Doctoral Thesis

Discrete Morse theory on magnitude homotopy types of finite graphs

（有限グラフのマグニチュードホモトピー型上の離散モース理論）

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Introduction

The magnitude is an invariant for metric spaces defined by Leinster [11, 12]. It measures a certain size of space. Hepworth-Willerton [8] and Leinster-Shulman [13] defined the magnitude homology as a categorification of the magnitude. The magnitude homology $\mathrm{MH}_k^\ell(G)$ is a bigraded module, and determined for a graph G, length ℓ and degree k. There are various studies about magnitude and magnitude homology [1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17].

Recently, Kaneta-Yoshinaga [9] introduced a simplicial complex for a mertic space with a certain condition whose homology group is isomorphic to the magnitude homology of the metric space. Asao-Izumihara [3] introduced a CW complex (we call the Asao-Izumihara complex) for a graph whose (reduced) homology is isomophic to the magnitude homology of the graph. Asao-Izumihara complex is constructed as the quotient of a pair of a simplicial complex and a subcomplex (for details see §2.1). It is important for the construction of Asao-Izumihara complex that vertices of the simplicial complex are defined with the information of distance between the vertex and the basepoint. Note that Asao-Izumihara complex captures only $\mathrm{MH}_k^\ell(G)$ for $\ell \geq 2$ and $k \geq 2$. To improve this, we introduce the notion of magnitude homotopy type $\mathcal{M}_\ell(G)$. The magnitude homotopy type is defined also for metric spaces.

By the construction, we can apply discrete Morse theory for both Asao-Izumihara complex and magnitude homotopy type. Both the Asao-Izumihara complex and the magnitude homotopy type have advantages. The Asao-Izumihara complex is useful to discribe the figure of it for lower dimension especially $\ell = 3, 4$. On the other hands, the magnitude homotopy type has advantages in theoretical aspects. For example, the manitude homotopy type can be used to prove Künneth formula and Mayer-Vietoris type theorem, etc.

We are interested in the relationship between magnitude homology groups and qualitative properties of graphs. In particular, we are interested in the following topics.

(a) Diagonality.

(b) Graph operations.

It is said that a graph G is diagonal if $\mathrm{MH}_k^\ell(G) = 0$, for $k \neq \ell$. Diagonal graph is an important class of graphs, because if a graph is diagonal then the rank of magnitude homology of the graph is determined by only the magnitude. It is expected that there are several relationships between diagonality and qualitative properties of graphs. For example, it is shown that "diagonality implies girth$= 3, 4$ or ∞" in [2]. However, we do not know when graphs are diagonal. About the topic (b), Leinster proved the invariance of magnitudes of graphs under a Whitney twist in [12]. Roff generalized the result for a sycamore twist. Moreover, Künneth formula and Mayer-Vietoris theorem for magnitude homology were proved in [8]. Recently, magnitude homotopy type version of these results were also proved in [17].

In this thesis, for the sake of conceptual simplicity, we focus on finite metric spaces defined by finite graphs. This thesis is organized as follows. In §1.1, we recall the definitions of magnitude and magnitude homology. In §1.2, we recall basic definitions and theorems on discrete Morse theory. In §2, we review the definition of the Asao-Izumihara complex and basic properties. In §3, we show that the Asao-Izumihara complex of a pawful graph (or some other diagonal graph) is homotopy equivalent to a wedge of spheres by using discrete Morse theory. In §4, we describe the Asao-Izumihara complex of the odd cycle graph. In §5, we show several results for the magnitude homotopy type of graphs using discrete Morse theory. In §5.1, we introduce a useful matching (called projecting matching) on magnitude homotopy type of graphs obtained by gluing two graphs. In §5.2, we show the Mayer-Vietoris type theorem for the magnitude homotopy type. In §5.3, we prove the invariance of magnitudes of graphs under a sycamore twist using the projecting matching.

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Contents

Chapter 1

Preliminaries

1.1 Magnitude and magnitude homology

We define a finite graph G as the pair $(V(G), E(G))$ of the following sets.

- $V(G)$ is a finite set.
- $E(G) \subseteq \{\{x, y\} \mid (x, y) \in V(G) \times V(G),\ x \neq y\}$.

The distance function on a graph G is defined as follows.

$$d_G \colon V(G) \times V(G) \to \mathbb{Z}_{\geq 0},$$
$$d_G(a, b) := \begin{cases} \min\{k \in \mathbb{Z}_{\geq 0} \mid \exists\{v_0, v_1\}, \{v_1, v_2\}, \cdots, \{v_{k-1}, v_k\} \in E(G), v_0 = a, v_k = b\} & (a \neq b), \\ 0 & (a = b). \end{cases}$$

If there does not exist such a sequence of edges for some pair of vertices (a, b), then we define $d_G(a, b) = \infty$. From now, we assume that $d_G(a, b) < \infty$ for any pairs of vertices (a, b) of G. We denote d_G simply as d.

Definition 1.1.1 (Magnitude of a graph)**.** Let q be a variable, and $m := |V(G)|$. Define the $m \times m$ matrix Z_G whose entries are elements of $\mathbb{Q}(q)$ as follows.

$$Z_G := \left(q^{d(x,y)}\right)_{x,y \in G}.$$

Then Z_G is invertible. We define the magnitude of G as follows.

$$\#G := \sum_{x,y \in G} Z_G^{-1}(x, y).$$

Definition 1.1.2 (Sequence)**.** We call $(x_0, \cdots, x_k) \in V(G)^{k+1}$ a sequence on G if it satisfies $x_i \neq x_{i+1}$ for each $i \in \{0, \cdots, k-1\}$. For a sequence $x = (x_0, \cdots, x_k)$, the subsequence of x is a sequence $y = (y_0, \cdots, y_m)$ that satisfies the following.

- There exists a sequence $0 \leq i_0 < i_1 < \cdots < i_m \leq k$ such that $y_j = x_{i_j}$ $(j = 0, 1, \cdots, m)$.

Then, we denote $y \prec x$. The length of sequence $x = (x_0, \cdots, x_k)$ is defined by $L(x) := d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{k-1}, x_k)$.

Definition 1.1.3 (Smooth point)**.** Let $x = (x_0, \cdots, x_k)$ be a sequence. We say that the point x_i $(i = 1, \cdots, k-1)$ is smooth in x if x_i satisfies $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$. If the point x_i $(i = 1, \cdots, k-1)$ is not smooth in x, we say x_i is singular in x. Denote by $x_{i-1} \prec x_i \prec x_{i+1}$ if x_i is smooth in x, and by $x_{i-1} \not\prec x_i \not\prec x_{i+1}$ if x_i is singular in x.

Definition 1.1.4 (Magnitude homology of a graph)**.** Fix $\ell \geq 0$. Define the abelian group $\mathrm{MC}_k^\ell(G)$ and the homomorphism ∂_k as follows.

$$\mathrm{MC}_k^\ell(G) := \bigoplus \mathbb{Z}\langle \text{sequence } (x_0, \cdots, x_k) \text{ on } G \mid L(x_0, \cdots, x_k) = \ell \rangle,$$

$$\partial_k \colon \mathrm{MC}_k^\ell(G) \to \mathrm{MC}_{k-1}^\ell(G), \ \partial_k := \sum_{i=1}^{k-1} (-1)^i \partial_{k,i},$$

$$\partial_{k,i}(x_0, \cdots, x_k) := \begin{cases} (x_0, \cdots, \widehat{x_i}, \cdots, x_k) & (\text{if } x_i \text{ is smooth in } x), \\ 0 & (\text{otherwise}). \end{cases}$$

Then $(\mathrm{MC}_*^\ell(G), \partial_*)$ is a chain complex and it is called the magnitude chain complex. The magnitude homology $\mathrm{MH}_k^\ell(G)$ is defined by $\mathrm{MH}_k^\ell(G) := H_k(\mathrm{MC}_*^\ell(G))$.

Theorem 1.1.5 ([8], Theorem 2.8.)**.** *Let q be a variable. Then, we have*

$$\#G = \sum_{\ell \geq 0} \left(\sum_{k \geq 0} (-1)^k \operatorname{rank} \mathrm{MH}_k^\ell(G) \right) q^\ell.$$

1.2 Discrete Morse theory

For elements a, b of a poset P, we denote $a \prec b$ if a and b satisfy $a < b$ and there does not exist $c \in P$ such that $a < c < b$.

Definition 1.2.1 (Partial matching)**.** Let P be a poset. A partial matching M on P is a subset of $P \times P$ satisfying the following.

- For any $(a, b) \in M$, $a \prec b$.
- Each $a \in P$ belongs to at most one element in M.

Denote by $a \vdash b$ for $(a, b) \in M$.

Definition 1.2.2 (Acyclic matching)**.** Let M be a partial matching on P. We say M is acyclic if there does not exist a cycle

$$b^1 \succ a^1 \prec b^2 \succ a^2 \prec \cdots \prec b^p \succ a^p \prec b^{p+1} = b^1$$

with $p \geq 2$, that satisfies $(a^i, b^{i+1}) \in M$ for each $i \in \{1, 2, \cdots, p\}$ and $b^i \neq b^j$ $(i \neq j)$ for every $i, j \in \{1, 2, \cdots, p\}$.

Definition 1.2.3 (Critical element)**.** Let M be a partial matching on P. An element of P that belongs to no element in M is called a critical element.

Theorem 1.2.4 ([10], Theorem 11.13.(a)(b))**.** *Let P be the face poset of a simplicial complex S. Assume that we have an acyclic matching on P.*

(a) *If the critical elements form a subcomplex S_c of S, then there exists a sequence of cellular collapse leading from S to S_c.*

(b) *Denote the number of critical i-dimensional simplex by c_i. Then, S is homotopy equivalent to a CW complex with c_i cells in dimension i.*

Definition 1.2.5 (Strong deformation retract)**.** Let A be a topological space, and $B \subset A$ be a subspace. A continuous map $F\colon A \times [0, 1] \to A$ is called a strong deformation retract if it satisfies the following conditions.

- For $x \in A$, $F(x, 0) = x$ and $F(x, 1) \in B$,
- For $x \in B$ and $t \in [0, 1]$, $F(x, t) = x$.

In the situation, we also say that there exists the strong deformation retract from A to B.

Remark 1.2.6. Theorem 1.2.4 (a) especially means that

(a′) If the critical elements form a subcomplex S_c of S, then there exists the strong deformation retract from S to S_c.

Chapter 2

Magnitude homotopy type

2.1 Asao-Izumihara complex

Definition 2.1.1 (Simplicial complex)**.** Let V be a set and $P(V)$ be the power set. If a subset $S \subset P(V)$ satisfies the following, then we call S a simplicial complex.

- For any element $A \in S$ and any subset $B \subset A$, B is an element of S.

Remark 2.1.2. In this thesis, we consider the empty set $\emptyset \subset V$ as a (-1)-dimensional simplex called empty simplex. Any simplex contain the empty simplex. The simplicial complex which has only the empty simplex is called the empty simplicial complex, and denoted by $\{\emptyset\}$. It is a subcomplex of any simplicial complex (other than the void). The simplicial complex which has no simplices is called the void and denoted by **void**.

Definition 2.1.3 (Path)**.** Let $x = (x_0, \cdots, x_k)$ be a sequence. We call x a path if x satisfies $d(x_i, x_{i+1}) = 1$ for any $i \in \{0, \cdots, k-1\}$. We define the set of paths on graph G with length ℓ as follows.

$$P_\ell(G; a, b) := \{\text{path } x = (x_0, \cdots, x_\ell) \text{ on } G \mid \ x_0 = a,\ x_\ell = b\},\ a, b \in V(G).$$

Definition 2.1.4 (Asao-Izumihara complex)**.** Let $\ell \geq 2$ and $a, b \in G$. We define the sets $K_\ell(G; a, b)$ and $K'_\ell(G; a, b)$ as follows.

$$\begin{aligned}
K_\ell(G; a, b) &:= \{\emptyset \neq \{(x_{i_1}, i_1), \cdots, (x_{i_{k-1}}, i_{k-1})\} \subset G \times \{1, 2, \cdots, \ell - 1\} \\
&\qquad\qquad \mid (a, x_{i_1}, \cdots, x_{i_{k-1}}, b) \prec \exists (a, x_1, \cdots, x_{\ell-1}, b) \in P_\ell(G; a, b)\}, \\
K'_\ell(G; a, b) &:= \{\{x_{i_1}, \cdots, x_{i_{k-1}}\} \in K_\ell(G; a, b) \mid L(a, x_{i_1}, \cdots, x_{i_{k-1}}, b) < \ell\}.
\end{aligned}$$

Then $K_\ell(G; a, b)$ is a simplicial complex and $K'_\ell(G; a, b)$ is the subcomplex of $K_\ell(G; a, b)$. We call the CW complex $K_\ell(G; a, b)/K'_\ell(G; a, b)$ Asao-Izumihara complex.

Remark 2.1.5. We denote a simplex $\{x_0, \cdots, x_k\} \in K_\ell(G; a, b)$ by $(x_0, \cdots, x_k)$. For $x, y \in K_\ell(G; a, b)$, let $y \prec x$ denote that y is the subsimplex of x.

Example 2.1.6. Let G be the cycle graph C_4 with vertices a, b, c, d in order. Let $\ell = 4$. We describe the Asao-Izumihara complex $K_4(G; a, c)/K'_4(G; a, c)$. First we compute the set of paths $P_4(G; a, c)$.

$$\begin{aligned} P_4(G; a, c) = \{&(a, b, a, b, c), (a, b, a, d, c), (a, b, c, b, c), (a, b, c, d, c), \\ &(a, d, a, b, c), (a, d, a, d, c), (a, d, c, b, c), (a, d, c, d, c)\}. \end{aligned}$$

From the paths we obtain maximal faces of $K_4(G; a, c)$ as follows.

$$\begin{gathered} ((b, 1), (a, 2), (b, 3)),\ ((b, 1), (a, 2), (d, 3)),\ ((b, 1), (c, 2), (b, 3)),\ ((b, 1), (c, 2), (d, 3)), \\ ((d, 1), (a, 2),\ (b, 3)), ((d, 1), (a, 2), (d, 3)),\ ((d, 1), (c, 2), (b, 3)),\ ((d, 1), (c, 2), (d, 3)). \end{gathered}$$

Next we compute $K'_4(G; a, c)$.

$$\begin{aligned} K'_4(G; a, c) = \{&(b, 1),\ (a, 2),\ (b, 3),\ (d, 3),\ (c, 2),\ (d, 1),\ ((a, 2), (b, 3)),\ ((b, 1), (b, 3)), \\ &((a, 2), (d, 3)),\ ((b, 1), (c, 2)),\ ((d, 1), (d, 3)),\ ((d, 1), (c, 2))\}. \end{aligned}$$

Therefore $K_4(G; a, c)$ and $K'_4(G; a, c)$ are as shown in Figure 2.1 (The part of red is $K'_4(G; a, c)$), and we have

$$K_4(G; a, c)/K'_4(G; a, c) \approx S^2 \vee S^2.$$

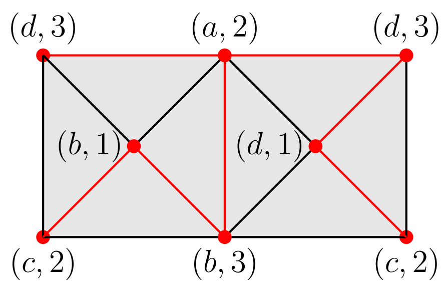


Figure 2.1: $K_4(C_4; a, c)$ and $K'_4(C_4; a, c)$.

Proposition 2.1.7. Let $\ell \geq 0$. Then,

$$\mathrm{MC}^\ell_*(G) = \bigoplus_{a,b \in G} \mathrm{MC}^\ell_*(G; a, b)$$

as chain complexes, where $\mathrm{MC}^\ell_*(G; a, b)$ is the subcomplex of $\mathrm{MC}^\ell_*(G)$ generated by sequences which start from a and end with b.

Theorem 2.1.8 ([3], Theorem 4.3.). *Let $\ell \geq 3$, $* \geq 0$ and $a, b \in G$. Then,*

$$C_*(K_\ell(G;a,b)/K'_\ell(G;a,b), pt) \cong C_*(K_\ell(G;a,b), K'_\ell(G;a,b)) \cong \mathrm{MC}^\ell_{*+2}(G;a,b),$$

as chain complexes.

Example 2.1.9. Let G be a cycle graph C_4 with vertices a, b, c, d in order. Let us compute the magnitude homology $\mathrm{MH}^4_*(G)$ using Asao-Izumihara complexes. We have $K_4(G;a,c)/K'_4(G;a,c) \approx S^2 \vee S^2$ in Example 2.1.6. Similarly we have $K_4(G;a,a)/K'_4(G;a,a) \approx S^2 \vee S^2 \vee S^2$. Since there is no paths from a to b with length 4, $K_4(G;a,b) = \emptyset$. Then,

$$\begin{aligned}\mathrm{MH}^4_k(G) = \bigoplus_{a,b\in G} \mathrm{MH}^4_k(G;a,b) &\cong \bigoplus_{a,b\in G} \widetilde{H}_{k-2}(K_4(G;a,b)/K'_4(G;a,b)) \\ &\cong (\widetilde{H}_{k-2}(S^2 \vee S^2))^{\oplus 4} \oplus (\widetilde{H}_{k-2}(S^2 \vee S^2 \vee S^2))^{\oplus 4}.\end{aligned}$$

Therefore $\mathrm{MH}^4_k(G) \cong \begin{cases} \mathbb{Z}^{20} & (k = 4), \\ 0 & (k \neq 4,\ k \geq 2). \end{cases}$

2.2 Magnitude homotopy type

Definition 2.2.1. Let $\ell \geq 0$ and $a, b \in G$. We define the sets $\Delta_\ell(G;a,b)$ and $\Delta'_\ell(G;a,b)$ as follows.

$$\begin{aligned}\Delta_\ell(G;a,b) &:= \{\{(x_{i_0}, i_0), \cdots, (x_{i_k}, i_k)\} \subset G \times \{0, 1, \cdots, \ell\} \\ &\qquad\qquad \mid (x_{i_0}, \cdots, x_{i_k}) \prec \exists (x_0, \cdots, x_\ell) \in P_\ell(G;a,b)\}, \\ \Delta'_\ell(G;a,b) &:= \{\{x_{i_0}, \cdots, x_{i_k}\} \in \Delta_\ell(G;a,b) \mid L(x_{i_0}, \cdots, x_{i_k}) < \ell\}.\end{aligned}$$

Then $\Delta_\ell(G;a,b)$ is a simplicial complex and $\Delta'_\ell(G;a,b)$ is a subcomplex of $\Delta_\ell(G;a,b)$.

Remark 2.2.2. We consider the cases that $\Delta_\ell(G;a,b)$ or $\Delta'(G;a,b)$ are empty set.

- In the case of $\ell = 0$, if $a = b$, then $\Delta_0(X;a,a) = \{(a,0)\}$. Since the empty simplex $\emptyset$ is contained in $\Delta_0(X;a,a)$, $\Delta'_0(X;a,a) = \{\emptyset\}$. If $a \neq b$, then $\Delta_0(X;a,b) = \Delta'_0(X;a,b) = \textbf{void}$.
- Consider the case of $\ell > 0$. If $\Delta_\ell(X;a,b) \neq \emptyset$, then $\Delta'_\ell(X;a,b) \neq \emptyset$ since $\{(a,0)\}$ is contained in it. If $d(a,b) > \ell$, then $\Delta_\ell(X;a,b) = \Delta'_\ell(X;a,b) = \textbf{void}$. If $d(a,b) = \ell$, then $\Delta_\ell(X;a,b) \neq \emptyset$. If $d(a,b) < \ell$ and $\Delta_\ell(X;a,b) = \emptyset$, then $\Delta_\ell(X;a,b) = \Delta'_\ell(X;a,b) = \{\emptyset\}$.

Definition 2.2.3 (Magnitude homotopy type of a graph)**.** The magnitude homotopy type of a graph G is defined as follows.

$$\mathcal{M}_\ell(G) := \bigvee_{a,b\in G} \mathcal{M}_\ell(G;a,b), \quad \text{where} \quad \mathcal{M}_\ell(G;a,b) := \Delta_\ell(G;a,b)/\Delta'_\ell(G;a,b).$$

Remark 2.2.4. As in the Asao-Izumihara complex, we denote a simplex $\{x_0, \cdots, x_k\} \in \Delta_\ell(G;a,b)$ by $(x_0, \cdots, x_k)$, and let $y \prec x$ denote that y is a subsimplex of x.

Example 2.2.5 ([17], Example 4.11.(1)(2))**.** (1) Let G be a tree. Then,

$$\mathcal{M}_\ell(G) \approx \begin{cases} S^0 \vee \cdots \vee S^0 \text{ (wedge of } |V(G)| \text{ spheres)} \approx \{(|V(G)|+1) \text{ points}\}, & \ell = 0, \\ S^\ell \vee \cdots \vee S^\ell \text{ (wedge of } 2|E(G)| \text{ spheres)}, & \ell \geq 1. \end{cases}$$

(2) Let G be a complete graph with m vertices ($m \geq 2$). Then,

$$\mathcal{M}_\ell(G) \approx S^\ell \vee \cdots \vee S^\ell \text{ (wedge of } m(m-1)^\ell \text{ spheres)}.$$

Proof. (1) We prove it in the end of §4.

(2) For $a, b \in V(G)$, all maximal faces of $\Delta_\ell(G;a,b)$ are ℓ-simplices and their subsimplices which consist boundaries are all in $\Delta'_\ell(G;a,b)$. There are $m(m-1)^\ell$ maximal faces in $\bigcup_{a,b\in V(G)} \Delta_\ell(G;a,b)$. Therefore we have the result. □

Theorem 2.2.6 ([17], Theorem 4.7.)**.** *For a graph G and $k, \ell \geq 0$,*

$$\widetilde{H}_k(\mathcal{M}_\ell(G)) \cong \mathrm{MH}^\ell_k(G).$$

Proof. We prove that $\widetilde{H}_k(\mathcal{M}_\ell(G;a,b)) \cong \mathrm{MH}^\ell_k(G;a,b)$ for $a, b \in V(G)$. Clealy,

$$\widetilde{H}_k(\mathcal{M}_\ell(G;a,b)) \cong H_k(\Delta_\ell(G;a,b), \Delta'_\ell(G;a,b)).$$

It is sufficient to show the isomorphism

$$C_*(\Delta_\ell(G;a,b), \Delta'_\ell(G;a,b)) \cong \mathrm{MC}^\ell_*(G;a,b)$$

as chain complexes. Define a homomorphism $\varphi_k \colon C_k(\Delta_\ell(G;a,b), \Delta'_\ell(G;a,b)) \to \mathrm{MC}^\ell_k(G;a,b)$ by $((x_{i_0}, i_0), \cdots, (x_{i_k}, i_k)) \mapsto (x_{i_0}, \cdots, x_{i_k})$. We can easily check the map φ_* is a chain map and isomorphism. □

Definition 2.2.7 (Suspension)**.** Let X, Y be CW complexes.

- We define the join of X and Y by

$$X * Y := X \times Y \times [0,1] \Big/ \sim,$$

where $\sim$ is the equivalence relation such that $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for any $x, x_1, x_2 \in X$, $y_1, y_2, y \in Y$. Especially, we denote by $\Gamma_\alpha(X)$ the cone of X with apex α:

$$\Gamma_\alpha(X) := \{\alpha\} * X.$$

Note that $\Gamma_\alpha(\mathbf{void}) = \mathbf{void}$ and $\Gamma_\alpha(\{\emptyset\}) = \{\alpha\}$.

- Let $A \subset X$ be a subcomplex. The (reduced) suspension of the pair (X, A) is defined by

$$\Sigma(X, A) := (\Gamma_\alpha(X), \Gamma_\alpha(A) \cup X).$$

Note that $\Sigma(X, \mathbf{void}) = (\Gamma_\alpha(X), X)$ and $\Sigma(\{\emptyset\}, \mathbf{void}) = (\{\alpha\}, \{\emptyset\})$.

Theorem 2.2.8 ([17], Theorem 4.21.)**.** *Let $\ell \geq 2$. The magnitude homotopy type $\mathcal{M}_\ell(X; a, b)$ of a graph X is homotpy equivalent to the double suspension of Asao-Izumihara complex $K_\ell(X; a, b)/K'_\ell(X; a, b)$.*

Proof. It is sufficient to show that

$$(|\Delta_\ell(X; a, b)|, |\Delta'_\ell(X; a, b)|) \simeq \Sigma^2(|K_\ell(X; a, b)|, |K'_\ell(X; a, b)|),$$

for $\ell \geq 2$, $a, b \in X$ with $d(a, b) \leq \ell$. Let $\ell \geq 0$, $a, b \in X$. Suppose that $d(a, b) \leq \ell$, and $K_\ell(X; a, b) \neq \emptyset$. From this point, for simplicity, we will write $K_\ell = K_\ell(X; a, b)$, $\Delta_\ell = \Delta_\ell(X; a, b)$, etc. Define the set $\widetilde{\Delta}_\ell = \widetilde{\Delta}_\ell(X; a, b)$ as follows.

$$\widetilde{\Delta}_\ell := \{\{(a, 0), (x_{i_1}, i_1), \cdots, (x_{i_k}, i_k)\} \mid \{(x_{i_1}, i_1), \cdots, (x_{i_k}, i_k)\} \in K_\ell\} \cup K_\ell.$$

Then, we can easily check that $\widetilde{\Delta}_\ell$ is a simplicial complex. Let $\alpha = (a, 0)$, then we have

$$|\widetilde{\Delta}_\ell| = \Gamma_\alpha(|K_\ell|).$$

Define the subset $\widetilde{\Delta'}_\ell \subset \widetilde{\Delta}_\ell$ by

$$\widetilde{\Delta'}_\ell = \{\{(x_{i_0}, i_0), \cdots, (x_{i_k}, i_k)\} \in \widetilde{\Delta}_\ell \mid L(x_{i_0}, \cdots, x_{i_k}, b) < \ell\}.$$

Then, $\widetilde{\Delta'}_\ell$ is the subcomplex of $\widetilde{\Delta}_\ell$, and we have

$$|\widetilde{\Delta'}_\ell| = \Gamma_\alpha(|K'_\ell|) \cup |K_\ell|.$$

By Definition 2.2.7, we have the following.

$$\begin{aligned}\Sigma(|K_\ell|, |K'_\ell|) &= (\Gamma_\alpha(|K_\ell|), \Gamma_\alpha(|K'_\ell|) \cup |K_\ell|)\\ &= (|\widetilde{\Delta}_\ell|, |\widetilde{\Delta}'_\ell|).\end{aligned}$$

Similarly, let an apex $\beta = (b, \ell)$, then

$$\begin{aligned}\Sigma^2(|K_\ell|, |K'_\ell|) &= \Sigma(|\widetilde{\Delta}_\ell|, |\widetilde{\Delta}'_\ell|)\\ &= (\Gamma_\beta(|\widetilde{\Delta}_\ell|), \Gamma_\beta(|\widetilde{\Delta}'_\ell|) \cup |\widetilde{\Delta}_\ell|)\\ &= (|\Delta_\ell|, |\Delta'_\ell|).\end{aligned}$$

$\square$

Chapter 3

Diagonality

3.1 Diagonal graph

Definition 3.1.1 (Diagonal graph)**.** Let G be a graph. We say that G is diagonal if G satisfies $\mathrm{MH}_k^\ell(G) = 0$ for $k \neq \ell$.

Example 3.1.2. Complete graphs and trees are diagonal ([8], Example 2.5, Corollary6.8). The join of two (non-empty) graphs are diagonal ([8], Theorem 7.5.). On the other hands, cycle graphs C_m ($m \geq 5$) are not diagonal.

3.2 Pawful graph

Y. Gu introduced pawful graphs in [5].

Definition 3.2.1 (Pawful graph)**.** Let G be a graph with the diameter ≤ 2. We call G a pawful graph if G satisfies the following condition.

- For any vertices $x, y, z \in G$ with $d(x, y) = 2$, $d(y, z) = 2$, and $d(x, z) = 1$, there exists the vertex $a \in G$ such that $d(a, x) = d(a, y) = d(a, z) = 1$.

Remark 3.2.2. The join of two (non-empty) graphs is a pawful graph.

Theorem 3.2.3 ([16], Theorem 3.4)**.** *Let G be a pawful graph. Let $\ell \geq 3$ and $a, b \in G$. Then the Asao-Izumihara complex $K_\ell(G; a, b)/K'_\ell(G; a, b)$ is empty or contractible or homotopy equivalent to the wedge of $(\ell - 2)$-spheres.*

Proof. For the face poset of a simplicial complex $K_\ell(a, b)$, let P be a (induced) subposet of the face poset whose elements are contained in $K_\ell(G; a, b) \setminus K'_\ell(G; a, b)$.

Let p be a map $p\colon \{(x,y,z) \in V(G)^3 \mid d(x,y)=2,\ d(y,z)=2,\ d(x,z)=1\} \to V(G)$ such that $p(x,y,z)=a$ satisfies $d(a,x)=d(a,y)=d(a,z)$. Such a map exists by Definition 3.2.1. We also fix a map $q\colon \{(x,y) \in V(G) \mid d(x,y)=2\} \to V(G)$ such that $q(x,y)=b$ satisfies $d(x,b)=d(b,y)=1$.

Next we define the set $\mathcal{S}$ as follows.

$$\begin{aligned}\mathcal{S} :=& \{(\alpha,\beta,\gamma,\delta) \in V(G)^4 \mid d(\alpha,\delta)=1, d(\beta,\delta)=2, d(\alpha,\beta)=1, \gamma=\alpha\}\\ &\sqcup \{(\alpha,\beta,\gamma,\delta) \in V(G)^4 \mid d(\alpha,\delta)=2, d(\beta,\delta)=2, d(\alpha,\beta)=1, \gamma=p(\alpha,\delta,\beta)\}\\ &\sqcup \{(\alpha,\beta,\gamma) \in V(G)^3 \mid d(\alpha,\gamma)=2, \beta=q(\alpha,\gamma)\}.\end{aligned}$$

For $x=(x_0,\cdots,x_k) \in P$, we denote the minimum $i \in \{1,2,\cdots,k-2\}$ such that

$$(x_{i-1},x_i,x_{i+1},x_{i+2}) \in \mathcal{S} \tag{3.2.1}$$

by $i(x)$. If $(x_0,x_1,x_2) \in \mathcal{S}$, then let $i(x)=0$. In the case that there does not exist $i \in \{1,2,\cdots,k-2\}$ satisfying (3.2.1) and $i(x) \neq 0$, then define $i(x)=\infty$. On the other hands, we denote the minimum $j \in \{0,1,\cdots,k-1\}$ such that $d(x_j,x_{j+1})=2$ by $j(x)$. If there does not exist such $j \in \{0,1,\cdots,k-1\}$, then define $j(x)=\infty$. Now we define the subsets A, P', $P'' \subset P$ as follows.

$$\begin{aligned}A &:= \{x \in P \mid i(x)=\infty \text{ and } j(x)=\infty\},\\ P' &:= \{x \in P \mid i(x) > j(x)\},\\ P'' &:= \{x \in P \mid i(x) < j(x)\}.\end{aligned}$$

Define the map $f\colon P' \to P''$ by

$$x=(x_0,\cdots,x_k) \mapsto (x_0,\cdots,x_{j(x)-1},x_{j(x)},z,x_{j(x)+1},\cdots,x_k),$$

where $z \in G$ is a vertex satisfying $(x_0,z,x_1) \in \mathcal{S}$ if $j(x)=0$, or $(x_{j(x)-1},x_{j(x)},z,x_{j(x)+1}) \in \mathcal{S}$ if $j(x) \geq 1$. Also we define the map $g\colon P'' \to P'$ by

$$y=(y_0,\cdots,y_k) \mapsto (y_0,\cdots,\widehat{y_{i(y)+1}},\cdots,x_k).$$

Since $g \circ f = \mathrm{id}_{P'}$ and $f \circ g = \mathrm{id}_{P''}$ hold, f and g are bijection.

We define the subset $M \subseteq P \times P$ by $M := \{(x,y) \in P' \times P'' \mid f(x)=y\}$. Then $x \prec y$ for any $(x,y) \in M$, and each $x \in P$ belongs at most one element in M since f is injective. Therefore M is a partial matching on M.

Next we prove that M is acyclic. Assume that there exists a cycle

$$y^1 \succ x^1 \prec y^2 \succ x^2 \prec \cdots \prec y^p \succ x^p \prec y^{p+1} = y^1$$

such that $(x^i, y^{i+1}) \in M$ for each $i \in \{1, 2, \cdots, p\}$ $(p \geq 2)$ and $y^i \neq y^j$ for every $i, j \in \{1, 2, \cdots, p\}$ $(i \neq j)$. For $y^t = (y^t_0, \cdots, y^t_k) \in P''$, let $(y^t_0, \cdots, \widehat{y_{i^t+1}}, \cdots, y^t_k) = x^t$. For $x^t \in P'$, we denote $j(x^t)$ by j^t. We can assume that j^1 is the minimum number in all j^t. Then we have the following.

- $j^t = i^t$ for $t \geq 1$,
- $j^t + 1 \geq j^{t+1}$ and $j^t \neq j^{t+1}$ for $t \geq 1$,
- $j^2 = j^1 + 1$.

Let $y^1 = (y_0, \cdots, y_k)$, then x^1, y^2, x^2, y^3 are as follows.

$x^1 = (y_0, \cdots, y_{i^1}, y_{i^1+2}, \cdots, y_k)$,
$y^2 = (y_0, \cdots, y_{i^1}, z_1, y_{i^1+2}, \cdots, y_k)$, where z_1 satisfies $(y_{i^1-1}, y_{i^1}, z_1, y_{i^1+2}) \in \mathcal{S}$,
$x^2 = (y_0, \cdots, y_{i^1}, z_1, y_{i^1+3}, \cdots, y_k)$,
$y^3 = (y_0, \cdots, y_{i^1}, z_1, z_2, y_{i^1+3}, \cdots, y_k)$, where z_2 satisfies $(y_{i^1}, z_1, z_2, y_{i^1+3}) \in \mathcal{S}$.

For $y^t = (y^t_0, \cdots, y^t_k) \in P''$ $(t \geq 3)$, we prove $d(y^t_{i^1}, y^t_{i^1+2}) \leq 1$ by induction.

(I) In the case of $t = 3$, by $(y_{i^1}, z_1, z_2, y_{i^1+3}) \in \mathcal{S}$, we have $z_2 = y_{i^1}$ or $z_2 = p(y_{i^1}, z_1, y_{i^1+3})$ for y^3. In both cases, $d(y^3_{i^1}, y^3_{i^1+2}) = d(y_{i^1}, z_2) \leq 1$.

(II) For y^t $(t \geq 3)$, we assume $d(y^t_{i^1}, y^t_{i^1+2}) \leq 1$. Then

$$y^{t+1} = (y^t_0, \cdots, y^t_{i^t-1}, y^t_{i^t}, z_t, y^t_{i^t+2}, \cdots, y^t_k),$$

where z_t satisfies $(y^t_{i^t-1}, y^t_{i^t}, z_t, y^t_{i^t+2}) \in \mathcal{S}$. By the assumption $d(y^t_{i^1}, y^t_{i^1+2}) \leq 1$, we have $i^t \neq i^1$. Therefore $i^t \geq i^1 + 1$.

(i) In the case of $i^t = i^1 + 1$, as in (I), $d(y^{t+1}_{i^1}, y^{t+1}_{i^1+2}) = d(y^t_{i^t-1}, z_t) \leq 1$ since $(y^t_{i^t-1}, y^t_{i^t}, z_t, y^t_{i^t+2}) \in \mathcal{S}$.

(ii) In the case of $i^t \geq i^1 + 2$, $d(y^{t+1}_{i^1}, y^{t+1}_{i^1+2}) = d(y^t_{i^1}, y^t_{i^1+2}) \leq 1$.

By (I) and (II), we have $d(y^t_{i^1}, y^t_{i^1+2}) \leq 1$ $(t \geq 3)$. Since $y^t_{i^1+1}$ is singular in y^t, $j^t = i^t \neq i^1 = j^1$ $(t \geq 3)$. By $j^2 \neq j^1$, we obtain $j^t \neq j^1$ $(t \geq 2)$. On the other hands, $j^1 = i^1 = i^{p+1} = j^{p+1}$ by $y^1 = y^{p+1}$. It contradicts. Now we have the matching M is acyclic. We consider the matching M on P as the matching on $K_\ell(G;a,b)$, then the set of critical elements is $A \sqcup K'_\ell(G;a,b)$. The elements of A are all $(\ell-2)$-simplices. By Theorem 1.2.4, $K_\ell(G : a,b)$ is homotopy equivalent to the CW complex which is constructed some $(\ell-2)$-cells attaching $K'_\ell(G;a,b)$. Therefore the Asao-Izumihara complex $K_\ell(G;a,b)/K'_\ell(G;a,b)$ is (empty or contractible or) homotopy equivalent to wedge of $(\ell-2)$-spheres. $\square$

Corollary 3.2.4 ([5], Theorem 4.4.)**.** *Pawful graphs are diagonal.*

3.3 Generalization

In this subsection, we generalize the result for pawful graphs (Theorem 3.2.3). Let G be a graph with diameter 2. We define the set X, Y, X', Y' as follows.

$$\begin{aligned}
X &:= \{(\alpha,\beta,\gamma) \in V(G)^3 \mid d(\alpha,\beta) = d(\beta,\gamma) = 1, d(\alpha,\gamma) = 2\},\\
Y &:= \{(\alpha,\beta,\gamma,\delta) \in V(G)^4 \mid d(\alpha,\beta) = d(\beta,\gamma) = d(\gamma,\delta) = 1, d(\beta,\delta) = 2\},\\
X' &:= \{(\alpha,\gamma) \in V(G)^2 \mid d(\alpha,\gamma) = 2\},\\
Y' &:= \{(\alpha,\beta,\delta) \in V(G)^3 \mid d(\alpha,\beta) = 1, d(\beta,\delta) = 2\}.
\end{aligned}$$

Let $\pi_1 \colon X \to X'$ and $\pi_2 \colon Y \to Y'$ be natural projections.

Definition 3.3.1. Assume that maps $f_1 \colon X' \to X$ and $f_2 \colon Y' \to Y$ satisfy the following.

(i) $\pi_1 \circ f_1 = \mathrm{id}_{X'}$ and $\pi_2 \circ f_2 = \mathrm{id}_{Y'}$.

(ii) If $(\alpha,\beta,\gamma,\delta) \in f_2(Y')$, then $(*,\alpha,\beta,\gamma) \notin f_2(Y')$ and $(\alpha,\beta,\gamma) \notin f_1(X')$.

(iii) Let $(\alpha,\beta,\gamma,\delta) \in f_2(Y')$. If $d(\alpha,\gamma) = 2$, then there does not exist $\gamma'(\neq \gamma) \in V(G)$ such that $d(\beta,\gamma') = d(\gamma',\delta) = 1$.

Then we define the set $\mathcal{S} \subseteq X \sqcup Y$ by $\mathcal{S} := f_1(X') \sqcup f_2(Y')$.

Theorem 3.3.2. *Let G be a graph with diameter 2. Assume that G has $\mathcal{S}$ as in Definition 3.3.1. Let $\ell \geq 3$ and $a, b \in V(G)$. Then the Asao-Izumihara complex $K_\ell(G;a,b)/K'_\ell(G;a,b)$ is homotopy equivalent to a wedge of $(\ell-2)$-spheres. In particular, G is diagonal.*

Proof. Similar to Theorem 3.2.3. $\square$

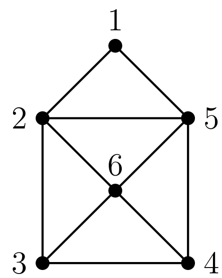


Figure 3.1: Non-pawful diagonal graph G_1.

Example 3.3.3. Let G_1 be a graph as in Figure 3.1. Note that G_1 is not pawful, because for vertices $1, 3, 4$ with $d(1,3) = d(1,4) = 2$ and $d(3,4) = 1$ there does not exist a vertix a such that $d(a,1) = d(a,3) = d(a,4) = 1$. For this graph, we can construct the set S as follows. Define the map $f_1\colon X' \to X$ by

$$f_1(X') = \left\{ \begin{array}{l} (1,2,3),(1,5,4),(1,2,6),(2,6,4),(3,2,1),(3,6,5), \\ (4,5,1),(4,6,2),(5,6,3),(6,2,1) \end{array} \right\}.$$

Define the map $f_2\colon Y' \to Y$ by

$$f_2(Y') = \left\{ \begin{array}{l} (2,1,2,3),(5,1,2,3),(2,1,5,4),(5,1,5,4),(2,1,2,6),(5,1,2,6), \\ (1,2,5,4),(3,2,6,4),(5,2,6,4),(6,2,6,4), \\ (2,3,2,1),(4,3,2,1),(6,3,2,1),(2,3,6,5),(4,3,6,5),(6,3,6,5), \\ (3,4,5,1),(5,4,5,1),(6,4,5,1),(3,4,6,2),(5,4,6,2),(6,4,6,2), \\ (1,5,2,3),(2,5,6,3),(4,5,6,3),(6,5,6,3), \\ (2,6,2,1),(3,6,2,1),(4,6,5,1),(5,6,5,1) \end{array} \right\}.$$

Then, since we have

$$\{(\alpha,\beta,\gamma,\delta) \in f_2(Y') \mid d(\alpha,\gamma) = 2\} = \{(4,3,2,1),(3,4,5,1)\},$$

it is easily seen that f_1 and f_2 satisfy the conditions (i), (ii) and (iii) of Definition 3.3.1. Thus by Theorem 3.3.2 we conclude that G_1 is diagonal.

Example 3.3.4. Let G_2 be a graph as in Figure 3.2. The graph G_2 is not pawful but diagonal graph. We can check G_2 is diagonal by using Mayer-Vietoris type Theorem by [8] (see Corollary 5.2.10). However, there does not exist $\mathcal{S}$ as in Definition 3.3.1. The reason why is as follows. For $(4,3,1) \in Y'$, $f_2((4,3,1)) = (4,3,2,1)$. On the other hands, for $(3,4,2) \in Y'$, we have $f_2((3,4,2)) \neq (3,4,5,2)$ since f_2 satisfies the condition (iii) in Definition 3.3.1. Then $(3,4,3,2) \in f_2(Y')$, and we can not satisfy the condition (ii).

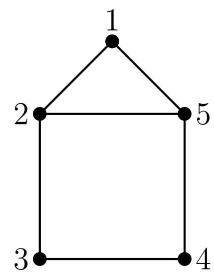


Figure 3.2: Non-pawful diagonal graph G_2 which does not have $\mathcal{S}$.

Example 3.3.5. Example 3.3.3 and Example 3.3.4 are cases of graphs whose each edge is contained in a cycle of length ≤ 4 and which are diagonal. However, it is

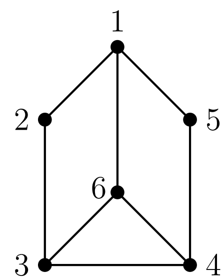


Figure 3.3: Non-diagonal graph G_3 .

not true that for a graph if each edge is contained in a cycle of length ≤ 4 then it is diagonal. Let G_3 be a graph as in Figure 3.3, and it is not diagonal. In fact, $\mathrm{MH}_2^3(G_3)$ does not vanish since $(2, 4, 5) \in \mathrm{MH}_2^3(G_3)$.

Chapter 4

Cycle graph

As an example of non-diagonal graph, we describe the Asao-Izumihara complex of odd cycle graphs. Y. Gu computed the magnitude homology of cycle graphs. The result for odd cycle graphs is as follows.

Theorem 4.0.1 ([5], Theorem 4.6.). *Let C_{2m-1} $(m \geq 3)$ be the cycle graph. Then the magnitude homology of C_{2m-1} is as follows.*

$$\mathrm{MH}_k^\ell(C_{2m-1}) = \begin{cases} \mathbb{Z}^{2m-1} & ((k,\ell) = (0,0)), \\ \mathbb{Z}^{4m-2} & ((k,\ell) = (1,1)), \\ \mathbb{Z}^{\operatorname{rank} \mathrm{MH}_{k-1}^{\ell-1}(C_{2m-1}) + 2 \operatorname{rank} \mathrm{MH}_{k-2}^{\ell-m}(C_{2m-1})} & (k \geq 0, \ell \geq 0, (k,\ell) \neq (0,0), (1,1)), \\ 0 & (\textit{otherwise}). \end{cases}$$

Example 4.0.2. The rank of the magnitude homology of cycle graph C_7 is as follows.

$\ell \backslash k$	0	1	2	3	4	5	6	7	8	9
0	7									
1		14								
2			14							
3				14						
4			14		14					
5				42		14				
6					70		14			
7						98		14		
8					28		126		14	
9						112		154		14

We describe the Asao-Izumihara complexes of odd cycle graphs.

Theorem 4.0.3. *Let $G = C_{2m-1}$ $(m \geq 3)$ be the cycle graph with $2m-1$ vertices. Fix $\ell \geq 3$ and $a, b \in G$. Then the Asao-Izumihara complex $K_\ell(G; a, b)/K'_\ell(G; a, b)$ is empty or contractible or homotopy equivalent to the wedge of spheres with various dimensions.*

Proof. Let $P := K_\ell(G; a, b) \setminus K'_\ell(G; a, b)$. We define the set $\mathcal{S}$ as follows.

$$\mathcal{S} := \{(\alpha, \beta, \gamma) \in V(G)^3 \mid d(\alpha, \beta) = 1, d(\alpha, \gamma) = d(\alpha, \beta) + d(\beta, \gamma)\}.$$

For $x = (x_0, \cdots, x_k) \in P$, we denote the minimum $i \in \{0, 1, \cdots, k-2\}$ such that $(x_i, x_{i+1}, x_{i+2}) \in \mathcal{S}$ by $i(x)$. In the case that there does not exist such $i \in \{0, 1, \cdots, k-2\}$, then define $i(x) = \infty$. On the other hands, we denote by $j(x)$ the minimum $j \in \{1, 2, \cdots, k-1\}$ satisfying either of the following.

- $d(x_{j-1}, x_j) = 1, d(x_j, x_{j+1}) \geq 2, d(x_j, x_{j+1}) = d(x_j, x_{j-1}) + d(x_{j-1}, x_{j+1})$,
- $d(x_{j-1}, x_j) \geq 2, d(x_j, x_{j+1}) \geq 2$.

In the case of $d(x_0, x_1) \geq 2$, define $j(x) = 0$. If there does not exist such $j \in \{1, 2, \cdots, k-1\}$ and $j(x) \neq 0$, then define $j(x) = \infty$. Now we define the subsets A, P', $P'' \subset P$ as follows.

$$\begin{aligned}
A &:= \{x \in P \mid i(x) = \infty \text{ and } j(x) = \infty\}, \\
P' &:= \{x \in P \mid i(x) > j(x)\}, \\
P'' &:= \{x \in P \mid i(x) < j(x)\}.
\end{aligned}$$

Define the map $f \colon P' \to P''$ by

$$x = (x_0, \cdots, x_k) \mapsto (x_0, \cdots, x_{j(x)}, z, x_{j(x)+1}, \cdots, x_k),$$

where $z \in G$ satisfies $(x_{j(x)}, z, x_{j(x)+1}) \in \mathcal{S}$. Note that f is well-defined since there is the only one shortest path between any two different vertices. Also we define the map $g \colon P'' \to P'$ by

$$y = (y_0, \cdots, y_k) \mapsto (y_0, \cdots, \widehat{y_{i(y)+1}}, \cdots, x_k).$$

We prove $f \circ g = \mathrm{id}_{P''}$. For $y = (y_0, \cdots, y_k) \in P''$,

$$f \circ g(y) = f((y_0, \cdots, y_{i(y)-1}, y_{i(y)}, y_{i(y)+2}, \cdots, y_k)).$$

If $d(y_{i(y)-1}, y_{i(y)}) \geq 2$, then $j(g(y)) = i(y)$. If $d(y_{i(y)-1}, y_{i(y)}) = 1$, then $y_{i(y)}$ satisfies $d(y_{i(y)-1}, y_{i(y)+1}) < d(y_{i(y)-1}, y_{i(y)}) + d(y_{i(y)}, y_{i(y)+1}) = 2$ since $(y_{i(y)-1}, y_{i(y)}, y_{i(y)+1}) \notin$

$\mathcal{S}$. The graph G has no 3-cycle, then $d(y_{i(y)-1}, y_{i(y)+1}) = 0$ i.e. $y_{i(y)-1} = y_{i(y)+1}$. Therefore, $j(g(y)) = i(y)$. In any cases we have $f \circ g(y) = y$.

Next we prove $g \circ f = \mathrm{id}_{P'}$. For $x = (x_0, \cdots, x_k) \in P'$,

$$g \circ f(x) = g((x_0, \cdots, x_{j(x)}, z, x_{j(x)+1}, \cdots, x_k)), \text{ where } (x_{j(x)}, z, x_{j(x)+1}) \in \mathcal{S}.$$

If $j(x) = 0$, then $i(f(x)) = j(x)$. If $j(x) \geq 1$, then $i(f(x)) = j(x)$ since $(x_{j(x)-1}, x_{j(x)}, z) \notin \mathcal{S}$. In any cases we have $g \circ f(x) = x$. Therefore f and g are bijection.

We define the subset $M \subseteq P \times P$ by $M := \{(x, y) \in P' \times P'' \mid f(x) = y\}$. Then $x \prec y$ for any $(x, y) \in M$, and each $x \in P$ belongs at most one element in M since f is injective. Therefore M is a partial matching on M.

Next we prove that M is acyclic. Assume that there exists a cycle such that

$$y^1 \succ x^1 \prec y^2 \succ x^2 \prec \cdots \prec y^p \succ x^p \prec y^{p+1} = y^1$$

with $p \geq 2$, that satisfies $(x^i, y^{i+1}) \in M$ for each $i \in \{1, 2, \cdots, p\}$ and $y^i \neq y^j$ $(i \neq j)$ for every $i, j \in \{1, 2, \cdots, p\}$. For $y^t = (y^t_0, \cdots, y^t_k) \in P''$, let $(y^t_0, \cdots, \widehat{y_{i^t+1}}, \cdots, y^t_k) = x^t$. For $x_t \in P'$, we denote $j(x_t)$ by j^t. We can assume that j^1 is the minimum number in all j^t. Then we have the following.

- $j^t = i^t$ for $t \geq 1$,
- $j^t + 1 \geq j^{t+1}$ and $j^t \neq j^{t+1}$ for $t \geq 1$,
- $j^2 = j^1 + 1$.

Let $y^1 = (y_0, \cdots, y_k)$, then x^1, y^2, x^2 are as follows.

$x^1 = (y_0, \cdots, y_{i^1}, y_{i^1+2}, \cdots, y_k)$,
$y^2 = (y_0, \cdots, y_{i^1}, z_1, y_{i^1+2}, \cdots, y_k)$, where z_1 satisfies $(y_{i^1}, z_1, y_{i^1+2}) \in \mathcal{S}$ in the case of $i^1 = 0$ or $d(y_{i^1-1}, y_{i^1}) \geq 2$, and $z_1 = y_{i^1-1}$ in the case of $d(y_{i^1-1}, y_{i^1}) = 1$,
$x^2 = (y_0, \cdots, y_{i^1}, z_1, y_{i^1+3}, \cdots, y_k)$.

Moreover $y^3 = (y_0, \cdots, y_{i^1}, z_1, y_{i^1}, y_{i^1+3}, \cdots, y_k)$ since $d(y_{i^1}, z_1) = 1$.
For $y^t = (y^t_0, \cdots, y^t_k) \in P''$ $(t \geq 3)$, we prove $d(y^t_{i^1}, y^t_{i^1+1}) = 1$, $y^t_{i^1} = y^t_{i^1+2}$ by induction.

(I) In the case of $t = 3$, $d(y^3_{i^1}, y^3_{i^1+1}) = d(y_{i^1}, z_1) = 1$, $y^3_{i^1} = y_{i^1} = y^3_{i^1+2}$.

(II) For y^t ($t \geq 3$), we assume $d(y^t_{i^1}, y^t_{i^1+1}) = 1$, $y^t_{i^1} = y^t_{i^1+2}$. Then we have $i^t \neq i^1$. Therefore $i^t \geq i^1 + 1$.

(i) In the case of $i^t = i^1 + 1$, for $y^{t+1} = (y^t_0, \cdots, y^t_{i^1}, y^t_{i^1+1}, z_t, y^t_{i^1+3}, \cdots, y^t_k)$, $y^{t+1}_{i^1+2} = z_t = y^t_{i^1} = y^{t+1}_{i^1}$ holds because of $d(y^{t+1}_{i^1}, y^{t+1}_{i^1+1}) = d(y^t_{i^1}, y^t_{i^1+1}) = 1$.

(ii) In the case of $i^t \geq i^1+2$, for $y^{t+1} = (y^t_0, \cdots, y^t_{i^1}, y^t_{i^1+1}, y^t_{i^1+2}, \cdots, z_t, \cdots, y^t_k)$, we have $d(y^{t+1}_{i^1}, y^{t+1}_{i^1+1}) = 1$ and $y^{t+1}_{i^1} = y^t_{i^1} = y^t_{i^1+2} = y^{t+1}_{i^1+2}$.

By (I) and (II), we have $y^t_{i^1} = y^t_{i^1+2}$ ($t \geq 3$). Since $y^t_{i^1+1}$ is singular in y^t, $j^t = i^t \neq i^1 = j^1$ ($i \geq 3$). By $j^2 \neq j^1$, we obtain $j^t \neq j^1$ ($t \leq 2$). On the other hands, $j^1 = i^1 = i^{p+1} = j^{p+1}$ by $y^1 = y^{p+1}$. It contradicts. Now we have that the matching M is acyclic. Remark that the set of critical elements is A.

For any $x \in A$, let us prove that x has no smooth point. First x satisfies $i(x) = j(x) = \infty$. It means $x = (x_0, \cdots, x_k)$ satisfies as follows.

(1) There does not exist $i \in \{1, \cdots, k-1\}$ such that $d(x_{i-1}, x_i) = 1$ and $x_{i-1} \prec x_i \prec x_{i+1}$.

(2) $d(x_0, x_1) = 1$.

(3) There does not exist $j \in \{2, \cdots, k-1\}$ such that $d(x_{j-1}, x_j) \geq 2$ and $d(x_j, x_{j+1}) \geq 2$.

(4) There does not exist $j \in \{2, \cdots, k-1\}$ such that $d(x_{j-1}, x_j) = 1$, $d(x_j, x_{j+1}) \geq 2$ and $x_j \prec x_{j-1} \prec x_{j+1}$.

Assume that x has a smooth point x_n ($n \in \{1, \cdots, k-1)\}$. By (1), we have $d(x_{n-1}, x_n) \geq 2$. Then $n \geq 2$ by (2). Furthermore, $d(x_{n-2}, x_{n-1}) = 1$ (otherwise it contradicts (3)). If x_{n-1} is smooth in x, then $(x_{n-2}, x_{n-1}, x_n) \in \mathcal{S}$. Therefore, x_{n-1} is singular in x. It means $x_{n-1} \prec x_{n-2} \prec x_n$. We know $d(x_{n-2}, x_{n-1}) = 1$ and $d(x_{n-1}, x_n) \geq 2$. It contradicts (4). Therefore, x has no smooth point. It means that the any subsimplex which consists boundaries of every critical element is contained in $K'(G; a, b)$.

We consider the matching M on P as the matching on $K_\ell(G; a, b)$, then the set of critical elements is $A \sqcup K'_\ell(G; a, b)$. By Theorem 1.2.4, $K_\ell(G; a, b)$ is homotopy equivalent to the CW complex which is constructed some cells attaching $K'_\ell(G; a, b)$. Therefore the Asao-Izumihara complex $K_\ell(G; a, b)/K'_\ell(G; a, b)$ is (empty or contractible or) homotopy equivalent to wedge of spheres with various dimensions. $\square$

Proof of Example 2.2.5(1). Let G be a tree. Let $\ell \geq 3$ and $a, b \in V(G)$. First we describe the Asao-Izumihara complexes for trees. As same as the proof of Theorem 4.0.3, we construct a matching M on $P := K_\ell(G; a, b) \setminus K'_\ell(G; a, b)$. Define the set $\mathcal{S}$, A, P', P'' and the map f, g by all the same as the proof of Theorem 4.0.3. Then we have the bijectivity of f by the same reason, and we can define a partial matching $M \subseteq P \times P$ as same. In the case of trees, we can show that the matching M is acyclic in exactly the same way as in the proof of Theorem 4.0.3. All critical elements are contained in A, and satisfy the condition (1), (2), (3) and (4) of the proof of Theorem 4.0.3. We can easily check that they have no smooth point. It means that the any subsimplex which consists the boundaries of every critical element is contained in $K'(G; a, b)$. Similar to the last part of the proof of Theorem 4.0.3, the Asao-Izumihara complex $K_\ell(G; a, b)/K'_\ell(G; a, b)$ is homotopy equivalent to wedge of spheres. Moreover, for any critical element $x = (x_0, \cdots, x_k) \in A$, we can check there does not exist $j \in \{0, \cdots, k-1\}$ such that $d(x_j, x_{j+1}) \geq 2$. If $(\alpha, \beta, \gamma) \in V(G)^3$ with $d(\alpha, \beta) = d(\beta, \gamma) = 1$ satisfies $\alpha \neq \gamma$, then β smooth in (α, β, γ). Therefore, any critical element $x = (x_0, \cdots, x_k) \in A$ satisfies

(i) $d(x_i, x_{i+1}) = 1$ for any i, and

(ii) $x_i = x_{i+2}$ for any i.

By (i), the Asao-Izumihara complex is consisted of only $(\ell - 2)$ dimensional spheres. By (i) and (ii), the number of spheres is equal to $2|E(G)|$. By Theorem 2.2.8, the magnitude homotopy type $\mathcal{M}_\ell(G; a, b)$ is homotopy equivalent to the double suspension of Asao-Izumihara complex, then we have the result for $\ell \geq 3$. The result for $0 \leq \ell \leq 2$ can be easily checked. □

Remark 4.0.4. For a even cycle graph, we can not construct an acycle matching by the same way for a odd cycle graph since there exist pairs of vertices which have more than one shortest paths.

Chapter 5

Discrete Morse theory on magnitude homotopy type

5.1 Projecting matching

Definition 5.1.1 (Induced subgraph)**.** Let G be a graph. An induced subgraph H is a graph with the vertex set $V(H) \subseteq V(G)$, and which satisfies $\{x, y\} \in E(H)$ if and only if $\{x, y\} \in E(G)$ for $x, y \in V(H)$. In this paper, we call a induced subgraph simply a subgraph.

Definition 5.1.2. Let G be a graph, and $H \subset G$ be a subgraph. We say that $x \in G$ projects to H if there exists $\pi(x) \in H$ such that $d_G(x, y) = d_G(x, \pi(x)) + d_G(\pi(x), y)$ for any $y \in H$.

Definition 5.1.3 (Convex subgraph)**.** Let G be a graph, and $H \subset G$ be a subgraph. If H satisfies the following, then we say H is convex.

- For any $x, y \in H$, $d_H(x, y) = d_G(x, y)$.

Through this section, our setting is as follows. Let G, H and K be conected graphs. Assume that there exists an isomorphism $i_G \colon K \to i_G(K) \subset G$ of K to an induced subgraph $i_G(K) \subset G$, and $i_G(K)$ is convex in G (similarly for H). Define a new graph $X = G \cup H$ by $G \sqcup H$ identified $i_G(k)$ and $i_H(k)$ for each $k \in K$. Denote by simply K or $G \cap H$ the subgraph $i_G(K)(= i_H(K)) \subset G \cup H$.

Definition 5.1.4. (Biased point) Define the set

$$H_* = \{y \in H \setminus K \mid y \text{ projects to } K\}.$$

We call an element of H_* a biased point. We denote the set $H_0 := H \setminus (K \cup H_*)$.

Let $\ell \geq 0$ and $a, b \in X$.

Definition 5.1.5. Let $x = (x_0, \cdots, x_k) \in \Delta_\ell(X; a, b)$.

(i) We say that x is flat if $x_0, \cdots, x_k \in G \cup H_0$ or $x_0, \cdots, x_k \in H$.

(ii) We say that $(x_i, x_{i+1}, \cdots, x_j)$ is sticky if $x_i \in \mathrm{int}(G)$ and $x_j \in H_*$, or vise versa, and $x_{i+1}, x_{i+2}, \cdots, x_{j-1} \in K$. For sticky subsequence $(x_i, x_{i+1}, \cdots, x_j)$, we say it is fillable if $x_{i+1} \neq \pi(x_i)$ $(x_i \in H_*)$ or $x_{j-1} \neq \pi(x_j)$ $(x_j \in H_*)$. Otherwise, we say it is removable.

(iii) We say x is twistable if there does not exist $i \in \{0, \cdots, k-1\}$ and $j \in \{i+1, \cdots, k\}$ such that $(x_i, x_{i+1}, \cdots, x_j)$ is sticky.

Definition 5.1.6. Let $x = (x_0, \cdots, x_k)$, $y = (y_0, \cdots, y_n) \in \Delta_\ell(X; a, b)$. Let $x' = (x_i, x_{i+1}, \cdots, x_j)$ be a subsequence of x, and $y' = (y_p, y_{p+1}, \cdots, y_q)$ be a subsequence of y. If $x_j = y_p$, define $x' * y' := (x_i, x_{i+1}, \cdots, x_j, y_{p+1}, \cdots, y_q)$.

Proposition 5.1.7 ([14], Proposition 5.9)**.** Let G, H, X, K be as above. Assume that K is convex in X, and H projects to K. Let $\ell \geq 0$, $a, b \in X$ and $x = (x_0, \cdots, x_k) \in \Delta_\ell(X; a, b)$. Then the following are equivalent.

(i) x is twistable.

(ii) x can be decomposed $x = x_1 * x_2 * \cdots * x_m$, where each x_i is flat and each point of concatenation is contained in H_0.

Proof. We prove (i) $\Longrightarrow$ (ii). Assume that x is twistable. If x is flat, x satisfies (ii). Let x be not flat, and the maximum $i \in \{1, 2, \cdots, k-1\}$ such that $(x_0, \cdots, x_i)$ is flat.

- If $x_0, \cdots, x_i \in G \cup H_0$, then $x_{i+1} \in H_*$ and $x_i \in H_0 \cup K$. If $x_i \in H_0$, then x is decomposed at x_i such that $(x_0, \cdots, x_i)$ is flat. Assume that $x_i \in K$. Take $j \in \{0, \cdots, i-1\}$ such that $x_j \in (G \cup H_0) \setminus K = \mathrm{int}(G) \cup H_0$, $x_{j+1}, \cdots, x_i \in K$. Since $(x_j, x_{j+1}, \cdots, x_i, x_{i+1})$ is not sticky, we have $x_j \in H_0$. Hence, x has a decomposition $x = (x_0, \cdots, x_j) * (x_j, \cdots, x_k)$ such that $(x_0, \cdots, x_j)$ is flat and the point of concatenation x_j is in H_0.

- If $x_0, \cdots, x_i \in H$, then $x_{i+1} \in \mathrm{int}(G)$ and $x_i \in H_0 \cup K$. We may assume that $x_i \in K$. Take $j \in \{0, \cdots, i-1\}$ such that $x_j \in \mathrm{int}(H)$, $x_{j+1}, \cdots, x_i \in K$. Since $(x_j, x_{j+1}, \cdots, x_i, x_{i+1})$ is not sticky, we have $x_j \in H_0$. Hence, x has a decomposition $x = (x_0, \cdots, x_j) * (x_j, \cdots, x_k)$ such that $(x_0, \cdots, x_j)$ is flat and the point of concatenation x_j is in H_0.

In both cases, we continue decomposition for $(x_j, \cdots, x_k)$, then we obtain the decomposition for x such as (ii). The converse (ii) $\Longrightarrow$ (i) is straightforward $\square$

Definition 5.1.8 (Projecting matching)**.** For $x = (x_0, \cdots, x_k) \in \Delta_\ell(X; a, b) \setminus \Delta'_\ell(X; a, b)$, we denote by $i(x)$ the minimum $i \in \{0, \cdots, k-1\}$ such that $(x_i, x_{i+1}, \cdots, x_j)$ $(j \geq i + 1)$ is sticky, and this j is denoted by $j(x)$. Then we define the projecting matching as follows.

(i) If $x_{i(x)} \in H_*$ and $x_{i(x)+1} \neq \pi(x_{i(x)})$, then

$$x \vdash (x_0, \cdots, x_{i(x)}, \pi(x_{i(x)}), x_{i(x)+1}, \cdots, x_k).$$

(ii) If $x_{i(x)} \in H_*$ and $x_{i(x)+1} = \pi(x_{i(x)})$, then

$$x \dashv (x_0, \cdots, \widehat{x_{i(x)+1}}, \cdots, x_k).$$

(iii) If $x_{j(x)} \in H_*$ and $x_{j(x)-1} \neq \pi(x_{j(x)})$, then

$$x \vdash (x_0, \cdots, x_{j(x)-1}, \pi(x_{j(x)}), x_{j(x)}, \cdots, x_k).$$

(iv) If $x_{j(x)} \in H_*$ and $x_{j(x)-1} = \pi(x_{j(x)})$, then

$$x \dashv (x_0, \cdots, \widehat{x_{j(x)-1}}, \cdots, x_k).$$

Proposition 5.1.9 ([17], Proposition 5.9.)**.** The projecting matching is acyclic.

Proof. Let $x = (x_0, \cdots, x_k) \in \Delta_\ell(X; a, b) \setminus \Delta'_\ell(X; a, b)$. A sequence x can be decomposed as

$$x = w_1 * w_2 * \cdots * w_m, \tag{5.1.1}$$

where each w_i is flat or sticky. Such a decomposition is constructed as follows. First, we pick up all sticky subsequence, then x is decomposed into sticky subsequences and other subsequences which do not have sticky subsequence. By Proposition 5.1.7, we can decompose non sticky parts into flat subsequence such that concatenations of flat subsequences are contained in H_0. Note that concatenations of a sticky subsequence and a flat subsequence in (5.1.1) are not contained in K since endpoints of sticky subsequence are containd in $H_* \cup \mathrm{int}(G)$.

We prove the matching is acyclic. Assume that there exists a cycle such that

$$y^1 \succ x^1 \prec y^2 \succ x^2 \prec \cdots \prec y^p \succ x^p \prec y^{p+1} = y^1$$

with $p \geq 2$, that satisfies $x^i \vdash y^{i+1}$ for each $i \in \{1, 2, \cdots, p\}$ and $y^i \neq y^j$ $(i \neq j)$ for every $i, j \in \{1, 2, \cdots, p\}$. For $x \in \Delta_\ell(X; a, b) \setminus \Delta'_\ell(X; a, b)$, we denote the number of points in $\mathrm{int}(G)$ by $|x|_G$, and as same in $\mathrm{int}(H)$ by $|x|_H$. We have $|y^1|_G \geq |x^1|_G = |y^2|_G \geq |x^2|_G = \cdots \geq |x^p|_G = |y^{p+1}|_G = |y^1|_G$, then it must be all equal. Similary, $|y^1|_H = |x^1|_H = |y^2|_H = |x^2|_H = \cdots = |x^p|_H = |y^{p+1}|_H = |y^1|_H$. Let $y^1 = (y_0, \cdots, y_k)$. Let y_{α_1} be a removal point such that $x_1 = (y_0, \cdots, \widehat{y_{\alpha_1}}, \cdots, y_k)$, then $y_{\alpha_1} \in K$. A decomposition of y such as (5.1.1) is denoted by

$$y = w_1 * w_2 * \cdots * w_m$$

with the first sticky subsequence $w_i = (y_q, y_{q+1}, \cdots, y_r)$. We can assume $y_q \in H_*$. By definition of projecting matching, $y_{\alpha_1} = y_{q+1}$. To remove a point in K does not bring out a new stickey subsequence. Then the first sticky subsequence of x^1 is $w'_i = (y_q, y_{q+2}, \cdots, y_r)$ and it is fillable. Therefore, $y^2 = y^1$ since $y^2 \dashv x^1$. It contradicts $p \geq 2$. $\square$

Definition 5.1.10. Denote by $T_\ell(X; a, b)$ the set of critical elements of projecting matching M on $\Delta_\ell(X; a, b) \setminus \Delta'_\ell(X; a, b)$, and define $T_\ell(X) := \sqcup_{a,b \in X} T_\ell(X; a, b)$.

Remark 5.1.11. Clearly,

$$T_\ell(X; a, b) = \{x \in \Delta_\ell(X; a, b) \setminus \Delta'_\ell(X; a, b) \mid x \text{ is twistable } \}.$$

5.2 Mayer-Vietoris type theorem

In this subsection, the setting is same as §5.1. Let us define the condition$(*)$ by

$(*)$ the graph H projects to $G \cap H$.

Theorem 5.2.1 ([17], Corollary 5.16.(1)). *Assume that the graph $X = G \cup H$ satisfies the condition$(*)$. Then,*

$$\mathcal{M}_\ell(G \cap H) \vee \mathcal{M}_\ell(X) \simeq \mathcal{M}_\ell(G) \vee \mathcal{M}_\ell(H).$$

Before the proof of Theorem 5.2.1, we need some preparation. Let $\ell \geq 0$ and $a, b \in X$. Define the subsets $S(G)$, $S(H)$, $S(G \cap H) \subset \Delta_\ell(X; a, b)$ as follows.

$$\begin{aligned} S(G) &:= \{(x_0, \cdots, x_k) \in \Delta_\ell(X; a, b) \mid x_0, \cdots, x_k \in G\}, \\ S(H) &:= \{(x_0, \cdots, x_k) \in \Delta_\ell(X; a, b) \mid x_0, \cdots, x_k \in H\}, \\ S(G \cap H) &:= \{(x_0, \cdots, x_k) \in \Delta_\ell(X; a, b) \mid x_0, \cdots, x_k \in G \cap H\}. \end{aligned}$$

Note that $S(G) \subset \Delta_\ell(X;a,b)$ is not equal to $\Delta_\ell(G;a,b)$. The inclusion $S(G) \supset \Delta_\ell(G;a,b)$ is true, however the converse $S(G) \subset \Delta_\ell(G;a,b)$ is not true. In fact, for $x \in S(G)$ with a path $p(x) \in P_\ell(X;a,b)$ such that $x \prec p(x)$, there does not necessarily exist a path $p'(x) \in P_\ell(G;a,b)$ such that $x \prec p'(x)$. Similarly for $S(H)$ and $S(G \cap H)$.

Moreover, we define the subset $S'(H) \subset S(H)$ by

$$S'(H) := \left\{ x = (x_0, \cdots, x_k) \in S(H) \;\middle|\; \begin{array}{l} x \in \Delta'_\ell(X;a,b) \text{ or there exists } i \in \{0, \cdots, k\} \\ \text{such that } x_i \in \mathrm{int}(H) \end{array} \right\}.$$

Lemma 5.2.2. *For any $a, b \in G$ and $x \in \mathrm{int}(H)$,*

$$d_X(a,b) < d_X(a,x) + d_X(x,b).$$

Proof. In this proof, the distance d means d_X. First we prove $g \prec \pi(h) \prec h$ for any $g \in G$ and $h \in \mathrm{int}(H)$. In the case of $g \in G \cap H$, it holds. Let $g \in \mathrm{int}(G)$. If $\{g,h\} \in E(X)$, then $\{g,h\} \in E(G)$ or $\{g,h\} \in E(H)$. However, since $g \in \mathrm{int}(G)$ and $h \in \mathrm{int}(H)$, then $\{g,h\} \notin E(X)$. Hence $d(g,h) \geq 2$. Let $(g, x_1, \cdots, x_k, h)$ be a shortest path on X. There exist $i \in \{1, \cdots, k\}$ such that $x_i \in G \cap H$. We have $d(g,h) = d(g,x_i) + d(x_i,h) = d(g,x_i) + d(x_i,\pi(h)) + d(\pi(h),h)$. Hence we have $g \prec \pi(h) \prec h$ for any $g \in G$ and $h \in \mathrm{int}(H)$. Therefore, for any $a, b \in G$ and $x \in \mathrm{int}(H)$, $d(a,x) + d(x,b) = d(a,\pi(x)) + d(\pi(x),x) + d(x,\pi(x)) + d(\pi(x),b) \geq d(a,\pi(x)) + d(\pi(x),b) + 2 \geq d(a,b) + 2 > d(a,b)$. □

Lemma 5.2.3. *The subset $S'(H) \subset S(H)$ is the subcomplex of $\Delta_\ell(X;a,b)$.*

Proof. Let $x = (x_0, \cdots, x_k) \in S'(H)$.

(I) In the case of $L(x) < \ell$, $x \in \Delta'_\ell(X;a,b)$. For any subsimplex $x' \subset x$, $x' \in \Delta'_\ell(X;a,b)$ holds since $\Delta'_\ell(X;a,b)$ is subcomplex of $\Delta_\ell(X;a,b)$.

(II) In the case of $L(x) = \ell$, there exists $i \in \{0, \cdots, k\}$ such that $x_i \in \mathrm{int}(H)$. Denote a subsimplex of x which obtained by removig x_i from x by x'. By Lemma 5.2.2, $L(x') < \ell$. Any subsimplex $y \subset x$ such that $L(y) = \ell$ has a point contained in $\mathrm{int}(H)$, we have $y \in S'(H)$. This completes the proof.

□

Definition 5.2.4. Define the subset $S(G,H) \subset \Delta_\ell(X;a,b)$ by

$$S(G,H) := \left\{ x = (x_0, \cdots, x_k) \in \Delta_\ell(X;a,b) \;\middle|\; \begin{array}{l} L(x) = \ell \text{ and, there exists } i, j \in \{0, \cdots, k\} \\ \text{such that } x_i \in \mathrm{int}(G) \text{ and } x_j \in \mathrm{int}(H) \end{array} \right\}.$$

Remark 5.2.5. As sets of simplices, we have

$$\Delta_\ell(X;a,b) = (\Delta'_\ell(X;a,b) \cup S(G) \cup S(H)) \sqcup S(G,H).$$

Lemma 5.2.6. *For the subsets of $\Delta_\ell(X;a,b)$, we have the following.*

(i) $S(G \cap H) \cap S'(H) \subset \Delta'_\ell(X;a,b)$.

(ii) $S(G) \cap S'(H) \subset \Delta'_\ell(X;a,b)$.

(iii) $S(G \cap H) \cup S'(H) = S(H)$.

(iv) $S(G) \cup S(H) = S(G) \cup S'(H)$.

Proof. (i) Let $x = (x_0, \cdots, x_k) \in S(G \cap H) \cap S'(H) \subset \Delta_\ell(X;a,b)$. Since $x_0, \cdots, x_k \in G \cap H$, x does not have a point in $\mathrm{int}(H)$. Moreover, since $x \in S'(H)$, we have $x \in \Delta'_\ell(X;a,b)$.

(ii) Clearly, $S(G \cap H) = S(G) \cap S(H)$ as sets. By (i), $S(G \cap H) \cap S'(H) = S(G) \cap S(H) \cap S'(H) = S(G) \cap S'(H) \subset \Delta'_\ell(X;a,b)$.

(iii) We prove $S(G \cap H) \cup S'(H) \supseteq S(H)$. Let $x = (x_0, \cdots, x_k) \in S(H)$. If $x \in \Delta'_\ell(X;a,b)$, then $s \in S'(H)$. Consider in the case of $x \notin \Delta'_\ell(X;a,b)$. If x has a point of $\mathrm{int}(H)$, we have $x \in S'(H)$. Otherwise, we have $x \in S(G \cap H)$. The converse inclusion is straightforward.

(iv) By (iii), $S(G) \cup S(H) = S(G) \cup S(G \cap H) \cup S'(H) = S(G) \cup S'(H)$.

□

Proposition 5.2.7. We have the following homotopy equivalence

$$\Delta_\ell(X;a,b) \simeq \Delta'_\ell(X;a,b) \cup S(G) \cup S(H).$$

Moreover there exists a strong deformation retract from left-hand side to right-hand side.

Proof. Consider the projecting matching M on $\Delta_\ell(X;a,b) \setminus \Delta'_\ell(X;a,b)$. By Proposition 5.1.9, M is acyclic. Note that

$$S(G,H) = \{x \in \Delta_\ell(X;a,b) \setminus \Delta'_\ell(X;a,b) \mid x \text{ belongs to an element of } M\},$$

and

$$\Delta_\ell(X;a,b) = (\Delta'_\ell(X;a,b) \cup S(G) \cup S(H)) \sqcup S(G,H).$$

Since $\Delta'_\ell(X;a,b) \cup S(G) \cup S(H)$ is a subcomplex of $\Delta_\ell(X;a,b)$, by Theorem 1.2.4(a) and Remark 1.2.6, we have the result. □

Proposition 5.2.8. We have the following.

(i)
$$\frac{\Delta_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \simeq \frac{S(G)\cup S(H)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)}.$$

(ii)
$$\frac{S(G)\cup S(H)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \approx \frac{\Delta_\ell(G;a,b)}{\Delta'_\ell(G;a,b)} \vee \frac{S'(H)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)}.$$

(iii)
$$\frac{\Delta_\ell(H;a,b)}{\Delta'_\ell(H;a,b)} \approx \frac{\Delta_\ell(G\cap H;a,b)}{\Delta'_\ell(G\cap H;a,b)} \vee \frac{S'(H)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)}.$$

In general, we know the following.

Fact 5.2.9. (I) Let A be a topological space and $C \subset B \subset A$ be subspaces. Assume that there exists a strong deformation retract from A to B. Then it induces the strong deformation retract from A/C to B/C.

(II) Let A_1, A_2 be topological spaces, and $X = A_1 \cup A_2$, $B = A_1 \cap A_2$. Then $X/B \approx (A_1/B) \vee (A_2/B)$.

(III) Let $X = A \cup B$ be a topological space and A, $B \subset X$ be closed subspace. Then $A/(A\cap B) \approx X/B$.

Proof of Proposition 5.2.8. (i) Let $A = \Delta_\ell(X;a,b)$, $B = S(G)\cup S(H)\cup\Delta'_\ell(X;a,b)$ and $C = \Delta'_\ell(X;a,b)$. By Proposition 5.2.7, there exists a strong deformation retract from A to B. Then A, B and C satisfy the assumption of Fact 5.2.9 (I), we have the result.

(ii) First, let $A = \Delta_\ell(G;a,b)$ and $B = \Delta'_\ell(X;a,b)$, then $A\cap B = \Delta'_\ell(G;a,b)$. Since A, B and $X = A\cup B$ satisfy the condition of Fact 5.2.9 (III),

$$\frac{\Delta_\ell(G;a,b)}{\Delta'_\ell(G;a,b)} \approx \frac{\Delta_\ell(G;a,b)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} = (\star). \tag{5.2.1}$$

For $x \in S(G)$, if $x \notin \Delta_\ell(G;a,b)$, then $x \in \Delta'_\ell(X;a,b)$. Hence,

$$(\star) = \frac{S(G)\cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)}. \tag{5.2.2}$$

Next, let $A_1 = S(G) \cup \Delta'_\ell(X; a, b)$, $A_2 = S'(H) \cup \Delta'_\ell(X; a, b)$. Let $X = A_1 \cup A_2$ and $B = A_1 \cap A_2$. By Lemma 5.2.6 (iv),

$$X = S(G) \cup S'(H) \cup \Delta'_\ell(X; a, b) = S(G) \cup S(H) \cup \Delta'_\ell(X; a, b).$$

By Lemma 5.2.6 (ii), We have $B = \Delta'_\ell(X; a, b)$. By applying Fact 5.2.9 (II),

$$\frac{S(G) \cup S(H) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(X; a, b)} \approx \frac{S(G) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(G; a, b)} \vee \frac{S'(H) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(X; a, b))}.$$

Therefore, by (5.2.1), (5.2.2), this completes the proof.

(iii) First, we show $S(H) \setminus \Delta_\ell(H; a, b) \subseteq \Delta'_\ell(X; a, b)$. For any $x \in S(H)$, we prove x satisfies $x \in \Delta_\ell(H; a, b)$ or $x \in \Delta'_\ell(X; a, b)$. Let $x = (x_{i_0}, \cdots, x_{i_k}) \in S(H)$, then there exists a path $(a, x_1, \cdots, x_{\ell-1}, b) \in P_\ell(X; a, b)$ such that $x \prec (a, x_1, \cdots, x_{\ell-1}, b)$. Denote the path by $p(x)$. If $L(x) < \ell$, then $x \in \Delta'_\ell(X; a, b)$. We consider in the case of $L(x) = \ell$. Note that $x_{i_0} = a$ and $x_{i_k} = b$. If the path $p(x)$ has a subpath $(x_{i_m}, x_{i_m+1}, \cdots, x_{i_{m+1}})$ ($m \in \{0, \cdots, k\}$) such that the ends points $x_{i_m}, x_{i_{m+1}} \in G \cap H$ and others $x_{i_m+1}, \cdots, x_{i_{m+1}-1} \in \mathrm{int}(G)$. By $x \in S(H)$, $x_{i_m+1}, \cdots, x_{i_{m+1}-1}$ does not belong to x. By $L(x) = \ell$, we have $d_X(x_{i_m}, x_{i_{m+1}}) = i_{m+1} - i_m$. Since $G \cap H$ is convex in X, $d_{G\cap H}(x_{i_m}, x_{i_{m+1}}) = i_{m+1} - i_m$. It means there exists a path on $G \cap H$ from x_{i_m} to $x_{i_{m+1}}$ with length $i_{m+1} - i_m$. We replace th subpath $(x_{i_m}, x_{i_m+1}, \cdots, x_{i_{m+1}})$ of $p(x)$ with the path on $G \cap H$, and we continue such replacement then we obtain the path $\tilde{p}(x) \in P_\ell(G \cap H; a, b)$ such that $x \prec \tilde{p}(x)$. Hence $x \in \Delta_\ell(H; a, b)$. Now we have

$$S(H) \setminus \Delta_\ell(H; a, b) \subseteq \Delta'_\ell(X; a, b). \tag{5.2.3}$$

Let $A_1 = S(G \cap H) \cup \Delta'_\ell(X; a, b)$ and $A_2 = S'(H) \cup \Delta'_\ell(X; a, b)$. Let $B = A_1 \cap A_2$ and $X = A_1 \cup A_2$. Then, $B = \Delta'_\ell(X; a, b)$ and

$$\begin{aligned} X &= S(G \cap H) \cup S'(H) \cup \Delta'_\ell(X; a, b) \\ &= S(H) \cup \Delta'_\ell(X; a, b) \\ &= \Delta_\ell(H; a, b) \cup \Delta'_\ell(X; a, b). \end{aligned}$$

The second equation holds by Lemma 5.2.6 (iii), and third equation holds by (5.2.3). Therefore, by Fact 5.2.9 (II),

$$\frac{\Delta_\ell(H; a, b) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(X; a, b)} \approx \frac{S(G \cap H) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(X; a, b)} \vee \frac{S'(H) \cup \Delta'_\ell(X; a, b)}{\Delta'_\ell(X; a, b)}. \tag{5.2.4}$$

By Fact 5.2.9 (III), the left-hand side of (5.2.4) is

$$\frac{\Delta_\ell(H;a,b) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \approx \frac{\Delta_\ell(H;a,b)}{\Delta'_\ell(H;a,b)}.$$

For the former part of right-hand side of (5.2.4), by (5.2.3) and Fact 5.2.9 (III),

$$\frac{S(G \cap H) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \approx \frac{\Delta_\ell(G \cap H;a,b) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \approx \frac{\Delta_\ell(G \cap H;a,b)}{\Delta'_\ell(G \cap H;a,b)}.$$

This completes the proof.

□

Proof of Theorem 5.2.1. By Proposition 5.2.8 (i) and (ii),

$$\mathcal{M}_\ell(X;a,b) \simeq \mathcal{M}_\ell(G;a,b) \vee \frac{S'(H) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)}.$$

Moreover, by Proposition 5.2.8 (iii), we have

$$\begin{aligned}\mathcal{M}_\ell(G \cap H;a,b) \vee \mathcal{M}_\ell(X;a,b) \vee \frac{S'(H) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)} \\ \simeq \mathcal{M}_\ell(G;a,b) \vee \mathcal{M}_\ell(H;a,b) \vee \frac{S'(H) \cup \Delta'_\ell(X;a,b)}{\Delta'_\ell(X;a,b)},\end{aligned}$$

for any $a, b \in X$. This completes the proof. □

Corollary 5.2.10 ([8], Theorem 6.6)**.** *Assume that the graph $X = G \cup H$ satisfies the condition$(*)$. Then, there exists the following short split exact sequence.*

$$0 \to \mathrm{MH}^\ell_k(G \cap H) \to \mathrm{MH}^\ell_k(G) \oplus \mathrm{MH}^\ell_k(H) \to \mathrm{MH}^\ell_k(X) \to 0.$$

5.3 Sycamore twist

In this subsection, the setting is also the same as §5.1. Denote the set of biased points by

$$H_* := \{x \in H \setminus i_H(K) \mid x \text{ projects to } i_H(K)\},$$

and the set of non-biased points by

$$H_0 := H \setminus (i_H(K) \sqcup H_*).$$

Definition 5.3.1 (Sycamore twist)**.** Let $\alpha\colon K \to K$ be a isometry. Assume that

$$d_H(h,k) = d_H(h,\alpha(k)), \tag{5.3.1}$$

for any $h \in H_0$ and $k \in K$. Define a new graph X by $G \sqcup H$ identified $i_G(k)$ and $i_H(k)$ for each $k \in K$. Another new graph Y is defined by $G \sqcup H$ identified $i_G(k)$ and $i_H(\alpha(k))$ for each $k \in K$. Then, X and Y are called graphs differ by a sycamore twist.

Remark 5.3.2. Under the above setting, note that H projects to $i_H(K)$ if and only if $H_0 = \emptyset$. Therefore, if $H_0 = \emptyset$, then (G,H,X,K) and (G,H,Y,K) satisfy the condition$(*)$ in §5.2 respectively. Hence, by Theorem 5.2.1, $\mathcal{M}_\ell(X) \simeq \mathcal{M}_\ell(Y)$ for $\ell \geq 0$.

Example 5.3.3 (Whitney twist)**.** Let G and H be graphs. Let g_+, g_- be vertices of G, and h_+, h_- be vertices of H. A new graph X is defined by $G \sqcup H$ identified $g_+ \sim h_+$ and $g_- \sim h_-$. Define another new graph Y by $G \sqcup H$ identified $g_+ \sim h_-$ and $g_- \sim h_+$. Then, X and Y are called graphs differ by a Whitney twist, and it is a special case of a sycamore twist. In [12], it is proved that the magnitudes of X and Y coincide under the assumption that $\{g_+, g_-\} \in E(G)$ and $\{h_+, h_-\} \in E(H)$.

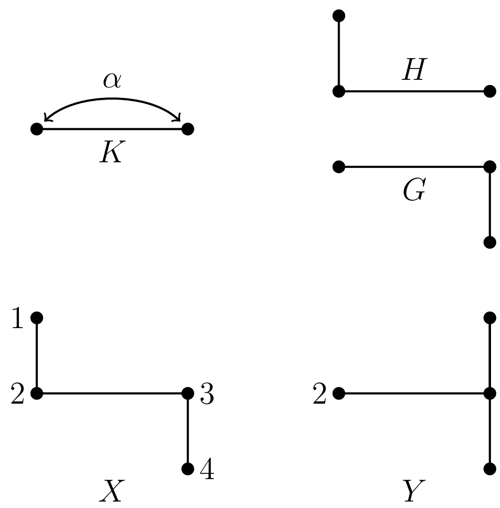
Theorem 5.3.4 ([17], Theorem 5.20.)**.** *Let X and Y be graphs differ by a sycamore twist. Then, there exists a bijection between $T_\ell(X)$ and $T_\ell(Y)$ which preserves the dimensions of critical elements.*

Proof. We can prove in a similar manner as [14, Proposition 5.6]. First let us define maps $\tau_G, \tau_H : X \longrightarrow Y$ as follows (see also Figure 5.1). (Note that, here, X and Y are defined as $G \sqcup H/\sim$, where $\sim$ is certain equivalence relation. Therefore, any point in X can be expressed as $\overline{x}$ with $x \in G \sqcup H$. One can easily check the following is well-defined on $i_G(K) \sqcup i_H(K)$.)

$$\tau_G(\overline{x}) = \begin{cases} \overline{x}, \text{ if } x \in G \smallsetminus i_G(K), \\ \overline{x}, \text{ if } x \in i_G(K), \\ \overline{x}, \text{ if } x \in H \smallsetminus i_H(K), \\ \overline{i_H(\alpha(i_H^{-1}(x)))}, \text{ if } x \in i_H(K), \end{cases} \qquad \tau_H(\overline{x}) = \begin{cases} \overline{x}, \text{ if } x \in G \smallsetminus i_G(K), \\ \overline{i_G(\alpha^{-1}(i_G^{-1}(x)))}, \text{ if } x \in i_G(K), \\ \overline{x}, \text{ if } x \in H \smallsetminus i_H(K), \\ \overline{x}, \text{ if } x \in i_H(K). \end{cases} \tag{5.3.2}$$

By Proposition 5.1.7, $x \in T_\ell(X)$ can be expressed as a concatenation

$$x = x_1 * x_2 * \cdots * x_m, \tag{5.3.3}$$



$\tau_G(1) = \tau_H(1) = 1$

$\tau_G(2) = 2, \tau_G(3) = 3, \tau_H(2) = 3, \tau_H(3) = 2$

$\tau_G(4) = \tau_H(4) = 4$

Figure 5.1: The maps τ_G and τ_H [17, Figure 13.].

of flat sequences x_i such that each point of concatenation is contained in H_0. If x_i is contained in $G \cup H_0$, then by the assumption (5.3.1) of the sycamore twist, $\tau_G(x_i)$ has the same length with x_i. Define $\tau(x_i)$ by

$$\tau(x_i) := \begin{cases} \tau_G(x_i) & (x \subset G \cup H_0), \\ \tau_H(x_i) & (\text{otherwise}), \end{cases} \tag{5.3.4}$$

and

$$\tau(x) := \tau(x_1) * \cdots * \tau(x_m). \tag{5.3.5}$$

This gives a desired bijection $T_\ell(X) \longrightarrow T_\ell(Y)$. $\square$

Corollary 5.3.5 ([14], Theorem 6.5)**.** *Let X and Y be graphs differ by a sycamore twist. Then, the magnitudes of X and Y coincide.*

Proof. By Theorem 1.1.5,

$$\begin{aligned} \#X &= \sum_{\ell \geq 0} \left(\sum_{k \geq 0} (-1)^k \operatorname{rank} \mathrm{MH}_k^\ell(X) \right) q^\ell \\ &= \sum_{\ell \geq 0} \left(\sum_{k \geq 0} (-1)^k \operatorname{rank} \mathrm{MC}_k^\ell(X) \right) q^\ell. \end{aligned} \tag{5.3.6}$$

By the definitions of $\mathrm{MC}_k^\ell(X)$, $\Delta_\ell(X;a,b)$ and $\Delta'_\ell(X;a,b)$,

$$\operatorname{rank}\mathrm{MC}_k^\ell(X) = \sum_{a,b\in X} |\Delta_{k,\ell}(X;a,b)|, \tag{5.3.7}$$

where $\Delta_{k,\ell}(X;a,b)$ is the set of all k-simplices of $\Delta_\ell(X;a,b)\setminus\Delta'_\ell(X;a,b)$. For any $a,b\in\Delta_\ell(X;a,b)\setminus\Delta'_\ell(X;a,b)$ such that $a\vdash b$, we have $\dim(a)+1=\dim(b)$. Hence, by (5.3.6) and (5.3.7), we have

$$\begin{aligned}\#X &= \sum_{\ell\geq 0}\left(\sum_{k\geq 0}(-1)^k\left(\sum_{a,b\in X}|\Delta_{k,\ell}(X;a,b)|\right)\right)q^\ell\\ &= \sum_{\ell\geq 0}\left(\sum_{k\geq 0}(-1)^k|T_{k,\ell}(X)|\right)q^\ell,\end{aligned}$$

where $T_{k,\ell}(X)$ is the set of all k-simplices of $T_\ell(X)$. Similarly, we have

$$\#Y = \sum_{\ell\geq 0}\left(\sum_{k\geq 0}(-1)^k|T_{k,\ell}(Y)|\right)q^\ell.$$

Therefore, by Theorem 5.3.4, $\#X$ and $\#Y$ coincide. $\square$

Remark 5.3.6. Let $\{g_+,g_-\}\in E(G)$ and $\{h_+,h_-\}\in E(H)$, and X, Y be as in Example 5.3.3. Denote by K a subgraph $\{g_+,g_-\}$ of X (or Y). By Corollary 5.3.5, $\#X=\#Y$. It is still open whether magnitude homology groups are isomorphic or not. However, if either the following (i) or (ii) holds, then magnitude homotopy types are equivalent.

(i) $H_* := \{x\in H\setminus K \mid x \text{ projects to } K\}=\emptyset$, or

(ii) $H_0 := H\setminus(K\cup H_*)=\emptyset$.

In the case of (i), X and Y are isomorphic as graphs. In the case of (ii), H projects to K. Then, by Mayer-Vietoris type theorem (Corollary 5.2.10), $\mathrm{MH}_*^\ell(X)\cong\mathrm{MH}_*^\ell(Y)$ for $\ell\geq 0$. Moreover, by Theorem 5.2.1, $\mathcal{M}_\ell(X)\simeq\mathcal{M}_\ell(Y)$ for $\ell\geq 0$.

Example 5.3.7. Let X and Y be graphs differ by a Whitney twist such as Figure 5.2. Then X and Y satisfy neither condition (i) nor (ii) in Remark 5.3.6. We do not know whether $\mathrm{MH}_*^\ell(X)$ and $\mathrm{MH}_*^\ell(Y)$ are isomorphic or not.

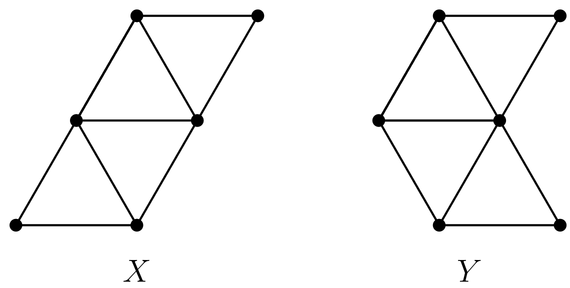


Figure 5.2: Graphs differ by a Whitney twist.

Example 5.3.8. Let X and Y be graphs differ by a Whitney twist such as Figure 5.3. Then X and Y satisfy neither condition (i) nor (ii) in Remark 5.3.6. However, in this case, we can show that $\mathrm{MH}^{\ell}_{*}(X) \cong \mathrm{MH}^{\ell}_{*}(Y)$ and $\mathcal{M}_{\ell}(X) \simeq \mathcal{M}_{\ell}(Y)$. In fact, X and Y have another decomposition (G', H') in Figure 5.3 such that G' projects to $G' \cap H'$. Then both of $\mathrm{MH}^{\ell}_{*}(X)$ and $\mathrm{MH}^{\ell}_{*}(Y)$ are determined only by $\mathrm{MH}^{\ell}_{*}(G')$, $\mathrm{MH}^{\ell}_{*}(H')$ and $\mathrm{MH}^{\ell}_{*}(G' \cap H')$. Hence, $\mathrm{MH}^{\ell}_{*}(X) \cong \mathrm{MH}^{\ell}_{*}(Y)$. Similary, we have $\mathcal{M}_{\ell}(X) \simeq \mathcal{M}_{\ell}(Y)$.

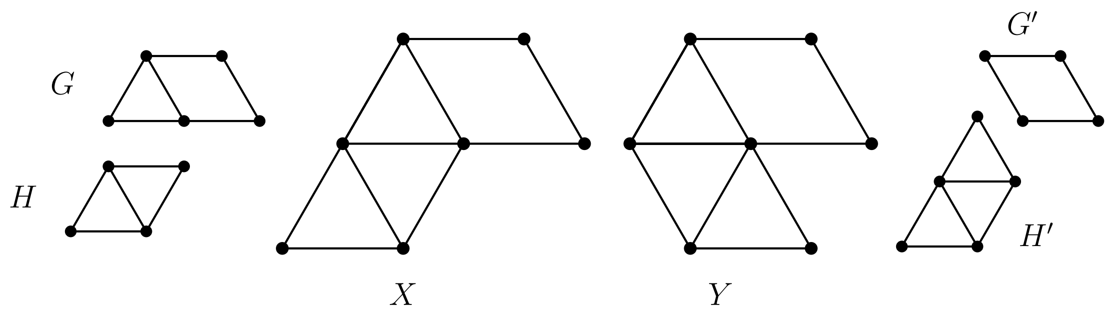


Figure 5.3: Graphs differ by a Whitney twist which have another decompositions.

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