Mechanics Research Communications.
Year

Publication Office:
Elsevier UK

Editor-in-Chief: A. Rosato New Jersey Institute of Technology, Newark, New Jersey, USA Anthony.Rosato@njit.edu

Summation Rules in Critical Self-buckling States of Cylinders

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Accepted:

Abstract

This study aimed to derive a critical height formula that would be uniformly applicable to hollow and solid cylinders, to clarify the effects of hollow cross-sections on self-buckling characteristics. The models used for our calculations took the cylindrical cantilevers with cavities of arbitrary radius. The governing equation was obtained from the equilibrium of forces. The eigenvalue problem obtained by applying appropriate boundary conditions to the general solution of the governing equation led to the derivation of a critical height formula. Furthermore, based on the derived theoretical formula, we found that the self-weight buckling properties of a hollow cylinder could be represented as a summation of the self-weight buckling properties of two solid cylinders, which can be derived more easily. The findings of this study have a wide range of applications, such as the simplification of complex buckling problems and the shape quantification of hollow plants.

Keywords: self-buckling, critical height, cylinders, summation rules, hollow plants

1. Introduction

Because of Earth's gravitational effect, it is necessary to understand the mechanical effects of gravity on artificial structures as accurately as possible and design them so that they are sufficiently resistant to the effects of gravity. However, when considering the mechanical behavior of structures, it is not uncommon to ignore the effect of the object weight (i.e., self-weight), a practice that is based on mechanically valid assumptions. A typical example is the general buckling problem of columns.

In the general buckling problem of columns, the effect of a column's own weight is often neglected because the effect of the compressive force on the top of the column is more dominant than the effects of self-weight [1,2]. However, in the case of tall and large columns, the effect of self-weight becomes too large to be ignored. In recent years, the use of tall columns in which the effect of self-weight is dominant has been increasing, particularly in civil engineering [3,4]. Therefore, in addition to the conventional buckling problems in which only concentrated loads are considered, the formulation of buckling problems that consider the effect of self-weight [5-18] is necessary.

The study of the buckling of a cylinder under its self-weight was initiated by Greenhill [5], whose theoretical analysis clarified that the critical height of a cylinder at which buckling under self-weight starts occurring is determined by the flexural rigidity and the weight of the cylinder. Moreover, Greenhill indicated that the theoretical critical height is proportional to 2/3 power of the radius and proposed that this power law might also apply to natural structures such as trees. The validity of this hypothesis was verified by McMahon, which confirmed that the power law is applicable to real trees [19]. Because of its simplicity, this scaling law has been applied widely in fields such as forest science and ecology [20-23]. Kanahama et al. [6] extended their study of the buckling problem from cylinders to cones, considering the tapered shape of trees, and derived an expression for the critical height for buckling under self-weight. In addition to the scaling laws for radius, which agree with those of Greenhill [5], a new scaling law linking the critical height with the taper of a cone was determined.

In previous studies of buckling under self-weight, it was claimed that the most effective ways to increase the critical height include increasing the bending stiffness and reducing the self-weight. Therefore, it is necessary for living organisms such as plants that require height for survival to reduce their self-weight as much as possible while ensuring high bending stiffness. However, for a solid cylinder of constant density, increasing the cross-sectional area to increase the flexural rigidity will necessarily increase the overall self-weight. Consequently, if the cross-sectional area is reduced to decrease the weight, the flexural rigidity will decrease. This conundrum may be solved at a low cost using a hollow cylindrical structure rather than a solid cylindrical structure to increase the self-

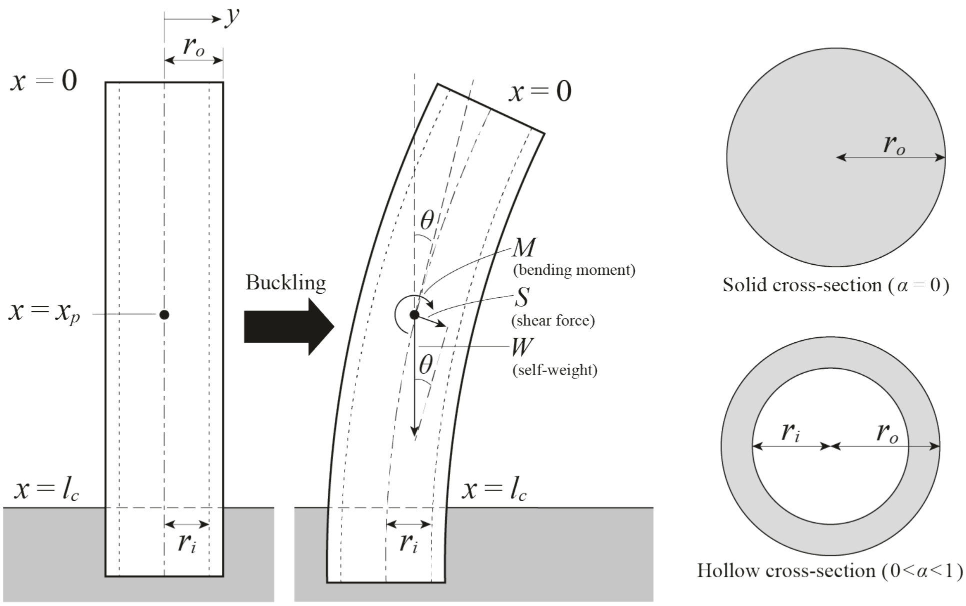


Fig. 1 Calculation model.

buckling resistance under self-weight while using a constant amount of material [24].

The cross-sections of many woody plants, such as trees, which have a large self-weight, are solid instead of hollow [25]. This is because plants that survive must be able to resist not only the burdens of self-weight but also various external forces such as wind, snow, and earthquakes in a well-balanced manner, and a plant with a hollow cross-section is less able to resist crushing and local buckling than a plant with a solid cross-section [26]. Nevertheless, plants with hollow cross-sections do exist, such as bamboo. By arranging the nodes at appropriate intervals along its height, the aforementioned weaknesses are ameliorated, and the bamboo can grow taller while composed of less material [27,28].

Thus, a hollow structure is very effective for achieving a greater critical height with respect to self-weight buckling, although it has some mechanical weaknesses. A theoretical investigation of the physical characteristics of the resistance to self-weight buckling of hollow structures would provide useful insights for the economical design of long columns and reveal mechanisms for mitigating the risk of self-weight buckling. Furthermore, in analogy with Greenhill's equation which has been widely applied to trees in the study of forest science, the derivation of a simple scaling law for bamboo, which is widely used as a structural material and in everyday commodities [29-32], will make a valuable contribution to forest science and ecology, the maintenance of bamboo forests, and the quantification of bamboo shapes.

This study aimed to derive a simple critical height formula for self-weight buckling that could be applied uniformly to solid and hollow cylindrical structures and to provide a theoretical clarification of the effect of hollow cross-sections on the self-weight buckling resistance of cylinders. The governing equation for a cantilever beam with a circular hollow cross-section was obtained from the equilibrium of forces at an arbitrary point, and the eigenvalue problem obtained by applying appropriate boundary conditions to the general solution was solved to obtain a simple critical height formula for self-weight buckling of hollow and solid cylindrical structures. Moreover, because there are more parameters to consider for a hollow cross-section than for a solid cross-section, the model for a hollow cross-section is more computationally complex. Therefore, by deriving various theoretical formulae linking the self-weight buckling characteristics of solid and hollow structures, we attempted to describe the self-weight buckling characteristics of hollow structures using solid structures, which are easy to formulate.

2. Critical height formula

2.1. Governing equation

The calculation model is the cantilever beam with a circular hollow cross-section depicted in **Fig. 1**, where the coordinate is along the neutral axis, with $x = 0$ at the free end and $x = l_c$ at the fixed end. We represent the outer radius as r_o and the inner radius as r_i, and we define the hollow ratio α as follows:

$$\alpha = \frac{r_i}{r_o}. \tag{1}$$

Therefore, the cross-section is solid when the hollow ratio $\alpha = 0$ and the volume of the cavity increases as α increases. Because the calculation model disappears when $\alpha = 1$, we define the range of values of the hollow ratio α as $0 \leq \alpha < 1$. Considering the equilibrium of forces at an arbitrary point x when this cylinder buckles under its self-weight, the shear force $S(x)$ is expressed as follows:

$$S(x) = \rho g A x \sin\theta = \gamma A x \sin\theta, \tag{2}$$

where ρ is density [kg/m^3], g is gravitational acceleration [m/s^2], γ is the unit volume weight [N/m^3], A is the cross-sectional area [m^2], and θ is the deflection angle. The bending moment $M(x)$ can be obtained from the elastic curve equation:

$$M(x) = -EI\frac{d^2y}{dx^2}, \tag{3}$$

where y is the deflection [m], and EI is the flexural rigidity [N·m^2]. If the deflection angle θ is very small, the shear force $S(x)$ and the bending moment $M(x)$ can be written as follows:

$$S(x) \approx \gamma Ax\theta, \tag{4}$$

$$M(x) \approx -EI\frac{d\theta}{dx}. \tag{5}$$

Using the relationship between the shear force and the bending moment ($S = dM/dx$), the governing equation can be obtained from Eqns. (4) and (5):

$$\frac{d^2\theta}{dx^2} + \frac{\gamma A}{EI}x\theta = 0. \tag{6}$$

2.2. General solution

The governing equation (6) is the same as that in Greenhill [5], and its general solution is

$$\begin{aligned}\theta(x) = J_{1/3}\left(\frac{2}{3}(\omega x)^{\frac{3}{2}}\right)\sqrt{x}\,c_1 \\ + J_{-1/3}\left(\frac{2}{3}(\omega x)^{3/2}\right)\sqrt{x}\,c_{2,}\end{aligned} \tag{7}$$

where $J_n(x)$ is the Bessel function of the first kind, and c_1 and c_2 are arbitrary constants. Furthermore, ω is expressed as follows:

$$\omega = \left(\frac{\gamma A}{EI}\right)^{1/3}. \tag{8}$$

The following boundary conditions are applied to the general solution:

$$\begin{cases} \dfrac{d\theta}{dx} = 0 & (at\ x = 0) \\ \theta = 0 & (at\ x = l_c) \end{cases}. \tag{9}$$

From the boundary condition at the free end ($x = 0$), we obtain $c_1 = 0$. The boundary condition at the fixed end ($x = l_c$) dictates that c_2 should not be zero if a non-trivial value of l_c is to be obtained. These choices for the constants allow the following result to be obtained:

$$J_{-1/3}\left(\frac{2}{3}(\omega l_c)^{3/2}\right) = 0. \tag{10}$$

We solve the above equation to obtain the critical height l_c:

$$l_c = \left(\frac{9}{4\omega^3}j_{-1/3,n}^2\right)^{1/3}, \tag{11}$$

where $j_{-1/3,n}$ is the nth zero point of the Bessel function of the first kind in Eqn. (10); its minimum value can be obtained using $n = 1$. Furthermore, the second moment of area I and the cross-sectional area A are expressed as follows:

$$I = \frac{(1-\alpha^4)\pi r_o^4}{4}, \tag{12}$$

$$A = (1-\alpha^2)\pi r_o^2. \tag{13}$$

From Eqns. (8), (11), (12), and (13), the critical height formula that is uniformly applicable to hollow and solid cylindrical columns can be obtained:

$$l_c = (1+\alpha^2)^{1/3}\left(k\frac{E}{\gamma}r_o^2\right)^{1/3}, \tag{14}$$

where k is given by:

$$k = \left(\frac{3}{4}j_{-1/3,1}\right)^2. \tag{15}$$

Because $k \approx 2$, the critical height formula can be approximated as follows:

$$l_c \approx (1+\alpha^2)^{1/3}\left(2\frac{E}{\gamma}r_o^2\right)^{1/3}. \tag{16}$$

Greenhill [5] derived the following critical height formula for solid cylindrical columns:

$$l_c \approx \left(2\frac{E}{\gamma}r_o^2\right)^{1/3}. \tag{17}$$

The critical height formula derived in this study (Eqn. (16)) corresponds to Greenhill's formula when the cross-section is solid ($\alpha = 0$). The critical height of a hollow cylindrical structure can be represented as the product of the coefficient $(1+\alpha^2)^{1/3}$ that expresses the effect of the hollow cross-section and the critical height of a solid cylinder.

Furthermore, the coefficient $(1+\alpha^2)^{1/3}$ can be rewritten, using the radius of gyration $\bar{r}_S$ of the solid cylinder with outer radius r_o and the radius of gyration $\bar{r}_H$ of the hollow cylinder with outer radius r_o and hollow ratio α:

$$(1+\alpha^2)^{1/3} = \left(\frac{\bar{r}_H}{\bar{r}_S}\right)^{2/3} = R_{\bar{r}}^{2/3}, \tag{18}$$

where $R_{\bar{r}}$ is the ratio of the radii of gyration in hollow and solid cylinders with the same radius r_o (depicted in **Fig. 1**). Therefore, Eq. (16) can be written as follows:

$$l_c \approx \left(2\frac{E}{\gamma}R_{\bar{r}}^2 r_o^2\right)^{1/3}. \tag{19}$$

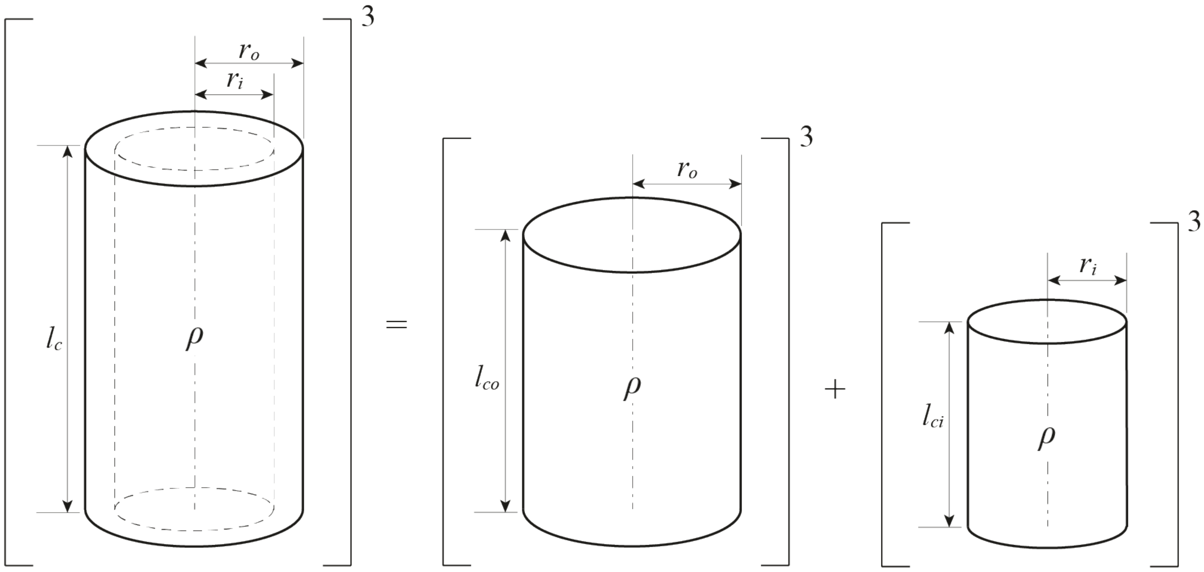


Fig. 2 "Sum of cubes" formula in critical height.

Consequently, the critical heights of both hollow and solid cylinders are proportional to the 2/3 power of the outer radii r_o (as in Greenhill's formula) and also proportional to the 2/3 power of the radii of gyration ratio $R_{\bar{r}}$.

3. Summation formulae in the self-buckling problem

In this section we use the aforementioned critical height formula to derive two important formulae that represent the relationship between the critical self-buckling characteristics of hollow and solid cylinders. We indicate that the complex self-buckling characteristics of a hollow cylinder can be written in terms of the simple self-buckling characteristics of a solid cylinder.

3.1. "Sum of cubes" formula in critical height

If we cube and expand both sides of Eqn. (16), we obtain the following:

$$l_c^3 = \left(2\frac{E}{\gamma}r_o^2\right) + \left(2\frac{E}{\gamma}(\alpha r_o)^2\right). \tag{20}$$

From Eqns. (1) and (17), we obtain the following "sum of cubes" formula for the critical height:

$$l_c^3 = l_{co}^3 + l_{ci}^3, \tag{21}$$

where l_c is the critical height of the hollow cylinder, l_{ci} is the critical height of the solid cylinder with radius r_i, and l_{co} is the critical height of the solid cylinder with radius r_o. As graphically illustrated in **Fig. 2**, Eqn. (21) indicates that the critical height of the hollow model can be represented as a summation containing the critical heights of solid cylinders that have the same elastic modulus E and the same unit volume weight γ.

3.2. "Sum of first power" formula in critical density

By rewriting Eqn. (21) as an expression for the critical density ρ_c, the following formula is obtained:

$$\rho_c = \rho_{ci} + \rho_{co}, \tag{22}$$

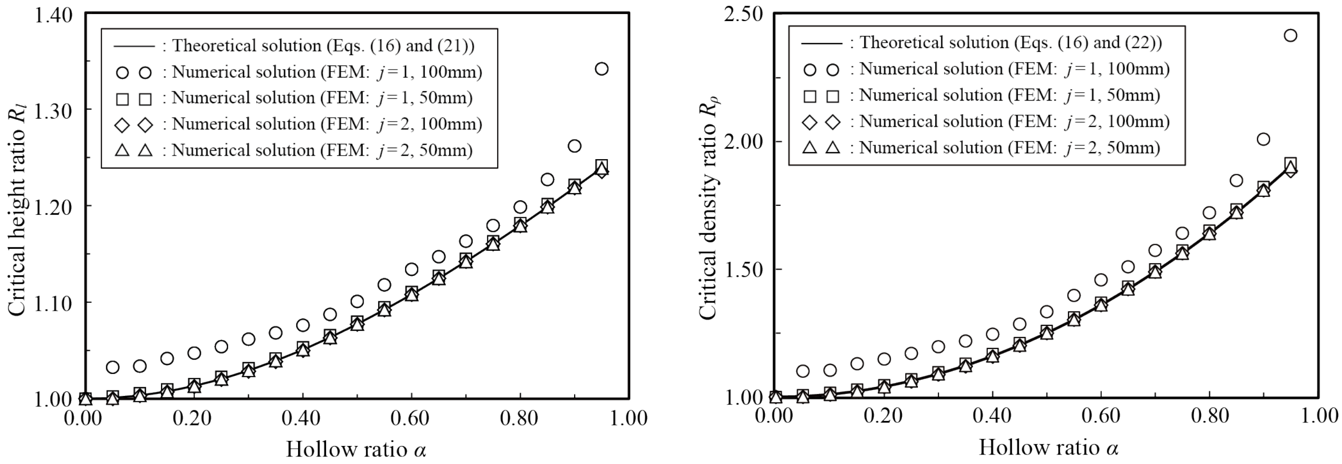
where ρ_c is the critical density in the hollow model, ρ_{ci} is the critical density in the solid model with radius r_i, and ρ_{co} is the critical density in the solid model with radius r_o. The critical density for a cylinder with a hollow cross-section is obtained by summing the critical densities of the solid cylinders having radii r_o and r_i, respectively.

3.3. Comparative verification of theoretical and FEM solutions

This study proceeds with a verification of the theoretical expressions for the critical height and critical density for a hollow cylindrical structure (both derived from Eqn. (16)) by comparing them with a numerical solution based on the finite element method (FEM), obtained by the method of Dargahi et al. [33]. We used ANSYS 2021 as analysis software and set the values of the parameters according to the results of Niklas [25] and Adam [8], as follows: outer radius $r_o = 0.23$ [m], modulus of elasticity $E = 1.1 \times 10^{10}$ [kN/m^2], model density $\rho_m = 526$ [kg/m^3], and model height $l_m = 60.8$ [m]. For this model, based on Dargahi's method [33], mesh partitioning was performed using tetrahedral elements of order $j = 1$ and 2, and mesh sizes of 100 [mm] and 50 [mm]. Note that we used the same analysis conditions for all hollow ratios.

To examine the effect of the hollowing of the cross-section on the critical height for self-weight buckling, we define the critical height ratio $R_l(\alpha)$ as follows:

$$R_l(\alpha) = \frac{l_c(\alpha)}{l_c(0)}, \tag{23}$$



(a) Relationship between critical height ratio and hollow ratio. (b) Relationship between critical density ratio and hollow ratio.

Fig. 3 Effect of the hollow ratio on the self-buckling characteristics.

where α is the hollow ratio in Eqn. (1). Therefore, Eqn. (23) expresses the ratio of the critical height for the hollow cylinder (parametrized by α) to the critical height for the solid cylinder. Using the relationship between the critical height and the critical density, the critical density ratio $R_\rho(\alpha)$ is defined as follows:

$$R_\rho(\alpha) = \frac{\rho_c(\alpha)}{\rho_c(0)} = R_l^3(\alpha). \tag{24}$$

Eqn. (24) expresses the ratio of the critical density of the hollow cylinder (parametrized by α) to the critical density of the solid cylinder. From Eqn. (14), the critical density ratio $R_\rho(\alpha)$ can be written as the third power of the critical height ratio $R_l(\alpha)$.

The results of the theoretical and finite element solutions of Eqn. (16), based on the aforementioned parameters, are illustrated in **Fig. 3**. The calculation results of the critical height ratio R_l are depicted in **Fig. 3 (a)**.

It can be observed that the FEM calculation results for a tetrahedral linear element and mesh size of 100 [mm] are significantly different from the theoretical solution. When the mesh size is reduced to 50 [mm], the obtained solution is in good agreement with the theoretical solution; however, the calculation speed decreases. In the case of a tetrahedral nonlinear element (2nd-order), the solution is in good agreement with the theoretical solution even with a mesh size of 100 [mm]. The critical height ratio increases with the hollow ratio α, reaching a maximum value of approximately 1.25

The calculation results of the critical density ratio $R_\rho(\alpha)$ are depicted in **Fig. 3 (b)**. It can be observed that the relation between the theoretical solution and FEM calculation is almost the same as in the case of the critical height ratio. The critical height ratio increases with the hollow ratio α, reaching a maximum value of approximately 2.00

3.4. Circular arc of safety factors

By transforming Eqn. (20) such that its left-hand side is expressed as the sum of the squares of the outer radius r_o and the inner radius r_i, the following equation is obtained:

$$r_{co}^2 + r_{ci}^2 = \beta^2, \tag{25}$$

where β has the following meaning:

$$\beta^2 = \frac{\gamma l^3}{2E}. \tag{26}$$

Equation (25) has the same form as the equation of a circle and can be illustrated in the plane defined by axes representing the outer diameter r_o and the inner diameter r_i, respectively. A graphical representation of Eqn. (25) is depicted in **Fig. 4**, with the outer diameter r_o on the horizontal axis, and the inner diameter r_i on the vertical axis.

A simple illustration of Eqn. (25) in the $r_o - r_i$ plane is shown in **Fig. 4 (a)**: The defining domains of the outer radius r_o and the inner radius r_i are expressed as $0 \le r_i < r_o$, so that the valid value domain of Eqn. (25) is limited to the section bounded by the r_o axis and the line $r_i = r_o$ (not including the boundary defined by the line $r_i = r_o$ itself).

Equation (25) expresses the combinations of internal and external diameters (r_o, r_i) for which self-weight buckling will occur in solid and hollow cylinders of height l made of materials with a certain unit volume weight γ and a certain elastic modulus E. The red circular arc shown in **Fig. 4 (a)** traces the boundary line between the dangerous and safe domains with respect to self-buckling; in the domain to the left of the arc (red area), self-buckling occurs, and in the domain to the right of the arc (blue area), no self-buckling occurs.

In addition, the angle Θ shown in the figure can be expressed as follows, in terms of the hollow ratio α:

$$\Theta = \tan^{-1} \alpha. \tag{27}$$

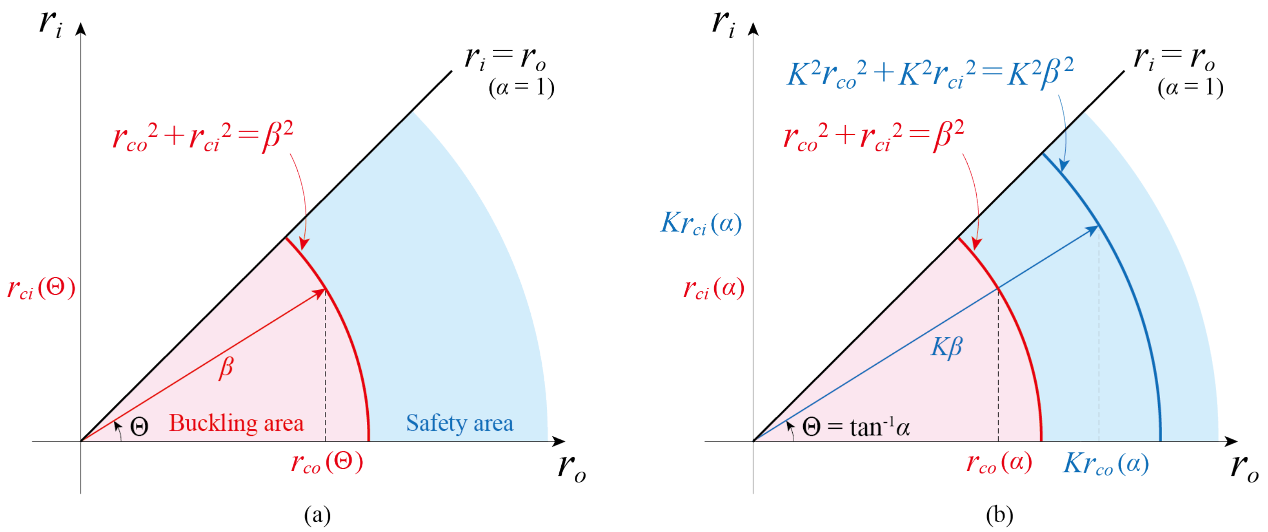


Fig. 4 Circular arc of safety factors.

This means that all the points on the circular arc with radius β and central angle Θ have an equivalent factor of safety against buckling under self-weight; however, the total volume of the system decreases as one moves from the r_o axis in the positive circumferential direction, which means that the required resistance to self-buckling can be achieved using less material. It should be noted that the present formulation is valid for buckling modes of beams, except for thin-walled structures where (r_o, r_i) is close to the line $r_o = r_i$. In such a case, local buckling may occur.

The circular arc with radius β in **Fig. 4 (a)** may be enlarged by a factor of K in the radial direction, as shown in **Fig. 4 (b).** Consequently, the safety factor $S_{F(l)}$ for the critical height l_c and the safety factor $S_{F(\rho)}$ for the critical density ρ_c against self-weight buckling can be written as follows in terms of K:

$$S_{F(l)} = K^{2/3}, \tag{28}$$
$$S_{F(\rho)} = K^2 = S_{F(l)}^3. \tag{29}$$

The outer radius and inner radius corresponding to the safety factors $S_{F(l)}$ and $S_{F(\rho)}$ are expressed as

$$r_o = S_{F(l)}^{3/2}\, r_{co}, \quad r_i = \alpha r_o, \tag{30}$$
$$r_o = S_{F(\rho)}^{1/2}\, r_{co}, \quad r_i = \alpha r_o. \tag{31}$$

In other words, the points on the circular arc (Kr_{co}, Kr_{ci}) shown in blue in **Fig. 4 (b)**, share a constant safety factor with respect to the points on the circular arc (r_{co}, r_{ci}) shown in red, which is the boundary of the occurrence of self-weight buckling. The safety factor S_F against self-weight buckling is uniformly determined only by the radial distance from the origin to the arc in the $r_o - r_i$ plane, and all the points on the arc have an equivalent resistance to self-weight buckling.

3.5. Scaling law of hollow plants

In this section, we consider how accurately the critical height formula derived in this study represents the scaling law in hollow plants. We transform the derived expression for the critical height (Eqn. (16)) to obtain the following equation:

$$l_c = \left(1 + \frac{A_i}{A_o}\right)^{1/3} \left(2\frac{E}{\gamma} r_o^2\right)^{1/3}, \tag{32}$$

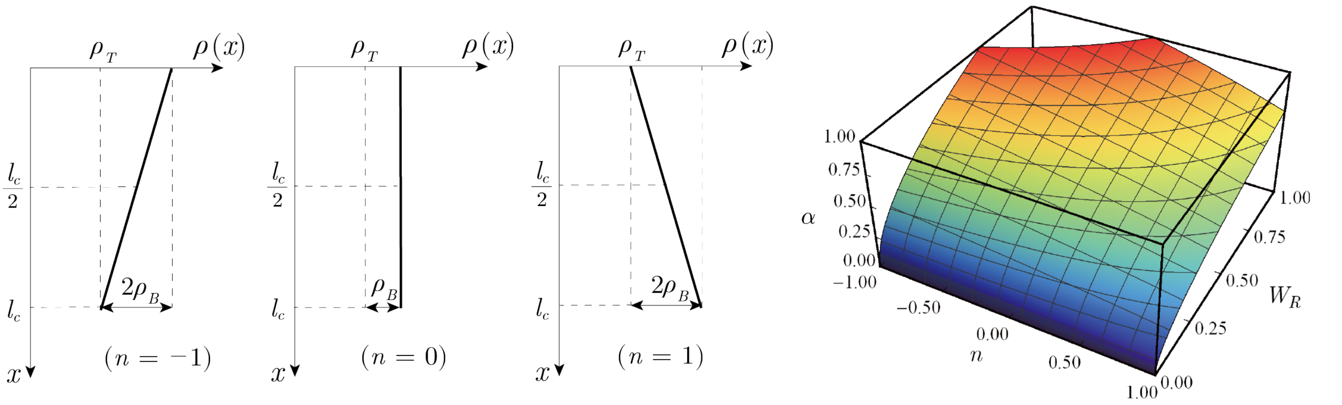
where A_i is the area of the circle with radius r_i, and A_o is the area of the circle with radius r_o. In this study, the outer radius r_o and the hollow ratio α are assumed to be constant in the vertical direction, for simplicity. However, wild bamboo in nature has a shape in which the wall thickness varies along the length of the bamboo [34,35]. Notwithstanding this, Inoue et al. [36] used real measurement data to show that the following shape law is valid for wild bamboo:

$$\frac{A_i}{A_o} = const. \tag{33}$$

Considering the scaling law of bamboo, the hollow ratio α can be assumed to be constant. Considering that the scaling law of Greenhill [5], which does not consider the taper, has been applicable to real trees with a taper, the establishment of Eqn. (33) suggests that the scaling law Eqn. (32) may also be applied to bamboo.

3.6. Influence of weight distribution in hollow plants

In this section, we consider the effect of the weight distribution, including branches and the weight ratio of branches to trunk, on the self-weight buckling resistance in hollow plants. Based on our previous study [13], we introduce a density function to the calculation model in **Fig. 1** as follows:



(a) Density functions (b) Relationship between hollow ratio and weight distribution

Fig. 5 Influence of weight distribution on self-buckling resistance.

$$\rho(x) = \left(\frac{2n}{l_c}x + (1-n)\right)\rho_B + \rho_T, \tag{34}$$

where n is the weight distribution parameter, ρ_B is the density of the branches [kg/m^3], and ρ_T is the density of the trunk [kg/m^3]. Eqn. (34) changes as shown in **Fig. 5 (a)** depending on the weight distribution parameter n.

By the same method in Section 2.1, the governing equations can be obtained as follows:

$$\frac{d^2\theta}{dx^2} + \left(\left(\frac{n}{l_c}x + (1-n)\right)W_R + 1\right)\frac{\rho_T g A}{EI}x\theta = 0, \tag{35}$$

where W_R is weight ratio ($= \rho_B/\rho_T$). Because it is difficult to solve Eqn. (35) exactly, Mathematica was used to obtain its series solution as follows:

$$\begin{aligned}\theta(x) = &\left(1 - \frac{1}{6}((1-n)W_R + 1)\eta^3 x^3 + \cdots\right)c_1 \\ &+ \left(x - \frac{1}{12}((1-n)W_R + 1)\eta^3 x^4 \cdots\right)c_2,\end{aligned} \tag{36}$$

where η is given by:

$$\eta = \left(\frac{\rho_T g A}{EI}\right)^{1/3}. \tag{37}$$

As in Section 2.2, from the boundary conditions (Eqn. (9)), the critical height equation can be obtained as follows:

$$\left(1 - \frac{1}{6}((1-n)W_R + 1)\eta^3 l_c^3 + \cdots\right) = 0. \tag{38}$$

By numerically solving Eqn. (38), the critical height considering the weight distribution can be obtained. Based on our previous study [13], sufficient convergence can be obtained using 25 expansion terms.

To examine the effect of the weight distribution and weight balance on the critical height for self-weight buckling in a hollow cylinder, we define the critical height ratio $R_l(\alpha, n, W_R)$ as follows:

$$R_l(\alpha, n, W_R) = \frac{l_c(\alpha, n, W_R)}{l_c(0,0,0)} = R_\rho^3(\alpha, n, W_R), \tag{39}$$

where $l_c(0,0,0)$ is the critical height when there are no branches or cavities, which can be obtained from Greenhill's formula [5]. By substituting Eqn. (39) into Eqn. (38), the critical height ratio formula can be obtained:

$$\left(1 - \frac{3((1-n)W_R + 1)}{8(1+\alpha^2)}j_{-1/3,1}^2 R_l^3 + \cdots\right) = 0. \tag{40}$$

We show the relationship between the hollow ratio α and the weight distribution when $R_l(\alpha, n, W_R) = 1$ in **Fig. 5 (b)**.

This figure shows that it is possible for a hollow cross-section with non-uniform weight to achieve the same critical height as one with uniform weight by adjusting the hollow ratio α. However, if the weight distribution parameter is negative and the weight ratio W_R is large, no hollow ratio α satisfying $R_l(\alpha, n, W_R) = 1$ exists. Nevertheless, in wild bamboo, we can observe that branches are typically concentrated high on the trunk. Light branches [37,38] and a hollow cross-section allows the bamboo may be able to acquire self-buckling resistance and absorb light at a high position simultaneously.

4. Conclusions

In this study, the effect of a hollow cross-section on self-weight buckling resistance was clarified by deriving a theoretical formula for the critical height, which can be applied uniformly to solid and hollow cylinders. Furthermore, various relational equations were derived to express the relationship between the self-weight buckling characteristics of solid and hollow cylinders. It was shown that the self-weight buckling characteristics of a hollow cylinder can be represented by the summation of the self-weight buckling characteristics of two solid cylinders. The main findings of this study are summarized below.

(1) The critical height of a cylinder is proportional to the 2/3 power of the radius, independent of the degree of hollowing of the cross-section, in agreement with Greenhill's scaling law. The critical height is also proportional to the 2/3 power

of the ratio of the radii of gyration in the solid and hollow states.

(2) The self-weight buckling characteristics of a hollow cylinder can be represented by the summation of the self-weight buckling characteristics of two solid cylinders with the inner and outer radii of the hollow cylinder, respectively. The third power of the critical height of a hollow cylinder is equal to the summation of the cubes of the critical heights of solid cylinders with the inner and outer radii of the hollow cylinder, respectively; the critical density of a hollow cylinder is equal to the summation of the critical densities of solid cylinders with the inner and outer radii of the hollow cylinder, respectively.

(3) The self-weight buckling characteristics of a solid cylinder, or a hollow cylinder, can be illustrated as a circular arc in the $r_i - r_o$ plane, and the safety factor for self-weight buckling is uniquely determined only by the radius of the circular arc.

(4) According to the results of the formulation considering the weight distribution in hollow cylinders, by carefully distributing the self-weight, the bamboo may be able to acquire self-buckling resistance and absorb light at a high position simultaneously.

In the future, we will apply the methodology of this study to formulate the critical height against buckling under self-weight in a hollow cone with a specific taper and to verify the scaling law for bamboo proposed in this study. Furthermore, to better model real plants, we will formulate the critical height considering the influence of the base rigidity based on the fact of the real plants not strictly fixed. We aim to consolidate these findings into a rational framework for next-generation structural design with high performance and material-saving advantages.

Acknowledgements

We acknowledge Professors T. Nakagaki, T. Obata, A. Inoue, and H. Shima for helpful discussions. This work was supported by JSPS KAKENHI, Japan (Grant Number 18H03818).

References

[1] Z.P. Bazant, L. Cedoliin, Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories, World Scientific Publishing Co Inc., Singapore, 2010.

[2] D.O. Brush, B.O. Almroth, Buckling of Bars, Plates and Shells, McGraw-Hill, New York, 1975.

[3] J.K. Lee, B.K. Lee, Buckling lengths of heavy column with various end conditions, Eng. solid mech., 8 (2020) 163–168. doi:/10.5267/j.esm.2019.9.005.

[4] B.K. Lee, J.K. Lee, Buckling of tapered heavy columns with constant volume, Mathematics. 9 (2021) 657–670. doi:/10.3390/math9060657.

[5] A.G. Greenhill, Determination of the greatest height consistent with stability that a vertical pole or mast can be made, and the greatest height to which a tree of given proportions can grow, Proc. Camb. Philol. Soc. 4 (1881) 62–73. doi:/10.1051/jphystap:018820010033701.

[6] T. Kanahama, T. Fujimura, M. Sato, Critical height for self-weight buckling in tapered trees (in Japanese), JJSCE A2, 77 (2021) 62–71. doi:/10.2208/jscejam.77.1_62. (in Japanese).

[7] T. von Karman, M.A. Biot, Mathematical Methods in Engineering, Cambridge University Press, Cambridge, 1940.

[8] Adam, J. A. Mathematics in Nature, Princeton Univ. Press, Princeton, 2003.

[9] N. Grishcoff, in: S.P. Timoshenko, J.M. Gere (Eds.), Theory of Elastic Stability, McGraw-Hill, New York, 1961, pp.101.

[10] C.-Y. Wang, B. Drachman, Stability of a heavy column with an end load, J. Appl. Mech. 48 (1981) 668–669. doi:/10.1115/1.3157696.

[11] Y.H. Chai, C.M. Wang, An application of differential transformation to stability analysis of heavy columns, Int. J. Struct. Stab. Dyn. 6 (2006) 317–332. doi:/10.1142/S0219455406001988.

[12] W.H. Duan, C.M. Wang, Exact solution for buckling of columns including self-weight, J. Eng. Mech. 134 (2008) 116–119. doi:/10.1061/(ASCE)0733-9399(2008)134:1(116).

[13] T. Kanahama, M. Sato, Mathematical modelling to determine the greatest height of trees, Sci. Rep., 12 (2022) 2039. doi.org/10.1038/s41598-022-06041-w.

[14] R.Frisch-Fay, Buckling of masonry pier under its own weight, Int. J. Solids. Struct., 16 (1980) 445-450. doi.org/10.1016/0020-7683(80)90042-6.

[15] D. Zorica, N. Challamel, M. Janev, T. M. Atanacković, Buckling and postbuckling of a heavy compressed nanorod on elastic foundation. J. Nanomech. Micromech. 7 (2017) 04017004_1-04017004-6. doi.org/10.1061/(ASCE)NM.2153-5477.0000124.

[16] M. A. Vaz, G. H. W. Mascaro, Post-buckling analysis of slender elastic rods subjected to terminal forces and self-weight, Int. J. Non Linear Mech., 38 (2003) 483-492. doi.org/10.1016/S0020-7462(01)00072-5.

[17] L. N. Virgin, R. H. Plaut, Postbuckling and vibration of linearly elastic and softening columns under self-weight, Int. J. Solids Struct., 41 (2004) 4989-5001. doi.org/10.1016/j.ijsolstr.2004.03.023.

[18] Li, Q. S. Exact solutions for buckling of non-uniform columns under axial concentrated and distributed loading. Eur. J. Mech. A/Solids 20, (2001) 485-500. doi.org/10.1016/S0997-7538(01)01143-3.

[19] T.A. McMahon, Size and shape in biology-elastic, Science. 179 (1973) 1201–1204. doi:/10.1126/science.179.4079.1201.

[20] N.M. Holbrook, F.E. Putz, Influence of neighbors on tree form: effects of lateral shading and prevention of sway on the allometry of Liquidambar styraciflua (sweet gum), Am. J. Bot. 76 (1989) 1740–1749. doi:/10.1002/j.1537-2197.1989.tb15164.x.

[21] T.D. Jackson, A.F. Shenkin, N. Majalap, J. Bin Jami, A. Bin Sailim, G. Reynolds, D.A. Coomes, C.J. Chandler, D.S. Boyd, A. Burt, P. Wilkes, M. Disney, Y. Malhi, The mechanical stability of the world's tallest broadleaf trees, Biotropica. 53 (2021) 110–120. doi:/10.1111/btp.12850.

[22] C. Goudenhooft, T. Alméras, A. Bourmaud, C. Baley, The remarkable slenderness of flax plant and pertinent factors affecting its mechanical stability, Biosyst. Eng. 178 (2019) 1–8. doi:/10.1016/j.biosystemseng.2018.10.015.

[23] M. Fournier, J. Dlouhá, G. Jaouen, T. Almeras, Integrative biomechanics for tree ecology: beyond wood density and strength, J. Exp. Bot. 64 (2013) 4793–4815. doi:/10.1093/jxb/ert279.

[24] S.P. Timoshenko, J.M. Gere, Mechanics of materials, Van Nonstrand Reinhold, New York, 1972.

[25] K.J. Niklas, Plant Allometory, The Scaling of Form and Process, the Univ. of Chicago Press, Chicago, 1994.

[26] C.R. Calladine, Theory of Shell Structures, Cambridge University Press, Cambridge, 1989.

[27] H. Shima, M. Sato, A. Inoue, Self-adaptive formation of uneven node spacings in wild bamboo, Phys. Rev. E. 93 (2016)_1-022406_9. doi:/10.1103/PhysRevE.93.022406.

[28] M. Sato, A. Inoue, H. Shima, Bamboo-inspired optimal design for functionally graded hollow cylinders, PLOS ONE. 12 (2017) e0175029. doi:/10.1371/journal.pone.0175029, http://www.ncbi.nlm.nih.gov/pubmed/0175029_1-0175029_14.

[29] N.N. Wai, H. Nanko, K. Murakami, A morphological study on the behavior of bamboo pulp fibers in the beating process, Wood Sci. Technol. 19 (1985) 211–222. doi:/10.1007/BF00392050.

[30] U. Wegst, H. Shercliff, M. Ashby, The Structure and Properties of Bamboo as an Engineering Material, University of Cambridge, Cambridge, 1993.

[31] K.F. Chung, W.K. Yu, Mechanical properties of structural bamboo for bamboo scaffoldings, Eng. Struct. 24 (2002) 429–442. doi:/10.1016/S0141-0296(01)00110-9.

[32] W. Liese, M. Kohl, Bamboo: the Plant and Its Uses, Springer, Berlin, 2015.

[33] M. Dargahi, T. Newson, J. Moore, Buckling behavior of trees under self-weight loading, Forestry. 92 (2019) 393–405. doi:/10.1093/forestry/cpz027.

[34] A. Inoue, M. Sato, H. Shima, A new taper index based on form-factor: application to three bamboo species (Phyllostachys spp.), Eur. J. For. Res. 140 (2021) 1533–1542. doi:/10.1007/s10342-021-01416-6.

[35] A. Inoue, Culm form analysis for bamboo, Phyllostachys pubescens, J. For. Res. 24 (2013) 525–530. doi:/10.1007/s11676-013-0383-4.

[36] A. Inoue, F. Kitahara, H. Suga, T. Wajima, Deriving a reduction factor for woody part of culm for bamboo Phyllostachys pubescens, Landsc. Ecol. Eng. 7 (2011) 153–157. doi:/10.1007/s11355-010-0121-4.

[37] D. King, O.L. Loucks, The theory of tree bole and branch form, Radiat. Environ. Biophys. 15 (1978) 141–165. doi:/10.1007/BF01323263.

[38] S. Ishii, Operating time of cutting and moisture content of phyllostachys pubescens, Bull. Okayama Prefect. Technol. Cent. Agric. For. Fish. Res. Inst. For. For. Prod. 26 (2010) 19–28. (In Japanese)