_{\gamma(s)}\gamma$ is a 2-dimensional and includes $T_{\gamma(s)}\gamma$. Thus the bundle $T^\perp\gamma$ can be decomposed into

$$T^\perp\gamma = T\gamma \oplus S(T^\perp\gamma)$$

for some spacelike line bundle $S(T^\perp\gamma)$, called a *screen bundle* of γ (see [19]). Therefore when we take a screen bundle of γ, the bundle $T\gamma$ can be decomposed orthogonally:

$$T_\gamma\mathrm{Nil}_3 = \left(S(T^\perp\gamma)\right)^\perp \oplus_{\text{orthogonal}} S(T^\perp\gamma).$$

Here, clearly, the plane bundle $\left(S(T^\perp\gamma)\right)^\perp$ includes the tangent vector $\frac{d\gamma}{ds}$ and gives Lorentzian plane at each point of γ. Therefore the existence of the $\mathfrak{nil}_3$-valued vector field B satisfying the condition $g_+(A, B) = 1$ is guaranteed by the following lemma.

Lemma 5.3.1 ([19])**.** *Let $\langle\ ,\ \rangle$ denote the Lorentzian metric in Minkowski 3-space $\mathbb{L}^3$. Moreover, fix a null vector $v \in \mathbb{L}^3$. For each 2-dimensional Lorentzian vector subspace $W \subset \mathbb{L}^3$ which includes v, there exists a unique null vector $w \in W$ which satisfies $\langle v, w\rangle = 1$.*

Defining a spacelike vector field C along γ as $C := A \times B$, we obtain the following proposition.

Proposition 5.3.2. *Every null curve $\gamma(s)$ has a frame (A, B, C) and real-valued functions k_i $(i = 0, 1, 2)$ satisfying the following conditions*:

$$\begin{gathered}
A = \gamma^{-1}\frac{d\gamma}{ds}, \quad g_+(A, B) = g_+(C, C) = 1, \\
g_+(A, A) = g_+(B, B) = g_+(A, C) = g_+(B, C) = 0, \\
(A', B', C') = (A, B, C)\begin{pmatrix} k_0 & 0 & -k_2 \\ 0 & -k_0 & -k_1 \\ k_1 & k_2 & 0 \end{pmatrix}.
\end{gathered} \tag{5.3.1}$$

Proof. For a null curve γ, take a frame (A, B, C) as discussed above. Then it is sufficient to show the frame (A, B, C) satisfies the last condition. Let us denote A', B', and C' as

$$A' = a_A A + b_A B + c_A C, \quad B' = a_B A + b_B B + c_B C,$$

and

$$C' = a_C A + b_C B + c_C C.$$

Clearly, we have $b_A = 0$, $a_B = 0$, and $c_C = 0$ since $g_+(A, A)$, $g_+(B, B)$, and $g_+(C, C)$ are constant. Moreover, since $g_+(A, B)$, $g_+(A, C)$, and $g_+(B, C)$ are constant, we have

$$a_A = g_+(A', B) = \frac{d}{ds} g_+(A, B) - g_+(A, B') = -g_+(A, B') = -b_B,$$

$$c_A = g_+(A', C) = \frac{d}{ds} g_+(A, C) - g_+(A, C') = -g_+(A, C') = -b_C,$$

$$c_B = g_+(B', C) = \frac{d}{ds} g_+(B, C) - g_+(B, C') = -g_+(B, C') = -a_C.$$

Therefore we obtain the last condition of the proposition by putting k_j as $k_0 = a_A = -b_B$, $k_1 = c_A = -b_C$, and $k_2 = c_B = -a_C$. □

Remark 5.3.3. We can assume $k_0 = 0$ from the beginning, if necessary, by reparametrizing the curve [3]. In the curve theory in Minkowski space, such a parameter is known as the *distinguished parameter*. Because we took the vector field B for a fixed screen bundle, the frame (A, B, C) depends on the choice of a screen bundle. Hence another choice of a screen bundle gives a change in k_j.

Definition 5.3.4. We call a frame (A, B, C) given by Proposition 5.3.2 with $k_0 = 0$ a *null frame* for a null curve γ and the functions k_1 and k_2 the *first curvature* and *second curvature* of γ with respect to (A, B, C).

Theorem 5.3.5. *For any real-valued functions k_1 and k_2, there exists a null curve γ in* Nil_3 *which has k_j $(j = 1, 2)$ as the first and second curvature with respect to some null frame.*

Proof. Let $\mathrm{O}(2, 1)$ denote the Lorentz group,

$$\mathrm{O}(2,1) = \left\{ X \in M_3\mathbb{R} \;\middle|\; X^\top \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Moreover, define a matrix F_0 by

$$F_0 = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is straightforward that for an arbitrary null frame (A, B, C) along a null curve in Nil_3, there exists a unique $\mathrm{O}(2, 1)$-valued map X such that $(A, B, C) = XF_0$. Conversely, the map XF_0 satisfies the conditions (5.3.1) with $k_0 = 0$ for every $\mathrm{O}(2, 1)$-valued map X. Therefore it is sufficient to prove the proposition with the following initial condition:

$$\gamma(0) = (0, 0, 0), \quad (A(0), B(0), C(0)) = F_0. \tag{5.3.2}$$

For any basis $(\sum_{j=1}^3 A^j e_j, \sum_{j=1}^3 B^j e_j, \sum_{j=1}^3 C^j e_j)$ of $\mathfrak{nil}_3$ which satisfies the null frame condition (5.3.1) with $k_0 = 0$, we consider the matrix

$$\begin{pmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{pmatrix}$$

and denote it as (A, B, C). Moreover, we identify the basis $(\sum_{j=1}^3 A^j e_j, \sum_{j=1}^3 B^j e_j, \sum_{j=1}^3 C^j e_j)$ and the matrix (A, B, C). First, take the solution of the following system of ordinary differential equations with the initial condition (5.3.2):

(5.3.3)

$$\frac{d}{ds}\begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = k_1 \begin{pmatrix} C^1 \\ C^2 \\ C^3 \end{pmatrix}, \quad \frac{d}{ds}\begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix} = k_2 \begin{pmatrix} C^1 \\ C^2 \\ C^3 \end{pmatrix}, \quad \frac{d}{ds}\begin{pmatrix} C^1 \\ C^2 \\ C^3 \end{pmatrix} = -k_2 \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} - k_1 \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}.$$

Then we should prove that the solution (A, B, C) satisfies the conditions (5.3.1). By a direct calculation using (5.3.3), we obtain

$$\frac{d}{ds}\left(A^i B^j + A^j B^i + C^i C^j\right) = 0.$$

Thus by substituting $s = 0$, we get

$$A^i B^j + A^j B^i + C^i C^j = \begin{cases} -1 & (i = j = 1) \\ 1 & (i = j \neq 1) \\ 0 & (i \neq j) \end{cases}.$$

Since the triplet $\{V_1, V_2, V_3\}$ where V_j are defined by

$$V_1 := \frac{1}{\sqrt{2}}(B - A), \quad V_2 := \frac{1}{\sqrt{2}}(B + A), \quad V_3 := C$$

becomes an orthonormal basis of $\mathfrak{nil}_3$ at each point s, the solution (A, B, C) satisfies the conditions (5.3.1). Furthermore, the null curve $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ which has the null frame (A, B, C) is given by

$$\gamma^1 = \int_0^s A^1 ds, \quad \gamma^2 = \int_0^s A^2 ds, \quad \gamma^3 = \int_0^s (A^3 - \frac{1}{2}\gamma^2 A^1 + \frac{1}{2}A^2\gamma^1) ds.$$

□

From the discussion about the relation between B-scrolls and B-scroll type minimal surfaces in Example 4.7.4, we obtain minimal null scrolls from an arbitrary function.

Theorem 5.3.6. *For a null curve γ in Nil_3 with a null frame (A, B, C), let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a null scroll over γ with the ruling $\widetilde{B} = \sum_{j=1}^3 B^j e_j$ where $B = B^1 e_1 + B^2 e_2 - B^3 e_3$, that is,*

$$f(s, t) = \gamma(s) \cdot \exp(t\widetilde{B}(s)).$$

If the second curvature of γ with respect to the null frame (A, B, C) is $1/2$, then the null scroll f is minimal.

Proof. Since the vector field B satisfies the condition (5.3.1) with $k_0 = 0$, we have $\beta = k_2$ where β is defined in Lemma 5.2.3. Then assuming $k_2 = 1/2$ derives $2\beta = 1 = g_+(A, B)$. Thus Theorem 5.2.4 implies f is minimal. $\square$

By Theorem 5.3.5 and Theorem 5.3.6 we obtain the following corollary immediately.

Corollary 5.3.7. *For an arbitrary real valued function k_1, there exists a minimal null scroll such that the base curve has the first curvature k_1 with respect to some null frame.*

In general, an another coordinate system (x, y) for a null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ is a null coordinate system if and only if the either of following two conditions holds:

$$\begin{aligned} s_x = 0 \quad &\text{and} \quad g_{11}s_y + 2g_{12}t_y = 0, \\ s_y = 0 \quad &\text{and} \quad g_{11}s_x + 2g_{12}t_x = 0. \end{aligned} \tag{5.3.4}$$

Here the subscripts x and y denote the partial differentials with respect to x and y, respectively. The following lemma clarifies the conditions for null coordinate systems (and conformal coordinate systems) of minimal null scrolls constructed in Theorem 5.3.6.

Lemma 5.3.8. *Let $f(s, t) = \gamma(s) \cdot \exp\left(t\widetilde{B}(s)\right)$ be a minimal null scroll constructed in Theorem 5.3.6, and (x, y) be a null coordinate system for f satisfying the condition* (5.3.4)*. Then coordinate systems (s, t) and (x, y) satisfy the following relation*:

$$s(x) = 8p(x), \quad t(x, y) = \frac{1}{p(x) + q(y)}$$

where one variable functions $p(x)$ and $q(y)$ meet the conditions $p(x)+q(y) \neq 0$ and $p_x q_y < 0$.

Proof. We will compute the support function h and the Abresch-Rosenberg differential Qdz^2 for minimal null scrolls defined in Theorem 5.3.6.

Since the para-complex number $\ell = (1 + i')/2$ has the properties, $\ell^2 = \ell$ and $\ell\bar{\ell} = 0$ a coordinate system $z = \ell x + \bar{\ell} y$ for a null coordinate (x, y) gives a conformal coordinate system, and then the first fundamental form I and the unit normal vector field N of a null scroll f in Theorem 5.3.6 are represented as

$$I = e^u dz d\bar{z} = e^u dx dy = 2g_{12}s_x t_y dx dy,$$

$$N = -i' \frac{\partial f \times \overline{\partial} f}{|g_+(\partial f \times \overline{\partial} f, \partial f \times \overline{\partial} f)|^{1/2}} = -(g_{12})^{-1} f_s \times f_t.$$

Thus we have the support function h of f:

$$\begin{aligned} h =& g_+(-e^{u/2} f^{-1} N, e_3) \\ =& \sqrt{2}|s_x t_y|^{1/2} |g_{12}|^{-1/2} g_+(f^{-1} f_s \times f^{-1} f_t, e_3). \end{aligned}$$

Since we have $B \times B' = \frac{1}{2}B$ and $g_+(A, B) = 1$, $g_+(f^{-1}f_s \times f^{-1}f_t, e_3)$ and g_{12} are computed as

$$\begin{aligned} g_+(f^{-1}f_s \times f^{-1}f_t, e_3) =& (A^1 + tB^{1\prime})B^2 - (A^2 + tB^{2\prime})B^1 \\ =& A^1 B^2 - A^2 B^1 + t\frac{1}{2}B^3, \end{aligned}$$

$$
\begin{aligned}
g_{12} =& g_+(A,\widetilde{B}) + t(A^1B^2 - A^2B^1)B^3 + t^2\frac{1}{2}(B^2B^{1\prime} - B^1B^{2\prime})B^3 \\
=& g_+(A,\widetilde{B}) - (A^1B^2 - A^2B^1)^2 + (A^1B^2 - A^2B^1 + t\frac{1}{2}B^3)^2 \\
=& g_+(A,\widetilde{B}) - g_+(A,B)g_+(A,\widetilde{B}) + (A^1B^2 - A^2B^1 + t\frac{1}{2}B^3)^2 \\
=& (A^1B^2 - A^2B^1 + t\frac{1}{2}B^3)^2.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
h =& \sqrt{2}|s_xt_y|^{1/2}|g_{12}|^{-1/2}g_+(f^{-1}f_s \times f^{-1}f_t, e_3) \\
=& \sqrt{2}\epsilon|s_xt_y|^{1/2}
\end{aligned}
$$

where $\epsilon \in \{\pm 1\}$ is the signature of $g_+(f^{-1}f_s \times f^{-1}f_t, e_3)$.

To obtain the Abresch-Rosenberg differential, we will calculate $\phi^3 = g_+(f^{-1}\partial f, e_3)$ and $g_+(\nabla_\partial f_z, N)$. Since the derivative ∂f of f with respect to z is

$$
\begin{aligned}
f^{-1}\partial f =& \ell(s_xf^{-1}f_s + t_xf^{-1}f_t) + \bar{\ell}(t_yf^{-1}f_t) \\
=& \Big(\ell(s_x(A^1 + tB^{1\prime}) + t_xB^1) + \bar{\ell}(t_yB^1)\Big)\, e_1 + \Big(\ell(s_x(A^2 + tB^{2\prime}) + t_xB^2) + \bar{\ell}(t_yB^2)\Big)\, e_2 \\
&+ \left(\ell(s_xD^3 + t_xB^3) + \bar{\ell}(t_yB^3)\right) e_3,
\end{aligned}
$$

we have

$$
\phi^3 = \ell(s_xD^3 + t_xB^3) + \bar{\ell}(t_yB^3)
$$

where D^3 is the e_3-component of $f^{-1}f_s$:

$$
D^3 = g_+(f^{-1}f_s, e_3) = A^3 + t(B^{3\prime} + A^1B^2 - A^2B^1) + t^2\frac{1}{2}(B^2B^{1\prime} - B^1B^{2\prime}).
$$

Besides, a straightforward computation shows

$$
\nabla_\partial \partial f = \ell\left(s_{xx}f_s + t_{xx}f_t + (s_x)^2\nabla_{\partial_s}f_s + 2s_xt_x\nabla_{\partial_t}f_s + (t_x)^2\nabla_{\partial_t}f_t\right) + \bar{\ell}\left(t_{yy}f_t + (t_y)^2\nabla_{\partial_t}f_t\right),
$$

and then, by using the conditions (5.2.2), (5.2.3), and (5.3.4), the coefficient of the Hopf differential can be computed as

$$
g_+(\nabla_\partial \partial f, N) = \ell\left((s_x)^2h_{11} - (t_x)^2(B^3)^2\right) + \bar{\ell}\left((t_y)^2(B^3)^2\right).
$$

Hence the Abresch-Rosenberg differential Qdz^2 of f is calculated into

$$
Q = \frac{\ell}{4}\left(-(s_x)^2h_{11} - (s_x)^2(D^3)^2 - 2s_xt_xD^3B^3\right), \tag{5.3.5}
$$

and it can be represented by using the frame conditions (5.3.1) with $k_0 = 0$ as,

$$
\begin{aligned}
Q =& \frac{\ell}{4}(s_x)^2(g_{12})^{-1}\left(k_1g_{12} + \frac{t^2}{8}\left(1 + 2A^3B^3 - (A^1B^2 - A^2B^1)^2\right)\right) \\
=& \frac{\ell}{4}(s_x)^2k_1.
\end{aligned}
$$

It should be noted that in the calculations of the Abresch-Rosenberg differential we don't use the frame conditions (5.3.1) with $k_0 = 0$ until (5.3.5). This means that the Abresch-Rosenberg differential Qdz^2 of minimal null scrolls, which are not necessarily defined in Theorem 5.3.6, satisfies $Q\overline{Q} = 0$.

Moreover, a solution w of the compatibility conditions (4.2.3) for timelike minimal surfaces satisfies

$$e^w = \frac{1}{16}h^2 = \frac{1}{8}s_x t_y.$$

On the other hand, it is known that the exact solution w of the Liouville equation

$$\frac{1}{2}w_{z\bar{z}} + e^w = 0$$

is given in the form of

$$w = \log\left(-\frac{p_x q_y}{(p(x)+q(y))^2}\right), \quad p(x)+q(y) \neq 0, \ p_x q_y < 0$$

for arbitrary one variable functions p and q of x and y, respectively. Therefore coordinate systems (s,t) and (x,y) are in the following relation:

$$\frac{s_x t_y}{8} = -\frac{p_x q_y}{(p+q)^2}.$$

Hence we obtain the explicit representation of the null scroll's coordinate systems (s,t) in terms of null coordinate systems (x,y) by separating variables

$$s(x) = 8p(x), \quad t = \frac{1}{p(x)+q(y)}.$$

$\square$

Furthermore, for a minimal null scroll constructed in Theorem 5.3.6, another minimal null scroll with the same or multiplied by -1 support function and the same Abresch-Rosenberg differential can be obtained by the following theorem.

Theorem 5.3.9. *Let f be a minimal null scroll constructed in Theorem 5.3.6 and F_0 be a matrix satisfying*

$$F_0^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det F_0 = 1.$$

Denote the support function and the Abresch-Rosenberg differential of f by h and Qdz^2 and define a timelike minimal surface $\tilde{f}$ so that

$$(\tilde{\phi}^2, \tilde{\phi}^1, i'\tilde{\phi}^3) = (\phi^2, \phi^1, i'\phi^3)F_0$$

where ϕ^j and $\tilde{\phi}^j$ are given by $f^{-1}\partial f = \sum \phi^j e_j$ and $\tilde{f}^{-1}\partial\tilde{f} = \sum \tilde{\phi}^j e_j$. Then $\tilde{f}$ has the support function $\pm h$ and the Abresch-Rosenberg differential Qdz^2, and it is also a minimal null scroll.

Proof. The proof that the surface $\tilde{f}$ has the support function $\pm h$ and the Abresch-Rosenberg differential Qdz^2 is given by Theorem 5.1.1. Moreover, the fact that B-scroll type minimal surfaces induce B-scrolls shows that $\tilde{f}$ is also a null scroll. $\square$

From the above discussion, we obtain the characterization of minimal null scrolls.

Theorem 5.3.10. *If a null scroll f is minimal, then the Abresch-Rosenberg differential Qdz^2 of f satisfies $Q\overline{Q} = 0$. Conversely, every timelike minimal surface with $Q\overline{Q} = 0$ is a null scroll.*

Proof. The first half of the claim is already proved. We prove the latter half. Let f be a timelike minimal surface of the Abresch-Rosenberg differential Qdz^2 with $Q\overline{Q} = 0$ and let h denote the support function of f. By Theorem 5.1.1 and Theorem 5.3.9, it is sufficient to construct a minimal null scroll which has the support function $\pm h$ and the Abresch-Rosenberg differential Qdz^2.

For a null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$, $(A, B, A \times B)$ is not always a frame. It is easy to check that (A, B, C) for $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ is not a frame if and only if A and B are linearly dependent, that is $g_+(A, B) = 0$. If $g_+(A, B) = 0$, minimal null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ must be a vertical plane (see Proposition 5.4.1 and Example 5.4.4). The Abresch-Rosenberg differential of the vertical plane is 0. From now on we assume f is not a vertical plane, and construct minimal null scrolls by using Corollary 5.3.7. The holomorphicity of Q means that Q can be separated into functions of x and y,

$$Q = \ell S(x) + \bar{\ell} T(y)$$

where $z = \ell x + \bar{\ell} y$, and S and T are one variable real functions of x and y, respectively. Then the condition $Q\overline{Q} = 0$ implies

$$S = 0 \quad \text{or} \quad T = 0.$$

We prove only the case of $T = 0$ for simplicity. The solutions of the compatibility conditions (4.2.3) for timelike minimal surfaces are given by

$$w = \log\left(-\frac{p_x q_y}{(p(x) + q(y))^2}\right)$$

where p and q are functions of x and y, respectively, such that $p(x) + q(y) \neq 0$ and $p_x q_y < 0$. Then we define functions $k_1(s)$ and $k_2(s)$ by

$$k_1(s(x)) = \frac{S(x)}{16{p_x}^2}, \quad k_2(s(x)) = \frac{1}{2} \tag{5.3.6}$$

where the parameter s is defined by $s(x) = 8p(x)$. From Proposition 5.3.5 we can obtain a null frame (A, B, C) and a null curve $\gamma(s)$ which has the first and second curvature $k_1(s)$ and $k_2(s)$ with respect to (A, B, C). Now, we consider the null scroll

$$\gamma(s(x)) \cdot \exp\left(\frac{1}{p(x) + q(y)} \widetilde{B}(s(x))\right). \tag{5.3.7}$$

Here, $\widetilde{B} = \sum_{i=1}^{3} B^i e_i$ is the curve in $\mathfrak{nil}_3$ determined from $B = B^1 e_1 + B^2 e_2 - B^3 e_3$. Direct computations show that for (5.3.7), the Abresch-Rosenberg differential is $\ell S(x) dz^2$ and the support function can be represented as $4\epsilon e^{w/2}$ where $\epsilon \in \{\pm 1\}$ is the signature of $g_+(f^{-1} f_s \times f^{-1} f_t, E_3)$. Since the null scroll (5.3.7) has the support function $4e^{w/2}$ or $-4e^{w/2}$ and the same Abresch-Rosenberg differential as f, timelike minimal surface f has to be a null scroll by Theorem 5.3.9. □

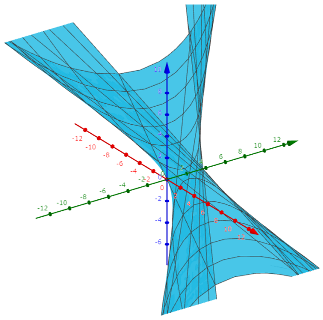

FIGURE 3. ([27]) Horizontal umbrella: Timelike minimal surface with the Abresch-Rosenberg differential vanishing anywhere.

Example 5.3.11. Let us construct timelike minimal surfaces with vanishing Abresch-Rosenberg differential except for vertical planes. Solve the differential equation (5.3.3) under the condition $k_1 = 0$, $k_2 = 1/2$ and the initial conditions $A^j(0) = A_0^j$, $B^j(0) = B_0^j$, and $C^j(0) = C_0^j$. Then the null frame $(\sum A^j e_j, \sum B^j e_j, \sum C^j e_j)$ is given by

$$A^j(s) = A_0^j, \quad B^j(s) = -\frac{s^2}{8}A_0^j + \frac{s}{2}C_0^j + B_0^j, \quad C^j(s) = -\frac{s}{2}A_0^j + C_0^j,$$

and the base curve γ is given by $\gamma(s) = (A_0^1 s, A_0^2 s, A_0^3 s)$. Hence the null scroll $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ obtained from $k_1 = 0$ and $k_2 = 1/2$ is written explicitly as follows:

$$\begin{aligned}
&(A_0^1 s,\ A_0^2 s,\ A_0^3 s)\cdot(t(-\frac{s^2}{8}A_0^1 + \frac{s}{2}C_0^1 + B_0^1),\ t(-\frac{s^2}{8}A_0^2 + \frac{s}{2}C_0^2 + B_0^2),\ t(\frac{s^2}{8}A_0^3 - \frac{s}{2}C_0^3 - B_0^3))\\
&\quad = \left((s - \frac{s^2 t}{8})A_0^1 + tB_0^1 + \frac{st}{2}C_0^1,\ (s - \frac{s^2 t}{8})A_0^2 + tB_0^2 + \frac{st}{2}C_0^2,\ (s - \frac{s^2 t}{8})A_0^3 - tB_0^3\right).
\end{aligned}$$

By (5.3.6), it is obvious that the condition $k_1 = 0$ means the vanishing of the Abresch-Rosenberg differential. Therefore surfaces given in Example 5.3.11 are horizontal umbrellas (FIGURE 3). Since, by $C \times A = A$ and $B \times C = B$, a simple computation shows

$$\det\begin{pmatrix} A_0^1 & B_0^1 & C_0^1 \\ A_0^2 & B_0^2 & C_0^2 \\ A_0^3 & -B_0^3 & 0 \end{pmatrix} = 0,$$

the vectors (A_0^1, A_0^2, A_0^3), $(B_0^1, B_0^2, -B_0^3)$, and $(C_0^1, C_0^2, 0)$ are linear dependent at each point.

5.4. **Construction of minimal null scrolls with prescribed rulings.** In Subsection 5.3, we observed that timelike minimal surfaces with $Q\overline{Q} = 0$ are obtained as minimal null scrolls, in particular, as null scrolls constructed from null curves with null frames and the second curvatures $k_2 = 1/2$ except for the vertical planes. However, the construction from curvatures is troublesome because of the need to solve a system of differential equations. In this subsection, we will construct the minimal null scrolls with prescribed rulings using only elementary computations. We should consider only the following three cases:

$$g_+(A, B) = 2\beta \text{ with } \beta = 0, \quad g_+(A, B) = 2\beta \text{ with } \beta \neq 0, \quad g_+(A, \widetilde{B}) = 0$$

because any minimal null scroll belongs to the class satisfying one of them by Theorem 5.2.4.

First, we consider the case of $g_+(A,B) = 2\beta$ with $\beta = 0$. The following proposition reveals that vertical planes are characterized by this condition.

Proposition 5.4.1. *Let γ be an affine null line in* Nil_3 *and $\widetilde{B} = \sum_{i=1}^{3} B^i e_i$ be a curve which takes values in the light cone in $\mathfrak{nil}_3$, satisfies $B^3 \neq 0$, and induces the null vector $B = B^1e_1 + B^2e_2 - B^3e_3$ linear dependent on $\gamma^{-1}\frac{d\gamma}{ds}$. Then the null scroll $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ is minimal and satisfies the minimality condition $g_+(A,B) = 2\beta$ with $\beta = 0$. Conversely, if a minimal null scroll has a ruling $\widetilde{B}$ with $\beta = 0$ and satisfies the minimality condition $g_+(A,B) = 2\beta$, the base curve is an affine null line and the ruling $\widetilde{B}$ induces a null vector field B linear dependent on the velocity.*

Proof. First, we take the minimal null scroll given by an affine null line $\gamma(s) = K_1(s)(c^1, c^2, c^3)$ and a ruling $\widetilde{B} = \sum_{i=1}^{3} B^i e_i$ which induces the null vector fields $B = B^1e_1 + B^2e_2 - B^3e_3$ linear dependent on the velocity $\gamma^{-1}\frac{d\gamma}{ds}$, that is,

$$g_+(\gamma^{-1}\frac{d\gamma}{ds}, B) = \frac{dK_1}{ds}\left(-c^1B^1 + c^2B^2 - c^3B^3\right) = 0.$$

Here K_1 is a function and c^j $(j = 1,2,3)$ are constants that are not 0 simultaneously and satisfy $-(c^1)^2+(c^2)^2+(c^3)^2 = 0$. By the linear dependence, the vector field B can be rewritten as $B = K_2(c^1e_1 + c^2e_2 - c^3e_3)$ for some smooth function K_2. Then a simple computation shows

$$B \times B' = 0,$$

that means $\beta = 0$. Hence the null scroll $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ fulfills the minimality condition $g_+(A,B) = 2\beta$ with $\beta = 0$. Conversely, we assume $g_+(A,B) = 2\beta$ and $\beta = 0$. In this case, the vector field $\widetilde{B}'$ is null, and then the condition

$$\widetilde{B} \times \widetilde{B}' = 0$$

means the vector field $\widetilde{B}'$ is linear dependent on $\widetilde{B}$ at each point. Therefore it follows the differential equations

$$B^{j\prime} = K_3 B^j \quad j = 1,2,3$$

for some function K_3. Hence the ruling $\widetilde{B}$ has to be of the form

$$\widetilde{B}(s) = e^{\int K_3 ds}\left(\sum c^j e_j\right) \tag{5.4.1}$$

where c^j $(j = 1,2,3)$ are some constants. Thus, from the assumption $g_+(A,B) = 2\beta$ and $\beta = 0$, the base curve γ is written explicitly

$$\begin{aligned} \gamma^{-1}\frac{d\gamma}{ds}(s) &= K_4(s)e^{\int K_3 ds}\left(c^1e_1 + c^2e_2 - c^3e_3\right), \\ \gamma(s) &= \int_0^s \left(K_4(s)e^{\int K_3 ds}\right) ds (c^1, c^2, -c^3) \end{aligned} \tag{5.4.2}$$

for some function K_4 under the initial condition $\gamma(0) = (0,0,0)$. Therefore the base curve γ draws an affine null line. $\square$

Remark 5.4.2. The first and second fundamental forms of minimal null scrolls defined from (5.4.1) and (5.4.2) can be computed as

$$I = -2g_{12}\left(tK_3(s)ds^2 + dsdt\right),$$

$$II = \frac{g_{12}}{|g_{12}|}e^{2\int K_3(s)ds}(c^3)^2\left(\frac{(K_4(s)+tK_3(s))(-K_4(s)+tK_3(s))}{K_4(s)}ds^2 + 2tK_3(s)dsdt + dt^2\right)$$

where $g_{12} = -2K_4(s)e^{2\int K_3(s)ds}(c^3)^2$. Hence the non-degeneracy of I means vanishing c^3 nowhere on the domain, and the first fundamental form degenerates if and only if the second fundamental form vanishes. Furthermore if the first fundamental form is non-degenerate then the second fundamental form is also non-degenerate.

Corollary 5.4.3. *A null scroll $f(s,t) = \gamma(s)\cdot\exp(t\widetilde{B}(s))$ satisfies the minimality condition $g_+(A,B) = 2\beta$ with $\beta = 0$ if and only if the null scroll is a part of a vertical plane.*

Proof. Proposition 5.4.1 implies that minimal null scrolls which satisfy the conditions $g_+(A,B) = 2\beta$ and $\beta = 0$ can be constructed from an arbitrary null vector field B which has $B^3 \neq 0$ and $\beta = 0$, by defining the velocity A of the base curves as $A = hB$. A direct computation shows the vector field E_3 is tangent to these surfaces, and then they are part of vertical surfaces, that is, Hopf cylinders. Therefore these minimal null scrolls contain the affine lines in x_3-axis direction. Furthermore, since the direction of the ruling (5.4.1) is independent to the parameter s, these minimal null scrolls are part of planes spanned by (c^1, c^2, c^3) and $(0,0,1)$. □

Example 5.4.4 (Vertical plane). Let $\theta \in (0,\pi)$ be a constant. Take a null vector field $\widetilde{B}$ as $\widetilde{B}(s) = se_1 + s\cos\theta e_2 + s\sin\theta e_3$, and define a null vector field A as $A = e_1 + \cos\theta e_2 - \sin\theta e_3$. Then the null scroll

$$\gamma(s)\cdot\exp(t\widetilde{B}(s)) = ((1+t)s, (1+t)s\cos\theta, (1+t)s\sin\theta - 2s\sin\theta)$$

is a timelike minimal surface (FIGURE 4).

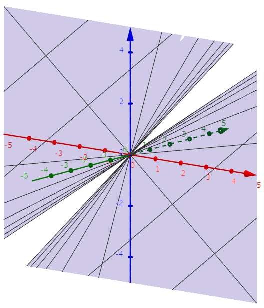

FIGURE 4. ([27]) Vertical plane: Minimal null scrolls with the minimality condition $g(A,B) = 2\beta$ and $\beta = 0$.

From now on, we will construct the minimal null scrolls which satisfy the minimality condition $g_+(A,B) = 2\beta$ with β vanishing nowhere. Without loss of generality, we can assume $\beta = 1/2$ if necessary replacing $\widetilde{B}$ with $\frac{1}{2\beta}\widetilde{B}$. For an arbitrary curve that takes values in the light cone in $\mathfrak{nil}_3$ with $\beta = 1/2$, the base curve γ can be constructed as follows.

Since the vector field $\gamma^{-1}\frac{d\gamma}{ds} \times B$ for a null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ is orthogonal to B at each point, it can be represented as

$$\gamma^{-1}\frac{d\gamma}{ds} \times B = aB' + bB$$

for some functions a and b. Therefore the equations

$$g_+(\gamma^{-1}\frac{d\gamma}{ds} \times B, \gamma^{-1}\frac{d\gamma}{ds}) = 0, \quad g_+(\gamma^{-1}\frac{d\gamma}{ds} \times B, B') = g_+(\gamma^{-1}\frac{d\gamma}{ds}, B \times B') = 1/2$$

derive $a = 2$ and $b = -2g_+(\gamma^{-1}\frac{d\gamma}{ds}, B')$ when the null scroll fulfills the minimality condition $g_+(\gamma^{-1}\frac{d\gamma}{ds}, B) = 2\beta$ with $\beta = 1/2$. This implies that determining the direction for $\gamma^{-1}\frac{d\gamma}{ds} \times B$ is equivalent to taking a function $g_+(\gamma^{-1}\frac{d\gamma}{ds}, B')$. On the other hand, by Lemma 5.3.1, taking a null vector $\gamma^{-1}\frac{d\gamma}{ds}(s)$ which satisfies

$$g_+(\gamma^{-1}\frac{d\gamma}{ds}(s), B(s)) = 1$$

coincides with taking the Lorentz plane including $B(s)$. Since determining a Lorentz plane is equivalent to determining a spacelike normal direction, we can prove the following proposition.

Proposition 5.4.5. *For any minimal null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ which satisfies the minimality condition $g_+(A, B) = 2\beta$ with $\beta = 1/2$, it follows*

$$\gamma^{-1}\frac{d\gamma}{ds} = -4\left(2g_+(B'', B'')B + B''\right) - 2b\left(B' + \frac{b}{4}B\right)$$

where $b = -2g_+(\gamma^{-1}\frac{d\gamma}{ds}, B')$. Conversely, for any curve $\widetilde{B}$ which takes values in the light cone in $\mathfrak{nil}_3$ with $\beta = 1/2$ and any real valued function b, define a $\mathfrak{nil}_3$-valued vector field A as

$$A = -4\left(2g_+(B'', B'')B + B''\right) - 2b\left(B' + \frac{b}{4}B\right).$$

Then the curve γ which has the velocity A is null and it follows that $g_+(A, B) = 1$. Thus the null scroll over γ with the ruling $\widetilde{B}$ is minimal.

Proof. Let $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ be a minimal null scroll which satisfies the minimality condition $g_+(A, B) = 2\beta$ with $\beta = 1/2$. Since we have

$$\begin{aligned} g_+(B, bB' + 2B'') &= 2g_+(B, B'') = -\frac{1}{2} \neq 0, \\ g_+(2B' + bB, B) &= 0, \\ g_+(2B' + bB, bB' + 2B'') &= 2bg_+(B', B') + 2bg_+(B, B'') = 0, \end{aligned}$$

it is straightforward that the velocity $\gamma^{-1}\frac{d\gamma}{ds}$ belongs to

$$(\gamma^{-1}\frac{d\gamma}{ds} \times B)^{\perp} = (2B' + bB)^{\perp} = span\{B, bB' + 2B''\}$$

at each point. Threfore the velocity can be represented as $\gamma^{-1}\frac{d\gamma}{ds} = uB + v(bB' + 2B'')$ for some functions u and v. It is easy to see that $u = -8g_+(B'', B'') - b^2/2$ and $v = -2$ by substituting the velocity into

$$g_+\left(\gamma^{-1}\frac{d\gamma}{ds}, \gamma^{-1}\frac{d\gamma}{ds}\right) = 0, \quad g_+\left(\gamma^{-1}\frac{d\gamma}{ds}, B\right) = 1.$$

Hence we obtain

$$\gamma^{-1}\frac{d\gamma}{ds} = -4\left(2g_+(B'', B'')B + B''\right) - 2b\left(B' + \frac{b}{4}B\right).$$

Conversely, for an arbitrary curve $\widetilde{B}$ which takes values in the light cone in $\mathfrak{nil}_3$ and derives $\beta = 1/2$, and an arbitrary function b, set a $\mathfrak{nil}_3$-valued vector field A as

$$A = -4\left(2g_+(B'', B'')B + B''\right) - 2b\left(B' + \frac{b}{4}B\right).$$

By tracing back the computations in the first half of this proof, it can be seen that A is the null vector field which belongs to $(2B' + bB)^{\perp} = span\{B, bB' + 2B''\}$ and satisfies $g_+(A, B) = 1$. Then by Theorem 5.2.4, the null scroll over the null curve which has the velocity A with the ruling $\widetilde{B}$ is minimal. $\square$

Remark 5.4.6. The null vector field A given in Proposition 5.4.5 with $b = 0$ and B define a null frame $(A, B, 2B')$, and then the curvatures are $k_1 = 4g_+(B'', B'')$ and $k_2 = 1/2$. Hence Example 5.3.11 shows if a curve $\widetilde{B}$ has $\beta = 1/2$ and B'' is null, $\widetilde{B}$ constructs a horizontal umbrella by Proposition 5.4.5 with $b = 0$ (see Example 5.4.9). Moreover, although it is hard to compute the Abresch-Rosenberg differential in general, it can be obtained easily when $b = 0$. Since we have (5.3.6) with the null frame $(A, B, 2B')$, we can see $S = 16{p_x}^2 k_1$. Therefore we have $Q = \ell 64{p_x}^2 g_+(B'', B'')$.

Example 5.4.7 (Circle ruling). Set a ruling $\widetilde{B} = \frac{1}{2}e_1 + \frac{1}{2}\cos(s)e_2 + \frac{1}{2}\sin(s)e_3$. This describes a circle (left of FIGURE 5). It is obvious that $\beta = 1/2$. Let us define a null vector field A by Proposition 5.4.5 with $b = 0$. Then we have

$$A = -e_1 + \cos(s)e_2 - \sin(s)e_3.$$

Thus we obtain a null curve γ with the initial condition $\gamma(0) = (0, 0, 0)$ from A:

$$\gamma(s) = \left(-s, \sin(s), -\frac{1}{2}s\sin(s)\right).$$

Hence the minimal null scroll $f(s,t)$ over γ with ruling $\widetilde{B}$ (right of FIGURE 5) is given by

$$f(s,t) = \left(-s + \frac{1}{2}t, \sin(s) + \frac{1}{2}t\cos(s), -\frac{1}{2}s\sin(s) + \frac{1}{4}t\sin(s) - \frac{1}{4}ts\cos(s)\right).$$

Example 5.4.8 (Hyperbola ruling). Set a ruling $\widetilde{B} = \frac{1}{2}\cosh(s)e_1 + \frac{1}{2}\sinh(s)e_2 - \frac{1}{2}e_3$. This describes a hyperbola (left of FIGURE 6). Then we have $\beta = \frac{1}{2}$. Define a null vector field A by using Proposition 5.4.5 with $b = 0$:

$$A = -\cosh(s)e_1 - \sinh(s)e_2 + e_3,$$

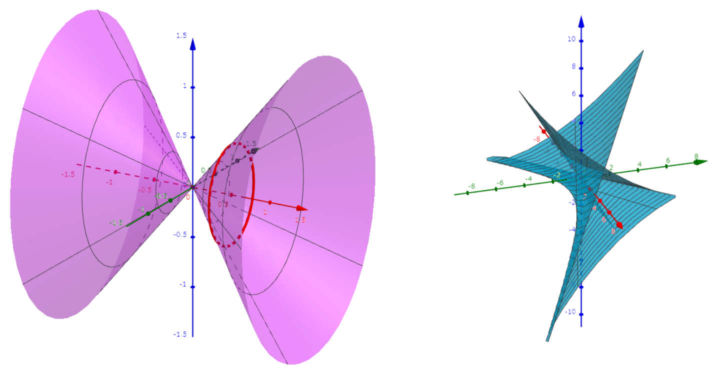

FIGURE 5. ([27]) Example of minimal null scrolls constructed from ruling valued in a circle (right) and the image of its ruling (left).

and then we obtain a null curve γ

$$\gamma(s) = \left(-\sinh(s), -\cosh(s) + 1, \frac{1}{2}s + \frac{1}{2}\sinh(s)\right)$$

with the initial condition $\gamma(0) = (0,0,0)$. Hence the minimal null scroll $f(s,t)$ over γ with ruling $\widetilde{B}$ (right of FIGURE 6) is given by

$$f(s,t) = \left(-\sinh(s) + \frac{1}{2}t\cosh(s), -\cosh(s) + 1 + \frac{1}{2}t\sinh(s), \frac{1}{2}s + \frac{1}{2}\sinh(s) - \frac{1}{4}t - \frac{1}{4}t\cosh(s)\right).$$

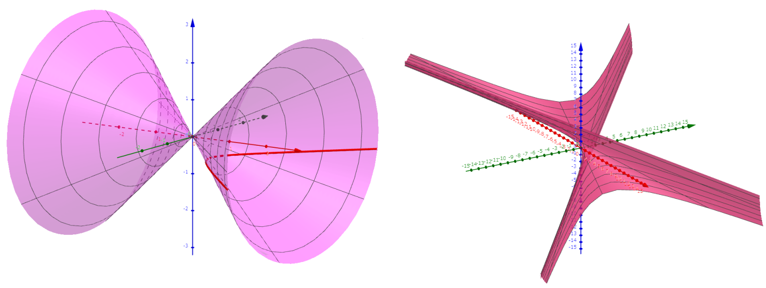

FIGURE 6. ([27]) Example of minimal null scrolls constructed from ruling valued in a hyperbola (right) and the image of its ruling (left).

Example 5.4.9 (Parabola ruling). Set a ruling $\widetilde{B} = \left(\frac{1}{8}s^2 + \frac{1}{2}\right)e_1 + \frac{1}{2}se_2 + \left(\frac{1}{8}s^2 - \frac{1}{2}\right)e_3$. Then we have $\beta = \frac{1}{2}$. This describes a parabola (left of FIGURE 7). By using Proposition 5.4.5 with a constant b we define a null vector field A by

$$A = \left(-\frac{b^2}{16}s^2 - \frac{b}{2}s - \frac{b^2}{4} - 1\right)e_1 - \left(\frac{b^2}{4}s + b\right)e_2 + \left(\frac{b^2}{16}s^2 + \frac{b}{2}s - \frac{b^2}{4} + 1\right)e_3.$$

Then we obtain a null curve γ with the initial condition $\gamma(0) = (0,0,0)$ which has the velocity A:

$$\gamma(s) = \left(-\frac{b^2}{48}s^3 - \frac{b}{4}s^2 - \left(\frac{b^2}{4}+1\right)s, -\frac{b^2}{8}s^2 - bs, -\frac{b^4}{3840}s^5 - \frac{b^3}{192}s^4 + \frac{b^4}{192}s^3 + \frac{b}{4}s^2 + \left(1-\frac{b^2}{4}\right)s\right).$$

Hence the minimal null scroll $f(s,t)$ over γ with ruling $\widetilde{B}$ is given by

$$f(s,t) = \begin{pmatrix} -\frac{b^2}{48}s^3 - \frac{b}{4}s^2 - \left(\frac{b^2}{4}+1\right)s + t\left(\frac{1}{8}s^2 + \frac{1}{2}\right), \\ -\frac{b^2}{8}s^2 - bs + \frac{st}{2}, \\ -\frac{b^4}{3840}s^5 - \frac{b^3}{192}s^4 + \frac{b^4}{192}s^3 + \frac{b}{4}s^2 + \left(1-\frac{b^2}{4}\right)s + t\left(\frac{b^2}{384}s^4 - \left(\frac{1}{8}+\frac{b^2}{32}\right)s^2 + \frac{b}{4}s - \frac{1}{2}\right) \end{pmatrix}^\top.$$

If $b = 0$, then the minimal null scroll f lies on a plane in $\mathbb{R}^3$ (right of FIGURE 7), and this is also an example of horizontal umbrellas (see Example 5.3.11).

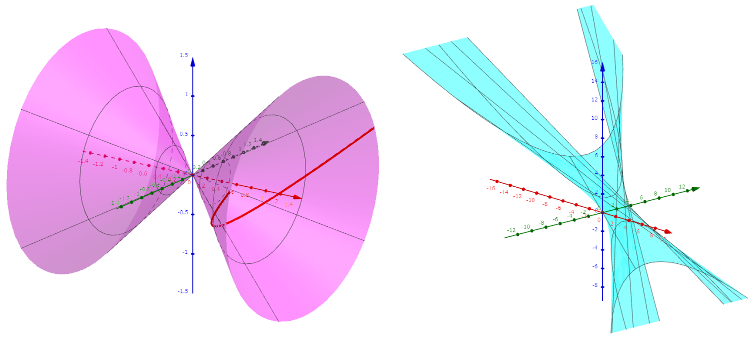

FIGURE 7. ([27]) Example of minimal null scrolls constructed from ruling valued in a hyperbola (right) and the image of its ruling (left).

Finally, we look into the case of $g_+(A, \widetilde{B}) = 0$. In this case, obviously, the rulings don't impose any conditions other than linear dependence on the velocity of the base curve. Thus we have the proposition.

Proposition 5.4.10. *Let $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ be a minimal null scroll which satisfy the minimality condition $g_+(\gamma^{-1}\frac{d\gamma}{ds}, \widetilde{B}) = 0$. Then it follows that $\beta \neq 0$ and $B^3 \neq 0$. Conversely, for a curve $\widetilde{B}$ in the light cone in $\mathfrak{nil}_3$ and a real valued function h, define a null vector field $A = \sum_{i=1}^3 A^i e_i$ by $A = h\widetilde{B}$, and γ as the curve which has the velocity A. If $\beta \neq 0$ and $B^3 \neq 0$ hold, then $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ is a minimal null scroll on $t \neq 0$.*

Proof. First, we assume that a minimal null scroll $\gamma(s)\cdot\exp(t\widetilde{B}(s))$ satisfies the condition $g_+(\gamma^{-1}\frac{d\gamma}{ds}, \widetilde{B}) = 0$, that is, the velocity is linear dependent on the ruling everywhere. Then we have

$$A^1B^2 - A^2B^1 = 0.$$

Therefore it can be seen that the $dsdt$-part of the first fundamental form is given by

$$g_{12} = t^2\frac{1}{2}\beta(B^3)^2. \tag{5.4.3}$$

Because of the non-degeneracy of a null scroll, we obtain $\beta \neq 0$ and $B^3 \neq 0$ everwhere. Conversely, let us consider the map $f(s,t) = \gamma(s) \cdot \exp(t\widetilde{B}(s))$, built from a curve $\widetilde{B}$ in the light cone in $\mathfrak{nil}_3$ and a curve γ which has the velocity linearly dependent on $\widetilde{B}$ at each point, The non-degeneracy on $t \neq 0$ of the first fundamental form is guaranteed from (5.4.3). Thus the map f is an immersion and minimal by Theorem 5.2.4. $\square$

It can be checked easily that a minimal null scroll $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ is of the form

$$\gamma(s) \cdot \exp(t\widetilde{B}(s)) = (\gamma^1(s) + \tilde{t}\gamma^{1\prime}(s), \gamma^2(s) + \tilde{t}\gamma^{2\prime}(s), \gamma^3(s) + \tilde{t}\gamma^{3\prime}(s))$$

when the null scroll fulfills the minimality condition $g_+(A, \widetilde{B}) = 0$. Hence all the minimal null scrolls that satisfy the minimality condition $g_+(A, \widetilde{B}) = 0$ are given by the following example.

Example 5.4.11 (Tangent surfaces). Let us consider a curve $\gamma(s) = (\gamma^1(s), \gamma^2(s), \gamma^3(s))$ in (semi-)Euclidean space $\mathbb{R}^3$. A surface $f(s,t)$ in $\mathbb{R}^3$ is said to be a *tangent surface* on γ if f is a ruled surface over γ, and its ruling is the velocity of γ, that is, the surface of the form

$$\begin{aligned} f(s,t) &= \gamma(s) + t\gamma'(s) \\ &= (\gamma^1(s) + t\gamma^{1\prime}(s), \gamma^2(s) + t\gamma^{2\prime}(s), \gamma^3(s) + t\gamma^{3\prime}(s)). \end{aligned} \tag{5.4.4}$$

Let us regard γ as a curve in Nil_3 by ignoring the multiplication of ambient space. When γ is null with respect to the Lorentzian metric g_+ in Nil_3 and satisfies $\gamma^{-1}\frac{d\gamma}{ds} \times (\gamma^{-1}\frac{d\gamma}{ds})' \neq 0$ and $g_+(\gamma^{-1}\frac{d\gamma}{ds}, e_3) \neq 0$, the map defined in (5.4.4) forms a null scroll in Nil_3. Moreover, it is minimal. In fact (5.4.4) can be represented as

$$\gamma(s) \cdot \exp(t(\gamma^{-1}\frac{d\gamma}{ds}(s))).$$

Since γ is null in Nil_3 and Proposition 5.4.10, tangent surface f in $\mathbb{R}^3$ can be regarded as a minimal null scroll in Nil_3.

Propositions 5.4.1, 5.4.5 and 5.4.10 can be summarised to obtain the construction theorem of minimal null scrolls with prescribed null rulings.

Theorem 5.4.12. *Let $\widetilde{B} = \sum_{i=1}^3 B^i e_i$ be a curve which takes values in the light cone in $\mathfrak{nil}_3$ except for the origin, and satisfies $\widetilde{B} \times \widetilde{B}' = -\beta\widetilde{B}$ with $\beta = 0$ or $1/2$. Define a vector field A if $\beta = 0$ and $B^3 \neq 0$ as*

$$A = \alpha B,$$

and if $\beta = 1/2$ as

$$A = -\frac{1}{2}b^2 B - 2bB' - 4\left(2g_+(B'', B'')B + B''\right).$$

Or else, if $\beta = 1/2$ and $B^3 \neq 0$, define A as

$$A = \alpha\widetilde{B}.$$

Here, $B = B^1 e_1 + B^2 e_2 - B^3 e_3$, b is an arbitrary function and α is a nowhere vanishing function. Moreover, let γ be the curve in Nil_3 which has the velocity A. Then the map $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ locally defines a minimal null scroll.

From Propositions 5.4.1, 5.4.5 and 5.4.10, we can construct every minimal null scroll locally by the Theorem 5.4.12. Furthermore, the following theorem implies it is sufficient to construct all minimal null scrolls that we consider only Propositions 5.4.1 and 5.4.5 with $b = 0$.

Theorem 5.4.13. *If a minimal null scroll is not a vertical plane then there exist a null curve γ in Nil_3 and a curve $\widetilde{B}$ in the light cone in $\mathfrak{nil}_3$ which satisfy*

$$\gamma^{-1}\frac{d\gamma}{ds} = -4(2g_+(B'', B'')B + B'') \tag{5.4.5}$$

and which define the map $\gamma(s) \cdot \exp(t\widetilde{B}(s))$ describing the original minimal null scroll.

Proof. We can obtain a null curve γ with a null frame (A, B, C) and the second curvature $1/2$ from the Abresch-Rosenberg differential and the support function of the original minimal null scroll as seen in the proof of Theorem 5.3.10. The null frame condition (5.3.1) derives the condition (5.4.5). □

Remark 5.4.14. Theorem 5.4.13 implies the class of minimal null scrolls with the minimality condition $g_+(A, \widetilde{B}) = 0$ is included in the one with the minimality condition $g_+(A, B) = 2\beta$ and $\beta \neq 0$. However, it is difficult to construct timelike minimal surfaces of the former class from the curvature of the base curves because no special features of the Abresch-Rosenberg differential and the support function are expected.

References

[1] U. Abresch, H. Rosenberg. A Hopf differential for constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. *Acta Math.* **193**(2): 141–174, 2004.

[2] U. Abresch, H. Rosenberg. Generalized Hopf differentials. *Mat. Contemp.* **28**: 1–28, 2005.

[3] K. L. Duggal, A. Bejancu. Lightlike Submanifolds of semi-Riemannian Manifolds and Applications. *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, 1996.

[4] D. A. Berdinskiĭ. Surfaces of constant mean curvature in the Heisenberg group (Russian). *Mat. Tr.* **13**(2): 3–9, 2010. English translation: *Siberian Adv. Math.* **22**(2): 75–79, 2012.

[5] D. A. Berdinskiĭ, I. A. Taĭmanov. Surfaces in three-dimensional Lie groups (Russian). *Sibirsk. Mat. Zh.* **46**(6): 1248–1264, 2005. English translation: *Siberian Math. J.* **46**(6): 1005–1019, 2005.

[6] O. P. Bhutani, M. H. M. Moussa, K. Vijayakumar. On the generalized forms of exact solutions to the Liouville equation via direct approach. *Int. J. Engng. Sci.* **32**(12): 1965–1969, 1994.

[7] A. I. Bobenko. Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. Harmonic maps and integrable systems, Aspects Math., E23, Friedr. Vieweg, Braunschweig, 83–127, 1994.

[8] W. B. Bonnor. Null curves in a Minkowski space-time. *Tensor (N. S.)* **20**: 229–242, 1969.

[9] D. Brander, S. P. Kobayashi. Maximal surfaces in the Lorentzian Heisenberg group. arXiv:2302.10559, 2023

[10] S. Cartier. Surfaces des espaces homogènes de dimension 3. *Thèse de Doctorat, Université Paris-Est Marne-la-Vallée*, 2011. http://perso-math.univ-mlv.fr/users/cartier.sebastien/.

[11] A. A. Cintra, I. I. Onnis. Enneper representation of minimal surfaces in the three-dimensional Lorentz–Minkowski space. *Ann. Mat. Pura Appl.* **197**(1): 21–39, 2018.

[12] P. A. Clarkson, M. D. Kruskal. New similarity reductions of the Boussinesq equation. *J. Math. Phys.* **30**(10): 2201–2213, 1989.

[13] J. Clelland. Totally quasi-umbilic timelike surfaces in $\mathbb{R}^{1,2}$. *Asian J. Math.*, **16**(2): 189–208, 2012.

[14] D. G. Crowdy. General solutions to the 2D Liouville equation. *Int. J. Engng. Sci.* **35**(2): 141–149, 1997.

[15] B. Daniel. The Gauss map of minimal surfaces in the Heisenberg group. *Int. Math. Res. Not. IMRN*, **3**: 674–695, 2011.

[16] J. F. Dorfmeister, J. Inoguchi, S.-P. Kobayashi. A loop group method for minimal surfaces in the three-dimensional Heisenberg group. *Asian J. Math.*, **20**(3), 409–448, 2016.

[17] J. F. Dorfmeister, J. Inoguchi, M. Toda. Weierstrass type representations of timelike surfaces with constant mean curvature. In: *Integrable Systems in Differential Geometry*, Contemp. Math. **308**: 77–99, Amer. Math. Soc., Providence, RI, 2002.

[18] J. F. Dorfmeister, F. Pedit, H. Wu. Weierstrass type representations of harmonic maps into symmetric spaces. *Comm. Anal. Geom.*, **6**(4): 633–668, 1998.

[19] K. L. Duggal, D. H. Jin. Null curves and hypersurfaces of semi-Riemannian manifolds. *World Scientific Publishing Company.* 2007.

[20] I. Fernandez, P. Mira. Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space. *Trans. Amer. Math. Soc.*, **361**(11): 5737–5752, 2009.

[21] A. Fujioka, J. Inoguchi. Timelike Bonnet surfaces in Lorentzian space forms. *Differential Geom. Appl.*, **18**(1): 103–111, 2003.

[22] L. Graves. Codimension one isometric immersions between Lorentz spaces. *Trans. Amer. Math. Soc.*, **252**: 367–392, 1979.

[23] J. Inoguchi. Surfaces in Minkowski 3-space and harmonic maps. *RIMS Kokyuroku* **995**: 58–69, 1997.

[24] J. Inoguchi. Timelike surfaces of constant mean curvature in Minkowski 3-space. *Tokyo J. Math.* **21**: 141–152, 1998.

[25] J. Inoguchi. Minimal surfaces in the 3-dimensional Heisenberg group. *Differ. Geom. Dyn. Syst.* (Electronic), **10**: 163–169, 2008.

[26] H. Kiyohara, S.-P. Kobayashi. Timelike minimal surfaces in the three-dimensional Heisenberg group. *J. Geom. Anal.*, **32**(8): Paper No. 225, 2022.

[27] H. Kiyohara. Minimal null scrolls in the three-dimensional Heisenberg group. arXiv preprint arXiv: 2308.08178, 2023.

[28] J. J. Konderak. A Weierstrass representation theorem for Lorentz surfaces. *Complex Var. Theory Appl.* **50**(5): 319–332, 2005.

[29] N. Klarreich. Smoothability of the conformal boundary of a Lorentz surface inplies 'global smoothability'. *Geom. Dedicata***89**: 59–107, 2002.

[30] A. Korn. Zwei Anwendungen der Methode der sukzessiven Annäherungen. *Mathematische Abhandlungen Hermann Amandus Schwarz.* Springer Berlin Heidelberg. 215–229, 1914.

[31] R. Kulkarni. An analogue of the Riemann mapping theorem for Lorentz metrics. *Proc. Roy. Soc. London Ser. A* **401**: 117–130, 1985.

[32] L. Lichtenstein. Zur Theorie der konformen Abbildung. Konforme Abbildung nichtanalytischer, singularitätenfreier Flächenstücke auf ebene Gebiete *Bull. Acad. Sci. Cracovie, Cl. Sci. Math. Nat. Ser. A*: 192–217, 1916.

[33] S. Lin, T. Weinstein. $\mathbb{E}^3$-complete timelike surfaces in $\mathbb{E}^3_1$ are globally hyperbolic. *Michigan Math. J.* **44**(3): 529–541, 1997.

[34] S. Lin, T. Weinstein. A better conformal Bernstein's theorem. *Geom. Dedicata* **81**(1–3): 53–59, 2000.

[35] J. Liouville. Sur l'equation aux différences partielles $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$. *J. Math. Pures Appl.* **18**: 71–72, 1853.

[36] H. Liu. Ruled surfaces with lightlike ruling in 3-Minkowski space. *J. Geom. Phys.*, **59**(1):74–78, 2009.

[37] M. Melko, I. Sterling. Application of soliton theory to the construction of pseudospherical surfaces in $\mathbb{R}^3$. *Ann. Glob. Anal. Geom.* **11**(1): 65–107, 1993.

[38] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.* **21**(3): 304–310, 1976.

[39] T. K. Milnor. A conformal analog of Bernstein's theorem in Minkowski 3-space. In: *The Legacy of Sonya Kovalevskaya*, Contemp. Math. **64**: 123–130, Amer. Math. Soc., Providence, RI, 1987.

[40] T. K. Milnor. Entire timelike minimal surfaces in $\mathbb{E}^3_1$. *Michigan Math. J.* **37**: 163–177, 1990.

[41] T. K. Milnor. Associate harmonic immersions in 3-space. *Michigan Math. J.* **37**: 179–190, 1990.

[42] C. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.* **43**: 126–166, 1938.

[43] K. Pohlmeyer. Integrable Hamiltonian systems and interactions through quadratic constraints. *Comm. Math. Phys.* **46**: 207–221, 1976.

[44] A. G. Popov. Exact formulas for constructing solutions of the Liouville equation $\Delta_2 u = e^u$ from solutions of the Laplace equation $\Delta_2 v = 0$. *Dokl. Akad. Nauk, Russian Academy of Sciences*: 440–441, 1993.

[45] A. Pressley, G. Segal. Loop groups. *Oxford Mathematical Monographs*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1986.

[46] S. Rahmani. Métriques de Lorentz sur les groupes de Lie unimodulaires, de dimension trois, [Lorentz metrics on three-dimensional unimodular Lie groups]. *J. Geom. Phys.* **9**(3): 295–302, 1992.

[47] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.* **56**: 401–487, 1983.

[48] B. A. Shipman, P. D. Shipman, D. Packard. Generalized Weierstrass–Enneper representations of Euclidean, spacelike, and timelike surfaces : a unified Lie-algebraic formulation. *J. Geom.* **108**(2): 545–563, 2017.

[49] Z. M. Sipus, L. P. Gajčić, I. Protrka. Null scrolls as B-scrolls in Lorentz-Minkowski 3-space. *Turkish J. Math.*, **43**(6): 2908–2920, 2019.

[50] R. W. Smyth, T. Weinstein. Conformally homeomorphic Lorentz surfaces need not be conformally diffeomorphic. *Proc. Amer. Math. Soc.* **123**: 3499–3506, 1995.

[51] R. W. Smyth, T. Weinstein. How many Lorentz surfaces are there. In: S. Grindikin (ed.), *Topics in Geometry in Memory of Joseph D'Atri*, Progr. Nonlinear Differential Equations Appl. **20**: 315–330, Birkhäuser, Boston, MA, 1996.

[52] A. Sym. Soliton surfaces and their applications (soliton geometry from spectral problems). Geometric Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Phys. **239**, Springer, Berlin, 154–231, 1985.

[53] I. A. Taĭmanov. Modified Novikov-Veselov equation and differential geometry of surfaces. *Amer. Math. Soc. Transl.* **2**(179): 133–151, 1997.

[54] I. A. Taĭmanov. Dirac operators and conformal invariants of tori in 3-space. *Proc. Steklov Inst. Math.* **244**: 233–263, 2004.

[55] I. A. Taĭmanov. Surfaces in three-dimensional Lie groups in terms of spinors (Geometry related to the theory of integrable systems). *RIMS Kôkyûroku* **1605**: 133–150, 2008. http://hdl.handle.net/2433/139943

[56] W. M. Thurston (S. Levy ed.). *Three-dimensional geometry and topology.* Vol. 1, Princeton Math. Series, Vol. 35, Princeton Univ. Press, 1997.

[57] T. Weinstein. An introduction to Lorentz surfaces. *De Gruyter Exp. Math.* **22**. Walter de Gruyter & Co., Berlin, 1996.