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Infinite-dimensionality of the second bounded cohomology of free products of racks
(ラックの自由積の 2 次有界コホモロジーの無限次元性)

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Abstract

Recently, the bounded cohomology of racks and quandles was introduced by Kędra. In this thesis, we study the infinite-dimensionality of the second bounded cohomology of racks and quandles in comparison with that of groups.

Our main theorem is that the second bounded cohomology of a free product of finitely generated racks is infinite-dimensional as a real vector space. As a corollary, the second bounded cohomology of a free rack is infinite-dimensional.

To prove the theorem, we give an explicit construction of quasimorphisms on the free product of racks using Rolli's quasimorphism on free products of groups. This gives rise to an injective linear map from an infinite-dimensional vector space to the second bounded cohomology of the free product of racks.

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Chapter 1

Introduction

Racks (see, for example [13]) and quandles (introduced independently in [22, 28]) are algebraic objects whose axioms reflect the Reidemeister moves of a knot diagram or the corresponding properties of conjugation in a group. Racks and quandles appear in many other branches of mathematics, such as set-theoretic solutions of the Yang-Baxter equation ([11, 26]), pointed Hopf algebras ([1]), the Hurwitz action of braid groups ([3, 24, 33]), and so on.

As in the case of groups, we can consider the cohomology of racks and quandles. Rack cohomology was introduced as a topological invariant of racks in [14]. Quandle cohomology was introduced as a modification of rack cohomology in [6], and was used to construct a knot invariant called the state-sum invariant. Rack and quandle cohomology sometimes exhibit results analogous to those in group cohomology; for instance, the cohomology of free racks and free quandles vanish in degree two and higher, just as in the case of free groups ([12]).

In this thesis, we will consider bounded cohomology. Henceforth, we restrict our attention to trivial real coefficients. Bounded group cohomology was first introduced by Johnson ([21]) and Trauber (unpublished, cf. [19]) in the context of cohomology of Banach algebras. The study of bounded group cohomology developed following Gromov's pioneering paper [19]. Bounded cohomology is hard to compute in general due to the lack of excision. Nevertheless, one can still prove for certain classes of groups that the bounded cohomology is trivial or infinite-dimensional in low degrees. For example, amenable groups have trivial bounded cohomology. On the other hand, the second bounded cohomology of a (nonabelian) free group is infinite-dimensional.

To prove infinite-dimensionality, one standard approach is to use quasimorphisms. Roughly speaking, a quasimorphism is a homomorphism 'allowing a bounded error'. This notion dates back to Poincaré's rotation number (cf. [5]). Brooks ([4])

first provided a way to construct infinitely many quasimorphisms on the free group on two generators, and then Mitsumatsu ([29]) gave a rigorous proof of infinite-dimensionality using these quasimorphisms. The viewpoint of geometric group theory provides tools to study such notions involving bounded errors, and allows us to extend Brooks's construction to various groups. Using those quasimorphisms, it is known that the second bounded cohomology is infinite-dimensional for hyperbolic groups ([9]), (amalgamated) free products ([17]), groups acting on a hyperbolic space in a 'good' manner ([16, 2]), and so on.

In [25], Kędra introduced the bounded cohomology, metric, and quasimorphisms for racks and quandles, and studied the relation of the geometry of the metric and amenability condition with the bounded cohomology. The rack metric is a metric on a connected component of a rack introduced in analogy with the word metric of a group. In relation to the metric, Kędra proved that the unboundedness of a rack and a quandle with respect to the metric is equivalent to the nontriviality of the second bounded cohomology. For finite racks, Etingof-Graña [10] proved that the cohomology is trivial in a certain sense. In relation to amenability, Kędra proved the analogous 'vanishing result' for racks whose inner automorphism groups are bounded and amenable.

We next ask when bounded cohomology is infinite-dimensional. As mentioned above, it is known that for free groups, the second (ordinary) cohomology is trivial but the second bounded cohomology is infinite-dimensional. In the cases of racks and quandles, Farinati-Guccione-Guccione [12] showed that the second (and higher) cohomology of free racks and free quandles vanishes. In [25], Kędra observed that a free quandle (and thus a free rack) is unbounded with respect to the rack metric (Corollary 3.16), and thus its second bounded cohomology does not vanish (Proposition 5.7). Therefore, it is expected that the second bounded cohomology of free racks and free quandles is infinite-dimensional. Moreover, since the free product of racks and quandles under certain condition is unbounded, the second bounded cohomology of free products is also expected to be infinite-dimensional.

As our main result, we give an affirmative answer to this question.

Theorem 1.1. Let X_s ($s \in S$) be a family of finitely generated racks, where $2 \leq |S| < \infty$, and $X = \ast_{s \in S} X_s$ the free product. Then the second bounded cohomology $H_b^2(X, \mathbb{R})$ is infinite-dimensional as a real vector space.

Since the free rack on a set S is the free product of trivial racks $\{s\}$ ($s \in S$), we immediately obtain the following corollary:

Corollary 1.1.1 (cf. 5.7 in Saraf-Singh [32]). Let X be the free rack of finite rank ≥ 2 . Then $H_b^2(X, \mathbb{R})$ is infinite-dimensional.

Since those results also hold for quandles, we will restrict to racks hereafter. Our strategy is the same as in the group case; we consider quasimorphisms and construct an injective linear map from an infinite-dimensional space to the second bounded cohomology via quasimorphisms.

Saraf-Singh [32] studied bounded quandle cohomology with application to knot theory in mind. They provide two sufficient conditions for the infinite-dimensionality; one is from homogeneous group quasimorphisms, and the other is related to stable commutator length. As applications, they prove the infinite-dimensionality of the second bounded quandle cohomology for link quandles and free quandles. The latter is obtained as the corollary of the infinite-dimensionality for the free product, similar to our approach. Although they also use quandle quasimorphisms, their proof of the sufficient condition for the free product depends on the assumption that at least one of the factors has an amenable inner automorphism group.

Our results suggest that the viewpoint of geometric group theory or its analogy to racks and quandles might be useful to study bounded cohomology of racks and quandles, and racks and quandles themselves. In response to introduction of rack metric by Kędra, the study of racks and quandles from a geometric viewpoint is also developing in Iwamoto-Kai-Kodama [20].

The organization of this thesis is as follows. We first recall basic concepts and facts in chapters 2 and 3. We will use quasimorphisms on racks to prove our results. Our main theorem is proved in chapter 4. Here we propose a method to construct a quasimorphism on the free product of racks using a quasimorphism on the free product of groups which was introduced by Rolli in [31]. This construction provides an injective linear map from an infinite-dimensional vector space to the second bounded cohomology. We will give another proof of Corollary 4.1.1 at the end of the chapter. A similar argument also works when we use a homogeneous group quasimorphism instead of the quasimorphism of Rolli. We obtain the injective linear map from the space of homogeneous quasimorphisms on the adjoint group of the free product of racks to the second bounded cohomology of the free product. The space of homogeneous quasimorphisms on the adjoint group of a free racks is also infinite-dimensional. Indeed, this follows from the facts that the adjoint group of the free rack is the free group, that the free group (of rank ≥ 2) is non-elementary word-hyperbolic, and that the second bounded cohomology of any non-elementary word-hyperbolic group is infinite-dimensional. We conclude the thesis with an observation on Kabaya's chain map which induces a homomorphism from bounded group cohomology to bounded rack cohomology. Although pulling back a group quasimorphism by this induced homomorphism is seemingly a more natural way to obtain a rack quasimorphism than our method, we observe that this gives rise to only a trivial

rack quasimorphism.

Chapter 2

Racks and Quandles

In this chapter, we recall the basics of racks and quandles. The adjoint group of a rack and the free product of racks are important in our argument. We refer to [13] and [30] for details. We conclude this chapter by summarizing facts on the rack metric and boundedness of a rack with respect to this metric which are studied in [25].

As mentioned in the Introduction, we omit the quandle cases. Unless otherwise mentioned, each statement about racks also holds for quandles.

2.1 Definition and examples

We first recall the definitions and examples of racks and quandles.

A *rack* is a set X together with a binary operation $\triangleleft: X \times X \rightarrow X$ satisfying the following axioms:

1. the *rack identity*: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ for any $x, y, z \in X$, and
2. a map $\sigma_y: X \rightarrow X$ defined by $\sigma_y(x) = x \triangleleft y$ is bijective for any $y \in X$.

A *quandle* is a rack $(X, \triangleleft)$ satisfying

3. $x \triangleleft x = x$ for any $x \in X$.

We sometimes call σ_y the *(point) symmetry* at y . We write $\sigma_y^n(x) = x \triangleleft^n y$ for each $n \in \mathbb{Z}$.

A *rack homomorphism* between racks is a map $f: X \rightarrow Y$ with $f(x \triangleleft y) = f(x) \triangleleft f(y)$. In the case of quandles, it is called a *quandle homomorphism*.

Here are examples of racks, which are also quandles.

Example 2.1. A set X together with the operation $x \triangleleft y = x$ is a rack, called the *trivial rack*. When we consider it as a quandle, we call it the *trivial quandle*.

Example 2.2. Let G be a group. A set G together with the operation $g \triangleleft h = h^{-1}gh$ is a rack, called the *conjugacy rack* (or the *conjugation rack*) of G . As a quandle, we call it the *conjugacy quandle*.

Example 2.3. Let R_n be the set of reflections in the dihedral group D_n of order $2n$. Then R_n is a quandle by the conjugation, called the *dihedral quandle* of order n . As a rack, we call it the *dihedral rack*.

Example 2.4. For a group G , there is another operation which makes G into a rack, moreover a quandle. The set G together with the operation $g \triangleleft h = hg^{-1}h$ is a rack, called the *core rack* of G .

Example 2.5. Let Λ be the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$ in the variable t . A Λ -module M together with the operation $a \triangleleft b = ta + (1 - t)b$ is a quandle, called the *Alexander quandle*.

There are racks which are not quandles. Here we give an example.

Example 2.6. The set $\mathbb{Z}/n$ together with the operation $x \triangleleft y = x + 1 \pmod n$ is a rack, called the *cyclic rack* of order n . Since $x \triangleleft x = x + 1$, this can not be a quandle.

2.2 Groups associated to racks

There are several groups constructed from racks. Here we recall the adjoint group and the inner automorphism group.

Let X be a rack. The *adjoint group* of X is a group $\text{Ad}(X)$ presented as

$$\langle e_x \ (x \in X) \mid e_x e_y = e_y e_{x \triangleleft y} \rangle. \quad (2.1)$$

Since a rack homomorphism $f: X \rightarrow Y$ induces a group homomorphism $f_{\sharp}: \text{Ad}(X) \rightarrow \text{Ad}(Y)$, this gives rise to the functor $\text{Ad}(-)$ from the category of racks to the category of groups. This is left adjoint to the functor $\text{Conj}(-)$ which makes a group into the conjugacy rack.

The adjoint group $\text{Ad}(X)$ acts on X by $x \cdot e_y = x \triangleleft y$. In this thesis, the *action* of $\text{Ad}(X)$ on X means this action. A *connected component* is an orbit of this action. We write the set of connected components of X by $\pi_0(X)$. A rack X is (*algebraically*) *connected* if the action of $\text{Ad}(X)$ is transitive.

Remark 2.7. Adjoint groups are also called *associated* groups, *enveloping* groups, or *structure* groups in several contexts.

Example 2.8. The adjoint group of a trivial rack T is the free abelian group generated by T . In particular, that of the trivial rack $\{*\}$ with the single element is $\mathbb{Z}$.

The following lemma is used to prove the infinite-dimensionality of the space we use later.

Lemma 2.9. The adjoint group for any nonempty rack is an infinite group. In particular, an element e_x has infinite order.

Proof. We consider the abelianization $p: \text{Ad}(X) \rightarrow \text{Ad}(X)^{\text{ab}}$.

The abelianization $\text{Ad}(X)^{\text{ab}}$ is isomorphic to the group presented as

$$\langle e_x(x \in X) \mid e_x = e_{x \triangleleft y} \rangle.$$

Thus it is isomorphic to $\mathbb{Z}^m$, where m is the number of connected components. Since the map p is surjective, $\text{Ad}(X)$ is infinite.

For each $e_x \in \text{Ad}(X)$, the image $p(e_x)$ has infinite order. Since the order is non-increasing with respect to a homomorphism, e_x also has infinite order. $\square$

The *automorphism group* of X is the group, $\text{Aut}(X)$, of the automorphisms on X . The *inner automorphism group* of X is the subgroup, $\text{Inn}(X)$, generated by the symmetries σ_x for all $x \in X$.

Since we have

$$(x \triangleleft y) \cdot e_z = (x \triangleleft y) \triangleleft z \quad \text{and} \quad (x \cdot e_z) \triangleleft (y \cdot e_z) = (x \triangleleft z) \triangleleft (y \triangleleft z),$$

$\text{Ad}(X)$ acts on X by inner automorphisms. For a corresponding group homomorphism $\sigma_X: \text{Ad}(X) \rightarrow \text{Inn}(X)$, $\sigma_X(e_x) = \sigma_x$, it is well-known that the adjoint group is a central extension of the inner automorphism group.

Proposition 2.10 (2.37 in [8]). There is a short exact sequence of groups

$$0 \rightarrow \ker \sigma_X \rightarrow \text{Ad}(X) \xrightarrow{\sigma_X} \text{Inn}(X) \rightarrow 1.$$

Moreover, the kernel is contained in the center of $\text{Ad}(X)$.

2.3 Free products

As in the case of groups, we have a free rack and a free product of racks. We give explicit constructions by following [12], [13] and [22].

Let S be a set and $F^G(S)$ the free group on S . We write its identity element by 1.

Let S be a set. The *free rack* on S is a set $F^R(S) = S \times F^G(S)$ together with the operation

$$(s, g) \triangleleft (t, h) = (s, gh^{-1}th). \quad (2.2)$$

The *free quandle* on S is the union of the conjugacy classes of elements of S inside $F^G(S)$, $F^Q(S) = \bigcup_{s \in S} \{g^{-1}sg \in F^G(S) \mid g \in F^G(S)\}$, with the operation $\alpha \triangleleft \beta = \beta^{-1}\alpha\beta$.

Both free racks and free quandles are also characterized by the universal property (cf. 1.3 in [12] for racks, 4.1 in [22] for quandles).

Proposition 2.11. For any rack X and any map $f: S \rightarrow X$, there is a unique homomorphism $\tilde{f}: F^R(S) \rightarrow X$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & X \\ \downarrow \iota & \nearrow \tilde{f} & \\ F^R(S) & & \end{array} \quad (2.3)$$

commutes, where $\iota: S \rightarrow F^R(S)$ is a natural inclusion $s \mapsto (s, 1)$. This also holds for the free quandles.

Remark 2.12. Since

$$(s, g) \triangleleft (s, g) = (s, sg)$$

for any $(s, g) \in F^R(S)$, the free rack is not a quandle.

The free product of groups is well-known. The free product of racks is introduced by Fenn-Rourke [13].

Let S be a set. For a family of racks X_s ($s \in S$), the *free product* is a rack $\ast_{s \in S} X_s$ consists of elements of the form (x, g) where $x \in X_t$ for some $t \in S$ and $g \in \ast_{s \in S} \text{Ad}(X_s)$ under the equivalence generated by

$$(x, gw) \sim (x \cdot g, w) \quad (2.4)$$

where $x \in X_t$, $g \in \text{Ad}(X_t)$ for each $t \in S$, and $w \in \ast_{s \in S} \text{Ad}(X_s)$. The operation is

$$(x, g) \triangleleft (y, h) = (x, gh^{-1}e_yh). \quad (2.5)$$

This is well-defined by the definition of the adjoint group. For any $(x, g) \in \ast_{s \in S} X_s$, we have

$$(x, g) \triangleleft (x, g) = (x \cdot e_x, g) = (x \triangleleft x, g).$$

Therefore, the free product is a quandle if and only if all the X_s are quandles.

The free product of racks satisfies the universal property.

Proposition 2.13 (2.4 in [13]). For any rack Y and any homomorphisms $f_s: X_s \rightarrow Y$, there is unique homomorphism $f: \ast X_s \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y \\ \downarrow \iota_s & \nearrow f & \\ \ast X_s & & \end{array} \quad (2.6)$$

commutes, where $\iota_s: X_s \hookrightarrow \ast X_s$ are natural inclusions $x \mapsto (x, 1)$.

Since the adjoint group of a trivial rack of the one element set $\{\ast\}$ is $\mathbb{Z}$, the free rack $F^R(S)$ is the free product of the trivial racks $\{s\}$ ($s \in S$). Since a left adjoint functor preserves colimits,

Proposition 2.14. The adjoint group of the free product is the free product of the adjoint groups: $\text{Ad}(\ast X_s) \cong \ast \text{Ad}(X_s)$.

In particular, the adjoint group of the free rack $F^R(S)$ is the free group $F^G(S)$.

Remark 2.15. The quotient $F^R(S)/\sim$ under the equivalence generated by $(s, g) \sim (s, sg)$ for any $s \in S$ and $g \in F^G(S)$ is a quandle. This quotient can be identified with the free quandle $F^Q(S)$ defined above by associating a conjugation $g^{-1}sg$ with the class represented by (s, g) . The above equivalence is descended from the equivalence (2.4). Thus, we can say that the free quandle on S is the free product of the trivial quandles $\{s\}$ as in the case of racks.

2.4 Finitely generated racks

A rack X is *finitely generated* if there is a finite subset $S \subset X$ such that, for any $x \in X$, there are an integer $n \geq 0$, elements $s_0, s_1, \dots, s_n \in S$, and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ such that

$$x = s_0 \triangleleft^{\varepsilon_1} s_1 \triangleleft \dots \triangleleft^{\varepsilon_n} s_n. \quad (2.7)$$

We shall use the following observation later.

Lemma 2.16. If a rack X is finitely generated by S , then the adjoint group $\text{Ad}(X)$ is finitely generated by $\{e_s \mid s \in S\}$.

Proof. By definition, the adjoint group $\text{Ad}(X)$ is generated by $\{e_x \mid x \in X\}$. Since a rack X is finitely generated by S , each generator e_x can be written in the form

$$e_x = e_{s_n}^{-\varepsilon_n} \cdots e_{s_1}^{-\varepsilon_1} \cdot e_{s_0} \cdot e_{s_1}^{\varepsilon_1} \cdots e_{s_n}^{\varepsilon_n} \quad (2.8)$$

for some $s_0, s_1, \dots, s_n \in S$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$. $\square$

2.5 Coset presentation

In [32], Saraf-Singh obtained sufficient conditions of infinite-dimensionality using coset presentation of quandles. Here we briefly recall coset presentation of racks and quandles, and establish a comparison between them.

Let G be a group, $S \subset G$ a subset, and $H_s \leq C_G(s)$ a subgroup for each $s \in S$. Then the set $\bigsqcup_{s \in S} H_s \backslash G$ together with the operation

$$H_s g \triangleleft H_t h = H_s g h^{-1} t h \quad (2.9)$$

is a rack, called the *coset rack*. We write this by $\bigsqcup_{s \in S} (H_s \backslash G, s)$.

Lemma 2.17. The coset rack $\bigsqcup_{s \in S} (H_s \backslash G, s)$ is a quandle if and only if $s \in H_s$ for each $s \in S$.

Proof. For any $H_s g \in \bigsqcup_{s \in S} H_s \backslash G$, we have

$$H_s g \triangleleft H_s g = H_s g g^{-1} s g = H_s s g = H_s g. \quad \square$$

If we define another operation by

$$H_s g \triangleleft_q H_t h = H_s s^{-1} g h^{-1} t h, \quad (2.10)$$

then this makes the cosets into a quandle. We call this the *coset quandle*. We write this by $\bigsqcup_{s \in S} (H_s \backslash G, s)_q$.

Proposition 2.18 ([22, 30, 25]). Every rack (resp. quandle) is isomorphic to a coset rack (resp. a coset quandle).

Proposition 2.19 (5.4 in [32]). A coset presentation of a free product of racks is a coset rack of a free product of groups.

For given triple of a group G , a subset $S \subset G$ and subgroups $H_s \leq C_G(s)$ ($s \in S$), the coset rack and the coset quandle are related in the following way. Recall Brieskorn's construction which makes a rack $(X, \triangleleft)$ into a quandle with the same underlying set X ([3], see also [1]). Let $\iota \in C_{\text{Aut}(X)}(\text{Inn}(X))$. Define an operation $\triangleleft_\iota: X \times X \rightarrow X$ by

$$x \triangleleft_\iota y := \iota(x) \triangleleft y.$$

Then $(X, \triangleleft_\iota)$ is again a rack *conjugated via ι* . Moreover, if ι satisfies $\iota(x) \triangleleft x = x$, then $(X, \triangleleft_\iota)$ is a quandle, called the quandle *associated to $(X, \triangleleft)$* .

Lemma 2.20. A coset quandle is the quandle associated to a coset rack.

Proof. When we define ι by

$$\iota(H_s g) := H_s s^{-1} g,$$

then this satisfies $\iota(H_s g) \triangleleft H_s g = H_s g$. Clearly, ι lies in the centralizer of the inner automorphism group, and the operation $\triangleleft_q$ coincides with $\triangleleft_\iota$. $\square$

2.6 Geometric viewpoint

In [25], Kędra introduced a metric on racks and studied the relation between the boundedness of a rack and its bounded cohomology. We conclude this chapter by summarizing the facts on metrics and boundedness of racks.

Let $X_0 \subset X$ be a connected component of a rack X . A *rack metric* on X_0 is defined by

$$d_{X_0}(x, y) = \min \{ n \mid y = x \triangleleft^\pm x_1 \triangleleft^\pm \cdots \triangleleft^\pm x_n \}.$$

A rack X is *bounded* if each component has finite diameter with respect to the rack metric. Otherwise, X is *unbounded*.

Remark 2.21 ([25]). This metric coincides with the graph metric of the Cayley graph of a rack X with respect to X itself as a generating set.

Lemma 2.22 (3.4 in [25]). If a group G is bounded, then any coset rack for G is bounded.

Example 2.23 ([25]). The cyclic rack $\mathbb{Z}$ is unbounded.

The following fact ensures that a free rack has nontrivial second bounded cohomology.

Example 2.24 (3.13 in [25]). Any coset rack from a free group FG_n of rank $n \geq 2$ and its finite normal generating set S is unbounded.

Proposition 2.25 (3.14 in [25]). Let X, Y be racks. If X and Y have finitely many connected components, and the maps $X \rightarrow \text{Ad}(X), Y \rightarrow \text{Ad}(Y)$ are injective, then every connected component of the free product $X * Y$ has infinite diameter.

Chapter 3

Bounded Cohomology and Quasimorphisms

In this chapter, we recall bounded cohomology and quasimorphisms of groups and racks. We first recall those of groups and summarize well-known facts which are related to our argument. As in the case of ordinary cohomology of racks and quandles, bounded cohomology of racks and quandles is also defined in a similar manner. Triviality for racks with amenable inner automorphism groups is obtained by Kędra [25] (Following the terminology used in the group case, he call this a ‘vanishing’ result), and sufficient condition for infinite-dimensionality for free products by Saraf-Singh [32]. For details or other results, see [5] and [15] on groups and [25, 32] on racks and quandles. Here we introduce the facts which are found in those materials without proofs.

For a function $f: Z \rightarrow \mathbb{R}$, we write $\|f\|_\infty = \sup \{ |f(z)| \mid z \in Z \}$. Since we will consider bounded cohomology and quasimorphisms, we restrict our attention to trivial real coefficients $\mathbb{R}$.

3.1 (Bounded) cohomology of groups

For a group G and a non-negative integer n , let $C^n(G; \mathbb{R}) = \text{Map}(G^n, \mathbb{R})$, the set of functions $G^n \rightarrow \mathbb{R}$. For $n < 0$, let $C^n(G; \mathbb{R}) = 0$. Here we understand G^0 to be a one-element set. We call an element $f \in C^n(G; \mathbb{R})$ a *group n -cochain*.

The coboundary map $\delta^n: C^n(G; \mathbb{R}) \rightarrow C^{n+1}(G; \mathbb{R})$ is defined by

$$\begin{aligned} \delta^n f(g_1, \dots, g_n, g_{n+1}) &= f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &\quad + (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned} \quad (3.1)$$

In the case of $n \leq 0$, we set $\delta^n = 0$. Thus, we obtain a cochain complex $C^*(G; \mathbb{R}) = (C^n(G; \mathbb{R}), \delta^n)$, called the (*inhomogeneous*) group cochain complex. Elements of submodules $Z^n(G; \mathbb{R}) = \ker \delta^n$ and $B^n(G; \mathbb{R}) = \operatorname{im} \delta^{n-1}$ are called *group n -cocycles* and *group n -coboundaries*, respectively.

The *group cohomology* is the cohomology of this complex:

$$H^n(G; \mathbb{R}) = Z^n(G; \mathbb{R}) / B^n(G; \mathbb{R}). \quad (3.2)$$

We call an element of $H^n(G; \mathbb{R})$ a (*group*) *n -cohomology class*.

Example 3.1. We demonstrate some computations.

$$\begin{aligned} \delta^1 f(g, h) &= f(h) - f(gh) + f(g) \\ \delta^2 f(g, h, k) &= f(h, k) - f(gh, k) + f(g, hk) - f(g, h). \end{aligned} \quad (3.3)$$

Notice that $\delta^1 f = 0$ if and only if f is a group homomorphism. Therefore, we have

$$H^1(G; \mathbb{R}) \cong \operatorname{Hom}(G, \mathbb{R}).$$

We recall the following basic fact for free groups.

Proposition 3.2. The second and higher cohomology of a free group is trivial.

For a group G , let $C_b^n(G; \mathbb{R})$ be the submodule of functions $G^n \rightarrow \mathbb{R}$ which are bounded with respect to $\|\cdot\|_\infty$. We call an element $f \in C_b^n(G; \mathbb{R})$ a *bounded (group) n -cochain*.

The coboundary maps δ may be restricted to $C_b^n(G; \mathbb{R})$, and $C_b^*(G; \mathbb{R}) = (C_b^n(G; \mathbb{R}), \delta^n)$ forms a cochain complex. Submodules $Z_b^n(G; \mathbb{R}) = \ker \delta^n$ and $B_b^n(G; \mathbb{R}) = \operatorname{im} \delta^{n-1}$ can also be defined, and their elements are called *bounded group n -cocycles* and *bounded group n -coboundaries*, respectively.

The *bounded group cohomology* $H_b^n(G; \mathbb{R})$ is the cohomology of this complex. We call an element of $H_b^n(G; \mathbb{R})$ a *bounded group n -cohomology class*.

The inclusion $C_b^n(G; \mathbb{R}) \hookrightarrow C^n(G; \mathbb{R})$ induces the map $c^n: H_b^n(G; \mathbb{R}) \rightarrow H^n(G; \mathbb{R})$ called the *comparison map*. The kernel of the comparison map c^n is called the *exact bounded cohomology*, and denoted by $EH_b^n(G; \mathbb{R})$.

We can easily see the following facts.

Proposition 3.3.

1. $H_b^1(G; \mathbb{R}) = 0$ for any group G .
2. If G is a finite group, then $H_b^n(G; \mathbb{R}) = 0$ for any $n > 0$.

By the latter fact, when we consider bounded cohomology, our interest is restricted to infinite groups.

A group G is amenable if it admits a left-invariant probability measure. The following vanishing result was first obtained by Trauber, then by Johnson in more general setting (cf. Grigorchuk [18]).

Theorem 3.4. The bounded cohomology of an amenable group vanishes in positive degrees.

3.2 Quasimorphisms of groups

A *group quasimorphism* on a group G is a function $\phi: G \rightarrow \mathbb{R}$ satisfying

$$D(\phi) = \sup_{g, h \in G} |\phi(g) + \phi(h) - \phi(gh)| < \infty. \quad (3.4)$$

The constant $D(\phi)$ is called the *defect* of ϕ . In terms of group cohomology, ϕ is a quasimorphism if and only if the cocycle $\delta^1 \phi$ is bounded. We write the vector space of quasimorphisms on G by $Q(G)$.

The space of quasimorphisms $Q(G)$ is related to the second exact bounded cohomology $EH_b^2(G; \mathbb{R})$ as follows.

Proposition 3.5 (cf. Proposition 2.8 in [15]). There exists an exact sequence

$$0 \rightarrow C_b^1(G; \mathbb{R}) \oplus \text{Hom}(G, \mathbb{R}) \hookrightarrow Q(G) \xrightarrow{\delta} EH_b^2(G; \mathbb{R}) \rightarrow 0,$$

where the map δ is induced from the coboundary map δ^1 . In particular,

$$EH_b^2(G; \mathbb{R}) \cong Q(G) / C_b^1(G; \mathbb{R}) \oplus \text{Hom}(G, \mathbb{R}).$$

Accordingly, we say that a group quasimorphism $f: G \rightarrow \mathbb{R}$ is *trivial* if $f \in C_b^1(G; \mathbb{R}) \oplus \text{Hom}(G, \mathbb{R})$.

A quasimorphism $\phi: G \rightarrow \mathbb{R}$ is *homogeneous* if

$$\phi(g^n) = n \cdot \phi(g)$$

for any $g \in G$ and $n \in \mathbb{Z}$. We write the subspace of homogeneous quasimorphisms by $Q^h(G)$. The following property of homogeneous quasimorphisms is well-known.

Lemma 3.6 (cf. 3.4 in [18]). A homogeneous quasimorphism is constant on each conjugacy class.

Remark 3.7. In [18], a quasimorphism is called a *quasicharacter*, and a homogeneous quasimorphism a *pseudocharacter*.

The space of homogeneous quasimorphisms $Q^h(G)$ is related to $Q(G)$ and $EH_b^2(G; \mathbb{R})$ as follows.

Proposition 3.8. There are isomorphisms

$$Q(G) \cong Q^h(G) \oplus C_b^1(G) \quad \text{and} \quad EH_b^2(G; \mathbb{R}) \cong Q^h(G)/\text{Hom}(G; \mathbb{R}).$$

Proposition 3.9. There is an exact sequence

$$0 \rightarrow H^1(G) \rightarrow Q^h(G) \rightarrow H_b^2(G) \xrightarrow{c^2} H^2(G).$$

In contrast to the ordinary cohomology, the second bounded cohomology of a free group is nontrivial (Proposition 2.8 in [21]); indeed, it is infinite-dimensional.

Proposition 3.10. If G is a free group, then $H_b^2(G; \mathbb{R})$ is infinite-dimensional.

There are many approaches to proving this statement. Brooks [4] first constructed an infinite family of quasimorphisms in the case of a free group of rank 2, and then Mitsumatsu [29] gave a rigorous proof of the infinite-dimensionality. Rolli [31] proposed another family of quasimorphisms to prove the infinite-dimensionality, and extended the construction to free products of groups, as explained in 4.2.

3.3 (Bounded) cohomology of racks

For a rack X and a non-negative integer n , let $C^n(X; \mathbb{R}) = \text{Map}(X^n, \mathbb{R})$, the set of functions $X^n \rightarrow \mathbb{R}$. For $n < 0$, let $C^n(X; \mathbb{R}) = 0$. Here we understand X^0 to be a one-element set. The coboundary operator $\delta^n: C^n(X; \mathbb{R}) \rightarrow C^{n+1}(X; \mathbb{R})$ is defined by

$$\begin{aligned} \delta^n f(x_1, \dots, x_n, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^i [f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \\ &\quad - f(x_1 \triangleleft x_i, \dots, x_{i-1} \triangleleft x_i, x_{i+1}, \dots, x_{n+1})]. \end{aligned} \tag{3.5}$$

In the case of $n \leq 0$, we define $\delta^n = 0$. Then we have the following as usual.

Lemma 3.11. $\delta^{n+1} \circ \delta^n = 0$ for any $n \geq 0$.

Thus, we obtain a cochain complex $C^*(X; \mathbb{R}) = (C^n(X; \mathbb{R}), \delta^n)$. The *rack cohomology* is the cohomology of this complex

$$H^n(X; \mathbb{R}) = \ker \delta^n / \operatorname{im} \delta^{n-1}. \quad (3.6)$$

If X is a quandle, the *quandle cochain complex* of X is the subcomplex $D^*(X; \mathbb{R})$ defined by

$$D^n(X; \mathbb{R}) = \{f \in C^n(X; \mathbb{R}) \mid f(x) = 0 \text{ if } x_i = x_{i+1} \text{ for some } i\} \quad (3.7)$$

for $n \geq 0$. The *quandle cohomology* of X is the cohomology of this complex.

Example 3.12 (Degree 0). Obviously, we have $H^0(X; \mathbb{R}) \cong \mathbb{R}$.

Example 3.13 (Coboundary maps in low degree). We demonstrate some computations.

$$\begin{aligned} \delta^1 f(x, y) &= f(x) - f(x \triangleleft y) \\ \delta^2 f(x, y, z) &= f(x, z) - f(x, y) - f(x \triangleleft y, z) + f(x \triangleleft z, y \triangleleft z) \end{aligned} \quad (3.8)$$

Example 3.14 (Triviality in degree 1: ordinary case). If $\mathbb{R}$ is equipped with the structure of a trivial rack, then $\delta^1 f = 0$ if and only if f is a rack homomorphism as in the case of groups. Moreover, we notice that this is also equivalent to f being constant on each connected component. Thus, we have

$$H^1(X; \mathbb{R}) \cong \operatorname{Hom}(X, \mathbb{R}) \cong \operatorname{Map}(\pi_0(X), \mathbb{R}).$$

Although the cohomology of ‘the’ trivial group vanishes, that of ‘a’ trivial rack does not literally vanish in general.

Example 3.15 (Trivial racks). Let T be a trivial rack. Then $\delta^n = 0$ for all n . Thus, we have

$$H^n(T; \mathbb{R}) \cong C^n(T; \mathbb{R}) = \operatorname{Map}(T^n, \mathbb{R}).$$

When T is a trivial quandle, similarly we have

$$H^n(T; \mathbb{R}) \cong D^n(T; \mathbb{R}).$$

On the other hand, the cohomology of free racks and free quandles vanishes in degree ≥ 2 as in the case of groups.

Theorem 3.16 (3.2 and 4.2 in [12] with $\mathbb{R}$ coefficients).

$$H^n(F^R(S); \mathbb{R}) = H^n(F^Q(S); \mathbb{R}) = \begin{cases} \mathbb{R}^{|S|} & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (3.9)$$

Remark 3.17. In [12], they use the complex $C_n(X)$ generated by X^{n+1} . Degrees in the above statement are shifted by +1 from the original statement in [12] to fit into our setting.

The bounded cohomology of a rack is defined similarly to that of a group. For a function $f: Z \rightarrow \mathbb{R}$, we write $\|f\|_\infty = \sup \{ |f(z)| \mid z \in Z \}$. For a rack X , let $C_b^n(X; \mathbb{R})$ be the submodule of functions $X^n \rightarrow \mathbb{R}$ which are bounded with respect to $\|\cdot\|_\infty$. The coboundary operators δ^n may be restricted to $C_b^n(X; \mathbb{R})$, and $C_b^*(X; \mathbb{R}) = (C_b^n(X; \mathbb{R}), \delta^n)$ forms a cochain complex. The *bounded rack cohomology* $H_b^n(X; \mathbb{R})$ is the cohomology of this complex. The *bounded quandle cohomology* is the quandle cohomology in this sense.

The *comparison map* $c^n: H_b^n(X; \mathbb{R}) \rightarrow H^n(X; \mathbb{R})$ and the *exact* bounded cohomology $EH_b^n(X; \mathbb{R})$ are also defined as in the case of groups.

Example 3.18 (Degree 0). Obviously, we have $H_b^0(X; \mathbb{R}) = H^0(X; \mathbb{R}) \cong \mathbb{R}$.

Example 3.19 (Triviality in degree 1: bounded case, 4.3 in [25]). Similar to the case of ordinary cohomology, the first bounded rack cohomology bijectively corresponds to the set of bounded functions on the set of connected components:

$$H_b^1(X; \mathbb{R}) \cong \text{Map}_b(\pi_0(X), \mathbb{R}). \quad (3.10)$$

Since, for groups, vanishing in degree one follows from the fact that the only bounded homomorphism is the trivial one, (3.10) can be viewed as the corresponding generalized triviality for first bounded rack cohomology.

This generalization of triviality is also seen in higher bounded cohomology for a trivial rack.

Example 3.20 (4.4 in [25]). Let T be a trivial rack. We immediately have

$$H_b^n(T; \mathbb{R}) \cong C_b^n(T; \mathbb{R}) = \text{Map}_b(T^n, \mathbb{R}).$$

Similar to the ordinary case, if T is a trivial quandle, then we have

$$H_b^n(T; \mathbb{R}) \cong D_b^n(T; \mathbb{R}).$$

In this sense, the trivializing effect of amenability is observed by Kędra.

Theorem 3.21 (6.3 (2) in [25]). If the inner automorphism group of a rack X is amenable and has finite diameter with respect to the word norm associated with the symmetries σ_x , then

$$H_b^n(X; \mathbb{R}) \cong \text{Map}_b(\pi_0(X)^n, \mathbb{R})$$

for any n .

3.4 Quasimorphisms of racks

A *rack quasimorphism* on a rack X is a function $\phi: X \rightarrow \mathbb{R}$ satisfying

$$D(\phi) = \sup_{x, y \in X} |\phi(x) - \phi(x \triangleleft y)| < \infty. \quad (3.11)$$

The constant $D(\phi)$ is called the *defect* of ϕ . As in the case of group quasimorphisms, ϕ is a rack quasimorphism if and only if $\delta^1 \phi$ is bounded by the definition of the rack coboundary map. We write the space of quasimorphisms on X by $Q(X)$.

There is a relation analogous to Proposition 3.5 between the space of rack quasimorphisms and the second exact bounded rack cohomology.

Proposition 3.22 (cf. 3.2 in [32]). There exists an exact sequence

$$0 \rightarrow C_b^1(X; \mathbb{R}) + \text{Hom}(X, \mathbb{R}) \hookrightarrow Q(X) \xrightarrow{\delta} EH_b^2(X; \mathbb{R}) \rightarrow 0,$$

where the map δ is induced from the coboundary map δ^1 . In particular,

$$EH_b^2(X; \mathbb{R}) \cong Q(X) / (C_b^1(X) + \text{Hom}(X, \mathbb{R})).$$

The same statement also holds for quandles (cf. 3.3 in [32]).

Accordingly, we shall say that a rack quasimorphism $f: X \rightarrow \mathbb{R}$ is *trivial* if $f \in C_b^1(X; \mathbb{R}) + \text{Hom}(X, \mathbb{R})$.

The following is the key observation that rack quasimorphisms give rise to bounded rack cohomology classes.

Lemma 3.23 (5.2 in [25]). If $\phi: X \rightarrow \mathbb{R}$ is a rack quasimorphism, then $\delta^1 \phi \in C_b^2(X; \mathbb{R})$ is a cocycle, and $[\delta^1 \phi] \in \ker c^2$. Moreover, if ϕ is unbounded on a connected component of X , then the bounded rack 2-cohomology class $[\delta^1 \phi]$ is non-trivial.

Example 3.24 (5.3 in [25]). Let FQ_2 be a free quandle on two generators x, y . For given homogeneous group quasimorphism $\phi: FG_2 \rightarrow \mathbb{R}$, we have a quandle quasimorphism $\hat{\phi}: FQ_2 \rightarrow \mathbb{R}$ defined by

$$\hat{\phi}(g^{-1}xg) = \phi(g'),$$

where g' is a reduced word ending in y and $g = x^n g'$ for some n , and $\hat{\phi}(g^{-1}yg)$ in the same way. Then $\hat{\phi}$ is unbounded since $\hat{\phi}(y^{-n}xy^n) = n \cdot \phi(y)$, and $\delta^1 \hat{\phi}$ represents a non-trivial bounded rack 2-cohomology class.

We conclude this chapter by presenting results on infinite-dimensionality obtained by Saraf-Singh [32].

Let G be a group. For an element g of the commutator subgroup $[G, G]$, the *commutator length*, $\text{cl}(g)$, is the least number of commutators in G whose product is equal to g , and the *stable commutator length* is the limit

$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n}.$$

Theorem 3.25 (4.5 in [32]). Let G be a group, $S \subset G$ a subset, $H_s \leq C_G(s)$ a subgroup for each $s \in S$, and $X = \bigsqcup (H_s \backslash G, s)$ the coset quandle. If G is finitely presented, $H_b^2(G; \mathbb{R})$ is infinite-dimensional, and there is $s_0 \in S$ such that H_{s_0} is finitely generated and $\text{scl}(h) = 0$ for each $h \in [H_{s_0}, H_{s_0}]$, then $H_b^2(X; \mathbb{R})$ is infinite-dimensional.

This is used to prove the infinite-dimensionality for link quandles.

Theorem 3.26 (5.1, 5.5 in [32]). Let L be an oriented link. If one of the following conditions holds, then the second bounded cohomology of the link quandle $\pi^Q(L)$ is infinite-dimensional:

1. L is non-splitting and the link group $\pi^G(L)$ is not solvable, or
2. L is splitting with more than one component.

The following is Saraf-Singh's sufficient condition for the infinite-dimensionality of the second bounded quandle cohomology of free products of quandles.

Proposition 3.27 (5.6 in [32]). Let X_s ($s \in S$, $|S| \geq 2$) be quandles. If there is at least one $s \in S$ such that $\text{Inn}(X_s)$ is amenable, then $H_b^2(\ast_{s \in S} X_s; \mathbb{R})$ is infinite-dimensional.

As a corollary, they also obtained the infinite-dimensionality for free quandles (Corollary 1.1.1).

Chapter 4

Infinite-Dimensionality for Free Products

In this chapter, we prove the main theorem that the second bounded cohomology of the free product of racks is infinite-dimensional. As a corollary, we obtain infinite-dimensionality for free racks. After stating our theorems, we first recall Rolli's construction of quasimorphisms on the free product of groups. We use these quasimorphisms to construct quasimorphisms of the free product of racks. This gives rise to a linear injection from an infinite-dimensional vector space to the second bounded rack cohomology. We conclude this chapter by giving another proof of the corollary of the theorem using homogeneous group quasimorphisms.

4.1 Main theorem

Our main theorem is infinite-dimensionality for the second bounded cohomology of free products of racks.

Theorem 4.1. Let $X = \ast_{s \in S} X_s$ be the free product of finitely generated racks with $2 \leq |S| < \infty$. Then the second bounded cohomology $H_b^2(X, \mathbb{R})$ is infinite-dimensional as a real vector space.

Since the free rack on a set S is the free product of trivial racks $\{s\}$ ($s \in S$), we immediately obtain the following corollary:

Corollary 4.1.1. Let X be the free rack of finite rank ≥ 2 . Then $H_b^2(X, \mathbb{R})$ is infinite-dimensional.

Remark 4.2. Using Proposition 3.27, Saraf-Singh have proved this as Corollary 5.7 in [32]. Our approach does not need amenability of inner automorphism groups.

4.2 Rolli's construction

Here we recall Rolli's construction of group quasimorphisms on the free product of groups, and then give some technical remarks on the elements in the free product of racks. We will use those observations in our argument.

Let G, H be groups. A sequence of elements $g_1, \dots, g_n$ of the free product $G * H$ is *reduced* if each $g_i \neq 1$, each g_i is in one of the factors G or H , and successive g_i, g_{i+1} are not in the same factor. We use the following fact (cf. Theorem 1.2 (II) in Section IV.1 of [27]).

Theorem 4.3 (Normal Form Theorem). Each element $g \neq 1$ in the free product $G * H$ can be uniquely expressed as a product $g = g_1 \cdots g_n$ where $g_1, \dots, g_n$ is a reduced sequence.

Such a product $g = g_1 \cdots g_n$ is called the *factorization* of g , and we will call each g_i a *syllable* in the factorization.

Let $\lambda = (\lambda_s)_{s \in S}$ be a uniformly bounded family of odd bounded functions, that is, each function $\lambda_s: \Gamma_s \rightarrow \mathbb{R}$ is bounded with respect to $\|\cdot\|_\infty$ and satisfies $\lambda_s(g^{-1}) = -\lambda_s(g)$ for any $g \in \Gamma_s$, and $\sup\{\|\lambda_s\|_\infty \mid s \in S\} < \infty$.

Proposition 4.4 (Proposition 4.1 in [31]). Let $\Gamma = \ast_{s \in S} \Gamma_s$ be the free product of (non-trivial) groups with $2 \leq |S|$, and λ a uniformly bounded family of odd bounded functions $\lambda_s: \Gamma_s \rightarrow \mathbb{R}$. Then a map $\phi_\lambda: \Gamma \rightarrow \mathbb{R}$ defined by

$$\phi_\lambda(g) = \sum_{i=1}^n \lambda_{s_i}(g_i), \quad (4.1)$$

where $g = g_1 \cdots g_n$ is the factorization, is a group quasimorphism.

Let $X = \ast_{s \in S} X_s$ be the free product of racks X_s . Observe that for each $t \in S$, if $x \in X_t$, $g \in \ast_{s \in S} \text{Ad}(X_s)$ with the factorization $g = g_0 \cdot g_1 \cdots g_n$, and $g_0 \in \text{Ad}(X_t)$, then the element (x, g) in X can be written in the form

$$(x \cdot g_0, g_1 \cdots g_n). \quad (4.2)$$

Therefore, we can assume that each element in the free product X has the form

$$(x, g_1 \cdots g_n) \quad (4.3)$$

where $x \in X_t$ and $g_1 \in \text{Ad}(X_{s_1})$ with $s_1 \neq t$.

Definition 4.5. We say that an element of a free product is *reduced* if it has the form (4.3).

Using a quasimorphism on the free product of the adjoint groups, we can construct a quasimorphism on the free product of racks as follows.

Proposition 4.6. Let $X = \ast_{s \in S} X_s$ be the free product of racks with $2 \leq |S|$, $\lambda = (\lambda_s)_{s \in S}$ be a uniformly bounded family of odd bounded functions $\lambda_s: \text{Ad}(X_s) \rightarrow \mathbb{R}$, and ϕ_λ be the group quasimorphism constructed in Proposition 4.4. Then a map $\hat{\phi}_\lambda: X \rightarrow \mathbb{R}$ defined by

$$\hat{\phi}_\lambda(x, g) = \phi_\lambda(g), \quad (4.4)$$

where (x, g) is reduced, is a rack quasimorphism. Moreover, if $\lambda \neq 0$ then $\hat{\phi}_\lambda$ is unbounded.

Proof. Assume that $(x, g), (y, h) \in X$ are reduced with the factorizations $g = g_1 \cdots g_m$ ($g_i \in \text{Ad}(X_{s_i})$) and $h = h_1 \cdots h_n$ ($h_j \in \text{Ad}(X_{t_j})$), and that $y \in X_{t_0}$ for some t_0 .

By construction, we have

$$\hat{\phi}_\lambda(x, g) - \hat{\phi}_\lambda(x, gh^{-1}e_yh) = \phi_\lambda(g) - \phi_\lambda(gh^{-1}e_yh). \quad (4.5)$$

Using the factorizations of g and h ,

$$gh^{-1}e_yh = (g_1 \cdots g_m) \cdot (h_n^{-1} \cdots h_1^{-1}) \cdot e_y \cdot (h_1 \cdots h_n) \quad (4.6)$$

and this may not be the factorization of $gh^{-1}e_yh$. Notice that neither cancellation nor concatenation occurs in $h_1^{-1}e_yh_1$ since $t_0 \neq t_1$ by the assumption that (y, h) is reduced. Let r be the number of pairs of syllables which are cancelled between g and $h^{-1}e_yh$. The possible cancellations fall into the following cases.

Case 1: $1 \leq r < n$, that is, $g_m \cdot h_n^{-1} = 1, \dots, g_{m-r+1} \cdot h_{n-r+1}^{-1} = 1$.

(i) If $s_{m-r} \neq t_{n-r}$, then no further cancellation occurs, and the factorization is of the form

$$g_1 \cdots g_{m-r} \cdot h_{n-r}^{-1} \cdots h_1^{-1} \cdot e_y \cdot h_1 \cdots h_n. \quad (4.7)$$

(ii) If $s_{m-r} = t_{n-r}$, let $u_r := s_{m-r} = t_{n-r}$ and $k_r := g_{m-r} \cdot h_{n-r}^{-1} \in \text{Ad}(X_{u_r})$. Then the factorization is of the form

$$g_1 \cdots g_{m-r-1} \cdot k_r \cdot h_{n-r-1}^{-1} \cdots h_1^{-1} \cdot e_y \cdot h_1 \cdots h_n, \quad (4.8)$$

since $u_r \neq s_{m-r-1}, t_{n-r-1}$.

Case 2: $r = n$, that is, $g_m \cdot h_n^{-1} = 1, \dots, g_{m-n+1} \cdot h_1^{-1} = 1$.

(i) If $s_{m-n} \neq t_0$, then

$$g_1 \cdots g_{m-n} \cdot e_y \cdot h_1 \cdots h_n. \quad (4.9)$$

(ii) If $s_{m-n} = t_0$, let $u_n := s_{m-n} = t_0$ and $k_n := g_{m-n} \cdot e_y$. Then

$$g_1 \cdots g_{m-n-1} \cdot k_n \cdot h_1 \cdots h_n. \quad (4.10)$$

Case 3: $r > n$, that is, $g_m \cdot h_n^{-1} = 1, \dots, g_{m-n+1} \cdot h_1^{-1} = 1, g_{m-n} \cdot e_y = 1, g_{m-n+1} \cdot h_1 = 1, \dots, g_{m-r+1} \cdot h_{r-n-1} = 1$.

(i) If $s_{m-r} \neq t_{r-n}$, then

$$g_1 \cdots g_{m-r} \cdot h_{r-n} \cdots h_n. \quad (4.11)$$

(ii) If $s_{m-r} = t_{r-n}$, let $u_r := s_{m-r} = t_{r-n}$ and $k_r := g_{m-r} \cdot h_{r-n}$. Then

$$g_1 \cdots g_{m-r-1} \cdot k_r \cdot h_{r-n+1} \cdots h_n. \quad (4.12)$$

Case 1-(ii) is the most critical for our estimation. Since the functions λ_s 's are odd functions, $\lambda_{s_m}(g_m) = \lambda_{t_n}(h_n), \dots, \lambda_{s_{m-r+1}}(g_{m-r+1}) = \lambda_{t_{n-r+1}}(h_{n-r+1})$ in this case, and we have

$$\begin{aligned} & |\hat{\phi}_\lambda(x, g) - \hat{\phi}_\lambda(x, gh^{-1}e_yh)| \\ & \leq \left| \sum_{i=1}^m \lambda_{s_i}(g_i) - \left(\sum_{i=1}^{m-r-1} \lambda_{s_i}(g_i) + \lambda_{u_r}(k_r) - \sum_{i=1}^{n-r-1} \lambda_{t_i}(h_i) + \lambda_{t_0}(e_y) + \sum_{i=1}^n \lambda_{t_i}(h_i) \right) \right| \\ & \leq |\lambda_{s_{m-r}}(g_{m-r})| + |\lambda_{u_r}(k_r)| + |\lambda_{t_0}(e_y)| + |\lambda_{t_{n-r}}(h_{n-r})|. \end{aligned} \quad (4.13)$$

Thus, in general,

$$|\hat{\phi}_\lambda(x, g) - \hat{\phi}_\lambda(x, gh^{-1}e_yh)| \leq 4 \cdot \|\lambda\|_\infty < \infty. \quad (4.14)$$

Therefore, $\hat{\phi}_\lambda$ is a rack quasimorphism.

Next, assume $\lambda \neq 0$. There is $s_0 \in S$ and $g_0 \in \text{Ad}(X_{s_0})$ such that $\lambda_{s_0} \neq 0$ and $\lambda_{s_0}(g_0) \neq 0$. Take $t \in S$ and $x \in X_t$ for $t \neq s_0$. Since λ_t is odd, at least one of $\lambda_{s_0}(g_0) + \lambda_t(e_x^{\pm 1}) \neq 0$, and we may assume $\lambda_{s_0}(g_0) + \lambda_t(e_x) \neq 0$ without loss of generality. For any $n > 0$, $(x, (g_0 e_x)^n)$ is reduced. Thus, we have

$$|\hat{\phi}_\lambda(x, (g_0 e_x)^n)| = |\phi_\lambda((g_0 e_x)^n)| = n \cdot |\lambda_{s_0}(g_0) + \lambda_t(e_x)| > 0. \quad (4.15)$$

Therefore, $\hat{\phi}_\lambda$ is unbounded. $\square$

4.3 Proof of Theorem 4.1

To prove Theorem 4.1, we will show the existence of an injective linear map from an infinite-dimensional vector space. The quasimorphisms constructed above provide an injective linear map from the space of uniformly bounded families of odd bounded functions to the second bounded cohomology. For the free product of nontrivial groups $\Gamma = \ast_{s \in S} \Gamma_s$, we write the space of uniformly bounded families of odd bounded functions by $V_0(\Gamma)$:

$$V_0(\Gamma) = \{\lambda = (\lambda_s)_{s \in S} \mid \text{each } \lambda_s: \Gamma_s \rightarrow \mathbb{R} \text{ is odd and bounded, and } \sup_{s \in S} \|\lambda_s\|_\infty < \infty\}. \quad (4.16)$$

Proposition 4.7. Let $X = \ast_{s \in S} X_s$ be the free product of finitely generated racks with $2 \leq |S| < \infty$. Then a map $\hat{\phi}: V_0(\text{Ad}(X)) \rightarrow H_b^2(X, \mathbb{R})$ defined by

$$\hat{\phi}(\lambda) = [\delta^1 \hat{\phi}_\lambda]_b \quad (4.17)$$

is an injective linear map.

Proof. Linearity is clear by construction.

To show injectivity, suppose $[\delta^1 \hat{\phi}_\lambda]_b = 0$. There is a bounded function $\beta \in C_b^1(X, \mathbb{R})$ such that $\delta^1 \beta = \delta^1 \hat{\phi}_\lambda$. Let $f = \hat{\phi}_\lambda - \beta$. Since f is a rack homomorphism and $\mathbb{R}$ is considered to be a trivial rack, f is constant on each connected component of X . Since $|S| < \infty$ and each X_s is finitely generated, the number of connected components of X is finite. Therefore, f is bounded, and then $\hat{\phi}_\lambda$ is bounded. By the latter half of Proposition 4.6, we can conclude that $\lambda = 0$. $\square$

The space $V_0(\text{Ad}(X))$ turns out to be infinite-dimensional by the nature of adjoint groups.

Proof of Theorem 4.1. By Proposition 4.7, it is enough to show that the space $V_0(\text{Ad}(X))$ is infinite-dimensional.

Let $\ell_{\text{odd}}^\infty(\mathbb{Z})$ be the space of odd bounded functions $\mathbb{Z} \rightarrow \mathbb{R}$.

The subgroup of the adjoint group generated by its generator e_x is infinite cyclic. Fix $s_0 \in S$ and $x_0 \in X_{s_0}$. Define a map $\iota: \ell_{\text{odd}}^\infty(\mathbb{Z}) \rightarrow V_0(\text{Ad}(X))$ by

$$\iota(\sigma) = (\iota(\sigma)_s)_{s \in S}, \quad \iota(\sigma)_s(g) = \begin{cases} \sigma(n) & \text{for } s = s_0 \text{ and } g = e_{x_0}^n \\ 0 & \text{otherwise} \end{cases}. \quad (4.18)$$

By construction, each function $\iota(\sigma)_s$ is odd and bounded, and the family $\iota(\sigma)$ is uniformly bounded. Thus, ι is well-defined. Clearly it is linear and injective. Therefore, the space $V_0(\text{Ad}(X))$ is infinite-dimensional. $\square$

Remark 4.8. Proposition 4.7 also holds for quandles. More precisely, if X_s 's are quandles in addition, then $\delta^1 \hat{\phi}_\lambda$ is a quandle cocycle. Indeed, $\delta^1 \hat{\phi}_\lambda$ is a rack cocycle by the above argument, and

$$\delta^1 \hat{\phi}_\lambda((x, g), (x, g)) = \phi_\lambda(g) - \phi_\lambda(g) = 0 \quad (4.19)$$

by the operation on $\ast_{s \in S} X_s$.

Remark 4.9 (cf. Corollary 4.3 in [31]). For the free product Γ of groups in general, the space $V_0(\Gamma)$ is not necessarily infinite-dimensional. For example, when $\Gamma = \text{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}/2 \ast \mathbb{Z}/3$, $\dim V_0(\Gamma) = 1$.

4.4 Another proof of Corollary 4.1.1

In this section, we give another proof of Corollary 4.1.1 using homogeneous group quasimorphisms.

It is known that a homogeneous quasimorphism $\phi: \Gamma \rightarrow \mathbb{R}$ is constant on conjugacy classes, that is, $\phi(h^{-1}gh) = \phi(g)$ for any $g, h \in \Gamma$. Analogous statements of Proposition 4.6 and 4.7 hold with slight modification of arguments.

We can construct a quandle quasimorphism on the free quandle on two generators using a homogeneous group quasimorphism on the free group on two generators (cf. Example 3.15 in [25]). This construction is generalized to the free product of racks as follows.

Proposition 4.10. Let $X = \ast_{s \in S} X_s$ be the free product of racks with $2 \leq |S| < \infty$, and $\phi: \text{Ad}(X) \rightarrow \mathbb{R}$ a homogeneous quasimorphism. If each X_s is finitely generated and $|S| < \infty$, then a map $\hat{\phi}: X \rightarrow \mathbb{R}$ defined by

$$\hat{\phi}(x, g) = \phi(g), \quad (4.20)$$

where (x, g) is reduced, is a rack quasimorphism.

Proof. Since ϕ is a homogeneous quasimorphism, we have

$$\begin{aligned} & |\hat{\phi}(x, g) - \hat{\phi}(x, gh^{-1}e_yh)| \\ & \leq |\phi(g) - \phi(gh^{-1}e_yh) + \phi(h^{-1}e_yh)| + |\phi(h^{-1}e_yh)| \\ & \leq D(\phi) + |\phi(e_y)|. \end{aligned} \quad (4.21)$$

Since each X_s is finitely generated and $|S| < \infty$, there is a constant $M \geq 0$ such that $|\phi(e_y)| \leq M$ for any $y \in \sqcup_{s \in S} X_s$. Therefore, $\hat{\phi}$ is a rack quasimorphism. $\square$

Proposition 4.11. Let $X = \ast_{s \in S} X_s$ be the free product of finitely generated racks with $2 \leq |S| < \infty$, and $\phi: \text{Ad}(X) \rightarrow \mathbb{R}$ a homogeneous quasimorphism on the adjoint group of X . Then a map $\eta: Q^h(\text{Ad}(X)) \rightarrow H_b^2(X, \mathbb{R})$ defined by

$$\eta(\phi) = [\delta^1 \hat{\phi}]_b \quad (4.22)$$

is an injective linear map.

Almost the same argument in the proof of Proposition 4.7 also works in this case.

Proof. Suppose $[\delta^1 \hat{\phi}]_b = 0$, and then $\hat{\phi}$ is bounded by the same argument in Proposition 4.7. Let $g \in \text{Ad}(X)$ with the factorization $g = g_1 \cdots g_m$ and $n \in \mathbb{Z}$. When $g_1 \in \text{Ad}(X_{s_1})$, we can choose $s_0 \neq s_1$ and fix arbitrary $x \in X_{s_0}$. Then we have

$$\hat{\phi}(x, g^n) = \phi(g^n) = n \cdot \phi(g). \quad (4.23)$$

Since $\hat{\phi}$ is bounded, $\phi = 0$. □

By combining the following results, we can conclude that the space $Q^h(\Gamma)$ is infinite-dimensional. The kernel of the comparison map $c^n: H_b^n(\Gamma, \mathbb{R}) \rightarrow H^n(\Gamma, \mathbb{R})$ is called the *exact bounded cohomology*, $EH_b^n(\Gamma, \mathbb{R})$. This is the subspace consisting of the classes represented by a bounded cocycle. The following relation between the space of homogeneous quasimorphisms and the second exact bounded cohomology is well-known (see for example [15]):

Lemma 4.12. $Q^h(\Gamma)/\text{Hom}(\Gamma, \mathbb{R}) \cong EH_b^2(\Gamma, \mathbb{R})$.

Recall that the adjoint group of the free rack is the free group and that the free group of rank ≥ 2 is non-elementary word-hyperbolic.

Proposition 4.13 (Theorem 3.1 in [9]). If a group Γ is non-elementary word-hyperbolic, then the exact second bounded cohomology $EH_b^2(\Gamma, \mathbb{R})$ is infinite-dimensional.

Thus, $Q^h(\text{Ad}(X))$ is infinite-dimensional, and we obtain another proof of Corollary 4.1.1.

Remark 4.14. Note that the free product of groups is not necessarily hyperbolic but is relatively hyperbolic.

4.5 Triviality for the Clauwens-Kabaya chain map

The following construction may seem more natural than (4.20), but it fails to obtain non-trivial rack quasimorphisms. The key fact is that a homogeneous group quasimorphism is constant on each conjugacy class. Recall that we say a rack quasimorphism is trivial if it is the sum of a bounded function and a function which is constant on each connected component.

Let X be a rack. We have a chain map $\tau: C_*(X) \rightarrow C_*(\text{Ad}(X))$ defined by

$$\tau_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) [e_{y_{\sigma,1}} | \dots | e_{y_{\sigma,n}}], \quad (4.24)$$

where $y_{\sigma,i}$ is defined by

$$y_{\sigma,i} = x_{\sigma(i)} \triangleleft x_{\sigma(j_1)} \triangleleft \dots \triangleleft x_{\sigma(j_k)} \quad (4.25)$$

for a permutation $\sigma \in S_n$, $i \in \{1, \dots, n\}$, and the maximal subset $j_1, \dots, j_k < i$ of $\{1, \dots, n\}$ satisfying $\sigma(i) < \sigma(j_1) < \dots < \sigma(j_k)$. For example, $\tau_1(x_1) = [e_{x_1}]$, $\tau_2(x_1, x_2) = [e_{x_1} | e_{x_2}] - [e_{x_2} | e_{x_1 \triangleleft x_2}]$, and

$$\begin{aligned} \tau_3(x_1, x_2, x_3) = & [e_{x_1} | e_{x_2} | e_{x_3}] - [e_{x_2} | e_{x_1 \triangleleft x_2} | e_{x_3}] - [e_{x_1} | e_{x_3} | e_{x_2 \triangleleft x_3}] \\ & + [e_{x_2} | e_{x_3} | e_{x_1 \triangleleft x_2 \triangleleft x_3}] + [e_{x_3} | e_{x_1 \triangleleft x_3} | e_{x_2 \triangleleft x_3}] - [e_{x_3} | e_{x_2 \triangleleft x_3} | e_{x_1 \triangleleft x_2 \triangleleft x_3}] \end{aligned} \quad (4.26)$$

(cf. Example 3.2 in [23]).

This gives rise to a cochain map from the bounded group cochain complex to the bounded rack cochain complex.

Proposition 4.15. For the induced cochain map $\tau^*: C^*(\text{Ad}(X), \mathbb{R}) \rightarrow C^*(X, \mathbb{R})$, we have $\tau^*(C_b^*(\text{Ad}(X), \mathbb{R})) \subset C_b^*(X, \mathbb{R})$. Therefore, τ^* restricts to a cochain map between the bounded cochain complexes $\tau^*: C_b^*(\text{Ad}(X), \mathbb{R}) \rightarrow C_b^*(X, \mathbb{R})$.

Proof. Let $\phi \in C_b^n(\text{Ad}(X), \mathbb{R})$ be a bounded cochain. Since

$$|\tau_n^* \phi(x_1, \dots, x_n)| = \left| \sum_{\sigma \in S_n} \text{sgn}(\sigma) \phi([e_{y_{\sigma,1}} | \dots | e_{y_{\sigma,n}}]) \right| \leq (n!) \cdot \|\phi\|_\infty \quad (4.27)$$

for any $(x_1, \dots, x_n) \in C_n(X)$, we have $\|\tau_n^* \phi\|_\infty \leq (n!) \cdot \|\phi\|_\infty < \infty$. $\square$

In general, a *quasicocycle* of a group Γ is a cochain $\phi \in C^n(\Gamma, \mathbb{R})$ such that $\delta\phi \in C_b^{n+1}(\Gamma, \mathbb{R})$. A group quasicocycle is *trivial* if it is the sum of an ordinary cocycle and a bounded cochain. In the same way, we can also define a rack quasicocycle.

Definition 4.16. A n -quasicocycle on a rack X is a rack n -cochain $\phi \in C^n(X, \mathbb{R})$ such that $\delta\phi \in C_b^{n+1}(X, \mathbb{R})$.

Rack quasicocycles have a property similar to rack quasimorphisms.

Lemma 4.17. If $\phi \in C^n(X, \mathbb{R})$ is a rack n -quasicocycle, then $\delta\phi$ is a bounded rack $(n+1)$ -cocycle and $[\delta\phi] \in \ker c^{n+1}$.

Proof. As in the case of quasimorphisms, it is obvious that $\delta\phi$ is a bounded cocycle and that the class $[\delta\phi]$ is in the kernel of the comparison map. $\square$

As a corollary of Proposition 4.15, we immediately obtain the following.

Corollary 4.17.1. If $\phi \in C^n(\text{Ad}(X), \mathbb{R})$ is a group quasicocycle, then $\tau_n^*\phi \in C^n(X, \mathbb{R})$ is a rack quasicocycle.

By this corollary, if ϕ is a group quasimorphism, then $\tau_1^*\phi$ is a rack quasimorphism. However, the pullback of a homogeneous group quasimorphism gives rise only to a trivial rack quasimorphism.

Proposition 4.18. If a group quasimorphism ϕ is homogeneous, then $\tau_1^*\phi$ is trivial.

Proof. By the defining equation of the adjoint group and the fact that a homogeneous group quasimorphism is constant on each conjugacy class, we have

$$\tau_1^*\phi(x \triangleleft y) = \phi(\tau_1(x \triangleleft y)) = \phi(e_{x \triangleleft y}) = \phi(e_y^{-1} e_x e_y) = \phi(e_x) = \tau_1^*\phi(x). \quad (4.28)$$

This shows that the pullback $\tau_1^*\phi$ is constant on each connected component, that is, trivial as a rack quasimorphism. $\square$

Remark 4.19. This chain map τ was introduced independently by Clauwens [7] for general $\square$ -sets and $\triangle$ -sets and by Kabaya [23] for a quandle and its adjoint group. Kabaya's construction is recovered by considering the cubical nerve of a rack and the simplicial nerve of the adjoint group.

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Bibliography

- [1] Nicolás Andruskiewitsch and Matías Graña. From racks to pointed Hopf algebras. *Adv. Math.*, 178(2):177–243, 2003.
- [2] Mladen Bestvina and Koji Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.*, 6:69–89, 2002.
- [3] Egbert V. Brieskorn. *Automorphic sets and braids and singularities*, volume 78 of *Contemp. Math.*, pages 45–115. Amer. Math. Soc., Providence, RI, 1988.
- [4] Robert Brooks. Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume No. 97 of *Ann. of Math. Stud.*, pages 53–63. Princeton Univ. Press, Princeton, NJ, 1981.
- [5] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.
- [6] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito. State-sum invariants of knotted curves and surfaces from quandle cohomology. *Electron. Res. Announc. Amer. Math. Soc.*, 5:146–156, 1999.
- [7] Frans Clauwens. The algebra of rack and quandle cohomology. *J. Knot Theory Ramifications*, 20(11):1487–1535, 2011.
- [8] Michael Eisermann. Quandle coverings and their Galois correspondence. *Fund. Math.*, 225(1):103–168, 2014.
- [9] David B. A. Epstein and Koji Fujiwara. The second bounded cohomology of word-hyperbolic groups. *Topology*, 36(6):1275–1289, 1997.
- [10] Pavel Etingof and Matías Graña. On rack cohomology. *J. Pure Appl. Algebra*, 177(1):49–59, 2003.

- [11] Pavel Etingof, Travis Schedler, and Alexandre Soloviev. Set-theoretical solutions to the quantum Yang-Baxter equation. *Duke Math. J.*, 100(2):169–209, 1999.
- [12] Marco A. Farinati, Jorge A. Guccione, and Juan J. Guccione. The homology of free racks and quandles. *Comm. Algebra*, 42(8):3593–3606, 2014.
- [13] Roger Fenn and Colin Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1(4):343–406, 1992.
- [14] Roger Fenn, Colin Rourke, and Brian Sanderson. James bundles and applications. *preprint*, 1996. <http://msp.warwick.ac.uk/%7ecpr/ftp/james.ps>.
- [15] Roberto Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.
- [16] Koji Fujiwara. The second bounded cohomology of a group acting on a Gromov-hyperbolic space. *Proc. London Math. Soc. (3)*, 76(1):70–94, 1998.
- [17] Koji Fujiwara. The second bounded cohomology of an amalgamated free product of groups. *Trans. Amer. Math. Soc.*, 352(3):1113–1129, 2000.
- [18] Rostislav Ivanovich Grigorchuk. Some results on bounded cohomology. In *Combinatorial and geometric group theory (Edinburgh, 1993)*, volume 204 of *London Math. Soc. Lecture Note Ser.*, pages 111–163. Cambridge Univ. Press, Cambridge, 1995.
- [19] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99, 1982.
- [20] Kohei Iwamoto, Ryoya Kai, and Yuya Kodama. Metrics for quandles. arXiv:math.GT/2505.07535, 2025.
- [21] Barry Edward Johnson. Cohomology in Banach algebras. No. 127:iii+96, 1972.
- [22] David Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23(1):37–65, 1982.
- [23] Yuichi Kabaya. Quandle cocycles from group cocycles. In *Intelligence of Low-dimensional Topology*, volume 1716, pages 25–36. Research Institute for Mathematical Sciences, Kyoto University, 2010.

- [24] Seiichi Kamada and Yukio Matsumoto. Certain racks associated with the braid groups. In *Knots in Hellas '98 (Delphi)*, volume 24 of *Ser. Knots Everything*, pages 118–130. World Sci. Publ., River Edge, NJ, 2000.
- [25] Jarek Kędra. On the geometry and bounded cohomology of racks and quandles. *J. Knot Theory Ramifications*, 33(5):Paper No. 2450020, 21, 2024.
- [26] Jiang-Hua Lu, Min Yan, and Yong-Chang Zhu. On the set-theoretical yang-baxter equation. *Duke Math. J.*, 104(1):1–18, 2000.
- [27] Roger C. Lyndon and Paul E. Schupp. *Combinatorial Group Theory*, volume Band 89 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1977.
- [28] Sergei V. Matveev. Distributive groupoids in knot theory. *Mat. Sb. (N.S.)*, 119(1):78–88, 1982.
- [29] Yoshihiko Mitsumatsu. Bounded cohomology and ℓ^1 -homology of surfaces. *Topology*, 23(4):465–471, 1984.
- [30] Takefumi Nosaka. *Quandles and Topological Pairs: Symmetry, Knots, and Cohomology*. SpringerBriefs in Mathematics. Springer, Singapore, 2017. Symmetry, knots, and cohomology.
- [31] Pascal Rolli. Quasi-morphisms on free groups. arXiv: math.GR/0911.4234, 2009.
- [32] Deepanshi Saraf and Mahender Singh. Second bounded cohomology of knot quandles. arXiv: math.GR/2502.04069, 2025.
- [33] David N. Yetter. Quandles and lefschetz fibrations. arXiv: math/0201270, 2002.