博士学位論文

Representations of the Hopf algebroid A_σ and construction of dynamical reflection maps (Hopf algebroid A_σ の表現と、ダイナミカル・リフレクション写像の構成)

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Contents

I Introduction 2

II Representation of the Hopf algebroid A_σ 4

1 Definition of A_σ and its representation 4
1.1 Definition of the Hopf algebroid A_σ 4
1.2 Representation of A_σ 7

2 Example of representations 11

III Construction of dynamical reflection maps 16

3 Dynamical reflection maps 16

4 Braided monoids and left modules of monoids 23
4.1 Braided monoids 23
4.2 Left modules of monoids 27

5 Left modules of twisted monoids 30

6 Main result 38

7 Proof of main result 39
7.1 Preliminaries 40
7.2 Step I 41
7.3 Proof of Proposition 7.3 42
7.4 Step II 52
7.5 Proof of Proposition 7.7 52
7.6 Step III 55
7.7 Proof of Proposition 7.10 56
7.8 Step IV 59
7.9 Proof of Proposition 7.12 59
7.10 Step V 61
7.11 Proof of Proposition 7.14 61

8 Examples of dynamical reflection maps 63
8.1 Examples 63
8.2 Conditions for dynamical reflection maps to be independent on the dynamical parameter 67

Abstract

In this paper, we treat two subjects. First, we construct representations of Hopf algebroids A_σ and give an example of representations of A_σ, in which we use a quasigroup. Second, we produce a way to construct dynamical reflection maps, solutions to the reflection equation in the tensor category Set_H. By using the above method, we give examples of dynamical reflection maps.

Part I

Introduction

The Yang-Baxter equation is of great interest in not only mathematics but also physics. For example, Hopf algebras, solvable lattice models, and quantum field theory are deeply related to the Yang-Baxter equation. In particular, Hopf algebras are still subjects of active research, and there are various generalization of them.

Weak bialgebras and weak Hopf algebras are the generalization of Hopf algebras. G. Böhm, F. Nill, and K. Szlachányi produced the notion of weak bialgebras [2]. The weak Hopf algebra is a weak bialgebra with a linear map called an antipode [2, 4], and this is the generalization of face algebras, which were introduced by T. Hayashi [11, 12]. The weak bialgebra and the weak Hopf algebra are both $\times_R$-bialgebras [30], while each $\times_R$-bialgebra, where R is Frobenius-separable, is a weak bialgebra [22].

Hopf algebroids, which were introduced by J. H. Lu [16], are the generalization of the Hopf algebras to non-commutative base algebras. G. Böhm and K. Szlachányi introduced a different definition of Hopf algebroids [3], which is given by considering the twist of Hopf algebras. To construct Hopf algebroids, dynamical Yang-Baxter maps are useful. The dynamical Yang-Baxter map was introduced by Y. Shibukawa and this is a solution to the set-theoretic dynamical Yang-Baxter equation [23, 24]. The tensor category Set_H (Definition 3.2) plays an essential role in obtaining the dynamical Yang-Baxter map. The dynamical Yang-Baxter map induces a Hopf algebroid A_σ [27, 28]. Y. Otsuto and Y. Shibukawa discuss the base ring of Hopf algebroid A_σ in [20]. In the construction of A_σ in [24], the base ring of A_σ was the $\mathbb{K}$-algebra composed of maps from a finite set to $\mathbb{K}$, where $\mathbb{K}$ is a field. In [20], this base ring was generalized to an arbitarary algebra L. Y. Otsuto and Y. Shibukawa also described the conditions of σ for A_σ to be a left bialgebroid and a right bialgebroid.

There are many studies on representations related to the Yang-Baxter equation. The reflection equation is one of them. The phrase "reflection equation" first emerged in the study of integrable systems [7]. In other context, J. Donin, P. P. Kulish, and A. I. Mudrov discussed reflection equations from the perspective that these are representations of reflection equation algebras, and introduced the universal K-matrix [9]. Similar to the set-theoretic Yang-Baxter equation, there are several studies investigating the set-theoretic reflection equation. For example, V. Caudrelier and Q. C. Zhang produced reflection maps, solutions to the set-theoretic reflection equation, by considering the factrization property on the half-line [6]. K. de Commer produced a way to construct solutions to the set-theoretic reflection equation by using actions of skew left braces

[8]. For other recent studies on set-theoretic reflection equation, for instance, see [5, 14]. There is also a study that introduce construction of solutions to the Yang-Baxter equation by means of those to the reflection equation [15].

In this paper, we focused on represions of algebras related to the Yang-Baxter equation. We first discuss representations of the Hopf algebroid A_σ. A_σ induces a tensor category of representations of A_σ, whose representations are called dynamical representations [27]. Refering to a non-trivial dynamical representation in [27, Remark 2.6], we construct representations of A_σ, where the base ring is an arbitrary ring L, if σ satisfies suitable conditions. We next consider solutions to the reflection equation in the tensor category Set_H. Motivated by the construction of solutions to the set-theoretic reflection equation in [8], we introduce a way to construct solutions to the reflection equation in Set_H, which we call dynamical reflection maps. This part is a joint work [1] with Y. Shibukawa.

This paper is organized as follows: Part II (Sections 1 and 2) deals with reprensentations of the Hopf algebroid A_σ. In Section 1, we introduce the definition of A_σ, and show that there uniquely exists a representation ρ_σ of A_σ under several conditions (Theorem 1.12). Section 2 discusses examples of representations of A_σ. At the end of Section 2, we also give an example of the base ring of A_σ that is not Frobenius-separable, which implies that we construct the Hopf algebroid A_σ that is not a weak Hopf algebra. Part III (Sections 3–8) is devoted to constructing the dynamical reflection maps. In Section 3, we provide a way to construct solutions to the reflection equation in the tensor category Set_H. Section 4 presents some preliminaries: left quasigroups; braided monoids; and left modules of monoids. The notion of left modules of twisted monoids, stated in Section 5, is crucial in the proof of Theorem 6.3. In Section 6 we state the main result Theorem 6.3. We now introduce it below.

Let $(L, \cdot, e_L)$ be a left quasigroup with a unit (Definition 4.4) and write $H = L$. Let G be a group isomorphic to L as sets. From the left quasigroup $(L, \cdot, e_L)$, we can produce a braided monoid (L, m, η, σ) (For more details, see Section 4). Let (X, m_X) be a left (L, m, η)-module in Set_H. We set $Y = L \otimes X \in Set_H$.

Theorem 0.1 (Theorem 6.3)**.** *There is a bijection between* (1) *and* (2):

(1) *Left (L, m, η)-modules (Y, m_Y) in Set_H satisfying*

$$\begin{aligned}
&m_X m_Y = m_X(1_A \otimes m_X),\\
&m_Y(1_A \otimes m_Y^{\mathrm{triv}}) = m_Y^{\mathrm{triv}}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1},\\
&m_Y(1_A \otimes m_Y^{\sigma}) = m_Y^{\sigma}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1}.
\end{aligned}$$

Here, m_Y^{triv} and m_Y^{σ} are the morphisms of Set_H defined by

$$m_Y^{\mathrm{triv}} = (m \otimes 1_X)a_{AAX}^{-1}; \quad m_Y^{\sigma} = (1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}.$$

(2) *Families of group homomorphisms $\{f_x : G \to G \mid x \in X\}$.*

Thanks to this theorem, we can obtain an explicit expression of dynamical reflection maps (See (8.1)). Section 7 is devoted to the proof of Theorem 6.3. In Section 8, which is the final section in this paper, we give examples of dynamical reflection maps. We also introduce a necessary and sufficient condition for the dynamical reflection maps to be independent on the dynamical parameter λ. Using this condition, we will give a solution to the set-theoretic reflection equation.

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Part II
Representation of the Hopf algebroid A_σ

In Part II, we construct representations of the Hopf algebroid A_σ (Definition 1.3) as k-algebras. Here, k is a commutative ring. We first recall the definition of A_σ and then we introduce its representation. We also give examples of representations of A_σ, where the base ring of A_σ is not necessarily Frobenius-separable.

1 Definition of A_σ and its representation

In this section, we introduce the Hopf algebroid A_σ. We will see that there exists a representation of A_σ as k-algebras.

1.1 Definition of the Hopf algebroid A_σ

Following [20, Section 2, 3, 4], we state the definition of the Hopf algebroid A_σ.

Let A and L be associative rings with units. We write the units 1_A and 1_L, respectively. If $A_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ satisfies the following conditions, then A_L is a left bialgebroid.

(i) $s_L : L \to A$ and $t_L : L^{op} \to A$ are ring homomorphisms that satisfy $s_L(f)t_L(g) = t_L(g)s_L(f)$ for all $f, g \in L$. Here, L^{op} is the opposite ring of L, and we denote by $\cdot_{op}$ the binary operation on L^{op}. The ring A has an L-bimodule structure as follows.

$$f \cdot a \cdot g = s_L(f)t_L(g)a \ (a \in A, f, g \in L).$$

We denote by ${}_LA$ and A_L the left L-module structure and the right L-module structure.

(ii) $\Delta_L : A \to A \otimes_L A$ and $\pi_L : A \to L$ are L-bimodule homomorphisms satisfying

$$(id_A \otimes \Delta_L) \circ \Delta_L = (\Delta_L \otimes id_A) \circ \Delta_L,$$
$$(\pi_L \otimes id_A) \circ \Delta_L = id_A = (id_A \otimes \pi_L) \circ \Delta_L.$$

We use Sweedler's notation: $\Delta_L(a) = a_{(1)} \otimes a_{(2)}$ $(a \in A)$.

(iii) For $f \in L$ and $a, b \in A$,

$$a_{(1)}t_L(f) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}s_L(f), \tag{1.1}$$
$$\Delta_L(ab) = \Delta_L(a)\Delta_L(b), \tag{1.2}$$
$$\Delta_L(1_A) = 1_A \otimes 1_A, \tag{1.3}$$
$$\pi_L(as_L(\pi_L(b))) = \pi_L(ab) = \pi_L(at_L(\pi_L(b))), \tag{1.4}$$
$$\pi_L(1_A) = 1_L. \tag{1.5}$$

We note that the right-hand-side of (1.2) is well-defined owing to (1.1).

We assume that $A_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ is a left bialgebroid, and S is an anti-automorphism of the ring A.

Let L' be a ring isomorphic to the opposite ring L^{op}. We denote by $\nu : L^{op} \to L'$ the isomorphism. By means of ν, we can equip the ring A with an L'-bimodule structure as follows:

$$f' \cdot a \cdot g' = aS(s_L(\nu^{-1}(g'))))s_L(\nu^{-1}(f')) \ (a \in A, f', g' \in L).$$

We write ${}^{L'}A$ and $A^{L'}$ as this left L'-module structure and the above right L'-module structure, respectively.

Proposition 1.1. (1) *If*

$$S \circ t_L = s_L, \tag{1.6}$$

then there uniquely exists a $\mathbb{Z}$-module map $S_{A\otimes_L A} : A_L \otimes {}_L A \to A^{L'} \otimes {}^{L'}A$ satisfying

$$S_{A\otimes_L A}(a \otimes b) = S(b) \otimes S(a),$$

and $S(a_{(1)})a_{(2)}$ makes sense.
(2) *If the maps S, s_L, and t_L satisfy* (1.6) *and*

$$S(a_{(1)})a_{(2)} = t_L \circ \pi_L \circ S(a) \tag{1.7}$$

for all $a \in A$, then there uniquely exists a $\mathbb{Z}$-module map $S_{A\otimes_{L'} A} : A^{L'} \otimes {}^{L'}A \to A_L \otimes {}_L A$ satisfying

$$S_{A\otimes_{L'} A}(a \otimes b) = S(b) \otimes S(a)$$

for all $a, b \in A$.

A proof of the above proposition is straightforward.

Let (A, S) be a pair of a left bialgebroid A and an anti-automorphism $S : A \to A$ of the ring A satisfying (1.6), (1.7), and

$$(\Delta_L \otimes id_A) \circ \Delta_{L'} = (id_A \otimes \Delta_{L'}) \circ \Delta_L;$$
$$(\Delta_{L'} \otimes id_A) \circ \Delta_L = (id_A \otimes \Delta_L) \circ \Delta_{L'}.$$

Here, $\Delta_{L'} = S_{A\otimes_L A} \circ \Delta_L \circ S^{-1}$.

Definition 1.2. If the $\mathbb{Z}$-module map $S_{A\otimes_{L'} A}$ has the inverse $S^{-1}_{A\otimes_{L'} A} : A_L \otimes {}_L A \to A^{L'} \otimes {}^{L'}A$ satisfying

$$\Delta_{L'} = S^{-1}_{A\otimes_{L'} A} \circ \Delta_L \circ S,$$

then we say that A_L is a Hopf algebroid.

We now state the definition of the Hopf algebroid A_σ (Cf. [20, 28]). Let k be a commutative ring with the unit 1_k. We denote by L a k-algebra with the unit 1_L, that is, L is a k-module with a bilinear and associative multiplication. Let G be a group and $T_\alpha : L \to L$ $(\alpha \in G)$ be k-algebra automorphisms satisfying

$$T_\alpha \circ T_{\alpha^{-1}} = id_L.$$

We write deg as a map from a finite set X to the group G.

We denote by $\langle Gen \rangle$ the monoid of all words of Gen, which is defined by

$$Gen = (L \otimes_k L^{op}) \bigsqcup \{L_{ab} \mid a, b \in X\} \bigsqcup \left\{(L^{-1})_{ab} \;\middle|\; a, b \in X\right\}.$$

Here L_{ab} and $(L^{-1})_{ab}$ are indeterminate elements for all $a, b \in X$. $\langle Gen \rangle$ is equipped with the empty word $\emptyset$, and the binary operation on $\langle Gen \rangle$ is a concatenation of words.

We denote by $k\langle Gen \rangle = \bigoplus_{w \in \langle Gen \rangle} kw$ the free k-algebra of $\langle Gen \rangle$.

Definition 1.3. We define A_σ as the quotient $k\langle Gen \rangle / I_\sigma$, where I_σ is the two-sided ideal of $k\langle Gen \rangle$ whose generators are (1) – (5) below:

(1) $\emptyset - 1_L \otimes 1_L$.

(2) $\xi + \xi' - (\xi + \xi')$, $\xi\xi' - (\xi\xi')$, $c\xi - (c\xi)$ $(\xi, \xi' \in L \otimes_k L^{op}, c \in k)$. Here the symbol " + " in $\xi + \xi'$ stands for the addition in $k\langle Gen \rangle$, and the symbol " + " in $(\xi + \xi')$ stands for the addition in $L \otimes_k L^{op}$. The notations of the multiplications and the scalar products are similarly defined.

(3) $\sum_{c \in X} L_{ac}(L^{-1})_{cb} - \delta_{ab}\emptyset$, $\sum_{c \in X} (L^{-1})_{ac} L_{cb} - \delta_{ab}\emptyset$ $(a, b \in X)$. Here δ_{ab} means Kronecker's delta symbol.

(4)

$$\begin{gathered}
(T_{deg(a)}(f) \otimes 1_L)L_{ab} - L_{ab}(f \otimes 1_L), \\
(1_L \otimes T_{deg(b)}(f))L_{ab} - L_{ab}(1_L \otimes f), \\
(f \otimes 1_L)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{deg(b)}(f) \otimes 1_L), \\
(1_L \otimes f)(L^{-1})_{ab} - (L^{-1})_{ab}(1_L \otimes T_{deg(a)}(f)). \\
(f \in L, a, b \in X)
\end{gathered}$$

(5) $\sum_{x, y \in X} (\sigma^{xy}_{ac} \otimes 1_L) L_{yd} L_{xb} - \sum_{x, y \in X} (1_L \otimes \sigma^{bd}_{xy}) L_{cy} L_{ax}$ $(a, b, c, d \in X)$. Here, σ^{ab}_{cd} $(a, b, c, d \in X)$ are elements in L.

We write $\sigma = (\sigma^{ab}_{cd})_{a,b,c,d \in X}$.

Definition 1.4 (rigid)**.** If there exists $x_{ab}, y_{ab} \in A_\sigma$ satisfying

$$\begin{aligned}
\sum_{c \in X} ((L^{-1})_{cb} + I_\sigma) x_{ac} &= \sum_{c \in X} x_{cb}((L^{-1})_{ac} + I_\sigma) \\
&= \sum_{c \in X} (L_{cb} + I_\sigma) y_{ac}
\end{aligned}$$

$$=\sum_{c\in X} y_{cb}(L_{ac}+I_\sigma)$$
$$=\delta_{ab}1_{A_\sigma}$$

for all $a, b \in X$, then we say that σ is rigid.

We define the left multipulication $\rho_l(f)$ and the right multiplication $\rho_r(f)$ by

$$\rho_l(f) : L \ni g \mapsto fg \in L;\ \rho_r(f) : L \ni g \mapsto gf \in L$$

for all $f \in L$. It is easily seen that $\rho_l : L \to End_k(L)$ and $\rho_r : L^{op} \to End_k(L)$ are ring homomorphisms and satisfy

$$\rho_l(f)\rho_r(g) = \rho_r(g)\rho_l(f) \tag{1.8}$$

for all $f, g \in L$. Here, $End_k(L)$ is the k-algebra that is composed of all k-module homomorphisms on L.

Theorem 1.5. *If σ is rigid and satisfies*

$$T_{deg(a)^{-1}} \circ T_{deg(c)^{-1}} \circ \rho_l(\sigma^{bd}_{ac}) = T_{deg(b)^{-1}} \circ T_{deg(d)^{-1}} \circ \rho_r(\sigma^{bd}_{ac}) \tag{1.9}$$

for all $a, b, c, d \in X$, then A_σ is a Hopf algebroid.

For a similar proof of Theorem 1.5, we refer to the proofs of [27, Proposition 3.3 and Theorem 3.9].

1.2 Representation of A_σ

From now on, we discuss representations of A_σ as k-algebras (Cf. [27, Remark 2.6]). We assume that σ is rigid and satisfies (1.9) for all $a, b, c, d \in X$, hence A_σ is a Hopf algebroid (Theorem 1.5).

Let X^σ denote the free left L-module whose base is X. For any $l \in X^\sigma$, we write

$$l = \sum_{v\in X} l_v v\ (l_v \in L).$$

Proposition 1.6. *We define the maps $s^{X^\sigma}_L : L \to End_k(X^\sigma)$ and $t^{X^\sigma}_L : L^{op} \to End_k(X^\sigma)$ by*

$$s^{X^\sigma}_L(f)(l) = \sum_{v\in X} T_{deg(v)}(f) l_v v;\ t^{X^\sigma}_L(f)(l) = \sum_{v\in X} l_v f v\ (f \in L, l \in X^\sigma).$$

Then $s^{X^\sigma}_L$ and $t^{X^\sigma}_L$ are k-algebra homomorphisms.

The proof of Proposition 1.6 is straightforward.

Proposition 1.7. *There exists a k-algebra homomorphism $\tilde{\zeta} : L \otimes_k L^{op} \to End_k(X^\sigma)$ satisfying*

$$\tilde{\zeta}(f \otimes g) = s^{X^\sigma}_L(f) t^{X^\sigma}_L(g) \quad (f, g \in L).$$

Since $s^{X^\sigma}_L : L \to End_k(X^\sigma)$ and $t^{X^\sigma}_L : L^{op} \to End_k(X^\sigma)$ are k-algebra homomorphisms, we can easily show Proposition 1.7.

Proposition 1.8. *For any $f, g \in L$,*

$$s_L^{X^\sigma}(f)t_L^{X^\sigma}(g) = t_L^{X^\sigma}(g)s_L^{X^\sigma}(f).$$

Moreover, $End_k(X^\sigma)$ has an L-bimodule structure through the following action:

$$f \cdot \phi \cdot g = s_L^{X^\sigma}(f)t_L^{X^\sigma}(g)\phi \quad (f, g \in L, \phi \in End_k(X^\sigma)).$$

Definition 1.9. We define the k-algebra homomorphism $\tilde{\rho}_\sigma : k\langle Gen \rangle \to End_k(X^\sigma)$ by

$$\begin{cases} \tilde{\rho}_\sigma(\xi) = \tilde{\zeta}(\xi), \\ \tilde{\rho}_\sigma(L_{ab})(l) = \sum_{v,w \in X} \sigma_{av}^{wb} T_{deg(b)}(l_w)v, \\ \tilde{\rho}_\sigma((L^{-1})_{ab})(l) = \sum_{v,w \in X} T_{deg(a)^{-1}}((\sigma^{-1})_{va}^{bw}) T_{deg(a)^{-1}}(l_w)v \end{cases}$$

for all $\xi \in L \otimes_k L^{op}, a, b \in X$, and $l = \sum_{v \in X} l_v v \in X^\sigma$. Here, $(\sigma^{-1})_{ab}^{cd}$ $(a, b, c, d \in X)$ are elements in L.

Proposition 1.10. *Under the following conditions, $\tilde{\rho}_\sigma(I_\sigma) = \{0\}$.*

$$\sum_{c,w \in X} \sigma_{av}^{wc}(\sigma^{-1})_{wc}^{by} = \sum_{c,w \in X} (\sigma^{-1})_{va}^{cw} \sigma_{cw}^{yb} = \delta_{ab}\delta_{vy}1_L, \tag{1.10}$$

$$\sum_{x,y,w \in X} T_{deg(v)}(\sigma_{ac}^{xy})\sigma_{yv}^{wd} T_{deg(d)}(\sigma_{xw}^{sb}) = \sum_{x,y,w \in X} \sigma_{cv}^{wy} T_{deg(y)}(\sigma_{aw}^{sx})\sigma_{xy}^{bd}, \tag{1.11}$$

$$\rho_l((\sigma^{-1})_{ac}^{bd}) \circ T_{deg(d)} \circ T_{deg(b)} = \rho_r((\sigma^{-1})_{ac}^{bd}) \circ T_{deg(c)} \circ T_{deg(a)}. \tag{1.12}$$

$$(a, b, c, d, s, v, y \in X)$$

Proof of Proposition 1.10. It suffices to show $\tilde{\rho}_\sigma(a) = 0$ for every generator a of I_σ. Let us check for each generator (1)–(5) of I_σ.
(1) For $g \in X^\sigma$,

$$\begin{aligned} &\tilde{\rho}_\sigma(\emptyset - 1_L \otimes 1_L)(g) \\ =&id_{X^\sigma}(g) - \sum_{v \in X} T_{deg(v)}(1_L)g_v v \\ =&0. \end{aligned}$$

(2) For $\xi, \xi' \in L \otimes_k L^{op}$,

$$\begin{aligned} \tilde{\rho}_\sigma(\xi + \xi' - (\xi + \xi')) &= \tilde{\zeta}(\xi) + \tilde{\zeta}(\xi') - \tilde{\zeta}(\xi + \xi') \\ &= \tilde{\zeta}(\xi + \xi') - \tilde{\zeta}(\xi + \xi') = 0. \end{aligned}$$

We can check for the multiplications and the scalar products similarly.
(3) From (1.10),

$$\begin{aligned} &\tilde{\rho}_\sigma(\sum_{c \in X} L_{ac}(L^{-1})_{cb} - \delta_{ab}\emptyset)(g) \\ =&\sum_{c \in X} \sum_{v,w \in X} \sigma_{av}^{wc} T_{deg(c)}(\tilde{\rho}_\sigma((L^{-1})_{cb})(g)_w)v - \delta_{ab}g \end{aligned}$$

$$
\begin{aligned}
&= \sum_{c\in X}\sum_{v,w\in X}\sum_{y\in X}\sigma_{av}^{wc}T_{deg(c)}(T_{deg(c)^{-1}}((\sigma^{-1})_{wc}^{by})T_{deg(c)^{-1}}(g_y))v - \delta_{ab}g \\
&= \sum_{c\in X}\sum_{v,w\in X}\sum_{y\in X}\sigma_{av}^{wc}(\sigma^{-1})_{wc}^{by}g_y v - \delta_{ab}g \\
&= \sum_{v\in X}\delta_{ab}g_v v - \delta_{ab}g \\
&= 0
\end{aligned}
$$

for all $a, b \in X$ and $g \in X^\sigma$. Hence $\tilde{\rho}_\sigma(\sum_{c\in X} L_{ac}(L^{-1})_{cb} - \delta_{ab}\emptyset) = 0$. We can show $\tilde{\rho}_\sigma(\sum_{c\in X}(L^{-1})_{ac}\, L_{cb} - \delta_{ab}\emptyset) = 0$ in much the same way.

(4) For the proof of (4), we use the following lemma.

Lemma 1.11. *The condition* (1.9) *implies*

$$
\rho_l(\sigma_{ac}^{bd}) \circ T_{deg(d)} \circ T_{deg(b)} = \rho_r(\sigma_{ac}^{bd}) \circ T_{deg(c)} \circ T_{deg(a)} \tag{1.13}
$$

for all $a, b, c, d \in X$.

For a proof of Lemma 1.11, see [20, The proof of Proposition 3.2].

Let us continue the proof for (4). From Lemma 1.11, (1.13) holds. On account of (1.13),

$$
\begin{aligned}
&\tilde{\rho}_\sigma((T_{deg(a)}(f)\otimes 1_L)L_{ab} - L_{ab}(f\otimes 1_L))(g) \\
&= \sum_{v\in X}(T_{deg(v)}\circ T_{deg(a)})(f)\tilde{\rho}_\sigma(L_{ab})(g)_v v - \sum_{v,w\in X}\sigma_{av}^{wb}T_{deg(b)}(\tilde{\rho}_\sigma(f\otimes 1_L)(g)_w)v \\
&= \sum_{v\in X}(T_{deg(v)}\circ T_{deg(a)})(f)(\sum_{w\in X}\sigma_{av}^{wb}T_{deg(b)}(g_w))v \\
&\quad - \sum_{v,w\in X}\sigma_{av}^{wb}T_{deg(b)}(T_{deg(w)}(f)g_w)v \\
&= 0
\end{aligned}
$$

for all $a, b \in X$, $f \in L$, and $g \in X^\sigma$. Also, from (1.12),

$$
\begin{aligned}
&\tilde{\rho}_\sigma((f\otimes 1_L)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{deg(b)}(f)\otimes 1_L))(g) \\
&= \sum_{v\in X}T_{deg(v)}(f)\tilde{\rho}_\sigma((L^{-1})_{ab})(g)_v v \\
&\quad - \sum_{v,w\in X}T_{deg(a)^{-1}}((\sigma^{-1})_{va}^{bw})T_{deg(a)^{-1}}(\tilde{\rho}_\sigma(T_{deg(b)}(f)\otimes 1_L)(g)_w)v \\
&= \sum_{v\in X}T_{deg(v)}(f)\sum_{w\in X}T_{deg(a)^{-1}}((\sigma^{-1})_{va}^{bw})T_{deg(a)^{-1}}(g_w)v \\
&\quad - \sum_{v,w\in X}T_{deg(a)^{-1}}((\sigma^{-1})_{va}^{bw})(T_{deg(a)^{-1}}\circ T_{deg(w)}\circ T_{deg(b)})(f)T_{deg(a)^{-1}}(g_w)v \\
&= 0
\end{aligned}
$$

for all $a, b \in X$, $f \in L$, and $g \in X^\sigma$. The rest of the proof is similar.

(5) From (1.11) and (1.13),

$$
\begin{aligned}
&\tilde{\rho}_\sigma(\sum_{x,y\in X}(\sigma_{ac}^{xy}\otimes 1_L)L_{yd}L_{xb} - \sum_{x,y\in X}(1_L\otimes\sigma_{xy}^{bd})L_{cy}L_{ax})(g)\\
=&\sum_{x,y\in X}\sum_{v\in X}T_{deg(v)}(\sigma_{ac}^{xy})\tilde{\rho}_\sigma(L_{yd}L_{xb})(g)_v v - \sum_{x,y\in X}\sum_{v\in X}\tilde{\rho}_\sigma(L_{cy}L_{ax})(g)_v\sigma_{xy}^{bd}v\\
=&\sum_{x,y,v\in X}T_{deg(v)}(\sigma_{ac}^{xy})(\sum_{w\in X}\sigma_{yv}^{wd}T_{deg(d)}(\tilde{\rho}_\sigma(L_{xb})(g)_w)v\\
&-\sum_{x,y,v\in X}(\sum_{w\in X}\sigma_{cv}^{wy}T_{deg(y)}(\tilde{\rho}_\sigma(L_{ax})(g)_w)\sigma_{xy}^{bd}v\\
=&\sum_{x,y,v\in X}T_{deg(v)}(\sigma_{ac}^{xy})\sum_{w\in X}\sigma_{yv}^{wd}\sum_{s\in X}T_{deg(d)}(\sigma_{xw}^{sb}T_{deg(b)}(g_s))v\\
&-\sum_{x,y,v\in X}\sum_{w\in X}\sigma_{cv}^{wy}(\sum_{s\in X}T_{deg(y)}(\sigma_{aw}^{sx}T_{deg(x)}(g_s)))\sigma_{xy}^{bd}v\\
=&\sum_{x,y,v,w,s\in X}T_{deg(v)}(\sigma_{ac}^{xy})\sigma_{yv}^{wd}T_{deg(d)}(\sigma_{xw}^{sb})(T_{deg(d)}\circ T_{deg(b)})(g_s)v\\
&-\sum_{x,y,v,w,s\in X}\sigma_{cv}^{wy}T_{deg(y)}(\sigma_{aw}^{sx})(T_{deg(y)}\circ T_{deg(x)})(g_s)\sigma_{xy}^{bd}v\\
=&\sum_{x,y,v,w,s\in X}\sigma_{cv}^{wy}T_{deg(y)}(\sigma_{aw}^{sx})\sigma_{xy}^{bd}(T_{deg(d)}\circ T_{deg(b)})(g_s)v\\
&-\sum_{x,y,v,w,s\in X}\sigma_{cv}^{wy}T_{deg(y)}(\sigma_{aw}^{sx})(T_{deg(y)}\circ T_{deg(x)})(g_s)\sigma_{xy}^{bd}v\\
=&0
\end{aligned}
$$

for all $a, b, c, d \in X$ and $g \in X^\sigma$. This completes the proof. □

From Proposition 1.10, we can induce the following theorem.

Theorem 1.12. *Under the conditions* (1.10), (1.11), *and* (1.12), *there exists a unique k-algebra homomorphism $\rho_\sigma : A_\sigma \to End_k(X^\sigma)$ satisfying*

$$
\begin{cases}
\rho_\sigma(\xi + I_\sigma) = \tilde{\zeta}(\xi),\\
\rho_\sigma(L_{ab} + I_\sigma)(l) = \sum_{v,w\in X}\sigma_{av}^{wb}T_{deg(b)}(l_w)v,\\
\rho_\sigma((L^{-1})_{ab} + I_\sigma)(l) = \sum_{v,w\in X}T_{deg(a)^{-1}}((\sigma^{-1})_{va}^{bw})T_{deg(a)^{-1}}(l_w)v
\end{cases}
$$

for all $\xi \in L\otimes_k L^{op}, a, b \in X$, and $l \in X^\sigma$. Moreover, ρ_σ is an L-bimodule homomorphism.

Proof. We only prove that ρ_σ is an L-bimodule homomorphism. We recall that $s_L : L \to A_\sigma$ and $t_L : L^{op} \to A_\sigma$ are given by

$$s_L(f) = f\otimes 1_L + I_\sigma;\ t_L(f) = 1_L\otimes f + I_\sigma \quad (f \in L).$$

For more details, see [20]. Since ρ_σ is a k-algebra homomorphism,

$$
\begin{aligned}
\rho_\sigma(f\cdot a\cdot g) =&\rho_\sigma(s_L(f)t_L(g)a)\\
=&\rho_\sigma(s_L(f))\rho_\sigma(t_L(g))\rho_\sigma(a)
\end{aligned}
$$

$$
\begin{aligned}
&=\tilde{\zeta}(f \otimes 1_L)\tilde{\zeta}(1_L \otimes g)\rho_\sigma(a)\\
&=s_L^{X^\sigma}(f)t_L^{X^\sigma}(g)\rho_\sigma(a)\\
&=f \cdot \rho_\sigma(a) \cdot g
\end{aligned}
$$

for all $f, g \in L$ and $a \in A^\sigma$. Hence ρ_σ is an L-bimodule homomorphism. □

2 Example of representations

In this section, by using Theorem 1.12, we construct representations of the Hopf algebroid A_σ as k-algebras (Cf. [20, Section 5]). We also demonstrate that there exists a k-algebra L that is not Frobenius-separable.

Definition 2.1 (quasigroup)**.** Let QG be a set with a binary operation $\cdot :$ $QG \times QG \ni (a, b) \to ab \in QG$. If QG satisfies the following conditions, we say that QG is a quasigroup.

(1) For any $a, c \in QG$, there uniquely exists the element $b \in QG$ such that $ab = c$.

(2) For any $b, c \in QG$, there uniquely exists the element $a \in QG$ such that $ab = c$.

Let X be a finite quasigroup, where $|X| > 2$. For $a, c \in X$, we denote by $a\backslash c$ the unique element $b \in X$ such that $ab = c$. We denote by M an abelian group isomorphic to X as sets, and define the ternary operation $\mu : M \times M \times M \to M$ by

$$
\mu(a, b, c) = a - b + c \tag{2.1}
$$

for all $a, b, c \in M$. We write $\pi : X \to M$ as the bijection.

We set $H = X$. Let k be a commutative ring with the unit 1_k and R a k-algebra with the unit 1_R. We denote by L the k-algebra of all maps from H to R, where the multiplication is $(fg)(\lambda) = f(\lambda)g(\lambda)$ $(f, g \in L, \lambda \in H)$. For $a, b, c, d \in X$, let ${\sigma_X}^{ab}_{cd} : H \to R$ denote the element of L defined by

$$
{\sigma_X}^{ab}_{cd}(\lambda) = \begin{cases} 1_R & (c = \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)a)))\backslash((\lambda b)a), \\ & \quad d = \lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)a)))), \\ 0_R & (\text{otherwise}) \end{cases} \tag{2.2}
$$

for all $\lambda \in H$.

In addition, for any $a, b, c, d \in X$, let $(\sigma_X^{-1})^{ab}_{cd} : H \to R$ be the element of L defined by

$$
(\sigma_X^{-1})^{ab}_{cd}(\lambda) = \begin{cases} 1_R & (c = \pi^{-1}(\mu(\pi((\lambda b)a), \pi(\lambda b), \pi(\lambda)))\backslash((\lambda b)a), \\ & \quad d = \lambda\backslash\pi^{-1}(\mu(\pi((\lambda b)a), \pi(\lambda b), \pi(\lambda)))), \\ 0_R & (\text{otherwise}) \end{cases} \tag{2.3}
$$

for all $\lambda \in H$.

Let G denote the opposite group of the symmetric group on H. For $a \in X$, we define $deg(a) \in G$ by $\lambda deg(a) = \lambda a$ $(\lambda \in H)$. Let us denote by $T_\alpha : L \to L$ $(\alpha \in G)$ the k-algebra automorphisms that are defined by

$$
T_\alpha(f)(\lambda) = f(\alpha(\lambda)) \ (\lambda \in H, f \in L).
$$

It is easily seen that $T_\alpha \circ T_{\alpha^{-1}} = id_L$ for all $\alpha \in G$.

Proposition 2.2. *For* $a, b, c, d \in X$*,* ${\sigma_X}^{ab}_{cd}$ *and* $(\sigma_X^{-1})^{ab}_{cd}$ *satisfy* (1.9)*,* (1.10)*,* (1.11)*, and* (1.12)*.*

Proof. We can show (1.9) in much the same way as the proof of [20, Theorem 5.2].

We prove (1.12). For $\lambda \in H, a, b, c, d \in X$ and $f \in L$,

$$\begin{aligned}(\rho_l((\sigma_X^{-1})^{bd}_{ac}) \circ T_{deg(d)} \circ T_{deg(b)})(f)(\lambda) =&(\sigma_X^{-1})^{bd}_{ac}(\lambda)(T_{deg(d)} \circ T_{deg(b)})(f)(\lambda)\\ =&(\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda d)b);\\ (\rho_r((\sigma_X^{-1})^{bd}_{ac}) \circ T_{deg(c)} \circ T_{deg(a)})(f)(\lambda) =&(T_{deg(c)} \circ T_{deg(a)})(f)(\lambda)(\sigma_X^{-1})^{bd}_{ac}(\lambda)\\ =&(\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda c)a).\end{aligned}$$

From (2.3), if $(\sigma_X^{-1})^{bd}_{ac}(\lambda) = 1_R$, then $(\lambda c)a = (\lambda d)b$. Hence $(\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda d)b) = (\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda c)a)$. And if $(\sigma_X^{-1})^{bd}_{ac}(\lambda) = 0_R$, it is obvious that $(\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda d)b) = (\sigma_X^{-1})^{bd}_{ac}(\lambda)f((\lambda c)a)$. Therefore (1.12) holds.

We next prove (1.10). Our goal is to show

$$\sum_{c,w\in X} ({\sigma_X}^{wc}_{av}(\sigma_X^{-1})^{by}_{wc})(\lambda) = \sum_{c,w\in X} ((\sigma_X^{-1})^{cw}_{va}{\sigma_X}^{yb}_{cw})(\lambda) = \delta_{ab}\delta_{vy}1_R.$$

for all $\lambda \in H$ and $a, b, v, y \in X$.

Lemma 2.3. (1) *For* $\lambda \in H$ *and* $a, b, v, y \in X$*,*

$$\sum_{c,w\in X} ({\sigma_X}^{wc}_{av}(\sigma_X^{-1})^{by}_{wc})(\lambda) = {\sigma_X}^{w_1c_1}_{av}(\lambda).$$

Here w_1 *and* c_1 *are the elements of* X *defined by*

$$\begin{cases} w_1 = \pi^{-1}(\mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda)))\backslash((\lambda y)b);\\ c_1 = \lambda\backslash\pi^{-1}(\mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda))).\end{cases} \tag{2.4}$$

(2) *For* $\lambda \in H$ *and* $a, b, v, y \in X$*,*

$$\sum_{c,w\in X} ((\sigma_X^{-1})^{cw}_{va}(\sigma_X)^{yb}_{cw})(\lambda) = (\sigma_X^{-1})^{c_2w_2}_{va}(\lambda).$$

Here w_2 *and* c_2 *are the elements of* X *defined by*

$$\begin{cases} c_2 = \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)y)))\backslash((\lambda b)y);\\ w_2 = \lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)y))).\end{cases} \tag{2.5}$$

Proof of Lemma 2.3. We first show (1). From (2.3), $(\sigma_X^{-1})^{by}_{wc}(\lambda) = 1_R$ if $w = \pi^{-1}(\mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda)))\backslash((\lambda y)b)$ and $c = \lambda\backslash\pi^{-1}(\mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda)))$, and otherwise $(\sigma_X^{-1})^{by}_{wc}(\lambda) = 0_R$ ($\lambda \in H, a, b, c, v, w, y \in X$). Consequently,

$$\begin{aligned}\sum_{c,w\in X} ({\sigma_X}^{wc}_{av}(\sigma_X^{-1})^{by}_{wc})(\lambda) =& \sum_{c,w\in X} {\sigma_X}^{wc}_{av}(\lambda)(\sigma_X^{-1})^{by}_{wc}(\lambda)\\ =&{\sigma_X}^{w_1c_1}_{av}(\lambda).\end{aligned}$$

Hence (1) holds.

We next show (2). By (2.2), ${\sigma_X}^{yb}_{cw}(\lambda) = 1_R$ if $c = \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)y)))\backslash((\lambda b)y)$ and $w = \lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)y)))$, and otherwise ${\sigma_X}^{yb}_{cw}(\lambda) = 0_R$ ($\lambda \in H, a, b, c, v, w, y \in X$). Hence we can show (2) in much the same way. ∎

In view of Lemma 2.3, our next task is to prove $\sigma_X{}^{w_1 c_1}_{av}(\lambda) = (\sigma_X^{-1})^{c_2 w_2}_{va}(\lambda) = \delta_{ab}\delta_{vy}1_R$. We first show $\sigma_X{}^{w_1 c_1}_{av}(\lambda) = \delta_{ab}\delta_{vy}1_R$. If $a = b$ and $v = y$,

$$
\begin{aligned}
&\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda c_1), \pi((\lambda c_1)w_1)))\backslash((\lambda c_1)w_1)\\
=&\pi^{-1}(\mu(\pi(\lambda), \mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda)), \pi((\lambda y)b)))\backslash((\lambda y)b)\\
=&(\lambda y)\backslash((\lambda y)b)\\
=&a;\\
&\lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda c_1), \pi((\lambda c_1)w_1)))\\
=&\lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \mu(\pi((\lambda y)b), \pi(\lambda y), \pi(\lambda)), \pi((\lambda y)b)))\\
=&\lambda\backslash(\lambda y)\\
=&v,
\end{aligned}
$$

and otherwise $\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda c_1), \pi((\lambda c_1)w_1)))\backslash((\lambda c_1)w_1) \neq a$ or $\lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda c_1), \pi((\lambda c_1)w_1))) \neq v$ because of (2.1) and (2.4). On account of (2.2), we see that $\sigma_X{}^{w_1 c_1}_{av}(\lambda) = \delta_{ab}\delta_{vy}1_R$.

From (2.1), (2.3), and (2.5), we can check $(\sigma_X^{-1})^{c_2 w_2}_{va}(\lambda) = \delta_{ab}\delta_{vy}1_R$ in much the same way. Therefore (1.10) holds.

We next prove (1.11). Our goal is to show

$$
\begin{aligned}
&\sum_{x,y,w\in X}(T_{deg(v)}(\sigma_X{}^{xy}_{ac})\sigma_X{}^{wd}_{yv}T_{deg(d)}(\sigma_X{}^{sb}_{xw}))(\lambda)\\
=&\sum_{x,y,w\in X}(\sigma_X{}^{wy}_{cv}T_{deg(y)}(\sigma_X{}^{sx}_{aw})\sigma_X{}^{bd}_{xy})(\lambda)
\end{aligned}
\tag{2.6}
$$

for all $\lambda \in H$ and $a, b, c, d, s, v \in X$. By the definition,

$$
\begin{aligned}
\text{LHS of (2.6)} &= \sum_{x,y,w\in X} T_{deg(v)}(\sigma_X{}^{xy}_{ac})(\lambda)\sigma_X{}^{wd}_{yv}(\lambda)T_{deg(d)}(\sigma_X{}^{sb}_{xw})(\lambda)\\
&= \sum_{x,y,w\in X} \sigma_X{}^{xy}_{ac}(\lambda v)\sigma_X{}^{wd}_{yv}(\lambda)\sigma_X{}^{sb}_{xw}(\lambda d);\\
\text{RHS of (2.6)} &= \sum_{x,y,w\in X} \sigma_X{}^{wy}_{cv}(\lambda)T_{deg(y)}(\sigma_X{}^{sx}_{aw})(\lambda)\sigma_X{}^{bd}_{xy}(\lambda)\\
&= \sum_{x,y,w\in X} \sigma_X{}^{wy}_{cv}(\lambda)\sigma_X{}^{sx}_{aw}(\lambda y)\sigma_X{}^{bd}_{xy}(\lambda).
\end{aligned}
$$

Lemma 2.4. (1) *For $\lambda \in H$ and $a, b, c, d, s, v \in X$,*

$$
\sum_{x,y,w\in X} \sigma_X{}^{xy}_{ac}(\lambda v)\sigma_X{}^{wd}_{yv}(\lambda)\sigma_X{}^{sb}_{xw}(\lambda d) = \delta_{vv_0}\sigma_X{}^{x_3 y_3}_{ac}(\lambda v). \tag{2.7}
$$

Here v_0, x_3, and y_3 are the elements of X defined by

$$
v_0 = \lambda\backslash\pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)) \tag{2.8}
$$

and

$$
\begin{cases}
x_3 = \pi^{-1}(\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s))\backslash(((\lambda d)b)s);\\
y_3 = \pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s))\backslash\\
\qquad \pi^{-1}(\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s))).
\end{cases}
\tag{2.9}
$$

(2) *For $\lambda \in H$ and $a, b, c, d, s, v \in X$,*

$$\sum_{x,y,w \in X} \sigma X_{cv}^{wy}(\lambda) \sigma X_{aw}^{sx}(\lambda y) \sigma X_{xy}^{bd}(\lambda) = \delta_{aa_0} \sigma X_{cv}^{w_4 y_4}(\lambda). \tag{2.10}$$

Here a_0, y_4, and w_4 are the elements of X defined by

$$a_0 = \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)) \backslash (((\lambda d)b)s) \tag{2.11}$$

and

$$\begin{cases} y_4 = \lambda \backslash \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi((\lambda d)b)); \\ w_4 = \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi((\lambda d)b)) \backslash \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)). \end{cases} \tag{2.12}$$

Proof of Lemma 2.4. We first show (1). From (2.1) and (2.2), $\sigma X_{yv}^{wd}(\lambda) = \sigma X_{xw}^{sb}(\lambda d) = 1_R$ if

$$\begin{aligned}
x =& \pi^{-1}(\mu(\pi(\lambda d), \pi((\lambda d)b), \pi(((\lambda d)b)s))) \backslash (((\lambda d)b)s) \\
=& \pi^{-1}(\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)) \backslash (((\lambda d)b)s); \\
w =& (\lambda d) \backslash \pi^{-1}(\mu(\pi(\lambda d), \pi((\lambda d)b), \pi(((\lambda d)b)s))) \\
=& (\lambda d) \backslash \pi^{-1}(\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)); \\
y =& \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda d), \pi((\lambda d)w))) \backslash ((\lambda d)w) \\
=& \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda d), \mu(\pi(\lambda d), \pi((\lambda d)b), \pi(((\lambda d)b)s)))) \backslash \\
& \quad \pi^{-1}(\mu(\pi(\lambda d), \pi((\lambda d)b), \pi(((\lambda d)b)s)))); \\
=& \pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)) \backslash \\
& \quad \pi^{-1}(\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)))); \\
v =& \lambda \backslash \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda d), \pi((\lambda d)w))) \\
=& \lambda \backslash \pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)),
\end{aligned}$$

and otherwise $\sigma X_{yv}^{wd}(\lambda) = 0_R$ or $\sigma X_{xw}^{sb}(\lambda d) = 0_R$ ($\lambda \in H, a, b, c, d, s, v, x, y, w \in X$). From (2.8) and (2.9), we see that (2.7) holds.

We next show (2). By (2.1) and (2.2), $\sigma X_{aw}^{sx}(\lambda y) = \sigma X_{xy}^{bd}(\lambda) = 1_R$ if

$$\begin{aligned}
x =& \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda d), \pi((\lambda d)b))) \backslash ((\lambda d)b) \\
=& \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi((\lambda d)b)) \backslash ((\lambda d)b) \\
y =& \lambda \backslash \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda d), \pi((\lambda d)b))) \\
=& \lambda \backslash \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi((\lambda d)b)); \\
a =& \pi^{-1}(\mu(\pi(\lambda y), \pi((\lambda y)x), \pi(((\lambda y)x)s))) \backslash (((\lambda y)x)s) \\
=& \pi^{-1}(\mu(\mu(\pi(\lambda), \pi(\lambda d), \pi((\lambda d)b)), \pi((\lambda d)b), \pi(((\lambda d)b)s))) \backslash (((\lambda d)b)s) \\
=& \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)) \backslash (((\lambda d)b)s); \\
w =& (\lambda y) \backslash \pi^{-1}(\mu(\pi(\lambda y), \pi((\lambda y)x), \pi(((\lambda y)x)s))) \\
=& \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi((\lambda d)b)) \backslash \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)),
\end{aligned}$$

and otherwise $\sigma X_{aw}^{sx}(\lambda y) = 0_R$ or $\sigma X_{xy}^{bd}(\lambda) = 0_R$ ($\lambda \in H, a, b, c, d, s, v, x, y, w \in X$). On account of (2.11) and (2.12), we see (2.10) holds. □

Lemma 2.5. *For $\lambda \in H$ and $a, b, c, d, s, v \in X$,*

$$\delta_{vv_0}\sigma_X{}^{x_3y_3}_{ac}(\lambda v) = \delta_{aa_0}\sigma_X{}^{w_4y_4}_{cv}(\lambda) = \delta_{aa_0}\delta_{cc_0}\delta_{vv_0}1_R.$$

Here c_0 is the element of X defined by

$$c_0 = \pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s))\backslash\pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)). \tag{2.13}$$

For the definitions of the elements $a_0, v_0, x_3, y_3, w_4, y_4 \in X$, see (2.8), (2.9), (2.11), *and* (2.12).

Proof of Lemma 2.5. We first show $\delta_{vv_0}\sigma_X{}^{x_3y_3}_{ac}(\lambda v) = \delta_{aa_0}\delta_{cc_0}\delta_{vv_0}1_R$. On account of (2.1), (2.2), (2.8), and (2.9), $\delta_{vv_0}\sigma_X{}^{x_3y_3}_{ac}(\lambda v) = 1_R$ if

$$\begin{aligned}
v =& v_0;\\
a =& \pi^{-1}(\mu(\pi(\lambda v), \pi((\lambda v)y_3), \pi(((\lambda v)y_3)x_3)))\backslash(((\lambda v)y_3)x_3)\\
=& \pi^{-1}((\pi(\lambda) - \pi(\lambda d) + (\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)))\\
&\quad - (\pi(\lambda d) - \pi((\lambda d)b) + \pi(((\lambda d)b)s)) + \pi(((\lambda d)b)s)))\backslash(((\lambda d)b)s)\\
=& \pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s))\backslash(((\lambda d)b)s);\\
c =& (\lambda v)\backslash\pi^{-1}(\mu(\pi(\lambda v), \pi((\lambda v)y_3), \pi(((\lambda v)y_3)x_3)))\\
=& \pi^{-1}(\pi(\lambda) - \pi((\lambda d)b) + \pi(((\lambda d)b)s))\backslash\pi^{-1}(\pi(\lambda) - \pi(\lambda d) + \pi(((\lambda d)b)s)),
\end{aligned}$$

and otherwise $\delta_{vv_0}\sigma_X{}^{x_3y_3}_{ac}(\lambda v) = 0_R$. From (2.11) and (2.13), $\delta_{vv_0}\ \sigma_X{}^{x_3y_3}_{ac}(\lambda v) = \delta_{aa_0}\delta_{cc_0}\ \delta_{vv_0}1_R$ holds.

On account of (2.1), (2.2), (2.8), (2.11), (2.12), and (2.13), we can show $\delta_{aa_0}\sigma_X{}^{w_4y_4}_{cv}(\lambda) = \delta_{aa_0}\delta_{cc_0}\delta_{vv_0}1_R$ in much the same way. □

On account of Lemmas 2.4 and 2.5, we can see that (2.6) holds. This proves the proposition. □

We write $\sigma_X = (\sigma_X{}^{ab}_{cd})_{a,b,c,d\in X}$.

Proposition 2.6. *σ_X is rigid.*

For a similar proof of Proposition 2.6, see [27, The proof of Proposition 4.7] and [19, The proof of Proposition 5.4].

On account of Theorems 1.5, 1.12, and Propositions 2.2, 2.6, σ_X induces the Hopf algebroid A_{σ_X}, and we have a k-algebra homomorphism $\rho_{\sigma_X} : A_{\sigma_X} \to End_k(X^{\sigma_X})$.

We next consider the k-algebra R. If R is not Frobenius-separable, then the Hopf algebroid A_{σ_X} is not a weak bialgebra (Cf. [22]). Making use of [29, Theorem IV.2.1], we will see that there exists a k-algebra R that is not Frobenius-separable.

Let $\mathbb{K}$ be a field which includes an element x satisfying $x^2 = -1$, and $M_2(\mathbb{K})$ the matrix algebra which consists of all 2×2 matrices. We write $R \subset M_2(\mathbb{K})$ as the $\mathbb{K}$-subspace whose base is

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & x \\ 0 & -1 \end{pmatrix}.$$

We can calculate that $A_1A_1 = A_1, A_1A_2 = A_2, A_2A_1 = -A_1, A_2A_2 = -A_2$, and that

$$\begin{aligned}&(aE + bA_1 + cA_2)(dE + eA_1 + fA_2)\\=&adE + (ae + bd + be - ce)A_1 + (af + cd + bf - cf)A_2\end{aligned} \tag{2.14}$$

for all $a, b, c, d, e, f \in \mathbb{K}$. Since $A_1A_2 = A_2 \neq -A_1 = A_2A_1$, the $\mathbb{K}$-algebra R is not commutative.

Proposition 2.7. *R is not Frobenius-separable.*

We can show Proposition 2.7 in much the same way as [20, The proof of Proposition 5.4].

Therefore $L \cong R^{|H|}$ is not Frobenius-separable [29, Proposition IV.2.4]. Consequently, the Hopf algebroid A_{σ_X} is not a weak bialgebra.

Part III
Construction of dynamical reflection maps

In Part III, we construct solutions to reflection equations in the tensor category Set_H, which we call dynamical reflection maps. After several preparations, we state the main result (Theorem 6.3). We also give examples of dynamical reflection maps. This is a joint work [1] with Y. Shibukawa.

3 Dynamical reflection maps

In this section, we prepare for the proof of Theorem 3.8, which implies that we can construct solutions to the reflection equation in the tensor category Set_H. We first recall the notion of Set_H and show Theorem 3.8 (Cf. [8, Section 6]). Throughout this paper, we follow the definition of tensor categories in [13, Chapter XI].

Let us recall the definition of a tensor category (Cf. [13, Definition XI.2.1]).

Definition 3.1. A tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ with:

(1) (tensor product)
a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

(2) (unit)
an object I;

(3) (associativity constraint)
a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$;

(4) (left unit constraint with respect to the unit I)
a natural isomorphim $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$;

(5) (right unit constraint with respect to the unit I)

a natural isomorphim $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$

such that the following relations are satisfied. For $U, V, W, X \in \mathcal{C}$,

$$(1_U \otimes a_{VWX})a_{UV\otimes WX}(a_{UVW} \otimes 1_X) = a_{UVW\otimes X}a_{U\otimes VWX}; \tag{3.1}$$
$$(1_V \otimes l_W)a_{VIW} = r_V \otimes 1_W. \tag{3.2}$$

The relations (3.1) and (3.2) are respectively called the pentagon axiom and the triangle axiom (See [13, Definition XI.2.1]).

Let $A \in \mathcal{C}$. If a morphism $\sigma : A \otimes A \to A \otimes A$ of $\mathcal{C}$ satisfies

$$\begin{aligned} &a_{AAA}(\sigma \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A) \\ =&(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}, \end{aligned} \tag{3.3}$$

then we say that σ satisfies a braid relation in $\mathcal{C}$.

From now on, we introduce the notion of Set_H (See [25, 26]).

Definition 3.2. Let H be a nonempty set. Set_H is the category consisting of the following.

(1) An object $(X, \cdot_X)$ is a pair of a set X with a map $\cdot_X : H \times X \to H$.

(2) A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ is a map $f : H \to Map(X, Y)$ satisfying

$$\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x$$

for all $\lambda \in H, x \in X$. Here, $Map(X, Y)$ is the set of all maps from X to Y.

(3) Let $f : (X, \cdot_X) \to (Y, \cdot_Y)$ and $g : (Y, \cdot_Y) \to (Z, \cdot_Z)$ be morphisms of Set_H. The composition $g \circ f : (X, \cdot_X) \to (Z, \cdot_Z)$ is defined by $(g \circ f)(\lambda) = g(\lambda) \circ f(\lambda)$ $(\lambda \in H)$.

(4) Let $(X, \cdot_X) \in Set_H$. The identity $1_{(X,\cdot_X)}$ is defined by $1_{(X,\cdot_X)}(\lambda)(x) = x$ $(\lambda \in H, x \in X)$.

Proposition 3.3. *Set_H is a tensor category (Definition 3.1).*

In fact, for any $X = (X, \cdot_X) \in Set_H$ and $Y = (Y, \cdot_Y) \in Set_H$, the tensor product $X \otimes Y$ is the object $(X \times Y, \cdot_{X\otimes Y})$ where the map $\cdot_{X\otimes Y} : H \times (X \times Y) \to H$ is defined by

$$\lambda \cdot_{X\otimes Y} (x, y) = (\lambda \cdot_X x) \cdot_Y y \quad (\lambda \in H, x \in X, y \in Y).$$

Obviously, $X \otimes Y$ is an object of Set_H.

Let $f : (X, \cdot_X) \to (Y, \cdot_Y)$ and $g : (X', \cdot_{X'}) \to (Y', \cdot_{Y'})$ be morphisms of Set_H. The tensor product $f \otimes g : (X, \cdot_X) \otimes (X', \cdot_{X'}) \to (Y, \cdot_Y) \otimes (Y', \cdot_{Y'})$ of f and g is defined by

$$(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda \cdot_X x)(y)) \quad (\lambda \in H, x \in X, y \in Y).$$

We see that $f \otimes g$ is a morphism of Set_H.

A unit $I = (I, \cdot_I)$ is a pair of a set $I = \{\bullet\}$ with the map $\cdot_I : H \times I \to H$ defined by $\lambda \cdot_I \bullet = \lambda$ $(\lambda \in H)$.

For $X, Y, Z \in Set_H$, the associativity constraint $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is defined by

$$a_{X,Y,Z}(\lambda)((x, y), z) = (x, (y, z)) \quad (\lambda \in H, x \in X, y \in Y, z \in Z).$$

For every $X \in Set_H$, the left unit constraint $l_X : I \otimes X \to X$ with respect to the unit I is defined by

$$l_X(\lambda)(\bullet, x) = x \quad (\lambda \in H, x \in X).$$

Similarly, for every $X \in Set_H$, the right unit constraint $r_X : X \otimes I \to X$ with respect to the unit I is defined by

$$r_X(\lambda)(x, \bullet) = x \quad (\lambda \in H, x \in X).$$

Then $(Set_H, \otimes, I, a, l, r)$ is actually a tensor category.

We now define the dynamical reflection map.

Definition 3.4. [dynamical reflection map] Let $A, X \in Set_H$. We assume that a morphism $\sigma : A \otimes A \to A \otimes A$ of Set_H satisfies the braid relation (3.3). If a morphism $k : A \otimes X \to A \otimes X$ of Set_H enjoys

$$\begin{aligned} &a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX} \\ =&(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k), \end{aligned} \tag{3.4}$$

then k is called a dynamical reflection map (we also say that k is a solution to the reflection equation associated with σ).

For a while, we focus on an arbitrary tensor category again. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category.

Notation. Let $A, X \in \mathcal{C}$, and write $Y = A \otimes X$. Let $m : A \otimes A \to A$, $\sigma : A \otimes A \to A \otimes A$, $m_X : A \otimes X \to X$, and $m_Y : A \otimes Y \to Y$ be morphisms of $\mathcal{C}$. We denote by m^{triv}_Y and m^{σ}_Y the following morphisms:

$$m^{\mathrm{triv}}_Y = (m \otimes 1_X)a^{-1}_{AAX} : A \otimes Y \to Y; \tag{3.5}$$

$$m^{\sigma}_Y = (1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX} : A \otimes Y \to Y. \tag{3.6}$$

Let $A, X \in \mathcal{C}$, and write $Y = A \otimes X$. Let $m : A \otimes A \to A$, $\sigma : A \otimes A \to A \otimes A$, $m_X : A \otimes X \to X$, $m_Y : A \otimes Y \to Y$, and $\eta : I \to A$ be morphisms of $\mathcal{C}$. We define the morphism $k : A \otimes X \to A \otimes X$ of $\mathcal{C}$ by

$$k = m_Y(1_A \otimes ((\eta \otimes 1_X)l^{-1}_X)). \tag{3.7}$$

Proposition 3.5. *If the morphisms m, σ, m_X, m_Y, and η satisfy*

$$m(1_A \otimes \eta)r^{-1}_A = 1_A, \tag{3.8}$$

$$m_Y(m \otimes 1_Y) = m_Y(1_A \otimes m_Y)a_{AAY}, \tag{3.9}$$

$$m_X(\eta \otimes 1_X) = l_X, \tag{3.10}$$

$$m_X m_Y = m_X(1_A \otimes m_X), \tag{3.11}$$
$$m_Y(1_A \otimes m_Y^{\mathrm{triv}}) = m_Y^{\mathrm{triv}}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1}, \tag{3.12}$$

then the morphism k (3.7) satisfies the following relations.

$$m_X k = m_X; \tag{3.13}$$
$$k(m \otimes 1_X) = (m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}. \tag{3.14}$$

Proof. We first prove (3.13). From (3.10) and (3.11),

$$\begin{aligned} m_X k =& m_X m_Y(1_A \otimes ((\eta \otimes 1_X)l_X^{-1})) \\ =& m_X(1_A \otimes m_X)(1_A \otimes ((\eta \otimes 1_X)l_X^{-1})) \\ =& m_X. \end{aligned}$$

For the proof of (3.14), we use the following lemma.

Lemma 3.6. *The condition* (3.8) *and* (3.12) *imply*

$$m_Y = (m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}. \tag{3.15}$$

We assume Lemma 3.6 for a while. On account of (3.9),

$$\begin{aligned} k(m \otimes 1_X) =& m_Y(1_A \otimes ((\eta \otimes 1_X)l_X^{-1}))(m \otimes 1_X) \\ =& m_Y(m \otimes 1_Y)(1_{A\otimes A} \otimes ((\eta \otimes 1_X)l_X^{-1})) \\ =& m_Y(1_A \otimes m_Y)(1_A \otimes (1_A \otimes ((\eta \otimes 1_X)l_X^{-1})))a_{AAX} \\ =& m_Y(1_A \otimes k)a_{AAX}. \end{aligned} \tag{3.16}$$

By means of (3.15), we can easily see that (3.14) holds. □

Proof of Lemma 3.6. Because of (3.5), (3.8), and (3.12),

$$\begin{aligned} m_Y =& m_Y(1_A \otimes ((m(1_A \otimes \eta)r_A^{-1}) \otimes 1_X)) \\ =& m_Y(1_A \otimes (m \otimes 1_X))(1_A \otimes (((1_A \otimes \eta)r_A^{-1}) \otimes 1_X)) \\ =& m_Y^{\mathrm{triv}}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1} \\ & (1_A \otimes a_{AAX})(1_A \otimes (((1_A \otimes \eta)r_A^{-1}) \otimes 1_X)) \\ =& m_Y^{\mathrm{triv}}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)(1_{A\otimes A} \otimes (\eta \otimes 1_X)) \\ & a_{AAI\otimes X}^{-1}(1_A \otimes (a_{AIX}(r_A^{-1} \otimes 1_X))). \end{aligned} \tag{3.17}$$

On account of the triangle axiom (3.2) and (3.7),

$$\begin{aligned} & \text{RHS of (3.17)} \\ =& m_Y^{\mathrm{triv}}(1_A \otimes (m_Y(1_A \otimes (\eta \otimes 1_X))))a_{AAI\otimes X}(\sigma \otimes 1_{I\otimes X}) \\ & a_{AAI\otimes X}^{-1}(1_A \otimes (1_A \otimes l_X^{-1})) \\ =& m_Y^{\mathrm{triv}}(1_A \otimes (m_Y(1_A \otimes ((\eta \otimes 1_X)l_X^{-1}))))a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1} \\ =& (m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}. \end{aligned}$$

Hence (3.15) holds. □

Proposition 3.7. *Let $A, X \in \mathcal{C}$, and write $Y = A \otimes X$. Let $m : A \otimes A \to A$, $m_X : A \otimes X \to X$, $m_Y : A \otimes Y \to Y$, and $\eta : I \to A$ be morphisms of $\mathcal{C}$. We assume that $\sigma : A \otimes A \to A \otimes A$ is an isomorphism satisfying* (3.3).

If the morphisms m, σ, m_X, m_Y, and η satisfy (3.12) *and*

$$m(\eta \otimes 1_A)l_A^{-1} = 1_A = m(1_A \otimes \eta)r_A^{-1}, \tag{3.18}$$

$$(m \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A) = \sigma(1_A \otimes m)a_{AAA}, \tag{3.19}$$

$$\sigma(1_A \otimes \eta) = (\eta \otimes 1_A)l_A^{-1}r_A, \tag{3.20}$$

$$m_Y(1_A \otimes m_Y^\sigma) = m_Y^\sigma(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1}, \tag{3.21}$$

then the morphism k (3.7) *satisfies the following relation.*

$$\begin{aligned}&k(1_A \otimes m_X)\\ =&(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}.\end{aligned} \tag{3.22}$$

Here, m_Y^σ is defined by (3.6).

Proof. From (3.18),

$$\begin{aligned}&(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}\\ =&((m(\eta \otimes 1_A)l_A^{-1}) \otimes 1_X)(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}\\ &(\sigma \otimes 1_X)a_{AAX}^{-1}.\end{aligned} \tag{3.23}$$

We note that, on account of the fact that $l_{A\otimes A} = (l_A \otimes 1_A)a_{IAA}^{-1}$ [13, Lemma XI.2.2],

$$(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_A)\sigma = a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_A).$$

Using this relation repeatedly, we deduce that

$$\begin{aligned}&\text{RHS of (3.23)}\\ =&(m \otimes m_X)a_{A\otimes AAX}((a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}) \otimes 1_X)a_{A\otimes AAX}^{-1}(1_{A\otimes A} \otimes k)\\ &a_{A\otimes AAX}((a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}) \otimes 1_X)((((\eta \otimes 1_A)l_A^{-1}) \otimes 1_A) \otimes 1_X)a_{AAX}^{-1}.\end{aligned} \tag{3.24}$$

From (3.3), (3.6), (3.19), (3.20), and the fact that σ is an isomorphism,

$$\begin{aligned}&\text{RHS of (3.24)}\\ =&(m \otimes m_X)a_{A\otimes AAX}((a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}) \otimes 1_X)a_{A\otimes AAX}^{-1}(1_{A\otimes A} \otimes k)a_{A\otimes AAX}\\ &((\sigma \otimes 1_A) \otimes 1_X)(((\sigma^{-1} \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)) \otimes 1_X)\\ &((((1_A \otimes \eta)r_A^{-1}) \otimes 1_A) \otimes 1_X)a_{AAX}^{-1}\\ =&(1_A \otimes m_X)a_{AAX}(((m \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)) \otimes 1_X)\\ &a_{A\otimes AAX}^{-1}(1_{A\otimes A} \otimes k)a_{A\otimes AAX}((a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}) \otimes 1_X)\\ &(((\sigma \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma^{-1})a_{AAA}) \otimes 1_X)\\ &((((1_A \otimes \eta)r_A^{-1}) \otimes 1_A) \otimes 1_X)a_{AAX}^{-1}\\ =&m_Y^\sigma a_{AAX}(((1_A \otimes m)a_{AAA}) \otimes 1_X)a_{A\otimes AAX}^{-1}(1_{A\otimes A} \otimes k)a_{A\otimes AAX}\end{aligned}$$

$$((a_{AAA}^{-1}(1_A\otimes\sigma)a_{AAA}(\sigma\otimes 1_A)a_{AAA}^{-1}(1_A\otimes\sigma^{-1})a_{AAA})\otimes 1_X)$$
$$((((1_A\otimes\eta)r_A^{-1})\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=m_Y^\sigma(1_A\otimes((m\otimes 1_X)a_{AAX}^{-1}(1_A\otimes k)a_{AAX}(\sigma\otimes 1_X)))a_{AA\otimes AX}$$
$$((a_{AAA}(\sigma\otimes 1_A)a_{AAA}^{-1}(1_A\otimes\sigma^{-1})a_{AAA})\otimes 1_X)$$
$$((((1_A\otimes\eta)r_A^{-1})\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}. \tag{3.25}$$

We note that Lemma 3.6 holds because of (3.12) and (3.18). Consequently, on account of (3.15) and (3.21),

RHS of (3.25)

$$=m_Y^\sigma(1_A\otimes m_Y)a_{AAY}(\sigma\otimes 1_Y)a_{A\otimes AAX}((a_{AAA}^{-1}(1_A\otimes\sigma^{-1})a_{AAA})\otimes 1_X)$$
$$((((1_A\otimes\eta)r_A^{-1})\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=(m\otimes 1_X)a_{AAX}^{-1}(1_A\otimes k)a_{AAX}(\sigma\otimes 1_X)a_{AAX}^{-1}(1_A\otimes(1_A\otimes m_X))$$
$$(1_A\otimes(a_{AAX}(\sigma\otimes 1_X)))a_{AA\otimes AX}(((1_A\otimes\sigma^{-1})a_{AAA})\otimes 1_X)$$
$$((((1_A\otimes\eta)r_A^{-1})\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=(m\otimes 1_X)a_{AAX}^{-1}(1_A\otimes k)a_{AAX}(\sigma\otimes 1_X)a_{AAX}^{-1}(1_A\otimes((1_A\otimes m_X)a_{AAX}))$$
$$a_{AA\otimes AX}(a_{AAA}\otimes 1_X)((((1_A\otimes\eta)r_A^{-1})\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=(m\otimes 1_X)a_{AAX}^{-1}(1_A\otimes k)a_{AAX}((\sigma(1_A\otimes\eta))\otimes 1_X)a_{AIX}^{-1}$$
$$(1_A\otimes((1_I\otimes m_X)a_{IAX}))a_{AI\otimes AX}(a_{AIA}\otimes 1_X)((r_A^{-1}\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}. \tag{3.26}$$

By using the triangle axiom (3.2), (3.18), (3.20), and the fact $l_{A\otimes X}=(l_A\otimes 1_X)a_{IAX}^{-1}$ [13, Lemma XI.2.2], we see that

RHS of (3.26)

$$=(m\otimes 1_X)a_{AAX}^{-1}(1_A\otimes k)a_{AAX}(((\eta\otimes 1_A)l_A^{-1}r_A)\otimes 1_X)a_{AIX}^{-1}$$
$$(1_A\otimes((1_I\otimes m_X)a_{IAX}))a_{AI\otimes AX}(a_{AIA}\otimes 1_X)((r_A^{-1}\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=((m(\eta\otimes 1_A))\otimes 1_X)a_{IAX}^{-1}(1_I\otimes k)a_{IAX}((l_A^{-1}r_A)\otimes 1_X)a_{AIX}^{-1}$$
$$(1_A\otimes((1_I\otimes m_X)a_{IAX}))a_{AI\otimes AX}(a_{AIA}\otimes 1_X)((r_A^{-1}\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=(l_A\otimes 1_X)a_{IAX}^{-1}(1_I\otimes k)a_{IAX}((l_A^{-1}r_A)\otimes 1_X)a_{AIX}^{-1}$$
$$(1_A\otimes((1_I\otimes m_X)a_{IAX}))a_{AI\otimes AX}(a_{AIA}\otimes 1_X)((r_A^{-1}\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=l_{A\otimes X}(1_I\otimes k)l_{A\otimes X}^{-1}(r_A\otimes 1_X)a_{AIX}^{-1}(1_A\otimes((1_I\otimes m_X)a_{IAX}))$$
$$a_{AI\otimes AX}(a_{AIA}\otimes 1_X)((r_A^{-1}\otimes 1_A)\otimes 1_X)a_{AAX}^{-1}$$
$$=k(1_A\otimes m_X).$$

Hence (3.22) holds. □

Propositions 3.5 and 3.7 yield the following theorem.

Theorem 3.8. *Let $A,X\in Set_H$, and $m:A\otimes A\to A, m_X:A\otimes X\to X,\sigma: A\otimes A\to A\otimes A$, and $k:A\otimes X\to A\otimes X$ be morpshims of Set_H. We assume that these morphisms satisfy* (3.13), (3.14), (3.22), *and*

$$m\sigma=m. \tag{3.27}$$

In addition, for any $\lambda \in H$ and $a \in A$, we assume that the maps $A \ni b \mapsto m(\lambda)(a,b) \in A$ are injective. Then k is a dynamical reflection map.

Proof. Let $\lambda \in H$, $a, b \in A$, and $x \in X$. We set $a_1, a_2, b_1, b_2 \in A$ and $x_1, x_2 \in X$ by

$$\begin{aligned}((a_1,b_1),x_1) = (&a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k) \\ &a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)); \end{aligned} \tag{3.28}$$

$$\begin{aligned}((a_2,b_2),x_2) = (&(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X) \\ &a_{AAX}^{-1}(1_A \otimes k))(\lambda)(a,(b,x)). \end{aligned} \tag{3.29}$$

It suffices to show $a_1 = a_2$, $b_1 = b_2$, and $x_1 = x_2$. For the proof, we need:

$$(m(\lambda)(a_1,b_1),x_1) = (m(\lambda)(a_2,b_2),x_2); \tag{3.30}$$

$$(a_1, m_X(\lambda a_1)(b_1,x_1)) = (a_2, m_X(\lambda a_2)(b_2,x_2)). \tag{3.31}$$

In fact, if (3.30) hold, then $m(\lambda)(a_1,b_1) = m(\lambda)(a_2,b_2)$ and $x_1 = x_2$. Similarly, if (3.31) hold, then $a_1 = a_2$. Hence

$$m(\lambda)(a_1,b_1) = m(\lambda)(a_1,b_2).$$

Since $A \ni b \mapsto m(\lambda)(a,b) \in A$ is injective, we obtain $b_1 = b_2$,

We first show (3.30). From (3.14) and (3.27),

$$\begin{aligned}&(m(\lambda)(a_1,b_1),x_1) \\ =&((m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX} \\ &(\sigma \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)) \\ =&(k(m \otimes 1_X)(\sigma \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)) \\ =&(k(m \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)) \\ =&((m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k))(\lambda)(a,(b,x)) \\ =&(m(\lambda)(a_2,b_2),x_2).\end{aligned}$$

The next task is to show (3.31). On account of (3.13) and (3.22),

$$\begin{aligned}&(a_1, m_X(\lambda a_1)(b_1,x_1)) \\ =&((1_A \otimes m_X)(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX} \\ &(\sigma \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)) \\ =&((1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1})(\lambda)(a,(b,x)) \\ =&(k(1_A \otimes m_X))(\lambda)(a,(b,x)) \\ =&(k(1_A \otimes m_X)(1_A \otimes k))(\lambda)(a,(b,x)) \\ =&((1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X) \\ &a_{AAX}^{-1}(1_A \otimes k))(\lambda)(a,(b,x)) \\ =&(a_2, m_X(\lambda a_2)(b_2,x_2)).\end{aligned}$$

Therefore (3.30) and (3.31) hold. □

4 Braided monoids and left modules of monoids

Section 4 deals with braided monoids and left modules of monoids in a tensor category. Also, we introduce a braided monoid in Set_H (Cf. [8, Sections 5 and 7]).

4.1 Braided monoids

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category.

Definition 4.1 (monoid)**.** Let A be an object of $\mathcal{C}$, together with morphisms $m : A \otimes A \to A$ and $\eta : I \to A$. If the morphisms m and η satisfy (3.18) and

$$m(m \otimes 1_A) = m(1_A \otimes m)a_{AAA}, \tag{4.1}$$

then we say that (A, m, η) is a monoid in $\mathcal{C}$.

Definition 4.2 (braided monoid)**.** Let (A, m, η) be a monoid in $\mathcal{C}$ with a morphism $\sigma : A \otimes A \to A \otimes A$. If the morphisms m, η, and σ satisfy (3.19), (3.20), and

$$(1_A \otimes m)a_{AAA}(\sigma \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma) = \sigma(m \otimes 1_A)a^{-1}_{AAA}; \tag{4.2}$$

$$\sigma(\eta \otimes 1_A) = (1_A \otimes \eta)r^{-1}_A l_A, \tag{4.3}$$

then we say that (A, m, η, σ) is braided.

Remark 4.3. Braided monoids in Set_H are equivalent to dynamical braided groups [25].

Let (A, m, η) be a monoid in Set_H and $\sigma : A \otimes A \to A \otimes A$ be a morphism of Set_H. Our goal is to show that (3.20) is equivalent to

$$(m \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}(1_{A\otimes A} \otimes \eta) = r_{A\otimes A}, \tag{4.4}$$

and that (4.3) is equivalent to

$$(1_A \otimes m)a_{AAA}(\sigma \otimes 1_A)a^{-1}_{AAA}(\eta \otimes 1_{A\otimes A}) = l_{A\otimes A}. \tag{4.5}$$

We only prove that (3.20) is equivalent to (4.4).

We first assume (3.20). On account of the triangle axiom (3.2), (3.18), (3.20), and the fact that $r_{A\otimes A} = (1_A \otimes r_A)a_{AAI}$ [13, Lemma XI.2.2],

$$\begin{aligned}
&(m \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}(1_{A\otimes A} \otimes \eta)\\
=&(m \otimes 1_A)a^{-1}_{AAA}(1_A \otimes ((\eta \otimes 1_A)l^{-1}_A r_A))a_{AAI}\\
=&((m(1_A \otimes \eta)) \otimes 1_A)a^{-1}_{AIA}(1_A \otimes (l^{-1}_A r_A))a_{AAI}\\
=&(r_A \otimes 1_A)a^{-1}_{AIA}(1_A \otimes (l^{-1}_A r_A))a_{AAI}\\
=&(1_A \otimes r_A)a_{AAI}\\
=&r_{A\otimes A}.
\end{aligned}$$

Conversely, we assume (4.4). From (3.18) and the fact that $l_{A\otimes A} = (l_A \otimes 1_A)a^{-1}_{IAA}$ [13, Lemma XI.2.2],

$$\sigma(1_A \otimes \eta) =((m(\eta \otimes 1_A)l^{-1}_A) \otimes 1_A)\sigma(1_A \otimes \eta)$$

$$
\begin{aligned}
&=(m\otimes 1_A)a^{-1}_{AAA}(\eta\otimes 1_{A\otimes A})l^{-1}_{A\otimes A}\sigma(1_A\otimes\eta)\\
&=(m\otimes 1_A)a^{-1}_{AAA}(1_A\otimes\sigma)(\eta\otimes 1_{A\otimes A}))l^{-1}_{A\otimes A}(1_A\otimes\eta)\\
&=(m\otimes 1_A)a^{-1}_{AAA}(1_A\otimes\sigma)a_{AAA}((\eta\otimes 1_A)\otimes 1_A)(l^{-1}_A\otimes 1_A)(1_A\otimes\eta)\\
&=(m\otimes 1_A)a^{-1}_{AAA}(1_A\otimes\sigma)a_{AAA}(1_{A\otimes A}\otimes\eta)((\eta\otimes 1_A)\otimes 1_I)(l^{-1}_A\otimes 1_I)
\end{aligned}
\tag{4.6}
$$

Because of (4.4), the right-hand-side of (4.6) is $r_{A\otimes A}((\eta\otimes 1_A)\otimes 1_I)(l^{-1}_A\otimes 1_I)$. Then,

$$
\begin{aligned}
\text{RHS of (4.6)} &=(\eta\otimes 1_A)r_{I\otimes A}(l^{-1}_A\otimes 1_I)\\
&=(\eta\otimes 1_A)(1_I\otimes r_A)a_{IAI}(l^{-1}_A\otimes 1_I)\\
&=(\eta\otimes 1_A)(1_I\otimes r_A)l^{-1}_{A\otimes I}\\
&=(\eta\otimes 1_A)l^{-1}_A r_A
\end{aligned}
$$

because $l_{A\otimes I}=(l_A\otimes 1_I)a^{-1}_{IAI}$ and $r_{I\otimes A}=(1_I\otimes r_A)a_{IAI}$ [13, Lemma XI.2.2]. Hence (3.20) is equivalent to (4.4).

From now on, we introduce a braided monoid in Set_H. Let us first define left quasigroups with a unit.

Definition 4.4 (left quasigroup)**.** Let L be a set with a binary operation $\cdot :$ $L\times L\ni(a,b)\mapsto ab\in L$. Let e_L be an element in L.

If $(L,\cdot,e_L)$ satisfies (1) in Definition 2.1 and the following condition, we say that $(L,\cdot,e_L)$ is a left quasigroup with a unit.

(3) For any $a\in L$, $ae_L=e_La=a$.

For $a,c\in L$, we denote by $a\backslash c$ the unique element $b\in L$ such that $ab=c$.

Remark 4.5. Let $(L,\cdot,e_L)$ be a left quasigroup with a unit. Then $a(a\backslash b)=b$ and $a\backslash a=e_L$ hold for all $a,b\in L$.

Example 4.6. Let $L=\{e_L,l_1,l_2,l_3,l_4,l_5,l_6,l_7\}$ be a set. We define the binary operation $\cdot : L\times L\to L$ by Table 1. For example, $l_2\cdot l_4=l_1$ and $l_4\cdot l_2=l_7$.

	e_L	l_1	l_2	l_3	l_4	l_5	l_6	l_7
e_L	e_L	l_1	l_2	l_3	l_4	l_5	l_6	l_7
l_1	l_1	l_4	l_5	l_2	l_7	l_3	e_L	l_6
l_2	l_2	l_3	e_L	l_6	l_1	l_4	l_7	l_5
l_3	l_3	l_7	l_1	e_L	l_5	l_6	l_4	l_2
l_4	l_4	l_6	l_7	l_5	l_2	e_L	l_3	l_1
l_5	l_5	l_2	l_6	l_4	l_3	l_7	l_1	e_L
l_6	l_6	l_5	l_3	l_7	e_L	l_1	l_2	l_4
l_7	l_7	e_L	l_4	l_1	l_6	l_2	l_5	l_3

Table 1: The binary operation on L

This $(L,\cdot,e_L)$ is a left quasigroup with a unit, but it is not associative because $(l_2\cdot l_3)\cdot l_4=e_L$ and $l_2\cdot(l_3\cdot l_4)=l_4$. Hence $(L,\cdot,e_L)$ is not a group.

Let $(L, \cdot, e_L)$ be a left quasigroup with a unit. We write $H = L$. By the definition, it is easily seen that $(L, \cdot) \in Set_H$.

Let G be a group isomorphic to L as sets. We denote by π the bijective map from L to G.

For $\lambda \in H$, we define the map $\sigma(\lambda) : L \times L \to L \times L$ by

$$\sigma(\lambda)(a,b) = (\xi_\lambda(a,b), \eta_\lambda(a,b)) \ (a, b \in L), \tag{4.7}$$

where we write $\xi_\lambda(a,b) = \lambda\backslash\pi^{-1}((\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b)))$ and $\eta_\lambda(a,b) = (\lambda\xi_\lambda(a,b))\backslash((\lambda a)b)$. By definition, we can check that $\sigma : L \otimes L \to L \otimes L$ defined by (4.7) is actually the morphism of Set_H. Moreover,

Proposition 4.7. *σ satisfies the braid relation* (3.3) *in Set_H.*

Proof. From (4.7),

$$\begin{aligned}
&(a_{LLL}(\sigma\otimes 1_L)a_{LLL}^{-1}(1_L\otimes\sigma)a_{LLL}(\sigma\otimes 1_L))(\lambda)((a,b),c)\\
=&(a_{LLL}(\sigma\otimes 1_L)a_{LLL}^{-1})(\lambda)(\xi_\lambda(a,b),\sigma(\lambda\xi_\lambda(a,b))(\eta_\lambda(a,b),c))\\
=&(a_{LLL}(\sigma\otimes 1_L)a_{LLL}^{-1})(\lambda)(\xi_\lambda(a,b),((\lambda\xi_\lambda(a,b))\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c)),\\
&\quad\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c))\backslash(((\lambda a)b)c)))\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c)),\\
&\quad(\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c)),\\
&\quad\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c))\backslash(((\lambda a)b)c)))
\end{aligned}$$

for $\lambda \in H$ and $a, b, c \in L$. Similarly,

$$\begin{aligned}
&((1_L\otimes\sigma)a_{LLL}(\sigma\otimes 1_L)a_{LLL}^{-1}(1_L\otimes\sigma)a_{LLL})(\lambda)((a,b),c)\\
=&((1_L\otimes\sigma)a_{LLL})(\lambda)(\sigma(\lambda)(a,\xi_{\lambda a}(b,c)),\eta_{\lambda a}(b,c))\\
=&((1_L\otimes\sigma)a_{LLL})(\lambda)((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c)),\\
&\quad\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))\backslash\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))),\\
&\quad\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))\backslash(((\lambda a)b)c))\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c)),\\
&\quad(\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c)),\\
&\quad\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c))\backslash(((\lambda a)b)c))).
\end{aligned}$$

Therefore σ is a solution to the braid relation (3.3) in Set_H. □

We define the morphisms $m : L \otimes L \to L$ and $\eta : I(= \{\bullet\}) \to L$ of Set_H by

$$m(\lambda)(a,b) = \lambda\backslash((\lambda a)b), \quad \eta(\lambda)(\bullet) = e_L \quad (\lambda \in H, a, b \in L). \tag{4.8}$$

Proposition 4.8. *(L, m, η, σ) is a braided monoid in Set_H.*

Proof. We first show that (L, m, η) is a monoid in Set_H. From (4.8),

$$\begin{aligned}
(m(m\otimes 1_L))(\lambda)((a,b),c) =& m(\lambda)(\lambda\backslash((\lambda a)b),c)\\
=&\lambda\backslash(((\lambda a)b)c)\\
=&\lambda\backslash((\lambda a)((\lambda a)\backslash(((\lambda a)b)c)))
\end{aligned}$$

$$=m(\lambda)(a,(\lambda a)\backslash(((\lambda a)b)c))$$
$$=m(\lambda)(a,m(\lambda a)(b,c))$$
$$=(m(1_L\otimes m)a_{LLL})(\lambda)((a,b),c)$$

for all $\lambda \in H$ and $a,b,c\in L$. In addition,

$$\begin{aligned}(m(\eta\otimes 1_L)l_L^{-1})(\lambda)(a) =&m(\lambda)(e_L,a)\\ =&\lambda\backslash(\lambda a)\\ =&a,\\ (m(1_L\otimes\eta)r_L^{-1})(\lambda)(a) =&m(\lambda)(a,e_L)\\ =&\lambda\backslash(\lambda a)\\ =&a\end{aligned}$$

for all $\lambda\in H$ and $a\in L$. Therefore (L,m,η) is a monoid in Set_H.

We next show (3.19). Because of (4.7) and (4.8),

$$\begin{aligned}&((m\otimes 1_L)a_{LLL}^{-1}(1_L\otimes\sigma)a_{LLL}(\sigma\otimes 1_L))(\lambda)((a,b),c)\\ =&((m\otimes 1_L)a_{LLL}^{-1})(\lambda)(\xi_\lambda(a,b),\sigma(\lambda\xi_\lambda(a,b))(\eta_\lambda(a,b),c))\\ =&(\lambda\backslash((\lambda\xi_\lambda(a,b))\xi_{\lambda\xi_\lambda(a,b)}(\eta_\lambda(a,b),c)),\eta_{\lambda\xi_\lambda(a,b)}(\eta_\lambda(a,b),c))\\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda\xi_\lambda(a,b))\pi((\lambda\xi_\lambda(a,b))\eta_\lambda(a,b))^{-1}\pi(((\lambda\xi_\lambda(a,b))\eta_\lambda(a,b))c)),\\ &\quad((\lambda\xi_\lambda(a,b))\xi_{\lambda\xi_\lambda(a,b)}(\eta_\lambda(a,b),c))\backslash(((\lambda\xi_\lambda(a,b))\eta_\lambda(a,b))c))\\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c)),\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((((\lambda a)b)c)\backslash(((\lambda a)b)c))))\end{aligned}$$

for all $\lambda\in H$ and $a,b,c\in L$. Similarly,

$$\begin{aligned}&(\sigma(1_L\otimes m)a_{LLL})(\lambda)((a,b),c)\\ =&\sigma(\lambda)(a,(\lambda a)\backslash(((\lambda a)b)c))\\ =&(\xi_\lambda(a,(\lambda a)\backslash(((\lambda a)b)c)),\eta_\lambda(a,(\lambda a)\backslash(((\lambda a)b)c)))\\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(((\lambda a)b)c)),\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((((\lambda a)b)c)\backslash(((\lambda a)b)c)))).\end{aligned}$$

Hence (3.19) holds. We can prove (4.2) in much the same way.

The next task is to show (3.20). From (4.8),

$$((\eta\otimes 1_L)l_L^{-1}r_L)(\lambda)(a,\bullet)=(e_L,a)$$

for $\lambda\in H$ and $a\in L$. On account of (4.7), (4.8), and the fact that L is a left quasigroup,

$$\begin{aligned}&(\sigma(1_L\otimes\eta))(\lambda)(a,\bullet)\\ =&(\xi_\lambda(a,e_L),\eta_\lambda(a,e_L))\\ =&(\lambda\backslash\lambda,\lambda\backslash(\lambda a))\\ =&(e_L,a).\end{aligned}$$

Hence (3.20) holds. We can prove (4.3) similarly. □

4.2 Left modules of monoids

Let $\mathcal{C}$ be a tensor category and (A, m, η) be a monoid in $\mathcal{C}$. We first introduce left modules of the monoid (A, m, η) in $\mathcal{C}$.

Definition 4.9 (left module of monoid)**.** Let X be an object in $\mathcal{C}$ with a morphism $m_X : A \otimes X \to X$ of $\mathcal{C}$. If the morphisms m, η, and m_X satisfy (3.10) and

$$m_X(m \otimes 1_X) = m_X(1_A \otimes m_X)a_{AAX}, \tag{4.9}$$

then we say that (X, m_X) is a left (A, m, η)-module in $\mathcal{C}$.

We give some examples of left modules of monoids. Let $(L, \cdot, e_L)$ be a left quasigroup with a unit, and write $H = L$. We define the morphisms $m : L \otimes L \to L$ and $\eta : I \to L$ by (4.8). Proposition 4.8 implies that (L, m, η) is a monoid in Set_H.

Example 4.10 (left regular module)**.** (L, m) is a left (L, m, η)-module because (L, m, η) is a monoid. We say that (L, m) is a left regular module.

Example 4.11. Let X be a nonempty set and let $i : L \times X \to X$ be a map satisfying that, for every $a \in L$, the map $X \ni x \to i(a, x) \in X$ is injective. For $y \in X$, we denote by $a\backslash y$ the unique element $x \in X$ such that $i(a, x) = y$. For simplicity of notation, we write ax instead of $i(a, x)$.

Let $f : X \to H$ be a map. We define the element $\lambda \cdot_X x \in H$ $(\lambda \in H, x \in X)$ by

$$\lambda \cdot_X x = f(e_L\backslash(\lambda x)) \ (\lambda \in H, x \in X). \tag{4.10}$$

Then $(X, \cdot_X)$ is an object of Set_H.

For every $\lambda \in H$, we define the map $m_X(\lambda) : L \times X \to X$ by

$$m_X(\lambda)(a, x) = \lambda\backslash((\lambda a)x) \ (a \in L, x \in X). \tag{4.11}$$

Then $m_X : L \otimes X \to X$ is a morphism of Set_H. In fact, by means of (4.10),

$$\begin{aligned} \lambda \cdot_X m_X(\lambda)(a, x) =& \lambda \cdot_X (\lambda\backslash((\lambda a)x)) \\ =& f(e_L\backslash((\lambda a)x)) \\ =& (\lambda a) \cdot_X x \\ =& \lambda \cdot_{L\otimes X} (a, x) \end{aligned}$$

for all $\lambda \in H, a \in L$, and $x \in X$.

In addition, (X, m_X) is a left (L, m, η)-module. In fact, by (4.8),

$$\begin{aligned} (m_X(m \otimes 1_X))(\lambda)((a, b), x) =& m_X(\lambda)(\lambda\backslash((\lambda a)b), x) \\ =& \lambda\backslash(((\lambda a)b)x) \\ =& \lambda\backslash((\lambda a)((\lambda a)\backslash(((\lambda a)b)x))) \\ =& m_X(\lambda)(a, m_X(\lambda a)(b, x)) \\ =& (m_X(1_L \otimes m_X)a_{LLX})(\lambda)((a, b), x) \end{aligned}$$

for $\lambda \in H, a, b \in L$, and $x \in X$. Also,

$$(m_X(\eta \otimes 1_X))(\lambda)(\bullet, x) = m_X(\lambda)(e_L, x)$$

$$
\begin{aligned}
&= \lambda\backslash(\lambda x) \\
&= x \\
&= l_X(\lambda)(\bullet, x)
\end{aligned}
$$

for $\lambda \in H$ and $x \in X$.

Example 4.12. We denote by $\mathrm{Map}(L, L)$ the set of all maps from L to L. For any $\lambda \in H$ and $f \in \mathrm{Map}(L, L)$, we define $\alpha_{\lambda,f} \in \mathrm{Map}(L, L)$ by

$$
\alpha_{\lambda,f}(a) = \pi^{-1}(\pi(\lambda)^{-1}\pi(\lambda f(\lambda\backslash\pi^{-1}(\pi(a)\pi(\lambda))))) \ (a \in L). \tag{4.12}
$$

Let g be a map from $\mathrm{Map}(L, L)$ to H. We denote by $\cdot_{\mathrm{Map}(L,L)}$ the element of $H \times \mathrm{Map}(L, L) \to H$ defined by

$$
\lambda \cdot_{\mathrm{Map}(L,L)} f = g(\alpha_{\lambda,f}) \ (\lambda \in H, f \in \mathrm{Map}(L, L)).
$$

It is easily seen that $(\mathrm{Map}(L, L), \cdot_{\mathrm{Map}(L,L)})$ is an object of Set_H. We now set $X = \mathrm{Map}(L, L)$.

For any $\lambda \in H$ and $a \in L$, let $\varphi^\lambda_a : L \to L$ denote the map defined by $\varphi^\lambda_a(b) = \xi_\lambda(a, b)$ $(b \in L)$ (for the definition of $\xi_\lambda(a, b)$, see (4.7)). We define the map $m_X(\lambda) : L \times X \to X$ $(\lambda \in H)$ by

$$
m_X(\lambda)(a, f) = \varphi^\lambda_a \circ f \circ \rho^{\lambda a}_{(\lambda a)\backslash\lambda} \quad (a, b \in L, f \in X). \tag{4.13}
$$

For the definition of $\rho^\lambda_b(\lambda \in H, b \in L)$, see (7.2).

$m_X : L \otimes X \to X$ is a morphism of Set_H. In addition, (X, m_X) is a left (L, m, η)-module.

We first show that $m_X : L \otimes X \to X$ is a morphism of Set_H. By the definition,

$$
\begin{aligned}
\lambda \cdot_X m_X(\lambda)(a, f) &= g(\alpha_{\lambda,m_X(\lambda)(a,f)}), \\
\lambda \cdot_{L\otimes X} (a, f) &= (\lambda a) \cdot_X f \\
&= g(\alpha_{\lambda a,f})
\end{aligned}
$$

for all $\lambda \in H, a \in L$, and $f \in X$. From (4.12) and (4.13),

$$
\begin{aligned}
\alpha_{\lambda,m_X(\lambda)(a,f)}(b) &= \pi^{-1}(\pi(\lambda)^{-1}\pi(\lambda m_X(\lambda)(a, f)(\lambda\backslash\pi^{-1}(\pi(b)\pi(\lambda))))) \\
&= \pi^{-1}(\pi(\lambda)^{-1}\pi(\lambda((\varphi^\lambda_a \circ f)((\lambda a)\backslash\pi^{-1}(\pi(b)\pi(\lambda a)))))) \\
&= \pi^{-1}(\pi(\lambda a)^{-1}\pi((\lambda a)f((\lambda a)\backslash\pi^{-1}(\pi(b)\pi(\lambda a))))) \\
&= \alpha_{\lambda a,f}(b)
\end{aligned}
$$

for all $b \in L$. Hence $\lambda \cdot_X m_X(\lambda)(a, f) = \lambda \cdot_{L\otimes X} (a, f)$ for all $\lambda \in H, a \in L$, and $f \in X$. We have thus proved that $m_X : L \otimes X \to X$ is a morphism of Set_H.

The next task is to show that (X, m_X) is a left (L, m, η)-module; that is, m_X satisfies (3.10) and (4.9). We note that $\varphi^\lambda_{e_L} = \rho^\lambda_{e_L} = 1_L$, $\varphi^\lambda_a \circ \varphi^{\lambda a}_b = \varphi^\lambda_{\lambda\backslash((\lambda a)b)}$, and $\rho^\lambda_a \circ \rho^{\lambda a}_b = \rho^\lambda_{\lambda\backslash((\lambda a)b)}$ for all $\lambda \in H$ and $a, b \in L$. Hence it follows that

$$
m_X(\lambda)(a, f) = \varphi^\lambda_a \circ f \circ (\rho^\lambda_a)^{-1}
$$

for all $a, b \in L, f \in X$. Consequently,

$$
(m_X(m \otimes 1_X))(\lambda)((a, b), f) = \varphi^\lambda_{\lambda\backslash((\lambda a)b)} \circ f \circ (\rho^\lambda_{\lambda\backslash((\lambda a)b)})^{-1}
$$

$$
\begin{aligned}
&=\varphi_a^{\lambda} \circ \varphi_b^{\lambda a} \circ f \circ (\rho_b^{\lambda a})^{-1} \circ (\rho_a^{\lambda})^{-1} \\
&=m_X(\lambda)(a, \varphi_b^{\lambda a} \circ f \circ (\rho_b^{\lambda a})^{-1}) \\
&=(m_X(1_L \otimes m_X)a_{LLX})(\lambda)((a,b),f)
\end{aligned}
$$

for all $\lambda \in H$ and $a, b \in L$. Hence (4.9) holds.

Since $\varphi_{e_L}^{\lambda} = \rho_{e_L}^{\lambda} = 1_L$ for all $\lambda \in H$, $(m_X(\eta \otimes 1_X))(\lambda)(\bullet, f) = \varphi_{e_L}^{\lambda} \circ f \circ (\rho_{e_L}^{\lambda})^{-1} = f = l_X(\lambda)(\bullet, f)$. Hence (3.10) holds. Therefore (X, m_X) is a left (L, m, η)-module.

Let (X, m_X) be a left (A, m, η)-module in $\mathcal{C}$. We set $Y = A \otimes X$.

Proposition 4.13. *(Y, m_Y^{triv}) is a left (A, m, η)-module in $\mathcal{C}$ (For the definition of m_Y^{triv}, see* (3.5)*).*

Proof. From the pentagon axiom (3.1), (3.5), and (4.1),

$$
\begin{aligned}
m_Y^{\mathrm{triv}}(m \otimes 1_Y) =&(m \otimes 1_X)((m \otimes 1_A) \otimes 1_X))a^{-1}_{A\otimes AAX} \\
=&((m(1_A \otimes m)) \otimes 1_X)(a_{AAA} \otimes 1_X)a^{-1}_{A\otimes AAX} \\
=&(m \otimes 1_X)a^{-1}_{AAX}(1_A \otimes (m \otimes 1_X))(1_A \otimes a^{-1}_{AAX})a_{AAY} \\
=&m_Y^{\mathrm{triv}}(1_A \otimes m_Y^{\mathrm{triv}})a_{AAY}.
\end{aligned}
$$

In view of (3.18) and the fact that $l_Y = (l_A \otimes 1_X)a^{-1}_{IAX}$ [13, Lemma XI.2.2],

$$
\begin{aligned}
m_Y^{\mathrm{triv}}(\eta \otimes 1_Y) =&(m \otimes 1_X)((\eta \otimes 1_A) \otimes 1_X)a^{-1}_{IAX} \\
=&l_Y.
\end{aligned}
$$

Hence (Y, m_Y^{triv}) is a left (A, m, η)-module in $\mathcal{C}$. □

Let (A, m, η, σ) be a braided monoid in $\mathcal{C}$ and (X, m_X) be a left (A, m, η)-module in $\mathcal{C}$. We set $Y = A \otimes X$.

Proposition 4.14. *(Y, m_Y^{σ}) is a left (A, m, η)-module in $\mathcal{C}$ (For the definition of m_Y^{σ}, see* (3.6)*).*

Proof. On account of the pentagon axiom (3.1), (3.6), (4.2), and (4.9),

$$
\begin{aligned}
&m_Y^{\sigma}(1_A \otimes m_Y^{\sigma})a_{AAY} \\
=&(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes ((1_A \otimes m_X)a_{AAX})) \\
&\quad (1_A \otimes ((\sigma \otimes 1_X)a^{-1}_{AAX}))a_{AAY} \\
=&(1_A \otimes (m_X(1_A \otimes m_X)))a_{AAY}(\sigma \otimes 1_Y)a^{-1}_{AAY} \\
&\quad (1_A \otimes (a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}))a_{AAY} \\
=&(1_A \otimes (m_X(m \otimes 1_X)a^{-1}_{AAX}))a_{AAY}(\sigma \otimes 1_Y)a^{-1}_{AAY} \\
&\quad (1_A \otimes (a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}))a_{AAY} \\
=&(1_A \otimes m_X)a_{AAX}(((1_A \otimes m)a_{AAA}(\sigma \otimes 1_A)a^{-1}_{AAA}(1_A \otimes \sigma)) \otimes 1_X) \\
&\quad a^{-1}_{AA\otimes AX}(1_A \otimes a^{-1}_{AAX})a_{AAY} \\
=&(1_A \otimes m_X)a_{AAX}((\sigma(m \otimes 1_A)a^{-1}_{AAA}) \otimes 1_X)a^{-1}_{AA\otimes AX}(1_A \otimes a^{-1}_{AAX})a_{AAY} \\
=&(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(m \otimes 1_Y)
\end{aligned}
$$

$$=m_Y^\sigma(m \otimes 1_Y).$$

By means of the triangle axiom (3.2), (3.6), (4.3), and the fact that $l_Y = (l_A \otimes 1_X)a_{IAX}^{-1}$ [13, Lemma XI.2.2],

$$\begin{aligned} m_Y^\sigma(\eta \otimes 1_Y) =&(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(\eta \otimes 1_Y) \\ =&(1_A \otimes m_X)a_{AAX}((\sigma(\eta \otimes 1_A)) \otimes 1_X)a_{IAX}^{-1} \\ =&(1_A \otimes m_X)a_{AAX}(((1_A \otimes \eta)r_A^{-1}l_A) \otimes 1_X)a_{IAX}^{-1} \\ =&(1_A \otimes (m_X(\eta \otimes 1_X)))a_{AIX}((r_A^{-1}l_A) \otimes 1_X)a_{IAX}^{-1} \\ =&(1_A \otimes l_X)a_{AIX}((r_A^{-1}l_A) \otimes 1_X)a_{IAX}^{-1} \\ =&l_Y. \end{aligned}$$

Hence (Y, m_Y^σ) is a left (A, m, η)-module. □

5 Left modules of twisted monoids

In this section, we discuss left modules of twisted monoids in tensor categories (Cf. [8, Sections 5 and 8]).

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category. For every braided monoid (A, m, η, σ) in $\mathcal{C}$ (Definition 4.2), we define two morphisms $m_{A\otimes A} : (A \otimes A) \otimes (A \otimes A) \to A \otimes A$ and $\eta_{A\otimes A} : I \to A \otimes A$ by

$$\begin{aligned} m_{A\otimes A} =&(m \otimes m)a_{A\otimes AAA}(a_{AAA}^{-1} \otimes 1_A)((1_A \otimes \sigma) \otimes 1_A) \\ &(a_{AAA} \otimes 1_A)a_{A\otimes AAA}^{-1}; \end{aligned} \tag{5.1}$$

$$\eta_{A\otimes A} =(\eta \otimes \eta)l_I^{-1}(= (\eta \otimes \eta)r_I^{-1}). \tag{5.2}$$

Proposition 5.1. *$(A \otimes A, m_{A\otimes A}, \eta_{A\otimes A})$ is a monoid in $\mathcal{C}$.*

Proof. We prove only for the case that the associativity constraint a, the left unit constraint l, and the right unit constraint r are all identities. In this case, for example, $m_{A\otimes A} = (m \otimes m)(1_A \otimes \sigma \otimes 1_A)$.

Let us first show

$$m_{A\otimes A}(\eta_{A\otimes A} \otimes 1_{A\otimes A}) = 1_{A\otimes A} = m_{A\otimes A}(1_{A\otimes A} \otimes \eta_{A\otimes A}). \tag{5.3}$$

From (3.18) and (4.3), we see that

$$\begin{aligned} &m_{A\otimes A}(\eta_{A\otimes A} \otimes 1_{A\otimes A}) \\ =&(m \otimes m)(1_A \otimes \sigma \otimes 1_A)(\eta \otimes \eta \otimes 1_{A\otimes A}) \\ =&(m \otimes m)(\eta \otimes 1_A \otimes 1_{A\otimes A})(1_I \otimes (\sigma(\eta \otimes 1_A)) \otimes 1_A) \\ =&(1_A \otimes m)(1_A \otimes (1_A \otimes \eta) \otimes 1_A) \\ =&1_{A\otimes A}. \end{aligned}$$

The rest of the equality in (5.3) can be proved similarly.

The next task is to show

$$m_{A\otimes A}(m_{A\otimes A} \otimes 1_{A\otimes A}) = m_{A\otimes A}(1_{A\otimes A} \otimes m_{A\otimes A}). \tag{5.4}$$

From (4.2),

$$
\begin{aligned}
&m_{A\otimes A}(m_{A\otimes A}\otimes 1_{A\otimes A})\\
=&(m\otimes m)(1_A\otimes\sigma\otimes 1_A)(((m\otimes m)(1_A\otimes\sigma\otimes 1_A))\otimes 1_{A\otimes A})\\
=&(m\otimes m)(1_A\otimes(\sigma(m\otimes 1_A))\otimes 1_A)(m\otimes 1_{A\otimes A\otimes A\otimes A})(1_A\otimes\sigma\otimes 1_{A\otimes A\otimes A})\\
=&(m\otimes m)(1_A\otimes((1_A\otimes m)(\sigma\otimes 1_A)(1_A\otimes\sigma))\otimes 1_A)(m\otimes 1_{A\otimes A\otimes A\otimes A})\\
&\quad(1_A\otimes\sigma\otimes 1_{A\otimes A\otimes A})\\
=&((m(m\otimes 1_A))\otimes 1_A)(1_{A\otimes A\otimes A}\otimes(m(m\otimes 1_A)))(1_{A\otimes A}\otimes\sigma\otimes 1_{A\otimes A})\\
&\quad(1_{A\otimes A\otimes A}\otimes\sigma\otimes 1_A)(1_A\otimes\sigma\otimes 1_{A\otimes A\otimes A}).
\end{aligned}
\tag{5.5}
$$

On account of (3.19) and (4.1),

$$
\begin{aligned}
&\text{RHS of } (5.5)\\
=&((m(1_A\otimes m))\otimes 1_A)(1_{A\otimes A\otimes A}\otimes(m(1_A\otimes m)))(1_{A\otimes A}\otimes\sigma\otimes 1_{A\otimes A})\\
&\quad(1_{A\otimes A\otimes A}\otimes\sigma\otimes 1_A)(1_A\otimes\sigma\otimes 1_{A\otimes A\otimes A})\\
=&(m\otimes 1_A)(1_{A\otimes A}\otimes(m(1_A\otimes m)))(1_A\otimes((m\otimes 1_A)(1_A\otimes\sigma))\otimes 1_{A\otimes A})\\
&\quad(1_A\otimes(\sigma\otimes 1_A)\otimes 1_{A\otimes A})(1_{A\otimes A\otimes A}\otimes\sigma\otimes 1_A).\\
=&(m\otimes 1_A)(1_{A\otimes A}\otimes(m(1_A\otimes m)))(1_A\otimes(\sigma(1_A\otimes m))\otimes 1_{A\otimes A})\\
&\quad(1_{A\otimes A\otimes A}\otimes\sigma\otimes 1_A)\\
=&(m\otimes m)(1_A\otimes\sigma\otimes 1_A)(1_{A\otimes A}\otimes((m\otimes m)(1_A\otimes\sigma\otimes 1_A)))\\
=&m_{A\otimes A}(1_{A\otimes A}\otimes m_{A\otimes A}).
\end{aligned}
$$

Hence (5.4) holds. □

Definition 5.2 (twisted monoid)**.** We call the monoid $(A\otimes A, m_{A\otimes A}, \eta_{A\otimes A})$ in Proposition 5.1 a twisted monoid, and denote it by $A\underset{\mathrm{tw}}{\otimes}A$.

Let (X, m_X) be a left (A, m, η)-module in $\mathcal{C}$ (for the definition of left modules of monoids, see Definition 4.9). We set $Y = A\otimes X$.

Theorem 5.3. *The followings are equivalent.*

(1) *Left (A, m, η)-modules (Y, m_Y) satisfying* (3.11) *and* (3.12). *Here, m_Y^{triv} is the morphism defined by* (3.5).
(2) *Left $A\underset{\mathrm{tw}}{\otimes}A$-modules (Y, θ_Y) satisfying*

$$m_X\theta_Y = m_X(1_A\otimes m_X)a_{AAX}(1_{A\otimes A}\otimes m_X), \tag{5.6}$$

$$\theta_Y((1_A\otimes\eta)\otimes 1_Y) = (m\otimes 1_X)a^{-1}_{AAX}(r_A\otimes 1_Y). \tag{5.7}$$

In addition, there is a bijection between (1) *and* (2).

Proof. **Step 1.** We assume that (Y, m_Y) is a left (A, m, η)-module satisfying (3.11) and (3.12). Let us define the morphism $\theta_Y : (A\otimes A)\otimes Y\to Y$ of $\mathcal{C}$ by

$$\theta_Y = m_Y^{\mathrm{triv}}(1_A\otimes m_Y)a_{AAY}. \tag{5.8}$$

Our goal is to show that (Y, θ_Y) is a left $A\underset{\mathrm{tw}}{\otimes}A$-module satisfying (5.6) and (5.7).

From (3.11), (4.9), and (5.8),

$$\begin{aligned}m_X(1_A\otimes m_X)a_{AAX}(1_{A\otimes A}\otimes m_X) =&m_X(1_A\otimes(m_X(1_A\otimes m_X)))a_{AAY}\\ =&m_X(1_A\otimes m_Xm_Y)a_{AAY}\\ =&m_X(1_A\otimes m_X)(1_A\otimes m_Y)a_{AAY}\\ =&m_X(m\otimes 1_X)a^{-1}_{AAX}(1_A\otimes m_Y)a_{AAY}\\ =&m_X\theta_Y.\end{aligned}$$

Hence (5.6) holds.

Because of the triangle axiom (3.2), (5.8), and the fact that (Y, m_Y) is a left (A, m, η)-module,

$$\begin{aligned}&\theta_Y((1_A\otimes\eta)\otimes 1_Y)\\ =&(m\otimes 1_X)a^{-1}_{AAX}(1_A\otimes m_Y)a_{AAY}((1_A\otimes\eta)\otimes 1_Y)\\ =&(m\otimes 1_X)a^{-1}_{AAX}(1_A\otimes(m_Y(\eta\otimes 1_Y)))a_{AIY}\\ =&(m\otimes 1_X)a^{-1}_{AAX}(1_A\otimes l_Y)a_{AIY}\\ =&(m\otimes 1_X)a^{-1}_{AAX}(r_A\otimes 1_Y).\end{aligned}$$

Therefore (5.7) holds.

The next task is to show that (Y, θ_Y) is a left $A\underset{\text{tw}}{\otimes}A$-module. We first prove that

$$\theta_Y(\eta_{A\otimes A}\otimes 1_Y) = l_Y. \tag{5.9}$$

Here, $\eta_{A\otimes A}$ is the morphism defined by (5.2). On account of (3.5), (3.18), (5.8), and the fact that (Y, m_Y) is a left (A, m, η)-module,

$$\begin{aligned}&\theta_Y(\eta_{A\otimes A}\otimes 1_Y)\\ =&m^{\text{triv}}_Y(1_A\otimes m_Y)a_{AAY}(((\eta\otimes\eta)r^{-1}_I)\otimes 1_Y)\\ =&((m(\eta\otimes 1_A))\otimes 1_X)a^{-1}_{IAX}(1_I\otimes(m_Y(\eta\otimes 1_Y)))a_{IIY}(r^{-1}_I\otimes 1_Y)\\ =&(l_A\otimes 1_X)a^{-1}_{IAX}(1_I\otimes l_Y)a_{IIY}(r^{-1}_I\otimes 1_Y).\end{aligned} \tag{5.10}$$

From the triangle axiom (3.2) and the fact that $l_Y = (l_A\otimes 1_X)a^{-1}_{IAX}$ [13, Lemma XI.2.2],

$$\begin{aligned}\text{RHS of (5.10)} &= l_Y(1_I\otimes l_Y)a_{IIY}(r^{-1}_I\otimes 1_Y)\\ &= l_Y.\end{aligned}$$

Hence (5.9) holds.

The next task is to show

$$\theta_Y(m_{A\otimes A}\otimes 1_Y) = \theta_Y(1_{A\otimes A}\otimes\theta_Y)a_{A\otimes AA\otimes AY}. \tag{5.11}$$

Here, $m_{A\otimes A}$ is the morphism defined by (5.1). From (3.12), (5.8), Proposition 4.13, and the fact that (Y, m_Y) is a left (A, m, η)-module,

$$\begin{aligned}&\theta_Y(1_{A\otimes A}\otimes\theta_Y)a_{A\otimes AA\otimes AY}\\ =&m^{\text{triv}}_Y(1_A\otimes m_Y)a_{AAY}(1_{A\otimes A}\otimes(m^{\text{triv}}_Y(1_A\otimes m_Y)a_{AAY}))a_{A\otimes AA\otimes AY}\end{aligned}$$

$$
\begin{aligned}
&= m_Y^{\mathrm{triv}}(1_A \otimes (m_Y(1_A \otimes m_Y^{\mathrm{triv}})))a_{AAA\otimes Y} \\
&\quad (1_{A\otimes A} \otimes ((1_A \otimes m_Y)a_{AAY}))a_{A\otimes AA\otimes AY} \\
&= m_Y^{\mathrm{triv}}(1_A \otimes (m_Y^{\mathrm{triv}}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1}))a_{AAA\otimes Y} \\
&\quad (1_{A\otimes A} \otimes ((1_A \otimes m_Y)a_{AAY}))a_{A\otimes AA\otimes AY} \\
&= m_Y^{\mathrm{triv}}(1_A \otimes m_Y^{\mathrm{triv}})(1_A \otimes (1_A \otimes (m_Y(1_A \otimes m_Y))))(1_A \otimes a_{AAA\otimes Y}) \\
&\quad (1_A \otimes (\sigma \otimes 1_{A\otimes Y}))(1_A \otimes a_{AAA\otimes Y}^{-1})a_{AAA\otimes(A\otimes Y)} \\
&\quad (1_{A\otimes A} \otimes a_{AAY})a_{A\otimes AA\otimes AY} \\
&= m_Y^{\mathrm{triv}}(m \otimes 1_Y)a_{AAY}^{-1}(1_A \otimes (1_A \otimes (m_Y(m \otimes 1_Y)a_{AAY}^{-1})))(1_A \otimes a_{AAA\otimes Y}) \\
&\quad (1_A \otimes (\sigma \otimes 1_{A\otimes Y}))(1_A \otimes a_{AAA\otimes Y}^{-1})a_{AAA\otimes(A\otimes Y)} \\
&\quad (1_{A\otimes A} \otimes a_{AAY})a_{A\otimes AA\otimes AY} \\
&= m_Y^{\mathrm{triv}}(1_A \otimes m_Y)(m \otimes 1_{A\otimes Y})a_{AAA\otimes Y}^{-1}(1_A \otimes (1_A \otimes ((m \otimes 1_Y)a_{AAY}^{-1}))) \\
&\quad (1_A \otimes a_{AAA\otimes Y})(1_A \otimes (\sigma \otimes 1_{A\otimes Y}))(1_A \otimes a_{AAA\otimes Y}^{-1})a_{AAA\otimes(A\otimes Y)} \\
&\quad (1_{A\otimes A} \otimes a_{AAY})a_{A\otimes AA\otimes AY}.
\end{aligned} \tag{5.12}
$$

On account of the pentagon axiom (3.1),

$$
\begin{aligned}
&\text{RHS of (5.12)} \\
&= \theta_Y a_{AAY}^{-1}(m \otimes 1_{A\otimes Y})a_{AAA\otimes Y}^{-1}(1_A \otimes (1_A \otimes ((m \otimes 1_Y)a_{AAY}^{-1}))) \\
&\quad (1_A \otimes a_{AAA\otimes Y})(1_A \otimes (\sigma \otimes 1_{A\otimes Y}))(1_A \otimes a_{AAA\otimes Y}^{-1})a_{AAA\otimes(A\otimes Y)} \\
&\quad (1_{A\otimes A} \otimes a_{AAY})a_{A\otimes AA\otimes AY} \\
&= \theta_Y((m \otimes m) \otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}a_{AA(A\otimes A)\otimes Y}^{-1}(1_A \otimes (1_A \otimes a_{AAY}^{-1})) \\
&\quad (1_A \otimes a_{AAA\otimes Y})(1_A \otimes (\sigma \otimes 1_{A\otimes Y}))(1_A \otimes a_{AAA\otimes Y}^{-1})a_{AAA\otimes(A\otimes Y)} \\
&\quad (1_{A\otimes A} \otimes a_{AAY})a_{A\otimes AA\otimes AY} \\
&= \theta_Y(((m \otimes m)a_{A\otimes AAA}(a_{AAA}^{-1} \otimes 1_A)((1_A \otimes \sigma) \otimes 1_A)) \otimes 1_Y) \\
&\quad ((a_{AAA} \otimes 1_A)a_{A\otimes AAA}^{-1}) \otimes 1_Y) \\
&= \theta_Y(m_{A\otimes A} \otimes 1_Y).
\end{aligned}
$$

Hence (5.11) holds. Therefore (Y, θ_Y) is a left $A \underset{\mathrm{tw}}{\otimes} A$-module.

Step 2. Conversely, we assume that (Y, θ_Y) is a left $A \underset{\mathrm{tw}}{\otimes} A$-module satisfying (5.6) and (5.7). Let us define the morphism $m_Y : A \otimes Y \to Y$ of $\mathcal{C}$ by

$$
m_Y = \theta_Y(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y). \tag{5.13}
$$

Our task is to show that (Y, m_Y) is a left (A, m, η)-module satisfying (3.11) and (3.12).

From (3.10), (5.6), (5.13), and the fact that $l_Y = (l_A \otimes 1_X)a_{IAX}^{-1}$ [13, Lemma XI.2.2],

$$
\begin{aligned}
m_X m_Y &= m_X\theta_Y(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y) \\
&= m_X(1_A \otimes m_X)a_{AAX}(1_{A\otimes A} \otimes m_X)(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y) \\
&= m_X(\eta \otimes 1_X)(1_I \otimes m_X)l_Y^{-1}(1_A \otimes m_X)
\end{aligned}
$$

$$=m_X(1_A\otimes m_X).$$

Hence (3.11) holds.

We next prove (3.12). We note that, by means of (3.5) and (5.7),

$$\begin{aligned}m_Y^{\mathrm{triv}} =&(m\otimes 1_X)a_{AAX}^{-1}\\ =&\theta_Y(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y).\end{aligned}\tag{5.14}$$

By means of (3.18), (5.1), (5.13), (5.14), and the fact that θ_Y is a left $A\underset{\mathrm{tw}}{\otimes}A$-module, we deduce that

$$\begin{aligned}&m_Y(1_A\otimes m_Y^{\mathrm{triv}})\\ =&\theta_Y(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y)(1_A\otimes(\theta_Y(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y)))\\ =&\theta_Y(1_{A\otimes A}\otimes\theta_Y)(((\eta\otimes 1_A)l_A^{-1})\otimes(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y))\\ =&\theta_Y(m_{A\otimes A}\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y))\\ =&\theta_Y(((m\otimes m)a_{A\otimes AAA}(a_{AAA}^{-1}\otimes 1_A)((1_A\otimes\sigma)\otimes 1_A))\otimes 1_Y)\\ &\quad(((a_{AAA}\otimes 1_A)a_{A\otimes AAA}^{-1})\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}\\ &\quad(((\eta\otimes 1_A)l_A^{-1})\otimes(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y))\\ =&\theta_Y(((m(\eta\otimes 1_A))\otimes(m(1_A\otimes\eta)))a_{I\otimes AAI}(a_{IAA}^{-1}\otimes 1_I)((1_I\otimes\sigma)\otimes 1_I)\otimes 1_Y)\\ &\quad(((a_{IAA}\otimes 1_I)a_{I\otimes AAI}^{-1})\otimes 1_Y)a_{I\otimes AA\otimes IY}^{-1}(l_A^{-1}\otimes(r_A^{-1}\otimes 1_Y))\\ =&\theta_Y(((l_A\otimes r_A)a_{I\otimes AAI}(a_{IAA}^{-1}\otimes 1_I)((1_I\otimes\sigma)\otimes 1_I))\otimes 1_Y)\\ &\quad(((a_{IAA}\otimes 1_I)a_{I\otimes AAI}^{-1})\otimes 1_Y)a_{I\otimes AA\otimes IY}^{-1}(l_A^{-1}\otimes(r_A^{-1}\otimes 1_Y))\\ =&\theta_Y(\sigma\otimes 1_Y)a_{AAY}^{-1}.\end{aligned}$$

On account of (5.13), (5.14), and the fact that θ_Y is a left $A\underset{\mathrm{tw}}{\otimes}A$-module,

$$\begin{aligned}&m_Y^{\mathrm{triv}}(1_A\otimes m_Y)a_{AAY}(\sigma\otimes 1_Y)a_{AAY}^{-1}\\ =&\theta_Y(((1_A\otimes\eta)r_A^{-1})\otimes 1_Y)(1_A\otimes(\theta_Y(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y)))\\ &\quad a_{AAY}(\sigma\otimes 1_Y)a_{AAY}^{-1}\\ =&\theta_Y(1_{A\otimes A}\otimes\theta_Y)(((1_A\otimes\eta)r_A^{-1})\otimes 1_{(A\otimes A)\otimes Y})\\ &\quad(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}(\sigma\otimes 1_Y)a_{AAY}^{-1}\\ =&\theta_Y(m_{A\otimes A}\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}(((1_A\otimes\eta)r_A^{-1})\otimes 1_{(A\otimes A)\otimes Y})\\ &\quad(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}(\sigma\otimes 1_Y)a_{AAY}^{-1}.\end{aligned}\tag{5.15}$$

From (4.3) and (5.1),

$$\begin{aligned}&\text{RHS of (5.15)}\\ =&\theta_Y(((m\otimes m)a_{A\otimes AAA}(a_{AAA}^{-1}\otimes 1_A)((1_A\otimes\sigma)\otimes 1_A))\otimes 1_Y)\\ &\quad(((a_{AAA}\otimes 1_A)a_{A\otimes AAA}^{-1})\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}((1_A\otimes\eta)\otimes 1_{(A\otimes A)\otimes Y})\\ &\quad(r_A^{-1}\otimes 1_{(A\otimes A)\otimes Y})(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}(\sigma\otimes 1_Y)a_{AAY}^{-1}\\ =&\theta_Y(((1_A\otimes m)a_{AAA})\otimes 1_Y)((((m\otimes 1_A)a_{AAA}^{-1})\otimes 1_A)\otimes 1_Y)\\ &\quad(((1_A\otimes(\sigma(\eta\otimes 1_A)))\otimes 1_A)\otimes 1_Y)(((a_{AIA}\otimes 1_A)a_{A\otimes IAA}^{-1})\otimes 1_Y)\end{aligned}$$

$$
\begin{aligned}
&a^{-1}_{A\otimes IA\otimes AY}(r^{-1}_A\otimes 1_{(A\otimes A)\otimes Y})(1_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y(((1_A\otimes m)a_{AAA})\otimes 1_Y)((((m\otimes 1_A)a^{-1}_{AAA})\otimes 1_A)\otimes 1_Y)\\
&(((1_A\otimes((1_A\otimes\eta)r^{-1}_A l_A))\otimes 1_A)\otimes 1_Y)(((a_{AIA}\otimes 1_A)a^{-1}_{A\otimes IAA})\otimes 1_Y)\\
&a^{-1}_{A\otimes IA\otimes AY}(r^{-1}_A\otimes 1_{(A\otimes A)\otimes Y})(1_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y((1_A\otimes(m(\eta\otimes 1_A)))\otimes 1_Y)(a_{AIA}\otimes 1_Y)((((m\otimes 1_I)a^{-1}_{AAI})\otimes 1_A)\otimes 1_Y)\\
&(((1_A\otimes r^{-1}_A l_A)\otimes 1_A)\otimes 1_Y)(((a_{AIA}\otimes 1_A)a^{-1}_{A\otimes IAA})\otimes 1_Y)\\
&a^{-1}_{A\otimes IA\otimes AY}(r^{-1}_A\otimes 1_{(A\otimes A)\otimes Y})(1_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}. \qquad (5.16)
\end{aligned}
$$

By means of the triangle axiom (3.2), (3.18), and the fact that $r^{-1}_{A\otimes A} = a^{-1}_{AAI}(1_A\otimes r^{-1}_A)$ [13, Lemma XI.2.2],

$$
\begin{aligned}
&\text{RHS of (5.16)}\\
=&\theta_Y((1_A\otimes l_A)\otimes 1_Y)(a_{AIA}\otimes 1_Y)((((m\otimes 1_I)a^{-1}_{AAI})\otimes 1_A)\otimes 1_Y)\\
&(((1_A\otimes r^{-1}_A l_A)\otimes 1_A)\otimes 1_Y)(((a_{AIA}\otimes 1_A)a^{-1}_{A\otimes IAA})\otimes 1_Y)\\
&a^{-1}_{A\otimes IA\otimes AY}(r^{-1}_A\otimes 1_{(A\otimes A)\otimes Y})(1_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y((r_A\otimes 1_A)\otimes 1_Y)((((m\otimes 1_I)r^{-1}_{A\otimes A})\otimes 1_A)\otimes 1_Y)(((r_A\otimes 1_A)\otimes 1_A)\otimes 1_Y)\\
&(a^{-1}_{A\otimes IAA}\otimes 1_Y)a^{-1}_{A\otimes IA\otimes AY}(r^{-1}_A\otimes 1_{(A\otimes A)\otimes Y})\\
&(1_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y(((m(1_A\otimes\eta))\otimes 1_A)\otimes 1_Y)(a^{-1}_{AIA}\otimes 1_Y)a^{-1}_{AI\otimes AY}(1_A\otimes(l^{-1}_A\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y((r_A\otimes 1_A)\otimes 1_Y)(a^{-1}_{AIA}\otimes 1_Y)a^{-1}_{AI\otimes AY}(1_A\otimes(l^{-1}_A\otimes 1_Y))\\
&a_{AAY}(\sigma\otimes 1_Y)a^{-1}_{AAY}\\
=&\theta_Y(\sigma\otimes 1_Y)a^{-1}_{AAY}.
\end{aligned}
$$

We thus proved (3.12).

The next task is to show that (Y, m_Y) is a left (A, m, η)-module. From (5.2), (5.13), and the fact that (Y, θ_Y) is a left $A \underset{\mathrm{tw}}{\otimes} A$-module,

$$
\begin{aligned}
m_Y(\eta\otimes 1_Y) =&\theta_Y(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y)(\eta\otimes 1_Y)\\
=&\theta_Y(\eta_{A\otimes A}\otimes 1_Y)\\
=&l_Y.
\end{aligned}
$$

We next show that

$$
m_Y(m\otimes 1_Y) = m_Y(1_A\otimes m_Y)a_{AAY}.
$$

From (5.13) and the fact that $l^{-1}_{A\otimes A} = a_{IAA}(l^{-1}_A\otimes 1_A)$ [13, Lemma XI.2.2],

$$
m_Y(m\otimes 1_Y)
$$

$$
\begin{aligned}
&=\theta_Y(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y)(m\otimes 1_Y)\\
&=\theta_Y((\eta\otimes 1_A)\otimes 1_Y)((1_I\otimes m)\otimes 1_Y)(l_{A\otimes A}^{-1}\otimes 1_Y)\\
&=\theta_Y((\eta\otimes m)\otimes 1_Y)((a_{IAA}(l_A^{-1}\otimes 1_A))\otimes 1_Y)\\
&=\theta_Y((\eta\otimes m)\otimes 1_Y)(a_{IAA}\otimes 1_Y)((1_{I\otimes A}\otimes l_A)\otimes 1_Y)\\
&\qquad((l_A^{-1}\otimes l_A^{-1})\otimes 1_Y)a_{AAY}^{-1}a_{AAY}. \qquad (5.17)
\end{aligned}
$$

On account of the triangle axiom (3.2), (3.18), and the fact that $r_{A\otimes A}=(1_A\otimes r_A)a_{AAI}$ [13, Lemma XI.2.2],

$$
\begin{aligned}
&\text{RHS of (5.17)}\\
&=\theta_Y((\eta\otimes m)\otimes 1_Y)(a_{IAA}\otimes 1_Y)(((r_{I\otimes A}\otimes 1_A)a_{I\otimes AIA}^{-1})\otimes 1_Y)\\
&\qquad a_{I\otimes AI\otimes AY}^{-1}(l_A^{-1}\otimes(l_A^{-1}\otimes 1_Y))a_{AAY}\\
&=\theta_Y((1_A\otimes m)\otimes 1_Y)(a_{AAA}\otimes 1_Y)((((r_A\otimes 1_A)a_{AIA}^{-1})\otimes 1_A)\otimes 1_Y)\\
&\qquad(((1_A\otimes l_A^{-1})\otimes 1_A)\otimes 1_Y)((((1_A\otimes r_A)a_{AAI})\otimes 1_A)\otimes 1_Y)\\
&\qquad(a_{A\otimes AIA}^{-1}\otimes 1_Y)a_{A\otimes AI\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes(l_A^{-1}\otimes 1_Y))a_{AAY}\\
&=\theta_Y((1_A\otimes m)\otimes 1_Y)(a_{AAA}\otimes 1_Y)((((m(1_A\otimes\eta))\otimes 1_A)\otimes 1_A)\otimes 1_Y)\\
&\qquad((a_{AIA}^{-1}\otimes 1_A)\otimes 1_Y)(((1_A\otimes l_A^{-1}r_A)\otimes 1_A)\otimes 1_Y)((a_{AAI}\otimes 1_A)\otimes 1_Y)\\
&\qquad(a_{A\otimes AIA}^{-1}\otimes 1_Y)a_{A\otimes AI\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes(l_A^{-1}\otimes 1_Y))a_{AAY}\\
&=\theta_Y((1_A\otimes m)\otimes 1_Y)(a_{AAA}\otimes 1_Y)((((m\otimes 1_A)a_{AAA}^{-1})\otimes 1_A)\otimes 1_Y)\\
&\qquad(((1_A\otimes((\eta\otimes 1_A)l_A^{-1}r_A))\otimes 1_A)\otimes 1_Y)((a_{AAI}\otimes 1_A)\otimes 1_Y)\\
&\qquad(a_{A\otimes AIA}^{-1}\otimes 1_Y)a_{A\otimes AI\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes(l_A^{-1}\otimes 1_Y))a_{AAY}. \qquad (5.18)
\end{aligned}
$$

Because of (3.20), (5.1), (5.13), and the fact that (Y,θ_Y) is a left $A\underset{\text{tw}}{\otimes}A$-module,

$$
\begin{aligned}
&\text{RHS of (5.18)}\\
&=\theta_Y((1_A\otimes m)\otimes 1_Y)(a_{AAA}\otimes 1_Y)((((m\otimes 1_A)a_{AAA}^{-1})\otimes 1_A)\otimes 1_Y)\\
&\qquad(((1_A\otimes(\sigma(1_A\otimes\eta)))\otimes 1_A)\otimes 1_Y)((a_{AAI}\otimes 1_A)\otimes 1_Y)\\
&\qquad(a_{A\otimes AIA}^{-1}\otimes 1_Y)a_{A\otimes AI\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes(l_A^{-1}\otimes 1_Y))a_{AAY}\\
&=\theta_Y(((m\otimes m)a_{A\otimes AAA})\otimes 1_Y)((a_{AAA}^{-1}\otimes 1_A)\otimes 1_Y)(((1_A\otimes\sigma)\otimes 1_A)\otimes 1_Y)\\
&\qquad(((a_{AAA}\otimes 1_A)a_{A\otimes AAA}^{-1})\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes 1_{(A\otimes A)\otimes Y})\\
&\qquad(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}\\
&=\theta_Y(m_{A\otimes A}\otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}(((\eta\otimes 1_A)l_A^{-1})\otimes 1_{(A\otimes A)\otimes Y})\\
&\qquad(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}\\
&=\theta_Y(1_{A\otimes A}\otimes\theta_Y)(((\eta\otimes 1_A)l_A^{-1})\otimes 1_{(A\otimes A)\otimes Y})\\
&\qquad(1_A\otimes(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y))a_{AAY}\\
&=\theta_Y(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y)(1_A\otimes(\theta_Y(((\eta\otimes 1_A)l_A^{-1})\otimes 1_Y)))a_{AAY}\\
&=m_Y(1_A\otimes m_Y)a_{AAY}.
\end{aligned}
$$

Therefore (Y,m_Y) is a left (A,m,η)-module in $\mathcal{C}$.

Step 3. From now on, we prove that the correspondence between the previous steps 1 and 2 is bijective. Let (Y,m_Y) be a left (A,m,η)-module in $\mathcal{C}$ satisfying

(3.11) and (3.12). We write $\theta_Y = m_Y^{\rm triv}(1_A \otimes m_Y)a_{AAY}$. From (3.5), (3.18) and the fact that $l_{A\otimes Y}^{-1} = a_{IAY}(l_A^{-1} \otimes 1_Y)$ [13, Lemma XI.2.2],

$$\begin{aligned}
&\theta_Y(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y)\\
=&m_Y^{\rm triv}(1_A \otimes m_Y)a_{AAY}(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y)\\
=&((m(\eta \otimes 1_A)) \otimes 1_X)a_{IAX}^{-1}(1_I \otimes m_Y)l_{A\otimes Y}^{-1}\\
=&((m(\eta \otimes 1_A)l_A^{-1}) \otimes 1_X)m_Y\\
=&m_Y.
\end{aligned}$$

Conversely, let (Y, θ_Y) be a left $A \underset{\rm tw}{\otimes} A$-module satisfying (5.6) and (5.7). We write $m_Y = \theta_Y(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y)$. From (3.5), (5.7), and the fact that (Y, θ_Y) is a left $A \underset{\rm tw}{\otimes} A$-module,

$$\begin{aligned}
&m_Y^{\rm triv}(1_A \otimes m_Y)a_{AAY}\\
=&\theta_Y(((1_A \otimes \eta)r_A^{-1}) \otimes 1_Y)(1_A \otimes (\theta_Y(((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y)))a_{AAY}\\
=&\theta_Y(1_{A\otimes A} \otimes \theta_Y)(((1_A \otimes \eta)r_A^{-1}) \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}\\
=&\theta_Y(m_{A\otimes A} \otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}(((1_A \otimes \eta)r_A^{-1}) \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}.
\end{aligned} \tag{5.19}$$

On account of (4.3) and (5.1),

$$\begin{aligned}
&\text{RHS of (5.19)}\\
=&\theta_Y(((m \otimes m)a_{A\otimes AAA}(a_{AAA}^{-1} \otimes 1_A)((1_A \otimes \sigma) \otimes 1_A)) \otimes 1_Y)\\
&(((a_{AAA} \otimes 1_A)a_{A\otimes AAA}^{-1}) \otimes 1_Y)a_{A\otimes AA\otimes AY}^{-1}\\
&(((1_A \otimes \eta)r_A^{-1}) \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}\\
=&\theta_Y(((1_A \otimes m)a_{AAA}) \otimes 1_Y)((((m \otimes 1_A)a_{AAA}^{-1}) \otimes 1_A) \otimes 1_Y)\\
&(((1_A \otimes (\sigma(\eta \otimes 1_A))) \otimes 1_A) \otimes 1_Y)((a_{AIA} \otimes 1_A) \otimes 1_Y)\\
&(a_{A\otimes IAA}^{-1} \otimes 1_Y)a_{A\otimes IA\otimes AY}^{-1}(r_A^{-1} \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}\\
=&\theta_Y(((1_A \otimes m)a_{AAA}) \otimes 1_Y)((((m \otimes 1_A)a_{AAA}^{-1}) \otimes 1_A) \otimes 1_Y)\\
&(((1_A \otimes ((1_A \otimes \eta)r_A^{-1}l_A)) \otimes 1_A) \otimes 1_Y)((a_{AIA} \otimes 1_A) \otimes 1_Y)\\
&(a_{A\otimes IAA}^{-1} \otimes 1_Y)a_{A\otimes IA\otimes AY}^{-1}(r_A^{-1} \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}\\
=&\theta_Y((1_A \otimes (m(\eta \otimes 1_A))) \otimes 1_Y)(a_{AIA} \otimes 1_Y)(((m \otimes 1_I) \otimes 1_A) \otimes 1_Y)\\
&((a_{AAI}^{-1} \otimes 1_A) \otimes 1_Y)(((1_A \otimes r_A^{-1}l_A) \otimes 1_A) \otimes 1_Y)((a_{AIA} \otimes 1_A) \otimes 1_Y)\\
&(a_{A\otimes IAA}^{-1} \otimes 1_Y)a_{A\otimes IA\otimes AY}^{-1}(r_A^{-1} \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}.
\end{aligned} \tag{5.20}$$

Because of the triangle axiom (3.2), (3.18), and the fact that $r_{A\otimes A}^{-1} = a_{AAI}^{-1}(1_A \otimes r_A^{-1})$ [13, Lemma XI.2.2],

$$\begin{aligned}
&\text{RHS of (5.20)}\\
=&\theta_Y((1_A \otimes l_A) \otimes 1_Y)(a_{AIA} \otimes 1_Y)(((m \otimes 1_I) \otimes 1_A) \otimes 1_Y)((a_{AAI}^{-1} \otimes 1_A) \otimes 1_Y)\\
&(((1_A \otimes r_A^{-1}l_A) \otimes 1_A) \otimes 1_Y)((a_{AIA} \otimes 1_A) \otimes 1_Y)\\
&(a_{A\otimes IAA}^{-1} \otimes 1_Y)a_{A\otimes IA\otimes AY}^{-1}(r_A^{-1} \otimes (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_Y))a_{AAY}
\end{aligned}$$

$$
\begin{aligned}
=&\theta_Y((r_A\otimes 1_A)\otimes 1_Y)(((m\otimes 1_I)\otimes 1_A)\otimes 1_Y)((r^{-1}_{A\otimes A}\otimes 1_A)\otimes 1_Y)\\
&\quad((((r_A\otimes 1_A)\otimes 1_A)a^{-1}_{A\otimes IAA})\otimes 1_Y)a^{-1}_{A\otimes IA\otimes AY}\\
&\quad(r^{-1}_A\otimes(((\eta\otimes 1_A)l^{-1}_A)\otimes 1_Y))a_{AAY}\\
=&\theta_Y(((m(1_A\otimes\eta))\otimes 1_A)\otimes 1_Y)(a^{-1}_{AIA}\otimes 1_Y)a^{-1}_{AI\otimes AY}(1_A\otimes(l^{-1}_A\otimes 1_Y))a_{AAY}\\
=&\theta_Y((r_A\otimes 1_A)\otimes 1_Y)(a^{-1}_{AIA}\otimes 1_Y)a^{-1}_{AI\otimes AY}(1_A\otimes(l^{-1}_A\otimes 1_Y))a_{AAY}\\
=&\theta_Y((1_A\otimes l_A)\otimes 1_Y)a^{-1}_{AI\otimes AY}(1_A\otimes(l^{-1}_A\otimes 1_Y))a_{AAY}\\
=&\theta_Y.
\end{aligned}
$$

We conclude the claim. □

6 Main result

In this section, we state our main result on the construction of dynamical reflection maps [23, 24]. We will prove it in Section 7.

Let $(L, \cdot, e_L)$ be a left quasigroup with a unit (for the definition, see Definition 4.4). We write $H = L$. By the definition, it is easily seen that $(L, \cdot) \in Set_H$.

Let G be a group isomorphic to L as sets, and we denote by π the bijection map. We define the morphism $\sigma : L \otimes L \to L \otimes L$ of Set_H by (4.7).

Proposition 6.1. *σ is an isomorphism in Set_H.*

Proof. (Cf. [23, Proposition 5.1]) We define a map $\sigma^{-1} : L \otimes L \to L \otimes L$ by

$$
\sigma^{-1}(\lambda)(a,b) = (\lambda\backslash c, c\backslash((\lambda a)b)), \tag{6.1}
$$

which is the inverse of σ. Here, $c = \pi^{-1}(\pi((\lambda a)b)\pi(\lambda a)^{-1}\pi(\lambda))$. From (4.7) and (6.1),

$$
\begin{aligned}
&(\sigma^{-1}\sigma)(\lambda)(a,b)\\
=&\sigma^{-1}(\lambda)(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b)),\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b))\backslash((\lambda a)b))\\
=&(\lambda\backslash\pi^{-1}(\pi((\lambda a)b)(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b))^{-1}\pi(\lambda)),\\
&\quad\pi^{-1}(\pi((\lambda a)b)(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b))^{-1}\pi(\lambda))\backslash((\lambda a)b))\\
=&(\lambda\backslash(\lambda a),(\lambda a)\backslash((\lambda a)b))\\
=&(a,b)
\end{aligned}
$$

for all $\lambda \in H$ and $a, b \in L$. In the same way, we can show $\sigma\sigma^{-1} = 1_{L\otimes L}$. Therefore σ is an isomorphism in Set_H. □

We define the morphisms $m : L \otimes L \to L$ and $\eta : I \to L$ of Set_H by (4.8). From Proposition 4.8, (L, m, η, σ) is a braided monoid. In addition,

Proposition 6.2. (1) *$m\sigma = m$.*
(2) *The map $L \ni b \mapsto m(\lambda)(a,b) \in L$ is injective for all $\lambda \in H$ and $a \in L$.*

Proof. (1) From (4.8) and the fact that L is a left quasigroup,

$$
\begin{aligned}
&(m\sigma)(\lambda)(a,b)\\
=&m(\lambda)(\xi_\lambda(a,b),\eta_\lambda(a,b))
\end{aligned}
$$

$$
\begin{aligned}
&=\lambda\backslash((\lambda\xi_\lambda(a,b))((\lambda\xi_\lambda(a,b))\backslash((\lambda a)b))) \\
&=\lambda\backslash((\lambda a)b) \\
&=m(\lambda)(a,b)
\end{aligned}
$$

for $\lambda \in H$ and $a, b \in L$. Hence $m\sigma = m$.
(2) We assume that $m(\lambda)(a,b) = m(\lambda)(a,b')$ ($\lambda \in H, a, b, b' \in L$). This is equivalent to $\lambda\backslash((\lambda a)b) = \lambda\backslash((\lambda a)b')$. Since L is a left quasigroup, this is also equivalent to $b = b'$. Therefore the map $L \ni b \mapsto m(\lambda)(a,b) = \lambda\backslash((\lambda a)b) \in L$ is injective for all $\lambda \in H$ and $a \in L$. □

Let (X, m_X) be a left (L, m, η)-module in Set_H. We set $Y = L \otimes X \in Set_H$. Let λ_0 denote the unique element of L such that the element $\pi(\lambda_0)$ is the unit of the group G.

The following theorem is our main result.

Theorem 6.3. *The followings are equivalent.*

(1) *Left (L, m, η)-modules (Y, m_Y) in Set_H satisfying* (3.11), (3.12) *and* (3.21). *For the definitions of m_Y^{triv} and m_Y^σ, see* (3.5) *and* (3.6).

(2) *Families of group homomorphisms $\{f_x^{\lambda_0} : G \to G \mid x \in X\}$.*

In addition, there is a bijection between (1) *and* (2).

We have constructed the braided monoid (L, m, η, σ), where σ is an isomorphism satisfying the braid relation (3.3) (Propositions 4.7, 4.8, and 6.1). In view of Propositions 3.5, 3.7, 6.2, and Theorem 6.3, we have a way to construct the dynamical reflection map $k : L \otimes X \to L \otimes X \in Set_H$ (3.7) in Theorem 3.8.

In the next section, we will give the proof of Theorem 6.3.

7 Proof of main result

In this section, our main result, Theorem 6.3, will be proved in several steps (Cf. [8, Section 8]).

Let $L = (L, \cdot, e_L)$ be a left quasigroup with a unit (For the definition, see Definition 4.4). We set $H = L$. As mentioned in Section 4, we can construct a braided monoid (L, m, η, σ) in Set_H (see (4.7), (4.8), and Proposition 4.8).

Notation. For $\lambda \in H, a, b \in L$, we denote by $a \underset{\lambda}{\cdot} b$ and $\rho_b^\lambda(a) \in L$ the elements:

$$a \underset{\lambda}{\cdot} b = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)); \tag{7.1}$$

$$\rho_b^\lambda(a) = \lambda\backslash\pi^{-1}(\pi((\lambda b)a)\pi(\lambda b)^{-1}\pi(\lambda)). \tag{7.2}$$

For $\lambda \in H$, let ι^λ denote the map from L to $L \times L$ defined by

$$\iota^\lambda(a) = (a, (\lambda a)\backslash\lambda) \quad (a \in L). \tag{7.3}$$

Then

$$\lambda\iota^\lambda(a) = (\lambda a)((\lambda a)\backslash\lambda) = \lambda \tag{7.4}$$

holds for all $\lambda \in H$ and $a \in L$.

7.1 Preliminaries

Let (X, m_X) be a left (L, m, η)-module and set $Y = L \otimes X$.

Lemma 7.1. *For $\lambda \in H$ and $a, b \in L$,*

$$m_{L\otimes L}(\lambda)(\iota^\lambda(a), \iota^\lambda(b)) = \iota^\lambda(a \underset{\lambda}{\cdot} b), \tag{7.5}$$

$$m_{L\otimes L}(\lambda)((b, e_L), \iota^{\lambda b}(a)) = m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (b, e_L)). \tag{7.6}$$

Here, $m_{L\otimes L}$ is the morphism defined by (5.1)*.*

Proof. By means of (4.7), (5.1), and (7.3),

$$\begin{aligned}
&m_{L\otimes L}(\lambda)(\iota^\lambda(a), \iota^\lambda(b))\\
=&m_{L\otimes L}(\lambda)((a, (\lambda a)\backslash\lambda), (b, (\lambda b)\backslash\lambda))\\
=&(m(\lambda)(a, \xi_{\lambda a}((\lambda a)\backslash\lambda, b)), m((\lambda a)\xi_{\lambda a}((\lambda a)\backslash\lambda, b))(\eta_{\lambda a}((\lambda a)\backslash\lambda, b), (\lambda b)\backslash\lambda))
\end{aligned} \tag{7.7}$$

for all $\lambda \in H$ and $a, b \in L$. From (4.8), (7.1), and the fact that $((\lambda a)\xi_{\lambda a}((\lambda a)\backslash\lambda, b))\eta_{\lambda a}((\lambda a)\backslash\lambda, b) = \lambda b$,

$$\begin{aligned}
&\text{RHS of (7.7)}\\
=&(\lambda\backslash((\lambda a)\xi_{\lambda a}((\lambda a)\backslash\lambda, b)), ((\lambda a)\xi_{\lambda a}((\lambda a)\backslash\lambda, b))\backslash((\lambda b)((\lambda b)\backslash\lambda)))\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)), \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))\backslash\lambda)\\
=&\iota^\lambda(a \underset{\lambda}{\cdot} b).
\end{aligned}$$

Hence (7.5) holds.

We next show (7.6). Because of (4.7), (4.8), and (5.1),

$$\begin{aligned}
&\text{LHS of (7.6)}\\
=&m_{L\otimes L}(\lambda)((b, e_L), (a, ((\lambda b)a)\backslash(\lambda b)))\\
=&(m(\lambda)(b, \xi_{\lambda b}(e_L, a)), m((\lambda b)\xi_{\lambda b}(e_L, a))(\eta_{\lambda b}(e_L, a), ((\lambda b)a)\backslash(\lambda b)))\\
=&(\lambda\backslash((\lambda b)\xi_{\lambda b}(e_L, a)), ((\lambda b)\xi_{\lambda b}(e_L, a))\backslash(\lambda b))\\
=&(\lambda\backslash((\lambda b)a), ((\lambda b)a)\backslash(\lambda b))
\end{aligned}$$

for all $\lambda \in H$ and $a, b \in L$. Similarly, by means of (4.7), (4.8), (5.1), (7.2), and (7.3),

$$\begin{aligned}
&\text{RHS of (7.6)}\\
=&m_{L\otimes L}(\lambda)((\rho_b^\lambda(a), (\lambda\rho_b^\lambda(a))\backslash\lambda), (b, e_L))\\
=&(m(\lambda)(\rho_b^\lambda(a), \xi_{\lambda\rho_b^\lambda(a)}((\lambda\rho_b^\lambda(a))\backslash\lambda, b)),\\
&\quad m((\lambda\rho_b^\lambda(a))\xi_{\lambda\rho_b^\lambda(a)}((\lambda\rho_b^\lambda(a))\backslash\lambda, b))(\eta_{\lambda\rho_b^\lambda(a)}((\lambda\rho_b^\lambda(a))\backslash\lambda, b), e_L))\\
=&(\lambda\backslash((\lambda\rho_b^\lambda(a))\xi_{\lambda\rho_b^\lambda(a)}((\lambda\rho_b^\lambda(a))\backslash\lambda, b)), ((\lambda\rho_b^\lambda(a))\xi_{\lambda\rho_b^\lambda(a)}((\lambda\rho_b^\lambda(a))\backslash\lambda, b))\backslash(\lambda b))\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda\rho_b^\lambda(a))\pi(\lambda)^{-1}\pi(\lambda b)), \pi^{-1}(\pi(\lambda\rho_b^\lambda(a))\pi(\lambda)^{-1}\pi(\lambda b))\backslash(\lambda b))\\
=&(\lambda\backslash((\lambda b)a), ((\lambda b)a)\backslash(\lambda b))
\end{aligned}$$

for all $\lambda \in H$ and $a, b \in L$. Therefore (7.6) holds. □

Lemma 7.2. *For every $\lambda \in H$, the map $F^\lambda : (L \times L) \times ((L \times L) \times Y) \to (L \times L) \times ((L \times L) \times Y)$ defined as follows is bijective: for any $a, b, c, d, f \in L$ and $x \in X$,*

$$\begin{aligned} &F^\lambda((a,b),((c,d),(f,x))) \\ =&(m_{L\otimes L}(\lambda)((a,e_L),\iota^{\lambda a}(b)),(m_{L\otimes L}(\lambda a)((c,e_L),\iota^{(\lambda a)c}(d)), \\ &(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))). \end{aligned} \tag{7.8}$$

Proof. For every $\lambda \in H$, we define the map $G^\lambda : (L \times L) \times ((L \times L) \times Y) \to (L \times L) \times ((L \times L) \times Y)$ by

$$\begin{aligned} &G^\lambda((a,b),((c,d),(f,x))) \\ =&((\lambda\backslash((\lambda a)b),((\lambda a)b)\backslash(\lambda a)),((((\lambda a)b)\backslash((((\lambda a)b)c)d), \\ &((((\lambda a)b)c)d)\backslash(((\lambda a)b)c)),(f,m_X((((\lambda a)b)c)d)(f,x)))) \end{aligned} \tag{7.9}$$

for $a, b, c, d, f \in L$, and $x \in X$. Then G^λ is the inverse of F^λ. We only show that $G^\lambda \circ F^\lambda$ is the identity map.

From (4.7), (4.8), (5.1), (7.8), (7.9), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned} &(G^\lambda \circ F^\lambda)((a,b),((c,d),(f,x))) \\ =&G^\lambda(m_{L\otimes L}(\lambda)((a,e_L),\iota^{\lambda a}(b)),(m_{L\otimes L}(\lambda a)((c,e_L),\iota^{(\lambda a)c}(d)), \\ &(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))) \\ =&G^\lambda((m(\lambda)(a,\xi_{\lambda a}(e_L,b)),m((\lambda a)\xi_{\lambda a}(e_L,b))(\eta_{\lambda a}(e_L,b),((\lambda a)b)\backslash(\lambda a))), \\ &((m(\lambda a)(c,\xi_{(\lambda a)c}(e_L,d)),m(((\lambda a)c)\xi_{(\lambda a)c}(e_L,d)) \\ &(\eta_{(\lambda a)c}(e_L,d),(((\lambda a)c)d)\backslash((\lambda a)c))), \\ &(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))) \\ =&G^\lambda((\lambda\backslash((\lambda a)b),((\lambda a)b)\backslash(\lambda a)),(((\lambda a)\backslash(((\lambda a)c)d),(((\lambda a)c)d)\backslash((\lambda a)c)), \\ &(f,m_X(((\lambda a)c)f)(((\lambda a)c)f)\backslash((\lambda a)c),x))) \\ =&((\lambda\backslash(\lambda a),(\lambda a)\backslash((\lambda a)b)),(((\lambda a)\backslash((\lambda a)c),((\lambda a)c)\backslash(((\lambda a)c)d)), \\ &(f,m_X((\lambda a)c)(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x))))) \\ =&((a,b),((c,d),(f,m_X((\lambda a)c)(((\lambda a)c)\backslash((\lambda a)c),x)))) \\ =&((a,b),((c,d),(f,m_X((\lambda a)c)(e_L,x)))) \\ =&((a,b),((c,d),(f,x))). \end{aligned}$$

Hence $G^\lambda \circ F^\lambda$ is the identity map. □

7.2 Step I

We proceed to prove Theorem 6.3.

Proposition 7.3. *The following conditions are equivalent.*
(1) *Left $L \underset{\mathrm{tw}}{\otimes} L$-modules (Y, θ_Y) satisfying* (5.6) *and* (5.7) (*For the definition of $L \underset{\mathrm{tw}}{\otimes} L$, see* Definition 5.2).

(2) *Elements* $a \overset{\lambda}{\underset{x}{\Box}} b \in L$ $(\lambda \in H, a, b \in L, x \in X)$ *satisfying*

$$\lambda\backslash((\lambda b)(a \overset{\lambda b}{\underset{x}{\Box}} c)) = \rho_b^\lambda(a) \overset{\lambda}{\underset{m_X(\lambda)(b,x)}{\Box}} (\lambda\backslash((\lambda b)c)), \tag{7.10}$$

$$(a \underset{\lambda}{\cdot} b) \overset{\lambda}{\underset{x}{\Box}} c = a \overset{\lambda}{\underset{x}{\Box}} (b \overset{\lambda}{\underset{x}{\Box}} c), \tag{7.11}$$

$$e_L \overset{\lambda}{\underset{x}{\Box}} b = b \tag{7.12}$$

for all $\lambda \in H, a, b, c \in L$, *and* $x \in X$.

In addition, there is a bijection between (1) *and* (2).

Remark 7.4. We recall that left $L \underset{\mathrm{tw}}{\otimes} L$-modules (Y, θ_Y) in (1) are equivalent to left (L, m, η)-modules (Y, m_Y) satisfying (3.11) and (3.12) (see Theorem 5.3).

7.3 Proof of Proposition 7.3

7.3.1 We first deduce (2) from (1). Let (Y, θ_Y) be a left $L \underset{\mathrm{tw}}{\otimes} L$-module satisfying (5.6) and (5.7). We define $a \overset{\lambda}{\underset{x}{\Box}} b \in L$ $(\lambda \in H, a, b \in L, x \in X)$ by the first component of $\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))$.

Lemma 7.5. *For* $\lambda \in H, a, b, c \in L$, *and* $x \in X$,

$$\begin{aligned}
&\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(a \overset{\lambda}{\underset{x}{\Box}} b, m_X(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash\lambda, x)), \qquad (7.13)\\
&\theta_Y(\lambda)((b, e_L), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=&(\lambda\backslash((\lambda b)c), m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x))). \qquad (7.14)
\end{aligned}$$

Proof of Lemma 7.5. We first show (7.13). Let us define $y \in X$ by the second component of $\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))$. From (4.8), (5.6), (7.4), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}
&(m_X\theta_Y)(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX}(1_{L\otimes L} \otimes m_X))(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), m_X(\lambda\iota^\lambda(a))(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), m_X(\lambda)(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), (m_X(1_L \otimes m_X))(\lambda)(b, ((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), (m_X(m \otimes 1_X)a_{LLX}^{-1})(\lambda)(b, ((\lambda b)\backslash\lambda, x))) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), m_X(\lambda)(e_L, x)) \\
=&(m_X(1_L \otimes m_X)a_{LLX})(\lambda)(\iota^\lambda(a), x) \\
=&(m_X(m \otimes 1_X))(\lambda)((a, (\lambda a)\backslash\lambda), x) \\
=&m_X(\lambda)(e_L, x) \\
=&x.
\end{aligned}$$

By the definition of y, we have $x = m_X(\lambda)(a \overset{\lambda}{\underset{x}{\Box}} b, y)$. Because of (4.8) and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}
y =& m_X(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))(e_L, y) \\
=& m_X(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b)), y) \\
=& (m_X(m \otimes 1_X)a_{LLX}^{-1})(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash\lambda, (a \overset{\lambda}{\underset{x}{\Box}} b, y)) \\
=& (m_X(1_L \otimes m_X))(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash\lambda, (a \overset{\lambda}{\underset{x}{\Box}} b, y)) \\
=& m_X(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash\lambda, m_X(\lambda)(a \overset{\lambda}{\underset{x}{\Box}} b, y)) \\
=& m_X(\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))((\lambda(a \overset{\lambda}{\underset{x}{\Box}} b))\backslash\lambda, x).
\end{aligned}$$

Hence (7.13) holds.

We next show (7.14). From (4.8) and (5.7),

$$\begin{aligned}
& \theta_Y(\lambda)((b, e_L), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=& (\theta_Y((1_L \otimes \eta) \otimes 1_Y))(\lambda)((b, \bullet), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x)))) \\
=& ((m \otimes 1_X)a_{LLX}^{-1}(r_L \otimes 1_Y))(\lambda)((b, \bullet), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x)))) \\
=& (m(\lambda)(b, c), m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x))) \\
=& (\lambda\backslash((\lambda b)c), m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x))).
\end{aligned}$$

Hence (7.14) holds. □

Now, we show (7.10), (7.11), and (7.12). To prove (7.10), we focus on the element

$$\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (b, e_L)), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x)))$$
$$(\lambda \in H, a, b, c \in L, x \in X).$$

On account of (7.6), (7.13), and the fact that (Y, θ_Y) is a left $L \underset{\mathrm{tw}}{\otimes} L$-module,

$$\begin{aligned}
& \theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (b, e_L)), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=& \theta_Y(\lambda)(m_{L\otimes L}(\lambda)((b, e_L), \iota^{\lambda b}(a)), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=& (\theta_Y(1_{L\otimes L} \otimes \theta_Y)a_{L\otimes L L\otimes L Y})(\lambda)((b, e_L), \iota^{\lambda b}(a)), \\
& \quad (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=& \theta_Y(\lambda)((b, e_L), \theta_Y(\lambda b)(\iota^{\lambda b}(a), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x)))) \\
=& \theta_Y(\lambda)((b, e_L), (a \overset{\lambda b}{\underset{x}{\Box}} c, m_X((\lambda b)(a \overset{\lambda b}{\underset{x}{\Box}} c))(((\lambda b)(a \overset{\lambda b}{\underset{x}{\Box}} c))\backslash(\lambda b), x))).
\end{aligned} \tag{7.15}$$

From (7.14), the first component of the right-hand-side of (7.15) is

$$\lambda\backslash((\lambda b)(a \overset{\lambda b}{\underset{x}{\Box}} c)).$$

Because of (7.4), (7.14), and the fact that (Y, θ_Y) is a left $L \underset{\text{tw}}{\otimes} L$-module,

$$\begin{aligned}
&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (b, e_L)), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=&(\theta_Y(1_{L\otimes L} \otimes \theta_Y)a_{L\otimes LL\otimes LY})(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (b, e_L)), \\
&\quad (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x))) \\
=&\theta_Y(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), \theta_Y(\lambda)((b, e_L), (c, m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b), x)))) \\
=&\theta_Y(\lambda)(\iota^\lambda(\rho_b^\lambda(a)), (\lambda\backslash((\lambda b)c), m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda, m_X(\lambda)(b, x)))). \quad (7.16)
\end{aligned}$$

By the definition, the first component of the right-hand-side of (7.16) is

$$\rho_b^\lambda(a) \overset{\lambda}{\underset{m_X(\lambda)(b,x)}{\square}} (\lambda\backslash((\lambda b)c)).$$

Hence (7.10) holds.

From (7.4), (7.5), (7.13), and the fact that (Y, θ_Y) is a left $L \underset{\text{tw}}{\otimes} L$-module,

$$\begin{aligned}
&(a\overset{\lambda}{\underset{x}{\square}}(b\overset{\lambda}{\underset{x}{\square}}c), m_X(\lambda(a\overset{\lambda}{\underset{x}{\square}}(b\overset{\lambda}{\underset{x}{\square}}c)))((\lambda(a\overset{\lambda}{\underset{x}{\square}}(b\overset{\lambda}{\underset{x}{\square}}c)))\backslash\lambda, x)) \\
=&\theta_Y(\lambda)(\iota^\lambda(a), (b\overset{\lambda}{\underset{x}{\square}}c, m_X(\lambda(b\overset{\lambda}{\underset{x}{\square}}c))((\lambda(b\overset{\lambda}{\underset{x}{\square}}c))\backslash\lambda, x))) \\
=&\theta_Y(\lambda)(\iota^\lambda(a), \theta_Y(\lambda)(\iota^\lambda(b), (c, m_X(\lambda c)((\lambda c)\backslash\lambda, x)))) \\
=&(\theta_Y(1_{L\otimes L} \otimes \theta_Y))(\lambda)(\iota^\lambda(a), (\iota^\lambda(b), (c, m_X(\lambda c)((\lambda c)\backslash\lambda, x)))) \\
=&(\theta_Y(m_{L\otimes L} \otimes 1_Y)a^{-1}_{L\otimes LL\otimes LY})(\lambda)(\iota^\lambda(a), (\iota^\lambda(b), (c, m_X(\lambda c)((\lambda c)\backslash\lambda, x)))) \\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(a), \iota^\lambda(b)), (c, m_X(\lambda c)((\lambda c)\backslash\lambda, x))) \\
=&\theta_Y(\lambda)(\iota^\lambda(a \underset{\lambda}{\cdot} b), (c, m_X(\lambda c)((\lambda c)\backslash\lambda, x))) \\
=&((a \underset{\lambda}{\cdot} b)\overset{\lambda}{\underset{x}{\square}}c, m_X(\lambda((a \underset{\lambda}{\cdot} b)\overset{\lambda}{\underset{x}{\square}}c))((\lambda((a \underset{\lambda}{\cdot} b)\overset{\lambda}{\underset{x}{\square}}c))\backslash\lambda, x)).
\end{aligned}$$

for $\lambda \in H, a, b, c \in L$, and $x \in X$. Comparing the first components, we deduce that (7.11) holds.

The next task is to show (7.12). From (5.2), (7.3), (7.13), and the fact that (Y, θ_Y) is a left $L \underset{\text{tw}}{\otimes} L$-module,

$$\begin{aligned}
&(e_L\overset{\lambda}{\underset{x}{\square}}b, m_X(\lambda(e_L\overset{\lambda}{\underset{x}{\square}}b))((\lambda(e_L\overset{\lambda}{\underset{x}{\square}}b))\backslash\lambda, x)) \\
=&\theta_Y(\lambda)(\iota^\lambda(e_L), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&\theta_Y(\lambda)((e_L, e_L), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(\theta_Y(\eta_{L\otimes L} \otimes 1_Y))(\lambda)(\bullet, (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&l_{L\otimes L}(\lambda)(\bullet, (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))) \\
=&(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)).
\end{aligned}$$

Comparing the first components, we see that (7.12) holds.

7.3.2 We next deduce (1) from (2). Let $a\overset{\lambda}{\underset{x}{\square}}b$ $(\lambda \in H, a, b \in L, x \in X)$ be elements of L satisfying (7.10), (7.11), and (7.12) for all $\lambda \in H, a, b, c \in L$, and

$x \in X$. For every $\lambda \in H$, we define the map $\theta_Y(\lambda) : (L \times L) \times Y \to Y$ by

$$\theta_Y(\lambda)((a,b),(c,x)) = (\lambda\backslash d, m_X(d)(d\backslash((\lambda a)b), m_X((\lambda a)b)(c,x))) \quad (a,b,c \in L, x \in X). \tag{7.17}$$

Here, $d = ((\lambda a)b)((((\lambda a)b)\backslash(\lambda a)) \underset{m_X((\lambda a)b)(c,x)}{\overset{(\lambda a)b}{\Box}} c)$.

The morphism $\theta_Y : (L \otimes L) \otimes Y \to Y$ defined by (7.17) is a morphism of Set_H. In fact, because of the fact that m_X is a morphism of Set_H,

$$\begin{aligned}\lambda\theta_Y(\lambda)((a,b),(c,x)) =& d m_X(d)(d\backslash(((\lambda a)b)c), x) \\ =& (d(d\backslash(((\lambda a)b)c)))x \\ =& (((\lambda a)b)c)x \\ =& \lambda((a,b),(c,x))\end{aligned}$$

for $\lambda \in H, a, b, c \in L$, and $x \in X$.

Lemma 7.6. *For $\lambda \in H, a, b, f \in L$, and $x \in X$,*

$$\begin{aligned}&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(a),(b,e_L)),(f, m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x))) \\ =&(a \underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}} (\lambda\backslash((\lambda b)f)), m_X(\lambda(a \underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}} (\lambda\backslash((\lambda b)f)))) \\ &((\lambda(a \underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}} (\lambda\backslash((\lambda b)f))))\backslash(\lambda b), x)).\end{aligned} \tag{7.18}$$

Proof. From (4.7), (4.8), (5.1), (7.3), (7.17), and the fact that (X, m_X) is a (L, m, η)-module,

$$\begin{aligned}&\text{LHS of (7.18)} \\ =&\theta_Y(\lambda)((m(\lambda)(a, \xi_{\lambda a}((\lambda a)\backslash\lambda, b)), m((\lambda a)\xi_{\lambda a}((\lambda a)\backslash\lambda, b))(\eta_{\lambda a}((\lambda a)\backslash\lambda, b), e_L)), \\ &(f, m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x))) \\ =&\theta_Y(\lambda)((\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)), \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))\backslash(\lambda b)), \\ &(f, m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x))) \\ =&(\lambda\backslash s, m_X(s)(s\backslash(\lambda b), m_X(\lambda b)(f, m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x)))) \\ =&(\lambda\backslash s, m_X(s)(s\backslash((\lambda b)f), m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x))) \\ =&(\lambda\backslash s, m_X(s)(s\backslash(\lambda b), x)).\end{aligned} \tag{7.19}$$

Here,

$$\begin{aligned}s =&(\lambda b)((((\lambda b)\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))) \underset{m_X(\lambda b)(f, m_X((\lambda b)f)((((\lambda b)f)\backslash(\lambda b), x))}{\overset{\lambda b}{\Box}} f) \\ =&(\lambda b)((((\lambda b)\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))) \underset{x}{\overset{\lambda b}{\Box}} f).\end{aligned}$$

On account of (7.2) and (7.10),

$$\lambda\backslash s = \lambda\backslash((\lambda b)((((\lambda b)\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))) \underset{x}{\overset{\lambda b}{\Box}} f)$$

$$=\rho_b^\lambda((\lambda b)\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)))\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f))$$
$$=(\lambda\backslash(\lambda a))\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f))$$
$$=a\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f)).$$

Consequently,

$$\begin{aligned}&\text{RHS of (7.19)}\\&=(\lambda\backslash s, m_X(\lambda(\lambda\backslash s))((\lambda(\lambda\backslash s))\backslash(\lambda b), x))\\&=(a\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f)), m_X(\lambda(a\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f))))\\&\quad((\lambda(a\underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}}(\lambda\backslash((\lambda b)f))))\backslash(\lambda b), x)).\end{aligned}$$

Therefore (7.18) holds. $\Box$

Now, we show that (Y, θ_Y) is a left $L \underset{\text{tw}}{\otimes} L$-module. On account of (5.2), (7.12), (7.17), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}&\theta_Y(\eta_{L\otimes L}\otimes 1_Y)(\lambda)(\bullet,(c,x))\\&=\theta_Y(\lambda)((e_L,e_L),(c,x))\\&=(\lambda\backslash(\lambda(e_L\underset{m_X(\lambda)(c,x)}{\overset{\lambda}{\Box}}c)), m_X(\lambda(e_L\underset{m_X(\lambda)(c,x)}{\overset{\lambda}{\Box}}c))\\&\quad((\lambda(e_L\underset{m_X(\lambda)(c,x)}{\overset{\lambda}{\Box}}c))\backslash\lambda, m_X(\lambda)(c,x)))\\&=(\lambda\backslash(\lambda c), m_X(\lambda c)((\lambda c)\backslash\lambda, m_X(\lambda)(c,x)))\\&=(c, m_X(\lambda c)(e_L,x))\\&=(c,x)\\&=l_Y(\lambda)(\bullet,(c,x))\end{aligned}$$

for $\lambda\in H, c\in L$, and $x\in X$. Hence we have $\theta_Y(\eta_{L\otimes L}\otimes 1_Y)=l_Y$.

We next show $\theta_Y(1_{L\otimes L}\otimes\theta_Y)=\theta_Y(m_{L\otimes L}\otimes 1_Y)a^{-1}_{L\otimes L L\otimes L Y}$. From Lemma 7.2, our goal is to show

$$\begin{aligned}&(\theta_Y(1_{L\otimes L}\otimes\theta_Y))(\lambda)(F^\lambda((a,b),((c,d),(f,x))))\\&=(\theta_Y(m_{L\otimes L}\otimes 1_Y)a^{-1}_{L\otimes L L\otimes L Y})(\lambda)(F^\lambda((a,b),((c,d),(f,x))))\end{aligned}\tag{7.20}$$

for $\lambda\in H, a,b,c,d,f\in L$, and $x\in X$.

Because of (7.4), (7.6), (7.8), and the fact that $L \underset{\text{tw}}{\otimes} L$ is a monoid,

$$\begin{aligned}&\text{RHS of (7.20)}\\&=(\theta_Y(m_{L\otimes L}\otimes 1_Y)a^{-1}_{L\otimes L L\otimes L Y})(\lambda)(m_{L\otimes L}(\lambda)((a,e_L),\iota^{\lambda a}(b)),\\&\quad(m_{L\otimes L}(\lambda a)((c,e_L),\iota^{(\lambda a)c}(d)),(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x))))\\&=(\theta_Y((m_{L\otimes L}(m_{L\otimes L}\otimes 1_{L\otimes L}))\otimes 1_Y))(\lambda)((((\iota^\lambda(\rho_a^\lambda(b)),(a,e_L)),\end{aligned}$$

$$
\begin{aligned}
&\quad m_{L\otimes L}(\lambda a)(\iota^{\lambda a}(\rho_c^{\lambda a}(d))),(c,e_L))),(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&(\theta_Y((m_{L\otimes L}(1_{L\otimes L}\otimes m_{L\otimes L})a_{L\otimes LL\otimes LL\otimes L})\otimes 1_Y))(\lambda)((((\iota^\lambda(\rho_a^\lambda(b)),(a,e_L)),\\
&\quad m_{L\otimes L}(\lambda a)(\iota^{\lambda a}(\rho_c^{\lambda a}(d))),(c,e_L))),(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),\\
&\quad (m_{L\otimes L}(1_{L\otimes L}\otimes m_{L\otimes L}))(\lambda)((a,e_L),(\iota^{\lambda a}(\rho_c^{\lambda a}(d)),(c,e_L)))),\\
&\quad (f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),\\
&\quad (m_{L\otimes L}(m_{L\otimes L}\otimes 1_{L\otimes L})a^{-1}_{L\otimes LL\otimes LL\otimes L})(\lambda)((a,e_L),(\iota^{\lambda a}(\rho_c^{\lambda a}(d)),(c,e_L))),\\
&\quad (f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),\\
&\quad m_{L\otimes L}(\lambda)(m_{L\otimes L}(\lambda)((a,e_L),\iota^{\lambda a}(\rho_c^{\lambda a}(d))),(c,e_L))),\\
&\quad (f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),m_{L\otimes L}(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),(a,e_L)),\\
&\quad (c,e_L))),(f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x))). \qquad (7.21)
\end{aligned}
$$

On account of (4.7), (4.8), (5.1), (7.4), (7.5), and the fact that $L\underset{\mathrm{tw}}{\otimes}L$ is a monoid,

$$
\begin{aligned}
&\quad m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),m_{L\otimes L}(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),(a,e_L)),(c,e_L)))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),(m_{L\otimes L}(m_{L\otimes L}\otimes 1_{L\otimes L}))(\lambda)\\
&\quad (\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),(a,e_L)),(c,e_L))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),(m_{L\otimes L}(1_{L\otimes L}\otimes m_{L\otimes L})a_{L\otimes LL\otimes LL\otimes L})(\lambda)\\
&\quad (\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),(a,e_L)),(c,e_L))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),m_{L\otimes L}(\lambda)((a,e_L),(c,e_L))))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),\\
&\quad (m(\lambda)(a,\xi_{\lambda a}(e_L,c)),m((\lambda a)\xi_{\lambda a}(e_L,c))(\eta_{\lambda a}(e_L,c),e_L))))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d))),(\lambda\backslash((\lambda a)c),e_L)))\\
=&(m_{L\otimes L}(m_{L\otimes L}\otimes 1_{L\otimes L})a^{-1}_{L\otimes LL\otimes LL\otimes L})(\lambda)\\
&\quad (\iota^\lambda(\rho_a^\lambda(b)),\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d)))),(\lambda\backslash((\lambda a)c),e_L)))\\
=&m_{L\otimes L}(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)),\iota^\lambda(\rho_a^\lambda(\rho_c^{\lambda a}(d)))),(\lambda\backslash((\lambda a)c),e_L))\\
=&m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)\underset{\lambda}{\cdot}\rho_a^\lambda(\rho_c^{\lambda a}(d))),(\lambda\backslash((\lambda a)c),e_L)). \qquad (7.22)
\end{aligned}
$$

Hence, by means of (7.18) and (7.22),

$$
\begin{aligned}
&\quad \text{RHS of } (7.21)\\
=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)(\iota^\lambda(\rho_a^\lambda(b)\underset{\lambda}{\cdot}\rho_a^\lambda(\rho_c^{\lambda a}(d))),(\lambda\backslash((\lambda a)c),e_L))\\
&\quad (f,m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c),x)))\\
=&((\rho_a^\lambda(b)\underset{\lambda}{\cdot}\rho_a^\lambda(\rho_c^{\lambda a}(d)))\overset{\lambda}{\underset{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)}{\square}}(\lambda\backslash(((\lambda a)c)f)),\\
&\quad m_X(\lambda((\rho_a^\lambda(b)\underset{\lambda}{\cdot}\rho_a^\lambda(\rho_c^{\lambda a}(d)))\overset{\lambda}{\underset{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)}{\square}}(\lambda\backslash(((\lambda a)c)f))))
\end{aligned}
$$

$$((\lambda((\rho_a^{\lambda}(b) \underset{\lambda}{\cdot} \rho_a^{\lambda}(\rho_c^{\lambda a}(d))) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)} (\lambda\backslash(((\lambda a)c)f))))\backslash((\lambda a)c),x)). \tag{7.23}$$

We note that

$$\begin{aligned}&(\rho_a^{\lambda}(b) \underset{\lambda}{\cdot} \rho_a^{\lambda}(\rho_c^{\lambda a}(d))) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)} (\lambda\backslash(((\lambda a)c)f))\\ =&\lambda\backslash((\lambda a)(b \mathop{\square}\limits^{\lambda a}_{m_X(\lambda a)(c,x)} ((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f))))).\end{aligned} \tag{7.24}$$

In fact, by means of the fact that $\rho_a^{\lambda}(\rho_c^{\lambda a}(d)) = \rho_{\lambda\backslash((\lambda a)c)}^{\lambda}(d)$,

$$\begin{aligned}&\text{LHS of (7.24)}\\ =&(\rho_a^{\lambda}(b) \underset{\lambda}{\cdot} \rho_{\lambda\backslash((\lambda a)c)}^{\lambda}(d)) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)} (\lambda\backslash(((\lambda a)c)f)).\end{aligned} \tag{7.25}$$

From (4.8), (7.10), (7.11), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}&\text{RHS of (7.25)}\\ =&\rho_a^{\lambda}(b) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)} (\rho_{\lambda\backslash((\lambda a)c)}^{\lambda}(d) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(\lambda\backslash((\lambda a)c),x)} (\lambda\backslash(((\lambda a)c)f)))\\ =&\rho_a^{\lambda}(b) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(a,m_X(\lambda a)(c,x))} (\lambda\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f)))\\ =&\rho_a^{\lambda}(b) \mathop{\square}\limits^{\lambda}_{m_X(\lambda)(a,m_X(\lambda a)(c,x))} (\lambda\backslash((\lambda a)((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f)))))\\ =&\lambda\backslash((\lambda a)(b \mathop{\square}\limits^{\lambda a}_{m_X(\lambda a)(c,x)} ((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f))))).\end{aligned}$$

Hence (7.24) holds. Consequently,

$$\begin{aligned}&\text{RHS of (7.23)}\\ =&(\lambda\backslash((\lambda a)(b \mathop{\square}\limits^{\lambda a}_{m_X(\lambda a)(c,x)} ((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f))))),\\ &m_X((\lambda a)(b \mathop{\square}\limits^{\lambda a}_{m_X(\lambda a)(c,x)} ((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f)))))\\ &(((\lambda a)(b \mathop{\square}\limits^{\lambda a}_{m_X(\lambda a)(c,x)} ((\lambda a)\backslash(((\lambda a)c)(d \mathop{\square}\limits^{(\lambda a)c}_{x} f)))))\backslash((\lambda a)c),x)).\end{aligned}$$

From (4.7), (4.8), (5.1), and (7.8),

$$\begin{aligned}&\text{LHS of (7.20)}\\ =&(\theta_Y(1_{L\otimes L} \otimes \theta_Y))(\lambda)(m_{L\otimes L}(\lambda)((a, e_L), \iota^{\lambda a}(b)),\\ &(m_{L\otimes L}(\lambda a)((c, e_L), \iota^{(\lambda a)c}(d)), (f, m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c), x))))\\ =&(\theta_Y(1_{L\otimes L} \otimes \theta_Y))(\lambda)((m(\lambda)(a, \xi_{\lambda a}(e_L, b)),\\ &m((\lambda a)\xi_{\lambda a}(e_L, b))(\eta_{\lambda a}(e_L, b), ((\lambda a)b)\backslash(\lambda a))), ((m(\lambda a)(c, \xi_{(\lambda a)c}(e_L, d)),\\ &m(((\lambda a)c)\xi_{(\lambda a)c}(e_L, d))(\eta_{(\lambda a)c}(e_L, d), (((\lambda a)c)d)\backslash((\lambda a)c))),\end{aligned}$$

$$
\begin{aligned}
&(f, m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c), x))))\\
=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b), ((\lambda a)b)\backslash(\lambda a)), \theta_Y(\lambda a)(((\lambda a)\backslash(((\lambda a)c)d),\\
&(((\lambda a)c)d)\backslash((\lambda a)c)), (f, m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c), x)))). \qquad (7.26)
\end{aligned}
$$

Since (X, m_X) is a left (L, m, η)-module, it follows that $m_X((\lambda a)c)(f, m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c), x)) = m_X((\lambda a)c)(e_L, x) = x$. By (7.17),

$$
\begin{aligned}
&\text{RHS of (7.26)}\\
=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b), ((\lambda a)b)\backslash(\lambda a)), ((\lambda a)\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f)),\\
&m_X(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))((((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\backslash(((\lambda a)c)f),\\
&m_X(((\lambda a)c)f)((((\lambda a)c)f)\backslash((\lambda a)c), x))))\\
=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b), ((\lambda a)b)\backslash(\lambda a)), ((\lambda a)\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f)),\\
&m_X(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))((((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\backslash((\lambda a)c), x))). \qquad (7.27)
\end{aligned}
$$

Since (X, m_X) is a left (L, m, η)-module, it follows that

$$
\begin{aligned}
&m_X(\lambda a)((\lambda a)\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f)), m_X(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\\
&\quad((((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\backslash((\lambda a)c), x))\\
=&m_X(\lambda a)((\lambda a)\backslash((\lambda a)c), x)\\
=&m_X(\lambda a)(c, x).
\end{aligned}
$$

Therefore, by means of (7.17) and the fact that (X, m_X) is a left (L, m, η)-module,

$$
\begin{aligned}
&\text{RHS of (7.27)}\\
=&(\lambda\backslash p, m_X(p)(p\backslash(\lambda a), m_X(\lambda a)((\lambda a)\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f)),\\
&\quad m_X(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))((((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\backslash((\lambda a)c), x))))\\
=&(\lambda\backslash p, m_X(p)(p\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f)),\\
&\quad m_X(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))((((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))\backslash((\lambda a)c), x)))\\
=&(\lambda\backslash p, m_X(p)(p\backslash((\lambda a)c), x)).
\end{aligned}
$$

Here, $p = (\lambda a)(b \overset{\lambda a}{\underset{m_X(\lambda a)(c,x)}{\square}} ((\lambda a)\backslash(((\lambda a)c)(d \overset{(\lambda a)c}{\underset{x}{\square}} f))))$. Hence (7.20) holds. We thus prove that (Y, θ_Y) is a left $L \underset{\text{tw}}{\otimes} L$-module.

The next task is to show that the morphism θ_Y satisfies (5.6) and (5.7). On account of (4.8), (7.17), and the fact that (X, m_X) is a left (L, m, η)-module,

$$
(m_X\theta_Y)(\lambda)((a, b), (c, x))
$$

$$
\begin{aligned}
&= m_X(\lambda)(\lambda\backslash d, m_X(d)(d\backslash((\lambda a)b), m_X((\lambda a)b)(c,x)))\\
&= (m_X(1_L \otimes m_X))(\lambda)(\lambda\backslash d, (d\backslash((\lambda a)b), m_X((\lambda a)b)(c,x)))\\
&= (m_X(m \otimes 1_X)a^{-1}_{LLX})(\lambda)(\lambda\backslash d, (d\backslash((\lambda a)b), m_X((\lambda a)b)(c,x)))\\
&= m_X(\lambda)(\lambda\backslash((\lambda a)b), m_X((\lambda a)b)(c,x))\\
&= m_X(\lambda)(m(\lambda)(a,b), m_X((\lambda a)b)(c,x))\\
&= (m_X(m \otimes 1_X))(\lambda)((a,b), m_X((\lambda a)b)(c,x))\\
&= (m_X(1_L \otimes m_X)a_{LLX})(\lambda)((a,b), m_X((\lambda a)b)(c,x))\\
&= (m_X(1_L \otimes m_X)a_{LLX}(1_{L\otimes L} \otimes m_X))(\lambda)((a,b),(c,x))
\end{aligned}
$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. Here, $d = ((\lambda a)b)((((\lambda a)b)\backslash(\lambda a)) \underset{m_X((\lambda a)b)(c,x)}{\overset{(\lambda a)b}{\Box}} c)$. Hence (5.6) holds.

Because of (3.10), (4.8), (7.12), and (7.17),

$$
\begin{aligned}
&\theta_Y((1_L \otimes \eta) \otimes 1_Y)(\lambda)((a,\bullet),(c,x))\\
&= \theta_Y(\lambda)((a,e_L),(c,x))\\
&= (\lambda\backslash((\lambda a)(e_L \underset{m_X(\lambda a)(c,x)}{\overset{\lambda a}{\Box}} c)), m_X((\lambda a)(e_L \underset{m_X(\lambda a)(c,x)}{\overset{\lambda a}{\Box}} c))\\
&\qquad (((\lambda a)(e_L \underset{m_X(\lambda a)(c,x)}{\overset{\lambda a}{\Box}} c))\backslash(\lambda a), m_X(\lambda a)(c,x)))\\
&= (\lambda\backslash((\lambda a)c), m_X((\lambda a)c)(e_L, x))\\
&= (\lambda\backslash((\lambda a)c), x)\\
&= ((m \otimes 1_X)a^{-1}_{LLX}(r_L \otimes 1_Y))(\lambda)((a,\bullet),(c,x))
\end{aligned}
$$

for all $\lambda \in H, a, c \in L$, and $x \in X$. Therefore (5.7) holds.

7.3.3 We are now in a position to prove that the correspondence of **7.3.1** is the inverse of that of **7.3.2**.

Let $a \underset{x}{\overset{\lambda}{\Box}} b$ ($\lambda \in H, a, b \in L, x \in X$) be elements of L satisfying (7.10), (7.11), and (7.12) for all $\lambda \in H, a, b, c \in L$, and $x \in X$. We define the morphism $\theta_Y : (L \otimes L) \otimes Y \to Y$ by

$$
\theta_Y(\lambda)((a,b),(c,x)) = (\lambda\backslash d, m_X(d)(d\backslash((\lambda a)b), m_X((\lambda a)b)(c,x))).
$$
$$
(\lambda \in H, a, b, c \in L, x \in X)
$$

Here, $d = ((\lambda a)b)((((\lambda a)b)\backslash(\lambda a)) \underset{m_X((\lambda a)b)(c,x)}{\overset{(\lambda a)b}{\Box}} c)$. From (7.3) and the fact that (X, m_X) is a left (L, m, η)-module,

$$
\begin{aligned}
&\theta_Y(\lambda)(\iota^\lambda(a),(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))\\
&= \theta_Y(\lambda)((a,(\lambda a)\backslash\lambda),(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))\\
&= (\lambda\backslash q, m_X(q)(q\backslash\lambda, m_X(\lambda)(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))).
\end{aligned}
$$

Here,

$$
q = \lambda((\lambda\backslash(\lambda a)) \underset{m_X(\lambda)(b, m_X(\lambda b)((\lambda b)\backslash\lambda, x))}{\overset{\lambda}{\Box}} b)
$$

$$=\lambda(a\overset{\lambda}{\underset{x}{\Box}}b).$$

Therefore the first component of $\theta_Y(\lambda)(\iota^\lambda(a),(b,m_X(\lambda b)((\lambda b)\backslash\lambda,x)))$ is exactly $a\overset{\lambda}{\underset{x}{\Box}}b$.

Conversely, let (Y,θ_Y) be a left $L\underset{\mathrm{tw}}{\otimes}L$-module satisfying (5.6) and (5.7). We denote by $a\overset{\lambda}{\underset{x}{\Box}}b$ $(\lambda\in H,a,b\in L,x\in X)$ the first component of $\theta_Y(\lambda)(\iota^\lambda(a),(b,m_X(\lambda b)((\lambda b)\backslash\lambda,x)))$. Our goal is to show

$$\begin{aligned}&(\lambda\backslash d',m_X(d')(d'\backslash((\lambda a)b),m_X((\lambda a)b)(c,x)))\\=&\theta_Y(\lambda)((a,b),(c,x)).\\&(\lambda\in H,a,b,c\in L,x\in X)\end{aligned}\tag{7.28}$$

Here, $d'=((\lambda a)b)((((\lambda a)b)\backslash(\lambda a))\overset{(\lambda a)b}{\underset{m_X((\lambda a)b)(c,x)}{\Box}}c)$.

Since (X,m_X) is a left (L,m,η)-module,

$$\text{LHS of (7.28)}=(\lambda\backslash d',m_X(d')(d'\backslash((\lambda a)b)c,x)).$$

We note that, by means of (4.7), (4.8), (5.1), and (7.3),

$$\begin{aligned}&m_{L\otimes L}(\lambda)((\lambda\backslash((\lambda a)b),e_L),\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a)))\\=&(m(\lambda)(\lambda\backslash((\lambda a)b),\xi_{(\lambda a)b}(e_L,((\lambda a)b)\backslash(\lambda a))),\\&\quad m(((\lambda a)b)\xi_{(\lambda a)b}(e_L,((\lambda a)b)\backslash(\lambda a)))(\eta_{(\lambda a)b}(e_L,((\lambda a)b)\backslash(\lambda a)),b))\\=&(\lambda\backslash(\lambda a),(\lambda a)\backslash((\lambda a)b))\\=&(a,b).\end{aligned}\tag{7.29}$$

On account of (7.29) and the fact that (Y,θ_Y) is a left $L\underset{\mathrm{tw}}{\otimes}L$-module,

$$\begin{aligned}&\text{RHS of (7.28)}\\=&\theta_Y(\lambda)(m_{L\otimes L}(\lambda)((\lambda\backslash((\lambda a)b),e_L),\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a))),(c,x))\\=&(\theta_Y(m_{L\otimes L}\otimes 1_Y))(\lambda)(((\lambda\backslash((\lambda a)b),e_L),\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a))),(c,x))\\=&(\theta_Y(1_{L\otimes L}\otimes\theta_Y)a_{L\otimes LL\otimes LY})(\lambda)(((\lambda\backslash((\lambda a)b),e_L),\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a))),(c,x))\\=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b),e_L),\theta_Y((\lambda a)b)(\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a)),(c,x))).\end{aligned}\tag{7.30}$$

Since (X,m_X) is a left (L,m,η)-module, it follows that $(c,x)=(c,m_X(((\lambda a)b)c)(((\lambda a)b)c)\backslash((\lambda a)b),m_X((\lambda a)b)(c,x))$. Hence, by means of (7.13), (7.14), and the fact that (X,m_X) is a left (L,m,η)-module,

$$\begin{aligned}&\text{RHS of (7.30)}\\=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b),e_L),\theta_Y((\lambda a)b)(\iota^{(\lambda a)b}(((\lambda a)b)\backslash(\lambda a)),\\&\quad(c,m_X(((\lambda a)b)c)(((\lambda a)b)c)\backslash((\lambda a)b),m_X((\lambda a)b)(c,x))))\\=&\theta_Y(\lambda)((\lambda\backslash((\lambda a)b),e_L),((((\lambda a)b)\backslash(\lambda a))\overset{(\lambda a)b}{\underset{m_X((\lambda a)b)(c,x)}{\Box}}c,\\&\quad m_X(((\lambda a)b)((((\lambda a)b)\backslash(\lambda a))\overset{(\lambda a)b}{\underset{m_X((\lambda a)b)(c,x)}{\Box}}c))\end{aligned}$$

$$
\begin{aligned}
&(((\lambda a)b)((((\lambda a)b)\backslash(\lambda a)) \underset{m_X((\lambda a)b)(c,x)}{\overset{(\lambda a)b}{\Box}} c))\backslash((\lambda a)b), m_X((\lambda a)b)(c,x))) \\
=&(\lambda\backslash d', m_X(d')(d'\backslash\lambda, m_X(\lambda)(\lambda\backslash((\lambda a)b), m_X((\lambda a)b)(c,x)))) \\
=&(\lambda\backslash d', m_X(d')(d'\backslash\lambda, m_X(\lambda)(\lambda\backslash(((\lambda a)b)c), x))) \\
=&(\lambda\backslash d', m_X(d')(d'\backslash(((\lambda a)b)c), x)).
\end{aligned}
$$

Therefore (7.28) holds.

This completes the proof of Proposition 7.3.

7.4 Step Ⅱ

Let (Y, θ_Y) be a left $L \underset{\mathrm{tw}}{\otimes} L$-module satisfying (5.6) and (5.7). We define $a \underset{x}{\overset{\lambda}{\Box}} b$ ($\lambda \in H, a, b \in L, x \in X$) by the first component of $\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash\lambda, x)))$. Let $m_Y : L \otimes Y \to Y$ be the morphism defined by (5.13).

Proposition 7.7. *The following two conditions are equivalent.*

(1) *m_Y satisfies* (3.21) *for $A = L$. Here, $m_Y^\sigma : L \otimes Y \to Y$ is the morphism defined by* (3.6).

(2) *For any $\lambda \in H, a, b, c \in L, x \in X$,*

$$
a \underset{x}{\overset{\lambda}{\Box}} (b \underset{\lambda}{\cdot} c) = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(a \underset{x}{\overset{\lambda}{\Box}} c))). \tag{7.31}
$$

7.5 Proof of Proposition 7.7

Lemma 7.8. *We set*

$$
(f_1, y_1) = (m_Y(1_L \otimes m_Y^\sigma))(\lambda)(a, (b, (c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b), x)))), \tag{7.32}
$$

$$
\begin{aligned}
(f_2, y_2) =& (m_Y^\sigma(1_L \otimes m_Y)a_{LLY}(\sigma \otimes 1_Y)a_{LLY}^{-1})(\lambda) \\
&(a, (b, (c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b), x)))) \\
&(\lambda \in H, a, b, c \in L, x \in X).
\end{aligned} \tag{7.33}
$$

If $f_1 = f_2$, then $y_1 = y_2$.

Remark 7.9. For every $x' \in X$,

$$
\begin{aligned}
x' =& m_X(((\lambda a)b)c)(e_L, x') \\
=& m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b), m_X((\lambda a)b)(c, x')) \\
&(\lambda \in H, a, b, c \in L)
\end{aligned}
$$

because (X, m_X) is a left (L, m, η)-module. Hence, in view of Lemma 7.8, m_Y satisfies (3.21) for $A = L$, if and only if $f_1 = f_2$ for all $\lambda \in H, a, b, c \in L$, and $x \in X$.

Proof of Lemma 7.8. From Proposition 7.3, we see that (7.17) holds. On account of (3.6), (4.7), (5.13), (7.17), (7.32), and the fact that (X, m_X) is a left (L, m, η)-module,

$$
(f_1, y_1)
$$

$$
\begin{aligned}
&=(m_Y(1_L \otimes (1_L \otimes m_X)))(\lambda)(a,(\xi_{\lambda a}(b,c),\\
&\quad(\eta_{\lambda a}(b,c), m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y(\lambda)(a,(\xi_{\lambda a}(b,c), m_X((\lambda a)\xi_{\lambda a}(b,c))(((\lambda a)\xi_{\lambda a}(b,c))\backslash(((\lambda a)b)c),\\
&\quad m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y(\lambda)(a,(\xi_{\lambda a}(b,c), m_X((\lambda a)\xi_{\lambda a}(b,c))(((\lambda a)\xi_{\lambda a}(b,c))\backslash((\lambda a)b),x)))\\
&=(\theta_Y(((\eta\otimes 1_L)l_L^{-1})\otimes 1_Y))(\lambda)\\
&\quad(a,(\xi_{\lambda a}(b,c), m_X((\lambda a)\xi_{\lambda a}(b,c))(((\lambda a)\xi_{\lambda a}(b,c))\backslash((\lambda a)b),x)))\\
&=\theta_Y(\lambda)((e_L,a),(\xi_{\lambda a}(b,c), m_X((\lambda a)\xi_{\lambda a}(b,c))(((\lambda a)\xi_{\lambda a}(b,c))\backslash((\lambda a)b),x)))\\
&=(\lambda\backslash p, m_X(p)(p\backslash(\lambda a), m_X(\lambda a)(\xi_{\lambda a}(b,c),\\
&\quad m_X((\lambda a)\xi_{\lambda a}(b,c))(((\lambda a)\xi_{\lambda a}(b,c))\backslash((\lambda a)b),x))))\\
&=(\lambda\backslash p, m_X(p)(p\backslash(\lambda a), m_X(\lambda a)(b,x)))\\
&=(\lambda\backslash p, m_X(p)(p\backslash((\lambda a)b),x))
\end{aligned}
$$

for all $\lambda \in H, a,b,c \in L$, and $x \in X$. Here, $p = (\lambda a)(((\lambda a)\backslash\lambda) \underset{m_X(\lambda a)(b,x)}{\overset{\lambda a}{\Box}} \xi_{\lambda a}(b,c))$.

Because of (7.17), (7.33), and the fact that (X, m_X) is a left (L,m,η)-module,

$$
\begin{aligned}
&(f_2,y_2)\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b), m_Y(\lambda\xi_\lambda(a,b))\\
&\quad(\eta_\lambda(a,b),(c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b),(\theta_Y(((\eta\otimes 1_L)l_L^{-1})\otimes 1_Y))(\lambda\xi_\lambda(a,b))\\
&\quad(((\lambda\xi_\lambda(a,b))\backslash((\lambda a)b),(c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b),\theta_Y(\lambda\xi_\lambda(a,b))\\
&\quad((e_L,(\lambda\xi_\lambda(a,b))\backslash((\lambda a)b)),(c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q,\\
&\quad m_X(q)(q\backslash((\lambda a)b), m_X((\lambda a)b)(c, m_X(((\lambda a)b)c)((((\lambda a)b)c)\backslash((\lambda a)b),x))))\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q, m_X(q)(q\backslash((\lambda a)b), m_X((\lambda a)b)(e_L,x)))\\
&=m_Y^\sigma(\lambda)(\xi_\lambda(a,b),((\lambda\xi_\lambda(a,b))\backslash q, m_X(q)(q\backslash((\lambda a)b),x))).
\end{aligned}
\tag{7.34}
$$

Here, $q = ((\lambda a)b)((((\lambda a)b)\backslash(\lambda\xi_\lambda(a,b))) \underset{x}{\overset{(\lambda a)b}{\Box}} c)$. From (3.6), (4.7), (5.13), and the fact that (X, m_X) is a left (L,m,η)-module,

$$
\begin{aligned}
&\text{RHS of (7.34)}\\
&=(1_L\otimes m_X)(\lambda)(\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q),\\
&\quad(\eta_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q), m_X(q)(q\backslash((\lambda a)b),x)))\\
&=(\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q), m_X(\lambda\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q))\\
&\quad((\lambda\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q))\backslash q, m_X(q)(q\backslash((\lambda a)b),x)))\\
&=(\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q), m_X(\lambda\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q))\\
&\quad((\lambda\xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q))\backslash((\lambda a)b),x)).
\end{aligned}
$$

Therefore, if $f_1 = f_2$, then

$$
\lambda\backslash p = \xi_\lambda(\xi_\lambda(a,b),(\lambda\xi_\lambda(a,b))\backslash q). \tag{7.35}
$$

By means of (7.35), we can easily see that $y_1 = y_2$. $\square$

We proceed to prove Proposition 7.7. We will show that (7.35) is equivalent to (7.31). From (7.2), (7.10), and the fact that (X, m_X) is a left (L, m, η)-module,

$$
\begin{aligned}
&\text{RHS of (7.35)}\\
=&\xi_\lambda(\xi_\lambda(a,b), (\lambda\xi_\lambda(a,b))\backslash(((\lambda a)b)((((\lambda a)b)\backslash(\lambda\xi_\lambda(a,b))) \underset{x}{\overset{(\lambda a)b}{\square}} c)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda\xi_\lambda(a,b))^{-1}\pi(((\lambda a)b)((((\lambda a)b)\backslash(\lambda\xi_\lambda(a,b))) \underset{x}{\overset{(\lambda a)b}{\square}} c)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}\\
&\quad\pi(\lambda(\lambda\backslash((\lambda(\lambda\backslash((\lambda a)b)))((((\lambda a)b)\backslash(\lambda\xi_\lambda(a,b))) \underset{x}{\overset{\lambda(\lambda\backslash((\lambda a)b))}{\square}} c)))))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}\\
&\quad\pi(\lambda(\rho^\lambda_{\lambda\backslash((\lambda a)b)}(((\lambda a)b)\backslash(\lambda\xi_\lambda(a,b))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}} (\lambda\backslash(((\lambda a)b)c)))))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}\\
&\quad\pi(\lambda((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}} (\lambda\backslash(((\lambda a)b)c))))).
\end{aligned}
$$

Similary,

$$
\begin{aligned}
\text{LHS of (7.35)} =&\lambda\backslash((\lambda a)(((\lambda a)\backslash\lambda) \underset{m_X(\lambda a)(b,x)}{\overset{\lambda a}{\square}} \xi_{\lambda a}(b,c)))\\
=&\rho^\lambda_a((\lambda a)\backslash\lambda) \underset{m_X(\lambda)(a,m_X(\lambda a)(b,x))}{\overset{\lambda}{\square}} (\lambda\backslash((\lambda a)\xi_{\lambda a}(b,c)))\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}}\\
&\quad(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c))).
\end{aligned}
$$

Hence (7.35) is equivalent to

$$
\begin{aligned}
&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}}\\
&\quad(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(((\lambda a)b)c)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}\\
&\quad\pi(\lambda((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}} (\lambda\backslash(((\lambda a)b)c))))). \quad (7.36)
\end{aligned}
$$

We prove that (7.36) is equivalent to (7.31). We first substitute $((\lambda a)b)\backslash(\lambda c)$ for c in (7.36). Then (7.36) becomes

$$
\begin{aligned}
&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\overset{\lambda}{\square}}\\
&\quad(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(\lambda c)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}
\end{aligned}
$$

$$\pi(\lambda((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \overset{\lambda}{\underset{m_X(\lambda)(\lambda\backslash((\lambda a)b),x)}{\square}} c)))).$$

We next substitute $m_X((\lambda a)b)(((\lambda a)b)\backslash\lambda, x)$ for x in the above equation. Since (X, m_X) is a left (L, m, η)-module, the above equation becomes

$$\begin{aligned}&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)))\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi((\lambda a)b)^{-1}\pi(\lambda c)))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)b)^{-1}\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)))\overset{\lambda}{\underset{x}{\square}}c))).\end{aligned} \tag{7.37}$$

We substitute $(\lambda a)\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda b)^{-1}\ \pi(\lambda a))$ for b in (7.37). Then

$$\begin{aligned}&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)))\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash\pi^{-1}(\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\\ &\quad\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda((\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)))\overset{\lambda}{\underset{x}{\square}}c))).\end{aligned}$$

Substituting $\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))$ for a, we obtain

$$\begin{aligned}&a\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash\pi^{-1}(\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(a\overset{\lambda}{\underset{x}{\square}}c))).\end{aligned} \tag{7.38}$$

From (7.1), we see that (7.38) is equivalent to (7.31).

Considering the above argument in reverse, we get (7.36) from (7.31). This completes the proof.

7.6 Step Ⅲ

Through Step I-II, we have shown that the following conditions are equivalent.

(1) Left (L, m, η)-modules (Y, m_Y) satisfying (3.11), (3.12), and (3.21).

(2) Elements $a\overset{\lambda}{\underset{x}{\square}}b \in L$ $(\lambda \in H, a, b \in L, x \in X)$ satisfying (7.10), (7.11), (7.12), and (7.31) for all $\lambda \in H, a, b, c \in L$, and $x \in X$.

In addition, there is a bijection between (1) and (2).

We proceed to the next step.

Proposition 7.10. *The elements* $a\overset{\lambda}{\underset{x}{\square}}b$ $(\lambda \in H, a, b \in L, x \in X)$ *in* (2) *is equivalent to:*

(3) *Elements* $\beta^\lambda_x(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ *satisfying*

$$(\lambda b)\beta^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)) = \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda\beta^\lambda_x(a))), \tag{7.39}$$

$$\beta_x^\lambda(a \underset{\lambda}{\cdot} b) = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda\beta_x^\lambda(b))\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))) \tag{7.40}$$

for all $\lambda \in H, a, b \in L$, *and* $x \in X$.

In addition, there is a bijection between (2) *and* (3).

7.7 Proof of Proposition 7.10

7.7.1 We first deduce (3) from (2). Let $a\overset{\lambda}{\underset{x}{\square}}b \in L$ ($\lambda \in H, a, b \in L, x \in X$) be elements satisfying (7.10), (7.11), (7.12), and (7.31).

Lemma 7.11. *The relation* (7.10) *is equivalent to*

$$a\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash((\lambda b)c)) = \lambda\backslash((\lambda b)(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)\overset{\lambda b}{\underset{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}{\square}}c)) \tag{7.41}$$

for all $\lambda \in H, a, b, c \in L$, *and* $x \in X$.

Proof. We assume (7.10). On account of (7.2), (7.10), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}
&\text{RHS of (7.41)}\\
=&\rho_b^\lambda(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))\overset{\lambda}{\underset{m_X(\lambda)(b,m_X(\lambda b)((\lambda b)\backslash\lambda,x))}{\square}}(\lambda\backslash((\lambda b)c))\\
=&\rho_b^\lambda((\lambda b)\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)))\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash((\lambda b)c))\\
=&(\lambda\backslash(\lambda a))\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash((\lambda b)c))\\
=&a\overset{\lambda}{\underset{x}{\square}}(\lambda\backslash((\lambda b)c)).
\end{aligned}$$

Conversely, we assume (7.41). From (7.2), (7.41), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}
&\text{RHS of (7.10)}\\
=&\lambda\backslash((\lambda b)(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(\rho_b^\lambda(a))\overset{\lambda b}{\underset{m_X(\lambda b)((\lambda b)\backslash\lambda,m_X(\lambda)(b,x))}{\square}}c))\\
=&\lambda\backslash((\lambda b)(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(\lambda\backslash\pi^{-1}(\pi((\lambda b)a)\pi(\lambda b)^{-1}\pi(\lambda)))\overset{\lambda b}{\underset{x}{\square}}c))\\
=&\lambda\backslash((\lambda b)(((\lambda b)\backslash((\lambda b)a))\overset{\lambda b}{\underset{x}{\square}}c))\\
=&\lambda\backslash((\lambda b)(a\overset{\lambda b}{\underset{x}{\square}}c)).
\end{aligned}$$

Therefore (7.10) is equivalent to (7.41). □

For any $\lambda \in H, a \in L$, and $x \in X$, we set

$$\beta_x^\lambda(a) = a\overset{\lambda}{\underset{x}{\square}}e_L. \tag{7.42}$$

Let us show that the elements $\beta_x^\lambda(a)$ satisfy (7.39) and (7.40) for all $\lambda \in H, a, b \in L$, and $x \in X$.

Let us substitute e_L for c in (7.31). Since $b \underset{\lambda}{\cdot} e_L =$ b for all $\lambda \in H$ and $b \in L$, (7.31) becomes

$$\begin{aligned} a\overset{\lambda}{\underset{x}{\Box}}b =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(a\overset{\lambda}{\underset{x}{\Box}}e_L))) \\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))). \end{aligned} \tag{7.43}$$

Similarly, we substitute e_L for c in (7.41). Then

$$\begin{aligned} a\overset{\lambda}{\underset{x}{\Box}}b =&\lambda\backslash((\lambda b)(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)\overset{\lambda b}{\underset{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}{\Box}}e_L)) \\ =&\lambda\backslash((\lambda b)\beta^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))). \end{aligned} \tag{7.44}$$

From (7.43) and (7.44), we can easily see that (7.39) holds.

We next show (7.40). In view of (7.1) and (7.43),

$$\begin{aligned} &(a \underset{\lambda}{\cdot} b)\overset{\lambda}{\underset{x}{\Box}}c \\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)))\overset{\lambda}{\underset{x}{\Box}}c \\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a \underset{\lambda}{\cdot} b))), \\ &a\overset{\lambda}{\underset{x}{\Box}}(b\overset{\lambda}{\underset{x}{\Box}}c) \\ =&a\overset{\lambda}{\underset{x}{\Box}}(\lambda\backslash\pi^{-1}(\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda\beta_x^\lambda(b)))) \\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda\beta_x^\lambda(b))\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))) \end{aligned}$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. On account of (7.11), we see that (7.40) holds.

7.7.2 We next deduce (2) from (3). Let $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ be elements satisfying (7.39) and (7.40). For any $\lambda \in H, a, b \in L$, and $x \in X$, we write

$$a\overset{\lambda}{\underset{x}{\Box}}b = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))). \tag{7.45}$$

We will check that the elements $a\overset{\lambda}{\underset{x}{\Box}}b$ satisfy (7.10), (7.11), (7.12), and (7.31) for all $\lambda \in H, a, b, c \in L$, and $x \in X$.

We note that, by means of (7.39),

$$a\overset{\lambda}{\underset{x}{\Box}}b = \lambda\backslash((\lambda b)\beta^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))). \tag{7.46}$$

holds for all $\lambda \in H, a, b \in L$, and $x \in X$. From (7.2), (7.46), and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned} \lambda\backslash((\lambda b)(a\overset{\lambda b}{\underset{x}{\Box}}c)) =&\lambda\backslash(((\lambda b)c)\beta^{(\lambda b)c}_{m_X((\lambda b)c)(((\lambda b)c)\backslash(\lambda b),x)}(\rho^{(\lambda b)c}_{((\lambda b)c)\backslash(\lambda b)}(a))) \\ =&\lambda\backslash(((\lambda b)c)\beta^{(\lambda b)c}_{m_X((\lambda b)c)(((\lambda b)c)\backslash\lambda,m_X(\lambda)(b,x))}(\rho^{(\lambda b)c}_{((\lambda b)c)\backslash\lambda}(\rho^\lambda_b(a)))) \end{aligned}$$

$$=\rho_b^\lambda(a) \underset{m_X(\lambda)(b,x)}{\overset{\lambda}{\Box}} (\lambda\backslash((\lambda b)c)).$$

Hence (7.10) holds.

From (7.45),

$$\begin{aligned}
&a\underset{x}{\overset{\lambda}{\Box}}(b\underset{x}{\overset{\lambda}{\Box}}c)\\
=&a\underset{x}{\overset{\lambda}{\Box}}(\lambda\backslash\pi^{-1}(\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda\beta_x^\lambda(b))))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda\beta_x^\lambda(b))\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a)))
\end{aligned}$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. By means of (7.1), (7.40), and (7.45),

$$\begin{aligned}
&(a\underset{\lambda}{\cdot}b)\underset{x}{\overset{\lambda}{\Box}}c\\
=&(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)))\underset{x}{\overset{\lambda}{\Box}}c\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a\underset{\lambda}{\cdot}b)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda b)^{-1}\pi(\lambda\beta_x^\lambda(b))\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))).
\end{aligned}$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. Hence (7.11) holds.

We next show (7.12). Let us substitute e_L for a and b in (7.40). Then

$$\beta_x^\lambda(e_L) = \lambda\backslash\pi^{-1}(\pi(\lambda\beta_x^\lambda(e_L))\pi(\lambda)^{-1}(\pi(\lambda\beta_x^\lambda(e_L))))$$

for all $\lambda \in H$ and $x \in X$. Hence $\beta_x^\lambda(e_L) = e_L$ $(\lambda \in H, x \in X)$. Because of this fact, together with (7.2) and (7.46),

$$\begin{aligned}
e_L\underset{x}{\overset{\lambda}{\Box}}b =&\lambda\backslash((\lambda b)\beta^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(e_L)))\\
=&\lambda\backslash((\lambda b)\beta^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}(e_L))\\
=&b
\end{aligned}$$

for all $\lambda \in H, b \in L$, and $x \in X$. Hence (7.12) holds.

The next task is to show (7.31). From (7.1) and (7.45),

$$\begin{aligned}
a\underset{x}{\overset{\lambda}{\Box}}(b\underset{\lambda}{\cdot}c) =&a\underset{x}{\overset{\lambda}{\Box}}(\lambda\backslash\pi^{-1}(\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))) \qquad (7.47)
\end{aligned}$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. Substituting c for b in (7.45), we have

$$\pi(\lambda\beta_x^\lambda(a)) = \pi(\lambda a)\pi(\lambda c)^{-1}\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda(a\underset{x}{\overset{\lambda}{\Box}}c)).$$

Hence

$$\text{RHS of (7.47)} =\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(a\underset{x}{\overset{\lambda}{\Box}}c))).$$

Therefore (7.31) holds.

7.7.3 We prove that the correspondence of **7.7.1** is the inverse of that of **7.7.2**.

We assume that $a \overset{\lambda}{\underset{x}{\Box}} b \in L$ $(\lambda \in H, a, b \in L, x \in X)$ are elements that satisfy (7.10), (7.11), (7.12), and (7.31). We write $\beta_x^\lambda(a) = a \overset{\lambda}{\underset{x}{\Box}} e_L$ for all $\lambda \in H, a \in L$, and $x \in X$.

From (7.1) and (7.31),

$$\begin{aligned}
&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a))) \\
=&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(a \overset{\lambda}{\underset{x}{\Box}} e_L))) \\
=&a \overset{\lambda}{\underset{x}{\Box}} (b \underset{\lambda}{\cdot} e_L) \\
=&a \overset{\lambda}{\underset{x}{\Box}} b.
\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$.

Conversely, we assume that $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ are elements that satisfy (7.39) and (7.40). We write

$$a \overset{\lambda}{\underset{x}{\Box}} b = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a)))$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. By the definition,

$$a \overset{\lambda}{\underset{x}{\Box}} e_L = \lambda\backslash(\lambda\beta_x^\lambda(a)) = \beta_x^\lambda(a)$$

for all $\lambda \in H, a \in L$, and $x \in X$. This concludes our claim.

7.8 Step Ⅳ

Proposition 7.12. *The following conditions are equivalent.*

(1) *Elements* $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ *satisfying* (7.39) *and* (7.40) *for all* $\lambda \in H, a, b \in L, x \in X$.

(2) *Elements* $\Pi_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ *satisfying*

$$\Pi_x^\lambda(a \underset{\lambda}{\cdot} b) = \Pi_x^\lambda(a) \underset{\lambda}{\cdot} \Pi_x^\lambda(b), \tag{7.48}$$

$$\begin{aligned}
&\Pi_x^\lambda(\lambda\backslash((\lambda a)b)) \\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)\Pi^{\lambda a}_{m_X(\lambda a)((\lambda a)\backslash\lambda, x)}(b))\pi(\lambda)^{-1}\pi(\lambda\Pi_x^\lambda(a)))
\end{aligned} \tag{7.49}$$

for all $\lambda \in H, a, b \in L, x \in X$.

In addition, there is a bijection between (1) *and* (2).

7.9 Proof of Proposition 7.12

We assume that $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ are elements that satisfy (7.40), and write

$$\Pi_x^\lambda(a) = \lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda\beta_x^\lambda(a))^{-1}\pi(\lambda a)) \tag{7.50}$$

for all $\lambda \in H, a \in L$, $x \in X$. From (7.1), (7.40), and (7.50),

$$\begin{aligned}\Pi_x^\lambda(a \underset{\lambda}{\cdot} b) =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda\beta_x^\lambda(a \underset{\lambda}{\cdot} b))^{-1}\pi(\lambda(a \underset{\lambda}{\cdot} b)))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda\beta_x^\lambda(a))^{-1}\pi(\lambda a)\pi(\lambda\beta_x^\lambda(b))^{-1}\pi(\lambda b))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda\Pi_x^\lambda(a))\pi(\lambda)^{-1}\pi(\lambda\Pi_x^\lambda(b)))\\ =&\Pi_x^\lambda(a) \underset{\lambda}{\cdot} \Pi_x^\lambda(b)\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. Hence (7.48) holds.

Conversely, we assume that $\Pi_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ are elements that satisfy (7.48). We write

$$\beta_x^\lambda(a) = \lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda\Pi_x^\lambda(a))^{-1}\pi(\lambda)) \tag{7.51}$$

for all $\lambda \in X, a \in L$, and $x \in X$. Because of (7.1), (7.48), and (7.51),

$$\begin{aligned}\beta_x^\lambda(a \underset{\lambda}{\cdot} b) =&\lambda\backslash\pi^{-1}(\pi(\lambda(a \underset{\lambda}{\cdot} b))\pi(\lambda\Pi_x^\lambda(a \underset{\lambda}{\cdot} b))^{-1}\pi(\lambda))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda\Pi_x^\lambda(b))^{-1}\pi(\lambda)\pi(\lambda\Pi_x^\lambda(a))^{-1}\pi(\lambda))\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda\beta_x^\lambda(b))\pi(\lambda a)^{-1}\pi(\lambda\beta_x^\lambda(a)))\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. Hence (7.40) holds.

The next task is to prove the following lemma.

Lemma 7.13. (1) *Suppose that $\beta_x^\lambda(a)$ $(\lambda \in H, a \in L, x \in X)$ are elements of L satisfying* (7.40)*, and define elements $\Pi_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ by* (7.50)*. If $\beta_x^\lambda(a)$ also satisfy* (7.39)*, then $\Pi_x^\lambda(a)$ enjoy* (7.49) *for all $\lambda \in H, a \in L$, and $x \in X$.*
(2) *Conversely, suppose that $\Pi_x^\lambda(a)$ $(\lambda \in H, a \in L, x \in X)$ are elements of L satisfying* (7.48)*. We define elements $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ by* (7.51)*. If $\Pi_x^\lambda(a)$ also satisfy* (7.49)*, then $\beta_x^\lambda(a)$ enjoy* (7.39) *for all $\lambda \in H, a \in L$, and $x \in X$.*

Proof. We first show (1). Let $\beta_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ be elements that satisfy (7.39) and (7.40). We define elements $\Pi_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ by (7.50). By means of (7.2) and (7.50),

$$\begin{aligned}&\text{LHS of (7.39)}\\ =&(\lambda b)((\lambda b)\backslash\pi^{-1}(\pi((\lambda b)\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))\pi((\lambda b)\Pi^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda, x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)))^{-1}\pi(\lambda b)))\\ =&\pi^{-1}(\pi((\lambda b)\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))\pi((\lambda b)\Pi^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda, x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)))^{-1}\pi(\lambda b))\\ =&\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi((\lambda b)\Pi^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda, x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a)))^{-1}\pi(\lambda b)).\end{aligned}$$

Similarly,

$$\begin{aligned}&\text{RHS of (7.39)}\\ =&\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda(\lambda\backslash\pi^{-1}(\pi(\lambda a)\pi(\lambda\Pi_x^\lambda(a))^{-1}\pi(\lambda)))))\\ =&\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda\Pi_x^\lambda(a))^{-1}\pi(\lambda)).\end{aligned}$$

Hence (7.39) becomes

$$\pi((\lambda b)\Pi^{\lambda b}_{m_X(\lambda b)((\lambda b)\backslash\lambda, x)}(\rho^{\lambda b}_{(\lambda b)\backslash\lambda}(a))) = \pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda\Pi_x^\lambda(a)).$$

Substituting $m_X(\lambda)(b,x)$ for x in the above equation, we have

$$\pi((\lambda b)\Pi_x^{\lambda b}(\rho_{(\lambda b)\backslash\lambda}^{\lambda b}(a))) = \pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda\Pi_{m_X(\lambda)(b,x)}^{\lambda}(a)),$$

because (X, m_X) is a left (L, m, η)-module. We respectively substitute λ and b for λb and $(\lambda b)\backslash\lambda$ in the above equation. Then

$$\pi(\lambda\Pi_x^{\lambda}(\rho_b^{\lambda}(a))) = \pi(\lambda)\pi(\lambda b)^{-1}\pi((\lambda b)\Pi_{m_X(\lambda b)((\lambda b)\backslash\lambda,x)}^{\lambda b}(a)). \tag{7.52}$$

As we showed in the beginning of this subsection, we can deduce (7.48) from (7.40). Hence, by means of (7.1), (7.2), (7.48), and (7.52),

$$\begin{aligned}
&\Pi_x^{\lambda}(\lambda\backslash((\lambda b)a)\\
=&\Pi_x^{\lambda}(\rho_b^{\lambda}(a) \underset{\lambda}{\cdot} b)\\
=&\Pi_x^{\lambda}(\rho_b^{\lambda}(a)) \underset{\lambda}{\cdot} \Pi_x^{\lambda}(b)\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda\Pi_x^{\lambda}(\rho_b^{\lambda}(a)))\pi(\lambda)^{-1}\pi(\lambda\Pi_x^{\lambda}(b)))\\
=&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda b)^{-1}\pi((\lambda b)\Pi_{m_x(\lambda b)((\lambda b)\backslash\lambda,x)}^{\lambda b}(a))\pi(\lambda)^{-1}\pi(\lambda\Pi_x^{\lambda}(b)))
\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. Exchanging a for b in the above equation, we see that $\Pi_x^{\lambda}(a)$ satisfy (7.49) for all $\lambda \in L, a, b \in L$, and $x \in X$. We thus prove (1).

We can also prove (2) by considering the above argument in reverse. □

In view of (7.50) and (7.51), it is easily seen that there is a bijection between (1) and (2) in Proposition 7.12. This completes the proof of Proposition 7.12.

7.10 Step V

We prove the proposition below, before ending this section.

Proposition 7.14. *The following conditions are equivalent.*

(1) *Elements* $\Pi_x^{\lambda}(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ *satisfying* (7.48) *and* (7.49) *for all* $\lambda \in H, a, b \in L, x \in X$.

(2) *Group homomorphisms* $f_x : G \to G$ $(x \in X)$.

In addition, there is a bijection between (1) *and* (2).

7.11 Proof of Proposition 7.14

7.11.1 Let $\Pi_x^{\lambda}(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ be elements that satisfy (7.48) and (7.49) for all $\lambda \in H, a, b \in L, x \in X$. We define the maps $f_x : G \to G$ $(x \in X)$ by

$$f_x(a) = \pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(a)))\ (a \in G). \tag{7.53}$$

Here, we denote by λ_0 the unique element of $L(= H)$ satisfying that $\pi(\lambda_0)$ is the unit element of the group G.

Let us show that $f_x : G \to G$ $(x \in X)$ are group homomorphisms, that is, f_x satisfy

$$f_x(ab) = f_x(a)f_x(b)\ (a, b \in G). \tag{7.54}$$

From (7.1), (7.48), (7.53), and the fact that $\pi(\lambda_0)$ is the unit element of the group G,

$$\begin{aligned}\text{LHS of (7.54)} =&\pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(ab)))\\ =&\pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(a\pi(\lambda_0)^{-1}b)))\\ =&\pi(\lambda_0\Pi_x^{\lambda_0}((\lambda_0\backslash\pi^{-1}(a)) \underset{\lambda_0}{\cdot} (\lambda_0\backslash\pi^{-1}(b))))\\ =&\pi(\lambda_0(\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(a)) \underset{\lambda_0}{\cdot} \Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(b))))\\ =&\pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(a)))\pi(\lambda_0)^{-1}\pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(b)))\\ =&f_x(a)f_x(b).\end{aligned}$$

Hence, for every $x \in X$, the map $f_x : G \to G$ satisfies (7.54).
7.11.2 Let $f_x : G \to G$ $(x \in X)$ be group homomorphisms. We write

$$\Pi_x^{\lambda}(a) = \lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda a))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1}) \tag{7.55}$$

for all $\lambda \in H, a \in L$, and $x \in X$.

On account of (7.1), (7.55), and the fact that the maps f_x $(x \in X)$ are group homomorphisms,

$$\begin{aligned}&\text{LHS of (7.48)}\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda(a \underset{\lambda}{\cdot} b)))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda a))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1}\\ &\quad f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda b))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda\Pi_x^{\lambda}(a))\pi(\lambda)^{-1}\pi(\lambda\Pi_x^{\lambda}(b)))\\ =&\Pi_x^{\lambda}(a) \underset{\lambda}{\cdot} \Pi_x^{\lambda}(b)\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. Hence (7.48) holds.

Because of (7.55) and the fact that (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}\text{RHS of (7.49)} =&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash(\lambda a),m_X(\lambda a)((\lambda a)\backslash\lambda,x))}(\pi((\lambda a)b))\\ &\quad f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi((\lambda a)b))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\Pi_x^{\lambda}(\lambda\backslash((\lambda a)b))\end{aligned}$$

for all $\lambda \in H, a, b \in L$, and $x \in X$. Hence (7.49) holds.
7.11.3 We next prove that the correspondence of **7.11.1** is the inverse of that of **7.11.2**. We first assume that $f_x : G \to G$ $(x \in X)$ are group homomorphisms, and define $\Pi_x^{\lambda}(a)$ $(\lambda \in H, a \in L, x \in X)$ by (7.55). Since $\pi(\lambda_0)$ is the unit element of the group G and (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned}&\pi(\lambda_0\Pi_x^{\lambda_0}(\lambda_0\backslash\pi^{-1}(a)))\\ =&\pi(\lambda_0)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda_0,x)}(\pi(\lambda_0(\lambda_0\backslash\pi^{-1}(a))))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda_0,x)}(\pi(\lambda_0))^{-1}\\ =&f_x(a)\end{aligned}$$

for all $a \in L$ and $x \in X$.

Conversely, let $\Pi_x^\lambda(a) \in L$ $(\lambda \in H, a \in L, x \in X)$ be elements satisfying (7.48) and (7.49) for all $\lambda \in H, a, b \in L$, and $x \in X$. We define the map $f_x : G \to G$ $(x \in X)$ by (7.53).

From (7.1) and the fact that $\pi(\lambda_0)$ is the unit element of the group G,

$$\begin{aligned}&\lambda\backslash\pi^{-1}(\pi(\lambda)f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda a))f_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\pi(\lambda))^{-1})\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda_0\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash(\lambda a)))\pi(\lambda_0\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash\lambda))^{-1})\end{aligned} \tag{7.56}$$

for all $\lambda \in H, a \in L$, and $x \in X$. We substitute respectively λ_0, $m_X(\lambda_0)(\lambda_0\backslash\lambda, x)$, $\lambda_0\backslash\lambda$, and a for λ, x, a, and b in (7.49). Then

$$\begin{aligned}\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash(\lambda a)) =&\lambda_0\backslash\pi^{-1}(\pi(\lambda)^{-1}\pi(\lambda\Pi^{\lambda}_{m_X(\lambda)(\lambda\backslash\lambda_0,m_X(\lambda_0)(\lambda_0\backslash\lambda,x))}(a))\\ &\pi(\lambda_0\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash\lambda))).\end{aligned}$$

Since (X, m_X) is a left (L, m, η)-module, the above equation is equivalent to

$$\pi(\lambda_0\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash(\lambda a)) = \pi(\lambda)^{-1}\pi(\lambda\Pi^\lambda_x(a))\pi(\lambda_0\Pi^{\lambda_0}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}(\lambda_0\backslash\lambda)). \tag{7.57}$$

By means of (7.57), we can easily see that the right-hand-side of (7.56) is $\Pi_x^\lambda(a)$.

This completes the proof.

8 Examples of dynamical reflection maps

Our main result implies a way to construct dynamical reflection maps. Let us construct examples of dynamical reflection maps in this section. We also give an example of reflection maps (Cf. [8, Section 3]).

8.1 Examples

We first establish the dynamical reflection maps by using the way that we have proved Theorem 6.3 (see Section 7).

Let $L = (L, \cdot, e_L)$ be a left quasigroup with a unit (Definition 4.4), and set $H = L$. As we showed in Section 4, by means of (4.7), (4.8), and Proposition 4.8, (L, m, η, σ) is a braided monoid in Set_H.

We can rewrite the dynamical reflection map $k : L \otimes X \to L \otimes X \in Set_H$ in (3.7) as

$$\begin{aligned}&k(\lambda)(a,x)\\ =&(\Pi^\lambda_{m_X(\lambda)(a,x)}(a), m_X(\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a))((\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a))\backslash(\lambda a), x))\\ &(\lambda \in H, a \in L, x \in X),\end{aligned} \tag{8.1}$$

where $\Pi_x^\lambda(a) \in L$ is the element that satisfies (7.48) and (7.49). In fact, from Propositions 3.7, 3.8, and the proof of Theorem 5.3, we can rewrite the morphism k as

$$k(\lambda)(a, x) =m_Y(\lambda)(a,(e_L, x))$$

$$=\theta_Y(\lambda)((e_L,a),(e_L,x)) \tag{8.2}$$

for all $\lambda \in H, a \in L$, and $x \in X$. On account of (7.17), (7.42), (7.51), and the fact that (X, m_X) is a left (L, m, η)-module (Definition 4.9),

$$\begin{aligned}\text{RHS of (8.2)} =&(\lambda\backslash c', m_X(c')(c'\backslash(\lambda a), m_X(\lambda a)(e_L,x)))\\ =&(\lambda\backslash c', m_X(c')(c'\backslash(\lambda a), x)). \end{aligned}\tag{8.3}$$

Here,

$$\begin{aligned} c' =&(\lambda a)(((\lambda a)\backslash\lambda) \underset{m_X(\lambda a)(e_L,x)}{\overset{\lambda a}{\square}} e_L)\\ =&(\lambda a)(((\lambda a)\backslash\lambda)\underset{x}{\overset{\lambda a}{\square}} e_L)\\ =&(\lambda a)\beta_x^{\lambda a}((\lambda a)\backslash\lambda)\\ =&(\lambda a)((\lambda a)\backslash\pi^{-1}(\pi(\lambda)\pi((\lambda a)\Pi_x^{\lambda a}((\lambda a)\backslash\lambda))^{-1}\pi(\lambda a)))\\ =&\pi^{-1}(\pi(\lambda)\pi((\lambda a)\Pi_x^{\lambda a}((\lambda a)\backslash\lambda))^{-1}\pi(\lambda a)). \end{aligned}\tag{8.4}$$

Substituting respectively $m_X(\lambda)(a,x)$ and $(\lambda a)\backslash\lambda$ for x and b in (7.49), we have

$$\begin{aligned}&\Pi^\lambda_{m_X(\lambda)(a,x)}(e_L)\\ =&\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)\Pi_x^{\lambda a}((\lambda a)\backslash\lambda))\pi(\lambda)^{-1}\pi(\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a))),\end{aligned}$$

because (X, m_X) is a left (L, m, η)-module. We note that, by means of (7.55), $\Pi_x^\lambda(e_L) = e_L$ for all $\lambda \in H$ and $x \in X$. Hence the above equation is equivalent to

$$\pi(\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a)) = \pi(\lambda)\pi((\lambda a)\Pi_x^{\lambda a}((\lambda a)\backslash\lambda))^{-1}\pi(\lambda a). \tag{8.5}$$

On account of (8.4) and (8.5),

$$\begin{aligned}&\text{RHS of (8.3)}\\ =&(\Pi^\lambda_{m_X(\lambda)(a,x)}(a), m_X(\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a))((\lambda\Pi^\lambda_{m_X(\lambda)(a,x)}(a))\backslash(\lambda a),x)).\end{aligned}$$

Hence (8.1) holds.

Making use of (7.55) and (8.1), we now consider several examples of dynamical reflection maps.

Let (X, m_X) be a left (L, m, η)-module in Set_H (Definition 4.9).

Example 8.1. Assuming that the group G is abelian, we define the group homomorphisms $f_x : G \to G$ $(x \in X)$ by $f_x(a) = a^{-1}$ $(a \in G)$. By means of (7.55), we can write $\Pi_x^\lambda(a) \in L$ as

$$\Pi_x^\lambda(a) = \lambda\backslash\pi^{-1}(\pi(\lambda)^2\pi(\lambda a)^{-1})$$

for all $\lambda \in H, a \in L$, and $x \in X$. Hence, from (8.1), the dynamical reflection map $k : L \otimes X \to L \otimes X$ is

$$\begin{aligned}&k(\lambda)(a,x)\\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda)^2\pi(\lambda a)^{-1}), m_X(\pi^{-1}(\pi(\lambda)^2\pi(\lambda a)^{-1}))(\pi^{-1}(\pi(\lambda)^2\pi(\lambda a)^{-1})\backslash(\lambda a),x)).\end{aligned}$$

Example 8.2. (Cf. [8, Example 3.4]) For every $x \in X$, let $f_x : G \to G$ denote the identity map on G. We see at once that f_x $(x \in X)$ is a group homomorphism of G. From (7.55),

$$\Pi_x^\lambda(a) = \lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)\pi(\lambda)^{-1}) \tag{8.6}$$

for all $\lambda \in H, a \in L$, and $x \in X$. Therefore the dynamical reflection map $k : L \otimes X \to L \otimes X$ is

$$\begin{aligned} &k(\lambda)(a,x)\\ =&(\lambda\backslash\pi^{-1}(\pi(\lambda)\pi(\lambda a)\pi(\lambda)^{-1}),\\ &\quad m_X(\pi^{-1}(\pi(\lambda)\pi(\lambda a)\pi(\lambda)^{-1}))(\pi^{-1}(\pi(\lambda)\pi(\lambda a)\pi(\lambda)^{-1})\backslash(\lambda a), x))\\ &(\lambda \in H, a \in L, x \in X), \end{aligned} \tag{8.7}$$

by means of (8.1).

If the group G is abelian, then k is trivial. In fact, from (8.6), the elements $\Pi_x^\lambda(a)$ are

$$\Pi_x^\lambda(a) = \lambda\backslash(\lambda a) = a$$

for all $\lambda \in L, a \in L$, and $x \in X$. Since (X, m_X) is a left (L, m, η)-module,

$$\begin{aligned} \text{RHS of (8.7)} =&(a, m_X(\lambda a)(e_L, x))\\ =&(a,x) \end{aligned}$$

for all $\lambda \in L, a \in L$, and $x \in X$.

Example 8.3. (Cf. [8, Examples 3.6 and 3.7]) For every $x \in X$, let g_x be an element of G. We define the homomorphisms $f_x : G \to G$ $(x \in X)$ by

$$f_x(a) = g_x^{-1} a g_x$$

for all $a \in G$. From (7.55),

$$\Pi_x^\lambda(a) = \lambda\backslash\pi^{-1}(\pi(\lambda)g^{-1}_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)}\pi(\lambda a)\pi(\lambda)^{-1}g_{m_X(\lambda_0)(\lambda_0\backslash\lambda,x)})$$

for all $\lambda \in H, a \in L$, and $x \in X$. Using (8.1), we have the dynamical reflection map

$$\begin{aligned} k(\lambda)(a,x) = (\lambda\backslash b, m_X(b)(b\backslash(\lambda a), x))\\ (\lambda \in H, a \in L, x \in X). \end{aligned} \tag{8.8}$$

Here, we write $b = \pi^{-1}(\pi(\lambda)(g_{m_X(\lambda_0)(\lambda_0\backslash(\lambda a),x)})^{-1}\pi(\lambda a)\pi(\lambda)^{-1}g_{m_X(\lambda_0)(\lambda_0\backslash(\lambda a),x)})$.

By means of examples in Section 4, we consider Example 8.3 in more detail. Let $(L, \cdot, e_L)$ denote the left quasigroup with a unit in Example 4.6, and set $H = L$.

Example 8.4. Let $X = \{x_1, x_2, x_3\}$. We define the map $i : L \times X \to X, (a,x) \mapsto i(a,x)$ by Table 2.

Table 2: The map $i : L \times X \ni (a, x) \mapsto i(a, x) \in X$

	x_1	x_2	x_3
e_L	x_1	x_2	x_3
l_1	x_2	x_1	x_3
l_2	x_3	x_1	x_2
l_3	x_2	x_1	x_3
l_4	x_1	x_3	x_2
l_5	x_3	x_1	x_2
l_6	x_2	x_3	x_1
l_7	x_3	x_2	x_1

It is immediate from Table 2 that, for any $a \in L$, the map $X \ni x \mapsto i(a, x) \in X$ is bijective. For simplicity of notaition, we write ax instead of $i(a, x)$ for all $a \in L$ and $x \in X$. This notation is the same as that in Example 4.11.

Let us define the map $f : X \to H$ by $f(x_1) = l_4, f(x_2) = l_1$, and $f(x_3) = l_6$. From Example 4.11, we can define $\lambda \cdot_X x \in L$ ($\lambda \in H, x \in X$) by (4.10), and define a left (L, m, η)-module (X, m_X) by (4.11). The table below shows the map $\cdot_X : H \times X \to H$.

Table 3: The map $\cdot_X : H \times X \to H$

	x_1	x_2	x_3
e_L	l_4	l_1	l_6
l_1	l_1	l_4	l_6
l_2	l_6	l_4	l_1
l_3	l_1	l_4	l_6
l_4	l_4	l_6	l_1
l_5	l_6	l_4	l_1
l_6	l_1	l_6	l_4
l_7	l_6	l_1	l_4

Let G denote the dihedral group $D_4 =< \nu, \tau \mid \nu^4 = \tau^2 = \text{id}, \nu^{-1} = \tau\nu\tau >$, and define a bijection $\pi : L \to G$ by:

$$\pi(e_L) = \text{id};\ \pi(l_1) = \nu;\ \pi(l_2) = \nu^2;\ \pi(l_3) = \nu^3;\ \pi(l_4) = \tau;$$
$$\pi(l_5) = \nu\tau;\ \pi(l_6) = \nu^2\tau;\ \pi(l_7) = \nu^3\tau.$$

We write $g_{x_1} = \nu, g_{x_2} = \nu\tau$, and $g_{x_3} = \nu^3$.

Applying the above situation to Example 8.3, we have the dynamical reflection map $k : L \otimes X \to L \otimes X$. Then k depends on the dynamical parameter $\lambda \in H$. Indeed, we see that $k(l_1)(l_6, x_2) = (l_3, x_3)$ and $k(l_5)(l_6, x_2) = (l_6, x_2)$.

We thus showed that we can construct an example of dynamical reflection maps that depends on $\lambda \in H$.

8.2 Conditions for dynamical reflection maps to be independent on the dynamical parameter

We now discuss dynamical reflection maps that are independent on the dynamical parameter $\lambda \in H$.

We assume that $L = (L, \cdot, e_L)$ is a group. Let $* : L \times L \to L$ denote the binary operation defined by

$$a * b = \pi^{-1}(\pi(a)\pi(b)) \tag{8.9}$$

for all $a, b \in L$. It is easily seen that $(L, *, \lambda_0)$ is a group, where $\lambda_0 \in H(= L)$ is the unique element satisfying that $\pi(\lambda_0)$ is the unit element of the group G. We denote by a^{-1} the inverse of $a \in L$ with respect to $*$. From (8.9), $a^{-1} = \pi^{-1}(\pi(a)^{-1})$ for all $a \in L$. Also, we write $\overline{a}$ as the inverse of $a \in L$ with respect to $\cdot$.

Proposition 8.5. *The following conditions are equivalent.*

(1) *The map $\sigma(\lambda)$ is independent on the dynamical parameter $\lambda \in H$ and $\pi(e_L)$ is the unit element of the group G ($\lambda_0 = e_L$).*

(2) *For any $a, b, c \in L$,*

$$a(b * c) = (ab) * a^{-1} * (ac). \tag{8.10}$$

Proof. Let us first deduce (1) from (2). We substitute e_L for a and b in (8.10), then

$$e_L * c = e_L * e_L^{-1} * c = c$$

for all $c \in L$. From the above equation, we can see that e_L is the unit element of the group $(L, *)$. Hence, by means of (8.9),

$$a = a * e_L = \pi^{-1}(\pi(a)\pi(e_L))$$

for all $a \in L$, which yields the fact that $\pi(e_L)$ is the unit element of the group G.

The next task is to show that the map $\sigma(\lambda)$ is independent on the dynamical parameter $\lambda \in H$. On account of (8.9), the relation (8.10) is equivalent to

$$a\pi^{-1}(\pi(b)\pi(c)) = \pi^{-1}(\pi(ab)\pi(a)^{-1}\pi(ac)) \tag{8.11}$$

for all $a, b, c \in L$. We respectively substitute $\lambda a, \overline{a}$, and b for a, b, and c in (8.11). Then $\lambda a\pi^{-1}(\pi(\overline{a})\pi(b)) = \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b))$. Hence the first component of $\sigma(\lambda)$ is

$$\begin{aligned}\xi_\lambda(a, b) =& \overline{\lambda}\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b))\\ =& \overline{\lambda}(\lambda a\pi^{-1}(\pi(\overline{a})\pi(b)))\\ =& a\pi^{-1}(\pi(\overline{a})\pi(b))\end{aligned}$$

for all $\lambda \in H$ and $a, b \in L$. Substituting respectively $\overline{a}$ and b for b and c in (8.11), we have $a\pi^{-1}(\pi(\overline{a})\pi(b)) = \pi^{-1}(\pi(a)^{-1}\pi(ab))$ by means of the fact that $\pi(e_L)$ is the unit element of G. Hence $\xi_\lambda(a, b)$ is independent on the dynamical parameter $\lambda \in H$. In addition, from the fact that $(L, \cdot)$ is a group, the second component of $\sigma(\lambda)$ is

$$\begin{aligned}\eta_\lambda(a,b) =&\overline{(\lambda\xi_\lambda(a,b))}\lambda ab\\ =&\overline{\xi_\lambda(a,b)}\ \overline{\lambda}\lambda ab\\ =&\overline{\xi_\lambda(a,b)}ab\end{aligned}$$

for all $\lambda \in H$ and $a, b \in L$. Since $\xi_\lambda(a,b)$ is independent on $\lambda \in H$, so is $\eta_\lambda(a,b)$. Consequently, $\sigma(\lambda)$ is independent on $\lambda \in H$.

Conversely, we deduce (2) from (1). On account of (8.9),

$$(ab) * a^{-1} * (ac) = \pi^{-1}(\pi(ab)\pi(a)^{-1}\pi(ac)) \tag{8.12}$$

for all $a, b, c \in L$. Since $\sigma(\lambda)$ is independent on the dynamical parameter $\lambda \in H$, $\xi_\lambda(a,b)$ is independent on λ. On account of the assumption that $\pi(e_L)$ is the unit element of G,

$$\xi_\lambda(a,b) = \xi_{e_L}(a,b) = \pi^{-1}(\pi(a)^{-1}\pi(ab))$$

for all $\lambda \in H$ and $a, b \in L$. Hence,

$$\overline{\lambda}\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi((\lambda a)b)) = \pi^{-1}(\pi(a)^{-1}\pi(ab)) \tag{8.13}$$

for all $\lambda \in H$ and $a, b \in L$.

Substituting respectively $ab, \overline{b}$, and c for λ, a, and b in (8.13), we have

$$\pi^{-1}(\pi(ab)\pi(a)^{-1}\pi(ac)) = ab\pi^{-1}(\pi(\overline{b})^{-1}\pi(\overline{b}c)).$$

We next respectively substitute $b, \overline{b}$, and c for λ, a, and b in (8.13). Since $\pi(e_L)$ is the unit element of G,

$$\overline{b}\pi^{-1}(\pi(b)\pi(c)) = \pi^{-1}(\pi(\overline{b})^{-1}\pi(\overline{b}c)).$$

Hence, by means of (8.9),

$$\begin{aligned}\text{RHS of (8.12)} =&ab(\overline{b}\pi^{-1}(\pi(b)\pi(c)))\\ =&a(b * c)\end{aligned}$$

for all $a, b, c \in L$. Therefore (8.10) holds. □

From the above arguement, we can rewrite the map $\sigma(\lambda)$ $(\lambda \in H)$ under the condition (1) (or (2)) in Proposition 8.5 as

$$\sigma(\lambda)(a,b) = (a^{-1} * (ab), \overline{a^{-1} * (ab)}ab) \quad (a, b \in L). \tag{8.14}$$

Definition 8.6 ([8, 10])**.** If two groups $(L, \cdot)$ and $(L, *)$ satisfy (8.10), we say that $(L, \cdot, *)$ is a skew left brace.

Remark 8.7. In view of Proposition 8.5, $\sigma(\lambda)$ is independent on the dynamical parameter $\lambda \in H$ if and only if L has a skew left brace structure.

To the end of this section, we assume that $(L, \cdot, *)$ is a skew left brace, where $*$ is the binary operation of L defined by (8.9).

Definition 8.8. Let X be a set with a map $L \times X \ni (a,x) \mapsto ax \in X$. If X satisfies the following conditions, then we say that X is a left $(L,\cdot)$-action.

(1) For any $a, b \in L$ and $x \in X$, $(ab)x = a(bx)$.

(2) For any $x \in X$, $e_L x = x$.

Let X be a left $(L,\cdot)$-action and $f : X \to H$ be an arbitrary map. We define the map $\cdot_X : H \times X \to H$ by

$$\lambda \cdot_X x = f(\lambda x) \ (\lambda \in H, x \in X). \tag{8.15}$$

By the definition, $(X, \cdot_X)$ is an object of Set_H. For every $\lambda \in H$, we define the map $m_X(\lambda) : L \times X \to X$ by

$$m_X(\lambda)(a,x) = ax \ (a \in L, x \in X). \tag{8.16}$$

Proposition 8.9. *$m_X : L \otimes X \to X$ is a morphism of Set_H. In addition, (X, m_X) is a left (L, m, η)-module in Set_H.*

Proof. From (8.15) and the fact that X is a left $(L,\cdot)$-action,

$$\begin{aligned}\lambda \cdot_X m_X(\lambda)(a,x) =& f(\lambda(ax)) \\ =& f((\lambda a)x) \\ =& \lambda \cdot_{L\otimes X} (a,x)\end{aligned}$$

Hence $m_X : L \otimes X \to X$ is a morphism of Set_H.

We note that

$$m(\lambda)(a,b) = \overline{\lambda}\lambda ab = ab \tag{8.17}$$

for all $\lambda \in H (= L)$ and $a, b \in L$ because of the fact that $(L,\cdot)$ is a group. Since X is a left $(L,\cdot)$-action,

$$m_X(\eta \otimes 1_X)(\lambda)(\bullet, x) = e_L x = x = l_X(\lambda)(\bullet, x)$$

for all $\lambda \in H$ and $x \in X$. Hence (3.10) holds. Similarly,

$$\begin{aligned}(m_X(m \otimes 1_X))(\lambda)((a,b),x) =& (ab)x \\ =& a(bx) \\ =& (m_X(1_L \otimes m_X)a_{LLX})(\lambda)((a,b),x)\end{aligned}$$

for all $\lambda \in H, a, b \in L$ and $x \in X$. Hence (4.9) holds. This is our claim. ☐

We now consider the dynamical reflection map $k(\lambda) : L \times X \to L \times X$ $(\lambda \in H)$ (8.1) associated with $\sigma(\lambda) : L \times L \to L \times L$ (8.14) in the case that X is a left $(L,\cdot)$-action.

Let $f_x : G \to G$ $(x \in X)$ be homomorphisms of the group G. On account of (7.55), we can write $k(\lambda)(a,b)$ $(\lambda \in H, a, b \in L)$ by means of f_x.

Theorem 8.10. *The dynamical reflection map $k(\lambda)$ is independent on $\lambda \in H$ if and only if*

$$\overline{\lambda}\pi^{-1}(\pi(\lambda) f_{\lambda(ax)}(\pi(\lambda a)) f_{\lambda(ax)}(\pi(\lambda))^{-1}) = \pi^{-1}(f_{ax}(\pi(a))) \tag{8.18}$$

for all $\lambda \in H, a \in L$, and $x \in X$.

Proof. Assume that $k(\lambda)$ $(\lambda \in H)$ is independent on λ. On account of (8.1) and (8.16), we can write $k(\lambda)(a,x)$ $(\lambda \in H, a \in L, x \in X)$ as

$$k(\lambda)(a,x) = (\Pi^{\lambda}_{ax}(a), (\overline{\Pi^{\lambda}_{ax}(a)}a)x).$$

Since $k(\lambda)$ is independent on $\lambda \in H$, $\Pi^{\lambda}_{ax}(a) = \Pi^{e_L}_{ax}(a)$ for all $a \in L$ and $x \in X$. We note that $\lambda_0 \in H(= L)$ in (7.55) is exactly e_L because of Proposition 8.5. Hence the relation $\Pi^{\lambda}_{ax}(a) = \Pi^{e_L}_{ax}(a)$ implies

$$\overline{\lambda}\pi^{-1}(\pi(\lambda)f_{\lambda(ax)}(\pi(\lambda a))f_{\lambda(ax)}(\pi(\lambda))^{-1}) = \pi^{-1}(f_{ax}(\pi(a)))$$

because of (7.55) and the fact that G is a group.

Considering the above arguement in reverse, we also see that (8.18) implies the fact that $k(\lambda)$ is independent on $\lambda \in H$. This completes the proof. □

Remark 8.11. In view of the proof of Theorem 8.10, if $k(\lambda)$ $(\lambda \in H)$ is independent on λ, we can rewrite $k(\lambda)$ as

$$k(\lambda)(a,x) = (\pi^{-1}(f_{ax}(\pi(a))), (\overline{\pi^{-1}(f_{ax}(\pi(a)))}a)x) \quad (a \in L, x \in X). \tag{8.19}$$

The map stated in [8, Theorem 8.5] is exactly the same as the above map $k(\lambda)$.

Example 8.12. We assume that $(L, *)$ is abelian. From (8.9), the group G is also abelian. Then we consider Example 8.1.

Since $f_x(a) = a^{-1}$ $(a \in G, x \in X)$, together with (8.9),

$$\begin{aligned} \text{LHS of (8.18)} =& \overline{\lambda}\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)) \\ =& \overline{\lambda}(\lambda * (\lambda a)^{-1} * \lambda). \end{aligned} \tag{8.20}$$

We respectively substitute $\lambda a, \overline{a}$, and $\overline{a}$ for a, b, and c in (8.10). Then $\lambda a(\overline{a} * \overline{a}) = \lambda * (\lambda a)^{-1} * \lambda$. Substituting respectively $\overline{a}$ for b and c in (8.10), we have $a(\overline{a} * \overline{a}) = (a\overline{a}) * a^{-1} * (a\overline{a}) = a^{-1}$ because of the fact that $\pi(e_L)$ is the unit element of G. Hence

$$\text{RHS of (8.20)} = a(\overline{a} * \overline{a}) = a^{-1}.$$

a^{-1} is exactly the right-hand-side of (8.18). In view of Theorem 8.10, the dynamical reflection map $k(\lambda)$ associated with σ (8.14) is hence independent on $\lambda \in H$.

We rewrite the map $k(\lambda)$ as k. On account of (8.19), the reflection map $k : L \times X \to L \times X$ is

$$k(a,x) = (a^{-1}, (\overline{(a^{-1})}a)x)$$

for all $a \in L$ and $x \in X$.

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