HARDER'S CONJECTURE AND MIYAWAKI LIFT

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Abstract. Let k, j and n be positive integers such that k is odd , and both j and n are even, satisfying $j \equiv n \mod 4$. Let f and g be primitive forms of weight $2k+j-2$ and $k+j/2-n/2-1$, respectively, for $\mathrm{SL}_2(\mathbb{Z})$. Then, we propose a conjecture on the congruence between the Klingen-Eisenstein lift of the Miyawaki lift of f and g of type II and a certain lift of a vector-valued Hecke eigenform of weight $(k+j,k)$ for $\mathrm{Sp}_2(\mathbb{Z})$. This conjecture implies Harder's conjecture. Through this formulation, we prove Harder's conjecture in some cases.

Contents

1. Introduction

Harder's conjecture is one of the most important and interesting conjectures in the arithmetic of automorphic forms. Let k and j be positive integers such that j is even. Then, Harder's conjecture predicts that the Fourier coefficients of a primitive form f of weight $2k+j-2$ for $\mathrm{SL}_2(\mathbb{Z})$ are related with those of a certain Hecke eigenform of weight $(k+j,k)$ for $\mathrm{Sp}_2(\mathbb{Z})$ modulo some prime ideal (cf. Conjecture 4.1).

One of main difficulties in treating this congruence arises from the fact that this is not concerning the congruence between Hecke eigenvalues of two Hecke eigenforms of the same weight. To overcome this issue, several approaches have been proposed (cf. [12], [13], [14], [4], [6]). See also [9] for a paramodular form version. In [2], H. Atobe, M. Chida, T. Ibukiyama, H. Katsurada and T. Yamauchi considered a conjecture concerning the congruence between two liftings to higher degree of Hecke eigenforms (of integral weight) of degree two in the case

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k is even. This implies Harder's conjecture. As a result, they proved Harder's congruence in some cases. Moreover, in [3], combining the result in cited above with Galois representation theoretic method, under certain mild conditions, they proved Harder' conjecture in the case k is even and $j \equiv 0 \mod 4$.

In this paper, we treat the case k is odd. We explain it more precisely. For a non-increasing sequence $\mathbf{k} = (k_1, \ldots, k_n)$ of non-negative integers we denote by $M_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ and $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ the spaces of modular forms and cusp forms of weight $\mathbf{k}$ (or, weight k, if $\mathbf{k} = (\overbrace{k, \ldots, k}^{n})$) for $\mathrm{Sp}_n(\mathbb{Z})$, respectively. (For the definition of modular forms, see Section 2). Let n be a positive even integer and suppose that $j \equiv n \mod 4$. For the f above and a primitive form g of weight $k + j/2 - n/2 - 1$ for $\mathrm{SL}_2(\mathbb{Z})$, let $\mathcal{M}_{n+1}(f, g) = \mathcal{M}_{n+1}^{k+\frac{j}{2}+\frac{n}{2}-1}(f, g)$ be the Miyawaki lift of g and f of type II to the space of cusp forms of weight $\frac{j}{2} + k + \frac{n}{2} - 1$ for $\mathrm{Sp}_{n+1}(\mathbb{Z})$ (cf. Theorem 4.4). For a sequence

$$\mathbf{k} = \Bigl(\overbrace{\frac{j}{2} + k + \frac{n}{2} - 1, \ldots, \frac{j}{2} + k + \frac{n}{2} - 1}^{n+1}, \overbrace{\frac{j}{2} + \frac{3n}{2} + 2, \ldots, \frac{j}{2} + \frac{3n}{2} + 2}^{n}\Bigr)$$

with $k \geq n+2$, let $[\mathcal{M}_{n+1}(f, g)]^{\mathbf{k}}$ be the Klingen-Eisenstein lift of $\mathcal{M}_{n+1}(f, g)$ to $M_{\mathbf{k}}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$. Then, we propose the following conjecture:

Conjecture. *(Conjecture 4.6) Let k, j and $\mathbf{k}$ be as above. Let $f(z) \in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form and $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(f)$. Then under certain assumptions, there exists a Hecke eigenform F in $S_{(k+j,k)}(\mathrm{Sp}_2(\mathbb{Z}))$ such that*

$$\lambda_{\mathcal{A}_{2n+1}^{\mathbf{k}}(F,g)}(T) \equiv \lambda_{[\mathcal{M}_{n+1}(f,g)]^{\mathbf{k}}}(T) \mod \mathfrak{p}'$$

for any integral Hecke operator T. Here, $\mathcal{A}_{2n+1}^{\mathbf{k}}(F, g)$ is a certain lift of g and F to $S_{\mathbf{k}}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, which will be defined in Theorem 4.3. (As for the definition of integral Hecke operators, see Section 3.)

This conjecture implies Harder's conjecture (cf. Theorem 4.8). Through this formulation, we confirm Harder's conjecture in some cases (cf. Corollaries 7.2 and 7.4).

This paper is organized as follows. In Section 2, we give a brief review of Siegel modular forms, especially about their $\mathbb{Q}$-structures and $\mathbb{Z}$-structures. In Section 3, we give a summary of several L-values. In Section 4, first we state Harder's conjecture. Next we introduce several lifts, and among other things define a certain lift of a primitive form and a vector-valued modular form, and propose a conjecture on the congruence between it and the Klingen-Eisenstein lift of the Miyawaki lift of type II, and explain how this conjecture implies Harder's conjecture. In Section 5, we consider the pullback formula of the Siegel Eisenstein series with differential operators. In Section 6, we consider the congruence for vector-valued Klingen-Eisenstein series, which is a generalization of [22] and [2]. Moreover, we give a formula for the Fourier coefficients of the Klingen-Eisenstein series, from which we can confirm some assumption in our main results. In Section 7, we state our main results, which confirm our conjecture, and so Harder's.

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Notation. Let R be a commutative ring. We denote by $R^\times$ the unit group of R. We denote by $M_{m,n}(R)$ the set of $m \times n$-matrices with entries in R. In particular put $M_n(R) = M_{n,n}(R)$. Put $\mathrm{GL}_m(R) = \{A \in M_m(R) \mid \det A \in R^\times\}$, where $\det A$ denotes the determinant of a square matrix A. For an $m \times n$-matrix X and an $m \times m$-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let $\mathrm{Sym}_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the subset of symmetric matrices of degree n with entries in the field of fractions of R whose (i,j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. We say that an element A of $M_n(R)$ is non-degenerate if $\det A \neq 0$. For a subset S of $M_n(R)$ we denote by S^{nd} the subset of S consisting of non-degenerate matrices. If S is a subset of $\mathrm{Sym}_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. positive semi-definite) matrices. The group $\mathrm{GL}_n(R)$ acts on the set $\mathrm{Sym}_n(R)$ by

$$\mathrm{GL}_n(R) \times \mathrm{Sym}_n(R) \ni (g, A) \longmapsto A[g] \in \mathrm{Sym}_n(R).$$

Let G be a subgroup of $\mathrm{GL}_n(R)$. For a G-stable subset $\mathcal{B}$ of $\mathrm{Sym}_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes use the same symbol $\mathcal{B}/G$ to denote a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/\mathrm{GL}_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Let R' be a subring of R. Then two symmetric matrices A and A' with entries in R are said to be equivalent over R' with each other and write $A \sim_{R'} A'$ if there is an element X of $\mathrm{GL}_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbb{Z}$ such that $D \equiv 0$ or $D \equiv 1 \bmod 4$, let $\mathfrak{d}_D$ be the discriminant of $\mathbb{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. We call an integer D a fundamental discriminant if it is the discriminant of some quadratic extension of $\mathbb{Q}$ or 1. For a fundamental discriminant D, let $\left(\frac{D}{*}\right)$ be the character corresponding to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Here we make the convention that $\left(\frac{D}{*}\right) = 1$ if $D = 1$. For an integer D such that $D \equiv 0$ or $\equiv 1 \bmod 4$, we define $\left(\frac{D}{*}\right) = \left(\frac{\mathfrak{d}_D}{*}\right)$. We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbb{Z}[p^{-1}]$.

Let K be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ we denote by $K_{\mathfrak{p}}$ and $\mathfrak{O}_{\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of K and $\mathfrak{O}$, respectively, and put $\mathfrak{O}_{(\mathfrak{p})} = \mathfrak{O}_{\mathfrak{p}} \cap K$. In the special case where $K = \mathbb{Q}$, $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$, we denote by $\mathrm{ord}_{\mathfrak{p}}(*)$ the additive valuation of $K_{\mathfrak{p}}$ normalized so that $\mathrm{ord}_{\mathfrak{p}}(\varpi) = 1$ for a prime element ϖ of $K_{\mathfrak{p}}$. Moreover for any element $a, b \in \mathfrak{O}_{\mathfrak{p}}$ we write $b \equiv a \pmod{\mathfrak{p}}$ if $\mathrm{ord}_{\mathfrak{p}}(a - b) > 0$.

2. Siegel modular forms

In this section, we review basic facts about Siegel modular forms in [2, Section 2] with a little modification. We denote by $\mathbb{H}_n$ the Siegel upper half-space of degree n, i.e.,

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) \mid Z = {}^tZ = X + \sqrt{-1}Y,\ X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any ring R and any positive integer n, we define the group $\mathrm{GSp}_n(R)$ by

$$\mathrm{GSp}_n(R) = \{g \in M_{2n}(R) \mid gJ_n{}^tg = \nu(g)J_n \text{ with some } \nu(g) \in R^\times\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We call $\nu(g)$ the symplectic similitude of g. We also define the symplectic group of degree n over R by

$$\mathrm{Sp}_n(R) = \{g \in \mathrm{GSp}_n(R) \mid \nu(g) = 1\}.$$

In particular, if R is a subfield of $\mathbb{R}$, we define

$$\mathrm{GSp}_n^+(R) = \{g \in \mathrm{GSp}_n(R) \mid \nu(g) > 0\}.$$

We put $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$ for the sake of simplicity.

Let $\lambda = (k_1, k_2, \ldots)$ be a finite or an infinite sequence of non-negative integers such that $k_i \geq k_{i+1}$ for all i and $k_m = 0$ for some m. We call this a dominant integral weight. We call the biggest integer m such that $k_m \neq 0$ a depth of λ and write it by $\mathrm{depth}(\lambda)$. It is well known that the set of dominant integral weights λ with $\mathrm{depth}(\lambda) \leq n$ corresponds bijectively to the isomorphism classes of irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. We denote this representation by $(\rho_{n,\lambda}, V_{n,\lambda})$. We also denote it by $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ with $\mathbf{k} = (k_1, \ldots, k_n)$ and call it an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\mathbf{k}$. Moreover, we write $\mathbf{k}' = (k_1 - k_n, \ldots, k_{n-1} - k_n, 0)$. Then, we have $\rho_{\mathbf{k}} \cong \det^{k_n} \otimes \rho_{\mathbf{k}'}$ with $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ an irreducible polynomial representation of highest weight $\mathbf{k}'$. Here we understand that $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ is the trivial representation on $\mathbb{C}$ if $k_1 = \cdots = k_{n-1} = k_n$. We fix a Hermitian inner product $\langle *, \, * \rangle$ on $V' = V_{\mathbf{k}'}$ such that

$$\langle \rho_{\mathbf{k}'}(g)v, w\rangle = \langle v, \rho_{\mathbf{k}'}({}^t\overline{g})w\rangle \quad \text{for } g \in \mathrm{GL}_n(\mathbb{C}), v, w \in V'.$$

Now we define vector-valued Siegel modular forms of $\Gamma^{(n)}$. For any V'-valued function F on $\mathbb{H}_n$, and for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{R})$, we put $J(g, Z) = CZ + D$ and

$$F|_{\rho_{\mathbf{k}}}[g] = \rho_{\mathbf{k}}(J(g, Z))^{-1} F(gZ).$$

From now on we identify $\rho_{\mathbf{k}}$ with $\det^{k_n} \otimes \rho_{\mathbf{k}'}$. We say that F is a C^∞-modular form of weight $\rho_{\mathbf{k}}$ or $\mathbf{k}$ with respect to $\Gamma^{(n)}$ if F is a C^∞-mapping from $\mathbb{H}_n$ to V' satisfying the following condition:

$$F|_{\rho_{\mathbf{k}}}[\gamma] = F \text{ for any } \gamma \in \Gamma.$$

We denote by $M_{\mathbf{k}}^\infty(\Gamma^{(n)}) = M_{\rho_{\mathbf{k}}}^\infty(\Gamma^{(n)})$ the space of C^∞-modular forms of weight $\rho_{\mathbf{k}}$ with respect to $\Gamma^{(n)}$. We say that an element F of $M_{\mathbf{k}}^\infty(\Gamma^{(n)})$ is a (holomorphic) modular form of weight $\rho_{\mathbf{k}}$ if F is a holomorphic mapping from $\mathbb{H}_n$ to V' which has the following Fourier expansion

$$F(Z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(T, F)\mathbf{e}(\mathrm{tr}(TZ)), \quad Z \in \mathbb{H}_n, \ a(T, F) \in V',$$

where $\mathrm{tr}(T)$ is the trace of a matrix T. We note that F has the above Fourier expansion automatically if $n \geq 2$. We denote by $M_{\mathbf{k}}(\Gamma^{(n)}) = M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ the space of modular forms of weight $\rho_{\mathbf{k}}$ with respect to $\Gamma^{(n)}$. We say that $F \in M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ is a cusp form if we have $a(T, F) = 0$ unless T is positive definite. We denote by $S_{\mathbf{k}}(\Gamma^{(n)}) = S_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ the subspace of $M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ consisting of cusp forms.

For $F, G \in M_\rho^\infty(\Gamma^{(n)})$ the Petersson inner product is defined by

$$(F, G) = \int_{\Gamma \backslash \mathbb{H}_n} \langle \rho(\sqrt{Y})F(Z), \rho(\sqrt{Y})G(Z)\rangle \det(Y)^{-n-1} dZ,$$

where $Y = \mathrm{Im}(Z)$ and $\sqrt{Y}$ is a positive definite symmetric matrix such that $\sqrt{Y}^2 = Y$. This integral converges if F and G are slowly increasing and at least one of them belongs to $S_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$. If $\mathbf{k} = (\overbrace{k, \ldots, k}^{n})$, we simply write $M_k(\Gamma^{(n)}) = M_{\mathbf{k}}(\Gamma^{(n)})$ and $S_k(\Gamma^{(n)}) = S_{\mathbf{k}}(\Gamma^{(n)})$.

We note that

$$M_{(k+j,k)}(\Gamma^{(2)}) = M_{\det^k \otimes \mathrm{Sym}^j}(\Gamma^{(2)}) \quad \text{and } S_{(k+j,k)}(\Gamma^{(2)}) = S_{\det^k \otimes \mathrm{Sym}^j}(\Gamma^{(2)}),$$

where Sym^j is the j-th symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$.

For a representation (ρ, V) of $\mathrm{GL}_n(\mathbb{C})$, we denote by $\mathfrak{F}(\mathbb{H}_n, V)$ the set of Fourier series F on $\mathbb{H}_n$ with values in V of the following form:

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F)\mathbf{e}(\mathrm{tr}(AZ)), \quad Z \in \mathbb{H}_n,\ a(A,F) \in V.$$

For $F \in \mathfrak{F}(\mathbb{H}_n, V)$ and a positive integer $r \leq n$ we define $\Phi(F) = \Phi_r^n(F)$ as

$$\Phi(F)(Z_1) = \lim_{\lambda \to \infty} F\left(\begin{pmatrix} Z_1 & O \\ O & \sqrt{-1}\lambda 1_{n-r}\end{pmatrix}\right), \quad Z_1 \in \mathbb{H}_r.$$

We make the convention that $\mathfrak{F}(\mathbb{H}_0, V) = V$ and $\Phi_0^n(F) = a(O_n, F)$. Then, $\Phi(F)$ belongs to $\mathfrak{F}(\mathbb{H}_r, V)$. For a representation (ρ, V) of $\mathrm{GL}_n(\mathbb{C})$, we denote by $\widetilde{\mathfrak{F}}(\mathbb{H}_n, V) = \widetilde{\mathfrak{F}}(\mathbb{H}_n, (\rho, V))$ the subset of $\mathfrak{F}(\mathbb{H}_n, V)$ consisting of elements F such that the following condition is satisfied:

(K0) $$a(A[g], F) = \rho(g)a(A, F) \text{ for any } g \in \mathrm{GL}_n(\mathbb{C}).$$

Now let $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ be a dominant integral weight of length n of depth m. Then we realize the representation space $V_{\boldsymbol{\ell}}$ in terms of bideterminants (cf. [17]). Let $U = (u_{ij})$ be an $m \times n$ matrix of variables. For a positive integer $a \leq m$ let $\mathcal{SI}_{n,a}$ denote the set of strictly increasing sequences of positive integers not greater than n of length a. For each $J = (j_1, \ldots, j_a) \in \mathcal{SI}_{n,a}$ we define U_J as

$$\begin{vmatrix} u_{1,j_1} & \ldots & u_{1,j_a} \\ \vdots & \ddots & \vdots \\ u_{a,j_1} & \ldots & u_{a,j_a} \end{vmatrix}.$$

Then we say that a polynomial $P(U)$ in U is a bideterminant of weight $\boldsymbol{\ell}$ if $P(U)$ is of the following form:

$$P(U) = \prod_{i=1}^{m} \prod_{j=1}^{l_i - l_{i+1}} U_{J_{ij}},$$

where $(J_{i1}, \ldots, J_{i,l_i-l_{i+1}}) \in \mathcal{SI}_{n,i}^{l_i - l_{i+1}}$. Here we make the convention that $\prod_{j=1}^{l_i-l_{i+1}} U_{J_{ij}} = 1$ if $l_i = l_{i+1}$. Let $\mathcal{BD}_{\boldsymbol{\ell}}$ be the set of all bideterminants of weight $\boldsymbol{\ell}$. Here we make the convention that $\mathcal{BD}_{\boldsymbol{\ell}} = \{1\}$ if $\boldsymbol{\ell} = (0, \ldots, 0)$. For a commutative ring R and an R-algebra S let $S[U]_{\boldsymbol{\ell}}$ denote the R-module of all S-linear combinations of $P(U)$ for $P(U) \in \mathcal{BD}_{\boldsymbol{\ell}}$. Then we can define an action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}[U]_{\boldsymbol{\ell}}$ as

$$\mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}[U]_{\boldsymbol{\ell}} \ni (g, P(U)) \mapsto P(Ug) \in \mathbb{C}[U]_{\boldsymbol{\ell}},$$

and we can take the $\mathbb{C}$-vector space $\mathbb{C}[U]_{\boldsymbol{\ell}}$ as a representation space $V_{\boldsymbol{\ell}}$ of $\rho_{\boldsymbol{\ell}}$ under this action.

Let $m \leq n-1$ be a non-negative integer and $U = (u_{ij})$ be an $m \times n$ matrix of variables. Let $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_1 \geq \cdots \geq k_m > k_{m+1} = \cdots = k_n$ and $\mathbf{k}' = (k_1 - k_{m+1}, \ldots, k_m - k_{m+1}, \overbrace{0, \ldots, 0}^{n-m})$. Here we make the convention that $\mathbf{k} = (k_1, \ldots, k_1)$ and $\mathbf{k}' = (0, \ldots, 0)$ if $m = 0$. Then under this notation and convention, $M_{\mathbf{k}}(\Gamma^{(n)})$ can be regarded as a $\mathbb{C}$-subspace of $\mathrm{Hol}(\mathbb{H}_n)[U]_{\mathbf{k}'}$, where $\mathrm{Hol}(\mathbb{H}_n)$ denotes the ring of holomorphic functions on $\mathbb{H}_n$.

We sometimes write $F(Z)(U)$ for $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ for $Z \in \mathbb{H}_n$ to highlight that F is $\mathbb{C}[U]_{\mathbf{k}'}$-valued. Moreover, the Fourier expansion of $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ can be expressed as

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $a(A, F) = a(A, F)(U) \in \mathbb{C}[U]_{\mathbf{k}'}$.

Let r be an integer such that $m \leq r \leq n$ and let $\mathbf{l} = (k_1, \ldots, k_{r-1}, k_r)$ and $\mathbf{l}' = (k_1 - k_{m+1}, \ldots, k_r - k_{m+1}, \overbrace{0, \ldots, 0}^{r-m})$. For the $m \times n$ matrix U, let $U^{(r)} = (u_{ij})_{1 \leq i \leq m, 1 \leq j \leq r}$ and put $W' = \mathbb{C}[U^{(r)}]_{\mathbf{l}'}$. Then we can define a representation (τ', W') of $\mathrm{GL}_r(\mathbb{C})$. The representations $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ and (τ', W') satisfy the following conditions:

(K1) $W' \subset V_{\mathbf{k}'}$.

(K2) $\rho_{\mathbf{k}'}\Big(\left(\begin{smallmatrix} g_1 & g_2 \\ O & g_4 \end{smallmatrix}\right)\Big)v = \tau'(g_1)v$ for $\left(\begin{smallmatrix} g_1 & g_2 \\ O & g_4 \end{smallmatrix}\right) \in \mathrm{GL}_n(\mathbb{C})$ with $g_1 \in \mathrm{GL}_r(\mathbb{C})$ and $v \in W'$.

(K3) If $v \in V_{\mathbf{k}'}$ satisfies the condition

$$\rho_{\mathbf{k}'}\Big(\left(\begin{smallmatrix} 1_r & O \\ O & h \end{smallmatrix}\right)\Big)v = v \text{ for any } h \in \mathrm{GL}_{n-r}(\mathbb{C}),$$

then v belongs to W'.

Let $F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F)\mathbf{e}(\mathrm{tr}(AZ)) \in \mathfrak{F}(\mathbb{H}_n, V_{\mathbf{k}'})$ Then, in a way similar to [1, (2.3.29)], we have

$$\Phi_r^n(F)(Z_1) = \sum_{A_1 \in \mathcal{H}_r(\mathbb{Z})_{\geq 0}} a\Big(\begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, F\Big)\mathbf{e}(\mathrm{tr}(A_1 Z_1)) \ (Z_1 \in \mathbb{H}_r).$$

Suppose that F belongs to $\widetilde{\mathfrak{F}}(\mathbb{H}_n, V_{\mathbf{k}'})$. Then, by (K0),

$$\rho_{\mathbf{k}'}\Big(\left(\begin{smallmatrix} 1_r & O \\ O & h \end{smallmatrix}\right)\Big)\Big(a(\left(\begin{smallmatrix} A_1 & O \\ O & O \end{smallmatrix}\right), F)\Big) = a(\left(\begin{smallmatrix} A_1 & O \\ O & O \end{smallmatrix}\right), F) \ \text{ for any } h \in \mathrm{GL}_{n-r}(\mathbb{C}).$$

Hence, by (K3), $a(\left(\begin{smallmatrix} A_1 & O \\ O & O \end{smallmatrix}\right), F)$ belongs to W' for any $A_1 \in \mathcal{H}_r(\mathbb{Z})_{\geq 0}$. This implies that $\Phi_r^n(F)$ belongs to $\mathfrak{F}(\mathbb{H}_r, W')$. We easily see that $\Phi_r^n(F)$ belongs to $\widetilde{\mathfrak{F}}(\mathbb{H}_r, W')$, and therefore Φ_r^n sends $\widetilde{\mathfrak{F}}(\mathbb{H}_n, V_{\mathbf{k}'})$ to $\widetilde{\mathfrak{F}}(\mathbb{H}_r, W')$. It is easily seen that it induces a mapping from $M_\rho(\Gamma^{(n)})$ to $M_\tau(\Gamma^{(r)})$, where $\rho = \det^{k_n} \otimes \rho_{\mathbf{k}'}$ and $\tau = \det^{k_n} \otimes \tau'$. Let $\Delta_{n,r}$ be the subgroup of $\Gamma^{(n)}$ defined by

$$\Delta_{n,r} := \left\{ \begin{pmatrix} * & * \\ O_{(n-r,n+r)} & * \end{pmatrix} \in \Gamma^{(n)} \right\}.$$

For $F \in S_\tau(\Gamma^{(r)})$ the Klingen-Eisenstein series $[F]_\tau^\rho(Z, s)$ of F associated to ρ is defined by

$$[F]_\tau^\rho(Z, s) := \sum_{\gamma \in \Delta_{n,r} \backslash \Gamma^{(n)}} \Big(\frac{\det \mathrm{Im}(Z)}{\det \mathrm{Im}(\mathrm{pr}_r^n(Z))}\Big)^s F(\mathrm{pr}_r^n(Z))|_\rho \gamma.$$

Here $\mathrm{pr}_r^n(Z) = Z_1$ for $Z = \begin{pmatrix} Z_1 & Z_2 \\ {}^tZ_2 & Z_4 \end{pmatrix} \in \mathbb{H}_n$ with $Z_1 \in \mathbb{H}_r, Z_4 \in \mathbb{H}_{n-r}, Z_2 \in M_{r,n-r}(\mathbb{C})$. We also write $[F]_\tau^\rho(Z, s)$ as $[F]_{\mathbf{l}}^{\mathbf{k}}(Z, s)$ or $[F]^{\mathbf{k}}(Z, s)$.

Suppose that k_n is even and $2\,\mathrm{Re}(s) + k_n > n + r + 1$. Then, $[F]_\tau^\rho(Z, s)$ converges absolutely and uniformly on $\mathbb{H}_n$. This is proved by [23] in the scalar-valued case, and can be proved similarly in general case. If $[F]^{\mathbf{k}}(Z, s)$ can be continued holomorphically in the neighborhood of 0 as a function of s, we put $[F]_\tau^\rho(Z) = [F]_\tau^\rho(Z, 0)$. If $[F]_\tau^\rho(Z)$ is holomorphic as a function of

Z, it belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$, and we say that it is the Klingen-Eisenstein lift of F to $M_{\mathbf{k}}(\Gamma^{(n)})$. In particular, if $k_n > n+r+1$, then $[F]_\tau^\rho(Z,s)$ is holomorphic at $s=0$ as a function of s, and $[F]_\tau^\rho(Z,0)$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$, and $\Phi_\tau^\rho([F]_\tau^\rho)=F$. We note that $[F]_\tau^\rho(Z)$ is not necessarily a holomorphic as a function of Z if $k_n \le n+r+1$.

For a positive integer k, we define $E_{n,k}(Z,s)$ as

$$E_{n,k}(Z,s)=\sum_{\gamma\in\Delta_{n,0}\backslash\Gamma^{(n)}}(\det \mathrm{Im}(Z))^s|_\gamma$$

and call it the Siegel-Eisenstein series of weight k with respect to $\Gamma^{(n)}$. The Siegel-Eisenstein series $E_{n,k}(Z,s)$ can be continued meromorphically to the whole s-plane as a function of s, holomorphic at $s=0$. We put $E_{n,k}(Z)=E_{n,k}(Z,0)$. Let $\mathbf{k}=(\overbrace{k+l,\ldots,k+l}^{m},\overbrace{k,\ldots,k}^{n-m})$ such that $k,l\ge 0$, and put $\rho_{\mathbf{k}}=\det^k\otimes\rho_{\mathbf{k}'}$ and $\tau=\det^k\otimes\rho_{\mathbf{l}'}$ with $\mathbf{k}'=(\overbrace{l,\ldots,l}^{m},0,\ldots,0)$ and $\mathbf{l}'=(\overbrace{l,\ldots,l}^{m})$. Then, for $F\in S_\tau(\Gamma^{(m)})$ we can define the Klingen-Eisenstein series $[F]_\tau^{\rho_{\mathbf{k}}}(Z,s)$ of F associated to $\rho_{\mathbf{k}}$ if k is even and $2\,\mathrm{Re}(s)+k>n+m+1$. We note that $\mathbb{C}[U^{(m)}]_{\mathbf{l}'}$ is a subspace of $\mathbb{C}[U]_{\mathbf{k}'}$ spanned by $(\det U^{(m)})^l$, and hence we have a natural isomorphism

$$\iota: S_{k+l}(\Gamma^{(m)})\ni f\mapsto \widetilde{f}:=(\det U^{(m)})^l f\in S_\tau(\Gamma^{(m)}).$$

We sometimes write $[f]^{\rho_{\mathbf{k}}}$ or $[f]^{\mathbf{k}}$ instead of $[\widetilde{f}]_\tau^{\rho_{\mathbf{k}}}$ for $f\in S_{k+l}(\Gamma^{(m)})$.

Let $\boldsymbol{\ell}=(l_1,\ldots,l_n)$ be a dominant integral weight of length n of depth m. Let $\widetilde{V}=\widetilde{V}_{\boldsymbol{\ell}}=\mathbb{Q}[U]_{\boldsymbol{\ell}}$. Then, $(\rho_{\boldsymbol{\ell}}|\mathrm{GL}_n(\mathbb{Q}),\widetilde{V})$ is a representation of $\mathrm{GL}_n(\mathbb{Q})$, and $\widetilde{V}\otimes\mathbb{C}=V_{\boldsymbol{\ell}}$. We consider a $\mathbb{Z}$-structure of $V_{\boldsymbol{\ell}}$. To do this, we fix a basis $\mathcal{S}=\mathcal{S}_{\boldsymbol{\ell}}=\{P\}$ of $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. We note here that the bideterminants are not linearly independent over $\mathbb{Z}$ and even over $\mathbb{C}$ in general, so the set $\mathcal{BD}_{\boldsymbol{\ell}}$ is not necessarily a basis of $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. Let R be a subring of $\mathbb{C}$. Since the set $\mathcal{S}$ is also linearly independent over $\mathbb{C}$, an element a of $R[U]_{\boldsymbol{\ell}}$ is uniquely written as

$$a=\sum_{P\in\mathcal{S}}a_P P \text{ with } a_P\in R.$$

Let K be a number field, and $\mathfrak{O}$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ and $a=a(U)=\sum_{P\in\mathcal{S}}a_P P\in K[U]_{\boldsymbol{\ell}}$ with $a_P\in K$, define

$$\mathrm{ord}_{\mathfrak{p}}(a)=\min_{P\in\mathcal{S}}\mathrm{ord}_{\mathfrak{p}}(a_P).$$

We say that $\mathfrak{p}$ divides a if $\mathrm{ord}_{\mathfrak{p}}(a)>0$.

Remark 2.1. *The definition of* $\mathrm{ord}_{\mathfrak{p}}$ *does not depend on the choice of a basis of* $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. *We note that* $\mathfrak{p}$ *does not divide* $a=a(U)$ *if* $\mathfrak{p}$ *does not divide* $a(U_0)$ *for some element* U_0 *of* $M_{m,n}(\mathfrak{O})$.

For a subring R of $\mathbb{C}$, we denote by $M_{\mathbf{k}}(\Gamma^{(n)})(R)$ the R-submodule of $M_{\mathbf{k}}(\Gamma^{(n)})$ consisting of all modular forms F such that $a(T,F)\in R[U]_{\mathbf{k}'}$ for all $T\in\mathcal{H}_n(\mathbb{Z})_{\ge 0}$. Here, $\mathbf{k}'=(k_1-k_{m+1},\ldots,k_m-k_{m+1},\overbrace{0,\ldots,0}^{n-m})$ for $\mathbf{k}=(k_1,\ldots,k_n)$ with $k_1\ge\cdots\ge k_m>k_{m+1}=\cdots=k_n$ as stated before.

We consider tensor products of modular forms, which will be used on and after Section 5. Let n_1 and n_2 be positive integers. Let $\mathbf{k}_1=(k_1,\ldots,k_m,k_{m+1},\ldots,k_{n_1})$ and $\mathbf{k}_2=$

$(k_1, \ldots, k_m, k_{m+1}, \ldots, k_{n_2})$ be non-increasing sequences of integers such that $k_m > k_{m+1} = \cdots = k_{n_i} = l$ for $i = 1, 2$. Then $(\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2}, V_1 \otimes V_2)$ is a representation of $\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$. Put $\mathbf{k}_1' = (k_1 - l, \ldots, k_m - l, \overbrace{0, \ldots, 0}^{n_1-m})$ and $\mathbf{k}_2' = (k_1 - l, \ldots, k_m - l, \overbrace{0, \ldots, 0}^{n_2-m})$. Then, $\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2} = (\det^l \otimes \rho_{\mathbf{k}_1'}) \otimes (\det^l \otimes \rho_{\mathbf{k}_2'})$ with $(\rho_{\mathbf{k}_i'}, V_i')$ a polynomial representation of highest weight $\mathbf{k}_i'$ for $i = 1, 2$. To make our formulation smooth, we sometimes regard a modular form of scalar weight k for $\Gamma^{(n)}$ as a function with values in the one-dimensional vector space spanned by $\det U^l$ with a non-negative integer $l \leq k$, where U is an $n \times n$ matrix of variables. Let U_1 and U_2 be $m \times n_1$ and $m \times n_2$ matrices, respectively, of variables and for a commutative ring R and an R-algebra S let

$$S[U_1, U_2]_{\mathbf{k}_1', \mathbf{k}_2'} = \Big\{ \sum_j P_j(U_1) P_j(U_2) \quad (\text{ finite sum }) \text{ with } P_j(U_i) \in S[U_i]_{\mathbf{k}_i'} \ (i = 1, 2) \Big\}.$$

Here we make the convention that $P_j(U_i) \in \langle (\det U_i)^{k_1 - l} \rangle_{\mathbb{C}}$ if $n_i = m$ and $k_1 = \cdots = k_m$ as stated above. Then, as a representation space $W = W_{\mathbf{k}_1', \mathbf{k}_2'}$ of $\rho_{\mathbf{k}_1'} \otimes \rho_{\mathbf{k}_2'}$ we can take $\mathbb{C}[U_1, U_2]_{\mathbf{k}_1', \mathbf{k}_2'}$. Let

$$\widetilde{W} = \widetilde{W}_{\mathbf{k}_1', \mathbf{k}_2'} = \mathbb{Q}[U_1, U_2]_{\mathbf{k}_1', \mathbf{k}_2'}.$$

Then $\widetilde{W} \cong \widetilde{V}_1' \otimes \widetilde{V}_2'$ and $\widetilde{W} \otimes_{\mathbb{Q}} \mathbb{C} = W$. Let

$$M = M_{\mathbf{k}_1', \mathbf{k}_2'} = \mathbb{Z}[U_1, U_2]_{\mathbf{k}_1', \mathbf{k}_2'}.$$

We note that

$$M = \left\{ \sum_{P_{\tau_1} \in \mathcal{S}_{\mathbf{k}_1'}, P_{\tau_2} \in \mathcal{S}_{\mathbf{k}_2'}} a_{\tau_1, \tau_2} P_{\tau_1}(U_1) P_{\tau_2}(U_2) \;\middle|\; a_{\tau_1, \tau_2} \in \mathbb{Z} \right\}.$$

Here we make the convention that $P_{\tau_i}(U_2) = (\det U_i)^{k_1 - l}$ if $n_i = m$ and $k_1 = \cdots = k_m$. Therefore, M is a lattice of $\widetilde{W}$ and $M \cong L_1 \otimes L_2$ with $L_i = \mathbb{Z}[U_i]_{\mathbf{k}_i'}$ $(i = 1, 2)$. Thus $(\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2}, V_1 \otimes V_2)$ has also a $\mathbb{Q}$-structure and $\mathbb{Z}$-structure and we can define $\mathrm{ord}_{\mathfrak{p}}(a \otimes b)$ for $a \otimes b \in \widetilde{W}_K$. If $\dim_{\mathbb{C}} V_1 = 1$, then we identify $V_1, \widetilde{V}_1$ and L_1 with $\mathbb{C}, \mathbb{Q}$ and $\mathbb{Z}$, respectively, and for $a, b \in V_1$ and $w \in V_2$, we write $a \otimes b$ and $a \otimes w$ as ab and aw, respectively through the identifications $V_1 \otimes V_1 \cong V_1$ and $V_1 \otimes V_2 \cong V_2 \otimes V_1 \cong V_2$. The tensor product $M_{\mathbf{k}_1}(\Gamma^{(n_1)}) \otimes M_{\mathbf{k}_2}(\Gamma^{(n_2)})$ is regarded as a $\mathbb{C}$-subspace of $(\mathrm{Hol}(\mathbb{H}_{n_1}) \otimes \mathrm{Hol}(\mathbb{H}_{n_2}))[U_1, U_2]_{\mathbf{k}_1', \mathbf{k}_2'}$.

3. Several automorphic L-functions and their special values

In this section we review several arithmetical properties of Hecke eigenvalues and L values of modular forms in [2, Section 2] without proof. Throughout this section, let $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_1 \geq \cdots \geq k_n \geq 0$. Let $\mathbf{L}_n = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$ be the Hecke algebra over $\mathbb{Z}$ associated to the Hecke pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$ and for a subring R of $\mathbb{C}$ put $\mathbf{L}_n(R) = \mathbf{L}_n \otimes_{\mathbb{Z}} R$. For an element $T = \Gamma^{(n)} g \Gamma^{(n)} \in \mathbf{L}_n(\mathbb{C})$ and $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ we can define $F|T$ as in [2, Section 3]. This defines an action of the Hecke algebra $\mathbf{L}_n(\mathbb{C})$ on $M_{\mathbf{k}}$. The operator $F \mapsto F|T$ with $T \in \mathbf{L}_n(\mathbb{C})$ is called the Hecke operator. We say that F is a Hecke eigenform if F is a common eigenfunction of all Hecke operators $T \in_n (\mathbb{C})$. Then we have

$$F|T = \lambda_F(T) F \text{ with } \lambda_F(T) \in \mathbb{C} \text{ for any } T \in \mathbf{L}_n(\mathbb{C}).$$

We call $\lambda_F(T)$ the Hecke eigenvalue of T with respect to F. For a Hecke eigenform F in $M_{\mathbf{k}}(\Gamma^{(n)})$, we denote by $\mathbb{Q}(F)$ the field generated over $\mathbb{Q}$ by all the Hecke eigenvalues $\lambda_F(T)$

with $T \in \mathbf{L}_n(\mathbb{Q})$ and call it the Hecke field of F. For two Hecke eigenforms F and G we sometimes write $\mathbb{Q}(F,G) = \mathbb{Q}(F)\mathbb{Q}(G)$. We say that an element $T \in \mathbf{L}_n(\mathbb{Q})$ is integral with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$ if $F|T \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$ for any $F \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$. We denote by $\mathbf{L}_n^{(\mathbf{k})}$ the subset of $\mathbf{L}_n(\mathbb{Q})$ consisting of all integral elements with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$. The following two propositions are due to [2, Section 4].

Proposition 3.1. *We have $\mathbf{L}_n \subset \mathbf{L}_n^{(\mathbf{k})}$ for any $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_n \geq n+1$.*

Proposition 3.2. *Let $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_n \geq n+1$. Let F be a Hecke eigenform in $S_{\mathbf{k}}(\Gamma^{(n)})$. Then $\lambda_F(T)$ belongs to $\mathfrak{O}_{\mathbb{Q}(F)}$ for any $T \in \mathbf{L}_n^{(\mathbf{k})}$.*

For a non-zero rational number a, we define an element $[a] = [a]_n$ of $\mathbf{L}_n$ by $[a]_n = \Gamma^{(n)}(a1_n)\Gamma^{(n)}$. For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1,\ldots,d_n,e_1,\ldots,e_n} \Gamma^{(n)}(d_1 \perp \cdots \perp d_n \perp e_1 \perp \cdots \perp e_n)\Gamma^{(n)},$$

where $d_1, \ldots, d_n, e_1, \ldots, e_n$ run over all positive integer satisfying

$$d_i | d_{i+1},\ e_{i+1} | e_i\ (i = 1, \ldots, n-1), d_n | e_n, d_i e_i = m\ (i = 1, \ldots, n).$$

Furthermore, for $i = 1, \ldots, n$ and a prime number p put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i)\Gamma^{(n)}.$$

As is well known, $\mathbf{L}_n(\mathbb{Q})$ is generated over $\mathbb{Q}$ by $T(p), T_i(p^2)$ $(i = 1, \ldots, n)$, and $[p^{-1}]_n$ for all p. We note that $T_n(p^2) = [p]_n$. We note that $\mathbf{L}_n$ is generated over $\mathbb{Z}$ by $T(p)$ and $T_i(p^2)$ $(i = 1, \ldots, n)$ for all p.

Let $\mathbf{L}_{n,p} = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}[p^{-1}]))$ be the Hecke algebra associated with the pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}[p^{-1}]))$. Then $\mathbf{L}_{n,p}$ can be considered as a subalgebra of $\mathbf{L}_n$, and is generated over $\mathbb{Q}$ by $T(p)$ and $T_i(p^2)$ $(i = 1, 2, \ldots, n)$, and $[p^{-1}]_n$.

We now review the Satake p-parameters of $\mathbf{L}_{n,p}$; let $\mathbf{P}_n = \mathbb{Q}[X_0^{\pm}, X_1^{\pm}, \ldots, X_n^{\pm}]$ be the ring of Laurent polynomials in $X_0, X_1, \ldots, X_n$ over $\mathbb{Q}$. Let $\mathbf{W}_n$ be the group of $\mathbb{Q}$-automorphisms of $\mathbf{P}_n$ generated by all permutations in variables $X_1, \ldots, X_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j\ (j \neq i).$$

Moreover, a group $\widetilde{\mathbf{W}}_n$ isomorphic to $\mathbf{W}_n$ acts on the set $T_n = (\mathbb{C}^\times)^{n+1}$ in a way similar to the above. Then there exists a $\mathbb{Q}$-algebra isomorphism $\Phi_{n,p}$, called the Satake isomorphism, from $\mathbf{L}_{n,p}$ to the $\mathbf{W}_n$-invariant subring $\mathbf{P}_n^{\mathbf{W}_n}$ of $\mathbf{P}_n$. Then for a $\mathbb{Q}$-algebra homomorphism λ from $\mathbf{L}_{n,p}$ to $\mathbb{C}$, there exists an element $(\alpha_0(p,\lambda), \alpha_1(p,\lambda), \ldots, \alpha_n(p,\lambda))$ of $\mathbf{T}_n$ satisfying

$$\lambda(\Phi_{n,p}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p,\lambda), \alpha_1(p,\lambda), \ldots, \alpha_n(p,\lambda))$$

for $F \in \mathbf{P}_n^{\mathbf{W}_n}$. The equivalence class of $(\alpha_0(p,\lambda), \alpha_1(p,\lambda), \ldots, \alpha_n(p,\lambda))$ under the action of $\widetilde{\mathbf{W}}_n$ is uniquely determined by λ. We call this the Satake parameters of $\mathbf{L}_{n,p}$ determined by λ. Now let F be a Hecke eigenform in $M_{\mathbf{k}}(\Gamma^{(n)})$. Then for each prime number p, F defines a $\mathbb{Q}$-algebra homomorphism $\lambda_{F,p}$ from $\mathbf{L}_{n,p}$ to $\mathbb{C}$ in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), \ldots, \alpha_n(p)$ the Satake parameters of $\mathbf{L}_{n,p}$ determined by F.

We write $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ as usual. Let

$$f(z) = \sum_{m=1}^{\infty} a(m,f)\mathbf{e}(mz)$$

be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$, that is, let f be a Hecke eigenform whose first coefficient is 1. For a prime number p let $\beta_{1,p}(f)$ and $\beta_{2,p}(f)$ be complex numbers such that $\beta_{1,p}(f)+\beta_{2,p}(f)=a(p,f)$ and $\beta_{1,p}(f)\beta_{2,p}(f)=p^{k-1}$. Then for a Dirichlet character χ we define the Hecke L-function twisted by χ as

$$L(s,f,\chi)=\prod_p \bigl((1-\beta_{1,p}(f)\chi(p)p^{-s})(1-\beta_{2,p}(f)\chi(p)p^{-s})\bigr)^{-1}.$$

We write $L(s,f,\chi)=L(s,f)$ if χ is the principal character.

Let $\{f_1,\ldots,f_d\}$ be a basis of $S_k(\Gamma^{(1)})$ consisting of primitive forms. Let K be an algebraic number field containing $\mathbb{Q}(f_1)\cdots\mathbb{Q}(f_d)$, and $\mathfrak{O}$ the ring of integers in K. Let f be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then Shimura [29] showed that there exist two complex numbers $c_\pm(f)$, uniquely determined up to multiplication by elements of $\mathbb{Q}(f)^\times$, such that the following property holds: The value $\dfrac{\Gamma_{\mathbb{C}}(l)\sqrt{-1}^l L(l,f,\chi)}{\tau(\chi)c_s(f)}$ belongs to $\mathbb{Q}(f)(\chi)$ for any positive integer $l\le k-1$ and a Dirichlet character χ, where $\tau(\chi)$ is the Gauss sum of χ, and $s=s(l,\chi)=+$ or $-$ according as $\chi(-1)=(-1)^l$ or $(-1)^{l-1}$.

We note that the above value belongs to $K(\chi)$. For short, we write

$$\mathbf{L}(l,f,\chi;c_s(f))=\frac{\Gamma_{\mathbb{C}}(l)\sqrt{-1}^l L(l,f,\chi)}{\tau(\chi)c_s(f)}.$$

We sometimes write $c_{s(l,\chi)}(f)=c_{s(l)}(f)$ and $\mathbf{L}(l,f,\chi;c_{s(l,\chi)}(f))=\mathbf{L}(l,f;c_{s(l)}(f))$ if χ is the principal character. We note that the value $\mathbf{L}(l,f,\chi;c_s(f))$ depends on the choice of $c_s(f)$, but if $(\chi\eta)(-1)=(-1)^{l+m}$, then $s:=s(l,\chi)=s(m,\eta)$ and, the ratio $\dfrac{\mathbf{L}(l,f,\chi;c_s(f))}{\mathbf{L}(m,f,\eta;c_s(f)}$ does not depend on $c_s(f)$, which will be denoted by $\dfrac{\mathbf{L}(l,f,\chi)}{\mathbf{L}(m,f,\eta)}$.

Let f be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Let $f_1,\ldots,f_d$ be a basis of $S_k(\mathrm{SL}_2(\mathbb{Z}))$ consisting of primitive forms with $f_1=f$ and let $\mathfrak{D}_f$ be the ideal of $\mathbb{Q}(f)$ generated by all $\prod_{i=2}^d(\lambda_{f_i}(T(m))-\lambda_f(T(m)))$'s ($m\in\mathbb{Z}_{>0}$). For a prime ideal $\mathfrak{p}$ of an algebraic number field, let $p_{\mathfrak{p}}$ be the prime number such that $(p_{\mathfrak{p}})=\mathbb{Z}\cap\mathfrak{p}$.

Let F be a Hecke eigenform in $M_{\mathbf{k}}(\Gamma^{(n)})$, and for a prime number p we take the p-Satake parameters $\alpha_0(p),\alpha_1(p),\ldots,\alpha_n(p)$ of F so that

$$\alpha_0(p)^2\alpha_1(p)\cdots\alpha_n(p)=p^{k_1+\cdots+k_n-n(n+1)/2}.$$

We define the polynomial $L_p(X,F,\mathrm{Sp})$ by

$$L_p(X,F,\mathrm{Sp})=(1-\alpha_0(p)X)\prod_{r=1}^n\prod_{1\le i_1<\cdots<i_r\le n}(1-\alpha_0(p)\alpha_{i_1}(p)\cdots\alpha_{i_r}(p)X)$$

and the spinor L function $L(s,F,\mathrm{Sp})$ by

$$L(s,F,\mathrm{Sp})=\prod_p L_p(p^{-s},F,\mathrm{Sp})^{-1}.$$

We note that $L(s,f,\mathrm{Sp})$ is the Hecke L-function $L(s,f)$ if f is a primitive form. In this case we write $L_p(s,f)$ for $L_p(s,f,\mathrm{Sp})$. We also define the polynomial $L_p(X,F,\mathrm{St})$ by

$$(1-X)\prod_{i=1}^{n}(1-\alpha_i(p)X)(1-\alpha_i(p)^{-1}X)$$

and the standard L-function $L(s,F,\mathrm{St})$ by

$$L(s,F,\mathrm{St})=\prod_{p}L_p(p^{-s},F,\mathrm{St})^{-1}.$$

For a Hecke eigenform $F\in S_k(\Gamma^{(r)})$ put

$$\mathbf{L}(s,F,\mathrm{St})=\Gamma_{\mathbb{C}}(s)\prod_{i=1}^{r}\Gamma_{\mathbb{C}}(s+k-i)\frac{L(s,F,\mathrm{St})}{(F,\ F)}.$$

Remark 3.3. *We note that for a positive integer $m\le k-r$*

$$\mathbf{L}(m,F,\mathrm{St})=A_{r,k,m}\frac{L(m,F,\mathrm{St})}{\pi^{r(k+m)+m-r(r+1)/2}(F,\ F)}$$

with an element $A_{r,k,m}\in\mathbb{Z}[2^{-1}]$ such that $\mathrm{ord}_p(A_{r,k,m})=0$ for any prime number $p\ge 2k-r-1$.

Proposition 3.4. *Let F be a Hecke eigenform in $S_k(\Gamma^{(r)})$. We define $n_0=3$ if $r\ge 5$ with $r\equiv 1 \bmod 4$ and $n_0=1$ otherwise. Let m be a positive integer $n_0\le m\le k-r$ such that $m\equiv r \pmod 2$. Then, $a(A,F)\overline{a(B,F)}\mathbf{L}(m,F,\mathrm{St})$ belongs to $\mathbb{Q}(F)$ for any $A,B\in\mathcal{H}_r(\mathbb{Z})_{>0}$.*

Proof. We note that the value $a(A,F)\overline{a(B,F)}\mathbf{L}(m,F,\mathrm{St})$ remains unchanged if we replace F by γF with any $\gamma\in\mathbb{C}^\times$. By the multiplicity one theorem for Hecke eigenforms (cf. [2, Appendix A]), we can take some non-zero complex number γ such that $\gamma F\in S_k(\Gamma^{(r)})(\mathbb{Q}(F))$. For this γ, we see $\mathbf{L}(m,\gamma F,\mathrm{St})\in\mathbb{Q}(F)$ by [28], Appendix A. This proves the assertion. □

4. Harder's conjecture and its modification

In this section, first we state the original Harder's conjecture in [10], and we treat a generalized version of this conjecture. Let R be a commutative ring, and $\mathfrak{a}$ an ideal of R. For two polynomials $P(X)=\sum_{i=0}^{n}a_iX^i$ and $Q(X)=\sum_{i=0}^{n}b_iX^i$ with coefficients in R, we write

$$P(X)\equiv Q(X)\mod \mathfrak{a}$$

if $a_i\equiv b_i \pmod{\mathfrak{a}}$ for any $0\le i\le m$. When R is a ring of integers in an algebraic number field and $\mathfrak{p}$ is a prime ideal of R, for two polynomial $P(X),Q(X)\in R_{\mathfrak{p}}[X]$ we sometimes write $P(X)\equiv Q(X)\mod \mathfrak{p}$ if $P(X)\equiv Q(X)\mod R_{\mathfrak{p}}\mathfrak{p}$. Now we will state Harder's conjecture.

Conjecture 4.1. *([10]) Let k and j be non-negative integers such that j is even and $k\ge 3$. Let $f=\sum a(n,f)\mathbf{e}(nz)\in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, and suppose that a "large" prime $\mathfrak{p}$ of $\mathbb{Q}(f)$ divides $\mathbf{L}(k+j,f;c_{s(k+j)})$. Then, there exists a Hecke eigenform $F\in S_{(k+j,k)}(\Gamma^{(2)})$, and a prime ideal $\mathfrak{p}'\mid\mathfrak{p}$ in (any field containing) $\mathbb{Q}(f)\mathbb{Q}(F)$ such that, for all primes p*

$$L_p(X,F,\mathrm{Sp})\equiv L_p(X,f)(1-p^{k-2}X)(1-p^{j+k-1}X)\pmod{\mathfrak{p}'}.$$

In particular,

$$\lambda_F(T(p))\equiv p^{k-2}+p^{j+k-1}+a(p,f)\pmod{\mathfrak{p}'}.$$

To avoid the ambiguity on choosing $c_{s(k+j)}$ (cf. [2, Remank 3.8. (2)]), we propose the following conjecture, which we also call Harder's conjecture.

Conjecture 4.2. *Let k and j be non-negative integers such that $k \geq 3$ and $j \geq 4$ is even. Let f be as that in Conjecture 4.1. Suppose that a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(f)$ satisfies $p_{\mathfrak{p}} > 2k+j-2$ and that $\mathfrak{p}$ divides $\dfrac{\mathbf{L}(k+j,f)}{\mathbf{L}(k_j,f)}$, where $k_j = k+j/2$ or $k+j/2+1$ according as $j \equiv 0 \pmod 4$ or $j \equiv 2 \pmod 4$. Then the same assertion as Conjecture 4.1 holds.*

The above conjecture does not address the congruence between the Hecke eigenvalues of two Hecke eigenforms in the same space, and this is one of the reasons that it is not easy to confirm it. To make it more approachable, we reformulate it in the case k is odd (cf. Conjecture 4.6). For even k, see [2], [3].

To do so, first, we consider several lifts. The first two theorems are special cases of [2, Theorem 4.2. (1)] and [2, Theorem 4.3], respectively.

Theorem 4.3. *Let k, j, n be positive integers such that j, n are even and k is odd. Suppose that $k \geq n+3, j \geq n+4$ and $j \equiv n \mod 4$. Put*

$$\mathbf{k} = \Big(\overbrace{\frac{j}{2}+k+\frac{n}{2}-1, \ldots, \frac{j}{2}+k+\frac{n}{2}-1}^{n+1}, \overbrace{\frac{j}{2}+\frac{3n}{2}+2, \ldots, \frac{j}{2}+\frac{3n}{2}+2}^{n}\Big).$$

Then, for a primitive form $g \in S_{k+j/2-n/2-1}(\mathrm{SL}_2(\mathbb{Z}))$ and a Hecke eigenform $G \in S_{(k+j,k)}(\Gamma^{(2)})$, there exists a Hecke eigenform $\mathcal{A}^{\mathbf{k}}_{2n+1}(G,g) \in S_{\mathbf{k}}(\Gamma^{(2n+1)})$ such that

$$L(s, \mathcal{A}^{\mathbf{k}}_{2n+1}(G,g), \mathrm{St}) = L(s,g,\mathrm{St}) \prod_{i=1}^{n} L(s+k+\frac{j}{2}+\frac{n}{2}-1-i, G, \mathrm{Sp})$$

Theorem 4.4. *Let k, n, d be positive integers such that $k > d$. Let f be a primitive form in $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and G a Hecke eigenform in $S_l(\Gamma^{(n)})$.*

(1) *Suppose that $k \equiv n+d \mod 2$ and that $l \geq k+n+d$. Then there exists a Hecke eigenform $\mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G) \in S_{\mathbf{k}'}(\Gamma^{(n+2d)})$ with $\mathbf{k}' = (\overbrace{l, \ldots, l}^{n}, \overbrace{k+n+d, \ldots, k+n+d}^{2d})$ such that*

$$L(s, \mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G), \mathrm{St}) = L(s,G,\mathrm{St}) \prod_{i=1}^{2d} L(s+k+d-i, f).$$

(2) *Suppose that $k \equiv d \mod 2$ and that $k+d \geq l$. Then, there exists a Hecke eigenform $\mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G) \in S_{\mathbf{k}'}(\Gamma^{(n+2d)})$ with $\mathbf{k}' = (\overbrace{k+d, \ldots, k+d}^{2d}, \overbrace{l, \ldots, l}^{n})$ such that*

$$L(s, \mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G), \mathrm{St}) = L(s,G,\mathrm{St}) \prod_{i=1}^{2d} L(s+k+d-i, f).$$

In (1) and (2), we make the convention that $L(s,G,\mathrm{St}) = \zeta(s)$ if $n = 0$.

We say that $\mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G)$ in Theorem 4.4 (1) (resp. (2)) is the Miyawaki lift of f and G of type I (resp, type II). We sometimes write $\mathcal{M}^{l}_{n+2d}(f,G)$ instead of $\mathcal{M}^{\mathbf{k}'}_{n+2d}(f,G)$ if $\mathbf{k}' = (l, \ldots, l)$. In this case, the Miyawaki lift of type I was constructed by Ikeda [19] under the non-vanishing condition. In the case $n=0$ in Theorem 4.4, we write $\mathcal{I}_{2d}(f)$ instead of $\mathcal{M}_{2d}^{\mathbf{k}'}(f,G)$, and we call it the Duke-Imamoglu-Ikeda lift of f (cf. [18]).

Theorem 4.5. (1) *Let k and l be positive even integers such that $k \geq l$, and put $\mathbf{k} = (k,k,k,k,l)$. Then, for a primitive form $f \in S_{2k-6}(\mathrm{SL}_2(\mathbb{Z}))$ and Hecke eigenform $G \in S_{(k,k,l-2)}(\Gamma^{(3)})$, there exists a Hecke eigenform $\mathcal{K}_5(f,G) \in S_{\mathbf{k}}(\Gamma^{(5)})$ such that*

$$L(s,\mathcal{K}_5(f,G),\mathrm{St}) = L(s,G,\mathrm{St})\prod_{i=1}^{2} L(s+k-2-i,f).$$

(2) *Let k and l be even positive integers such that $k \geq l$. Then, for a primitive $f \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ and a Hecke eigenform $G \in S_{(k,l-2)}(\Gamma^{(2)})$, there exists a Hecke eigenform $\mathcal{A}_4^{II}(f,G) \in S_{(k,k,k,l)}(\Gamma^{(4)})$ such that*

$$L(s,\mathcal{A}_4^{II}(f,G),\mathrm{St}) = L(s,G,\mathrm{St})\prod_{i=1}^{2} L(s+k-1-i,f).$$

Proof. The assertion (2) for $l=k$ has been proved in [2, Theorem 4.2 (2)], and another case can also be proved similarly. From now on we use the notation in [2, Appendix A]. To prove (1), put

$$\psi = \psi_G \boxplus \pi_f[2],$$

where ψ_G is the Arthur parameter associated with G, and π_f is the irreducible (unitary cuspidal automorphic self-dual) representation of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ such that

$$L^\infty(s,\pi_f) = L(s+\frac{2k-7}{2},f).$$

Then, ψ_G is one of the following forms:

- (i) $\psi_G = \pi_G[1]$ with π_G an irreducible representation of $\mathrm{PGL}_7(\mathbb{A}_\mathbb{Q})$ such that
$$L^\infty(s,\pi_G) = L(s,G,\mathrm{St}).$$
- (ii) $\psi_G = \pi_0[1] \boxplus \pi_1[2]$ with π_0 and π_1 irreducible representations of $\mathrm{PGL}_3(\mathbb{A}_\mathbb{Q})$ and $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$, respectively, such that
$$L^\infty(s,\pi_0) = L(s,g,\mathrm{St}) \text{ with } g \in S_{l-4}(\mathrm{SL}_2(\mathbb{Z})),$$
$$L^\infty(s,\pi_1) = L(s+\frac{2k-3}{2},f_1) \text{ with } f_1 \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})).$$

Suppose that (i) holds. We note that the sets of positive eigenvalues of the infinitesimal characters of $\pi_{G,\infty}$ and $\pi_{f,\infty}$ are $\{(2k-7)/2\}$ and $\{k-1,k-2,l-5\}$, respectively. Therefore, we easily see that ψ satisfies the conditions in [2, Theorem A.1] except (f). Moreover, by [2, Remark A.2 (4)], we have

$$\varepsilon(\pi_f \times \pi_G)^{\min(2,1)} = \varepsilon(\pi_f \times \pi_G) = -1 = (-1)^{\frac{2\cdot 2}{4}},$$

and ψ also satisfies the condition (f). Therefore the assertion follows from [2, Theorem A.1].

Suppose that (ii) holds. We note that the sets of positive eigenvalues of the infinitesimal characters of $\pi_{0,\infty}$ and $\pi_{1,\infty}$ are $\{l-5\}$ and $\{(2k-3)/2\}$, respectively. Therefore, by [2, Remark A.2 (4)] we have

$$\varepsilon(\pi_f \times \pi_0)^{\min(2,1)}\varepsilon(\pi_f \times \pi_1)^{\min(2,2)} = \varepsilon(\pi_f \times \pi_0) = -1 = (-1)^{\frac{2\cdot 2}{4}},$$

and
$$\varepsilon(\pi_1 \times \pi_0)^{\min(2,1)}\varepsilon(\pi_1 \times \pi_f)^{\min(2,2)} = \varepsilon(\pi_1 \times \pi_0) = -1 = (-1)^{\frac{2\cdot 2}{4}}.$$
Thus the assertion has been proved similarly to (i). $\square$

Let F and G be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma^{(n)})$ and $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(F)$. We say that F is Hecke congruent to G modulo $\mathfrak{p}$ if there is a prime ideal $\mathfrak{p}'$ of $\mathbb{Q}(F)\cdot\mathbb{Q}(G)$ lying above $\mathfrak{p}$ such that
$$\lambda_G(T) \equiv \lambda_F(T) \pmod{\mathfrak{p}} \text{ for any } T \in \mathbf{L}_n^{(\mathbf{k})}.$$
We denote this property by
$$G \equiv_{\mathrm{ev}} F \pmod{\mathfrak{p}}.$$

Conjecture 4.6. *Let k, j and n be positive integers. Suppose that*

(a) $n \equiv k-1 \equiv j \equiv 0 \mod 2$ *and* $j \equiv n \mod 4$.
(b) $k > n+1$ *and* $j > n-1$.

Put
$$\mathbf{k} = \Bigl(\overbrace{\frac{j}{2}+k+\frac{n}{2}-1,\ldots,\frac{j}{2}+k+\frac{n}{2}-1}^{n+1},\overbrace{\frac{j}{2}+\frac{3n}{2}+2,\ldots,\frac{j}{2}+\frac{3n}{2}+2}^{n}\Bigr).$$
Let f and g be primitive forms in $S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ and in $S_{k+\frac{j}{2}-\frac{n}{2}-1}(\mathrm{SL}_2(\mathbb{Z}))$, respectively. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(f)$ such that $p_{\mathfrak{p}} > 2k+j-2$ and suppose that $\mathfrak{p}$ divides $\dfrac{\mathbf{L}(k+j,f)}{\mathbf{L}(j/2+k+n/2-1,f)}$. Then, there exists a Hecke eigenform $F \in S_{(k+j,k)}(\Gamma^{(2)})$ such that
$$\mathcal{A}^{\mathbf{k}}_{2n+1}(F,g) \equiv_{\mathrm{ev}} [\mathcal{M}_{n+1}(f,g)]^{\mathbf{k}} \pmod{\mathfrak{p}}.$$

Remark 4.7. *Since we have $j+3n/2+2 > 3n/2+1$, $[\mathcal{M}_{n+1}(f,g)]^{\mathbf{k}}$ belongs to $M_{\mathbf{k}}(\Gamma^{(2n+1)})$ by* [2, Propoition 2.1, (2)].

Theorem 4.8. *Let the notation be as in Conjecture 4.6.*

(1) *Conjecture 4.2 holds for the case $j \equiv 2 \mod 4$ if Conjecture 4.6 holds for $n=2$.*
(2) *Suppose that $2k+j-2 \geq 20$. Then Conjecture 4.2 holds for the case $j \equiv 0 \mod 4$ if Conjecture 4.6 holds for $n=4$.*

Proof. The assertion can be proved in the same way as [2, Theorem 4.8]. $\square$

5. Pullback formula

In this section, we review the pullback formula for the Siegel Eisenstein series with differential operators in [2, Section 5], and give a generalization of [2, Theorem 5.8]. We also give an explicit differential operator which is used in the proof of our main results.

Now for an integer $n \geq 2$, fix a partition (n_1, n_2) with $n = n_1 + n_2$ with $n_i \geq 1$. Let λ be a dominant integral weight with $\mathrm{depth}(\lambda) \leq \min(n_1, n_2)$. For $i = 1, 2$, let $(\rho_{n_i,\lambda}, V_{n_i,\lambda})$ be the representation of $\mathrm{GL}_{n_i}(\mathbb{C})$ defined in Section 2. Put $V_{\lambda,n_1,n_2} = V_{n_1,\lambda} \otimes V_{n_2,\lambda}$. We regard $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$ as a subset of $\mathbb{H}_n$ by the diagonal embedding.

We consider V_{λ,n_1,n_2}-valued differential operators $\mathbb{D}$ on scalar-valued functions of $\mathbb{H}_n$, satisfying Condition $\mathbf{C}(k, \lambda, n_1, n_2)$ below on automorphy.

For irreducible representations (ρ_i, V_i) of $\mathrm{GL}_{n_i}(\mathbb{C})$ for $i = 1, 2$, a $(V_1 \otimes V_2)$-valued function f on $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$, and $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathrm{Sp}_{n_i}(\mathbb{R})$, we write

$$(f|_{\rho_1,\rho_2}[g_1, g_2])(Z_1, Z_2) = \left(\rho_1(C_1Z_1 + D_1)^{-1} \otimes \rho_2(C_2Z_2 + D_2)^{-1}\right) f(g_1Z_1, g_2Z_2), \quad Z_i \in \mathbb{H}_{n_i}.$$

We regard $\mathrm{Sp}_{n_1}(\mathbb{R}) \times \mathrm{Sp}_{n_2}(\mathbb{R})$ as a subgroup of $\mathrm{Sp}_n(\mathbb{R})$ by

$$\iota(g_1, g_2) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \qquad (g_i \in \mathrm{Sp}_{n_i}(\mathbb{R}) \text{ for } i = 1, 2).$$

For variables $Z = (z_{ij})$ of $\mathbb{H}_n$, we denote by ∂_Z the following $n \times n$ symmetric matrix of partial derivations

$$\partial_Z = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}}\right)_{1 \le i,j \le n}.$$

From now on, for an $m \times n$ matrix $U = (u_{ij})_{1 \le i \le m, 1 \le j \le n}$ of variables, we say that $Q(U)$ is a polynomial in U if it is a polynomial in u_{ij} $(1 \le i \le m, 1 \le j \le n)$. In particular if U is an $n \times n$ symmetric matrix of variables, we say that $Q(U)$ is a polynomial in U if it is a polynomial in u_{ij} $(1 \le i \le j \le n)$.

Fix k, λ, n_1, and n_2 with $\mathrm{depth}(\lambda) \le \min(n_1, n_2)$. Let $\mathbb{D} = P(\partial_Z)$ for a V_{λ,n_1,n_2}-valued polynomial $P(T)$ in an $n \times n$ symmetric matrix T. Assume that for any holomorphic function F on $\mathbb{H}_n$ and any $(g_1, g_2) \in \mathrm{Sp}_{n_1}(\mathbb{R}) \times \mathrm{Sp}_{n_2}(\mathbb{R})$, the operator $\mathbb{D}$ satisfies

$$(\mathbf{C}(k, \lambda, n_1, n_2)) \qquad \mathrm{Res}(\mathbb{D}(F|_k[\iota(g_1, g_2)])) = (\mathrm{Res}\ \mathbb{D}(F))|_{\det^k \otimes \rho_{n_1,\lambda}, \det^k \otimes \rho_{n_2,\lambda}}[g_1, g_2],$$

where Res means the restriction of a function on $\mathbb{H}_n$ to $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$. In such a case, we say that the operator $\mathbb{D}$ satisfies Condition $\mathbf{C}(k, \lambda, n_1, n_2)$.

For $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in \mathbb{H}_n$ with $Z_1 \in \mathbb{H}_{n_1}$, $Z_2 \in \mathbb{H}_{n_2}$, and $Z_{12} \in M_{n_1,n_2}(\mathbb{C})$, we sometimes write $\mathbb{D}(F)\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}$ instead of $\mathrm{Res}\ \mathbb{D}(F(Z))$. This condition on $\mathbb{D}$ can be roughly described as the requirement that if F is a Siegel modular form of degree n of weight k, then $\mathrm{Res}(\mathbb{D}(F))$ is a Siegel modular form of weight $\det^k \otimes \rho_{n_i,\lambda}$ for each variable Z_i for $i = 1, 2$. Here, if $2k \ge n$, the condition that ρ_1 and ρ_2 correspond to the same λ is a necessary and sufficient condition for the existence of $\mathbb{D}$ ([11]). We note that such a differential operator is uniquely determined up to constant if $k \ge n_1 + n_2$.

Now, we consider some special type of λ. We assume that $\lambda = (\overbrace{l, \dots, l}^{m}, 0, \dots, 0)$. We assume that $\lambda = (\overbrace{l, \dots, l}^{m}, 0, \dots, 0)$. Let S be a $2m \times 2m$ symmetric matrix of variables. Let $\mathcal{D}_{m,\alpha}^{\nu}$ be the differential operator in [5, (1.14)]. Then, for any holomorphic function F on $\mathbb{H}_n$ and any $(g_1, g_2) \in \mathrm{Sp}_m(\mathbb{R}) \times \mathrm{Sp}_m(\mathbb{R})$,

$$\mathrm{Res}\ (\mathcal{D}_{m,k}^{l}(F|_k[\iota(g_1, g_2)])) = (\mathrm{Res}\ \mathcal{D}_{m,k}^{l}(F))|_{\det^{k+l}, \det^{k+l}}[g_1, g_2],$$

and there exists a polynomial $\widetilde{P}_{m,k,k+l}$ such that $\mathcal{D}_{m,k}^{l} = \widetilde{P}_{m,k,k+l}(\partial_W)$, where $W = (w_{ij})$ denotes the variables of $\mathbb{H}_{2m}$.

Now we review realization of representations of $\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ by bideterminants. Let U, V be $m \times n_1$ and $m \times n_2$ matrices of independent variables respectively. Let $\lambda = (l, \dots, l, 0, \dots, 0)$ such that $\mathrm{depth}(\lambda) = m$. For integers n_1 and n_2 such that $n_1, n_2 \geq m$, put $\mathbf{k}_1{}' = (\overbrace{l, \dots, l}^{m}, \overbrace{0, \dots, 0}^{n_1 - m})$ and $\mathbf{k}_2{}' = (\overbrace{l, \dots, l}^{m}, \overbrace{0, \dots, 0}^{n_2 - m})$, and let $\mathbb{C}[U, V]_{\mathbf{k}_1', \mathbf{k}_2'}$ be the vector space defined in Section 2. Then, we can take $\mathbb{C}[U, V]_{\mathbf{k}_1', \mathbf{k}_2'}$ as a representation space of $\rho_{n_1,\lambda} \otimes \rho_{n_2,\lambda}$ as explained in Section 2. We denote by $\mathbb{U}$ the following $2m \times n$ matrix, where $n = n_1 + n_2$:

$$\mathbb{U} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Then, by [2, Proposition 5.2], we obtain the following proposition.

Proposition 5.1. *Notation being as above, consider $\lambda = (l, \dots, l, 0, \dots, 0)$ such that $\mathrm{depth}(\lambda) = m$. For a partition (n_1, n_2) of $n = n_1 + n_2$, we assume that $m \leq \min(n_1, n_2)$. Let T be an $n \times n$ symmetric matrix. Then for $Q_{k,\lambda,n_1,n_2}(T) = P_{m,k,k+l}(\mathbb{U} T {}^t\mathbb{U})$, the differential operator $\mathbb{D}_{k,\lambda,n_1,n_2} = Q_{k,\lambda,n_1,n_2}(\partial_Z)$ satisfies Condition $\mathbf{C}(k, \lambda, n_1, n_2)$.*

Remark 5.2. *By Proposition 5.1, the operator $\mathrm{Res}\ \mathbb{D}_{k,\lambda,n_1,n_2}$ sends $M_k^\infty(\Gamma^{(n_1+n_2)})$ (resp. $M_k(\Gamma^{(n_1+n_2)})$) to $M^\infty_{\det^k \otimes \rho_{n_1,\lambda}}(\Gamma^{(n_1)}) \otimes M^\infty_{\det^k \otimes \rho_{n_2,\lambda}}(\Gamma^{(n_2)})$ (resp. $M_{\det^k \otimes \rho_{n_1,\lambda}}(\Gamma^{(n_1)}) \otimes M_{\det^k \otimes \rho_{n_2,\lambda}}(\Gamma^{(n_2)})$). In particular, $\mathbb{D}_{k,\lambda,n_1,n_2} E_{n_1+n_2,k}\left(\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}, s\right) = (\mathrm{Res}\ \mathbb{D}_{k,\lambda,n_1,n_2} E_{n_1+n_2,k}(*, s))(Z_1, Z_2)$ belongs to $M^\infty_{\det^k \otimes \rho_{n_1,\lambda}}(\Gamma^{(n_1)}) \otimes M^\infty_{\det^k \otimes \rho_{n_2,\lambda}}(\Gamma^{(n_2)})$, and is slowly increasing as function of Z_1 and Z_2. Moreover, if $l > 0$ and $n_1 = m$, $(\mathrm{Res}\ \mathbb{D}_{k,\lambda,m,n_2})(M_k(\Gamma^{(m+n_2)})) \subset S_{\det^k \otimes \rho_{m,\lambda}}(\Gamma^{(m)}) \otimes M_{\det^k \otimes \rho_{n_2,\lambda}}(\Gamma^{(n_2)})$.*

For our later purpose, we give an explicit formula for Q_{k,λ,n_1,n_2} in the case $\lambda = (2, 2, 2, 0, \dots, 0)$ and $\min(n_1, n_2) \geq 3$. Let $T = \begin{pmatrix} R & W \\ {}^tW & S \end{pmatrix}$ be a symmetric matrix of variables of size 6. Define

$$P_0(T) = -(\det W)^2,$$

$$P_1(T) = \sum_{i_1=1}^{3} \sum_{\substack{4 \leq i_2 < i_3 \leq 6, 1 \leq i_4 < i_5 < i_6 \leq 6 \\ \{i_4, i_5, i_6\} \cap \{i_1, i_2, i_3\} = \emptyset}} (-1)^{i_1+i_2+i_3} \det\left(T\begin{pmatrix} i_1 & i_2 & i_3 \\ 1 & 2 & 3 \end{pmatrix}\right) \det\left(T\begin{pmatrix} i_4 & i_5 & i_6 \\ 4 & 5 & 6 \end{pmatrix}\right),$$

$$P_3(T) = \det R \det S,$$

$$P_2(T) = \det T - P_0(T) - P_1(T) - P_3(T),$$

and

$$Q^2_{3,k}(T) = \frac{2(k-1)(2k-3)(k-2)}{3} P_0(T) + \frac{(k-1)(2k-3)}{3} P_1(T) + \frac{2(k-1)}{3} P_2(T) + P_3(T).$$

Remark 5.3. *There is a misprint in [16]. The inequality '$4 < i_2 < i_3 \leq 6$' on page 15, line 12 should read '$4 \leq i_2 < i_3 \leq 6$'.*

Then, by [16, Section 4], we have the following lemma.

Lemma 5.4. *Let $\lambda = (2, 2, 2, 0, \dots, 0)$. Let U and V be $3 \times n_1$ and $3 \times n_2$ matrices of variables, respectively. Put $\mathbb{U} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. Then we have*

$$Q^2_{3,k} = c(k) P_{3,k,k+2},$$

and therefore,

$$Q_{3,k}^2(\mathbb{U}T^t\mathbb{U}) = c(k)Q_{k,\lambda,n_1,n_2}(T),$$

where $c(k)$ is a non-zero rational number and in particular, $c(k)$ belongs to $\mathbb{Z}_{(p)}^\times$ for any prime number $p > 2k+4$.

Let n_1, n_2 be positive integers such that $n_1 \le n_2$. Let λ be a dominant integral weight such that $\mathrm{depth}(\lambda) \le n_1$. For an integer r such that $\mathrm{depth}(\lambda) \le r$, we put $\rho_r = \det^k \otimes \rho_{r,\lambda}$. For a Hecke eigenform $f \in S_{\rho_r}(\Gamma^{(r)})$ we define $D(s,f)$ as

$$D(s,f) = \zeta(s)^{-1}\prod_{i=1}^{r}\zeta(2s-2i)^{-1}L(s-r,f,\mathrm{St}).$$

For any polynomial $Q(U)$ with complex coefficients, we denote by $\overline{Q}(U) = \overline{Q(U)}$ the polynomial obtained by changing the coefficients of $Q(U)$ by the complex conjugates. For any function f, we write $(\theta f)(Z) = \overline{f(-\overline{Z})}$. This means that if f is a Fourier series of the following form

$$f(Z) = \sum_T a(T)\mathbf{e}(\mathrm{tr}(TZ))$$

with $a(T) = a(T)(U)$ a polynomial in U, then we have

$$(\theta f)(Z) = \sum_T \overline{a(T)}\mathbf{e}(\mathrm{tr}(TZ)).$$

So if we take $a(T)$ to be real, we just have $\theta f = f$.

The next theorem is a pullback formula due to [2, Theorem 5.6].

Theorem 5.5. *Let $\lambda = (l,\dots,l,0,\dots,0), n_1, n_2, k$ and $\mathbb{D}_{k,\lambda,n_1,n_2}$ be those in Proposition 5.1. Besides we assume that k is even and $n_2 \ge n_1$. Let $s \in \mathbb{C}$ such that $2\,\mathrm{Re}(s)+k > n_1+n_2+1$. Then for any Hecke eigenform $f \in S_{\rho_{n_1}}(\Gamma^{(n_1)})$ we have*

$$\left(f, \mathbb{D}_{k,\lambda,n_1,n_2}E_{n_1+n_2,k}\left(\begin{pmatrix} * & O \\ O & -\overline{W}\end{pmatrix}, \overline{s}\right)\right) = c(s,\rho_{n_1})D(2s+k,f)[f]_{\rho_{n_1}}^{\rho_{n_2}}(W,s),$$

where $c(s,\rho_{n_1})$ is a function of s depending on ρ_{n_1} but not on n_2.

Then we have a weak type of the pullback formula. Let k and l be non-negative integers. For the dominant integral $\lambda = (\overbrace{l,\dots,l}^{m},0,\dots,0)$ of depth m_0 and integers n_1, n_2 such that $m_0 \le n_1 \le n_2$, let $\rho_{n_1} = \det^k \otimes \rho_{n_1,\lambda}$ and $\rho_{n_2} = \det^k \otimes \rho_{n_2,\lambda}$ be the representations of $\mathrm{GL}_{n_1}(\mathbb{C})$ and $\mathrm{GL}_{n_2}(\mathbb{C})$, respectively, as above. We note that $m_0 = 0$ or $m_0 = m$ according as $l = 0$ or $l > 0$. Moreover, let $\mathbb{D}_{k,\lambda,n_1,n_2}$ be the differential operator corresponding to the polynomial Q_{k,λ,n_1,n_2} in Proposition 5.1. The following theorem can be proved in the same way as [2, Theorem 5.7] using Theorem 5.5 (see also [15]).

Theorem 5.6. *Let the notation be as above. We define a subspace $\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)})$ of $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ as*

$$\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)}) = \{F \in M_{\rho_{n_1}}(\Gamma^{(n_1)}) \mid \Phi_m^{n_1}(F) \in S_{\rho_m}(\Gamma^{(m)})\}$$

or $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ according as $l > 0$ or $l = 0$. Let $\{f_{m,j}\}_{1\le j\le d(m)}$ be a basis of $S_{\rho_m}(\Gamma^{(m)})$ consisting of Hecke eigenforms, and take Hecke eigenforms $\{F_j\}_{d(m)+1\le j\le d}$ so that $\{[f_{m,j}]_{\rho_m}^{\rho_{n_1}}$ $(1 \le j \le d(m))$, F_j $(d(m)+1 \le j \le d)\}$ forms a basis of $\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)})$. Suppose that $k \ge$

$\max((n_1+n_2+1)/2, [3m/2+2])$ *and that neither* $k=(n_1+n_2+2)/2 \equiv 2 \bmod 4$ *nor* $k=(n_1+n_2+3) \equiv 2 \bmod 4$. *Then*

$$\mathbb{D}_{k,\lambda,n_1,n_2} E_{n_1+n_2,k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix} = \pi^{m(m+1)/2} \widetilde{c}(0,\rho_m) \sum_{j=1}^{d(m)} \frac{D(k, f_{m,j})}{(f_{m,j},\ f_{m,j})} [f_{m,j}]_{\rho_m}^{\rho_{n_1}}(Z)(U)[\theta f_{m,j}]_{\rho_m}^{\rho_{n_2}}(W)(V)$$

$$+ \sum_{j=d(m)+1}^{d} F_j(Z)(U) G_j(W)(V) \qquad (Z \in \mathbb{H}_{n_1}, W \in \mathbb{H}_{n_2}),$$

where G_j *is a certain element of* $M_{\rho_{n_2}}(\Gamma^{(n_2)})$. *Here,* U *and* V *are* $m \times n_1$ *and* $m \times n_2$ *matrices of variables, respectively, and we regard* $[f_{m,j}]_{\rho_m}^{\rho_{n_1}}$ *and* F_j *(resp.* $[\theta f_{m,j}]_{\rho_m}^{\rho_{n_2}}$ *and* G_j*) as elements of* $\mathrm{Hol}[U]_{\mathbf{k}'_{n_1}}$ *(resp.* $\mathrm{Hol}[V]_{\mathbf{k}'_{n_2}}$*). Moreover* $c(0,\rho_m)$ *is a rational number, and in particular it is* p*-adic unit for a prime number* p *such that* $p > 2(k+l)$.

6. Congruence for Klingen-Eisenstein lifts

To explain why Conjecture 4.6 is reasonable, we consider congruence for Klingen-Eisenstein series, which is a generalization of [2, Section 6] and [22]. For $\lambda = (\overbrace{k-l,\ldots,k-l}^{m_0}, 0, 0\ldots)$ and a positive integer m such that $k \geq l$ and $m \geq m_0$, put $\mathbf{k}'_m = (\overbrace{k-l,\ldots,k-l}^{m_0}, \overbrace{0,\ldots,0}^{m-m_0})$. Let $(\rho_{m,\lambda}, V_{m,\lambda})$ be an irreducible polynomial representation of $\mathrm{GL}_m(\mathbb{C})$ of highest weight $\mathbf{k}'_m$ and $\rho_m = \det^l \otimes \rho_{m,\lambda}$.

Let U and V be $m_0 \times n_1$ and $m_0 \times n_2$ matrices of variables, respectively, where $\min(n_1,n_2) \geq m_0$. Then we can take $V_{n_1,\lambda} = \mathbb{C}[U]_{\mathbf{k}'_{n_1}}$, $V_{n_2,\lambda} = \mathbb{C}[V]_{\mathbf{k}'_{n_2}}$ and every element F of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)})$ is expressed as

$$F(Z_1,Z_2) = \sum_{A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}, A_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}} c(A_1,A_2;F)(U,V)\mathbf{e}(\mathrm{tr}(A_1Z_1 + A_2Z_2))$$

with $c(A_1,A_2;F)(U,V) \in \mathbb{C}[U,V]_{\mathbf{k}'_{n_1},\mathbf{k}'_{n_2}}$. For a subring R of $\mathbb{C}$, we denote by $(M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)}))(R)$ the submodule of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)})$ consisting of all F's such that $c(A_1,A_2;F)(U,V) \in R[U,V]_{\mathbf{k}'_{n_1},\mathbf{k}'_{n_2}}$ for all $A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}$, $A_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}$. We also note that every element F of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda}$ is expressed as

$$F(Z_1) = \sum_{A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}} c(A_1;F)(U,V)\mathbf{e}(\mathrm{tr}(A_1Z_1))$$

with $c(A_1;F)(U,V) \in \mathbb{C}[U,V]_{\mathbf{k}'_{n_1},\mathbf{k}'_{n_2}}$. We then define a submodule $(M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(R)$ of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda}$ consisting of all F's such that $c(A_1;F)(U,V) \in R[U,V]_{\mathbf{k}'_{n_1},\mathbf{k}'_{n_2}}$ for all $A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}$.

For positive integers n and l, put

$$Z(n,l) = \zeta(1-l) \prod_{j=1}^{[n/2]} \zeta(1+2j-2l).$$

We define $\widetilde{E}_{n,l}$ as

$$\widetilde{E}_{n,l}(Z) = Z(n,l)E_{n,l}(Z)$$

and we set

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1,Z_2)=(k-l)!(2\pi\sqrt{-1})^{-m_0(k-l)}\mathbb{D}_{l,\lambda,n_1,n_2}\widetilde{E}_{n_1+n_2,l}\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}.$$

Moreover, for positive integers m,l and a Hecke eigenform $F\in S_k(\Gamma^{(m_0)})$ put

$$\mathcal{C}_{m,l}(F)=\frac{Z(m,l)}{Z(2m_0,l)}\mathbf{L}(l-m_0,F,\mathrm{St}).$$

We also use the same symbol $\mathcal{C}_{m,l}(f)$ to denote the value $\dfrac{Z(m,l)}{Z(2m_0,l)}\mathbf{L}(l-m_0,F,\mathrm{St})$ for a Hecke eigenform $f\in S_{\rho_{m_0}}(\Gamma^{(m_0)})$. As stated before, we have the following isomorphism:

$$\iota : S_k(\Gamma^{(m_0)})\ni F\mapsto \widetilde{F}:=(\det U)^{k-l}F\in S_{\rho_{m_0}}(\Gamma^{(m_0)}), \tag{1}$$

where U is $m_0\times m_0$ matrix of variables. Then we note that $\mathcal{C}_{m,l}(\widetilde{F})=\mathcal{C}_{m,l}(F)$ for a Hecke eigenform $F\in S_k(\Gamma^{(m_0)})$.

Now, for our later purpose, we rewrite a special case of Theorem 5.6 as follows.

Proposition 6.1. *Let n_1,n_2 be integers such that $m_0\le n_1\le n_2$ and let k,l be even positive integers such that $k\ge l$. Then we have*

$$\begin{aligned}\mathcal{E}_{k,l,n_1,n_2}(Z_1,Z_2)=\gamma_{m_0}&\sum_{j=1}^{d(m_0)}\mathcal{C}_{n_1+n_2,l}(f_{m_0,j})[f_{m_0,j}]_{\rho_{m_0}}^{\rho_{n_1}}(Z_1)(U)[\theta f_{m_0,j}]_{\rho_{m_0}}^{\rho_{n_2}}(Z_2)(V)\\ &+\sum_{j=d(m_0)+1}^{d}F_j(Z_1)(U)\widetilde{G_j}(Z_2)(V),\end{aligned}$$

where γ_{m_0} is a certain rational number which is p-unit for any prime number $p>2k$, and $\widetilde{G}_j(Z_2)(V)$ is an element of $M_{\rho_{n_2}}(\Gamma^{(n_2)})$.

We write $\mathcal{E}_{k,l,n_1,n_2}(Z_1,Z_2)$ as

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1,Z_2)=\sum_{N\in\mathcal{H}_{n_2}}g^{(n_1)}_{(k,l,n_1,n_2),N}(Z_1)\mathbf{e}(\mathrm{tr}(NZ_2)).$$

Then $g^{(n_1)}_{(k,l,n_1,n_2),N}$ belongs to $M_{\rho_{n_1}}(\Gamma^{(n_1)})\otimes V_{n_2,\lambda}$. To consider congruence between Klingen-Eisenstein lift and another modular form of the same weight, we rewrite the above proposition as follows:

Corollary 6.2. *Under the same notation and the assumption as above, let $N\in\mathcal{H}_{n_2}(\mathbb{Z})_{>0}$. Then,*

$$\begin{aligned}g^{(n_1)}_{(k,l,n_1,n_2),N}(Z_1)=\gamma_{m_0}&\sum_{j=1}^{d(m_0)}\mathcal{C}_{n_1+n_2,l}(f_{m_0,j})[f_{m_0,j}]_{\rho_{m_0}}^{\rho_{n_1}}(Z_1)(U)\overline{a(N,[f_{m_0,j}]_{\rho_{m_0}}^{\rho_{n_2}})(V)}\\ &+\sum_{j=d(m_0)+1}^{d}F_j(Z_1)(U)a(N,\widetilde{G}_j)(V).\end{aligned}$$

Observe that the first term on the right-hand side of the above is invariant if we multiply $f_{m_0,j}$ by an element of $\mathbb{C}^\times$.

For $T_1 \in \mathcal{H}_{n_1}$ and $T_2 \in \mathcal{H}_{n_2}$, put

$$\epsilon_{k,l,n_1,n_2}(T_1,T_2)(U,V) = \sum_{R \in M_{n_1,n_2}(\mathbb{Z})} a\Bigl(\begin{pmatrix} T_1 & R/2 \\ {}^tR/2 & T_2 \end{pmatrix}, \widetilde{E}_{n_1+n_2,l}\Bigr) \times Q_{l,\lambda,n_1,n_2}\Bigl(\begin{pmatrix} T_1 & R/2 \\ {}^tR/2 & T_2 \end{pmatrix}, U, V\Bigr), \tag{2}$$

where Q_{l,λ,n_1,n_2} is the polynomial in Proposition 5.1 corresponding to $\mathbb{D}_{l,\lambda,n_1,n_2}$. Then we have

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1,Z_2) = \sum_{T_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}, T_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}} \epsilon_{k,l,n_1,n_2}(T_1,T_2)(U,V)\mathbf{e}(\mathrm{tr}(T_1Z_1+T_2Z_2)),$$

and therefore

$$g^{(n_1)}_{(k,l,n_1,n_2),N}(Z_1) = \sum_{T \in \mathcal{H}_{n_1}(\mathbb{Z})} \epsilon_{k,l,n_1,n_2}(T,N)(U,V)\mathbf{e}(\mathrm{tr}(TZ_1)).$$

Therefore, in view of [2, Proposition 6.5], similarly to [2, Corollary 6.7], we prove the following proposition:

Proposition 6.3. *For each $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$ let $g_N^{(n_1)}$ be that defined above. Then*

$$g^{(n_1)}_{(k,l,n_1,n_2),N}(Z_1) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(\mathbb{Q})$$

and moreover

$$g^{(n_1)}_{(k,l,n_1,n_2),N}(Z_1) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(\mathbb{Z}_{(p)})$$

for any prime number $p > 2k$.

The following propositions can be proved in the same way as [2, Proposition 6.8] and [2, Proposition 6.9], respectively.

Proposition 6.4. *Let the notation and the assumptions be as in Theorem 5.6, and let $m_0 \leq m$. Then for any Hecke eigenform f in $S_{\rho_{m_0}}(\Gamma^{(m_0)})(\mathbb{Q}(f))$, $[f]^{\rho_m}_{\rho_{m_0}} \in M_{\rho_m}(\Gamma^{(m)})(\mathbb{Q}(f))$.*

Proposition 6.5. *Let the notation and the assumption be as in Proposition 6.1. Let f be a Hecke eigenform in $S_k(\Gamma^{(m_0)})$. Then, for any $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$ and $N_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{>0}$, the value $\mathcal{C}_{n_2+m_0,l}(f)\overline{a(N,[\widetilde{f}]^{\rho_{n_2}}_{\rho_{m_0}})(V)}a(N_1,f)$ belongs to $\mathbb{Q}(f)[V]_{\mathbf{k}'_{m_0}}$, where $\widetilde{f}$ is that in (1).*

Theorem 6.6. *Let k and l be positive even integers such that $k \geq l \geq [3m_0/2+2]$ and put $\mathbf{k} = (\overbrace{k,\ldots,k}^{m_0},\overbrace{l,\ldots,l}^{n_2-m_0})$ and $\widetilde{M}_{\mathbf{k}}(\Gamma^{(n_2)}) = \widetilde{M}_{\rho_{n_2}}(\Gamma^{(n_2)})$. Let $F \in S_k(\Gamma^{(m_0)})$ be a Hecke eigenform, and $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(F)$ with $p_{\mathfrak{p}} > 2k$. Suppose that $\mathfrak{p}$ divides $|a(A_1,F)|^2\mathbf{L}(l-m_0,F,\mathrm{St})$ and does not divide*

$$\mathcal{C}_{2n_2,l}(F)a(A_1,F)\overline{a(A,[F]^{\mathbf{k}})}$$

for some $A_1 \in \mathcal{H}_{m_0}(\mathbb{Z})_{>0}$ and $A \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}$, where $[F]^{\mathbf{k}} = [\widetilde{F}]^{\rho_{n_2}}_{\rho_{m_0}}$ as stated in Section 2. Then there exists a Hecke eigenform $G \in \widetilde{M}_{\mathbf{k}}(\Gamma^{(n_2)})$ such that G is not a constant multiple of $[F]^{\mathbf{k}}$ and

$$G \equiv_{\mathrm{ev}} [F]^{\mathbf{k}} \pmod{\mathfrak{p}}.$$

Proof. The assertion in the case $k = l$ has been proved in [22] in more general setting and the other case can also be proved using the same argument as in the proof of [2, Theorem 6.11]. $\square$

Remark 6.7. *The case $n_1 = 1, n_2 = 2$ and $m_0 = 1$ is investigated by [32] in more precise way.*

Let $\mathbf{k} = (\overbrace{k,\ldots,k}^{m_0},\overbrace{l,\ldots,l}^{n_2-m_0})$ and $\mathbf{k}' = (\overbrace{k-l,\ldots,k-l}^{m_0},\overbrace{0,\ldots,0}^{n_2-m_0})$ as above. To confirm the condition in Theorem 6.6, we give a formula for computing $\mathbf{L}(l-m_0,F,\mathrm{St})a(T,F)\overline{a(N,[F]^{\mathbf{k}})}$ for a Hecke eigenform F in $S_k(\Gamma^{(m_0)})$, $T \in \mathcal{H}_{m_0}(\mathbb{Z})_{>0}$ and $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$. Let $\epsilon_{k,l,m_0,n_2}(T,N)(U,V)$ be as in (2) and put $g_N = g^{(m_0)}_{(k,l,m_0,n_2),N}$. Recall that U and V are $m_0\times m_0$ and $m_0\times n_2$ matrices, respectively, of variables. We note that $\epsilon_{k,l,m_0,n_2}(T,N)(U,V)$ can be expressed as

$$\epsilon_{k,l,m_0,n_2}(T,N)(U,V) = (\det U)^{k-l}\epsilon_{k,\mathbf{k}}(T,N)(V) \tag{3}$$

with $\epsilon_{k,\mathbf{k}}(T,N) = \epsilon_{k,\mathbf{k}}(T,N)(V) \in \mathbb{C}[V]_{\mathbf{k}'}$. Then g_N is expressed as

$$g_N(W) = \sum_{T\in\mathcal{H}_{m_0}(\mathbb{Z})} (\det U)^{k-l}\epsilon_{k,\mathbf{k}}(T,N)\mathbf{e}(\mathrm{tr}(TW)).$$

Now, for a positive integer m, let $T(m)$ be the element of $\mathbf{L}_{m_0}$ defined in Section 3. For a positive integer $m = p_1\cdots p_r$ with p_i a prime number, we define the Hecke operator $T^{(m)} = T(p_1)\cdots T(p_r)$. We make the convention that $T^{(1)} = T(1)$. We note that $T^{(m)} = T(m)$ if $p_1,\ldots,p_r$ are distinct, but in general it is not. For each $m \in \mathbb{Z}_{>0}$ and $N \in \mathcal{H}_{m_0}(\mathbb{Z})_{>0}$, write $g_N|T^{(m)}(W)$ as

$$g_N|T^{(m)}(W) = \sum_{T\in\mathcal{H}_{m_0}(\mathbb{Z})_{>0}} (\det U)^{k-l}\epsilon_{k,\mathbf{k}}(m,T,N)\mathbf{e}(\mathrm{tr}(TW))$$

with $\epsilon_{k,\mathbf{k}}(m,T,N) \in \mathbb{C}[V]_{\mathbf{k}'}$.

Let $\mathcal{M}_{k,l} = M_k(\Gamma^{(m_0)})$ or $S_k(\Gamma^{(m_0)})$ according as $k = l$ or not, and let $\{F_j\}_{j=1}^d$ be a basis of $\mathcal{M}_{k,l}$ consisting of Hecke eigenforms. Furthermore write

$$F_j|T^{(m)}(z) = \lambda_{j,m}F_j(z).$$

Then the following proposition follows from Corollary 6.2.

Proposition 6.8. *Notation being as above, we have*

$$\epsilon_{k,\mathbf{k}}(m,T,N) = \sum_{j=1}^d \lambda_{j,m}a(T,F_j)B(F_j)$$

for any $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$, $T \in \mathcal{H}_{m_0}(\mathbb{Z})_{>0}$ and $m \in \mathbb{Z}_{>0}$, where $B(F_j)$ is a certain element of $\mathbb{C}[V]_{\mathbf{k}'}$, and in particular we have

$$B(F_j) = \gamma_{m_0}\mathcal{C}_{m_0+n_2,l}(F_j)\overline{a(N,[F_j]^{\mathbf{k}})}$$

if $F_j \in S_k(\Gamma^{(m_0)})$. Here, γ_{m_0} is the rational number in Proposition 6.1.

We note that $\mathcal{C}_{2n_2,l}(F) = \prod_{i=[(m_0+n_2)/2]+1}^{n_2}\zeta(2i+1-2l)\mathcal{C}_{m_0+n_2,l}(F)$ for a Hecke eigenform F in $S_k(\Gamma^{(m_0)})$. Hence by the above proposition, we have the following formula:

Proposition 6.9. *For $N_1 \in \mathcal{H}_{m_0}(\mathbb{Z})_{>0}, N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$ let $e_m = \epsilon_{k,\mathbf{k}}(m, N_1, N)$. Let F be a Hecke eigenform in $S_k(\Gamma^{(m_0)})$ and $\{F_j\}_{j=1}^d$ a basis of $\mathcal{M}_{k,l}$ consisting of Hecke eigenforms such that $F_1 = F$. For positive integers $m_1, \ldots, m_d$ put $\Delta = \Delta(m_1, \ldots, m_d) = \det(\lambda_{j,m_i})_{1 \le i,j \le d}$. Then,*

$$\Delta \gamma_{m_0} \mathcal{C}_{2n_2,l}(F) a(N_1, F) \overline{a(N, [F]^{\mathbf{k}})} = \prod_{i=[(m_0+n_2)/2]+1}^{n_2} \zeta(2i+1-2l) \begin{vmatrix} e_1 & \lambda_{1,2} & \ldots & \lambda_{1,d} \\ \vdots & \vdots & \vdots & \vdots \\ e_d & \lambda_{d,2} & \ldots & \lambda_{d,d} \end{vmatrix}.$$

Corollary 6.10. *Let the notation and the assumption as above. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(F)$ such that $p_{\mathfrak{p}} > 2k$. Suppose that $\mathfrak{p}$ divides neither $\prod_{i=[(m_0+n_2)/2]+1}^{n_2} \zeta(2i+1-2l)$ nor $\begin{vmatrix} e_1 & \lambda_{1,2} & \ldots & \lambda_{1,d} \\ \vdots & \vdots & \vdots & \vdots \\ e_d & \lambda_{d,2} & \ldots & \lambda_{d,d} \end{vmatrix}$. Then, $\mathfrak{p}$ does not divide $\mathcal{C}_{2n_2,l}(F) a(N_1, F) \overline{a(N, [F]^{\mathbf{k}})}$.*

Proof. By Proposition 3.2, Δ is an algebraic integer, and by the assumption, γ_{m_0} is a $\mathfrak{p}$-unit. Thus the assertion holds. $\square$

The following lemma will be used in the next section.

Lemma 6.11. *Let $N \in \mathcal{H}_5(\mathbb{Z})_{>0}$. Then for any $T \in \mathcal{H}_3(\mathbb{Z})_{>0}$ and a prime number p, we have the following formula for $\epsilon_{k,\mathbf{k}}(m, T, N)$:*

$$\begin{aligned} \epsilon_{k,\mathbf{k}}(p, T, N) =& \epsilon_{k,\mathbf{k}}(pT, N) + p^{3k-6} \epsilon_{k,\mathbf{k}}(p^{-1}T, N) \\ &+ p^{k-3} \sum_{D \in \mathrm{GL}_2(\mathbb{Z}) \mathrm{diag}(1,p,p) \mathrm{GL}_2(\mathbb{Z}) / \mathrm{GL}_2(\mathbb{Z})} \epsilon_{k,\mathbf{k}}(p^{-1}T[D], N) \\ &+ p^{2k-5} \sum_{D \in \mathrm{GL}_2(\mathbb{Z}) \mathrm{diag}(1,1,p) \mathrm{GL}_2(\mathbb{Z}) / \mathrm{GL}_2(\mathbb{Z})} \epsilon_{k,\mathbf{k}}(p^{-1}T[D], N). \end{aligned}$$

Here we make the convention that $\epsilon_{k,\mathbf{k}}(S, N) = 0$ if S is not half-integral.

Proof. The assertion follows from [1, Exercise 4.2.10]. $\square$

Let V be the 3×5 matrix of variables stated above.

Theorem 6.12. *Let $\mathbf{k} = (k, k, k, l, l)$ with $l = k$ or $k-2$. Let $A_0 \in \mathcal{H}_3(\mathbb{Z})_{>0}$ and $A_1 \in \mathcal{H}_5(\mathbb{Z})_{>0}$. Moreover, put*

$$P_{\mathbf{k}}\Big(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}\Big)(V) = \begin{cases} 1 & \text{if } l = k \\ c(k-2)^{-1} Q^2_{3,k-2}\Big(\begin{pmatrix} A_0 & R\,{}^tV/2 \\ V\,{}^tR/2 & VA_1{}^tV \end{pmatrix}\Big) & \text{if } l = k-2. \end{cases}$$

Here, $Q^2_{3,k-2}$ is the polynomial defined before Lemma 5.4, and $c(k-2)$ is the non-zero rational number in Lemma 5.4. Then

$$\epsilon_{k,\mathbf{k}}(A_0, A_1)(V) = \sum_{\substack{R \in M_{3,5}(\mathbb{Z}) \\ \begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix} \ge 0}} P_{\mathbf{k}}\Big(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}\Big)(V) a\Big(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}, \widetilde{E}_{8,l}\Big).$$

Proof. We prove the assertion in the case $l = k-2$. Let $\lambda = (2,2,2,0,0)$. Let $Q_{k-2,\lambda,3,5}$ be the polynomial in (2) with $m_0 = n_1 = 3$ and n_2=5. By Lemma 5.4,

$$c(k-2)Q_{k-2,\lambda,3,5}(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}, U, V) = Q^2_{3,l}(\mathbb{U}T{}^t\mathbb{U}),$$

where $\mathbb{U} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. Then, by (3) we have

$$c(k-2)Q_{k-2,\lambda,3,5}(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}, U, V) = (\det U)^{k-l}P_{\mathbf{k}}\Big(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}\Big)(V).$$

□

We have an explicit formula [21] for $a(T, \widetilde{E}_{l,m})$ for any semi-positive definite half-integral matrix T over $\mathbb{Z}$, and a Mathematica package [25] based on [8] for computing it. Therefore, by using Proposition 6.9 and Theorem 6.12 we can compute the Fourier coefficients of the Klingen-Eisenstein series in question.

7. Main results

In this section, we prove some cases of Conjecture 4.6, thereby proving the corresponding case of Harder's conjecture. For $l = 12, 16, 20, 22, 26$ let ϕ_l be the unique primitive form in $S_l(\mathrm{SL}_2(\mathbb{Z}))$. We have the following Fourier coefficients for these forms: $a(2,\phi_{12}) = -24$, $a(2,\phi_{16}) = 216$, $a(2,\phi_{20}) = 456$, $a(2,\phi_{22}) = -288$, and $a(2,\phi_{26}) = -48$. Let $\phi_{24}^{\pm}$ be the unique primitive form in $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ such that $a(2,\phi_{24}^{\pm}) = 12(45 \pm \sqrt{144169})$. Let $\mathfrak{O}$ be the ring of integers in $\mathbb{Q}(\phi_{24}^{+})$. Then we have $\mathfrak{O} = \mathbb{Z}[\theta]$, where $\theta = \frac{1+\sqrt{144169}}{2}$.

From the numerical tables in [31] based on [30] (see also [7]), we have

$$\begin{aligned} \dim_{\mathbb{C}} S_{14}(\Gamma^{(1)}) &= 0, & \dim_{\mathbb{C}} S_{(14,14,14)}(\Gamma^{(3)}) &= 1, \\ \dim_{\mathbb{C}} S_{(14,10)}(\Gamma^{(2)}) &= 1, & \dim_{\mathbb{C}} S_{(14,14,14,12)}(\Gamma^{(4)}) &= 2, \\ \dim_{\mathbb{C}} S_{(14,14)}(\Gamma^{(2)}) &= 1, & \dim_{\mathbb{C}} S_{(14,14,14,14)}(\Gamma^{(4)}) &= 3, \\ \dim_{\mathbb{C}} S_{(21,7)}(\Gamma^{(2)}) &= 1, & \dim_{\mathbb{C}} S_{(14,14,14,12,12)}(\Gamma^{(5)}) &= 2, \\ \dim_{\mathbb{C}} S_{(23,5)}(\Gamma^{(2)}) &= 1, & \dim_{\mathbb{C}} S_{(14,14,14,14,14)}(\Gamma^{(5)}) &= 3. \end{aligned} \tag{4}$$

For each $(k,j) = (5,18), (7,14)$, or $(10,4)$ let $G_{k+j,k}$ be a Hecke eigenform in $S_{(k+j,k)}(\Gamma^{(2)})$ uniquely determined up to constant multiple. Let $G_{14,14,12}$ be a Hecke eigenform in $S_{(14,14,12)}(\Gamma^{(3)})$ uniquely determined up to constant multiple.

7.1. **The** $(k,j) = (7,14)$ **case.** We consider Conjecture 4.6 with $(k,j) = (7,14)$ and $n = 2$. Let $f = \phi_{26}$ and $g = \phi_{12}$. The prime number 97 divides the ratio

$$\frac{\mathbf{L}(k+j,f)}{\mathbf{L}(k+j/2+1,f)} = \frac{\mathbf{L}(21,f)}{\mathbf{L}(15,f)} = 5 \cdot 97$$

(cf. [26, p. 383] and [33, p. 240]; note that there appears to be a misprint for the value r_4 in [26] regarding the exponent of 5).

Let $\mathbf{k} = (14,14,14,12,12)$. Then, by (4) and Theorems 4.3, 4.4 and 4.5, we have

$$S_{14}(\Gamma^{(3)}) = \langle \mathcal{M}_3^{14}(f,g)\rangle_{\mathbb{C}},$$

$$S_{(14,14,14,12)}(\Gamma^{(4)}) = \langle \mathcal{A}_4^{II}(\phi_{24}^{+}, G_{14,10}), \mathcal{A}_4^{II}(\phi_{24}^{-}, G_{14,10})\rangle_{\mathbb{C}},$$

and

$$S_{(14,14,14,12,12)}(\Gamma^{(5)}) = \langle \mathcal{M}_5^{\mathbf{k}}(\phi_{16}, \mathcal{M}_3^{14}(f,g)), \mathcal{A}_5^{\mathbf{k}}(G_{21,7}, g)\rangle_{\mathbb{C}}.$$

Hence we have $\widetilde{M_{\mathbf{k}}}(\Gamma^{(5)}) = \langle \mathcal{S}_{\mathbf{k}}\rangle_{\mathbb{C}}$, where

$$\mathcal{S}_{\mathbf{k}} = \big\{\mathcal{M}_5^{\mathbf{k}}(\phi_{16}, \mathcal{M}_3^{14}(f,g)), \mathcal{A}_5^{\mathbf{k}}(G_{21,7}, g), [\mathcal{A}_4^{II}(\phi_{24}^{+}, G_{14,10})]^{\mathbf{k}},$$
$$[\mathcal{A}_4^{II}(\phi_{24}^{-}, G_{14,10})]^{\mathbf{k}}, [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}\big\}$$

(cf. Theorems 5.6 and 6.6 for the definition of $\widetilde{M_{\mathbf{k}}}$). Here, we give a list of the standard L-functions for $F \in \mathcal{S}_{\mathbf{k}}$.

F	$L(s,F,\mathrm{St})$
$\mathcal{M}_5^{\mathbf{k}}(\phi_{16}, \mathcal{M}_3^{14}(f,g))$	$L(s,g,\mathrm{St})\prod_{i=1}^{2} L(s+14-i,f)\prod_{i=1}^{2} L(s+9-i,\phi_{16})$
$\mathcal{A}_5^{\mathbf{k}}(G_{21,7}, g)$	$L(s,g,\mathrm{St})\prod_{i=1}^{2} L(s+14-i,G_{21,7},\mathrm{Sp})$
$[\mathcal{A}_4^{II}(\phi_{24}^{+}, G_{14,10})]^{\mathbf{k}}$	$L(s,G_{14,10},\mathrm{St})\prod_{i=1}^{2} L(s+13-i,\phi_{24}^{+})\zeta(s-7)\zeta(s+7)$
$[\mathcal{A}_4^{II}(\phi_{24}^{-}, G_{14,10})]^{\mathbf{k}}$	$L(s,G_{14,10},\mathrm{St})\prod_{i=1}^{2} L(s+13-i,\phi_{24}^{-})\zeta(s-7)\zeta(s+7)$
$[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}$	$L(s,g,\mathrm{St})\prod_{i=1}^{2} L(s+14-i,f)\prod_{i=1}^{2}(\zeta(s-9+i)\zeta(s+9-i))$

TABLE 1. Standard L-functions for $F \in \mathcal{S}_{\mathbf{k}}$, $\mathbf{k} = (14,14,14,12,12)$.

Let $B_1 = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$. Put

$$\eta = \sum_{\substack{R \in M_{3,3}(\mathbb{Z}) \\ \begin{pmatrix} B_1 & R/2 \\ {}^tR/2 & B_1 \end{pmatrix} \geq 0}} Q_{3,k}^2\Big(\begin{pmatrix} B_1 & R/2 \\ {}^tR/2 & B_1 \end{pmatrix}\Big) a\Big(\begin{pmatrix} B_1 & R/2 \\ {}^tR/2 & B_1 \end{pmatrix}, \widetilde{E}_{3,12}\Big).$$

Then, by [16, Theorem 4.8], we have

$$|a(B_1, \mathcal{M}_3^{14}(f,g)|^2 \mathbf{L}(9, \mathcal{M}_3^{14}(f,g), \mathrm{St}) = d\eta$$

with $d \in \mathbb{Z}_{(97)}^{\times}$. Then, by a computation with Mathematica [34], we have

$$\eta = -6063676416 \equiv 0 \mod 97,$$

and we see that 97 divides

$$|a(B_1, \mathcal{M}_3^{14}(f,g)|^2 \mathbf{L}(9, \mathcal{M}_3^{14}(f,g), \mathrm{St}).$$

Let $A = \begin{pmatrix} 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 1 \end{pmatrix}$. Then, substituting V for $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ in Theorem 6.12, by a computation with Mathematica

$$c(12)\epsilon_{14,\mathbf{k}}(B_1, A)\Big(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\Big) = -\frac{1656777024}{5} \not\equiv 0 \mod 97.$$

Here $c(12)$ is the non-zero rational number in Lemma 5.4. We note that $c(12)$ belongs to $\mathbb{Z}_{(97)}^\times$. Moreover, 97 does not divide $\zeta(-13)$. This implies that 97 does not divide

$$\mathcal{C}_{10,12}(\mathcal{M}_3^{14}(f,g))a(B_1,\mathcal{M}_3^{14}(f,g)\overline{a(A,[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}})},$$

and, therefore by applying Theorem 6.6 with $\mathbf{k}=(14,14,14,12,12)$, $m_0=3$ and $n_2=5$, there exists a Hecke eigenform $H\in\mathcal{S}_{\mathbf{k}}$ such that $H\neq[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}$ and

$$H\equiv_{\mathrm{ev}}[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}\mod 97.$$

Then, we have

$$L_2(X,H,\mathrm{St})\equiv L_2(X,[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}},\mathrm{St})\mod 97. \tag{5}$$

Here, $L_p(X,F,\cdot)$ denotes the p-th Euler factor for $L(s,F,\cdot)$ as defined in Section 3.

F	$L_2(X,F,\mathrm{St})$
$\mathcal{M}_5^{\mathbf{k}}(\phi_{16},\mathcal{M}_3^{14}(f,g))$	$\left(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3\right)\left(1-\frac{27X}{32}+\frac{X^2}{2}\right)\left(1+\frac{3X}{2^9}+\frac{X^2}{2}\right)$ $\times\left(1-\frac{27X}{16}+2X^2\right)\left(1+\frac{3X}{2^8}+2X^2\right)$
$\mathcal{A}_5^{\mathbf{k}}(G_{21,7},g)$	$\left(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3\right)L_2(2^{-12}X,G_{21,7},\mathrm{Sp})L_2(2^{-13}X,G_{21,7},\mathrm{Sp})$
$[\mathcal{A}_4^{II}(\phi_{24}^+,G_{14,10})]^{\mathbf{k}}$	$(1-2^7X)\left(1-\frac{X}{2^7}\right)\left(1-\frac{3(22+\theta)X}{2^9}+\frac{X^2}{2}\right)\left(1-\frac{3(22+\theta)X}{2^8}+2X^2\right)L_2(X,G_{14,10},\mathrm{St})$
$[\mathcal{A}_4^{II}(\phi_{24}^-,G_{14,10})]^{\mathbf{k}}$	$(1-2^7X)\left(1-\frac{X}{2^7}\right)\left(1+\frac{3(-23+\theta)X}{2^9}+\frac{X^2}{2}\right)\left(1+\frac{3(-23+\theta)X}{2^8}+2X^2\right)L_2(X,G_{14,10},\mathrm{St})$
$[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}$	$(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3)(1-2^8X)(1-2^7X)(1-\frac{X}{2^7})(1-\frac{X}{2^8})$ $\times(1+\frac{3X}{2^9}+\frac{X^2}{2})(1+\frac{3X}{2^8}+2X^2)$

TABLE 2. Euler 2-factors $L_2(X,F,\mathrm{St})$ for $F\in\mathcal{S}_{\mathbf{k}}$, $\mathbf{k}=(14,14,14,12,12)$.

Let $\mathfrak{p}_1'$ and $\mathfrak{p}_2'$ be prime ideals of $\mathfrak{O}=\mathbb{Z}[\theta]$ such that $\mathfrak{p}_1'\mathfrak{p}_2'=(97)$, defined by $\langle 97,\theta+33\rangle$ and $\langle 97,\theta-34\rangle$ respectively. Recall that $\theta=\frac{1+\sqrt{144169}}{2}$.

From Table 2, it is straightforward to identify the following irreducible factors of $L_2(X,F,\mathrm{St})$ for $F\in S_{\mathbf{k}}$, when considered modulo a prime of $\mathbb{Z}$ or $\mathbb{Z}[\theta]$:

$$\begin{aligned}L_2(X,\mathcal{M}_5^{\mathbf{k}}(\phi_{16},\mathcal{M}_3^{14}(f,g)),\mathrm{St})\equiv{}&(1-X)(3-X)(41-X)(45-X)(65-X)(69-X)\\&\times(71-X)(2+65X+X^2)(49+81X+X^2)\mod 97,\end{aligned}$$

$$L_2(X,\mathcal{A}_5^{\mathbf{k}}(G_{21,7},g),\mathrm{St})\equiv(1-X)(3-X)(65-X)\times(\text{other factors})\mod 97,$$

$$\begin{aligned}L_2(X,[\mathcal{A}_4^{II}(\phi_{24}^+,G_{14,10})]^{\mathbf{k}},\mathrm{St})\equiv{}&(31-X)(72-X)(49+12X+X^2)(2+24X+X^2)\\&\times(\text{other factors})\mod \mathfrak{p}_1',\end{aligned}$$

$$\begin{aligned}L_2(X,[\mathcal{A}_4^{II}(\phi_{24}^+,G_{14,10})]^{\mathbf{k}},\mathrm{St})\equiv{}&(31-X)(35-X)(61-X)(70-X)(72-X)(79-X)\\&\times(\text{other factors})\mod \mathfrak{p}_2',\end{aligned}$$

$$\begin{aligned}L_2(X,[\mathcal{A}_4^{II}(\phi_{24}^-,G_{14,10})]^{\mathbf{k}},\mathrm{St})\equiv{}&(31-X)(35-X)(61-X)(70-X)(72-X)(79-X)\\&\times(\text{other factors})\mod \mathfrak{p}_1',\end{aligned}$$

$$L_2(X,[\mathcal{A}_4^{II}(\phi_{24}^-,G_{14,10})]^{\mathbf{k}},\mathrm{St}) = (31-X)(72-X)(49+12X+X^2)(2+24X+X^2)$$
$$\times\,(\text{other factors}) \mod \mathfrak{p}_2',$$

and

$$L_2(X,[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}},\mathrm{St}) \equiv (1-X)(3-X)(31-X)(36-X)(41-X)(45-X)$$
$$\times\,(62-X)(65-X)(69-X)(71-X)(72-X) \mod 97.$$

This leads to the conclusion that $F \in S_{\mathbf{k}}\backslash\{\mathcal{A}_5^{\mathbf{k}}(G_{21,7},g),[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}\}$ cannot be H. Consequently, it must be the case that H is identical to $\mathcal{A}_5^{\mathbf{k}}(G_{21,7},g)$.

Theorem 7.1. *We have*

$$\mathcal{A}_5^{\mathbf{k}}(G_{21,7},g) \equiv_{\mathrm{ev}} [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}} \mod 97.$$

By Theorem 4.8, we obtain the following:

Corollary 7.2. *Conjecture 4.2 holds for* $(k,j)=(7,14)$ *with* $f=\phi_{26}$*,* $F=G_{21,7}$ *and* $\mathfrak{p}=(97)$*.*

7.2. **The** $(k,j)=(5,18)$ **case.** We consider Conjecture 4.6 with $(k,j)=(5,18)$ and $n=2$. Let $f=\phi_{26}$ and $g=\phi_{12}$. The prime number 43 divides

$$\frac{\mathbf{L}(k+j,f)}{\mathbf{L}(k+j/2+1,f)} = \frac{\mathbf{L}(23,f)}{\mathbf{L}(15,f)} = 2^2\cdot 3\cdot 11\cdot 43$$

(cf. [26, p. 383]).

Let $\mathbf{k}=(14,14,14,14,14)$ and $\mathbf{k}'=(14,14,14)$. Then, by (4) and by Theorems 4.3, 4.4 and 4.5, we have

$$S_{14}(\Gamma^{(1)})=\{0\},\quad S_{14}(\Gamma^{(2)})=\langle \mathcal{I}_2(f)\rangle_{\mathbb{C}},\quad S_{14}(\Gamma^{(3)})=\langle \mathcal{M}_3^{14}(f,g)\rangle_{\mathbb{C}},$$
$$S_{14}(\Gamma^{(4)})=\langle \mathcal{M}_4^{14}(\phi_{22},\mathcal{I}_2(f)),\mathcal{I}_4(\phi_{24}^+),\mathcal{I}_4(\phi_{24}^-)\rangle_{\mathbb{C}},$$

and

$$S_{14}(\Gamma^{(5)})=\langle \mathcal{M}_5^{\mathbf{k}}(\phi_{20},\mathcal{M}_3^{14}(f,g)),\mathcal{K}_5(\phi_{22},G_{14,14,12}),\mathcal{A}_5^{\mathbf{k}}(G_{23,5},g)\rangle_{\mathbb{C}}.$$

Hence we have

$$M_{\mathbf{k}'}(\Gamma^{(3)})=M_{14}(\Gamma^{(3)})=\langle E_{3,14},[\mathcal{I}_2(f)]^{\mathbf{k}'},\mathcal{M}_3^{14}(f,g)\rangle_{\mathbb{C}}$$

and $\widetilde{M}_{\mathbf{k}}(\Gamma^{(5)})=M_{\mathbf{k}}(\Gamma^{(5)})=\langle \mathcal{S}_{\mathbf{k}}\rangle_{\mathbb{C}}$, where

$$\mathcal{S}_{\mathbf{k}}=\big\{E_{5,14},[\mathcal{I}_2(f)]^{\mathbf{k}},[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}},[\mathcal{M}_4^{14}(\phi_{22},\mathcal{I}_2(f))]^{\mathbf{k}},[\mathcal{I}_4(\phi_{24}^+)]^{\mathbf{k}},[\mathcal{I}_4(\phi_{24}^-)]^{\mathbf{k}},$$
$$\mathcal{M}_5^{\mathbf{k}}(\phi_{20},\mathcal{M}_3^{14}(f,g)),\mathcal{K}_5(\phi_{22},G_{14,14,12}),\mathcal{A}_5^{\mathbf{k}}(G_{23,5},g)\big\}$$

(cf. Theorems 5.6 and 6.6 for the definition of $\widetilde{M}_{\mathbf{k}}$). Here, we give a list of the standard L-functions for $F\in\mathcal{S}_{\mathbf{k}}$.

Let B_1 and A be as in (I), and

$$B_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \text{ and } B_3=\begin{pmatrix}1&1/2&0\\1/2&1&0\\0&0&2\end{pmatrix}.$$

F	$L(s,F,\mathrm{St})$
$E_{5,14}$	$\zeta(s)\prod_{i=1}^{5}(\zeta(s-14+i)\zeta(s+14-i))$
$[\mathcal{I}_2(f)]^{\mathbf{k}}$	$\prod_{i=1}^{2}L(s+14-i,f)\zeta(s)\prod_{i=1}^{3}(\zeta(s-12+i)\zeta(s+12-i))$
$[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}$	$L(s,g,\mathrm{St})\prod_{i=1}^{2}L(s+14-i,f)\prod_{i=1}^{2}(\zeta(s-11+i)\zeta(s+11-i))$
$[\mathcal{M}_4^{14}(\phi_{22},\mathcal{I}_2(f))]^{\mathbf{k}}$	$\prod_{i=1}^{2}L(s+14-i,f)\zeta(s)\prod_{i=1}^{2}L(s+12-i,\phi_{22})\zeta(s-9)\zeta(s+9)$
$[\mathcal{I}_4(\phi_{24}^{+})]^{\mathbf{k}}$	$\zeta(s)\prod_{i=1}^{4}L(s+14-i,\phi_{24}^{+})\zeta(s-9)\zeta(s+9)$
$[\mathcal{I}_4(\phi_{24}^{-})]^{\mathbf{k}}$	$\zeta(s)\prod_{i=1}^{4}L(s+14-i,\phi_{24}^{-})\zeta(s-9)\zeta(s+9)$
$\mathcal{M}_5^{\mathbf{k}}(\phi_{20},\mathcal{M}_3^{14}(f,g))$	$L(s,g,\mathrm{St})\prod_{i=1}^{2}L(s+14-i,f)\prod_{i=1}^{2}L(s+11-i,\phi_{20})$
$\mathcal{K}_5(\phi_{22},G_{14,14,12})$	$L(s,G_{14,14,12},\mathrm{St})\prod_{i=1}^{2}L(s+12-i,\phi_{22})$
$\mathcal{A}_5^{\mathbf{k}}(G_{23,5},g)$	$L(s,g,\mathrm{St})\prod_{i=1}^{2}L(s+14-i,G_{23,5},\mathrm{Sp})$

TABLE 3. Standard L-functions for $F\in\mathcal{S}_{\mathbf{k}}$, $\mathbf{k}=(14,14,14,14,14)$.

Let $F_1=\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}'}, F_2=E_{3,14}, F_3=[\mathcal{I}_2(f)]^{\mathbf{k}'}$. For a prime number q and $i=1,2,3$, put $\lambda_i(q)=\lambda_{F_i}(T(q))$, and $\Delta=\begin{vmatrix}1&1&1\\ \lambda_1(2)&\lambda_2(2)&\lambda_3(2)\\ \lambda_1(3)&\lambda_2(3)&\lambda_3(3)\end{vmatrix}$. By [1, Exercise 4.3.17] and [27, (8.1)], we have

$$\lambda_1(2)=-293760,\lambda_1(3)=486349920,$$
$$\lambda_2(2)=(1+2^{13})(1+2^{12})(1+2^{11}),\lambda_2(3)=(1+3^{13})(1+3^{12})(1+3^{11}),$$
and
$$\lambda_3(2)=(-48+2^{13}+2^{12})(1+2^{11}),\ \lambda_3(3)=(-195804+3^{13}+3^{12})(1+3^{11}).$$

Therefore we have $\Delta\not\equiv 0 \mod 43$.

For $B\in\mathcal{H}_3(\mathbb{Z})_{>0}$ put

$$\eta(B_1,B)=\sum_{\substack{R\in M_{3,3}(\mathbb{Z})\\ \begin{pmatrix}B_1&R/2\\ {}^tR/2&B\end{pmatrix}\geq 0}} a\Big(\begin{pmatrix}B_1&R/2\\ {}^tR/2&B\end{pmatrix},\widetilde{E}_{3,14}\Big).$$

$$\eta_1=\eta(B_1,B_1),\ \eta_2=\eta(B_1,2B_1)+2^{11}\eta(B_1,B_2),$$

and

$$\eta_3=\eta(B_1,3B_1)+4\times 3^{11}\eta(B_1,B_3).$$

Then, by using the same method as in [16, Theorem 4.8] combined with Lemma 6.11, we have

$$|a(B_1,\mathcal{M}_3^{14}(f,g)|^2\mathbf{L}(11,\mathcal{M}_3^{14}(f,g),\mathrm{St})=d_1\begin{vmatrix}\eta_1&1&1\\ \eta_2&\lambda_2(2)&\lambda_3(2)\\ \eta_3&\lambda_2(3)&\lambda_3(3)\end{vmatrix}\Delta^{-1}$$

with $d_1\in\mathbb{Z}_{(43)}^{\times}$. By using Mathematica,

$$\eta(B_1,B_1)=-\frac{2687696148060}{23},$$

$$\eta(B_1,2B_1)=-\frac{9488866468721603486166 0}{23},$$

$$\eta(B_1, 3B_1) = -\frac{20584550764251558762318463536 0}{23},$$
$$\eta(B_1, B_2) = -\frac{970186595803740}{23},$$
$$\eta(B_1, B_3) = -\frac{38538038613826004 40}{23},$$

and

$$\begin{vmatrix} \eta_1 & 1 & 1 \\ \eta_2 & \lambda_2(2) & \lambda_3(2) \\ \eta_3 & \lambda_2(3) & \lambda_3(3) \end{vmatrix} = -34737454170740873865242974365941760 00 \equiv 0 \mod 43.$$

This implies that 43 divides

$$|a(B_1, \mathcal{M}_3^{14}(f,g)|^2 \mathbf{L}(11, \mathcal{M}_3^{14}(f,g), \mathrm{St}).$$

Let

$$e_{1,\mathbf{k}} = \epsilon_{14,\mathbf{k}}(B_1, A),$$
$$e_{2,\mathbf{k}} = \epsilon_{14,\mathbf{k}}(2B_1, A) + 2^{11}\epsilon_{14,\mathbf{k}}(B_2, A),$$
$$e_{3,\mathbf{k}} = \epsilon_{14,\mathbf{k}}(3B_1, A) + 4 \times 3^{11}\epsilon_{14,\mathbf{k}}(B_3, A).$$

Then, by Proposition 6.9, we have

$$\Delta\mathcal{C}_{10,14}(\mathcal{M}_3^{14}(f,g))a(B_1, \mathcal{M}_3^{14}(f,g)\overline{a(A, [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}})}$$
$$= d_1\zeta(-17) \begin{vmatrix} e_{1,\mathbf{k}} & 1 & 1 \\ e_{2,\mathbf{k}} & \lambda_2(2) & \lambda_3(2) \\ e_{3,\mathbf{k}} & \lambda_2(3) & \lambda_3(3) \end{vmatrix}$$

with $d_1 \in \mathbb{Z}_{(43)}^{\times}$. By a computation with Mathematica,

$$\epsilon_{14,\mathbf{k}}(B_1, A) = \frac{1226172627792}{5},$$
$$\epsilon_{14,\mathbf{k}}(2B_1, A) = -\frac{17546694889588705038 24}{55},$$
$$\epsilon_{14,\mathbf{k}}(3B_1, A) = -\frac{36098215382451102927610717 44}{55} \tag{6}$$
$$\epsilon_{14,\mathbf{k}}(B_2, A) = \frac{952461422270064}{55}$$
$$\epsilon_{14,\mathbf{k}}(B_3, A) = -\frac{50292943071075936}{5},$$

and

$$\begin{vmatrix} e_{1,\mathbf{k}} & 1 & 1 \\ e_{2,\mathbf{k}} & \lambda_2(2) & \lambda_3(2) \\ e_{3,\mathbf{k}} & \lambda_2(3) & \lambda_3(3) \end{vmatrix} = -\frac{130636022011235199560130213445632000 00}{11} \not\equiv 0 \mod 43.$$

Moreover 43 does not divide $\zeta(-17)$. This implies that 43 does not divide

$$\mathcal{C}_{10,14}(\mathcal{M}_3^{14}(f,g))a(B_1, \mathcal{M}_3^{14}(f,g))\overline{a(A, [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}})}.$$

Therefore, by Theorem 6.6 with $\mathbf{k} = (14, 14, 14, 14, 14)$, $m_0 = 3$ and $n_2 = 5$, there is a Hecke eigenform $H \in \mathcal{S}_{\mathbf{k}}$ such that $H \neq [\mathcal{M}_3^{14}(f, g)]^{\mathbf{k}}$ and

$$H \equiv_{\mathrm{ev}} [\mathcal{M}_3^{14}(f, g)]^{\mathbf{k}} \mod 43.$$

We note that $\mathfrak{p} := 43\mathfrak{O}$ is a prime ideal of $\mathfrak{O}$. We have

$$L_3(X, H, \mathrm{St}) \equiv L_3(X, [\mathcal{M}_3^{14}(f, g)]^{\mathbf{k}}, \mathrm{St}) \mod \mathfrak{p}.$$

F	$L_2(X, F, \mathrm{St})$
$E_{5,14}$	$(1-X)\prod_{i=9}^{13}(1-2^iX)(1-2^{-i}X)$
$[\mathcal{I}_2(f)]^{\mathbf{k}}$	$(1+\frac{3X}{2^9}+\frac{X^2}{2})(1+\frac{3X}{2^8}+2X^2)(1-X)\prod_{i=9}^{11}(1-2^iX)(1-2^{-i}X)$
$[\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}$	$(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3)(1+\frac{3X}{2^9}+\frac{X^2}{2})(1+\frac{3X}{2^8}+2X^2)\prod_{i=9}^{10}(1-2^iX)(1-2^{-i}X)$
$[\mathcal{M}_4^{14}(\phi_{22},\mathcal{I}_2(f))]^{\mathbf{k}}$	$(1-2^9X)(1-X)(1-\frac{X}{2^9})(1+\frac{3X}{2^9}+\frac{X^2}{2})(1+\frac{9X}{64}+\frac{X^2}{2})$ $\times(1+\frac{3X}{2^8}+2X^2)(1+\frac{9X}{32}+2X^2)$
$[\mathcal{I}_4(\phi_{24}^+)]^{\mathbf{k}}$	$(1-2^9X)(1-X)(1-\frac{X}{2^9})\prod_{i=1}^{4}L_2(2^{-14}X,\phi_{24}^+)$
$[\mathcal{I}_4(\phi_{24}^-)]^{\mathbf{k}}$	$(1-2^9X)(1-X)(1-\frac{X}{2^9})\prod_{i=1}^{4}L_2(2^{-14}X,\phi_{24}^-)$
$\mathcal{M}_5^{\mathbf{k}}(\phi_{20},\mathcal{M}_3^{14}(f,g))$	$(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3)(1-\frac{57X}{2^7}+\frac{X^2}{2})(1+\frac{3X}{2^9}+\frac{X^2}{2})$ $\times(1-\frac{57X}{64}+2X^2)(1+\frac{3X}{2^8}+2X^2)$
$\mathcal{K}_5(\phi_{22},G_{14,14,12})$	$(1+\frac{9X}{64}+\frac{X^2}{2})(1+\frac{9X}{32}+2X^2)L_2(X,G_{14,14,12},\mathrm{St})$
$\mathcal{A}_5^{\mathbf{k}}(G_{23,5},g)$	$(1+\frac{23X}{32}-\frac{23X^2}{32}-X^3)L_2(2^{-12}X,G_{23,5},\mathrm{Sp})L_2(2^{-13}X,G_{23,5},\mathrm{Sp})$

TABLE 4. Euler 2-factors $L_2(X, F, \mathrm{St})$ for $F \in \mathcal{S}_{\mathbf{k}}$, $\mathbf{k} = (14, 14, 14, 14, 14)$.

From Table 4, it is straightforward to identify the following irreducible factors of $L_2(X, F, \mathrm{St})$ for $F \in S_{\mathbf{k}}$, when considered modulo $\mathfrak{p}$:

$$\begin{aligned} L_2(X, E_{5,14}, \mathrm{St}) \equiv\ & (1-X)(2-X)(4-X)(8-X)(11-X)(16-X) \\ & \times (22-X)(27-X)(32-X)(35-X)(39-X) \mod \mathfrak{p}, \end{aligned}$$

$$\begin{aligned} L_2(X, [\mathcal{I}_2(f)]^{\mathbf{k}}, \mathrm{St}) \equiv\ & (1-X)(8-X)(16-X)(27-X)(32-X)(35-X) \\ & \times (39-X)(22+10X+X^2)(2+20X+X^2) \mod \mathfrak{p}, \end{aligned}$$

$$\begin{aligned} L_2(X, [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}, \mathrm{St}) \equiv\ & (1-X)(16-X)(20-X)(28-X)(32-X)(35-X) \\ & \times (39-X)(22+10X+X^2)(2+20X+X^2) \mod \mathfrak{p}, \end{aligned}$$

$$\begin{aligned} L_2(X, [\mathcal{M}_4^{14}(\phi_{22}, \mathcal{I}_2(f))]^{\mathbf{k}}, \mathrm{St}) \equiv\ & (31-X)(35-X)(61-X)(70-X)(72-X)(79-X) \\ & \times (\text{other factors}) \mod \mathfrak{p}, \end{aligned}$$

$$L_2(X, [\mathcal{I}_4(\phi_{24}^+)]^{\mathbf{k}}, \mathrm{St}) \equiv$$
$$\begin{cases} (1-X)(32-X)(39-X) \times (4 \text{ quad. factors}) \mod \mathfrak{p} & \text{if } L_2(X, \phi_{24}^+) \text{ is irreducible} \mod \mathfrak{p} \\ (1-X)(32-X)(39-X) \times (8 \text{ linear factors}) \mod \mathfrak{p} & \text{otherwise,} \end{cases}$$

$$L_2(X, [\mathcal{I}_4(\phi_{24}^-)]^{\mathbf{k}}, \mathrm{St}) \equiv \begin{cases} (1-X)(32-X)(39-X) \times (4 \text{ quad. factors}) \mod \mathfrak{p} & \text{if } L_2(X, \phi_{24}^-) \text{ is irreducible} \mod \mathfrak{p} \\ (1-X)(32-X)(39-X) \times (8 \text{ linear factors}) \mod \mathfrak{p} & \text{otherwise,} \end{cases}$$

$$L_2(X, \mathcal{M}_5^{\mathbf{k}}(\phi_{20}, \mathcal{M}_3^{14}(f,g)), \mathrm{St}) = (1-X)(20-X)(28-X)(22+10X+X^2) \\ \times (22+14X+X^2)(2+20X+X^2)(2+28X+X^2) \mod \mathfrak{p},$$

$$L_2(X, \mathcal{K}_5(\phi_{22}, G_{14,14,12}), \mathrm{St}) = (5-X)(10-X)(13-X)(26-X) \\ \times (\text{other factors}) \mod \mathfrak{p},$$

$$L_2(X, \mathcal{A}_5^{\mathbf{k}}(G_{23,5}, g), \mathrm{St}) = (1-X)(20-X)(28-X) \\ \times (\text{other factors}) \mod \mathfrak{p}.$$

By comparing the irreducible factors of $L_2(X, [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}}, \mathrm{St})$ with those of $F \in S_{\mathbf{k}}$ modulo $\mathfrak{p}$, we can conclude that $H = \mathcal{A}_5^{\mathbf{k}}(G_{23,5}, g)$.

Hence, we have the following theorem.

Theorem 7.3. *We have*

$$\mathcal{A}_5^{\mathbf{k}}(G_{23,5}, g) \equiv_{\mathrm{ev}} [\mathcal{M}_3^{14}(f,g)]^{\mathbf{k}} \mod 43.$$

By Theorem 4.8, we obtain the following:

Corollary 7.4. *Conjecture 4.2 holds for $(k,j) = (5,18)$ with $f = \phi_{26}$, $F = G_{23,5}$ and $\mathfrak{p} = (43)$.*

Remark 7.5.
(1) *If we use the Galois representation theoretic method, we can shorten the verification of non-congruences in Subsections 7.1 and 7.2.*
(2) *Since we have*

$$L(j/2+2, \mathcal{M}_3^{k+j/2}(f,g), \mathrm{St}) = L(j/2+2, g, \mathrm{St}) L(k+j+1, f) L(k+j, f),$$

it is expected that the divisibility of $\mathbf{L}(j/2+2, \mathcal{M}_3^{k+j/2}(f,g), \mathrm{St})$ by a prime ideal $\mathfrak{p}$ follows from the divisibility of $\mathbf{L}(k+j, f)$ by $\mathfrak{p}$. In the Saito-Kurokawa lift case, such a result was given using the period relation due to Kohnen and Skoruppa [24] *(cf.* [2, Proposition 6.12]*). However, such a result has not been given in the Miyawaki lift of type II because we have no such a period relation at present. We note that such a period relation was conjectured in the case of Miyawaki lift of type I by Ikeda* [19].
(3) *Harder's conjecture for $(k,j) = (5,18)$ has been already proved by Ibukiyama* [13] *combined with Ishimoto* [20].
(4) *To compute* (6) *using Theorem 6.12, it is necessary to sum over all $R \in M_{3,5}(\mathbb{Z})$ such that $\begin{pmatrix} 3B_1 & R/2 \\ {}^tR/2 & A \end{pmatrix} \geq 0$. We note that there are 25,912,907 matrices R that satisfy this condition, which indicates the significant computational complexity involved. This particular computation is the most challenging one discussed in this paper.*

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