Proceedings of 51st Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, K. Kita, H. Kubo
H. Kuroda, S. Masaki, T. Ozawa, and K. Tsutaya

Series #192, 2026

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

#175 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Proceedings of 43rd Sapporo Symposium on Partial Differential Equations, 99 pages. 2018.

#176 I. Fukuda, M. Aoki, Y. Ueda, Y. Kamijima, T. Yabu, T. Niimura, N. Satoh, H. Toyokawa, K. Matsuzaka, S. Yamagata, K. Yoshida, 第 15 回数学総合若手研究集会, 657 pages. 2019.

#177 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 44th Sapporo Symposium on Partial Differential Equations, 115 pages. 2019.

#178 H. Toyokawa, Y. Otsuto, N. Satoh, K. Matsuzaka, S. Yamagata, K. Yoshida, R. Ashikaga, S. Okida, M. Oyama, K. Kuwata K. Misu, H. Ishii, T. Sukekawa, 第 16 回数学総合若手研究集会, 692 pages. 2020.

#179 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, K. Tsutaya, Proceedings of 45th Sapporo Symposium on Partial Differential Equations, 106 pages. 2020.

#180 K. Misu, S. Okida, R. Ashikaga, M. Oyama, K. Kuwata, H. Ishii, I. Kudo, T. Sukekawa, M. Seki, A. Hasegawa, K. Hatano, 第 17 回数学総合若手研究集会, 595 pages. 2021.

#181 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 46th Sapporo Symposium on Partial Differential Equations, 116 pages. 2021.

#182 Tsubasa Sukekawa(Representative organizer), Hiroshi Ishii, Isamu Kudo, Motoki Seki, Aoi Hasegawa, Kohei Hatano, Keisuke Abiko, Yu Tajima, Katsunori Fujie, Shinpei Makida, 第 18 回数学総合若手研究集会, 862 pages. 2022.

#183 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 47th Sapporo Symposium on Partial Differential Equations, 120 pages. 2022.

#184 Keisuke Abiko(Representative organizer), Katsunori Fujie, Yuta Takada, Yu Tajima, Sakumi Sugawara, Yu Ohno, Takuya Saito, Taichi Togashi, Kotaro Hata, Kazuya Hirose, 第 19 回数学総合若手研究集会, 820 pages. 2023.

#185 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 48th Sapporo Symposium on Partial Differential Equations, 136 pages. 2023.

#186 Sakumi Sugawara(Representative organizer), Yu Ohno, Takuya Saito, Ryo Negishi, Shimpei Makida, Ryu Ueno, MasatoTanabe, Taichi Togashi, Kazuya Hirose, Yuhei Kogo, Mao Nagamine, Eitaro Noda, 第 20 回数学総合若手研究集会, 960 pages. 2024.

#187 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo,H. Kuroda, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 49th Sapporo Symposium on Partial Differential Equations, 106 pages. 2024.

#188 Hiroyuki Tasaki, コンパクト対称空間の極大対蹠集合, 94 pages. 2024.

#189 Kazuya Hirose(Representative organizer), Taichi Togashi(Deputy Representative Organizer), Ryu Ueno, Masato Tanabe, Mamoru Aihara, Kaito Kayo, Yuhei Kogo, Ryunosuke Nakano, Mao Nagamine, Issei Noda, Hisato Matsukawa, Koji Wada, Ayato Kubo, 第 21 回数学総合若手研究集会, 822 pages. 2025.

#190 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, K. Kita, H. Kubo, H. Kuroda, S. Masaki, T. Ozawa, and K. Tsutaya, Proceedings of 50th Sapporo Symposium on Partial Differential Equations, 121 pages. 2025.

#191 Kaito Kayo (Representative organizer), Yuhei Kogo (Deputy representative organizer), Mamoru Aihara, Ryunosuke Nakano, Mao Nagamine, Issei Noda, Kohei Fujita, Hisato Matsukawa, Kango Ito, Yuta Kishimoto, Masahiro Sakamoto, So Toshima, Koji Wada, Kaoru Ishi, Haruto Tsujimoto 第 22 回数学総合若手研究集会 894 pages. 2026.

Proceedings of 51st Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, K. Kita, H. Kubo
H. Kuroda, S. Masaki, T. Ozawa, and K. Tsutaya

Sapporo, 2026

Partially supported by Grant-in-Aid for Scientific Research,
the Japan Society for the Promotion of Science.

日本学術振興会科学研究費補助金（基盤研究 A 課題番号 24H00183）
日本学術振興会科学研究費基金（基盤研究 B 課題番号 24K00529）
日本学術振興会科学研究費基金（基盤研究 C 課題番号 23K03175）

Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 5 through August 7 in 2026 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium almost 50 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Josai University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
K. Kita (Hokkaido University)
H. Kubo (Hokkaido University)
H. Kuroda (Hokkaido University)
S. Masaki (Hokkaido University)
T. Ozawa (Waseda University)
K. Tsutaya (Hirosaki University)

Contents

The 51st Sapporo Symposium on Partial Differential Equations

第 51 回偏微分方程式論札幌シンポジウム

Period	August 5, 2026 – August 7, 2026
Place	5-203, Faculty of Science Bldg. #5
Organizers	Satoshi Masaki, Kosuke Kita
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Kosuke Kita, Hideo Kubo, Hirotoshi Kuroda, Satoshi Masaki, Tohru Ozawa, Kimitoshi Tsutaya
URL	https://www.math.sci.hokudai.ac.jp/sympo/sapporo/program260805.html

August 5, 2026 (Wednesday)

13:20-13:30 Opening

13:30-14:30 中村周（学習院大学） Shu Nakamura (Gakushuin University)
Topics on the analysis of simple neural networks

14:40-15:10 佐藤卓弥（東京大学） Takuya Sato (The University of Tokyo)
Quantitative homogenization for obstacle problems of convex Hamilton-Jacobi equations

15:10-15:40 (break/discussion)

15:40-16:40 柳澤卓（早稲田大学/奈良女子大学）
Taku Yanagisawa (Waseda University/Nara Women's University)
Boundary value problems of magnetohydrodynamics in 3D bounded domains

16:50-17:30 寺本有花（愛媛大学） Yuka Teramoto (Ehime University)
Global solvability of simplified Ericksen-Leslie system with thermal effects

August 6, 2026 (Thursday)

10:00-11:00 Jacopo Bellazzini (University of Pisa)
NLS with mass-subcritical combined nonlinearities: small mass-scattering

11:10-11:40 駒田洸一（早稲田大学） Koichi Komada (Waseda University)
Scattering of group-invariant solutions for nonlinear fourth-order Schrödinger equation

11:40-13:40 (break/discussion)

13:40-14:00 (Free discussion with speakers in the tea room)

14:00-15:00	高橋仁（東京科学大学） Jin Takahashi (Institute of Science Tokyo) Blow-up rate for the subcritical semilinear heat equation in non-convex domains
15:10-15:40	岡優丞（東北大学） Yusuke Oka (Tohoku University) Grow up rate of exponential reaction diffusion equations with inhomogeneous terms
15:40-16:20	(break/discussion)
16:20-16:50	Yu Zhou (Hokkaido University) Lifespan estimates for classical solutions of nonlinear wave equations with nonlinear boundary condition in one dimension
17:00-18:00	堤誉志雄（京都大学） Yoshio Tsutsumi (Kyoto University) Well-posedness of the Cauchy problem in low regularity for kinetic derivative NLS on $\mathbf{T}$
19:00-	Banquet

August 7, 2026 (Friday)

10:00-11:00	下條昌彦（東京都立大学） Masahiko Shimojo (Tokyo Metropolitan University) Stability of traveling waves in non-cooperative systems with nonlocal dispersal via an entropy framework
11:10-11:50	田中吉太郎 （公立はこだて未来大学） Yoshitaro Tanaka (Future University Hakodate) Vanishing viscosity and quasi-steady state limit for a chemo-repulsion system in one-dimensional space
11:50-12:00	Closing

Topics on the analysis of simple neural networks*

Shu Nakamura

Abstract

We discuss simple models of neural networks, of which we can prove the convergence of the learning process with the standard gradient descent.

In Section 1, we start with a short introduction to the neural network (for mathematicians). We refer [Simon2023] for more information. In Section 2, we discuss a shallow (2 layer) neural network with 1 dimensional input data and the ReLU activation function. We modify the model, which we call a neural network with fixed biases, and show that the parameters converges to the unique solution. In Section 3, we propose an alternative activation function, and show how it simplifies the argument, and it also sheds light to the phenomena called spectral bias.

If we have enough time, we will also discuss neural network with fixed biases, with n-dimensional data (not included in this abstract).

This presentation is based on our preprint [MN2026].

1 A very short introduction to neural networks

1.1 Shallow (simple) neural network

We here consider simple neural networks defined as follows. Let $\mathbb{R}^N$ be the space of the input data, and for $x \in \mathbb{R}^N$, we set

$$y_k = \sum_{\ell=1}^{L} W^{(1)}_{k,\ell} h^{(1)}_\ell + b^{(1)}_k, \quad h^{(1)}_\ell = a\biggl(\sum_{n=1}^{N} W^{(0)}_{\ell,n} x_n + b^{(0)}_\ell\biggr),$$

where $k = 1, \dots, K$, and $y \in \mathbb{R}^K$ is the output. $W^{(*)}_{*,*}$ are called **weights**; $b^{(*)}_*$ are called **biases**; and $\{h^{(*)}_*\}$ is called a **hidden layer**. $a(\cdot)$ is a given non-linear function, which is called an **activation function**. Thus, by the map:

$$x \mapsto h^{(1)} \mapsto y \qquad (x \in \mathbb{R}^N, h^{(1)} \in \mathbb{R}^L, y \in \mathbb{R}^K)$$

a two-layer neural network is defined. It is sometimes called a **shallow** neural network. We denote the map as

$$y = F(\Phi; x), \quad \Phi = (\{W^{(0)}_{k,\ell}\}, \{b^{(0)}_k\}, \{W^{(1)}_{\ell,n}\}, \{b^{(1)}_\ell\}), \quad x \in \mathbb{R}^N,$$

*Based on joint works with Fabricio Macià (Universidad Politécnica de Madrid) where $\Phi \in \mathbb{R}^{NL} \times \mathbb{R}^{L} \times \mathbb{R}^{LK} \times \mathbb{R}^{K}$.

The purpose of the neural networks: For a given function $f : \mathbb{R}^N \to \mathbb{R}^K$, we wish to approximate f by $F(\Phi; \cdot)$. Moreover, we wish to generate the parameter Φ from f automatically. It would be nice if we can generate Φ from a finite set of data: $\{(x, f(x)) \mid x \in D\}$. This procedure is called the **learning** or the **training**.

1.2 Activation function

A neural network consists mostly of affine transforms (linear maps + translations). But iterations of affine transforms are also affine transforms, and these cannot approximate complicated functions. The introduction of the activations function $a(\cdot)$ makes it possible for neural networks to approximate arbitrary function $f(x)$.

Usually one of the following functions is used as the activation function:

Sigmoind function $a(x) = \dfrac{1}{1+e^{-x}}$

ReLU (Rectified Linear Unit function) $a(x) = (x)_+ = \begin{cases} x & (x > 0) \\ 0 & (x \geq 0) \end{cases}.$

Historically the Sigmoid function had been widely used as the standard activation function, probably since it can be considered as a regularized switching function: $H(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \geq 0) \end{cases}.$

In recent years, the ReLU function is used in most neural networks because of the fast convergence of learning and better accuracy of the approximations (it seems). There has been a vast research on finding better (or the best) activation functions, but almost all are based on numerical experiments, and few theoretical analysis.

For the moment, we use the ReLU activation function.

1.3 Deep neural network (DNN)

If we iterate the shallow network, then we have a map called a **deep neural network** (DNN). More specifically, we set $h^{(\nu)} \in \mathbb{R}^{L_\nu}$, $\nu = 1, \ldots, M$, be the dimensions of the hidden layers, and we write $L_0 = N$ and $h^{(0)} = x$. Then we set a map $x \mapsto y$ by

$$y_k = \sum_{\ell=1}^{L} W_{k,\ell}^{(M)} h_\ell^{(M)} + b_k^{(M)}, \quad k = 1, \ldots, K,$$

$$h_\ell^{(\nu+1)} = a\biggl(\sum_{n=1}^{L_\nu} W_{\ell,n}^{(\nu)} h_n^{(\nu)} + b_\ell^{(\nu)}\biggr), \quad \ell = 1, \ldots, L_{\nu+1},\ \nu = 0, \ldots, M-1.$$

The set of parameters are $\Phi = (\{W^{(0)}_{*,*}\}, \{b^{(0)}_*\}, \dots, \{W^{(M)}_{*,*}\}, \{b^{(M)}_*\})$, which consits of $M+1$ weight matrices and $M+1$ bias vectors. Such maps are called DNN with $M+1$ layers.

If we denote denote $y = F(\Phi; x)$, then the goal of this DNN is also finding a parameter Φ such that $F(\Phi; x)$ approximate a given map $f : \mathbb{R}^N \to \mathbb{R}^K$.

1.4 Universal approximation theorem

About the ability of neural networks to approximate arbitrary functions, the universal approximation theorems are well-known. We consider a shallow network $F(\Phi; x) : \mathbb{R}^N \to \mathbb{R}^K$ with the L dimensional hidden layer. We suppose the activation function is the Sigmoid or ReLU.

Theorem (Universal Approximation Theorem)**.** *For any continuous map $f : \mathbb{R}^N \to \mathbb{R}^K$, any bounded set $D \subset \mathbb{R}^N$, and for any $\varepsilon > 0$, we can find L large enough that*

$$\sup_{x \in D} |f(x) - F(\Phi; x)| < \varepsilon$$

with a set of parameters Φ.

Thus we know the existence of Φ such that $F(\Phi; x)$ approximates f as close as possible. So the remaining (big) question is "How to find such a Φ?" This is the problem of the learning, or more specifically the convergence of the learning.

1.5 Loss function and the gradient descent method

We measure the error, or the difference between the target function f and the approximation $F(\Phi; x)$ by a **loss function**. In this talk, we use the standard L^2 norm, or the mean square error function. Namely, we set the loss function by

$$L(f, g) = \|f - g\|^2_{L^2} = \int_{\mathbb{R}^N} |f(x) - g(x)|^2 dx$$

(In practice, we use various loss functions, and usually we cannot use integration over the data space. Thus we use the sum over the finite data set, but here we consider the model with an idealized setting.)

As an optimization scheme, we use the standard (Cauchy's) gradient descent method. Namely, with a $\Phi = \Phi_0$ as the initial parameter, we use the iteration:

$$\Phi_{m+1} = \Phi_m - \varepsilon \frac{\partial L(f, F(\Phi))}{\partial \Phi}\Big|_{\Phi = \Phi_m}, \quad m = 0, 1, 2, \dots,$$

to decide a series of parameters Φ_m, $m = 1, 2, \dots$. Here $\varepsilon > 0$ is the **learning coefficient**, which should be sufficiently small.

When $\Phi_\infty = \lim \Phi_m$ exists, and when it gives the minimum of the loss function, the we can declare that the network learned f, or the learning is successful.

1.6 Difficulties and practical recipes

There are several obvious difficulties concerning the convergence of the learning (gradient descent iterations).

Symmetries and corresponding degeneracies We here consider a shallow network (SNN). Since the network (function) is invariant under the exchange of the hidden layer variables, the network has (at least) the symmetry of the symmetry group: $\mathfrak{S}_L$, where L is the dimension of the hidden layer. Namely, $\mathfrak{S}_N$ acts on the space of parameters $\mathbb{W} = \{\Phi\}$ invariantly.

- If there is no elements in $\mathfrak{S}_L$ which leaves Φ_∞ invariant (except for the unit), then there are (at least) $|\mathfrak{S}_L|$ minima, and the minima is not unique, obviously.
- If there is (non-trivial) subgroup of $\mathfrak{S}_L$ which leaves Φ_j invariant, then Φ_∞ is also invariant with respect to this subgroup. In particular, if Φ_0 is invariant with respect to $\mathfrak{S}_L$, i.e., it is constant with respect to the hidden-layer index, then the hidden layer becomes essentially one-dimensional. Thus we need to be careful about the choice of the initial parameter Φ_0.
- There may be more symmetries, and the corresponding degeneracies (always there are), especially for the DNNs.

Finiteness of the training data In our discussion of the gradient descent, we use all the data, i.e, we use integration over the parameter space, but in practice, we have only sets of finite data and the results. In order to address this problem, we use a method called **stochastic gradient descent (SGD)**. In this case, the existence of non-minimal local minima and critical points are known, due to the biased data sets. We do not address this problem in this talk.

Thus, we cannot expect unconditional convergence of the learning, and usually the following "recipes" are employed in order to achieve the convergence.

1. We choose the initial weights: $\{W^{(*)}_{*,*}\}$ random, with the normal distribution. The averages are zero, and the variance is chosen so that the weight matrices have the norm of order 1. We also set the initial biases as 0.
2. We use the SGD to introduce more randomness in the learning process.
3. When the learning seems diverging, we use "normalization" (*batch normalization*) (for the DNNs).

2 Shallow (toy) model with one-dimensional data

2.1 The first idea

In order to understand the convergence of the learning somehow, we start with the simplest model possible: the input dimension $N = 1$, and the output dimension $K = 1$, and the network is shallow. We use ReLU as the activation function, and we write $a(x) = \mathrm{ReLU}(x) = (x)_+ = Y(x)$. Now our network is

$$y(x) = \sum_{j=1}^{L} W_j^{(1)} Y(W_j^{(0)} x + b_j^{(0)}) + b^{(1)}, \quad x \in \mathbb{R}.$$

Here the dimension of the hidden layer is L, and the dimension of the parameters is $3L + 1$. We may suppose $W_j^{(0)} \neq 0$ for all j, since if $W_j^{(0)} = 0$ then $Y(W_j^{(0)} x + b_j^{(0)}) = Y(b_j^{(0)})$ is constant. We note $Y(x)$ is homogeneous of order 1, and hence we can write

$$y(x) = \sum_{j=1}^{L} (W_j^{(1)} |W_j^{(0)}|) Y(\mathrm{sgn}(W_j^{(0)}) x + (b_j^{(0)} / |W_j^{(0)}|)) + b^{(1)}$$

Noting $Y(-x) = Y(x) - x$, and changing the notations, we may rewrite the network as

$$y(x) = \sum_{j=1}^{L} \psi_j Y(x - b_j) + b + cx.$$

Here we denote $c = -\sum_{\{j | W_j^{(0)} < 0\}} W_j^{(1)} |W_j^{(0)}|$. Now the dimension of the parameters is reduned to $2L + 2$.

With this simplified representation, it is clearly observed that the network is simply a linear combination of $Y(x - b_j)$, $j = 1, \ldots, L$ with an additional linear function $bx + c$. Namely the space of functions represented by the network is the linear space spanned by the the non-linear functions:

$$Y(x - b_j) = (x - b_j)_+, \quad j = 1, \ldots, L$$

with 1 and x.

The biases $\{b_j\}$ are learned parameters, at first, but if you look the space as above, it might work as well with a given set of constant biases. For example, maybe we can choose the biases evenly distributed numbers in an interval? Such model might be called a **neural network with fixed biases**.

Now, we may consider the continuous analogue of the network by making the biases a continuous parameter. Namely, we set (formally)

$$y(x) = \int \psi(t) Y(x - t) dt + b + cx \tag{1}$$

where $\psi(x)$ is a weight function on $\mathbb{R}$ (or on an interval). We call this network (function) as a **continuous neural network model with fixed biases**.

Remark: This network may be considered as a generalization of the original shallow neural network by extending the space of the weights ψ to the distributions. Namely, if we set $\psi(t) = \sum \psi_j \delta(t - b_j)$, then we have the above discrete neural network.

We now note that (1) is simply a convolution operator, which is very familiar to us (analysts). We now differentiate the both sides of (1) twice to obtain

$$y''(x) = \int \psi(t) Y''(x - t) dt = \int \psi(t) \delta(x - t) dt = \psi(x).$$

Here we have used $Y''(x) = \delta(x)$, i.e., Y is a fundamental solution to $\frac{d^2}{dx^2}$.

This observation implies, in particular, $\psi(x)$ is determined uniquely by $y(x)$, at least when y is C^2-class. Moreover, if we set $\psi = y''$, $b = y(0)$, $c = y'(0)$, then (1) holds obviously.

These observations are mathematically trivial. Then, can we make the network learn to attain these parameters (ψ, b, c) by the gradient descent?

2.2 Learning by the gradient descent for the continuous model

For simplicity, we consider the space of functions on $I = [0, 1]$. For a set of parameters $\varphi = (\psi, b, c) \in \mathbb{W} = L^2(I) \times \mathbb{R} \times \mathbb{R}$, we set

$$T\varphi(x) = g(x) = \int_0^1 \psi(z) Y(x - z) dz + b + cx, \quad x \in I.$$

It is easy to see T is a bounded operator from $\mathbb{W}$ to $L^2(I)$.

We set the loss function as L^2-norm of the difference, namely,

$$L(f, g) = \|f - g\|_{L^2}^2 = \int_0^1 |f(x) - g(x)|^2 dx, \quad f, g \in L^2(I).$$

Then the Fréchet derivative of $L(f, T\varphi)$ in φ is given by

$$\frac{\delta L(f, T\varphi)}{\delta \varphi} = -2T^*(f - T\varphi).$$

Thus the iteration procedure of the gradient descent is given by

$$g_n = T\varphi_n, \quad \varphi_{n+1} = \varphi_n + 2\varepsilon T^*(f - g_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $\varphi_0 \in \mathbb{W}$ is the initial parameters. Here $\varepsilon > 0$ is the learning coefficient, and we choose it sufficiently small.

Theorem 1. *For $f \in L^2(I)$, $\varphi_0 \in \mathbb{W}$, g_n converge to f with respect to the loss function, i.e.,* $\lim_{n \to \infty} \|f - g_n\|_{L^2} = 0$.

Sketch of Proof. Applying T to the both sides of (2) and subtracting f, we have

$$f - g_{n+1} = (1 - 2\varepsilon TT^*)(f - g_n).$$

Hence we have

$$f - g_n = (1 - 2\varepsilon TT^*)^n (f - g_0)$$

We note TT^* is a non-negative bounded symmetric operator on $L^2(I)$. We set $\varepsilon < (2\|TT^*\|)^{-1}$ so that the right hand side is uniformly bouded in n. Moreover, we can easily show T^* has no zero eigenfunction, and hence we learn $(1 - 2\varepsilon TT^*)^n \to 0$ strongly as $n \to \infty$. $\square$

It is a bit confusing but Theorem 1 does not imply the convergence of the parameters φ_n. In fact, if $f \notin H^2(I)$, then $\psi = f'' \notin L^2(I)$, and thus φ_n does not converges in our parameter space $\mathbb{W}$. If $f \in H^2(I)$, then we can show the convergence of φ_n.

Theorem 2. *Suppose $f \in H^2(I)$, $\varphi_0 \in \mathbb{W}$. Then there is $\varphi \in \mathbb{W}$ such that $\lim_{n\to\infty} \|\varphi - \varphi_n\|_{L^2} = 0$.*

Sketch of Proof. The proof is quite analogous to the proof of the previous theorem. We set $\varphi = (f''(x), f(0), f'(0)) \in \mathbb{W}$ so that $f = T\varphi$. Then, by (2), we have

$$\varphi - \varphi_n = (1 - 2\varepsilon T^*T)^n (\varphi - \varphi_0), \quad n = 1, 2, \dots.$$

We note $T^*T > 0$, and $\|T^*T\| = \|T\|^2$. Then $\|\varphi = \varphi_n\|_{L^2} \to 0$ follows immediately. $\square$

2.3 A discrete network with fixed biases, and the convergence of learning

Here we consider functions on $I = [0, 1]$ again, but we divide I into N small intervals. We set $t_j = j/N$, $j = 0, 1, 2, \dots, N$, and we consider the function space on $\{t_j \mid j = 0, \dots, N\}$. Namely, we set

$$\mathfrak{X} = \mathbb{R}^{N+1} = \{(f(t_0), f(t_1), \cdots, f(t_N))\} = \ell^2(\{t_j\}_{j=0}^N; \mathbb{R}).$$

We may identify an element in $\mathfrak{X}$ with a piecewise linear function on I by the linear interpolation.

We set the parameter space as $\mathbb{W}_N = \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}$, and for $\varphi = ((w(t_j))_{j=1}^{N-1}, b, c) \in \mathbb{W}_N$ we define the network T by

$$T\varphi(x) = g(x) = \frac{1}{N} \sum_{j=1}^{N-1} w(t_j) Y(x - t_j) + b + cx, \quad x \in I.$$

We note the discrete (difference) Laplacian on $\{t_0, \dots, t_N\}$ is defined by

$$\triangle_N f(t_j) = N^2(f(t_{j-1}) - 2f(t_j) + f(t_{j+1})), \quad j = 1, \dots, N-1.$$

By easy computations, we have

$$\triangle_N Y(z) = \begin{cases} N & \text{if } z = 0 \\ 0 & \text{if } z \neq 0 \end{cases}$$

By virtue of this formula, we learn $\triangle_N T\varphi(t_j) = w(t_j)$, where $\varphi = (\{w(t_j)\}, b, c) \in \mathbb{W}_N$. In fact, we also have

$$g(x) = \frac{1}{N}\sum_{j=1}^{N-1}(\triangle_N g)(t_j)Y(x - t_j) + g(t_0) + g'(t_0)x, \quad x \in I$$

for any $g \in \mathfrak{X}$, where we set $g'(t_0) = N(g(t_1) - g(t_0))$. Thus, all functions in $\mathfrak{X}$ can be represented by this network.

Learning: We use the function space $\mathfrak{X}$ with the usual Euclidean distance (devided by N). Namely, we consider $\mathfrak{X}$ with the ℓ^2-norm as follows. The loss function is defined by

$$L(f, g) = \frac{1}{N}\sum_{j=0}^{N}(f(t_j) - g(t_j))^2 = \|f - g\|_{\ell^2}^2.$$

The derivative of $L(f, T\varphi)$ in the parameters are computed as before:

$$\frac{\partial L}{\partial \varphi}(f, T\varphi) = -2T^*(f - T\varphi).$$

Thus the iteration of the gradient descent is computed by

$$g_n = T\varphi_n, \quad \varphi_{n+1} = \varphi_n + 2\varepsilon T^*(f - g_n), \quad n = 0, 1, 2, \ldots$$

and we can prove the following theorem as well.

Theorem 3. *Suppose $f \in \mathfrak{X}$ and $\varphi_0 \in \mathbb{W}_N$. Then $\lim_{n\to\infty} \|f - g_n\|_{\ell^2} = 0$. Moreover if we set $\varphi = (\triangle_N f, f(0), f'(0))$, then $\lim_{n\to\infty} \|\varphi - \varphi_n\|_{\mathbb{W}_N} = 0$*

Remark: Since the function space $\mathfrak{X}$ is finite dimensional, the convergence holds with respect to the norm.

3 A new activation function and the spectral bias

3.1 Alternative activation function?

Activation functions introduces non-linearity to neural networks, and they are essential to approximate arbitrary functions by learning. It is known that we can employ a wide variety of functions as activation functions in the sense that the universal approximation theorem holds.

However, it is also known, empirically, that some functions are more effective, i.e., it gives us better approximation results, and/or, faster convergence of learning. As we discussed before, the Sigmoid function was originally used widely, but mostly replaced by the ReLU function, because of these advantages.

There is a vast literature on finding better activation functions, but it seems ReLU is so far the best or close to the best in that more complicated functions give us only marginal improvements. Most of these results are based on numerical experiments, and we are aware of very few theoretical arguments what does make better activation functions.

In the discussions of the last section, we pointed out that one important property of the ReLU function is that it is a fundamental solution to the one-dimensional Laplacian. When we consider these simple network models with fixed biases, we learned that the representation by the network is unique, and the learning converges by virtue of this property.

This observation leads us to the idea that we may try to employ the fundamental solution to some different second-order (elliptic) operator as an activation function. It is better to have a convolution-type operator, and it requires the coefficients are constants. Since we use the inverse of the operator (fundamental solution), it is natural to use an elliptic operator, especially second-order differential operators. The simplest candidate might be

$$H_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + 1\right)$$

and its fundamental solution is given by

$$Z(x) = e^{-|x|}.$$

Here we choose the coefficients of H_0 to make $Z(x)$ as simple as possible.

$Z(t) = e^{-|x|}$ is very simple, and we are familiar with it in the PDE theory, but it would be nice to have a specific name. Analogously to the ReLU (Rectified Linear Unit) function, we may call it the **FReX** (Full-wave Rectified Exponential) function. By the definition, we have

$$H_0 Z(x) = \delta(x).$$

Hence, for any function f, we also have

$$H_0(Z * f) = f.$$

Since $H_0^{-1} = Z*$ is bounded in $L^2(\mathbb{R})$, we can carry out the same argument on the whole line $\mathbb{R}$ (instead of an interval), and this simplifies the argument considerably.

3.2 Continuous network model with the FReX activation, and the convergence of learning

We employ the function space $L^2(\mathbb{R})$, and we set

$$T\varphi(x) = Z * \varphi(x) = \int_{\mathbb{R}} Z(x-z)\varphi(z)dz, \quad z \in \mathbb{R},\ \varphi \in L^2(\mathbb{R}).$$

$T = H_0^{-1}$ and T is bounded in $L^2(\mathbb{R}) \to H^2(\mathbb{R}) \underset{\to}{\subset} L^2(\mathbb{R})$. We consider a network with a parameter $\varphi \in L^2(\mathbb{R})$ defined by

$$g(x) = T\varphi(x), \quad x \in \mathbb{R}.$$

The loss function is given by the L^2-norm:

$$L(f,g) = \|f-g\|_{L^2}^2, \quad f, g \in \mathcal{H},$$

and the Fréchet derivative of $L(f, T\varphi)$ in φ is also given by

$$\frac{\delta L(f, T\varphi)}{\delta\varphi} = -2T^*(f - T\varphi).$$

The iteration procedure of the gradient descent with the initial parameters $\varphi_0 \in L^2(\mathbb{R})$ is given by

$$g_n = T\varphi_n, \quad \varphi_{n+1} = \varphi_n + 2\varepsilon T^*(f - Tg_n), \quad n = 0, 1, 2, \ldots.$$

Thus we have $\varphi_n = (1 - 2\varepsilon T^*T)^n \varphi_0$, and we can show $\|f - g_n\|_{L^2} \to 0$, $n \to \infty$ as well as the ReLU activation case.

Theorem 4. *Suppose $f \in L^2(\mathbb{R})$ and $\varphi_0 \in L^2(\mathbb{R})$. Then $\lim_{n\to\infty} \|f - g_n\|_{L^2} = 0$. Moreover, if $f \in H^2(\mathbb{R})$, $\varphi_0 \in L^2(\mathbb{R})$, then $\lim_{n\to\infty} \|H_0 f - \varphi_n\|_{L^2} = 0$.*

We note that $T^*T = H_0^{-2}$, and it can be written explicitly using the Fourier transform:

$$\mathcal{F}T^*T\mathcal{F}^*\psi(\xi) = \mathcal{F}TT^*\mathcal{F}^*\psi(\xi) = 4(1 + (2\pi\xi)^2)^{-2}\psi(\xi),$$

where the Fourier transform is defined by

$$\mathcal{F}\psi(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi}\psi(x)dx.$$

This implies, in particular, $0 < T^*T, TT^* \leq 4$. and we may choose the learning coefficients as any $\varepsilon > 0$ so that $0 < \varepsilon < 1/4$. We also note the spectrum of TT^* (and T^*T) is absolutely continuous.

Remark: In general, if $f \in L^2(\mathbb{R})$ then (obviously) $H_0 f \in H^{-2}(\mathbb{R})$. Under this condition, we have $\lim_{n\to\infty} \|H_0 f - \varphi_n\|_{H^{-2}} = 0$. If we set

$$\varphi(x) = \sum_{j=1}^{N} w_j \delta(x - b_j) \in H^{-2}(\mathbb{R})$$

then we have the usual discrete neural network

$$f(x) = T\varphi(x) = \sum_{j=1}^{N} w_j Z(x - b_j).$$

We note, however, that the gradient descent iterations are not contained in the set of network of this form.

3.3 Discrete neural network with the FReX activation

We can also construct discrete neural network with fixed biases using the FReX activation function, and we can show the convergence of the learning as well as the ReLU function.

In this subsection, we set $\mathbb{W}_N = \ell^2(N^{-1}\mathbb{Z})$, and we use the usual ℓ^2 norm:

$$\|\varphi\|_{\mathbb{W}_N}^2 = \frac{1}{N} \sum_{z \in N^{-1}\mathbb{Z}} |\varphi(z)|^2.$$

The network $T : \mathbb{W}_N \to \mathbb{W}_N$ is defined by

$$g(x) = T\varphi(z) = \frac{1}{N} \sum_{t \in N^{-1}\mathbb{Z}} \varphi(t) Z(z - t), \quad z \in N^{-1}\mathbb{Z}.$$

Let $\triangle_N$ be the discrete Laplacian as before, and by direct computations we have

$$\triangle_N Z(x) = -(Na_N + b_N)\delta_{0,x} + b_N Z(x),$$

where

$$a_N = 2e^{-1/2N}(2N \sinh(1/2N)), \quad b_N = (2N \sinh(1/2N))^2.$$

We note $a_N \sim 2$, $b_N \sim 1$ で if N is large, and we define H_0 by

$$H_0 = c_N(-\triangle_N + b_N), \quad c_N = (a_N + b_N/N)^{-1}.$$

By the construction, we have $H_0 Z(x) = N\delta_{x,0}$, and we can carry out the same arguments as in the ReLU activation case (but simpler):

Theorem 5. *Suppose* $0 < \varepsilon < 1/2(b_N)^2$. *For* $f, \varphi_0 \in \mathbb{W}_N$ *we set the iteration procedure by*

$$g_n = T\varphi_n, \quad \varphi_{n+1} = \varphi_n + 2\varepsilon T^*(f - g_n), \quad n = 0, 1, 2, \ldots.$$

Then

$$\|f - g_n\|_{\mathbb{W}_N} \to 0, \quad \|H_0 f - \varphi_n\|_{\mathbb{W}_N} \to 0 \quad \text{as } n \to \infty.$$

Remark: We can represent TT^* and T^*T explicitly using the Fourier series expansion (or the so-called discrete Fourier transform), as well as the continuous model case.

3.4 Spectral bias

In this subsection, we consider the continuous model with the FReX activation function. Then the approximation error of f by $g_n = T\varphi_n$ by the gradient descent is given by

$$e_n = f - g_n = \mathcal{F}^*\big[r_\varepsilon(\cdot)^n \mathcal{F}(f - T\varphi_0)\big], \quad r_\varepsilon(\xi) = 1 - \frac{8\varepsilon}{(1+(2\pi\xi)^2)^2}$$

where $\mathcal{F}$ is the Fourier transform.

We observe that if $\xi \sim 0$, then $r_\varepsilon(\xi) \sim 1 - 8\varepsilon < 1$, and hence the error decays exponentially as $n \to \infty$. On the other hand, if $|\xi| \gg 1$, then $r_\varepsilon \sim 1$, and thus decay of the error is slow. Thus, the error decays faster for slowly oscillating components of the function $f - T\varphi_0$, and slower for the rapidly oscillating components. In other words, the convergence of the learning is faster for the slow frequency area.

This property is called **spectral bias**, and it has been observed numerically by several models. This property may be important since the noise may have high frequencies, and we may have better results by stopping the gradient descent procedure early. The scarcity of learning data can be considered as a high frequency noise, and this property might be one possible explanation of the "generalization" properties.

In our model, we can estimate the error rigorously (and explicitly) in the Fourier space. If $r_\varepsilon(\xi) \sim 1$, and if moreover $r_\varepsilon(\xi)^n \sim 1$, then by the Taylor expansion, we expect $r_\varepsilon(\xi)^n \sim 1 - n(1 - r_\varepsilon(x))$, and thus $r_\varepsilon(\xi)^n$ is measurably small when $n(1 - r_\varepsilon(\xi)) \approx 1$. Recalling $1 - r_\varepsilon(\xi) = \dfrac{8\varepsilon n}{(1+(2\pi\xi)^2)^2}$, we learn that the decay of the error is significant if and only if $|\xi| \lesssim O(n^{1/4})$.

Thus, after nth step of the gradient descent, the approximation is effective up to the frequency of order $|\xi| \lesssim O(n^{1/4})$. The approximation is not effective above this frequency.

We can show similar properties for the ReLU activation function case, but the analysis is especially simpler for the FReX case, since everything is explicitly expressed using the Fourier transform.

For the discrete network cases, the Fourier space is the torus, and it is compact. Hence the convergence of the learning is always uniform theoretically, but we may observe such spectral bias quantitatively.

References

[MN2026] Fabricio Maciá, Shu Nakamura: Mathematical analysis of one-layer neural network with fixed biases, a new activation function and other observations. https://arxiv.org/abs/2604.07715

[Prince2023] Simon J. D. Prince: *Understanding Deep Learning.* The MIT Press, 2023.

QUANTITATIVE HOMOGENIZATION FOR OBSTACLE PROBLEMS OF CONVEX HAMILTON–JACOBI EQUATIONS

TAKUYA SATO (THE UNIVERSITY OF TOKYO)

This talk is based on joint work with Jingcheng Ye (KPMG Consulting Co., Ltd.).

1. Introduction

We study the homogenization problem of Hamilton–Jacobi equations with obstacles. For each $\varepsilon > 0$, consider the viscosity solution $u^\varepsilon : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ to the obstacle problem of an evolutionary equation of the form:

$$\begin{cases} \max\left\{u_t^\varepsilon + H\left(\dfrac{x}{\varepsilon}, Du^\varepsilon\right),\ u^\varepsilon - \psi\left(x, \dfrac{x}{\varepsilon}\right)\right\} = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{OP$_\varepsilon$}$$

where $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is a given Hamiltonian, $\psi \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is a given obstacle function and $u_0 \in C(\mathbb{R}^n)$ is an initial condition. We assume that $H = H(y,p)$ is $\mathbb{Z}^n$-periodic in y and convex and coercive in p, and $\psi = \psi(x,y)$ is $\mathbb{Z}^n$-periodic in y. Our main purpose is to study the singular limit of unique viscosity solution u^ε to (OP$_\varepsilon$) as $\varepsilon \to +0$.

Obstacle problems for Hamilton–Jacobi equations. Obstacle problems for Hamilton–Jacobi equations arise as PDEs satisfied by the value functions of optimal stopping problems in control theory. Let $L = L(y,q)$ be the Lagrangian determined by the following Legendre transform of H:

$$L(y,q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(y,p)\}, \quad (y,q) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Since $H(y,p)$ is $\mathbb{Z}^n$-periodic in y and convex and coercive in p, it follows that $L(y,q)$ is also $\mathbb{Z}^n$-periodic in y and convex and coercive in q. For each $\varepsilon > 0$ and $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we consider the minimizing problem of the cost functional

$$J^\varepsilon(x,t,\gamma,\theta) := \int_\theta^t L\left(\frac{\gamma(s)}{\varepsilon}, \dot{\gamma}(s)\right) ds + \mathbf{1}_{\{\theta=0\}} u_0(\gamma(0)) + \mathbf{1}_{\{\theta>0\}} \psi\left(\gamma(\theta), \frac{\gamma(\theta)}{\varepsilon}\right),$$

where the controller can choose any pair (γ,θ) of an admissible trajectry and a stopping time with

$$\gamma \in \mathcal{A}(x,t) := \{\gamma \in \mathrm{AC}([0,t];\mathbb{R}^n) \mid \gamma(t) = x\} \quad \text{and} \quad \theta \in [0,t].$$

A distinctive feature of this optimal control problem is that the controller may either continue to optimize the trajectories satisfying the terminal condition or stop the process at any time by paying the stopping cost ψ. Then, we consider the value function u^ε for this control problem determined by

$$u^\varepsilon(x,t) = \inf_{\substack{\gamma \in \mathcal{A}(x,t) \\ \theta \in [0,t]}} J^\varepsilon(x,t,\gamma,\theta). \tag{1}$$

One can show that this value function u^ε satisfies dynamic programming principle, that is, u^ε satisfies

$$u^\varepsilon(x,t) = \inf_{\substack{\gamma\in\mathcal{A}(x,t)\\ \theta\in[0,t]}} \left\{ \int_{\theta\vee s}^t L\left(\frac{\gamma(r)}{\varepsilon}, \dot\gamma(r)\right) dr + \mathbf{1}_{\{\theta\le s\}} u^\varepsilon(\gamma(s),s) + \mathbf{1}_{\{\theta>s\}}\psi\left(\gamma(\theta), \frac{\gamma(\theta)}{\varepsilon}\right)\right\}$$

for all $(x,t)\in\mathbb{R}^n\times[0,\infty)$ and $s\in[0,t]$. Moreover, this dynamic programming principle equality implies that u^ε is a viscosty solution to $(\mathrm{OP}_\varepsilon)$.

We now recall the definition of viscosity solutions to obstacle problems.

Definition 1. We say that

(a) the function $w\in\mathrm{USC}(\mathbb{R}^n\times[0,\infty))$ is a viscosity subsolution to $(\mathrm{OP}_\varepsilon)$ if $w(\cdot,0)\le u_0$ on $\mathbb{R}^n$, and, for any $(x_0,t_0)\in\mathbb{R}^n\times(0,\infty)$ and for any $\phi\in C^1(\mathbb{R}^n\times(0,\infty))$ such that $w-\phi$ attains a local maximum at (x_0,t_0), we have

$$\max\left\{\phi_t(x_0,t_0) + H\left(\frac{x_0}{\varepsilon}, D\phi(x_0,t_0)\right),\ w(x_0) - \psi\left(x_0, \frac{x_0}{\varepsilon}\right)\right\} \le 0; \tag{2}$$

(b) the function $v\in\mathrm{LSC}(\mathbb{R}^n\times[0,\infty))$ is a viscosity supersolution to $(\mathrm{OP}_\varepsilon)$ if $v(\cdot,0)\ge u_0$ on $\mathbb{R}^n$, and, for any $(x_0,t_0)\in\mathbb{R}^n\times(0,\infty)$ and for any $\phi\in C^1(\mathbb{R}^n\times(0,\infty))$ such that $v-\phi$ attains a local minimum at (x_0,t_0), we have

$$\max\left\{\phi_t(x_0,t_0) + H\left(\frac{x_0}{\varepsilon}, D\phi(x_0,t_0)\right),\ v(x_0) - \psi\left(x_0, \frac{x_0}{\varepsilon}\right)\right\} \ge 0; \tag{3}$$

(c) the function $w\in C(\mathbb{R}^n\times[0,\infty))$ is a viscosity solution to $(\mathrm{OP}_\varepsilon)$ if w is both a viscosity sub and supersolution to $(\mathrm{OP}_\varepsilon)$.

Remark 2. Inequality (2) is equivalent to the condition that

$$\phi_t(x_0,t_0) + H\left(\frac{x_0}{\varepsilon}, D\phi(x_0,t_0)\right) \le 0 \quad\text{and}\quad w(x_0)\le\psi\left(x_0,\frac{x_0}{\varepsilon}\right).$$

Similarly, (3) is equivalent to the condition that

$$\phi_t(x_0,t_0) + H\left(\frac{x_0}{\varepsilon}, D\phi(x_0,t_0)\right) \ge 0 \quad\text{or}\quad v(x_0)\ge\psi\left(x_0,\frac{x_0}{\varepsilon}\right).$$

Thus, $(\mathrm{OP}_\varepsilon)$ can be written as

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) \le 0 & \text{in } \mathbb{R}^n\times(0,\infty),\\ u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) \ge 0 & \text{in } \left\{u^\varepsilon < \psi\left(x,\frac{x}{\varepsilon}\right)\right\},\\ u^\varepsilon \le \psi\left(x,\frac{x}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n\times[0,\infty), & u^\varepsilon(x,0)=u_0(x) \quad \text{on } \mathbb{R}^n. \end{cases}$$

From the viewpoint of control, the obstacle represents the cost that the controller need to pay for stopping the process, and the associated Hamilton–Jacobi variational inequality of obstacle type describes the competition between continuation and stopping. Optimal stopping problems are a fundamental class of optimal control problems whose value functions satisfy variational inequalities. In particular, in the absence of control acting on the dynamics, such problems were studied via dynamic programming approach before the development of the viscosity solution theory, and the analysis relied on the free boundary separating the continuation and stopping regions. We refer to [Men80, CDM81, Men82] for classical results and further developments. For the viscosity solution theory of Hamilton–Jacobi equations arising from optimal stopping problems, we refer to [BP87, BCD97].

Homogenization for obstacle problems. Our main results are as follows:

- Under appropriate assumptions for H, ψ and u_0, the unique viscosity solution u^ε to (OP_ε) converges locally uniformly, as $\varepsilon \to +0$, to a function u, which solves a certain limitimg equation.
- Moreover, if the obstacle function ψ is Lipschitz continuous, then the convergence is achieved with the optimal rate
$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} = O(\varepsilon) \quad \text{as } \varepsilon \to +0.$$
- On the other hand, if ψ is not Lipschitz continuous, the convergence rate may deteriorate. In fact, we construct an example of (OP_ε) for which the convergence rate is not $O(\varepsilon)$.

By considering the behavior of the cost functional and minimizing control when ε is small, it is expected that u^ε converges to another value function u defined by
$$u(x,t) = \inf_{\substack{\gamma \in \mathcal{A}(x,t) \\ \theta \in [0,t]}} \overline{J}(x,t,\gamma,\theta)$$
where $\overline{J}$ is defined by
$$\overline{J}(x,t,\gamma,\theta) := \int_\theta^t \overline{L}\left(\dot\gamma(s)\right) ds + \mathbf{1}_{\{\theta=0\}} u_0(\gamma(0)) + \mathbf{1}_{\{\theta>0\}} \overline{\psi}\left(\gamma(\theta)\right).$$
and $\overline{L}$ and $\overline{\psi}$ is defined by
$$\overline{L}(q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - \overline{H}(p)\} \quad \text{and} \quad \overline{\psi}(x) = \min_{y \in \mathbb{R}^n} \psi(x,y),$$
respectively. Here, the convex function $\overline{H} \in C(\mathbb{R}^n)$ is the effective Hamiltonian determined by the usual periodic cell problem
$$H(y, p + Dv) = \overline{H}(p) \quad \text{in } \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n \tag{Cell_p}$$
for each $p \in \mathbb{R}^n$ (more precisely, $\overline{H}(p)$ is the unique constant such that (Cell_p) has a $\mathbb{Z}^n$-periodic viscosity solution v). Furthermore, the value function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a viscosity solution to the effective obstacle problem:
$$\begin{cases} \max\left\{u_t + \overline{H}(Du),\ u - \overline{\psi}(x)\right\} = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{$\overline{\mathrm{OP}}$}$$

Throughout this talk, we assume, for (OP_ε), the Hamiltonian $H = H(y,p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies:

(A1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and the map $y \mapsto H(y,p)$ is $\mathbb{Z}^n$-periodic for each $p \in \mathbb{R}^n$,

(A2) H is coercive in p, that is,
$$\lim_{|p|\to\infty} \inf_{y \in \mathbb{R}^n} H(y,p) = +\infty,$$

(A3) the map $p \mapsto H(y,p)$ is convex for each $y \in \mathbb{R}^n$.

Additionaly, we assume the obstacle function $\psi = \psi(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the initial datum $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy:

(A4) $\psi \in C(\mathbb{R}^n \times \mathbb{R}^n)$, ψ is bounded below and the map $y \mapsto \psi(x,y)$ is $\mathbb{Z}^n$-periodic for each $x \in \mathbb{R}^n$,

(A5) $u_0 \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$ and satisfies
$$u_0(x) \le \min_{y \in \mathbb{R}^n} \psi(x,y) \ \left(= \overline{\psi}(x) \right) \quad \text{for all } x \in \mathbb{R}^n.$$

Finally, we mention several simplifications that can be made without loss of generality. Under assumptions (A1)–(A5), equation (OP_ε) admits a unique viscosity solution u^ε, which is Lipschitz continuous in space and time, for each $\varepsilon > 0$. Moreover, there exsits a constant $C > 0$ independent of ε such that

$$\|u_t^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} + \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le C.$$

Therefore, the value of Hamiltonian $H(y,p)$ for $|p| > C$ are irrelevant to analyze (OP_ε). Modifying the value of $H(y,p)$ on $\mathbb{R}^n \times (\mathbb{R}^n \setminus B(0,R))$ for sufficiently large $R > 0$ if necessary, we may assume that there exists a constant $K_0 > 0$ such that

$$\frac{1}{2}|p|^2 - K_0 \le H(y,p) \le \frac{1}{2}|p|^2 + K_0 \quad \text{for all } (y,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4}$$

By this argument, we may also assume the Lagrangian L satisfies the same growth condition

$$\frac{1}{2}|q|^2 - K_0 \le L(y,q) \le \frac{1}{2}|q|^2 + K_0 \quad \text{for all } (y,q) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and thus optimal control representation (1) remains valid under growth condition (A2) for H, without requiring the superlinear growth condition.

2. Previous literatures and methods for quantitative homogenization

Effective equations and cell problems. Here, we briefly recall some standard facts from the homogenization theory of Hamilton–Jacobi equations. The homogenization of Hamilton–Jacobi equations in periodic media was initiated in the seminal work of Lions, Papanicolaou and Varadhan [LPV87], and has become an important topic in the theory of viscosity solutions with connections to optimal control, calculus of variations and dynamical systems. We consider the following Cauchy problem for a Hamilton–Jacobi equation:

$$\begin{cases} u_t^\varepsilon + H\left(\dfrac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{5}$$

Under suitable assumptions on H such as (A1) and (A2), and on u_0, it is known that the solution u^ε to (5) converges, as $\varepsilon \to +0$, locally uniformly to a function u which solves an effective Hamilton–Jacobi equation with a homogenized Hamiltonian $\overline{H}$:

$$\begin{cases} u_t + \overline{H}\,(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{6}$$

Here, the function $\overline{H}$, called the effective Hamiltonian, is defined as follows: for each $p \in \mathbb{R}^n$, we consider a constant c_p and a function v satisfying the equation

$$H(y, p + Dv) = c_p \quad \text{in } \mathbb{T}^n. \tag{7}$$

Equation (7) is called the cell problem and it is known that, under suitable conditions on H, there exists a unique constant $c_p \in \mathbb{R}$ such that equation (7) admits a periodic viscosity solution $v \in C(\mathbb{T}^n)$. We denote $\overline{H}(p) = c$. It is also known that the function $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is continuous and coercive, and if H is convex, then $\overline{H}$ is also convex. It was proved in [LPV87] and [Eva92] that the unique viscosity solution u^ε to (5) converges locally uniformly to the unique viscosity solution u to (6). The convergence can be established by a PDE approach based on Evans' perturbed test function method, a viscosity solution technique which combines the doubling-of-variables argument with the correctors v obtained from the cell problem.

Quantitative homogenization for convex Hamilton–Jacobi equations. Next, we review some representative results on quantitative homogenization theory for Hamilton–Jacobi equations. Quantitative homogenization aims to estimate the rate of convergence of u^ε to a homogenized solution u. For general coercive Hamiltonians, Capuzzo-Dolcetta and Ishii [CDI01] obtained the convergence rate $O(\varepsilon^{1/3})$ for the homogenization of (5). In the convex setting, Mitake, Tran, and Yu [MTY19] developed a quantitative approach based on representation formulas from optimal control theory and fine properties of minimizing curves, making essential use of backward characteristics. From the same viewpoint, Tran and Yu [TY25] further refined the analysis and established the optimal convergence rate $O(\varepsilon)$ for periodic homogenization of convex Hamilton–Jacobi equations. A key ingredient in [TY25] is the use of a metric-type quantity $m(t,x,z)$, defined as

$$m(t,x,z) = \inf\left\{ \int_0^t L(\eta(s),\dot\eta(s))ds \;:\; \eta \in \mathrm{AC}([0,t];\mathbb{R}^n),\ \eta(0)=x,\ \eta(t)=z \right\}. \tag{8}$$

From the viewpoint of optimal control, $m(t,x,z)$ represents the minimum cost traveling from $x \in \mathbb{R}^n$ to $z \in \mathbb{R}^n$ in a fixed time $t > 0$. They established an approximate sub and superadditivity property for m. More precisely, they proved that for each $M > 0$, there exists $C > 0$ depending only on n, L and M such that

$$2m(t,x,z) - C \le m(2t,2x,2z) \le 2m(t,x,z) + C$$

holds for every x, z and t with $|x - z| \le Mt$. Subadditivity of m implies, via Fekete's lemma, that the homogenized limit

$$\overline{m}(t,x,z) := \lim_{k\to\infty} \frac{1}{k} m(kt,kx,kz)$$

exists for each x, z and t. Combining this with superadditivity, the following quantitative estimate for this limit holds:

Theorem 3 (see [TY25])**.** *For each $x, z \in \mathbb{R}^n$ and $t > 0$, the limit*

$$\overline{m}(t,x,z) = \lim_{\varepsilon\to+0} \varepsilon m\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\frac{z}{\varepsilon}\right)$$

exists. Moreover, for every $M > 0$, there exists a constant $C > 0$ depending only on n, L and M such that

$$\left|\varepsilon m\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\frac{z}{\varepsilon}\right) - \overline{m}(t,x,y)\right| \le C\varepsilon \tag{9}$$

holds for every $x, z \in \mathbb{R}^n$ and $t > 0$ with $|x - z| \le Mt$.

The proof of the superadditivity property relies on a geometric construction, often referred to as a curve surgery argument: roughly speaking, the idea is to cut an optimal trajectory via a curve cutting lemma due to topologial Burago [Bur94], shift the resulting pieces using periodicity, and patch them back together to build admissible competitors.

Using the optimal control representation for Cauchy problems together with a change of variables, the unique viscosity solution u^ε to (5) can be rewritten in the form

$$u^\varepsilon(x,t) = \inf_{z\in\mathbb{R}^n}\left\{ \varepsilon m\left(\frac{t}{\varepsilon},\frac{z}{\varepsilon},\frac{x}{\varepsilon}\right) + u_0(z)\right\}. \tag{10}$$

Combining the homogenization result for (5) with the Hopf–Lax formula for (6), one obtains the following representation formula for the homogenized solution u of (6):

$$u(x,t) = \inf_{z\in\mathbb{R}^n}\left\{ t\overline{L}\left(\frac{x-z}{t}\right) + u_0(z)\right\} = \inf_{z\in\mathbb{R}^n}\left\{\overline{m}(t,z,x) + u_0(z)\right\}. \tag{11}$$

The estimate $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le C\varepsilon$ then follows by applying (9) to (10) and (11). This observation forms the basis of the proof of the optimal convergence rate $O(\varepsilon)$.

Building on the optimal control approach, sharp convergence rates have recently been established in a variety of homogenization settings for convex Hamilton–Jacobi equations. The quantitative theory has subsequently been extended to Cauchy problems with multi-scale Hamiltonians $H(x, x/\varepsilon, p)$ [Tu21, HJ23], t/ε-variable dependent Hamiltonians [NT24], u/ε-variable dependent Hamiltonians [MNT25], weakly coupled Hamilton–Jacobi systems [MN25], viscous Hamilton–Jacobi equations [QSTY24], among others. Quantitative homogenization for initial-boundary value problems on perforated domains has also been investigated. The metric approach of [TY25] was extended to state-constraint, Neumann and Dirichlet boundary conditions on perforated domains in [HJMT25], [MN26] and [HT25], respectively. On the other hand, quantitative homogenization and sharp error estimates for obstacle problems are still scarce, although Hamilton–Jacobi variational inequalities of obstacle-type arise naturally in optimal control theory. The present work is motivated by this gap in the literature, as well as by the recent quantitative studies on perforated domains [HJMT25, MN26, HT25].

3. Main results

We assume (A1)–(A5). By using modification (4) if necessary, we can apply optimal control formulation for the solution u^ε to (OP_ε) and arguments for metric function. By using the notation (8), the unique solution u^ε to (OP_ε) can be rewritten as

$$\begin{aligned}
&u^\varepsilon(x,t)\\
&= \inf_{\substack{\eta\in\mathcal{A}(x/\varepsilon,t/\varepsilon)\\ \theta\in[0,t]}} \left\{ \varepsilon\int_{\theta/\varepsilon}^{t/\varepsilon} L(\eta(s),\dot\eta(s))ds + \mathbf{1}_{\{\theta=0\}} u_0(\varepsilon\eta(0)) + \mathbf{1}_{\{\theta>0\}}\psi\left(\varepsilon\eta\left(\frac{\theta}{\varepsilon}\right), \eta\left(\frac{\theta}{\varepsilon}\right)\right)\right\}\\
&= \inf_{\substack{z\in\mathbb{R}^n\\ \theta\in(0,t]}} \left\{ \varepsilon m\left(\frac{t-\theta}{\varepsilon}, \frac{z}{\varepsilon}, \frac{x}{\varepsilon}\right) + \mathbf{1}_{\{\theta=0\}}u_0(z) + \mathbf{1}_{\{\theta>0\}}\psi\left(z,\frac{z}{\varepsilon}\right)\right\}\\
&= \min\left\{ \inf_{\substack{z\in\mathbb{R}^n\\ \theta\in(0,t]}} \left\{\varepsilon m\left(\frac{t-\theta}{\varepsilon}, \frac{z}{\varepsilon}, \frac{x}{\varepsilon}\right) + \psi\left(z,\frac{z}{\varepsilon}\right)\right\},\ \inf_{z\in\mathbb{R}^n}\left\{\varepsilon m\left(\frac{t}{\varepsilon},\frac{z}{\varepsilon},\frac{x}{\varepsilon}\right)+u_0(z)\right\}\right\},
\end{aligned} \tag{12}$$

and the unique viscosity solution u to ($\overline{\mathrm{OP}}$) has the formula

$$\begin{aligned}
u(x,t) &= \inf_{\substack{z\in\mathbb{R}^n\\ \theta\in[0,t]}} \left\{ (t-\theta)\overline{L}\left(\frac{x-z}{t-\theta}\right) + \mathbf{1}_{\{\theta=0\}}u_0(z) + \mathbf{1}_{\{\theta>0\}}\psi\left(z,\frac{z}{\varepsilon}\right)\right\}\\
&= \min\left\{ \inf_{\substack{z\in\mathbb{R}^n\\ \theta\in(0,t]}} \left\{\overline{m}\left(t-\theta, z, x\right) + \overline{\psi}\left(z\right)\right\},\ \inf_{z\in\mathbb{R}^n}\left\{\overline{m}\left(t,z,x\right)+u_0(z)\right\}\right\}.
\end{aligned}$$

Estimating the errors $\varepsilon m(t/\varepsilon, z/\varepsilon, x/\varepsilon) - \overline{m}(t,z,x)$ and $\psi(z,z/\varepsilon) - \overline{\psi}(z)$, we obtain the following homogenization result. We state our main result:

Theorem 4. *Assume* (A1)–(A5). *Let u^ε and u be the unique viscosity solutions to* (OP_ε) *for each $\varepsilon > 0$ and* ($\overline{\mathrm{OP}}$), *respectively. Then, as $\varepsilon \to +0$, u^ε converges locally uniformly on $\mathbb{R}^n \times [0,\infty)$ to u. Moreover, if $\psi \in \mathrm{Lip}(\mathbb{R}^n\times\mathbb{R}^n)$, then there exists $C > 0$ depending only on H, ψ and u_0 such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} \le C\varepsilon \quad \textit{for all } \varepsilon > 0.$$

In the remainder of this section, we give a brief overview of the quantitative part of the proof of this main theorem. The difficulty to prove theorem 4 lies in controlling the stopping time $\theta \in [0, t]$ and estimating the limit behavior of $\psi(z, z/\varepsilon)$ for "optimal" z.

To apply theorem 3 by Tran and Yu, it is necessary to show that, in representation formula (12), the quantity $|x - z|/(t - \theta)$ is uniformly bounded for every $\varepsilon > 0$, every (x, t) and every pair (z, θ) attaining the infimum on the right-hand side of (12). For Cauchy problems without obstacles, the quadratic growth condition of the Lagrangian implies that, in the representation formula (10), the quantity $|x-z|/t$ is uniformly bounded for every $\varepsilon > 0$, every (x, t) and every z attaining the infimum. In our setting, however, representation formula (12) involves the additional stopping time $\theta \in [0, t]$, so such an estimate is no longer immediate. To overcome this difficulty, we return to original optimal control formula (1) and analyze the minimizing control (γ, θ) attaining the infimum. The Lipschitz estimate for the optimal trajectory in the optimal stopping problem is as follows.

Proposition 5. *Assume* (A1)–(A5)*. Then, for each $\varepsilon > 0$ and $(x, t) \in \mathbb{R}^n \times [0, \infty)$, there exists a minimizer $(\gamma, \theta) \in \mathcal{A}(x, t) \times [0, t]$ which attains the infimum of optimal control formula* (1)*. Moreover, there exists a constant $M > 0$ depending only on n, L and $\|Du_0\|_{L^\infty(\mathbb{R}^n)}$ such that $|\dot{\gamma}(s)| \leq M$ for a.e. $s \in [\theta, t]$.*

As a consequence, in representation formula (12), there exists a universal constant $M > 0$ such that $|x - z| \leq M(t - \theta)$ for every ε, every (x, t), and every pair (z, θ) attaining the infimum on the right-hand side of (12). This argument provides the desired estimate for the running cost term in (12).

It remains to estimate the stopping cost term. For each $\varepsilon > 0$ and $z \in \mathbb{R}^n$, we introduce the "nearest minimizer" $\tilde{z} \in \mathbb{R}^n$, namely a point satisfying

$$\frac{z - \widetilde{z}}{\varepsilon} \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n \quad \text{and} \quad \psi\left(z, \frac{\widetilde{z}}{\varepsilon}\right) = \min_{y \in \mathbb{R}^n} \psi(z, y) \ (= \overline{\psi}(z)).$$

By construction, the inequality $|z - \widetilde{z}| \leq \sqrt{n}\varepsilon$ holds. Hence, if ψ is Lipschitz continuous, then $\overline{\psi}(z) = \psi(z, \widetilde{z}/\varepsilon) \approx \psi(\widetilde{z}, \widetilde{z}/\varepsilon) + C\sqrt{n}\varepsilon$ for each ε and z. Consequently, if, for some ε, (x, t) and (z, θ),

$$u^\varepsilon(x, t) = \varepsilon m\left(\frac{t - \theta}{\varepsilon}, \frac{z}{\varepsilon}, \frac{x}{\varepsilon}\right) + \psi\left(z, \frac{z}{\varepsilon}\right)$$

holds, then

$$\begin{aligned} u^\varepsilon(x, t) &\geq \varepsilon m\left(\frac{t - \theta}{\varepsilon}, \frac{\widetilde{z}}{\varepsilon}, \frac{x}{\varepsilon}\right) + \psi\left(\widetilde{z}, \frac{\widetilde{z}}{\varepsilon}\right) + O(\varepsilon) \\ &\geq \overline{m}(t - \theta, \widetilde{z}, x) + \overline{\psi}(\widetilde{z}) + O(\varepsilon) \geq u(x, t) + O(\varepsilon) \end{aligned}$$

follows. Likewise, if

$$u(x, t) = \overline{m}(t - \theta, z, x) + \overline{\psi}(z)$$

holds, then

$$\begin{aligned} u(x, t) &= \overline{m}(t - \theta, z, x) + \psi\left(z, \frac{\widetilde{z}}{\varepsilon}\right) = \overline{m}(t - \theta, \widetilde{z}, x) + \psi\left(\widetilde{z}, \frac{\widetilde{z}}{\varepsilon}\right) + O(\varepsilon) \\ &\geq \varepsilon m\left(\frac{t - \theta}{\varepsilon}, \frac{\widetilde{z}}{\varepsilon}, \frac{x}{\varepsilon}\right) + \psi\left(\widetilde{z}, \frac{\widetilde{z}}{\varepsilon}\right) + O(\varepsilon) \geq u^\varepsilon(x, t) + O(\varepsilon) \end{aligned}$$

follows. Combining these inequalities, we obtain the optimal convergence rate $O(\varepsilon)$.

Finally, in this talk, we also discuss the case where the obstacle function ψ is merely α-Hölder continuous rather than Lipschitz continuous. In this setting, we present an example of (OP_ε) for which the convergence rate of homoginization deteriorates to $O(\varepsilon^\alpha)$.

References

[BCD97] Martino Bardi and Italo Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[BP87] G. Barles and B. Perthame. Discontinuous solutions of deterministic optimal stopping time problems. *RAIRO Modél. Math. Anal. Numér.*, 21(4):557–579, 1987.

[Bur94] D. Burago. Periodic metrics. In *Seminar on Dynamical Systems (St. Petersburg, 1991)*, volume 12 of *Progr. Nonlinear Differential Equations Appl.*, pages 90–95. Birkhäuser, Basel, 1994.

[CDI01] I. Capuzzo-Dolcetta and H. Ishii. On the rate of convergence in homogenization of Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 50(3):1113–1129, 2001.

[CDM81] I. Capuzzo Dolcetta and M. Matzeu. On the dynamic programming inequalities associated with the deterministic optimal stopping problem in discrete and continuous time. *Numer. Funct. Anal. Optim.*, 3(4):425–450, 1981.

[Eva92] Lawrence C. Evans. Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(3-4):245–265, 1992.

[HJ23] Yuxi Han and Jiwoong Jang. Rate of convergence in periodic homogenization for convex Hamilton-Jacobi equations with multiscales. *Nonlinearity*, 36(10):5279–5297, 2023.

[HJMT25] Yuxi Han, Wenjia Jing, Hiroyoshi Mitake, and Hung V. Tran. Quantitative homogenization of state-constraint Hamilton-Jacobi equations on perforated domains and applications. *Arch. Ration. Mech. Anal.*, 249(2):Paper No. 18, 53, 2025.

[HT25] Yuxi Han and Son Tu. Quantitative homogenization of Hamilton–Jacobi equations on perforated domains with Dirichlet boundary conditions, Preprint is available from arXiv:2510.27099, 2025.

[LPV87] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan. Homogenization of Hamilton–Jacobi equations, Unpublished work, 1987.

[Men80] J.-L. Menaldi. On the optimal stopping time problem for degenerate diffusions. *SIAM J. Control Optim.*, 18(6):697–721, 1980.

[Men82] José-Luis Menaldi. Le problème de temps d'arrêt optimal déterministe et l'inéquation variationnelle du premier ordre associée. *Appl. Math. Optim.*, 8(2):131–158, 1982.

[MN25] Hiroyoshi Mitake and Panrui Ni. Rate of convergence for homogenization of nonlinear weakly coupled Hamilton-Jacobi systems. *J. Differential Equations*, 440:Paper No. 113442, 24, 2025.

[MN26] Hiroyoshi Mitake and Panrui Ni. Quantitative homogenization of convex Hamilton-Jacobi equations with Neumann type boundary conditions. *Calc. Var. Partial Differential Equations*, 65(5):Paper No. 154, 41, 2026.

[MNT25] Hiroyoshi Mitake, Panrui Ni, and Hung. V. Tran. Quantitative homogenization of convex Hamilton–Jacobi equations with u/ε-periodic Hamiltonians, Preprint is available from arXiv:2507.00663, 2025.

[MTY19] Hiroyoshi Mitake, Hung V. Tran, and Yifeng Yu. Rate of convergence in periodic homogenization of Hamilton-Jacobi equations: the convex setting. *Arch. Ration. Mech. Anal.*, 233(2):901–934, 2019.

[NT24] Hoang Nguyen-Tien. Optimal convergence rate for homogenization of convex Hamilton-Jacobi equations in the periodic spatial-temporal environment. *Asymptot. Anal.*, 138(1-2):135–150, 2024.

[QSTY24] Jianliang Qian, Timo Sprekeler, Hung V. Tran, and Yifeng Yu. Optimal rate of convergence in periodic homogenization of viscous Hamilton-Jacobi equations. *Multiscale Model. Simul.*, 22(4):1558–1584, 2024.

[Tu21] Son N. T. Tu. Rate of convergence for periodic homogenization of convex Hamilton-Jacobi equations in one dimension. *Asymptot. Anal.*, 121(2):171–194, 2021.

[TY25] Hung V. Tran and Yifeng Yu. Optimal convergence rate for periodic homogenization of convex Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 74(3):555–573, 2025.

(Takuya Sato) Affiliation: Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan. E-mail: satoh-t@g.ecc.u-tokyo.ac.jp.

Boundary value problems of magnetohydrodynamics in 3D bounded domains

Taku Yanagisawa

Institute of Mathematical Science, Waseda University, Tokyo, Japan
Shinjuku-ku, Tokyo, 169-8555, Japan
taku@aoni.waseda.jp

Keywords: boundary value problems, Magnetohydrodynamics, Helmholtz-Weyl decomposition

1 Introduction

Magneto-Hydro-Dynamics (MHD in short) represents one of the simplest self-consistent models describing the macroscopic behaviour of electrically conducting fluids such as plasmas. Amongst many mathematical problems relating to MHD, I am concern about the boundary value problems of stationary incompressible MHD equations, in particular, the issues on the existence, uniqueness and regularity of their weak solutions. All results presented here are based on the collaboration with Hideo Kozono (Waseda University) and Senjo Shimizu (Kyoto University).

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the boundary value problems of incompressible stationary MHD equations in Ω:

$$\text{(MHD)} \qquad \begin{cases} -\Delta u + u\cdot\nabla u - B\cdot\nabla B + \nabla(\frac{1}{2}|B|^2 + p) = f \quad \text{in } \Omega, \\ \operatorname{rot}(\operatorname{rot} B) + \operatorname{rot}(B\times u) = g \quad \text{in } \Omega, \\ \operatorname{div} u = 0, \qquad \operatorname{div} B = 0 \quad \text{in } \Omega, \\ u = 0, \;\; B\cdot\nu = 0, \;\; \operatorname{rot} B\times\nu = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where $u = u(x) = (u_1(x), u_2(x), u_3(x))$, $p = p(x)$ and $B = B(x) = (B_1(x), B_2(x), B_3(x))$ are the velocity vector of the fluid, the pressure of the fluid and the magnetic vector field at point $x = (x_1, x_2, x_3)$, respectively, while $f = f(x) = (f_1(x), f_2(x), f_3(x))$ and $g = g(x) = (g_1(x), g_2(x), g_3(x))$ are the fluiddynamic and magnetic external forces, respectively. Here $\nu = \nu(x)$ denotes the unit outer normal vector at $x\in\partial\Omega$.

We firstly give an outline of the magnetohydrodynamic approximation in the next section. This process seems important to investigate, from physical point of view, proper assumptions imposed on the external forces f, g in (MHD). Whereas, since our study regarding the relationship between the topological structure of underlying domains Ω and the mathematical structure of (MHD) essentially depends on the Helmholtz-Weyl decomposition of vector fields, we briefly summarize the decomposition in §3. In $ 4, we state our main theorems.

2 Magnetohydrodymamic approximation

We begin by outlining the magnetohydrodynamic (MHD) approximation for models of incompressible, electrically conducting fluids, in which the Navier-Stokes equations are coupled to Maxwell's equations through Ohm's law (see [2]).

The basic equations for the motion of an electrically conducting incompressible fluid in the presence of eletromagnetic fields are the Navier-Stokes equations:

$$\text{(NS)} \qquad \begin{cases} (1) \quad \dfrac{\partial}{\partial t}u + u\cdot\nabla u = \kappa\Delta u - \nabla p + f, \\ (2) \quad \operatorname{div} u = 0, \end{cases}$$

and Maxwell's equations:

$$(\text{Maxwell}) \qquad \begin{cases} (3) \quad \operatorname{rot} E + \dfrac{\partial B}{\partial t} = 0, \\ (4) \quad \operatorname{rot} H - \dfrac{\partial D}{\partial t} = J, \\ (5) \quad \operatorname{div} D = \rho_e, \\ (6) \quad \operatorname{div} B = 0, \end{cases}$$

where, in (NS), $u = u(t, x)$ is the velocity vector of the time and spatial variables t and x, $p = p(t, x)$ the pressure, κ the coefficient of viscosity which is assumed to be a positive constant, and $f = f(t, x)$ the body force, the specific form of which is given in (10) below, whereas, in (Maxwell), $E = E(t, x)$ is the electric vector field, $B = B(t, x)$ the magnetic induction, $H = H(t, x)$ the magnetic vector field, $D = D(t, x)$ the electric displacement, $J = J(t, x)$ the electric current density, and $\rho_e = \rho_e(t, x)$ the electric charge density. We further assume that the following relations hold:

$$(\text{R}) \qquad \begin{cases} (7) \quad D = \epsilon E, \quad B = \mu H, \\ (8) \quad J = \rho_e u + j + j_0, \\ (9) \quad j = \sigma(E + u \times B), \end{cases}$$

where ϵ is the dielectric constant, μ the magnetic permeability, and σ the electric conductivity. The j_0 in (8) is the external current density. The relations (7) are valid, if we consider ordinary substances which are not ferromagnetic nor ferroelectric. The $\rho_e u$, j and j_0 in the relation (8) are called the convection current, the conduction current and the external current density, respectively. The equation (9) represents Ohm's law. When we suppose that the Hall effect is negligible, we can assume that σ is a scalar [1]. For simplicity, we assume that ϵ, μ, σ are positive constants.

Since electromagnetic vector fields exert forces on the electric charge density and the electric current density, the body force f in (1) is given by

$$f = f_{(\text{fluid})} + \rho_e E + J \times B, \tag{10}$$

where $f_{(\text{fluid})}$ is the fluiddynamic external force.

Here are two assumptions in the magnetohydrodynamic approximation:

Assumption I (Negligible displacement current) : The fluid is a good conductor, which means that

$$\frac{\partial D}{\partial t} = 0. \tag{11}$$

Assumption II (Quasineutrality) : The fluid is almost neutral, which means that $\rho_e \fallingdotseq 0$. This means that the size of ρ_e is much smaller than the other quantities. More precisely, the following equalities are assumed to hold:

$$J = j + j_0, \tag{12}$$

[1] If the Hall effect is taken into account, the equation (9) must be replaced by

$$(9)' \quad j = \sigma_0(E + u \times B) - \frac{\sigma_0}{n_e e} j \times B,$$

where σ_0 is the electric conductivity in the absence of magnetic vector fields, n_e the electron number density, and e the charge of an electron.

and

$$f = f_{(\text{fluid})} + J \times B. \tag{13}$$

The equality (12) means that the magnitude $\|\rho_e u\|$ is negligible compared to $\|j\|$ and $\|j_0\|$ in (8), while the equality (13) means that $\|\rho_e E\|$ is negligible compared to $\|f_{(\text{fluid})}\|$ and $\|J \times B\|$ in (10).

Under the assumptions I and II, we are going to derive MHD equations from (1)–(10). Firstly, we see from (11), (4) that

$$\operatorname{rot} H = J. \tag{14}$$

Combining (12), (9) with (14), we have

$$\operatorname{rot} H = J = j + j_0 = \sigma\,(E + u \times B) + j_0. \tag{15}$$

Then, taking rot of the both sides of (15), by (3) we see that

$$\begin{aligned}\operatorname{rot}\operatorname{rot} H &= \sigma\,(\operatorname{rot} E + \operatorname{rot}\,(u \times B)) + \operatorname{rot}\, j_0\\ &= \sigma\left(-\frac{\partial B}{\partial t} - \operatorname{rot}\,(B \times u)\right) + \operatorname{rot}\, j_0,\end{aligned}$$

so that we have

$$\frac{\partial B}{\partial t} + \operatorname{rot}\,(B \times u) = -\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot} B + \frac{1}{\sigma}\operatorname{rot} j_0, \tag{16}$$

which is called the equation of induction. On the other hand, it follows from (13), (14), (7) that

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \kappa \Delta u - \nabla p + f_{(\text{fluid})} + \frac{1}{\mu}\,(\operatorname{rot} B \times B)\,. \tag{17}$$

The equations (16), (17) combined with $\operatorname{div}\, u = \operatorname{div}\, B = 0$ are incompressible nonstationary MHD equations.

Putting $\sigma = \mu = \kappa = 1$ in (16) and (17), utilizing the vector calculus identity that $\operatorname{rot}\, B \times B = B \cdot \nabla B - \frac{1}{2}\nabla |B|^2$, assuming that u, B, p, and $f_{(\text{fluid})}$, j_0 are all independent of the time variable, we now reach incompressible stationary MHD equations:

$$\begin{cases} -\Delta u + u \cdot \nabla u - B \cdot \nabla B + \nabla(\frac{1}{2}|B|^2 + p) = f_{(\text{fluid})},\\ \operatorname{rot}\,(\operatorname{rot} B) + \operatorname{rot}\,(B \times u) = \operatorname{rot} j_0,\\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0.\end{cases} \tag{18}$$

By setting $f = f_{(\text{fluid})}$ and $g = \operatorname{rot} j_0$ in (18), the equations in (MHD) in §1 are derived.

3 Helmholtz-Weyl decomposition

In this section, we summarize the Helmholtz-Weyl decomposition theorem (see [5]). Let Ω be a bounded domain in $\mathbb{R}^3$ with C^∞-boundary $\partial\Omega$. We introduce

$$\begin{aligned} X_{\text{har}}(\Omega) &= \{h \in C^\infty(\overline{\Omega}) \mid \operatorname{div} h = 0, \operatorname{rot} h = 0,\ h \cdot \nu = 0\},\\ V_{\text{har}}(\Omega) &= \{h \in C^\infty(\overline{\Omega}) \mid \operatorname{div} h = 0, \operatorname{rot} h = 0,\ h \times \nu = 0\},\end{aligned}$$

where $X_{\rm har}(\Omega)$ is the space of tangential harmonic vector fields on Ω and $V_{\rm har}(\Omega)$ the space of normal harmonic vector fields on Ω. Let $1 \le r \le \infty$. We define two spaces $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ by

$$\begin{array}{ll} X^r_\sigma(\Omega) & = \{ w \in W^{1,r}(\Omega) \mid \operatorname{div} w = 0, \quad w \cdot \nu = 0 \}, \\ V^r_\sigma(\Omega) & = \{ w \in W^{1,r}(\Omega) \mid \operatorname{div} w = 0, \quad w \times \nu = 0 \}. \end{array}$$

Then the Helmholtz-Weyl decomposition of L^r-vector fields on Ω is stated as

Proposition 3.1 *Let $1 < r < \infty$. We have two kinds of decompositions of the vector-valued space $L^r(\Omega)$:*

$$\begin{array}{l} L^r(\Omega) = X_{har}(\Omega) \oplus \operatorname{rot} V^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega), \\ L^r(\Omega) = V_{har}(\Omega) \oplus \operatorname{rot} X^r_\sigma(\Omega) \oplus \nabla W^{1,r}_0(\Omega). \end{array}$$

Here $\oplus$ denotes the direct sum.

Remark 1 *The spaces of harmonic vector fields $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are both finite dimensional. If the domain Ω satisfies the following topological conditions, the dimensions of $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are given explicitly (see [1]): Suppose that the boundary $\partial\Omega$ has $L+1$ connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_L$ of C^∞-surfaces such that $\Gamma_1, \ldots, \Gamma_l$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \phi$ for $i \ne j$, and $\partial\Omega = \cup_{j=0}^L \Gamma_j$. Moerover, we assume that there are N C^∞-surfaces $\Sigma_1, \ldots, \Sigma_N$ transversal to $\partial\Omega$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \ne j$, and such that $\Omega \backslash \cup_{j=1}^N \Sigma_j$ is a simply connected domain.*

Then it holds that

$$\dim X_{har}(\Omega) = N \quad \textit{and} \quad \dim V_{har}(\Omega) = L.$$

Remark 2 *We define $L^r_\sigma(\Omega)$ by the closure of $C^\infty_{0,\sigma}(\Omega)$ in L^r-norm $\|\cdot\|_r$, where $C^\infty_{0,\sigma}(\Omega)$is the set of all C^∞-vector valued functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with compact support in Ω such that $\operatorname{div}\ \varphi = 0$. Then, the first decomposition in Proposition 3.1 can be written as*

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega), \quad \textit{where} \quad L^r_\sigma(\Omega) = X_{har}(\Omega) \oplus \operatorname{rot} V^r_\sigma(\Omega).$$

In the next section, we will introduce the space $\mathscr{X}^r_\sigma(\Omega)$ by

$$\mathscr{X}^r_\sigma(\Omega) = \operatorname{rot} V^r_\sigma(\Omega).$$

The space $\mathscr{X}^r_\sigma(\Omega)$ plays a crucial role in our study on boundary value problems for MHD equations. We also define the projection R_r along the first decomposition in Proposition 3.1 by

$$R_r : L^r(\Omega) \to \mathscr{X}^r_\sigma(\Omega).$$

It is easy to see that $R_r^2 = R_r$, $R_r^* = R_{r'}$ for $1/r + 1/r' = 1$, R_r^* denoting the adjoint operator on $L^{r'}(\Omega)$ of R_r.

4 Main theorems

Our first result on existence of weak solutions to (MHD) in §1 now reads

Theorem 1 *For every $\{f, g\} \in H^1_0(\Omega)^* \times \mathscr{X}^{\frac{6}{5}}_\sigma(\Omega)$, there exists as least a pair*

$$\{u, B\} \in H^1_0(\Omega) \times \left(H^1(\Omega) \cap \mathscr{X}^2_\sigma(\Omega) \right)$$

such that

$$(19) \qquad (\nabla u, \nabla\varphi) + (u \cdot \nabla u - B \cdot \nabla B, \nabla\varphi) =< f, \varphi > \quad \textit{for all } \varphi \in H^1_{0,\sigma}(\Omega),$$

$$(20) \qquad (\operatorname{rot} B,\, \operatorname{rot} \Psi) + (\operatorname{rot}(B \times u),\, \Psi) = (g,\, \Psi) \ \textit{for all}\ \Psi \in H^1(\Omega),$$

where $(\cdot,\,\cdot)$ *denotes the* L^2*-inner product and* $< \cdot,\, \cdot >$ *the duality pairing between* $H_0^1(\Omega)^*$ *and* $H_0^1(\Omega)$*. Here* $H_0^1(\Omega)$ *is the closure of the vector valued* $C_0^\infty(\Omega)$ *in* $W^{1,2}(\Omega)$ *and* $H_{0,\sigma}^1(\Omega) \equiv \{u \in H_0^1(\Omega) \mid \operatorname{div} u = 0\}$*.* $\mathscr{X}_\sigma^r(\Omega)$ *is the same space as in Remark 2 in the preceding section.*

Remark 3 *Weak forms (19) and (20) imply that the pair* $\{u,\, B\}$ *satisfies (MHD) in the sense of distributions in* Ω*. It is easy to see that if* $B \in H^2(\Omega)$ *in addition, then it holds that* $\operatorname{rot} B \times \nu = 0$ *on* $\partial\Omega$*. It should be noted that the above existence theorem of weak solutions to (MHD) holds in all bounded smooth domains with arbitrary geometry. Indeed , we do not need to impose any topological restrictions on* Ω *such as simple connectivity.*

Remark 4 *The assumption for the magnetic external force that* $g \in \mathscr{X}^{\frac{6}{5}}(\Omega)$ *seems restrictive. However, such a restriction may be regarded as a compatibility condition for the existence of the strong solution* $\{u,\, B\} \in \big(H^2(\Omega) \cap H_{0,\sigma}^1(\Omega)\big) \times \big(H^2(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$*. Indeed, it follows from [Lemma 2.9 in [3]] that such a* $\{u,\, B\}$ *necessarily satisfies* $\operatorname{rot}(\operatorname{rot} B) + \operatorname{rot}(B \times u) \in \mathscr{X}_\sigma^2(\Omega)$*. Furthermore, from a viewpoint of magnetohydrodynamic approximation discussed in §2, it seems reasonable to assume that* $g \in \mathscr{X}^{\frac{6}{5}}$*. I will touch on this remark in my talk.*

Next we consider uniqueness and regularity of the weak solutions given in Theorem 1.

Theorem 2 *(i) There is a positive constant* δ *such that if the pair* $\{u,\, B\} \in H_0^1(\Omega) \times \big(H^1(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$ *satisfies (19) and (20) with*

$$\|u\|_{3,\infty} + \|B\|_{3,\infty} \le \delta,$$

then $\{u,\, B\}$ *is unique, where* $\|\cdot\|_{3,\infty}$ *denotes the weak-*L^3 *norm on* Ω*, i.e.,*

$$\|u\|_{3,\infty} \equiv \sup_{t>0} t\ \big(m(\{x \in \Omega \mid |u(x)| > t\})\big)^{1/3}$$

with m *denoting the Lebesgue measure in* $\mathbb{R}^3$*.*

Moreover, there is a positive constant δ' *such that if* $\{f,\, g\} \in H_0^1(\Omega)^* \times \mathscr{X}_\sigma^{\frac{6}{5}}(\Omega)$ *satisfies*

$$\|f\|_{H^{-1}} + \|g\|_{\frac{6}{5}} \le \delta',$$

then there exists a unique pair $\{u,\, B\} \in H_{0,\sigma}^1 \times \big(H^1(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$ *such that (19) and (20) are satisfied. Here* $\|\cdot\|_p$ *denotes the norm of* $L^p(\Omega)$*,* $1 \le p \le \infty$*, and* $\|\cdot\|_{H^{-1}}$ *the norm of* $H_0^1(\Omega)^*$*.*

(ii) If $\{f,\, g\} \in L^2(\Omega) \times \mathscr{X}_\sigma^2(\Omega)$*, then the pair* $\{u,\, B\}$ *given in Theorem 1 satisfies* $\{u,\, B\} \in \big(H^2(\Omega) \cap H_{0,\sigma}^1(\Omega)\big) \times \big(H^2(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$*, and fulfills (MHD) almost everywhere in* Ω*. More generally, if* $\{f,\, g\} \in H^m(\Omega) \times \big(H^m(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$ *for* $m = 1, 2, \ldots$*, then it holds that* $\{u,\, B\} \in \big(H^{m+2}(\Omega) \cap H_{0,\sigma}^1(\Omega)\big) \times \big(H^{m+2}(\Omega) \cap \mathscr{X}_\sigma^2(\Omega)\big)$*.*

Remark 5 *It should be noted that the weak-*L^3 *space is scaling invariant for (MHD). The above theorem states unique existence of weak solutions to (MHD) under small external forces* f *and* g*.*

Acknowledgement. The research of T. Yanagisawa is partially supported by JST [Moonshot R&D] [Grant Number JPMJMS25A5].

References

[1] Foias, C and Temam, R.: *Remarque sur les équations de Navier-Stokes staionnaires et les phénomènes successifs de bifurcation.* Ann. Scoula Norm. Sup. Pisa, Cl. Sci (4). **5**, 28–63 (1978).

[2] Imai,I.: *General principles of Magneto-Fluid Dynamics.* Supplement of Progr. of Theoret. Phys. **24**, 1–34 (1962).

[3] Kozono, H., Shimizu, S. and Yanagisawa, T.: *Stability of harmonic vector fields as an equilibrium of 3D MHD equations in bounded domains with arbitrary geometry.* Math. Ann. **393**, 923–952 (2025).

[4] Kozono, H., Shimizu, S. and Yanagisawa, T.: *Stationary MHD equations in 3D bounded domains with arbitrary geometry.* Submitted.

[5] Kozono, H. and Yanagisawa, T.: *L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains.* Indiana Univ. Math. J. **58**, 1853–1920 (2009).

Global solvability of simplified Ericksen-Leslie system with thermal effects

Yuka Teramoto
Ehime University Graduate school of Science and Engineering
2-5 Bunkyo-cho, Matsuyama 790-8577, Japan

1 Introduction

We consider the flow of compressible nematic liquid crystal in an infinite layer, which is governed by the following system based on simplified Ericksen-Leslie system:

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) &= 0, &(1)\\ \partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) &= \operatorname{div}\mathbb{S}_C, &(2)\\ \partial_t(\rho\theta) + \operatorname{div}(\rho\theta\boldsymbol{u}) + \operatorname{div}\boldsymbol{q} &= \mathbb{S}_C : \nabla\boldsymbol{u}, &(3)\\ \partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} &= \tau^*(\Delta\boldsymbol{d} + |\nabla\boldsymbol{d}|^2\boldsymbol{d}). &(4) \end{aligned}$$

Here the unknown variables ρ, $\boldsymbol{u}$, θ denote the density, velocity and temperature of fluid, respectively, $\boldsymbol{d}$ the orientation field for averaged macroscopic molecular directions of nematic liquid crystal, and hence, $|\boldsymbol{d}| = 1$. $\boldsymbol{q}$ is the heat flux given by $\boldsymbol{q} = -\kappa^*\nabla\theta$. Moreover, $\mathbb{S}_C$is the Cauchy stress tensor defined by

$$\mathbb{S}_C = \mathbb{S}_N - \eta^*(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d} - \frac{1}{2}|\nabla\boldsymbol{d}|^2\mathbb{I}) - P\mathbb{I},$$

where $P = P(\rho,\theta)$ is pressure which satisfies $\partial_\rho P \geq 0$, $\partial_\theta P \geq 0$, and $\mathbb{S}_N$ is the conventional Newtonian viscous stress tensor,

$$\mathbb{S}_N = \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top) + \mu'(\operatorname{div}\boldsymbol{u})\mathbb{I}.$$

The positive constants η^*, κ^* and τ^* describe the competition between kinetic and potential energy, ratio of the heat conductivity coefficient over the heat capacity and microscopic elastic relaxation time, respectively. μ and μ' are the shear viscosity and the second viscosity coefficients which are both constants satisfying $\mu > 0$ and $\mu + \frac{2}{3}\mu' \geq 0$. $\mathbb{I}$ is the identity matrix. We set $(\boldsymbol{u}\otimes\boldsymbol{u})_{ij} = u^iu^j$, $(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_{ij} = \partial_{x_i}\boldsymbol{d}\cdot\partial_{x_j}\boldsymbol{d}$ and $\mathbb{S}_C : \nabla\boldsymbol{u} = \sum_{i,j}(\mathbb{S}_C)_{ij}\partial_{x_i}u^j$. The system (1)–(4) is considered in the infinite layer Ω_h:

$$\Omega_h = \{x = (x', x_3);\ x' = (x_1, x_2),\ 0 < x_3 < h\}.$$

The boundary condition is

$$\boldsymbol{u}|_{x_3=0,h} = \mathbf{0}, \tag{5}$$

$$\theta|_{x_3=0} = \theta_0^*,\ \theta|_{x_3=h} = \theta_1^*, \tag{6}$$

$$\frac{\partial \boldsymbol{d}}{\partial x_3}|_{x_3=0,h} = \mathbf{0}, \tag{7}$$

for given positive constants θ_0^*, θ_1^*. We consider density ρ and temperature θ around ρ^* and θ_0^*, respectively, where ρ^* is given positive constant.

The Ericksen-Leslie system is produced by Ericksen [1, 2, 3] and Leslie [4, 5]. For the simplified isothermal model, Lin et al. [9] first investigated the existence of global weak solutions for Dirichlet boundary conditions on $\boldsymbol{d}$ in 2-dimensions. The decay rates were obtained by Wu [10] in order $\mathcal{O}(1+t)^{-\frac{\gamma}{1-2\gamma}}$ for some parameter $\gamma \in (0, \frac{1}{2})$ on a bounded domain. Hieber et al. [**?**] obtained the global well-poseness for simplified system subject to Neumann conditions. Later, Wang [11] proved the global well-posedness with small initial data in $BMO^{-1} \times BMO$ in whole space. In 3-dimensions, Huang et al. [14] gave examples of small initial data on Dirichlet boundary conditions for which one has finite time blow up of $(\boldsymbol{u}, \boldsymbol{d})$. For whole space $\mathbb{R}^3$, Dai [12] proved the asymptotic behavior of the classical solution which decays like a 3-dimensional heat equation for $\boldsymbol{d}$ and Navier-Stokes equations with a force decaying in $\mathcal{O}(t^{-\frac{3}{4}})$ for $\boldsymbol{u}$, respectively.

On the other hand, there are few results for non-isothermal model of nematic liquid crystal for global classical solutions. Hieber and Prüss [15] got the presentation of thermodinamically consistent modeling of nematic liquid crystals and analyzed non-isothemal simplified model in L^p-L^q setting. Guo et al. [16] established the existence of global in time smooth solutions and their decay rates in $\mathbb{R}^3$ with initial data which are close to steady state. The isothermal simplified model is first proposed by Lin [6] and many works have done for this isothermal model.

2 Existence of global classical solutions for small data and asymptotic behavior

The problem (1)–(5) has a stationary solution $u_s = {}^\top(\rho_s, \boldsymbol{u}_s, \theta_s, \boldsymbol{d}_s) = {}^\top(\rho_s, \mathbf{0}, \theta_s, \boldsymbol{d}^*)$. Here ρ_s and θ_s are smooth function of x_3 satisfying $|\rho_s - \rho^*| \ll 1$ and $|\theta_s - \theta_0^*| \ll 1$ and $\boldsymbol{d}^*$ is a constant vector. The perturbation equation from this motionless state u_s is written in the following nondimensionalized form:

$$
\begin{aligned}
\partial_t\phi + \mathrm{div}\,((\rho_s-1)\boldsymbol{u}) + \mathrm{div}\,\boldsymbol{u} &= f, &(8)\\
\partial_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} - (\nu+\nu')\nabla\mathrm{div}\,\boldsymbol{u} + \nabla\theta + \nabla\phi &= \boldsymbol{g}, &(9)\\
\partial_t\theta - \kappa\Delta\theta + \beta(P(\rho_s,\theta_s)-1)\mathrm{div}\,\boldsymbol{u} + \beta\mathrm{div}\,\boldsymbol{u} &= h, &(10)\\
\partial_t\boldsymbol{d} - \tau\Delta\boldsymbol{d} &= \boldsymbol{k}, &(11)\\
\boldsymbol{u}|_{x_3=0,1} = \theta|_{x_3=0,1} = 0,\ \frac{\partial\boldsymbol{d}}{\partial x_3}|_{x_3=0,1} &= \boldsymbol{0}, &(12)\\
u|_{t=0} &= u_0, &(13)
\end{aligned}
$$

where $u_0 = {}^\top(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{d}_0)$ and f, $\boldsymbol{g}$, h, $\boldsymbol{k}$ are nonlinear terms.

Theorem 1. *There exist positive constants δ_1, δ_2, δ_3 and ϵ_0 such that if $u_0 \in (H^3(\Omega))^3 \times H^4(\Omega)$ satisfies some compatibility conditions and if $\|u_0\|_{H^3}$, $\|\boldsymbol{d}_0\|_{H^4} \le \epsilon_0$, $\|\rho_s - 1\|_{H^6}^2 \le \delta_1$, $\|\rho_s\theta_s - 1\|_{H^6}^2 \le \delta_2$ and $\|\theta_s - 1\|_{H^6}^2 \le \delta_3$, then (8)–(13) has a unique solution $u(t)$ and it satisfies the estimates*

$$\|u(t)\|_{(H^3(\Omega))^3\times H^4(\Omega)} \le C\|u_0\|_{(H^3(\Omega))^3\times H^4(\Omega)}.$$

We also consider the asymptotic behavior for this model.

Theorem 2. *In addition to the assumption of Theorem 1, we assume that $u_0 \in (L^1(\Omega))^4$, then the solution $u(t)$ satisfies the following estimates:*

(i)

$$\|\partial_{x_3}^l\partial_{x'}^{l'}u(t)\|_{L^2(\Omega)} \le C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

for $0 \le l$, $|l'| \le 1$. Furthermore,

$$\|\partial_{x_3}^l\partial_{x'}^{l'}(\boldsymbol{u},\theta)(t)\|_{L^2(\Omega)} \le C(1+t)^{-1}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

$$\|\partial_{x_3}^l\partial_{x'}^{l'}\boldsymbol{d}(t)\|_{L^2(\Omega)} \le C(1+t)^{-\frac{1}{2}-\frac{l+|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)}$$

for $0 \le l$, $|l'| \le 1$, $l + |l'| \le 1$.

(ii)

$$\|u(t) - \tilde{\sigma}_0(t)\|_{L^2(\Omega)} = o(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty,$$

where

$$\tilde{\sigma}_0(t) = {}^\top(\tilde{\phi}_{low}(t), \boldsymbol{0}, 0, \tilde{\boldsymbol{d}}_{low}(t)),$$

$$\tilde{\phi}_{low}(t) = \alpha_0 G_0(x',t),\ \tilde{\boldsymbol{d}}_{low} = {}^\top(\beta_1 G_1(x',t), \beta_2 G_2(x',t), \beta_3 G_3(x',t)),$$

$$\alpha_0 = \left(\int_\Omega \phi_0(y)\,dy\right)\frac{1}{\partial_\rho P(\rho_s,\theta_s)},\ \beta_j = \int_\Omega d_0^j(y)\,dy + \int_0^\infty\int_\Omega k^j(\boldsymbol{u},\boldsymbol{d})(y,s)\,dyds,$$

$$G_j(t) = (4\pi\kappa_j t)^{-1}\mathrm{e}^{-\frac{|x'|^2}{4\kappa_j t}} \quad (j = 0,1,2,3,\ \kappa_j > 0).$$

3 Existence of global weak solutions for large initial data

Here we consider the existence of global weak solutions of simplified Ericksen-Leslie system with $\rho = 1$ for the simplicity:

$$\begin{aligned}
\operatorname{div} \boldsymbol{u} &= 0, && (14)\\
\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) &= \operatorname{div}(\nu(\theta)\nabla \boldsymbol{u} - \eta(\theta)\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} - P\mathbb{I}), && (15)\\
\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta - \kappa(\theta)\Delta\theta &= \beta(\operatorname{div}(\nu(\theta)\nabla \boldsymbol{u} - \eta(\theta)\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} && (16)\\
&\quad -P\mathbb{I}) : \nabla \boldsymbol{u}, && (17)\\
\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} &= \Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}. && (18)
\end{aligned}$$

In addition to the boundary condition (12), we impose the peropdic boundary condition in x'.

We introduce the Ginzburg-Lamdau approximation corresponding to (14)–(18):

$$\begin{aligned}
\operatorname{div} \boldsymbol{u} &= 0, && (19)\\
\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) &= \operatorname{div}(\nu(\theta)\nabla \boldsymbol{u} - \eta(\theta)\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} - P\mathbb{I}), && (20)\\
\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta - \kappa(\theta)\Delta\theta &= \beta(\operatorname{div}(\nu(\theta)\nabla \boldsymbol{u} - \eta(\theta)\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} && (21)\\
&\quad -P\mathbb{I}) : \nabla \boldsymbol{u}, && (22)\\
\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} &= \Delta \boldsymbol{d} + \frac{1}{\epsilon^2}(1 - |\boldsymbol{d}|^2) \boldsymbol{d}. && (23)
\end{aligned}$$

The Ginzburg-Landau approximation was first systematically used by Lin and Liu [7, 8] to study the simplified Ericksen-Leslie system. In their works, the unit-length constraint $|\boldsymbol{d}| = 1$ is relaxed via a Ginzburg-Landau potential $\frac{1}{\epsilon^2}(1 - |\boldsymbol{d}|^2)\boldsymbol{d}$. For this approximate system, they proved the existence of unique global strong solutions in 2 or 3-dimension, under the assumption of sufficiently large viscosity. In the non-isothermal setting, Feireisl, Frémond, Rocca, and Schimperna [13] studied a Ginzburg-Landau type non-isothermal model and proved the existence of global weak solutions for large data in three dimension. Since the simplified Ericksen-Leslie system is formally obtained by taking the limit $\epsilon \to 0$ of (19)–(23), so we construct weak solution of (14)–(18) via that of (19)–(23).

We introduce a weak formulation of (14)–(18):

$$\int_\Omega \boldsymbol{u} \cdot \nabla \varphi \, dx = 0,$$

$$\int_0^T \int_\Omega \boldsymbol{u} \partial_t \varphi + (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \varphi \, dxdt = \int_0^T \int_\Omega (\nu(\theta)\nabla \boldsymbol{u} - \eta(\theta)\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) : \nabla \varphi \, dxdt,$$

$$\int_0^T\int_\Omega \boldsymbol{d}\partial_t b\varphi + (\boldsymbol{u}\otimes\boldsymbol{d}) : \nabla\varphi\, dxdt = \int_0^T\int_\Omega (\nabla\boldsymbol{d} : \nabla\varphi + |\nabla\varphi|^2\boldsymbol{d}\varphi)\, dxdt - \int_\Omega \boldsymbol{d}_0\varphi(0,\cdot)\, dx,$$

$$\int_0^T\int_\Omega H(\theta)\partial_t\varphi + (H(\theta)\boldsymbol{u} + H'(\theta)\kappa(\theta)\nabla\theta)\cdot\nabla\varphi\, dxdt$$
$$\le -\int_0^T\int_\Omega \{H'(\theta)(\mu(\theta)|\nabla\boldsymbol{u}|^2 + |\Delta\boldsymbol{d} + |\nabla\boldsymbol{d}|^2\boldsymbol{d}|^2) + H''(\theta)\kappa(\theta)|\nabla\theta|^2\}\varphi\, dxdt$$
$$+\int_0^T\int_\Omega H'(\theta|_{\partial\Omega})\kappa(\theta)\nabla\theta\cdot\boldsymbol{n}\varphi\, dSdt - \int_\Omega H(\theta_0)\varphi(\cdot,0)\, dx,$$

$$\int_\Omega \frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\boldsymbol{d}|^2 + \theta\, dx \le \int_\Omega (\frac{1}{2}|\boldsymbol{u}_0|^2 + \frac{1}{2}|\nabla\boldsymbol{d}_0|^2 + \theta_0)\, dx + \int_0^t\int_{\partial\Omega}\kappa(\theta)\nabla\theta\cdot\boldsymbol{u}\, dSds,$$

where H is smooth, non-decreasing and concave function, $\varphi \in C_0^\infty(\Omega \times [0,T])$.

Theorem 3. *There exists positive constant K such that if* $0 < \nu(\theta), \nu'(\theta), \eta(\theta) < K$ *for any* $\theta \ge 0$ *and if* $\boldsymbol{u}_0 \in L^2(\Omega)$, $\operatorname{div}\boldsymbol{u}_0 = 0$, $\theta_0 \in L^1(\Omega)$, $\operatorname{essinf}_\Omega\theta_0 > 0$, $\boldsymbol{d}_0 \in L^\infty(\Omega)$, *then system (14)–(18) with boundary condition has weak solution $u(t)$ and it belongs to the class*

$$\boldsymbol{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1),$$
$$\theta \in L^\infty(0,T;L^1) \cap L^p(0,T;W^{1,p}),\ \ 1 \le p < \frac{5}{4},$$
$$\boldsymbol{d} \in L^\infty(0,T;H^1)$$

for any $T > 0$.

References

[1] J. L. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheology **5** (1961). pp. 23–34.

[2] J. L. Ericksen, Hydrostatic theory of liquid crystals, Arch. Ratio. Mech. Anal., **9** (1962), pp. 371–378.

[3] J. Ericksen, Continuum theory of nematic liquid crystals, Res. Mechanica, **21** (1987), pp. 381–392.

[4] F. M. Leslie, Some constitutive equations for liquid crystals. Arch. Ratio. Mech. Anal., **28** (1968), pp.265–283.

[5] F. M. Leslie, Theory of flow phenomena in liquid crystals, Advances in Liquid Crystals, Vol. 4, G. Brown, ed., Academic Press, New York, 1979, pp. 1–81.

[6] F. H. Lin, Nonlinear theory of defects in nematic liquid crystals: Phase transition and flow phenomena, Comm. Pure Appl. Math., **42** (1989), pp. 789–814.

[7] F. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals. Comm. Pure Appl. Math. **48**(1995), 501-537.

[8] F. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals. Discrete Contin. Dyn. Syst. **2** (1996), 1-22.

[9] F. H. Lin, J. Y. Lin and C. Y. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal., **197** (2010), pp. 297–336.

[10] H. Wu, Long-time bahavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst., **26** (2010), pp. 379–396.

[11] C. Y. Wang, Well-posedness for the hest flow of harmonic maps and the liquid crystal flow with rough initial data, Arch. Ration. Mech. Anal., **200** (2011), pp. 1 –19.

[12] M. M. Dai, J. Qing and M. E. Schonbek, Asymptotic behavior of solutions to the liquid crystals system in $\mathbb{R}^3$, Comm. Partial Differential Equations, **37** (2012), pp. 2138–2164.

[13] E. Feireisl, M. Frémond, E. Rocca, G. Shimperna, A new approch to nonisothermal models for nematic liquid crystals. Arch. Rational Mech. Anal. **205** (2012), 651-672.

[14] T. Huang, F. H. Lin, C. Liu and C. Y. Wang, Finite time singularities of the nematic liquid crystal flow in dimension three, Arch. Ration. Mech. Anal., **221** (2016), pp. 1223–1254.

[15] M. Hieber, J. Prüss, Thermodynamical Consistent Modeling and Analysis of Nematic Liquid Crystal Flows, Mathematical Fluid Dynamics, Present and Future, Springer Proc. Math. and Statistics, **183** (2016), pp. 433–459.

[16] B. Guo, X. Xi, B. Xie, Global well-posedness and decay of smooth solutions to the non-isothermal model for compressible nematic liquid crystals, J. Differential Equations, **262** (2017), pp. 1413–1460.

NLS WITH MASS-SUBCRITICAL COMBINED NONLINEARITIES: SMALL MASS L^2-SCATTERING

JACOPO BELLAZZINI, LUIGI FORCELLA, AND VLADIMIR GEORGIEV

ABSTRACT. We prove small data scattering in the mass-subcritical regime for the NLS equation with double nonlinearities, where a focusing leading term is perturbed by a lower order defocusing nonlinear term. Our proof relies on the pseudo-conformal transformation in conjunction with a general variational argument used to obtain the positivity of certain modified energies. Moreover, the smallness assumption is only on the mass of the initial data, and not on the whole Σ-norm.

In this paper, we consider the Cauchy problem for the nonlinear Schrödinger equation (NLS) with combined nonlinearities (see [5])

$$\begin{cases} i\partial_t\psi + \Delta_x\psi = |\psi|^{q-1}\psi - |\psi|^{p-1}\psi \\ \psi(0) = \psi_0 \in \Sigma \end{cases}, \tag{0.1}$$

where $\psi = \psi(t,x)$, $\psi : \mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{C}$, Δ_x is the standard Laplace operator with respect to the space variables x, and

$$1 + \frac{2}{d} < q < p < 1 + \frac{4}{d}.$$

The space Σ is defined as the subspace of $H^1(\mathbb{R}^d)$ with finite variance:

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \text{ s.t. } |x|u \in L^2(\mathbb{R}^d)\},$$

endowed with norm $\|f\|_\Sigma^2 = \|f\|_{H^1(\mathbb{R}^d)}^2 + \||x|u\|_{L^2(\mathbb{R}^d)}^2$. Taking into account the signs in front of the nonlinear terms in (0.1), we are dealing with a mixed nonlinearity with focusing mass-subcritical term of order p, and a defocusing term of order $q \in (1+\frac{2}{d}, p)$. The critical power $p_0 = 1 + \frac{4}{d}$ is defined by borrowing from the usual scaling invariant equation with one nonlinearity

$$i\partial_t\psi + \Delta_x\psi = \pm|\psi|^{p-1}\psi,$$

2020 *Mathematics Subject Classification.* 35Q55; 35P25, 35J20.
Key words and phrases. Nonlinear Schrödinger equation, Scattering, mixed nonlinearities.

and refers to the conservation of the L^2-norm of a solution under the scaling invariance of the equation given by $\psi_\lambda(t,x) = \lambda^{\frac{2}{p_0-1}}\psi(\lambda^2 t, \lambda x)$.
The exponent $q_0 = 1 + \frac{2}{d}$ is instead known as the exponent separating the so-called short-range and long-range nonlinearities. See the discussion below.

A function $\psi(t)$, with $\psi \in C((-T_{\min}, T_{\max}); \Sigma)$ is said a mild solution to (0.1) if it satisfies the integral equation

$$\psi(t) = U(t)\psi_0 - i\int_0^t U(t-s)g(\psi(s))ds,$$

where $U(t) = e^{it\Delta_x}$ is the free Schrödinger propagator and $g(\psi) = |\psi|^{q-1}\psi - |\psi|^{p-1}\psi$.

At a formal level, equation (0.1) preserves the following quantities: the mass,

$$M(\psi(t)) = \int_{\mathbb{R}^d} |\psi(t,x)|^2 dx,$$

the energy

$$\begin{aligned} E(\psi(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(t,x)|^2 dx + \frac{1}{q+1}\int_{\mathbb{R}^d} |\psi(t,x)|^{q+1} dx \\ - \frac{1}{p+1}\int_{\mathbb{R}^d} |\psi(t,x)|^{p+1} dx, \end{aligned} \tag{0.2}$$

and the momentum

$$P(\psi(t)) = \operatorname{Im}\int_{\mathbb{R}^d} \psi(t,x)\nabla\bar{\psi}(t,x)dx.$$

In this paper, we are interested in the scattering properties of solutions to (0.1). In particular, once it is known that a solution exists globally-in-time, we ask whether the solution behaves linearly for large times.

We state our main result.

Theorem 1. Let $1 + \frac{2}{d} < q < p < 1 + \frac{4}{d}$. There exists a positive mass $\rho^\star$ such that for any $\psi_0 \in \Sigma$ with $\|\psi_0\|_{L^2(\mathbb{R}^d)} < \rho^\star$ a solution to (0.1) scatters in $L^2(\mathbb{R}^d)$.

Remark 0.1. We emphasize that our smallness assumption is only on the mass of the initial datum, and not in the whole Σ-norm. This is a major difference with respect to the classical scattering results as in [2–4] for the NLS equation with only one power-like nonlinearity.

As the existence of standing waves may represent an obstruction for scattering, we start by considering the possible existence of standing waves for our mixed model

(0.1). Specifically, we look at solutions to (0.1) of the form $\psi(t,x) = e^{i\omega t}u(x)$, with $\omega \in \mathbb{R}$ and $u(x) \in \mathbb{C}$ is a time-independent $H^1(\mathbb{R}^d)$-function which satisfies

$$-\Delta u + \omega u + |u|^{q-1}u - |u|^{p-1}u = 0. \tag{0.3}$$

Since the mass is a conserved physical quantity along the flow of (0.1), a natural approach to finding solutions u to (0.3) is to seek critical points of the energy functional, constrained to the L^2-spheres in $H^1(\mathbb{R}^d)$, which are defined by

$$S_\rho = \{u \in H^1(\mathbb{R}^d) \ : \ \|u\|_{L^2(\mathbb{R}^d)} = \rho\}.$$

Thus, a solution to (0.3) is understood as a pair $(\omega_\rho, u_\rho) \in \mathbb{R} \times H^1(\mathbb{R}^d)$, where ω_ρ serves as the Lagrange multiplier corresponding to the critical point u_ρ on the constraint set S_ρ.For a fixed a mass $\rho > 0$, we define the *ground state energy* the quantity I_{ρ^2}, defined as the infimum of the energy functional over S_ρ, namely,

$$\mathrm{I}_{\rho^2} = \inf_{S_\rho} E(u),$$

where E is defined as in (0.2).

We have the following existence and non-existence result for the existence of ground states.

Theorem 2. Let $d \geq 1$ and $1 < q < p < 1 + \frac{4}{d}$. Then there exists a strictly positive mass $\rho_{0,E}$ such that:
(i) $\mathrm{I}_{\rho^2} = 0$ for all $\rho \in (0, \rho_{0,E}]$;
(ii) $\mathrm{I}_{\rho^2} < 0$ for all $\rho \in (\rho_{0,E}, \infty)$.
Moreover, there are no constrained minimizers for $0 < \rho < \rho_{0,E}$, and for all $\rho \in [\rho_{0,E}, \infty)$ there exists $u_\rho \in S_\rho$ such that $\mathrm{I}_{\rho^2} = E(u_\rho)$.

We will refer to the minimal mass $\rho_{0,E}$ as the ground state threshold mass.

Remark 0.2. Below, we will provide an upper bound on the mass $\rho^\star$ by means of the threshold mass yielding the existence of ground states for equation (0.3).

A natural question that arises after proving the existence of a threshold mass for the existence of ground states, is whether there exists a smaller threshold mass for the existence of generic standing waves. In order to answer this question, we recall that for any standing wave $\psi(t,x) = e^{i\omega t}u(x)$ we have $G(u) = 0$, where $G(u)$ is the following Pohozaev functional

$$G(u) = \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{d(q-1)}{2(q+1)} \int_{\mathbb{R}^d} |u|^{q+1} dx - \frac{d(p-1)}{2(p+1)} \int_{\mathbb{R}^d} |u|^{p+1} dx.$$

Note that the Pohozaev functional has the same structure as the energy functional (0.2), the two being different only by the constants appearing in front of the integrals. Then, it becomes relevant to study the existence of minimizers for a general defocusing-focusing energy functional of the type

$$E^{\alpha,\beta,\gamma}(u) := \alpha\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \beta\|u\|^{q+1}_{L^{q+1}(\mathbb{R}^d)} - \gamma\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}$$

constrained to the manifold S_ρ, with positive constants α, β, γ and $1 < q < p < 1 + \frac{4}{d}$. Thus, we consider the general problem

$$\mathrm{I}^{\alpha,\beta,\gamma}_{\rho^2} := \inf_{S_\rho} E^{\alpha,\beta,\gamma}(u),$$

and we prove the following existence and non-existence result of minimizers.

Theorem 3. Let $d \geq 1$, $1 < q < p < 1 + \frac{4}{d}$, and $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Then there exists a strictly positive mass

$$\rho_0 = \rho_0(\alpha, \beta, \gamma)$$

such that:
(i) $\mathrm{I}^{\alpha,\beta,\gamma}_{\rho^2} = 0$ for all $\rho \in (0, \rho_0]$;
(ii) $\mathrm{I}^{\alpha,\beta,\gamma}_{\rho^2} < 0$ for all $\rho \in (\rho_0, \infty)$.
Moreover, there are no constrained minimizers for $0 < \rho < \rho_0$, and for all $\rho \in [\rho_0, \infty)$ there exists $u_\rho \in S_\rho$ such that $\mathrm{I}^{\alpha,\beta,\gamma}_{\rho^2} = E^{\alpha,\beta,\gamma}(u_\rho)$.

It is clear that Theorem 2 is just a specific case of Theorem 3 when $\alpha = \frac{1}{2}, \beta = \frac{1}{q+1}, \gamma = \frac{1}{p+1}$, namely $\rho_0 = \rho_{0,E}$, and that Theorem 3 also guarantees that $G(u) > 0$ for $u \in S_\rho$ when $\rho < \rho_{0,SW}$ where

$$\rho_{0,SW} = \rho_0\left(1, \frac{d(q-1)}{2(q+1)}, \frac{d(p-1)}{2(p+1)}\right).$$

As a consequence, we have the following.

Corollary 1. Standing waves solutions $\psi(t,x) = e^{i\omega t}u(x)$, $\omega \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^d)$ with mass smaller than $\rho_{0,SW}$ cannot exist.

The strategy to prove Theorem 2 and Theorem 3 is purely variational and it has been inspired by the recent paper [1], in particular it will be crucial to show (in the case of Theorem 2) that

$$A_\varepsilon = \{u \in H^1 \setminus \{0\} \ : \ E(u) \leq 0, \ G(u) = 0, \ \|\nabla u\|_2 \leq \varepsilon\} = \emptyset.$$

This condition will imply that the ground state energy is nonnegative for small mass and that the ground state with small mass cannot exist.
Going back to our main goal, i.e., the long-time dynamics of solutions to (0.1), and having in mind Theorem 2 and Corollary 1, it is clear that the long-time dynamics of solutions to (0.1) shall be rather different from those of the NLS equation with a single power-like nonlinear term. Indeed, we are in a scenario where standing waves may exist or not, depending on the mass. We can re-state Theorem 1 in a more precise and quantitative way. To this aim, we introduce the following exponents:

$$\delta(q) = \frac{4-d(q-1)}{2} \quad \text{and} \quad \delta(p) = \frac{4-d(p-1)}{2}.$$

Note that both exponents above are in the range $(0,1)$.

Theorem 4. Let $d \geq 1$, $1+\frac{2}{d} < q < p < 1+\frac{4}{d}$, and $\rho_{0,E} > 0$ be the threshold mass for the existence of ground states given in Theorem 2. Then there exists a positive mass $\rho^\star$, given by

$$\rho^\star = \rho_0\left(\frac{1}{2}, \frac{(1-\delta(q))(1-\delta(p))^{-\frac{\delta(q)}{\delta(p)}}}{q+1}, \frac{1}{p+1}\right) < \rho_{0,E},$$

such that for any $\psi_0 \in \Sigma$ with $\|\psi_0\|_{L^2(\mathbb{R}^d)} < \rho^\star$ there exist $\psi_\pm \in L^2(\mathbb{R}^d)$ satisfying

$$\lim_{t\to\pm\infty} \|\psi(t) - U(t)\psi_\pm\|_{L^2(\mathbb{R}^d)} = 0$$

with $\psi(t)$ solution to (0.1).

Remark 0.3. In view of the content of Theorem 4, it is natural to ask whether the threshold mass $\rho^\star$ is optimal or not. Alternatively, we may ask if we can push $\rho^\star$ up to $\rho_{0,E}$ defined in Theorem 2. We conjecture that this is not possible as we suspect the existence of standing waves which are not global minima for the energy functional, and have a mass strictly smaller than $\rho_{0,E}$.

Let us briefly illustrate the main ideas behind the proof of Theorem 4. First, we introduce the pseudo-conformal transformation

$$\varphi(\tau,\xi) = \psi\left(\frac{\tau}{1-\tau}, \frac{\xi}{1-\tau}\right)(1-\tau)^{-d/2} e^{-\frac{i|\xi|^2}{4(1-\tau)}}.$$

It is known that $e^{-it\Delta_x}\psi(t)$ has a strong limit in the L^2-topology as $t \to +\infty$ if and only if $\varphi(\tau)$ has a strong limit in the L^2-topology as $\tau \to 1^-$. In the new variables,

$\varphi(\tau)$ is a solution to the following NLS with time-dependent coefficients

$$i\partial_\tau\varphi + \Delta_\xi\varphi = (1-\tau)^{-\delta(q)}|\varphi(\tau)|^{q-1}\varphi(\tau) - (1-\tau)^{-\delta(p)}|\varphi(\tau)|^{p-1}\varphi(\tau),$$

and to perform a Tsutsumi and Yajima argument [6] it will be crucial to have suitable controls in terms of $(1-\tau)$ for the quantities $\|\nabla_\xi\varphi(\tau)\|_{L^2(\mathbb{R}^d)}$, $\|\varphi(\tau)\|_{L^{q+1}(\mathbb{R}^d)}$, and $\|\varphi(\tau)\|_{L^{p+1}(\mathbb{R}^d)}$. At this point, in order to control the aforementioned norms, we introduce the modified energy

$$\begin{aligned} E_A(\tau,\varphi(\tau)) := \frac{(1-\tau)^A}{2}\|\nabla_\xi\varphi(\tau)\|^2_{L^2(\mathbb{R}^d)} + \frac{(1-\tau)^{A-\delta(q)}}{q+1}\|\varphi(\tau)\|^{q+1}_{L^{q+1}(\mathbb{R}^d)} \\ - \frac{(1-\tau)^{A-\delta(p)}}{p+1}\|\varphi(\tau)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}. \end{aligned}$$

The modified energy above is parametrized by a non-negative real number A that satisfies $\delta(q) < A < 1$ (which will be chosen properly later in the paper). Specifically, the modified energy E_A is not conserved along the flow, so the crucial point will be to exploit a bound on the time evolution of the modified energy in terms of $E_A(0,\varphi(0))$, provided that the Cauchy datum has L^2-norm strictly lower than a certain mass $\rho_1(A)$. The variational analysis developed in Theorem 3 will be the key tool for this purpose. Once the time evolution of the modified energy is controlled, we will get the estimates

$$\begin{aligned} \|\nabla_\xi\varphi(\tau)\|^2_{L^2(\mathbb{R}^d)} &\lesssim (1-\tau)^{-A}, \\ \|\varphi(\tau)\|^{q+1}_{L^{q+1}(\mathbb{R}^d)} &\lesssim (1-\tau)^{\delta(q)-A}, \\ \|\varphi(\tau)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} &\lesssim (1-\tau)^{\delta(p)-A}. \end{aligned}$$

that enable us to prove the long-time dynamical results by exploiting the Tsutsumi and Yajima strategy. We therefore prove scattering below the threshold mass $\rho_1(A)$ parametrized by A. Eventually, we will show the monotonicity of the mass $\rho_1(A)$ as $A \to 1^-$ and we compute the largest possible threshold mass given by

$$\rho^\star = \rho_0\left(\frac{1}{2}, \frac{(1-\delta(q))(1-\delta(p))^{-\frac{\delta(q)}{\delta(p)}}}{q+1}, \frac{1}{p+1}\right).$$

References

[1] J. Bellazzini and L. Forcella, Mass-subcritical Half-Wave equation with mixed nonlinearities: existence and non-existence of ground states and traveling waves, Discrete Contin. Dyn. Syst. A 53: 131-145 2026

[2] T. Cazenave and F. B. Weissler, Rapidly decaying solutions of the nonlinear Schroedinger equation. Comm. Math. Phys., 147(1):75-100, 1992

[3] J. Ginibre, T. Ozawa, and G. Velo, On the existence of the wave operators for a class of nonlinear Schroedinger equations. Ann. Inst. Henri Poincare, Phys. Theor., 60(2), 211-239, 1994

[4] K. Nakanishi and T. Ozawa, Remarks on scattering for nonlinear Schroedinger equations, NoDEA Nonlinear Differential Equations Appl., 9(1), 45-68, 2002

[5] T. Tao, M. Visan, and X. Zhang, The nonlinear Schroedinger equation with combined power-type nonlinearities, Commun. Partial Differ. Equations, 32(8), 1281–1343, 2007

[6] Y. Tsutsumi and K. Yajima. The asymptotic behavior of nonlinear Schroedinger equations. Bull. Amer. Math. Soc., 11(1):186–188, 1984

(J. Bellazzini) Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo, 5, 56127, Pisa, Italy

Email address: jacopo.bellazzini@unipi.it

(L. Forcella) Dipartimento di Matematica, Università Degli Studi di Pisa, Largo Bruno Pontecorvo, 5, 56127, Pisa, Italy

Email address: luigi.forcella@unipi.it

(V. Georgiev) Dipartimento di Matematica, Università Degli Studi di Pisa, Largo Bruno Pontecorvo, 5, 56127, Pisa, Italy, and Faculty of Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan, and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria

Email address: vladimir.simeonov.gueorguiev@unipi.it

SCATTERING OF GROUP-INVARIANT SOLUTIONS FOR NONLINEAR FOURTH-ORDER SCHRÖDINGER EQUATION

KOICHI KOMADA[1]

[1] *Research Institute for Science and Engineering, Waseda University, Ohkubo, Shinjuku-ku, Tokyo 169–8555.*
E-mail: k-komada@aoni.waseda.jp

1. Introduction

We consider the focusing nonlinear fourth-order Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta^2 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued unknown function and p satisfies

$$1 + \tfrac{8}{d} < p < \begin{cases} \infty & \text{if } 1 \le d \le 4, \\ 1 + \tfrac{8}{d-4} & \text{if } d \ge 5. \end{cases} \tag{1.2}$$

(1.1) is locally well-posed in $H^2(\mathbb{R}^d)$ and has three conserved quantities; the mass $M(u)$, the energy $E(u)$ and the momentum $\vec{P}(u)$:

$$\begin{aligned} M(u(t)) &:= \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0), \\ E(u(t)) &:= \tfrac{1}{2} \int_{\mathbb{R}^d} |\Delta u(t,x)|^2 dx - \tfrac{1}{p+1} \int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx = E(u_0), \\ \vec{P}(u(t)) &:= \operatorname{Im} \int_{\mathbb{R}^d} \overline{u}(t,x) \nabla u(t,x) dx = \vec{P}(u_0). \end{aligned}$$

(1.1) is invariant under the scaling transformation. Let u be a solution to (1.1). Then for all $\lambda > 0$,

$$u_\lambda(t,x) := \lambda^{\frac{4}{p-1}} u(\lambda^4 t, \lambda x)$$

also solves (1.1). For all $\lambda > 0$, we have

$$\|u_\lambda(0)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s - \frac{d}{2} + \frac{4}{p-1}} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)}$$

for all $s \in \mathbb{R}$. This implies that the critical exponent of homogeneous Sobolev spaces is given by

$$s_c := \tfrac{d}{2} - \tfrac{4}{p-1}.$$

We say that (1.1) is L^2-supercritical and $\dot{H}^2$-subcritical if $0 < s_c < 2$. This condition holds if and only if p satisfies (1.2).

In this article, we study the global dynamics for (1.1). There are many results on the global dynamics for the nonlinear fourth-order Schrödinger equation. For the defocusing nonlinear fourth-order Schrödinger equation

$$i\partial_t u - \Delta^2 u - |u|^{p-1}u = 0,$$

Pausader [21] proved global well-posedness and scattering for all radially symmetric initial data in $H^2(\mathbb{R}^d)$ when $d \geq 5$ and $1 < p \leq 1 + 8/(d-4)$. The radial assumption is not necessary when $p < 1 + 8/(d-4)$ and is removed later in [20, 23] in the remaining case $p = 1+8/(d-4)$ when $d \geq 8$. For the low dimensional and L^2-supercritical case: $1 \leq d \leq 4$ and $p > 8/d$, the scattering result was obtained in [25]. For the high dimensional and L^2-critical case: $d \geq 5$ and $p = 1 + 8/d$, the global well-posedness and scattering was proved in [24].

For the focusing equation (1.1), the ground state solutions play an important role in describing global dynamics. Miao–Xu–Zhao [19] and Pausader [22] proved the global well-posedness and the scattering of radial solutions below the ground state for (1.1) in the energy-critical case $p = 1 + 8/(d-4)$ when $d \geq 5$. In [19, 22], the proofs are based on concentration compactness argument in the spirit of Kenig–Merle [15].

For the equation (1.1) in the inter-critical case (1.2), Guo [11] proved the scattering of radially symmetric solutions below the ground state when $d \geq 2$. The proof in [11] is based on the concentration compactness argument. Dinh [5] provides an alternative proof, avoiding the use of the concentration compactness argument. The proof in [5] relies on the approach introduced by Dodson–Murphy [8].

For $\omega > 0$, (1.1) has standing wave solutions $u(t,x) = e^{i\omega t}\phi(x)$, where ϕ is a solution of the fourth-order elliptic equation

$$-\omega\phi - \Delta^2\phi + |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^d. \tag{1.3}$$

For each $\omega > 0$, we define a ground state Q_ω as a solution of (1.3) such that

$$S_\omega(Q_\omega) = \inf\{S_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ \phi \text{ solves } (1.3)\},$$

where S_ω is the action functional defined by

$$S_\omega(\phi) := E(\phi) + \tfrac{\omega}{2}M(\phi).$$

It is known that there exists a ground state solution to (1.3) for all $1 < p < 8/(d-4)_+$ and for all $\omega > 0$ (see [2, 3], for instance). Let K be the virial functional defined by

$$\begin{aligned} K(\phi) &:= \tfrac{\partial}{\partial\lambda}E(\lambda^{\frac{d}{2}}\phi(\lambda x))|_{\lambda=1} \\ &= 2\int_{\mathbb{R}^d} |\Delta\phi(x)|^2 dx - \tfrac{d(p-1)}{2(p+1)}\int_{\mathbb{R}^d} |\phi(x)|^{p+1} dx. \end{aligned}$$

Then we see that every solution ϕ to (1.3) satisfies $K(\phi) = 0$. Moreover, if p satisfies (1.2), then for all $\omega > 0$,

$$S_\omega(Q_\omega) = \inf\{S_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ K(\phi) = 0\}. \tag{1.4}$$

In [5, 11], the following result was obtained.

Theorem 1.1 ([5, 11])**.** *Let $d \geq 2$ and p satisfy (1.2). If $u_0 \in H^2(\mathbb{R}^d)$ is radial and satisfies*

$$S_\omega(u_0) < S_\omega(Q_\omega) \ \text{ for some } \ \omega > 0, \quad \text{and} \ \ K(u_0) \geq 0, \tag{1.5}$$

then there exists a unique global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ to (1.1) with initial data u_0. Moreover, u scatters forward and backward in time, i.e., there exist $\phi^\pm \in H^2(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{-it\Delta^2}\phi^\pm\|_{H^2} = 0,$$

where $e^{-it\Delta^2}$ is the free evolution operator of the linear fourth-order Schrödinger equation.

Remark 1.2. Let $Q := Q_1$ be the ground state solution to (1.3) for $\omega = 1$. Since $Q_\omega(x) = \omega^{1/(p-1)}Q(\omega^{1/4}x)$ is a ground state solution to (1.3) for each $\omega > 0$, the condition (1.5) is equivalent to the following condition:

$$\begin{cases} M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < M(Q)^{\frac{2-s_c}{s_c}} E(Q), \\ \|u_0\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_{L^2} < \|Q\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta Q\|_{L^2}. \end{cases}$$

Remark 1.3. The existence of radially symmetric ground state solutions to (1.3) implies that the bound on action in (1.5) is sharp. Boulenger–Lenzmann [4] proved the existence of radial ground states for (1.3) when p is odd integer and satisfies $1 < p < 1 + 8/(d-4)_+$. However, if p is not odd, it remains unknown due to the inapplicability of classical rearrangement techniques caused by the presence of the fourth-order term. In [4], the proof relies on the symmetric-decreasing rearrangement in Fourier space, which is limited to the case where the nonlinearity is a polynomial and hence applicable only when p is odd.

Remark 1.4. When $d \geq 2$, p satisfies (1.2) and $p \leq 9$, Boulenger–Lenzmann [4] proved that if $u_0 \in H^2(\mathbb{R}^d)$ is radial and satisfies

$$S_\omega(u_0) < S_\omega(Q_\omega) \ \text{ for some } \ \omega > 0, \quad \text{and} \ \ K(u_0) < 0,$$

then the corresponding solution u to (1.1) blows up in finite time.

It is worth noting that in Theorem 1.1, the radial assumption is retained, and the one-dimensional case is excluded. Our aim is to extend the scattering results for radial solutions in Theorem 1.1 to the result for group-invariant solutions.

2. Scattering of Group-invariant Solutions

In this section, we introduce the definition of group-invariance and state our main result. We refer to [13, 14], in which the global dynamics of group-invariant solutions considered for the nonlinear Schrödinger equation. Let $M_d(\mathbb{R})$ be the set of $d \times d$ matrices with real entries. Let $O(d)$ denote the set of $d \times d$ orthogonal matrices, i.e.

$$O(d) := \{\mathcal{R} \in M_d(\mathbb{R}) \mid \mathcal{R}^T\mathcal{R} = \mathcal{I}_d\},$$

where the transpose of a matrix $\mathcal{A}$ is written by $\mathcal{A}^T$ and $\mathcal{I}_d$ denotes the $d \times d$ identity matrix. Note that $O(1) = \{\pm 1\}$. We consider a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, where

$\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ is a group with the binary operation $(+, \cdot)$. Throughout this paper, we assume that for any $(\theta_1, \mathcal{G}_1), (\theta_2, \mathcal{G}_2) \in G$, $\theta_1 = \theta_2$ if $\mathcal{G}_1 = \mathcal{G}_2$. This assumption reads as θ-part is given as a function of $\mathcal{G}$, and hence we can use the notation $\mathcal{G}$ without confusion to denote not only a matrix but also an element of G. In the sequel, we freely use this identification.

Definition 2.1 (G-invariance)**.** Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. A function $f : \mathbb{R}^d \to \mathbb{C}$ is G-invariant if $\mathcal{G}f = f$ for all $\mathcal{G} \in G$, where for $\mathcal{G} = (\theta, \mathcal{G}) \in G$ the group action $\mathcal{G}f$ is defined by $\mathcal{G}f(x) := e^{-i\theta}(f \circ \mathcal{G}^{-1})(x) = e^{-i\theta} f(\mathcal{G}^{-1}x)$.

We remark that the above assumption of G is a natural one since the only G-invariant function is zero if it fails. We also remark that the radial case corresponds to the case: $d \geq 2$ and $G_{\text{rad}} = \{0\} \times O(d)$. Other simple examples of G in $d = 1$ are $G_{\text{even}} = \{(0, 1), (0, -1)\}$ and $G_{\text{odd}} = \{(0, 1), (\pi, -1)\}$, which give even symmetry and odd symmetry of functions, respectively.

Note that since the Cauchy problem of (1.1) with p satisfying (1.2) is well-posed in $H^2(\mathbb{R}^d)$, it is obvious that if the initial data $u_0 \in H^2(\mathbb{R}^d)$ is G-invariant , then the corresponding solution u to (1.1) is also G-invariant, i.e., $u(t) = \mathcal{G}u(t)$ for all $\mathcal{G} \in G$, due to the fact that the operators Δ and Δ^2 are invariant for group actions by $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and the nonlinear term of (1.1) is gauge invariant.

Our main result is stated as follows.

Theorem 2.2. *Let $d \geq 1$ and p satisfy* (1.2)*. Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ such that*

$$\inf_{|x|=1} \#Gx = \inf_{|x|=1} \#\{\mathcal{G}x \mid \mathcal{G} = (\theta, \mathcal{G}) \in G\} \geq 4^{\frac{2}{p-1}}. \tag{2.1}$$

If $u_0 \in H^2(\mathbb{R}^d)$ is G-invariant and satisfies (1.5)*, then there exists a unique global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ to* (1.1) *with initial data u_0. Moreover, u scatters forward and backward in time.*

Remark 2.3. The assumption (2.1) for a subgroup G comes from the use of Theorem 4.1 in the proof of Theorem 2.2 (see Section 4 for details).

Remark 2.4. When $d \geq 2$, $\inf_{|x|=1} \#G_{\text{rad}}x = \infty$. Therefore, G_{rad} satisfies (2.1).

Remark 2.5. When $d = 1$, the condition (1.2) leads to $p > 9$, and then, $\inf_{|x|=1} \#G_{\text{even}}x = \inf_{|x|=1} \#G_{\text{odd}}x = 2 > 4^{\frac{2}{p-1}}$. Therefore, Theorem 2.2 proves the scattering of even or odd solutions below the ground state in the one dimensional case.

3. Scattering Threshold

In this section, to explain the outline of the proof of Theorem 2.2, we define the scattering thresholds of action for general solutions and group-invariant solutions. We note that if an initial data u_0 satisfies (1.5), then by using the conservation laws and by applying the variational analysis, we see that the corresponding solution u exists globally in time and $\|u\|_{L_t^\infty H_x^2(\mathbb{R}\times\mathbb{R}^d)} < \infty$. Hence, to complete the proof of Theorem 2.2, it suffices to prove the scattering part. For any time interval $I \subset \mathbb{R}$, we define a scattering size by

$$\|u\|_{X(I)} := \|u\|_{L_t^{q_1} L_x^r(I\times\mathbb{R}^d)},$$

where $r = p+1, q_1 = \frac{4(p-1)(p+1)}{8-(d-4)(p-1)}$. By using the Strichartz type estimate for the fourth-order Schrödinger equation, we see that if a global solution u to (1.1) satisfies the boundedness of the scattering size

$$\|u\|_{X(\mathbb{R})} < \infty, \tag{3.1}$$

then u scatters forward and backward in time. For $\omega > 0$ and for $L \geq 0$, we define

$$C_\omega(L) := \sup\{\|u\|_{X(I)} \mid S_\omega(u_0) \leq L,\ K(u_0) \geq 0\},$$

where the supremum is taken over all solutions u to (1.1) defined on $I \times \mathbb{R}^d$ with initial data $u_0 \in H^2(\mathbb{R}^d)$. By using the small data scattering theory and the long-time perturbation theory, we see that $C_\omega(L)$ is sublinear for small $L > 0$ and that $L \mapsto C_\omega(L)$ is non-decreasing and continuous whenever $C_\omega(L)$ is finite. Let

$$L^*_\omega := \sup\{L \in [0,\infty) \mid C_\omega(L) < \infty\}.$$

Then $L^*_\omega \geq S_\omega(Q_\omega)$ for all $\omega > 0$ implies that for all solutions u to (1.1) with initial data u_0 satisfying (1.5), (3.1) holds and so u scatters. Note that $L^*_\omega \leq S_\omega(Q_\omega)$ follows from $C_\omega(S_\omega(Q_\omega)) = \infty$, which is a consequence of the fact that $\|e^{i\omega t}Q_\omega\|_{X(\mathbb{R})} = \infty$. For $\omega > 0$ and for $L \geq 0$, we also define the criterion for G-invariant solutions by

$$C_{G,\omega}(L) := \sup\{\|u\|_{X(I)} \mid S_\omega(u_0) \leq L,\ K(u_0) \geq 0\},$$

where the supremum is taken over all G-invariant solutions u to (1.1) defined on $I \times \mathbb{R}^d$ with initial data $u_0 \in H^2(\mathbb{R}^d)$ that is G-invariant. Let

$$L^*_{G,\omega} := \sup\{L \in [0,\infty) \mid C_{G,\omega}(L) < \infty\}.$$

Then $L^*_{G,\omega} \geq S_\omega(Q_\omega)$ for all $\omega > 0$ implies that for all G-invariant solutions u to (1.1) with initial data u_0 satisfying (1.5), (3.1) holds. We note that by the definition, $C_{G,\omega}(L) \leq C_\omega(L)$ holds and hence $L^*_{G,\omega} \geq L^*_\omega$ for all subgroups G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$.

For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and for $\omega > 0$, we define

$$m_{G,\omega} := \inf_{|x|=1} (\#Gx) L^*_{G_x,\omega},$$

where $\#Gx = \#\{\mathcal{G}x \mid \mathcal{G} \in G\}$ and $G_x = \{\mathcal{G} \in G \mid \mathcal{G}x = x\}$. Note that G_x is a subgroup of G. In the previous works [16, 17], the following partial result for scattering of group-invariant solutions was obtained.

Theorem 3.1 ([16, 17]). *Let $d \geq 1$ and p satisfy (1.2). Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. For all $\omega > 0$, $L^*_{G,\omega} \geq S_\omega(Q_\omega)$ holds true under the hypothesis $L^*_{G,\omega} < m_{G,\omega}$.*

Remark 3.2. If $m_{G,\omega} < \infty$, then $L^*_{G,\omega} \leq m_{G,\omega}$ holds in general (see [16]). Hence, the hypothesis $L^*_{G,\omega} < m_{G,\omega}$ in Theorem 3.1 reads as the failure of the identity. If $m_{G,\omega} = \infty$, then the hypothesis is not necessary in Theorem 3.1 (see also [16]).

Remark 3.3. In the case $L^*_{G,\omega} = m_{G,\omega}$, we have

$$L^*_{G,\omega} = m_{G,\omega} \geq \left(\inf_{|x|=1} \#Gx\right) L^*_\omega.$$

From the small data scattering theory, we see that there exists $\delta > 0$ such that $L^*_\omega \geq \delta$. Therefore, if $\inf_{|x|=1} \#Gx$ is sufficiently large depending on δ, we obtain $L^*_{G,\omega} \geq S_\omega(Q_\omega)$.

In [16, 17], the proof of Theorem 3.1 is based on the concentration compactness argument introduced by Kenig–Merle [15] for proving global well-posedness and scattering of radial solutions below the ground state for the focusing energy-critical nonlinear Schrödinger (NLS) equation. We also refer to Holmer–Roudenko [12] for the focusing 3D cubic NLS in the radial case. Roughly speaking, the argument begins by contradiction and reformulating the problem as a variational problem. First, we suppose that the threshold for scattering is strictly below the ground state. Then, we can construct a critical non-scattering solution standing exactly at the threshold. If the problem is restricted to the radial case, we can show that the critical solution is spatially localized near the origin uniformly in time. This uniform localization enable us to establish the local virial identity, which leads to a contradiction. In [11], the proof of Theorem 1.1 follows this strategy.

We recall that the scattering results for NLS is extended to the nonradial case (see [1, 6, 7, 9, 10, 18]). In the nonradial case, the critical solution is localized near $x(t)$ for some function $x(t)$. For NLS, by using the Galilean invariance of the equation, one can control the behavior of $x(t)$, which enables the contradiction argument via the virial identity. However, for the fourth-order Schrödinger equation (1.1), the Galilean invariance does not hold. Due to this lack of the Galilean invariance, the local virial argument does not work for (1.1) in the nonradial case.

In the proof of Theorem 3.1, the hypothesis $L^*_{G,\omega} < m_{G,\omega}$ makes it possible to construct a critical solution localized near the origin. To construct a critical solution, we apply the profile decomposition for G-invariant functions to a sequence $u_{n,0}$ optimizing $L^*_{G,\omega}$. In this decomposition, each profile ϕ^j is G_{w^j}-invariant for some $w^j \in \mathbb{R}^d$ and appears in the following form:

$$\sum_{k=1}^{\#Gw^j} \mathcal{G}^j_k \left[\phi^j(x - x^j_n)\right] := \psi^j_n(x),$$

where $\mathcal{G}^j_k \in G$ are such that $G = \bigsqcup_{k=1}^{\#Gw^j} \mathcal{G}^j_k G_{w^j}$ and ψ^j_n is G-invariant. By applying the concentration compactness argument, we can show that the optimizing sequence $u_{n,0}$ is consist of only one ψ^1_n. If $|x^1_n| \to \infty$, we see that $w^1 \in \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$ and $L^*_{G,\omega} = \lim_{n\to\infty} S_\omega(u_{n,0}) = (\#Gw^1) S_\omega(\phi^1)$. Then $S_\omega(\phi^1) < L^*_{G_{w^1},\omega}$ under the hypothesis $L^*_{G,\omega} < m_{G,\omega}$. This implies that the solution to (1.1) with initial data ϕ^1 scatters, which leads to a contradiction. If x^1_n is bounded, we see that $w^1 = 0$ and $\psi^1_n = \phi^1$. In this case, we obtain a critical solution localized near the origin.

Since the first case mentioned above cannot be excluded without the hypothesis $L^*_{G,\omega} < m_{G,\omega}$, to prove $L^*_{G,\omega} \geq S_\omega(Q_\omega)$ in Theorem 3.1, we need to assume that hypothesis. However, since we do not yet know what $L^*_{G,\omega}$ and $L^*_{G_x,\omega}$ are, it is difficult to verify whether the hypothesis holds for a given subgroup G.

4. Non-optimal Scattering Threshold and Proof of Theorem 2.2

In this section, we establish the non-optimal scattering result for general solutions, where the threshold of action is less than certain fraction of the action of the ground state. Using this result, we can show that $m_{G,\omega} > S_\omega(Q_\omega)$ for all subgroups G satisfying (2.1), and then, $L^*_{G,\omega} > S_\omega(Q_\omega)$ holds in the remaining case $L^*_{G,\omega} = m_{G,\omega}$ in Theorem 3.1.

For the L^2-critical fourth-order Schrödinger equation

$$i\partial_t u - \Delta^2 u + |u|^{\frac{8}{d}} u = 0, \tag{4.1}$$

Pausader–Shao [24] obtained the non-optimal scattering result. They proved that if $d \geq 5$ and $u_0 \in L^2(\mathbb{R}^d)$ satisfies

$$M(u_0) \leq \left(\tfrac{1}{4}\right)^{\frac{d}{4}} M(Q),$$

where Q is a ground state solution to $\Delta^2 Q + Q = |Q|^{\frac{8}{d}} Q$, then the solution u to (4.1) with initial data u_0 globally exists and scatters. We emphasize that they do not assume that u_0 is radial. For the inter-critical equation (1.1) with p satisfying (1.2), we obtain the similar non-optimal scattering result as follows.

Theorem 4.1. *Let $d \geq 1$ and p satisfy (1.2). If $u_0 \in H^2(\mathbb{R}^d)$ satisfies*

$$S_\omega(u_0) \leq \left(\tfrac{1}{4}\right)^{\frac{2}{p-1}} S_\omega(Q_\omega) \quad \text{for some} \quad \omega > 0, \quad \text{and} \quad K(u_0) \geq 0, \tag{4.2}$$

then there exists a unique global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ to (1.1) with initial data u_0. Moreover, u scatters forward and backward in time.

Remark 4.2. Theorem 4.1 is equivalent to the lower bound

$$L^*_\omega > \left(\tfrac{1}{4}\right)^{\frac{2}{p-1}} S_\omega(Q_\omega).$$

Although the bound on action in the condition (4.2) does not reach the ground state, by using Theorem 4.1, we can prove the scattering of G-invariant solutions with initial data satisfying (1.5) under the assumption (2.1).

Proof of Theorem 2.2. If $L^*_{G,\omega} < m_{G,\omega}$, then by Theorem 3.1, we have $L^*_{G,\omega} \geq S_\omega(Q_\omega)$. On the other hand, if $L^*_{G,\omega} = m_{G,\omega}$, by Theorem 4.1, we obtain

$$L^*_{G,\omega} = m_{G,\omega} \geq \left(\inf_{|x|=1} \#Gx\right) L^*_\omega > \left(\inf_{|x|=1} \#Gx\right) \left(\tfrac{1}{4}\right)^{\frac{2}{p-1}} S_\omega(Q_\omega).$$

Therefore, if $\inf_{|x|=1} \#Gx \geq 4^{\frac{2}{p-1}}$, we have $L^*_{G,\omega} \geq S_\omega(Q_\omega)$ whether $L^*_{G,\omega} < m_{G,\omega}$ holds or not. This completes the proof of Theorem 2.2. ☐

Let us give the outline of the proof of Theorem 4.1. The proof follows the argument analogous to that in [24]. Let $u \in H^2(\mathbb{R}^d)$ be such that $S_\omega(u) < S_\omega(Q_\omega)$ and $K(u) > 0$. Then from (1.4) we see that

$$K(u) > \left\{ \left(\tfrac{S_\omega(Q_\omega)}{S_\omega(u)}\right)^{\frac{p-1}{2}} - 1 \right\} \left(2\|\Delta u\|^2_{L^2} - K(u)\right).$$

Furthermore, for any real-valued function $\phi(x)$,

$$K(e^{i\phi(x)} u) > \left\{ \left(\tfrac{S_\omega(Q_\omega)}{S_\omega(u)}\right)^{\frac{p-1}{2}} - 1 \right\} \left(2\|\Delta u\|^2_{L^2} - K(u)\right). \tag{4.3}$$

Assume that $L^*_\omega < S_\omega(Q_\omega)$. By the concentration compactness argument, we can construct a critical non-scattering global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ such that $S_\omega(u) = L^*_\omega$ and $K(u(0)) > 0$. Moreover, $\{u(t, \cdot + x(t)) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$ for some function $x(t)$. From the interaction virial identity, we have the following lemma.

Lemma 4.3. *Let $d \geq 1$ and p satisfy* (1.2). *Let u be a global solution to* (1.1) *such that $\|u\|_{L^\infty_t H^2_x(\mathbb{R}\times\mathbb{R}^d)} < \infty$ and $\{u(t, \cdot + x(t)) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$ for some function $x(t)$. Then for any $\varepsilon > 0$,*

$$\left| M(u)\int_I K(u(t))dt - 2\vec{P}(u) \cdot \int_I \operatorname{Im}\int_{\mathbb{R}^d} \Delta u(t,x)\nabla\overline{u}(t,x)dxdt \right| \leq \varepsilon|I| \tag{4.4}$$

holds for all sufficiently large intervals $I \subset \mathbb{R}$.

By the momentum conservation, without loss of generality, we may assume that $\vec{P}(u) = P(u)e_1$ with a constant $P(u) \geq 0$. For $X \in \mathbb{R}$, we have

$$K(e^{iXx_1}u(t)) = K(u(t)) + 8C(u(t))X + 12G(u(t))X^2 + 8P(u)X^3 + 2M(u)X^4 \tag{4.5}$$

for all $t \in \mathbb{R}$, where

$$C(u) := \operatorname{Im}\int_{\mathbb{R}^d} \Delta u \partial_1 \overline{u} dx, \quad G(u) := \int_{\mathbb{R}^d} \tfrac{1}{3}|\nabla u|^2 + \tfrac{2}{3}|\partial_1 u|^2 dx.$$

Let I_k be a sequence of intervals such that (4.4) holds for $\varepsilon = 1/k$. Since $\|u\|_{L^\infty_t H^2_x(\mathbb{R}\times\mathbb{R}^d)} < \infty$, all coefficients of the fourth-degree polynomial in X on the right-hand side in (4.5) are uniformly bounded in $t \in \mathbb{R}$. Therefore, passing to a subsequence if necessary, we may assume that the time averages of those coefficients on I_k converge to finite values. Then from (4.5), the average of $K(e^{iXx_1}u(t))$ on I_k converges pointwise to a polynomial, that is,

$$\langle K(e^{iXx_1}u)\rangle = \langle K(u)\rangle + 8\langle C(u)\rangle X + 12\langle G(u)\rangle X^2 + 8P(u)X^3 + 2M(u)X^4,$$

where $\langle F\rangle := \lim_{k\to\infty} \frac{1}{|I_k|}\int_{I_k} F(t)dt$ for functions $F(t)$. From Lemma 4.3, we have

$$M(u)\langle K(u)\rangle = 2P(u)\langle C(u)\rangle.$$

Using this and taking $X = X_0 := -\left(\langle K(u)\rangle/2M(u)\right)^{\frac{1}{4}}$, we obtain

$$\begin{aligned}\langle K(e^{iX_0x_1}u)\rangle &= \langle K(u)\rangle \left\{ 6\left(\frac{\langle G(u)\rangle}{\sqrt{\frac{1}{2}\langle K(u)\rangle M(u)}} - 1\right) - 4\left(\sqrt{\alpha} - \tfrac{1}{\sqrt{\alpha}}\right)^2 \right\} \\ &\leq 6\left(\sqrt{2\langle \|\Delta u\|^2_{L^2}\rangle\langle K(u)\rangle} - \langle K(u)\rangle \right),\end{aligned} \tag{4.6}$$

where $\alpha = P(u)/\left(\frac{1}{2}\langle K(u)\rangle M(u)\right)^{\frac{1}{4}}$ and we used the estimate $G(u) \leq \|\Delta u\|_{L^2}\|u\|_{L^2}$. Therefore, from (4.3) and (4.6) we have

$$\left(\frac{S_\omega(Q_\omega)}{S_\omega(u)}\right)^{\frac{p-1}{2}} < 1 + \frac{6\left(\sqrt{2\langle\|\Delta u\|^2_{L^2}\rangle\langle K(u)\rangle} - \langle K(u)\rangle\right)}{2\langle\|\Delta u\|^2_{L^2}\rangle - \langle K(u)\rangle} \leq 4.$$

This implies

$$L^*_\omega = S_\omega(u) > \left(\tfrac{1}{4}\right)^{\frac{2}{p-1}} S_\omega(Q_\omega).$$

References

[1] Akahori, T., Nawa, H.: Blowup and scattering problems for the nonlinear Schrödinger equations. Kyoto J. Math. **53**, 629–672 (2013).

[2] Bonheure, D., Castéras, J-B., Santos, E. M., Nascimento, R.: Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation. SIAM J. Math. Anal. **50**, 5027–5071 (2018).

[3] Bonheure, D., Nascimento, R.: Waveguide solutions for a nonlinear Schrödinger equation with mixed dispersion. Progr. Nonlinear Differential Equations Appl. **87**, 31–53 (2015).

[4] Boulenger, T., Lenzmann, E.: Blowup for biharmonic NLS. Ann. Sci. Éc. Norm. Supér. **50**, 503–544 (2017).

[5] Dinh, V. D.: Dynamics of radial solutions for the focusing fourth-order nonlinear Schrödinger equations. Nonlinearity **34**, 776–821 (2021).

[6] Dodson, B.: Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. Adv. Math. **285**, 1589–1618 (2015).

[7] Dodson, B.: Global well-posedness and scattering for the focusing, cubic Schrödinger equation in dimension $d = 4$. Ann. Sci. Éc. Norm. Supér. **52**, 139–180 (2019).

[8] Dodson, B., Murphy, J.: A new proof of scattering below the ground state for 3D radial focusing cubic NLS. Proc. Amer. Math. Soc. **145**, 4859–4867 (2017).

[9] Duyckaerts, T., Holmer, J., Roudenko, S.: Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. Math. Res. Lett. **15**, 1233–1250 (2008).

[10] Fang, D. Y., Xie, J., Cazenave, T.: Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. Sci. China Math. **54**, 2037–2062 (2011).

[11] Guo, Q.: Scattering for the focusing L^2-supercritical and $\dot{H}^2$-subcritical biharmonic NLS equations. Comm. Partial Differential Equations **41**, 185–207 (2016).

[12] Holmer, J., Roudenko, S.: A sharp condition for scattering of the radial 3D cubic nonlinear Schödinger equation. Comm. Math. Phys. **282**, 435–467 (2008).

[13] Inui, T.: Global dynamics of solutions with group invariance for the nonlinear Schrödinger equation. Commun. Pure Appl. Anal. **16**, 557–590 (2017).

[14] Inui, T.: Remarks on the global dynamics for solutions with an infinite group invariance to the nonlinear Schrödinger equation. Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, B70, Res. Inst. Math. Sci. (RIMS), Kyoto, 1–32 (2018).

[15] Kenig, C. E., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math. **166**, 645–675 (2006).

[16] Komada, K., Masaki, S.: Scattering of solutions with group invariance for the fourth-order nonlinear Schrödinger equation. Nonlinearity **37**, 085003 (2024).

[17] Komada, K., Masaki, S.: Scattering for the focusing, L^2-supercritical fourth-order NLS in one dimension. Discrete Contin. Dyn. Syst. **45**, 480–510 (2025).

[18] Killip, R., Visan, M.: The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Amer. J. Math. **132**, 361-424 (2010).

[19] Miao, C., Xu, G., Zhao, L.: Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case. J. Differential Equations **246**, 3715–3749 (2009).

[20] Miao, C., Xu, G., Zhao, L.: Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$. J. Differential Equations **251**, 3381–3402 (2011).

[21] Pausader, B.: Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. Dyn. Partial Differ. Equ. **4**, 197–225 (2007).

[22] Pausader, B.: The focusing energy-critical fourth-order Schrödinger equation with radial data. Discrete Contin. Dyn. Syst. **24**, 1275–1292 (2009).

[23] Pausader, B.: The cubic fourth-order Schrödinger equation. J. Funct. Anal. **256**, 2473–2517 (2009).

[24] Pausader, B., Shao, S.: The mass-critical fourth-order Schrödinger equation in high dimensions. J. Hyperbolic. Differ. Equ. **07**, 651–705 (2010).

[25] Pausader, B., Xia, S.: Scattering theory for the fourth-order Schrödinger equation in low dimensions. Nonlinearity **26**, 2175–2191 (2013).

BLOW-UP RATE FOR THE SUBCRITICAL SEMILINEAR HEAT EQUATION IN NON-CONVEX DOMAINS

JIN TAKAHASHI

Department of Mathematical and Computing Science, Institute of Science Tokyo, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, Japan

1. Introduction

This talk is based on a joint work with Professors Hideyuki Miura (Institute of Science Tokyo) and Erbol Zhanpeisov (Tohoku University).

We consider the following semilinear heat equation:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $u_0 \in L^\infty(\Omega)$, Ω is a domain in $\mathbf{R}^n$ with $n \geq 1$ and the boundary condition is not imposed if $\Omega = \mathbf{R}^n$. This is a prototypical model for the study of the blow-up phenomena and a fundamental problem is to specify the blow-up rate.

In the Sobolev subcritical range

$$1 < p < p_S := \begin{cases} \dfrac{n+2}{n-2} & \text{for } n \geq 3, \\ \infty & \text{for } n = 1, 2, \end{cases}$$

Weissler [15, 16] first obtained the blow-up rate

$$\sup_{0<t<T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{1.2}$$

for particular classes of nonnegative solutions. Here, (1.2) is referred to as type I blow-up. If (1.2) fails, the blow-up is said to be of type II. Friedman–McLeod [5] also obtained the type I rate for nonnegative solutions in a bounded convex domain under the assumption $u_t \geq 0$. By Giga–Kohn [6], the type I rate holds in convex domains for either $p < p_S$ and $u \geq 0$, or $p < (3n+8)/(3n-4)(< p_S)$.

After the above pioneering works, up to now, the type I rate for $p < p_S$ has been proved in the following cases:

- Sign-changing solutions in convex domains (Giga–Matsui–Sasayama [7, 8]);
- Nonnegative solutions in general domains (Quittner [12]).

E-mail address: takahashi.j.cf61@m.isct.ac.jp.

The author was supported in part by JSPS KAKENHI Grant Numbers 22KK0035, 23K12998 and 23K22402.

We note that the range $p < p_S$ cannot be extended, since there exist type II blow-up solutions [3, 4, 9, 14] for $p = p_S$ with $3 \le n \le 6$. Our goal is to establish the type I blow-up rate in the remaining case $p < p_S$ for solutions that may change sign in possibly unbounded and non-convex domains.

Theorem 1.1 ([10]). *Let $n \ge 1$, $1 < p < p_S$, Ω be any uniformly $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 < \alpha < 1$ and u be a classical solution of* (1.1) *with $u_0 \in L^\infty(\Omega)$. If the maximal existence time $T > 0$ is finite, then*

$$\sup_{0<t<T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

The proof of Theorem 1.1 is based on the analysis of the Giga–Kohn weighted energy [6] of solutions in the backward similarity variables. The main difficulty comes from non-convexity. In non-convex domains, the usual monotonicity of the Giga–Kohn energy can fail due to the lack of a definite sign of the boundary integral in the following estimate:

$$E_{(\tilde{x},\tilde{t})}(t_2) \le E_{(\tilde{x},\tilde{t})}(t_1) - \frac{1}{4}\int_{t_1}^{t_2} B_{(\tilde{x},\tilde{t})}(t)dt$$

for $0 < t_1 < t_2 < T$, where $E_{(\tilde{x},\tilde{t})}$ is the Giga–Kohn weighted energy, $B_{(\tilde{x},\tilde{t})}$ is the boundary integral given by

$$B_{(\tilde{x},\tilde{t})}(t) := (\tilde{t}-t)^{\frac{2}{p-1}} \int_{\partial\Omega} ((x-\tilde{x})\cdot\nu_x)|\nabla u|^2 K_{(\tilde{x},\tilde{t})}(x,t)dS(x)$$

with the exterior unit normal ν_x at $x \in \partial\Omega$ and $K_{(\tilde{x},\tilde{t})}$ is the backward heat kernel centered at $(\tilde{x},\tilde{t})$. For details of the definitions, see Subsection 2.1.

To overcome the difficulty arising from non-convexity, we decompose the boundary integral $B_{(\tilde{x},\tilde{t})}$ into its positive and negative parts. Then, we estimate each of them by using a boundary integral quantity

$$\beta(r) := r^{\frac{2(p+1)}{p-1}-n-1} \sup_{\substack{\tilde{x}\in\Omega,\\ \mathrm{dist}(\tilde{x},\partial\Omega)<r}} \int_{\tilde{t}-2r^2}^{\tilde{t}-r^2} \int_{\{|x-\tilde{x}|<r\}\cap\partial\Omega} |\nabla u(x,t)|^2 dS(x)dt.$$

We note that this idea is due to Blatt–Struwe [1]. However, while they [1] assume the boundedness of the domain and localize an analog of the Giga–Kohn energy, we address possibly unbounded domains and do not localize the energy at this stage.

The above procedure gives a uniform bound for the energy without assuming convexity. The subsequent steps are based on a bootstrap scheme of Giga–Matsui–Sasayama [7, 8] type. Then, we obtain an $L^\infty_t L^q_x$ estimate for all $p > 1$ and $q < p+1$. This together with the parabolic regularity with the help of $p < p_S$, equivalently of $q_c < p+1$, yields the type I blow-up rate. In what follows, we give a more detailed outline of the proof.

2. Outline of proof

2.1. **Definition of Giga–Kohn energy.** Let $p > 1$ and let u be a solution of (1.1) with blow-up time $0 < T < \infty$. Since Ω is uniformly $C^{2+\alpha}$, we can assume without loss of generality that

$$\begin{cases} u \in L^\infty((0,T_2);L^\infty(\Omega)),\ \nabla u,\ \nabla^2 u \text{ and } u_t \text{ are continuous} \\ \text{and bounded on } \overline{\Omega}\times[T_1,T_2] \text{ for each } 0<T_1<T_2<T. \end{cases} \tag{2.1}$$

We note that the uniformity of the regularity of the domain will also be used to derive the geometric inequality (2.7) below. Let $\tilde{x} \in \Omega$ and $0 < \tilde{t} \le T$. We define the Giga–Kohn weighted energy by

$$E_{(\tilde{x},\tilde{t})}(t) := (\tilde{t}-t)^{\frac{p+1}{p-1}} \int_\Omega \left(\frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1} + \frac{|u|^2}{2(p-1)(\tilde{t}-t)} \right) \times K_{(\tilde{x},\tilde{t})}(x,t)dx \tag{2.2}$$

for $0 < t < \tilde{t}$, where $K_{(\tilde{x},\tilde{t})}(x,t) = (\tilde{t}-t)^{-n/2}e^{-|x-\tilde{x}|^2/(4(\tilde{t}-t))}$ is the backward heat kernel and u and ∇u in the integral are evaluated at (x,t).

As preliminaries, we define the backward rescaled solution and the backward similarity variables by

$$\begin{aligned} &w_{(\tilde{x},\tilde{t})}(\eta,\tau) := e^{-\frac{1}{p-1}\tau} u(\tilde{x} + e^{-\frac{1}{2}\tau}\eta, \tilde{t} - e^{-\tau}) = (\tilde{t}-t)^{\frac{1}{p-1}} u(x,t), \\ &\eta := (\tilde{t}-t)^{-\frac{1}{2}}(x-\tilde{x}), \quad \tau := -\log(\tilde{t}-t). \end{aligned} \tag{2.3}$$

We see that w solves

$$\begin{cases} w_\tau = \Delta w - \dfrac{\eta}{2}\cdot\nabla w - \dfrac{w}{p-1} + |w|^{p-1}w, & \eta\in\Omega(\tau),\ \tau\in(\tilde{\tau}_0,\infty), \\ w = 0, & \eta\in\partial\Omega(\tau),\ \tau\in(\tilde{\tau}_0,\infty), \\ w(\eta,\tilde{\tau}_0) = \tilde{t}^{\frac{1}{p-1}} u_0(\tilde{x} + \tilde{t}^{\frac{1}{2}}\eta), & \eta\in\Omega(\tilde{\tau}_0), \end{cases}$$

where $\tilde{\tau}_0 := -\log\tilde{t}$ and

$$\Omega(\tau) := \{\eta\in\mathbf{R}^n; e^{-\frac{\tau}{2}}\eta + \tilde{x}\in\Omega\} = e^{\frac{\tau}{2}}(\Omega-\tilde{x}).$$

Here and below, we omit the subscripts $(\tilde{x},\tilde{t})$ if they are clear from the context. Note that w also satisfies, with $\rho(\eta) := e^{-|\eta|^2/4}$,

$$w_\tau\rho = \nabla\cdot(\rho\nabla w) - \frac{1}{p-1}w\rho + |w|^{p-1}w\rho.$$

In the backward similarity variables, $E_{(\tilde{x},\tilde{t})}$ can be written as

$$E_{(\tilde{x},\tilde{t})}(t) = \mathcal{E}_{(\tilde{x},\tilde{t})}(\tau), \quad \tau = -\log(\tilde{t}-t),\ \tau > \tilde{\tau}_0,$$

where $\mathcal{E}_{(\tilde{x},\tilde{t})}$ is the Giga–Kohn energy in the similarity variables, that is,

$$\mathcal{E}_{(\tilde{x},\tilde{t})}(\tau) := \int_{\Omega(\tau)} \left(\frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right)\rho d\eta. \tag{2.4}$$

2.2. **Quasi-monotonicity.** We examine the monotonicity of the energy. From Giga–Kohn [6, Proposition 2.1], it follows that

$$\frac{d}{d\tau}\mathcal{E}_{(\tilde{x},\tilde{t})}(\tau) = -\int_{\Omega(\tau)} |w_\tau|^2\rho d\eta - \frac{1}{4}\int_{\partial\Omega(\tau)} (\eta\cdot\nu_\eta)|\nabla w\cdot\nu_\eta|^2\rho dS(\eta)$$

for $\tau > \tilde{\tau}_0$, where ν_η is the exterior unit normal at $\eta\in\partial\Omega(\tau)$ and $dS = dS(\eta)$ is the surface area element. Since the tangential derivatives of w vanish by $w = 0$ on $\partial\Omega(\tau)$, we have $\nabla w = (\nabla w\cdot\nu_\eta)\nu_\eta$ and $|\nabla w|^2 = |\nabla w\cdot\nu_\eta|^2$ on $\partial\Omega(\tau)$. Then,

$$\begin{aligned} \mathcal{E}_{(\tilde{x},\tilde{t})}(\tau_2) = \mathcal{E}_{(\tilde{x},\tilde{t})}(\tau_1) &- \int_{\tau_1}^{\tau_2}\int_{\Omega(\tau)} |w_\tau|^2\rho d\eta d\tau \\ &- \frac{1}{4}\int_{\tau_1}^{\tau_2}\int_{\partial\Omega(\tau)} (\eta\cdot\nu_\eta)|\nabla w|^2\rho dS(\eta)d\tau \end{aligned} \tag{2.5}$$

for $\tilde{\tau}_0 < \tau_1 < \tau_2 < \infty$. If the domain Ω is convex, then we can easily check the monotonicity of $\tau \mapsto \mathcal{E}_{(\tilde{x},\tilde{t})}(\tau)$ by $\eta \cdot \nu_\eta \geq 0$ on $\partial\Omega(\tau)$. However, that is not the case when Ω is possibly non-convex. To overcome this difficulty, we proceed with the analysis based on Blatt–Struwe [1, Section 4].

Scaling back to the original variables shows that

$$E_{(\tilde{x},\tilde{t})}(t_2) \leq E_{(\tilde{x},\tilde{t})}(t_1) - \frac{1}{4}\int_{t_1}^{t_2} B_{(\tilde{x},\tilde{t})}(t)dt \tag{2.6}$$

for $0 < t_1 < t_2 < \tilde{t}$ with $\tau_1 = -\log(\tilde{t} - t_1)$ and $\tau_2 = -\log(\tilde{t} - t_2)$, where

$$B_{(\tilde{x},\tilde{t})}(t) := (\tilde{t} - t)^{\frac{2}{p-1}} \int_{\partial\Omega} ((x - \tilde{x}) \cdot \nu_x)|\nabla u|^2 K_{(\tilde{x},\tilde{t})}(x,t)dS(x)$$

for $0 < t < \tilde{t}$ and ν_x is the exterior unit normal at $x \in \partial\Omega$. For estimating the boundary integral, we set

$$B^{\pm}_{(\tilde{x},\tilde{t})}(t) := (\tilde{t} - t)^{\frac{2}{p-1}} \int_{\partial\Omega} ((x - \tilde{x}) \cdot \nu_x)_{\pm}|\nabla u|^2 K_{(\tilde{x},\tilde{t})}(x,t)dS(x)$$

for $0 < t < \tilde{t}$, where $((x - \tilde{x}) \cdot \nu_x)_{\pm} \geq 0$ are the positive and negative parts of $x \mapsto (x - \tilde{x}) \cdot \nu_x$. We note that the main obstacle here is the negative term $B^{-}_{(\tilde{x},\tilde{t})}$.

Since Ω is uniformly $C^{2+\alpha}$ (in particular, uniformly C^2), there exists $C_\Omega > 0$ depending only on Ω such that

$$(x - \tilde{x}) \cdot \nu_x \geq \operatorname{dist}(\tilde{x}, \partial\Omega) - C_\Omega|x - \tilde{x}|^2 \quad \text{for } x \in \partial\Omega,\ \tilde{x} \in \Omega. \tag{2.7}$$

See [1, (4.1)] for the proof. Then, (2.7) implies that

$$\begin{aligned} &((x - \tilde{x}) \cdot \nu_x)_+ \geq \operatorname{dist}(\tilde{x}, \partial\Omega) - C_\Omega|x - \tilde{x}|^2, \\ &((x - \tilde{x}) \cdot \nu_x)_- \leq C_\Omega|x - \tilde{x}|^2, \end{aligned}$$

for $x \in \partial\Omega$ and $\tilde{x} \in \Omega$. These inequalities suggest that $B^{-}_{(\tilde{x},\tilde{t})}$ admits a better estimate than $B^{+}_{(\tilde{x},\tilde{t})}$ near the boundary. Indeed, by some technical computations, we can prove that for $\tilde{x} \in \Omega$ and $T/2 < \tilde{t} < T$,

$$\sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^{+}_{(\tilde{x},\tilde{t})}(t)dt \geq C^{-1} \sup_{0<r<\delta/\sqrt{2}} \beta(r), \tag{2.8}$$

$$\sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^{-}_{(\tilde{x},\tilde{t})}(t)dt \leq C\delta \sup_{0<r<\delta/\sqrt{2}} \beta(r), \tag{2.9}$$

where $\delta > 0$ is a small constant depending only on n, p, Ω and T and $C > 1$ is a large constant depending only on n, p and Ω. Moreover, $\beta(r)$ is defined by

$$\beta(r) := r^{\frac{2(p+1)}{p-1} - n - 1} \sup_{\substack{\tilde{x}\in\Omega, \\ \operatorname{dist}(\tilde{x},\partial\Omega)<r}} \int_{\tilde{t}-2r^2}^{\tilde{t}-r^2} \int_{B_r(\tilde{x})\cap\partial\Omega} |\nabla u(x,t)|^2 dS(x)dt$$

for $0 < r < \sqrt{\tilde{t}/2}$, where $B_r(\tilde{x}) := \{x \in \mathbf{R}^n; |x - \tilde{x}| < r\}$ and the idea of using this kind of function is due to Blatt–Struwe [1, p. 2278].

The estimates (2.8) and (2.9) enable us to estimate the main obstacle term $B^-_{(\tilde{x},\tilde{t})}$ by using $B_{(\tilde{x},\tilde{t})} = B^+_{(\tilde{x},\tilde{t})} - B^-_{(\tilde{x},\tilde{t})}$ as follows. For $\tilde{x} \in \Omega$ and $T/2 < \tilde{t} < T$,

$$\begin{aligned}\sup_{0<r<\delta/\sqrt{2}} \beta(r) &\le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^+_{(\tilde{x},\tilde{t})} dt \le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})} dt + C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^-_{(\tilde{x},\tilde{t})} dt \\ &\le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})} dt + C\delta \sup_{0<r<\delta/\sqrt{2}} \beta(r)\end{aligned}$$

with a small constant $0 < \delta < 1$, and so

$$\sup_{0<r<\delta/\sqrt{2}} \beta(r) \le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})}(t) dt.$$

Then by (2.9) and $\delta < 1$,

$$\sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^-_{(\tilde{x},\tilde{t})}(t) dt \le C\delta \sup_{0<r<\delta/\sqrt{2}} \beta(r) \le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})}(t) dt.$$

We see from (2.6) that the integral of $B_{(\tilde{x},\tilde{t})}$ can be estimated by $E_{(\tilde{x},\tilde{t})}(\tilde{t}-\delta^2)$, where the important point is that the time $\tilde{t}-\delta^2$ is strictly before $t = \tilde{t}$. More precisely, (2.6) implies that for $0 < t_1 < t_2 < \tilde{t} < T$,

$$\int_{t_1}^{t_2} B_{(\tilde{x},\tilde{t})}(t) dt \le 4E_{(\tilde{x},\tilde{t})}(t_1) - 4E_{(\tilde{x},\tilde{t})}(t_2).$$

Since $\tilde{t} < T$ is prior to the blow-up time $t = T$, the boundedness in (2.1) together with $\int_\Omega K_{(\tilde{x},\tilde{t})} dx \le (4\pi)^{n/2}$ and (2.2) implies that $-4E_{(\tilde{x},\tilde{t})}(t_2) \to 0$ as $t_2 \to \tilde{t}$. Then by taking $t_1 = \tilde{t} - \delta^2$, we see that

$$\int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})}(t) dt \le 4E_{(\tilde{x},\tilde{t})}(\tilde{t}-\delta^2).$$

Hence we obtain the estimate

$$\sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^-_{(\tilde{x},\tilde{t})}(t) dt \le C \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B_{(\tilde{x},\tilde{t})}(t) dt \le 4C \sup_{\tilde{x}\in\Omega} E_{(\tilde{x},\tilde{t})}(\tilde{t}-\delta^2).$$

We recall (2.6). Then, for $\tilde{t} - \delta^2 \le t' < t < \tilde{t}$,

$$E_{(\tilde{x},\tilde{t})}(t) \le E_{(\tilde{x},\tilde{t})}(t') + \frac{1}{4} \sup_{\tilde{x}\in\Omega} \int_{\tilde{t}-\delta^2}^{\tilde{t}} B^-_{(\tilde{x},\tilde{t})}(t) dt \le E_{(\tilde{x},\tilde{t})}(t') + C \sup_{\tilde{x}\in\Omega} E_{(\tilde{x},\tilde{t})}(\tilde{t}-\delta^2),$$

where $\tilde{x} \in \Omega$ and $T/2 < \tilde{t} < T$. By estimating the second term in the right-hand side (from the definition of $E_{(\tilde{x},\tilde{t})}$ and the estimate (2.1)), and then by taking the limit $\tilde{t} \to T$, we obtain the following quasi-monotonicity of the Giga–Kohn energy in possibly non-convex domains:

Proposition 2.1. *Let $p > 1$. Then there exists $\tilde{C} > 0$ depending only on n, p and Ω such that for all $\tilde{x} \in \Omega$ and $T - \delta^2 \le t' < t < T$,*

$$\begin{aligned}E_{(\tilde{x},T)}(t) \le E_{(\tilde{x},T)}(t') &+ \tilde{C}\delta^{\frac{2(p+1)}{p-1}+1} \|\nabla u\|^2_{L^\infty(\Omega\times(\frac{T}{2}-\delta^2, T-\delta^2))} \\ &+ \tilde{C}\delta^{\frac{2(p+1)}{p-1}-1} \|u\|^2_{L^\infty(\Omega\times(\frac{T}{2}-\delta^2, T-\delta^2))},\end{aligned} \tag{2.10}$$

where $0 < \delta < \sqrt{T/2}$ depends only on n, p, Ω and T.

We remark that the second and third terms in the right-hand side of (2.10) are finite, since $t = T - \delta^2$ is strictly prior to the blow-up time.

2.3. **Estimates via quasi-monotonicity.** We give several estimates by using the quasi-monotonicity. Fix $0 < \delta < 1$ and $\tilde{C} > 0$ as in Proposition 2.1 and set

$$\begin{cases} \tau_* := -\log(\delta^2), \\ C_* := \tilde{C}\delta^{\frac{2(p+1)}{p-1}+1}\|\nabla u\|^2_{L^\infty(\Omega\times(\frac{T}{2}-\delta^2, T-\delta^2))} \\ \qquad + \tilde{C}\delta^{\frac{2(p+1)}{p-1}-1}\|u\|^2_{L^\infty(\Omega\times(\frac{T}{2}-\delta^2, T-\delta^2))}, \\ C'_* := \dfrac{1}{2}(4\pi)^{\frac{n}{2}}\delta^{\frac{2(p+1)}{p-1}}\|\nabla u(\cdot, T-\delta^2)\|^2_{L^\infty(\Omega)} \\ \qquad + \dfrac{1}{2(p-1)}(4\pi)^{\frac{n}{2}}\delta^{\frac{2(p+1)}{p-1}-2}\|u(\cdot, T-\delta^2)\|^2_{L^\infty(\Omega)}, \end{cases} \tag{2.11}$$

where these constants are independent of $\tilde{x} \in \Omega$. In the rest of this subsection, the constants $C > 0$ depend only on n, p, Ω, R, τ_*, C_* and C'_*. Proposition 2.1 in the backward similarity variables gives

$$\mathcal{E}_{(\tilde{x},T)}(\tau) \le \mathcal{E}_{(\tilde{x},T)}(\tau') + C \quad \text{for } \tilde{x} \in \Omega,\ \tau_* \le \tau' < \tau < \infty.$$

This together with a similar argument to [6, Proposition 2.2] deduces that

$$-C \le \mathcal{E}_{(\tilde{x},T)}(\tau) \le C \quad \text{for } \tilde{x} \in \Omega,\ \tau_* \le \tau' < \tau < \infty.$$

Then by (2.5) and the boundedness of the boundary integral,

$$\int_{\tau_*}^{\infty}\int_{\Omega(\tau)} |\partial_\tau w_{(\tilde{x},T)}|^2 \rho\, d\eta\, d\tau \le C.$$

The same computations as in [6, Proposition 2.1] yields

$$\frac{1}{2}\frac{d}{d\tau}\int_{\Omega(\tau)} |w|^2 \rho\, d\eta = -2\mathcal{E}_{(\tilde{x},T)}(\tau) + \frac{p-1}{p+1}\int_{\Omega(\tau)} |w|^{p+1}\rho\, d\eta \tag{2.12}$$

for $\tau_* \le \tau' < \tau < \infty$. This identity and the above estimates suggest that we can control the $L^{p+1}_{\eta,\tau}$ norm and the $L^\infty_\tau L^2_\eta$ norm of the backward rescaled solution w. Indeed, the $L^{p+1}_{\eta,\tau}$ norm can be estimated as follows. By $w = 0$ on $\partial\Omega(\tau)$, we see that for $\tau \ge \tau_* + 2$,

$$\begin{aligned} \int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} |w|^{p+1}\rho\, d\eta\, d\sigma &= \frac{p+1}{p-1}\int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} w w_\tau \rho\, d\eta\, d\sigma + \frac{2(p+1)}{p-1}\int_{\tau-2}^{\tau}\mathcal{E}\, d\sigma \\ &\le \frac{1}{2}\int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} |w|^{p+1}\rho\, d\eta\, d\sigma + C\int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} |w_\tau|^{\frac{p+1}{p}}\rho\, d\eta\, d\sigma + C, \end{aligned}$$

and so the Hölder inequality yields

$$\int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} |w|^{p+1}\rho\, d\eta\, d\sigma \le C\left(\int_{\tau-2}^{\tau}\int_{\Omega(\sigma)} |w_\tau|^2 \rho\, d\eta\, d\sigma\right)^{\frac{p+1}{2p}} + C \le C. \tag{2.13}$$

As for the $L^\infty_\tau L^2_\eta$ norm, we set

$$W(\tau) := \left(\int_{\Omega(\tau)} |w_{(\tilde{x},T)}(\eta,\tau)|^2 \rho(\eta)\, d\eta\right)^{\frac{1}{2}}.$$

We note from $w = 0$ on $\partial\Omega(\tau)$ that

$$|W'(\tau)| = \frac{1}{W(\tau)} \left| \int_{\Omega(\tau)} w w_\tau \rho d\eta \right| \le \left(\int_{\Omega(\tau)} |w_\tau|^2 \rho d\eta \right)^{\frac{1}{2}},$$

Then, by the 1-dimensional Sobolev inequality and (2.13),

$$\begin{aligned} W^2(\tau) &\le \left(\sup_{\sigma\in[\tau-2,\tau]} W(\sigma) \right)^2 \le C(\|W\|^2_{L^2(\tau-2,\tau)} + \|W'\|^2_{L^2(\tau-2,\tau)}) \\ &\le C + C\|W'\|^2_{L^2(\tau-2,\tau)} \le C + C\int_{\tau-2}^{\tau} \int_{\Omega(\sigma)} |w_\tau|^2 \rho d\eta d\sigma \le C \end{aligned}$$

for $\tau \ge \tau_* + 2$. Hence we obtain the desired $L^\infty_\tau L^2_\eta$ estimate

$$\sup_{\tau\ge\tau_*+2} \int_{\Omega(\tau)} |w_{(\tilde{x},T)}(\eta,\tau)|^2 \rho(\eta) d\eta \le C. \tag{2.14}$$

By using (2.14), we can improve (2.13). Indeed, (2.12) and (2.14) yield

$$\int_{\Omega(\tau)} |w|^{p+1} \rho d\eta \le C \int_{\Omega(\tau)} |w||w_\tau| \rho d\eta + C \ \le C \left(\int_{\Omega(\tau)} |w_\tau|^2 \rho d\eta \right)^{\frac{1}{2}} + C,$$

and so

$$\int_{\tau-2}^{\tau} \left(\int_{\Omega(\sigma)} |w|^{p+1} \rho d\eta \right)^2 d\sigma \le C \quad \text{for } \tau \ge \tau_* + 2. \tag{2.15}$$

2.4. **Inductive estimate.** To obtain further improvements of (2.15), we localize the spatial integrals in the same spirit of Giga–Matsui–Sasayama [8]. Let $0 < R < 1$ be a sufficiently small constant (see [10, (4.12)] for details). For $r > 0$, we write $B_r := B_r(0) = \{y \in \mathbf{R}^n; |y| < r\}$. For $k \ge 1$, let $\psi_k \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \le \psi_k \le 1$ in $\mathbf{R}^n$, $\psi_k = 1$ in $B_{R/2^k}$, $\psi_k = 0$ in $\mathbf{R}^n \setminus \overline{B_{R/2^{k-1}}}$ and

$$|\nabla\psi_k(\eta)| \le \frac{2^k C}{R}, \quad |\nabla^2\psi_k(\eta)| \le \frac{4^k C}{R^2} \quad \text{for } \eta \in \mathbf{R}^n,$$

with an absolute constant $C > 0$. We define the localization of $\mathcal{E}_{(\tilde{x},T)}$ by

$$\mathcal{E}^{\psi_k}_{(\tilde{x},T)}(\tau) := \int_{\Omega(\tau)} \left(\frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho \psi_k^4 d\eta$$

for $\tilde{x} \in \Omega$, where the original Giga–Kohn energy $\mathcal{E}_{(\tilde{x},T)}$ has been defined in (2.4) with $\tilde{t} = T$. The boundedness of $\mathcal{E}_{(\tilde{x},T)}$ gives

$$\sup_{\tau\ge\tau_*+2} \mathcal{E}^{\psi_k}_{(\tilde{x},T)}(\tau) \le C_k.$$

Here, $C_k > 0$ is independent of $\tilde{x} \in \Omega$ and may depend on k. Since our inductive steps consist of only finitely many steps, the dependence of C_k on k is harmless.

By using the estimates of the original Giga–Kohn energy $\mathcal{E}_{(\tilde{x},T)}$,

$$\int_{\Omega(\tau)} |w|^{p+1} \rho \psi_k^4 d\eta \le C_k \int_{\Omega(\tau)} |w w_\tau| \rho \psi_k^4 d\eta + C_k$$

for $\tau \ge \tau_* + 2$ and $k \ge 1$, where this corresponds to (2.12). In a similar way to the derivation of (2.15) with the help of Cazenave–Lions [2] type interpolation (see [11, (2.9)] and also [13, (51.6)]) and the maximal regularity in Giga–Matsui–Sasayama [8, Theorem 2.1], we can prove the following inductive estimate:

Proposition 2.2. *Let $\tilde{x} \in \Omega$. Suppose that there exist $q \geq 2$, $k \geq 1$ and $C'_k > 0$ independent of $\tilde{x} \in \Omega$ and satisfying*

$$\sup_{\tau \geq \tau_* + 2} \int_{\tau-1-2^{-(k-1)}}^{\tau} \left(\int_{\Omega(\sigma)} |w_{(\tilde{x},T)}|^{p+1} \rho \psi_{k-1}^4 d\eta \right)^q d\sigma \leq C'_k.$$

Then there exists $C''_k > 0$ depending only on k, n, p, q, Ω, R, τ_*, C_*, C'_* and C'_k (in particular, independent of $\tilde{x} \in \Omega$) such that

$$\sup_{\tau \geq \tau_* + 2} \int_{\tau-1-2^{-k}}^{\tau} \left(\int_{\Omega(\sigma)} |w_{(\tilde{x},T)}|^{p+1} \rho \psi_k^4 d\eta \right)^{q+\frac{1}{p+1}} d\sigma \leq C''_k.$$

2.5. **Type I estimate.** By combining the base case (2.15) and the inductive step in Proposition 2.2, we obtain the following $L^\infty_\tau L^\lambda_\eta$ and $L^\infty_t L^\lambda_x$ estimate:

Proposition 2.3. *Let $p > 1$ and $1 < \lambda < p + 1$. Then there exist $0 < R_* < 1$ depending only on p and λ and $C > 0$ depending only on n, p, λ, Ω, τ_*, C_* and C'_* such that for $\tilde{x} \in \Omega$,*

$$\sup_{\tau \geq \tau_* + 1} \int_{\Omega(\tau) \cap B_{R_*}(0)} |w_{(\tilde{x},T)}(\eta, \tau)|^\lambda \rho d\eta \leq C,$$

equivalently,

$$\sup_{T-\delta^2/e \leq t < T} (T-t)^{\frac{\lambda}{p-1}} \int_{\Omega \cap B_{R_*\sqrt{T-t}}(\tilde{x})} |u(x,t)|^\lambda K_{(\tilde{x},T)}(x,t) dx \leq C,$$

where τ_*, C_* and C'_* are given in (2.11), δ is given in Proposition 2.1 and they are independent of $\tilde{x} \in \Omega$.

Remark 2.4. Proposition 2.3 holds for all $p > 1$. Up to this point, the assumption $p < p_S$ has not been used.

Let $p < p_S$ and $\tilde{x} \in \Omega$. In what follows, $C > 0$ is independent of $\tilde{x} \in \Omega$. We prove Theorem 1.1 by deducing an L^∞-estimate of w. Since $p < p_S$ is equivalent to $n(p-1)/2 < p + 1$, we can choose $q, r > 1$ such that

$$\max\left\{\frac{n}{2}, 1\right\} < q < \frac{p+1}{p-1}, \quad \frac{1}{r} + \frac{n}{2q} < 1.$$

Then by (2.14) and Proposition 2.3 with $\lambda = (p-1)q$,

$$\int_{\tau'-1}^{\tau'} \int_{\Omega(\tau) \cap B_{R_*}} |w_{(\tilde{x},T)}(\eta,\tau)|^2 \rho d\eta d\sigma \leq C,$$

$$\int_{\tau'-1}^{\tau'} \left(\int_{\Omega(\sigma) \cap B_{R_*}} \left(|w_{(\tilde{x},T)}(\eta,\tau)|^{p-1} \right)^q \rho d\eta \right)^{\frac{r}{q}} d\sigma \leq C,$$

for $\tau' \geq \tau_* + 2$. By the same argument as in Giga–Kohn [6, Subsection 3A] based on the interior and boundary regularity (more precisely, see line 9 on page 14 through line 16 on page 16 and Remark 3.6 in [6]),

$$|w_{(\tilde{x},T)}(0,\tau)| \leq C$$

for $\tau' - 1/2 < \tau < \tau'$ and $\tau' \geq \tau_* + 2$. Thus, from (2.3) and from $u \in L^\infty((0, T - e^{-(\tau_*+2)}); L^\infty(\Omega))$ by (2.1), it follows that

$$(T-t)^{\frac{1}{p-1}} |u(\tilde{x}, t)| \leq C$$

for $\tilde{x} \in \Omega$ and $0 < t < T$, where $C > 0$ is independent of $\tilde{x}$. Hence u satisfies the desired type I blow-up estimate.

References

[1] S. Blatt, M. Struwe, Boundary regularity for the supercritical Lane-Emden heat flow. Calc. Var. Partial Differential Equations 54 (2015), no. 2, 2269–2284.

[2] T. Cazenave, P.-L. Lions, Solutions globales d'équations de la chaleur semi linéaires. Comm. Partial Differential Equations 9 (1984), no. 10, 955–978.

[3] M. del Pino, M. Musso, J. Wei, Type II blow-up in the 5-dimensional energy critical heat equation. Acta Math. Sin. (Engl. Ser.) 35 (2019), no. 6, 1027–1042.

[4] M. del Pino, M. Musso, J. Wei, Q. Zhang, Y. Zhou, Type II Finite time blow-up for the three dimensional energy critical heat equation, preprint, arXiv:2002.05765.

[5] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations. Indiana Univ. Math. J. 34 (1985), no. 2, 425–447.

[6] Y. Giga, R. V. Kohn, Characterizing blowup using similarity variables. Indiana Univ. Math. J. 36 (1987), no. 1, 1–40.

[7] Y. Giga, S. Matsui, S. Sasayama, Blow up rate for semilinear heat equations with subcritical nonlinearity. Indiana Univ. Math. J. 53 (2004), no. 2, 483–514.

[8] Y. Giga, S. Matsui, S. Sasayama, On blow-up rate for sign-changing solutions in a convex domain. Math. Methods Appl. Sci. 27 (2004), no. 15, 1771–1782.

[9] J. Harada, A type II blowup for the six dimensional energy critical heat equation. Ann. PDE 6 (2020), no. 2, Paper No. 13, 63 pp.

[10] H. Miura, J. Takahashi, E. Zhanpeisov, Blow-up rate for the subcritical semilinear heat equation in non-convex domains, preprint, arXiv:2510.17229.

[11] P. Quittner, Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems. Houston J. Math. 29 (2003), no. 3, 757–799.

[12] P. Quittner, Optimal Liouville theorems for superlinear parabolic problems. Duke Math. J. 170 (2021), no. 6, 1113–1136.

[13] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states. Second edition. Birkhäuser/Springer, Cham, 2019.

[14] R. Schweyer, Type II blow-up for the four dimensional energy critical semi linear heat equation. J. Funct. Anal. 263 (2012), no. 12, 3922–3983.

[15] F. B. Weissler, Single point blow-up for a semilinear initial value problem. J. Differential Equations 55 (1984), no. 2, 204–224.

[16] F. B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation. Comm. Pure Appl. Math. 38 (1985), no. 3, 291–295.

Grow-up rate of exponential reaction-diffusion equations with inhomogeneous terms*1

Yusuke Oka

Graduate School of Science, Tohoku University, 6-3,
Aramaki Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan.

This is a joint work with Kenta Kumagai (The University of Tokyo).

1 Introduction

We consider the following semilinear heat equation in the unit ball $B_1 \subset \mathbb{R}^N$:

$$\begin{cases} \partial_t u - \Delta u = \lambda e^u - f(r) & \text{in } B_1 \times (0,T), \\ u = 0 & \text{on } \partial B_1 \times (0,T), \\ u(x,0) = u_0(x) \in C^0(\overline{B_1}) & \text{in } B_1 \end{cases} \tag{1.1}$$

and its stationary counterpart

$$-\Delta v = \lambda e^v - f(r) \quad \text{in } B_1, \quad v = 0 \quad \text{on } \partial B_1. \tag{1.2}$$

Here, $N \in \mathbb{Z}_{\geqslant 3}$, $T > 0$, $r := |x|$, $\lambda > 0$ and $f(r) \in \mathrm{Lip}[0,1]$. Throughout of this talk, we assume*2 that $u_0 \geqslant \varphi_0$, where φ_0 is the solution of

$$-\Delta \varphi_0 = -f(r) \quad \text{in } B_1, \quad \varphi_0 = 0 \quad \text{on } \partial B_1. \tag{1.3}$$

Note that $\varphi_0 = 0$ if $f = 0$. The aim of this talk is to explain large-time asymptotics of solutions to (1.1), in terms of the solutions to (1.2); we determine whether the solution blows up in finite time $T > 0$ (i.e., $\limsup_{t\to T} \|u\|_{L^\infty(B_1)} = \infty$) or exists globally, and in the latter case, the grow up (i.e., $\limsup_{t\to\infty} \|u\|_{L^\infty(B_1)} = \infty$) occurs or not. Furthermore, in the case where u grows up, we obtain the precise grow up rate (Theorem 1.5). Let us first review the literature for the classical case $f \equiv 0$.

1.1 Classical case ($f \equiv 0$)

Here, we put our starting point on the study by Peral and Vázquez [9]. They showed that the stable*3 solution to (1.2), denoted by $\underline{v}_\lambda$, is realized as the large-time limit of the solution of (1.1) for the case $u_0 \leqslant \underline{v}_\lambda$. Their result reads as follows. (Hereafter, we set $\lambda^* := \sup\{\lambda \mid \text{ (1.2) admits a radial solution}\} \in \mathbb{R}_{>0}$ and $v^* := \{\text{the solution to (1.2) with } \lambda = \lambda^*\}$ (see e.g., [1]). For (ii), see also [1].)

*1This work was supported in part by JSPS KAKENHI Grant Number 26KJ0018.

*2It is not very restrictive, for all stable stationary solutions are allowed in this setting.

*3We say that a solution v to (1.2) is *stable* if $\int_{B_1} |\nabla \xi|^2 - \lambda e^v \xi^2 \, dx \geqslant 0$ holds for all $\xi \in C_0^1(B_1)$.

(i) If $\lambda < \lambda^*$ and $u_0 \leqslant \underline{v}_\lambda$ in B_1, there exists a unique global-in-time solution u of (1.1) such that $u \to \underline{v}_\lambda$ in $C^2(B_1)$ as $t \to \infty$.

(ii) If $\lambda^* < \lambda$, there exists a unique local-in-time solution u which blows up in finite time.

(iii) If $\lambda = \lambda^*$ and $u_0 \leqslant v^*$ in B_1, $\exists$ a unique global-in-time solution u such that

$(N \leqslant 9)$ $u \to v^*$ in $C^2(B_1)$ as $t \to \infty$;

$(N \geqslant 10)$ $u \to v^* = -2\log|x|$ in $L^2(B_1) \cap C^2_{\mathrm{loc}}(\overline{B_1} \setminus \{0\})$ and $\|u\|_{L^\infty(B_1)} \to \infty$ as $t \to \infty$ (grow-up). In this case, $\lambda^* = 2(N-2)$ holds.

We remark here that the following fact underlies the above properties:

$$\text{the singular solution } -2\log|x| \text{ is stable if, and only if } N \geqslant 10$$

(when $\lambda = 2(N-2)$). For details, see [2,6] and the subsequent literature. The case $u_0 \not\leqslant \underline{v}_\lambda$ being more delicate, we do not go into the detail in this talk.

In the case of $\lambda = \lambda^*$, $u_0 \leqslant v^*$ and $N \geqslant 11$ ($\Rightarrow$ u grows up), Dold, Galaktionov, Lacey and Vázquez [3] determined the grow-up rate $\|u\|_{L^\infty(B_1)} = H\gamma^{-1}t + O(1)$. For the definition of H and γ, see Proposition 1.2 and Theorem 1.5, respectively. For the case $N = 10$, Galaktionov and King [4] conjectured that the grow-up rate is $2H\gamma^{-1}t + O(\log t)$ by a formal computation.

Motivated by the classical case $f = 0$, we arrive at the fundamental question: how the inhomogeneous term f affects the structure of solutions to (1.2) and the large-time behavior of the solution of (1.1)?

1.2 Inhomogeneous stationary problem

We state our results. We first introduce the following parameterization result.

Proposition 1.1. The following assertions hold for problem (1.2).

1. The set of radial solutions of (1.2) is an unbounded analytic curve[*4] emanating from $(0, \varphi_0)$ and described by $\{(\lambda(\beta), v(r, \alpha(\beta)); \beta \in \mathbb{R}\}$ with $\alpha(\beta) := v(0, \beta) = \beta - \log\lambda(\beta)$. Moreover, (1.2) has no (even non-radial or weak) solution for $\lambda > \lambda^*$.

2. There exists the unique solution $v^* \in H^1_0(B_1)$ with $\lambda = \lambda^*$. The set of stable (possibly non-radial weak) solutions coincides with

$$\{(\lambda(\beta), v(r, \alpha(\beta))); -\infty < \beta < v^*(0) + \log\lambda^*\} \cup \{(\lambda^*, v^*)\}$$

Along this branch, $(\lambda(\beta), v(r, \alpha(\beta)))$ increases monotonically from $(0, \varphi_0)$ and converges to (λ^*, v^*) as $\beta \to v^*(0) + \log\lambda^* \in (0, \infty]$. In particular, stable solution is unique for each $\lambda \in (0, \lambda^*]$. So we let $\underline{v}_\lambda$ denote it.

[*4] We call the curve $\{(\lambda(\beta), \alpha(\beta)); \beta \in \mathbb{R}\}$ the *bifurcation curve* to problem (1.2).

Following the idea of [7], we explicitly construct a one-parameter family of the singular solutions $(\lambda, v) = (\lambda_h, V_h)$ to (1.2) with a parameterized inhomogeneous term $f = f_h$. They enable us to study the Morse index of V_h (denoted by $m(V_h)$).

Proposition 1.2. There exists the unique radial singular solution (λ_*, V_*) of (1.2) such that $\lambda(\beta) \to \lambda_*$ and $v(r, \alpha(\beta)) \to V_*$ in $C^2_{\mathrm{loc}}(0,1] \cap H^1_0(B_1)$ as $\beta \to \infty$, and $V_* = -2\log r + \log(2N-4) - \log\lambda_* + o(1)$ as $r \to 0$. Moreover, for

$$f_h(r) := h\left(8(N-2)a^2(r) + 2(N-2)a(r) + 1\right), \text{ where } a(r) := (2N-4+hr^2)^{-1}$$

with $h > 0$, the singular solution is represented by

$$(\lambda_h, V_h) = \left(2N-4+h, -2\log r - \log a(r) - \log\left[2N-4+h\right]\right).$$

The Morse index $m(V_h) = \infty$ if $3 \leqslant N \leqslant 9$; $m(V_h) = 0$ and V_h is stable if $N \geqslant 10$ and $h \leqslant H$; and $1 \leqslant m(V_h) < \infty$ if $N \geqslant 10$ and $h > H$, where

$$H := \left\{\text{the first eigenvalue of } -\Delta_D - (2N-4)|x|^{-2} \text{ in } B_1^N \subset \mathbb{R}^N\ (N \geqslant 10)\right\} > 0.$$

1.3 Inhomogeneous parabolic problem

The change in the bifurcation structure for (1.2) induced by f, as shown in §1.2, leads to qualitative changes in the large-time asymptotics of solutions to (1.1).

Theorem 1.3. Let $N \geqslant 3$ and $f \in C^0[0,\infty)$. Then,

(i) If $\lambda < \lambda^*$ and $u_0 \leqslant \underline{v}_\lambda$ in B_1, there exists a unique global-in-time solution u of (1.1) such that $u \to \underline{v}_\lambda$ in $L^\infty(B_1)$ as $t \to \infty$.

(ii) If $\lambda > \lambda^*$, there exists a unique local-in-time solution u which blows up in finite time.

(iii) If $\lambda = \lambda^*$ and $u_0 \leqslant v^*$, a unique global-in-time solution u exists. In addition,

$(N \leqslant 9)$ $u \to v^*$ in $L^\infty(B_1)$ as $t \to \infty$;

$(N \geqslant 10$ and $f \leqslant f_H)$ $u \to v^*$ in $L^2(B_1) \cap L^\infty_{\mathrm{loc}}(B_1 \setminus \{0\})$ and $\|u\|_{L^\infty(B_1)} \to \infty$ as $t \to \infty$;

$(N \geqslant 10$, $f \geqslant f_H$ and $f \not\equiv f_H)$ $u \to v^*$ in $L^\infty(B_1)$ as $t \to \infty$.

Remark 1.4. When $u_0 \not\leqslant \underline{v}_\lambda$, we can show that v^*, V_*, and any unstable solutions v are thresholds in the following sense: if $u_0 \leqslant v$ and $u_0 \not\equiv v$, then u converges to $\underline{v}_\lambda$ in L^∞ as $t \to \infty$; whereas if $u_0 \geqslant v$ and $u_0 \not\equiv v$, then u blows up in the finite time. Furthermore, so-called instantaneous blow-up occurs if $u_0 \geqslant V_*, \not\equiv V_*$.

Theorem 1.3 shows that the grow-up phenomena cease to occur beyond the threshold $f = f_H$. This leads us to expect that the grow-up rate decreases as $f \uparrow f_H$. We prove the expectation quantitatively by determining the grow-up rate.

Theorem 1.5. Let $N \geqslant 11$ and set $\gamma := \frac{1}{2}\left(N-2-\sqrt{(N-2)(N-10)}\right)$. Assume that $\varphi_0 \leqslant u_0 \leqslant V_*$ and $\lambda > \lambda_*$ ($\Rightarrow u$ grows up). Then we have

1. $\|u\|_{L^\infty} = \frac{2\mu_1}{\gamma}t + O(1)$ if $0 \leqslant f \leqslant f_H$ and $f \neq f_H$;
2. $\|u\|_{L^\infty} = \frac{2}{\gamma}\log t + O(1)$ if $f = f_H$ and $N \geqslant 12$;
3. $\|u\|_{L^\infty} = \frac{2}{\gamma}(\log t + \log\log t) + O(1)$ if $f = f_H$ and $N = 11$,

where μ_1 is the first eigenvalue of $\mathcal{L} = -\Delta - \lambda_* e^{V_*}$ in $H^1(B_1)$.

Remark 1.6. We can verify that $\mu_1 = H - h$ when $f = f_h$. Thus our result contains [3] as a special case ($h = 0$). We can also show that $\mu_1 > 0$ when $f \leqslant f_H$, $\not\equiv f_H$ in B_1, and $\mu_1 \downarrow 0$ as $f \uparrow f_H$, which reflects the decline of grow up rate quantitatively. At the threshold case $f = f_H$, u exhibits a logarithmic grow-up, which is slower than the case $f \leqslant f_H$ and $f \neq f_H$. When $N = 11$, the log-log type second order term arises from the singularity of the first eigenfunction φ_1 of $\mathcal{L}$.

2 Sketch of Proof

In this section, we explain Theorem 1.5 by formal calculation.

When $f \equiv 0$, the authors [3] assume without loss of generality that $\alpha(t) := \|u\|_{L^\infty} = u(0)$ by considering the radial and decreasing initial data and by using the moving plane method [5]. Then, by using the scale invariance of the equation and intersection method, they showed that $u = y_0(1+o(1))$ in B_{r_t} and $u \geqslant y_0$ in B_1 for large t, where $r_t := e^{-\alpha(t)/2}$ and $y_0 = y_0(r)$ is the solution of the problem

$$-\Delta y = \lambda_* e^y - f \quad \text{in } B_1, \quad y(0) = \alpha(t), \quad y'(0) = 0 \tag{2.1}$$

with $\lambda_* = \lambda_0 := 2N-4$ and $f = f_0 = 0$. On the other hand, when $f \neq 0$, the scaling invariance is lost. So instead of proving $u = y_0(1+o(1))$, we establish in [8] an alternative tool.

Next, for fixed $0 < \varepsilon \ll 1$, we describe the asymptotic behavior of y in terms of $\alpha(t)$ near a matching point $r = \varepsilon$. When $f = 0$, by the self-similarity of (2.1), we can deduce that y_0 satisfies $y_0 = \alpha(t) + \underline{y}(r/r_t)$, where $\underline{y}$ is the solution of

$$-\Delta \underline{y} = (2N-4)e^{\underline{y}} \quad \text{in } \mathbb{R}^N, \quad \underline{y}(0) = \underline{y}'(0) = 0.$$

Since the solution $\underline{y}$ satisfies

$$\underline{y}(r) = V_0(r) - b_N r^{-\gamma}(1+o(1)) \quad \text{as } r \to \infty$$

for some $b_N > 0$ (see (3.4), (3.5) in [3]), $u = y_0(1+o(1))$ means

$$V_0 - u \leqslant V_0 - y_0 \simeq e^{-\gamma\alpha(t)/2} r^{-\gamma} \quad \text{in } B_1 \setminus B_{r_t} \tag{2.2}$$

for $t \gg 1$. So we can estimate y_0 in terms of $\alpha(t)$. When $f \neq 0$, the above argument fails due to the lack of scaling invariance. Nevertheless, we can provide at the matching point $r = \varepsilon$.

Now, Theorem 1.5 follows from the following one.

Proposition 2.1. Define $\Phi := V_* - u$ and set Q as

$$Q(t) := e^{-\mu_1 t} \text{ if } f \neq f_H, \quad Q(t) := \begin{cases} t^{-1} & \text{if } f = f_H,\ N \geqslant 12, \\ t^{-1}(\log t)^{-1} & \text{if } f = f_H,\ N = 11. \end{cases}$$

There exists a positive constant C, independent of t, r and u such that $\Phi \leqslant CQ(t)\varphi_1$ in B_1 for sufficiently large t. Moreover, for any $\varepsilon > 0$, there exist $c > 0$ and $T > 0$ such that $\Phi \geqslant cQ(t)\varphi_1$ in $B_1 \setminus B_\varepsilon$ for all $t > T$.

Here, we provide a formal derivation of the result. We note that Φ satisfies

$$\Phi_t + \mathcal{L}\Phi = -\lambda_* e^{V_*} F(\Phi) \quad \text{in } B_1 \times (0, \infty), \quad \text{where } F(u) := e^{-u_+} - 1 + u_+ \tag{2.3}$$

and $\Phi = 0$ on ∂B_1. Assume formally that the large time behavior in the outer region is governed by the φ_1-mode, and write $\Phi = c(t)\varphi_1(1 + o(1))$. Then, we have

$$c'(t) + \mu_1 c(t) = -(\lambda_* e^{V_*} F(\Phi), \varphi_1)_{L^2}.$$

We remark that the following relations hold:

$$0 \leqslant F'(u) \leqslant \min\{u_+, 1\}, \qquad \begin{cases} \frac{u^2}{2e^2} \leqslant F(u) \leqslant \frac{u^2}{2}, & 0 \leqslant u < 2, \\ \frac{u}{2} \leqslant F(u) \leqslant u & u \geqslant 2; \end{cases} \tag{2.4}$$

$$N - 2 - 3\gamma = 0 \quad \text{if } N = 11, \qquad N - 2 - 3\gamma > 0 \quad \text{if } N \geqslant 12. \tag{2.5}$$

Set $\delta(t) := c(t)^{1/\gamma}$. In view of $\varphi_1 \simeq r^{-\gamma}$ near $r = 0$ and Proposition 1.2, we obtain

$$\begin{aligned} (\lambda_* e^{V_*} F(\Phi), \varphi_1)_{L^2} &\simeq \int_0^{\delta(t)} r^{N-3} \Phi \varphi_1 \, dr + \int_{\delta(t)}^1 r^{N-3} \Phi^2 \varphi_1 \, dr \\ &\simeq \begin{cases} c(t)\delta(t)^{N-2-2\gamma} + c(t)^2 & (N \geqslant 12) \\ c(t)\delta(t)^{N-2-2\gamma} - c(t)^2 \log \delta(t) & (N = 11) \end{cases} \simeq \begin{cases} c(t)^2 & (N \geqslant 12) \\ c(t)^2 |\log c(t)| & (N = 11). \end{cases} \end{aligned}$$

This calculation suggests that $c \simeq Q$ (we recall that $0 < \mu_1 = \mu_1(f) \downarrow 0$ as $f \uparrow f_H$).

A rigorous verification of Proposition 2.1 for the case $f \neq f_H$ proceeds similarly as in [3, Section 5]. On the other hand, when $f = f_H$, the nonlinear term governs the grow-up rate, and therefore the construction of a super/sub-solutions yielding the optimal rate is nontrivial.

References

[1] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow-up for* $u_t - \Delta u = g(u)$ *revisited*, Advances in Diff. Eq. **1** (1996), 73–90.

[2] H. Brezis and J. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid **10** (1997), 443–469.

[3] B. Dold, V. A. Galaktionov, A. A. Lacey and J. L. Vázquez, *Rate of approach to a singular steady state in quasilinear reaction-diffusion equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998) 663–687.

[4] V. A. Galaktionov and J.R. King, *Stabilization to a singular steady state for the Frank-Kamenetskii equation in a critical dimension*, Proc. Roy. Soc. Edinburgh Sect. A **135** (2005) 777–787.

[5] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. **68** (1979), 209–243.

[6] D.D. Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.

[7] K. Kumagai, *Classification of bifurcation diagrams for semilinear elliptic equations in the critical dimension*, J. Differential Equations **398** (2024) 290–318.

[8] K. Kumagai and Y. Oka, *Large-time behavior and grow-up rates of inhomogeneous semilinear heat equations*, in preparation.

[9] I. Peral and J. Vázquez, *On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term*, Arch. Rational Mech. Anal. **129** (1995) 201–224.

[10] P. Quittner and P. Souplet, *Superlinear parabolic problems*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, Cham, 2019.

LIFESPAN ESTIMATES FOR CLASSICAL SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH NONLINEAR BOUNDARY CONDITION IN ONE DIMENSION

YU ZHOU

This is a joint work with Professor H. Kubo.

1. Introduction

Before introducing our problem, we briefly review two types of related works. The first one concerns the so-called combined effect in the initial value problem:

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = A\,|\partial_t u|^p\,|u|^q\,(t,x) + B|u|^r(t,x), & (t,x) \in (0,T)\times(0,\infty), \\ u(0,x) = \varepsilon f(x),\ \partial_t u(0,x) = \varepsilon g(x), & x \in [0,\infty)\,, \end{cases} \tag{1.1}$$

where f and g are compactly supported initial data. It is known that solutions to (1.1) blow up in finite time in several cases. Kato [4] proved blow-up when $A = 0$, and Zhou [9] treated the case $B = 0$ with $q = 0$. Therefore, it is natural to study the lifespan $\widetilde{T}(\varepsilon)$ of solutions. Recently, Kido–Sasaki–Takamatsu–Takamura [5] showed that

$$\widetilde{T}(\varepsilon) \lesssim \varepsilon^{-(p+q)(r-1)/(r+1)}$$

for sufficiently small ε, under the condition $A, B > 0$ and $\int_{\mathbb{R}} g(x)\,dx = 0$. On the other hand, it is also known that

$$\widetilde{T}(\varepsilon) \gtrsim \varepsilon^{-(p+q-1)} \quad (B = 0), \qquad \widetilde{T}(\varepsilon) \gtrsim \varepsilon^{-r(r-1)/(r+1)} \quad \left(A = 0 \text{ and } \int_{\mathbb{R}} g(x)\,dx = 0\right)$$

by Li–Yu–Zhou [6] and Zhou [8]. Note that when $(r+1)/2 < p+q < r$,

$$(p+q)(r-1)/(r+1) < p+q-1 \quad \text{and} \quad (p+q)(r-1)/(r+1) < r(r-1)/(r+1)$$

hold. Therefore, the results mentioned in the above imply that if $(r+1)/2 < p+q < r+1$ and $\int_{\mathbb{R}} g(x)\,dx = 0$, then $\widetilde{T}(\varepsilon)$ is actually shorter than the corresponding lifespan in the cases $A = 0$ or $B = 0$ for sufficiently small ε. Hence, the interaction between different kinds of nonlinearities $|\partial_t u|^p|u|^q$ and $|u|^r$ produces such a peculiar phenomenon called combined effect (see also [7] for a special case $q = 0$).

Another one concerns the initial-boundary value problem for the homogeneous wave equation with a nonlinear boundary condition:

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, & (t,x) \in (0,T)\times(0,\infty), \\ -\partial_x u(t,0) = h(u(t,0)), & t \in [0,T], \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), & x \in [0,\infty)\,, \end{cases} \tag{1.2}$$

where $h(u)$ is a continuous function. Ackleh and Deng [1] studied the initial-boundary value problem (1.2) and derived a criterion for global existence and nonexistence. In particular, when $h(u) \geq B|u|^r$ with $r > 1$, it was shown that the solution of the corresponding integral equation blows up in finite time under suitable positivity assumptions on the initial data. Similar initial-boundary value problems with nonlinear Neumann boundary conditions have also been extensively studied for other equations, such as the heat equation and the Schrödinger equation. We note that in [2], the Schrödinger equation with both a nonlinear boundary condition and an interior nonlinearity was treated (see also [3]). These studies are expected to provide useful insights into the analysis of evolution equations with nonlinear boundary conditions. However, to the best of our knowledge, the corresponding setting for wave equations has not been sufficiently studied. Motivated by these previous works, we study the corresponding initial-boundary value problem for the wave equation with both interior and boundary nonlinearities.

We consider the initial-boundary value problem for nonlinear wave equations with nonlinear Neumann boundary conditions on the half-line:

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = A\,|\partial_t u|^p\,|u|^q\,(t,x), & (t,x) \in (0,T) \times (0,\infty), \\ -\,\partial_x u(t,0) = B\,|u(t,0)|^r\,, & t \in [0,T], \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), & x \in [0,\infty)\,, \end{cases} \tag{1.3}$$

where A, B are real constants, p, q, $r > 1$, and $T > 0$. Subsequently, we make the following assumptions on the initial data

$$(\varphi, \psi) \in C_0^2([0,\infty)) \times C_0^1([0,\infty)), \tag{1.4}$$

$$\operatorname{supp}\varphi, \operatorname{supp}\psi \subset \{x \in [0,\infty)\,|x \le R\}\ \ (R \ge 1). \tag{1.5}$$

For simplicity, we take $R = 1$. Moreover, we assume that

$$\lim_{x\to 0^+} \varphi'(x) = \lim_{x\to 0^+} \psi'(x) = 0, \tag{1.6}$$

so that the even extensions of φ and ψ are smooth functions, and that

$$\varphi(0) = 0 \tag{1.7}$$

holds for avoiding singularity emanating from the origin. Besides, we denote the size of the initial data by

$$\varepsilon := \sum_{\alpha=0}^{2} \|\phi^{(\alpha)}\|_{L^\infty([0,\infty))} + \sum_{\beta=0}^{1} \|\psi^{(\beta)}\|_{L^\infty([0,\infty))}. \tag{1.8}$$

We are interested in the lifespan $T(\varepsilon)$, that is, the maximal existence time of classical solutions $u \in C^2([0,T] \times [0,\infty))$ to the initial-boundary value problem (1.3).

Finally, we introduce the following notation. For $\xi \in \mathbb{R}$, $\langle\xi\rangle := 1 + |\xi|$ and $(\xi)^+ := \max\{\xi, 0\}$; $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$; $f(0^\pm) := \lim_{x\to 0^\pm} f(x)$; $\chi_D(x)$ denotes the characteristic function which equals 1 when $x \in D$ and 0 otherwise; $\operatorname{sgn}(x)$ denotes the sign function, taking the value 1 for $x > 0$, 0 for $x = 0$, and -1 for $x < 0$.

2. Main Results

The following two theorems give lower bounds for the lifespan of solutions.

Theorem 2.1. *Assume* (1.4)–(1.8). *Then there exists a positive constant* $\varepsilon_1 = \varepsilon_1(p,q,r,A,B)$ *such that*

$$T(\varepsilon) \ge \begin{cases} C^*\,\varepsilon^{-(p+q-1)}, & \textit{if}\ \ p+q \le r, \\ C^*\,\varepsilon^{-(r-1)}, & \textit{if}\ \ r \le p+q \end{cases} \tag{2.1}$$

for $0 < \varepsilon \le \varepsilon_1$, *where* C^* *is a positive constant independent of* ε.

We also show that the lower bound of the lifespan can be improved when $r < p+q$. More precisely, this improvement holds under the condition $\int_0^1 \psi(x)\,dx = 0$.

Theorem 2.2. *Let $p+q \geq r$. Assume* (1.4)–(1.8) *and*

$$\int_0^1 \psi(x)\,dx = 0. \tag{2.2}$$

Then there exists a positive constant $\varepsilon_2 = \varepsilon_2(p,q,r,A,B)$ such that

$$T(\varepsilon) \geq \begin{cases} C^*\,\varepsilon^{-\frac{(r-1)(p+q)}{r}}, & \text{if } r \leq p+q \leq r^2, \\ C^*\,\varepsilon^{-r(r-1)}, & \text{if } r^2 \leq p+q \end{cases} \tag{2.3}$$

for $0 < \varepsilon \leq \varepsilon_2$, where C^* is a positive constant independent of ε.

Therefore, if $p+q > r$ and $\int_0^1 \psi(x)\,dx = 0$, we obtain

$$\varepsilon^{-(r-1)} < \min\left\{\varepsilon^{-\frac{(r-1)(p+q)}{r}},\, \varepsilon^{-r(r-1)}\right\}. \tag{2.4}$$

This implies that the lower bound of the lifespan can be improved.

The following blow-up estimates show that the lower bounds of the lifespan in Theorems 2.1 and 2.2 are sharp.

Theorem 2.3. *Let $A > 0$ and $B > 0$. Assume that* (1.4)–(1.7) *hold and*

$$\varphi(x) = 0 \text{ for } x \in [0,1], \qquad \psi(x) > \psi_0\varepsilon \text{ for } x \in (0, \tfrac{1}{2}), \tag{2.5}$$

with a positive constant ψ_0 and a positive parameter ε.

(*i*) *If $\int_0^1 \psi(x)\,dx \geq 0$, then there exists a positive number $\varepsilon_3 = \varepsilon_3(\psi_0,p,q,r,A,B)$ such that*

$$T(\varepsilon) \leq C \min\left\{\varepsilon^{-(p+q-1)},\, \varepsilon^{-\frac{(r-1)(p+q)}{r}},\, \varepsilon^{-r(r-1)}\right\} \text{ for } 0 < \varepsilon \leq \varepsilon_3, \tag{2.6}$$

where C is a positive constant independent of ε.

(*ii*) *If $\psi(x) \geq 0$ for $x \in [0,1]$, then there exists a positive number $\varepsilon_4 = \varepsilon_4(\psi_0,r,B)$ such that*

$$T(\varepsilon) \leq C\,\varepsilon^{-(r-1)} \text{ for } 0 < \varepsilon \leq \varepsilon_4, \tag{2.7}$$

where C is a positive constant independent of ε.

Based on the above, we give the following detailed classification of the order of $T(\varepsilon)$:

$$T(\varepsilon) \approx \begin{cases} p+q \leq r : \varepsilon^{-(p+q-1)}, \\ p+q > r \begin{cases} \int_0^1 \psi(x)\,dx \neq 0 : \varepsilon^{-(r-1)}, \\ \int_0^1 \psi(x)\,dx = 0 \begin{cases} r < p+q < r^2 & : \varepsilon^{-\frac{(r-1)(p+q)}{r}} \\ p+q \geq r^2 & : \varepsilon^{-r(r-1)} \end{cases} \end{cases} \end{cases}$$

Hence, we conclude that the results of Theorems 2.1 and 2.2 are sharp. Moreover, we have

$$T(\varepsilon) \approx \varepsilon^{-(p+q-1)},$$

when only the interior nonlinearity is present ($A > 0$, $B = 0$), and

$$T(\varepsilon) \approx \varepsilon^{-(r-1)}, \text{ if } \int_0^1 \psi(x)\,dx \neq 0, \qquad T(\varepsilon) \approx \varepsilon^{-r(r-1)}, \text{ if } \int_0^1 \psi(x)\,dx = 0,$$

when only the nonlinear boundary condition is present ($A = 0$, $B > 0$). It can be seen that the combined effect is also observed for the initial-boundary value problem (1.3) in the case $r < p+q < r^2$ and $\int_0^1 \psi(x)\,dx = 0$. Namely, the lifespan for (1.3) with A, $B > 0$ is shorter than the corresponding lifespan in the cases $A = 0$ or $B = 0$.

3. Outline of the proof

We briefly outline the proof for the case in which the combined effect phenomenon occurs, that is

$$r < p + q < r^2 \quad \text{and} \quad \int_0^1 \psi(x)\,dx = 0 \tag{3.1}$$

as an example. To consider the initial-boundary value problem on the half-line with a nonlinear Neumann boundary condition, we extend the problem (1.3) to the whole line. First, we define the even extensions of the initial data by $\Phi(x) = \varphi(|x|)$ and $\Psi(x) = \psi(|x|)$ for $x \in \mathbb{R}$, and introduce $\Omega_0^j := \left\{ z(x) \in C_0^j(\mathbb{R}) \;\middle|\; z \text{ is even, } \operatorname{supp} z \subset \{\, x \in \mathbb{R} \mid |x| \le 1 \,\} \right\}$ for $j \in \mathbb{N}$. From the assumptions on (φ, ψ) in (1.4)–(1.7), it follows that

$$(\Phi, \Psi) \in \Omega_0^2 \times \Omega_0^1 \quad \text{and} \quad \Phi(0) = 0. \tag{3.2}$$

The corresponding problem on the whole line is given by

$$\begin{cases} \partial_t^2 U(t,x) - \partial_x^2 U(t,x) = F(t,x) & \text{in } (0,T) \times \mathbb{R}_*, \\ -\,\partial_x U(t, 0^+) = G(t), & t \in [0,T], \\ U(0,x) = \Phi(x), \quad \partial_t U(0,x) = \Psi(x), & x \in \mathbb{R}, \end{cases} \tag{3.3}$$

with $F(t,x) = A|\partial_t U|^p\,|U|^q(t,x)$ and $G(t) = B|U(t,0)|^r$.

For the homogeneous case $F = 0$ and $G = 0$, the solution of (3.3) is expressed as

$$U^0(t,x) = \frac{1}{2}\big[\Phi(x+t) + \Phi(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} \Psi(y)\,dy \quad \text{in } [0,T] \times \mathbb{R}. \tag{3.4}$$

Moreover, we introduce the following notation. For $G(t) \in C^1([0,T])$, we set

$$N[G](t,x) := \int_0^{(t-|x|)^+} G(s)\,ds \quad \text{in } [0,T] \times \mathbb{R}, \tag{3.5}$$

and for $F(t,x) \in \Omega_T^1$, we put

$$\Lambda[F](t,x) := \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} F(s,y)\,dy\,ds \quad \text{in } [0,T] \times \mathbb{R}. \tag{3.6}$$

Then, the corresponding integral equation to (3.3) is the following:

$$U(t,x) = U^0(t,x) + \Lambda[A|\partial_t U|^p\,|U|^q](t,x) + N[B|U(t,0)|^r](t,x). \tag{3.7}$$

We look for a solution of (3.7) in the following function space:

$$\begin{aligned} \Omega_T^1 := \Big\{ v(t,x) \in C([0,T] \times \mathbb{R}) \;\Big|\; & v \text{ is even in } x,\ \partial_x v \in C([0,T] \times \mathbb{R}_*), \\ & \partial_x v(t, 0^+) \text{ exists for each } t \in [0,T], \text{ and } \operatorname{supp} v \subset \{(t,x) \in [0,T] \times \mathbb{R} \mid |x| \le t+1\} \Big\}. \end{aligned} \tag{3.8}$$

If there exists a solution $U \in \Omega_T^1$ to (3.7) satisfying $\partial_t U \in \Omega_T^1$, then its restriction to $[0,T] \times (0,\infty)$, denoted by u, is a classical solution of (1.3).

The local existence of solutions to (3.7) appears to be self-evident, whereas the main difficulty lies in introducing an optimal sharp estimate. To this end, we divide the support of the solution into the following two regions:

$$D := \{(t,x) \in [0,T] \times \mathbb{R} \mid |t - |x|| \le 1\}, \quad E := \{(t,x) \in [0,T] \times \mathbb{R} \mid |x| < t - 1\}.$$

It is well known that if the integral of the initial velocity vanishes, the strong Huygens principle holds even in one spatial dimension. It follows that the homogeneous solution U^0 vanishes identically in the region E, so that one can expect improved upper bound estimates for the solution in the region E. In the case of vanishing integral of the initial velocity, Takamura et al. [5] employ different weighted L^∞ spaces on E and D, respectively. By exploiting the Huygens principle, they improve the lower bound of the lifespan for the Cauchy problem when $p+q>\frac{r+1}{2}$. We extend this idea to the initial–boundary value problem in order to improve the bound for U in E from $\mathcal{O}(\varepsilon)$ to $\mathcal{O}(\varepsilon^r)$, where ε is defined in (1.8) and is assumed to be sufficiently small. We choose α as in (3.11) to be defined later, thereby improving the lower bound of the lifespan from $\varepsilon^{-(r-1)}$ to $\varepsilon^{-\frac{(r-1)(p+q)}{r}}$ in the case (3.1).

Next, we apply the successive approximation method to (3.7). We define a sequence $\{U_j\}_{j\in\mathbb{N}}$ by

$$U_{j+1}=U^0+N[B\,|U_j(t,0)|^r]+\Lambda[A\,|\partial_t U_j|^p\,|U_j|^q]\ \ (j\in\mathbb{N}),\quad U_1=U^0. \tag{3.9}$$

We introduce a weighted L^∞ space Y suitable for the convergence of the sequence $\{U_j\}_{j\in\mathbb{N}}$:

$$Y:=\left\{U\in\Omega_T^1\ \middle|\ \partial_t U\in\Omega_T^1,\ \|U\|_Y<\infty\right\}, \tag{3.10}$$

with the norm

$$\begin{aligned}\|U\|_Y:=&\left\|\{\chi_D+(1-\chi_D)\langle t\rangle^{-\alpha}\}\,U\right\|_{L_T^\infty(\mathbb{R})}+\left\|\{\chi_D+(1-\chi_D)\langle t\rangle^{1-\alpha}\}\,\partial_t U\right\|_{L_T^\infty(\mathbb{R})}\\&+\sum_{j=0}^{1}\|\partial_x\partial_t^j U\|_{L_T^\infty(\mathbb{R}_*)},\end{aligned}$$

where $\|v\|_{L^\infty([0,T]\times\Omega)}:=\sup_{(t,x)\in[0,T]\times\Omega}|v(t,x)|$, and α is given by

$$\alpha=\frac{r^2-(p+q)}{(p+q)(r-1)}. \tag{3.11}$$

It follows from (2.2) that $U^0=0$ in E. Indeed, $U^0\in Y$ and satisfies

$$\|U^0\|_Y\le 4\varepsilon, \tag{3.12}$$

when ε is sufficiently small.

The following lemma provides a priori estimates in the case (3.1).

Lemma 3.1. *Let $0<\alpha<1,\ \lambda>1$ and $G(t)\in C^1([0,T])$ with $G(0)=G'(0)=0$. Then*

$$\|\langle t\rangle^{-\alpha}N[G]\|_{L_T^\infty(E)}\le\|G\|_{L^\infty([0,1])}+(1+T)^{\alpha(\lambda-1)+1}\,\|\langle t\rangle^{-\alpha\lambda}G\|_{L^\infty([1,T])},$$

$$\|\langle t\rangle^{1-\alpha}\partial_t N[G]\|_{L_T^\infty(E)}\le(1+T)^{\alpha(\lambda-1)+1}\,\|\langle t\rangle^{-\alpha\lambda}G\|_{L^\infty([1,T])},$$

$$\|\partial_x N[G]\|_{L_T^\infty(E_*)}\le(1+T)^{\alpha(\lambda-1)+1}\,\|\langle t\rangle^{-\alpha\lambda}G\|_{L^\infty([1,T])},$$

$$\|\partial_x\partial_t N[G]\|_{L_T^\infty(E_*)}\le(1+T)^{\alpha(\lambda-1)+1}\,\|\langle t\rangle^{-\alpha(\lambda-1)+(1-\alpha)}G'\|_{L^\infty([1,T])}.$$

Moreover, for let $0<\alpha<1,\ \mu,\nu>1$ and $F(t,x)\in\Omega_T^1$, we have

$$\|\langle t\rangle^{-\alpha}\Lambda[F]\|_{L_T^\infty(E)}\lesssim(1+T)^{1-\alpha}\|F\|_{L_T^\infty(D)}+(1+T)^{\alpha(\mu+\nu-1)+1}\|\langle t\rangle^{(1-\alpha)\mu-\alpha\nu}F\|_{L_T^\infty(E)},$$

$$\|\langle t\rangle^{1-\alpha}\partial_t\Lambda[F]\|_{L_T^\infty(E)}\lesssim(1+T)^{1-\alpha}\|F\|_{L_T^\infty(D)}+(1+T)^{\alpha(\mu+\nu-1)+1}\|\langle t\rangle^{(1-\alpha)\mu-\alpha\nu}F\|_{L_T^\infty(E)},$$

$$\|\partial_x\Lambda[F]\|_{L_T^\infty(E)}\lesssim\|F\|_{L_T^\infty(D)}+(1+T)^{\alpha(\mu+\nu-1)+1}\|\langle t\rangle^{(1-\alpha)\mu-\alpha\nu}F\|_{L_T^\infty(E)},$$

$$\|\partial_x\partial_t\Lambda[F]\|_{L_T^\infty(E)}\lesssim\|\partial_x F\|_{L_T^\infty(D_*)}+(1+T)^{\alpha(\mu+\nu-1)+1}\|\langle t\rangle^{(1-\alpha)(\mu-1)-\alpha\nu}\partial_x F\|_{L_T^\infty(E_*)},$$

$$\|\partial_x\partial_t\Lambda[F]\|_{L_T^\infty(E)}\lesssim\|\partial_x F\|_{L_T^\infty(D_*)}+(1+T)^{\alpha(\mu+\nu-1)+1}\|\langle t\rangle^{(1-\alpha)\mu-\alpha(\nu-1)}\partial_x F\|_{L_T^\infty(E_*)},$$

where $D_*:=D\setminus\{0\}$ and $E_*:=E\setminus\{0\}$.

Now, we prove $U_j \in Y$ and

$$\|U_j\|_Y \le C_0\varepsilon \tag{3.13}$$

for $j \in \mathbb{N}$, under suitable conditions on T and ε. Here C_0 is a positive constant independent of T and ε. We set

$$P(U) := \sum_{j=0}^{1}\Big\{\|\partial_t^j U\|_{L_T^\infty(D)} + \|\partial_t^j\partial_x U\|_{L_T^\infty(D_*)}\Big\},$$

$$Q(U) := \Big\|\langle t\rangle^{-\alpha}U\Big\|_{L_T^\infty(E)} + \Big\|\langle t\rangle^{1-\alpha}\partial_t U\Big\|_{L_T^\infty(E)} + \sum_{j=0}^{1}\|\partial_x\partial_t^j U\|_{L_T^\infty(E_*)},$$

note that $\|U\|_Y \le P(U)+Q(U)$. In what follows, C denotes a positive constant independent of T and ε, which may change from line to line.

Since $U_1 = U^0$, we have $P(U_1) \le 4\varepsilon$ and $Q(U_1) = 0$. Moreover, by using Lemma 3.1 with $\lambda = r$, $\mu = p$ and $\nu = q$, we have

$$P(U_{j+1}) \le 4\varepsilon + C\Big\{(P(U_j))^r + (1+T)\,(P(U_j))^{p+q}\Big\},$$

and

$$\begin{aligned} Q(U_{j+1}) \le C_1\,\big((P(U_j))^r + \,(P(U_j))^{p+q}\big) + C\Big\{&(1+T)^{\alpha(r-1)+1}\,(Q(U_j))^r \\ &+ (1+T)^{\alpha r}\,(Q(U_j))^r + (1+T)^{1-\alpha}\,(P(U_j))^{p+q} \\ &+ (1+T)^{\alpha(p+q-1)+1}\,(Q(U_j))^{p+q} + (1+T)^{\alpha(p+q)}\,(Q(U_j))^{p+q}\Big\}, \end{aligned}$$

where C_1 is a positive constant independent of T and ε. For $r \le p+q \le r^2$ and $0 < \varepsilon \le 1$, suppose that

$$C\varepsilon^{r-1} \le 1, \quad C(1+T)\varepsilon^{\beta} \le 1, \tag{3.14}$$

where $\beta = (p+q)(r-1)/r$. By induction, we have

$$P(U_j) \le 5\varepsilon, \quad Q(U_j) \le 3C_1\varepsilon^r,$$

for all $j \in \mathbb{N}$. Taking $C_0 = 5 + 3C_1$, we obtain (3.13) provided that (3.14) holds. Similarly, we have

$$P\big(U_j - U_{j-1}\big) \le C\big(2^{-j} + j\,2^{-ja}\big), \quad Q\big(U_j - U_{j-1}\big) \le C\big(2^{-j} + j\,2^{-ja}\big),$$

under (3.14), where $a = \min\{1,\, p-1,\, q-1,\, r-1\} \in (0,1]$. Now, if we set $\varepsilon_0 = \min\Big\{1, (2C)^{-\frac{1}{r-1}}, (2C)^{-\frac{1}{\beta}}\Big\}$, then for $0 < \varepsilon \le \varepsilon_0$, $T \le (2C)^{-1}\varepsilon^{-\beta}$ assures (3.14). Under this condition, $\{U_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in Y, so that its limit exists in Y and satisfies (3.7).

Finally, we give a brief description of the blow-up estimate under the combined effect case. We suppose that the initial-boundary value problem (1.3) admits a classical solution u in $[0,T]\times[0,\infty)$. Then its even extension U satisfies (3.7) with $U, \partial_t U \in \Omega_T^1$. Note that (2.5) implies $\Phi(x) = 0$ for $x \in \mathbb{R}$ and $\Psi(x) > \psi_0\varepsilon$ for $x \in (0,1/2)$. Here, Φ and Ψ are even extensions of ϕ and ψ, respectively.

Let A, $B > 0$ and $T \ge 2$. Since $U, U_t \ge 1/8\psi_0\varepsilon$ for $(t,x) \in \Sigma$, we have from (3.7)

$$\begin{aligned} U(t,0) &\ge A \iint\limits_{\Sigma\cap K_0} |\partial_t U(s,y)|^p\,|U(s,y)|^q\,dy\,ds + B\int_2^t |U(s,0)|^r\,ds \\ &\ge C\left[t\varepsilon^{p+q} + \int_2^t |U(s,0)|^r\,ds\right], \quad t \in [2,T], \end{aligned}$$

where $\Sigma = \{(t,x) \in [0,T]\times\mathbb{R} \mid t+x \geq 1,\ 1/4 \leq t-x < 1/2\}$, $K_0 = \{(s,y) \in [0,t]\times[0,\infty) \mid\ y \leq t-s\}$, and C is a positive constant independent of T and ε. If we define $Z(t)$ by

$$Z(t) = C\left[t\varepsilon^{p+q} + \int_2^t |Z(s)|^r\,ds\right], \quad t \geq 2,$$

then we have $U(t,0) \geq Z(t)$ for $t \in [2,T]$, by the comparison principle. Since $Z(t)$ satisfies

$$\begin{cases} Z'(t) = C\varepsilon^{p+q} + CZ^r(t), \quad t \geq 2, \\ Z(2) = 2C\varepsilon^{p+q}. \end{cases}$$

denoting by $\mathcal{T}$ the lifespan of $Z(t)$, we have

$$\mathcal{T} \leq \frac{1}{C}\int_{Z(2)}^{Z(t)} \frac{d\xi}{\varepsilon^{p+q}+\xi^r} + 2 \leq \frac{2}{C}\varepsilon^{-\frac{(p+q)(r-1)}{r}}\int_0^\infty \frac{d\eta}{1+\eta^r} < \infty, \quad t \in [2,T),$$

provided that ε is sufficiently small, where $\eta = \varepsilon^{-\frac{p+q}{r}}\,\xi$. Hence, we obtain $T \leq \mathcal{T} \leq C\,\varepsilon^{-\frac{(r-1)(p+q)}{r}}$ and (2.6) is proved for $r < p+q < r^2$, completing the proof.

References

[1] A. S. Ackleh and K. Deng, Existence and nonexistence of global solutions of the wave equation with a nonlinear boundary condition, *Q. Appl. Math.* **59** (1) (2001) 153–158.

[2] A. Batal and T. Özsari, Nonlinear Schrödinger equations on the half-line with nonlinear boundary conditions, *Electron. J. Differ. Equ.* **2016** (2016) 1–20.

[3] N. Hayashi, T. Ogawa and T. Sato, The initial-boundary value problem for the nonlinear Schrödinger equation with the nonlinear Neumann boundary condition on the half-line, *J. Differ. Equ.* **422** (2025) 355–385.

[4] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* **33** (1980) 501–505.

[5] R. Kido, T. Sasaki, S. Takamatsu and H. Takamura, The generalized combined effect for one dimensional wave equations with semilinear terms including product type, *J. Differ. Equ.* **403** (2024) 576–618.

[6] T.-T. Li, X. Yu and Y. Zhou, Life-span of classical solutions to one-dimensional nonlinear wave equations, *Chin. Ann. Math. Ser. B* **13** (3) (1992) 266–279.

[7] K. Morisawa, T. Sasaki and H. Takamura, The combined effect in one space dimension beyond the general theory for nonlinear wave equations, *Commun. Pure Appl. Anal.* **22** (5) (2023) 1629–1658.

[8] Y. Zhou, Life span of classical solutions to $u_{tt} - u_{xx} = |u|^{1+\alpha}$, *Chin. Ann. Math. Ser. B* **13** (2) (1992) 230–243.

[9] Y. Zhou, Blow up of solutions to the Cauchy problem for nonlinear wave equations, *Chin. Ann. Math. Ser. B* **22** (3) (2001) 275–280.

Y. Zhou

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan

Email address: yu.zhou.x5@elms.hokudai.ac.jp

Well-posedness of the Cauchy Problem in Low Regularity for Kinetic DNLS on $\mathbf{T}$

Yoshio TSUTSUMI
Institute for Liberal Arts and Sciences, Kyoto University
Yoshida Nihonmatsu-cho, Sakyo-ku,
Kyoto 606-8501, JAPAN

1 Introduction

We consider the well-posedness of the Cauchy problem in low regularity for the following Kinetic Derivative NLS (KDNLS):

$$\partial_t u - i\partial_x^2 u - \beta\partial_x\big[H\big(|u|^2\big)u\big] + \delta\partial_x\big(|u|^2 u\big) = 0, \quad (t,x) \in (0,T)\times\mathbf{T}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbf{T}. \tag{1.2}$$

where $T > 0$, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, δ is a real constant and β is a negative constant (ratio of plasma pressure to magnetic pressure). The Hilbert transform is defined as follows.

$$f \;\mapsto\; \frac{1}{\sqrt{2\pi}}\sum_{k\neq 0}\frac{-ik}{|k|}\hat{f}(k)e^{-ikx}.$$

Equation (1.1) takes the resonant interaction between the wave modulation and the ions into account, while it is ignored in the derivative NLS.

$$\partial_t u - i\partial_x^2 u + \delta\partial_x\big(|u|^2 u\big) = 0. \tag{DNLS}$$

The word "kinetic" implies that the collective motion of ions in a plasma is modeled by the Vlasov equation and not by the fluid equation, namely,

Maxwell eqs. $+$ Euler eqs. $\Longrightarrow$ DNLS,
Maxwell eqs. $+$ Vlasov eq. $\Longrightarrow$ KDNLS.

For more details of the physical background, see Mjølhus and Wyller [MW1] and [MW2]. In this paper, we consider the following problem.

Problem: For what s is there a unique local solution of the Cauchy problem (1.1)-(1.2) in Sobolev spaces in H^s?

The result in this paper is a part of the joint work [KT1]-[KT4] with Nobu Kishimoto, RIMS, Kyoto University.

It is instructive to review what is difficult in equation (DNLS). It is well known that "High $\times$ Low $\to$ High" interaction is the hardest difficulty that (1.1) and (DNLS) have in common. To look at this point, we define the so-called resonance function as follows.

$$\begin{aligned} R(k_1,k_2,k_3) =&(\tau+k^2)-(\tau_1+k_1^2)+(\tau_2+k_2^2)-(\tau_3+k_3^2)\\ =&2(k_1-k_2)(-k_2+k_3), \end{aligned}$$

where

$$\tau=\tau_1-\tau_2+\tau_3, \quad k=k_1-k_2+k_3.$$

We take the nonlinear interaction of (DNLS):

$$\partial_x\big(u\bar{u}u\big)=2(\partial_x u)\bar{u}u+u\partial_x\bar{u}u.$$

The "High $\times$ Low $\to$ High" interaction is divided into the following two cases.

Case 1. Suppose that $(\partial_x u)\bar{u}u$ and $|k_1|\gg|k_2|,|k_3|$.

$$\begin{aligned} &\mathcal{F}\big[(\partial_x u)\bar{u}u\big](k)=\sum_{k=k_1-k_2+k_3} ik_1\hat{u}(k_1)\hat{\bar{u}}(k_2)\hat{u}(k_3),\\ &R(k_1,k_2,k_3)\ \sim\ |k_1| \text{ if } -k_2+k_3\neq 0. \end{aligned}$$

Case 2. Suppose that $u(\partial_x\bar{u})u$ and $|k_2|\gg|k_1|,|k_3|$.

$$\begin{aligned} &\mathcal{F}\big[u(\partial_x\bar{u})u\big](k)=\sum_{k=k_1-k_2+k_3} \hat{u}(k_1)ik_2\hat{\bar{u}}(k_2)\hat{u}(k_3),\\ &R(k_1,k_2,k_3)\ \sim\ |k_2|^2. \end{aligned}$$

Remark 1 *(i) The Fourier restriction norm method yields the factor $|R|^{-1/2}$. Therefore, no loss of derivative occurs in Case 2, while $1/2$ derivative is still not recovered in Case 1.*

(ii) A pair of frequencies (k_1,k_2,k_3) is said to be resonant if $R(k_1,k_2,k_3)=0$, i.e., $k_1=k_2$ or $k_2=k_3$.

To overcome this difficulty, the gauge transformation was introduced in prior research.

(Gauge transformation)

$$\mathcal{G}_0 u = u \exp\Big[-i\frac{\delta}{4\pi}\int_0^{2\pi}\int_\theta^x (|u(y)|^2 - \frac{1}{2\pi}\|u\|_{L^2}^2)\,dyd\theta\Big]. \tag{1.3}$$

u; solution of (DNLS), $v := \mathcal{G}_0 u$.
Then, v satisfies the following equation.

$$\partial_t v - i\partial_x^2 v - \delta v^2 \partial_x \bar{v} = \text{(harmless terms)}.$$

For (DNLS), we solve the equation resulting from the gauge transformation (1.3) to obtain LWP in H^s, $s \geq 1/2$, which is due to Takaoka [Tak] for the case of $\mathbf{R}$ and to Herr [Herr] for the case of $\mathbf{T}$.

Remark 2 *(i) The gauge transformation* (1.3) *does not work for* (1.1) *as well as for* (DNLS) *because of the presence of the Hilbert transform. The lower bound $s \geq 1/2$ is optimal for (DNLS).*

(ii) The well-posedness in low regularity sapces has recently been improved by using the complete integrability of (DNLS) (see, e.g., Bahouri and Perelman [BP] and Harrop-Griffiths, Killop, Ntekoume and Vişan [HGKNV]).

We use the dissipative nture of (1.1) to full extent. For simplicity, we may assume $\delta = 0$ throughout this paper. Now we look at the dissipativeness of (1.1). Multiply (1.1) by $\bar{u}$ and integrate the resulting equation to have

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\beta \int_0^{2\pi} \big(D^{1/2}|u|^2\big)^2\,dx, \ t > 0,$$
$$Du = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbf{Z}} |k|\hat{u}(k)e^{-ikx}.$$

The above identity implies that if $\beta < 0$, the L^2 norm of the solution is decreasing in time. We make close investigation into the dissipative nature of (1.1).

Put $F = \partial_x[H(|u|^2)u]$. Take the Fourier transform of F in the spatial variable to have

$$
\begin{aligned}
(1.4)\qquad \hat{F}(k) =& ik \sum_{\substack{k_1-k_2+k_3=k\\ k_1-k_2\neq 0}} \Big(-i\,\frac{k_1-k_2}{|k_1-k_2|}\hat{u}(k_1)\bar{\hat{u}}(k_2)\Big)\hat{u}(k_3)\\
=&\Big(\sum_{k_3\neq k}|\hat{u}(k_3)|^2\Big)|k|\hat{u}(k)\\
&+\sum_{k_3\neq k}\big(|k-k_3|-|k|\big)|\hat{u}(k_3)|^2\hat{u}(k)\\
&+\sum_{k_3\neq k}\Big(-i\frac{k-k_3}{|k-k_3|}ik_3|\hat{u}(k_3)|^2\Big)\hat{u}(k)\\
&+ik\sum_{k_3\neq k}\Big(\sum_{\substack{k_1-k_2=-k_3+k\\ k_2\neq k_3}} -i\,\frac{k_1-k_2}{|k_1-k_2|}\hat{u}(k_1)\bar{\hat{u}}(k_2)\Big)\hat{u}(k_3).
\end{aligned}
$$

The last sum on the right hand side of (1.4) does not contain resonant frequencies. Set

$$
h(k) = -i\frac{k}{|k|}\ (k\neq 0), \quad h(0)=0.
$$

Consider the following equation from now on.

$$
\begin{aligned}
(1.5)\qquad & \partial_t\hat{u}(t,k) + ik^2\hat{u}(t,k)\\
&\qquad - \beta\Big(\sum_{k_3\in\mathbf{Z}}|\hat{u}_0(k_3)|^2\Big)|k|\hat{u}(k)\\
=&\ \beta\sum_{k_3\in\mathbf{Z}}\big(|\hat{u}(k_3)|^2-|\hat{u}_0(k_3)|^2\big)|k|\hat{u}(k)\\
&+\beta\sum_{k_3\in\mathbf{Z}}\underbrace{\big(|k-k_3|-|k|\big)}_{|\cdots|\leq|k_3|}|\hat{u}(k_3)|^2\ \hat{u}(k)\\
&+\beta\sum_{k_3\neq k}\Big(h(k-k_3)ik_3|\hat{u}(k_3)|^2\Big)\hat{u}(k)\\
&-\beta ik\sum_{k_3\neq k}\Big(\sum_{\substack{k_1-k_2=-k_3+k\\ k_2\neq k_3}} h(k_1-k_2)\\
&\qquad\times\hat{u}(k_1)\bar{\hat{u}}(k_2)\Big)\hat{u}(k_3), \quad t>0,
\end{aligned}
$$

where u_0 is the initial datum prescribed in the initial condition (1.2).

Remark 3 *At a formal level, the operator*

$$-ik^2 + \beta|k| \sum_{k_3 \in \mathbf{Z}} |\hat{u}_0(k_3)|^2$$

generates the semi-group

$$e^{-t\left(ik^2 - \beta|k| \sum_{k_3 \in \mathbf{Z}} |\hat{u}_0(k_3)|^2\right)}, \quad t \geq 0.$$

This suggests that the Caucy problem of (1.1) *might be ill-posed in the Sobolev space for* $\beta > 0$*. In this respect, it seems natural to assume that* $\beta < 0$*.*

2 Notation and Theorem

Assume $\beta = -1$ and $u_0 \neq 0$ and $u_0 \in H^s$, $s > \frac{1}{4}$. Put $\langle a \rangle = (1 + |a|^2)^{\frac{1}{2}}$ for $a \in \mathbf{C}$ and $\mu = \left(\sum_{l \in \mathbf{Z}} |\hat{u}_0(l)|^2\right) = \|u_0\|_{L^2}^2$. Denote the Fourier transform of u in t and x by $\tilde{u}(\tau, k)$. We define the modified Fourier restriction norm and space for s, $b \in \mathbf{R}$ as follows.

$$\begin{aligned}
\|u\|_{X^{s,b}} &= \Big(\sum_{k \in \mathbf{Z}} \int_{\mathbf{R}} \langle k \rangle^{2s} \langle \tau + k^2 - i\mu|k| \rangle^{2b} \\
&\qquad \times |\tilde{u}(\tau, k)|^2 \, d\tau\Big)^{1/2}, \\
X^{s,b} &= \big\{u \in \mathscr{S}'(\mathbf{R}^2) \big| \; u(t, x + 2\pi) = u(t, x), \\
&\qquad \|u\|_{X^{s,b}} < \infty\big\}, \\
\|u\|_{Y^s} &= \Big[\sum_{k \in \mathbf{Z}} \Big(\int_{\mathbf{R}} \langle k \rangle^s |\tilde{u}(\tau, k)| \, d\tau\Big)^2\Big]^{1/2}, \\
Y^s &= \big\{u \in \mathscr{S}'(\mathbf{R}^2) \big| \; u(t, x + 2\pi) = u(t, x), \\
&\qquad \|u\|_{Y^s} < \infty\big\},
\end{aligned}$$

Remark 4 *(i) Such a norm mixing dispersion and dissipativeness as above was intoroduced by Molinet and Ribaud [MR] for the KdV- Burgers equation. Tsugawa [Tsug] and Tanaka and Tsugawa [TT] observed the parabolicity induced by nonlinearity for the fifth order semilinear dispersive equations and the higher order Schrödinger equations. The above norm takes the perturbation by the nonlinear interaction into account.*

(ii) We easily see that

$$L^2(\mathbf{R}; H^{1/2}(\mathbf{T})) \subset X^{0,1/2},$$

which implies the $X^{0,1/2}$ *norm yields the gain of* $\frac{1}{2}$ *derivative in the spatial variable.*

We next introduce the ocalized space on $(0,T)$ for $X^{s,b}$ and Y^s for $T>0$.

$$\|u\|_{X^{s,b}_{(0,T)}} = \inf\big\{\|v\|_{X^{s,b}}\big|\ v\in X^{s,b},\ v(t)=u(t) \text{ on } (0,T)\big\},$$
$$X^{s,b}_{(0,T)} = \big\{u\in \mathscr{D}'((0,T)\times \mathbf{T})\big|\ \|u\|_{X^{s,b}_{(0,T)}}<\infty\big\}.$$

Furtermore, $Y^s_{(0,T)}$ is defined in the same way as $X^{s,b}_{(0,T)}$.

We state our main theorem in this paper.

Theorem 1 (Kishimoto-Y.T [KT4]) *Assume that $s>\frac{1}{4}$, $u_0\in H^s$ and $u_0\neq 0$. Then, there exists a $\exists T=T(\|u_0\|_{H^s},\|u_0\|_{L^2}^{-1})>0$ such that the Cauchy problem of* (1.1)-(1.2). *has the unique solution u satisfying*

$$u\in X^{s,1/2}_{(0,T)}\cap Y^s_{(0,T)}\subset C([0,T];H^s).$$

Remark 5 *(i) The solution u given by Theorem 1 can be extended to all positive times and has a smoothing property, i.e.,*

$$u\in C^\infty((0,\infty)\times\mathbf{T}).$$

(ii) When $u_0=0$, (1.1)-(1.2) *has a trivial solution. In fact, the continuous dependence of solutions on initial data holds.*

We consider the integral equation associated with (1.1)-(1.2)

$$\begin{aligned}
u(t) =&\psi(t)\Big[U(t)u_0 - i\chi_{\mathbf{R}_+}(t)\int_0^t U(t-\sigma)\\
\times\Big\{&\sum_{k,k_3\in\mathbf{Z}}\big(|\hat{u}(k_3)|^2-|\hat{u}_0(k_3)|^2\big)|k|\hat{u}(k)e^{ikx}\\
&+\sum_{k_3\in\mathbf{Z}}\underbrace{\big(|k-k_3|-|k|\big)}_{|\cdots|\le|k_3|}|\hat{u}(k_3)|^2\ \hat{u}(k)e^{ikx}\\
&+\sum_{\substack{k\in\mathbf{Z}\\ k_3\neq k}}\Big(h(k,k_3)ik_3|\hat{u}(k_3)|^2\Big)\hat{u}(k)e^{ikx}\\
&+\psi_T^3(\sigma)F_1(\sigma)\Big\}\ d\sigma\Big],
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
\big[\widehat{U}(t)\big](k) =&e^{-(itk^2+|t||k|\sum_{k_3\in\mathbf{Z}}|\hat{u}_0(k_3)|^2)},\\
\hat{F}_1 = \psi_T^3(t)&\Big[ik\sum_{k_3\neq k}\Big(\sum_{\substack{k_1-k_2=-k_3+k\\ k_2\neq k_3}} h(k_1-k_2)\\
&\times\hat{u}(k_1)\bar{\hat{u}}(k_2)\Big)\hat{u}(k_3)\Big],\quad t>0,
\end{aligned}$$

Here and hereafter,

$$\psi(t) \in C_0^\infty(\mathbf{R}),$$
$$\psi(t) = 1 \ (|t| < 1), \quad \psi(t) = 0 \ (|t| > 2),$$
$$\psi_T(t) = \psi(t/T),$$

Remark 6 *(i) Since we consider the Fourier transformation in time variable, we need to define the semigruop $U(t)$ on the whole real line $(-\infty, \infty)$.*

(ii) The coefficient of the dissipative part of the generator for $U(t)$ is $\|u_0\|_{L^2}^2$. Each time we gain 1/2 derivative by the parabolic effect, the unpleasant factor $\|u_0\|_{L^2}^{-1}$ appears.

3 Idea of Proof of Theorem 1

We first illustrate how to estimate the second and third terms on the right hand side of (2.1), because these are resonant terms. The second and third terms are estimated as follows.

$$\begin{aligned}
&\big|\big(|k|^s \sum_{k_3 \in \mathbf{Z}} |k_3||\hat{u}(k_3)|^2\big)\hat{u}(k)\big| \\
\le &C \sum_{|k_3| \le |k|} \big(|k_3|^{1/4}|\hat{u}(k_3)|\big)^2 \times |k|^{s+1/2}|\hat{u}(k)| \\
+ &\sum_{|k_3| > |k|} \big(|k_3|^{1/4+(s+1/2)}|\hat{u}(k_3)|^2\big) \times |k|^s|\hat{u}(k)| \\
&\le C\|u\|_{H^s}^2 \, \|u\|_{H^{s+1/2}} \quad \text{if } s \ge \frac{1}{4}.
\end{aligned}$$

Hence, if $s \ge 1/4$, the inequality is completed, since 1/2-derivative can be gained by the parabolic effect (see Remark 4 (ii)).

Remark 7 *When we use the dissipative effect of* (1.1)*, the unpleasant factor $\|u_0\|_{L^2}^{-1}$ appears in the estimate. To avoid this difficulty, we employ the short-time Fourier restriction norm to estimate solutions in a neighborhood of the origin.*

We next illustrate how to estimate the last term F_1 on the right hand side of (2.1) The last term F_1 is easier to handle than the other terms as long as $s \ge 1/2$, since it is nonresonant. But this is the worst term when $1/2 > s > 1/4$. Indeed, F_1 includes

$$|u_{low}|^2 \, \partial_x u_{high}, \tag{3.1}$$

where u_{high} and u_{low} denote the high and the low frequency parts of u, respectively.

Remark 8 *The term* (3.1) *is just the High $\times$ Low $\to$ High interaction. It is possible to estimate this kind of interaction by using both the dispersive and the dissipative effects of (KDNLS). However, in the estimate of F_1, a positive power of time variable, i.e., T^α, $\alpha > 0$ does not appear. Therefore, without the gauge transformation, we need to assume that the initial datum is small, which ensures that the solution is small.*

To remove this worst interaction, use the gauge transformation, following Tao [Ta] and Kenig and Takaoka [KT].

$$\begin{aligned} v_\pm &:= e^{\rho^\pm[u]} P_\pm u, \\ \rho^\pm[u] &:= \partial_x^{-1} P_\pm(|u|^2), \end{aligned}$$

where

$$P_\pm f = \frac{1}{\sqrt{2\pi}} \sum_{\pm k > 0} \hat{f}(k) e^{ikx}.$$

Remark 9 *The above guge transformation removes the high $\times$ low $\to$ high interaction and does not break the dissipative nature of KDNLS* (1.1) *(for the details, see [KT4]).*

References

[BP] H. Bahouri and G. Perelman, Global well-posedness for the derivative nonlinear Schrödinger equation, *Invent. Math.*, **229** (2022), 639–688.

[HGKNV] B. Harrop-Griffiths, R. Killip, M. Ntekoume and M. Vişan, Global well-posedness for the derivative nonlinear Schrödinger equation in $L^2(\mathbb{R})$, *J. Eur. Math. Soc.*, (2024).

[Herr] S. Herr, On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition, *Int. Math. Res. Not.*, **2006**, Art. ID 96763, 33 pages.

[IK] A. D. Ionescu and C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces, *J. Amer. Math. Soc.*, **20** (2007), 753–798.

[KT] C. Kenig and H. Takaoka, Global well-posedness for the modified Benjamin-Ono equation with initial data $\mathbb{H}^{1/2}$, *Int. Math. Res. Not.*, (2006) 95702.

[KT1] N. Kishimoto and Y. Tsutsumi, A priori bounds for the kinetic DNLS. *2019-20 MATRIX annals*, 797–803, MATRIX Book Ser., 4, Springer, Cham, [2021].

[KT2] N. Kishimoto -Y.T, Low regularity a priori estimate for KDNLS via the shor-time Fourier restriction method, *Advances in Continuous Discrete Models*, (2023) **2023:10**, 29 pages, DOI: https://doi.org/10.1186/s13662-023-03756-6

[KT3] N. Kishimoto -Y.T, Well-posedness of the Cauchy problem for the kinetic DNLS on **T**, *J. Hyperbolic Differ. Equ.*, **20** (2023), 1–49.

[KT4] N. Kishimoto-Y.T., Gage transformation for the kinetic derivative Schrödinger equation on the torus, J. Differential Equations, **453** (2026), part 1, Paper No. 113792, 47pp.

[MW1] E. Mjølhus and J. Wyller, Alfvén solitons, *Physica Scripta*, **33** (1986), 442–451.

[MW2] E. Mjølhus and J. Wyller, Nonlinear Alfvén waves in a finite-beta plasma, *J. Plasma Physics*, **40** (1988), 299–318.

[MR] L. Morinet and F. Ribaud, On the low regularity of the Korteweg-de Vries-Burgers equation, *Int. Math. Res. Not.*, **2003**(37) (2002), 1979–2005.

[Tak] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrodinger equation with the dirivative nonlinearity, *Adv. Differential Equations*, **4** (1999), 561–580.

[TT] T. Tanaka and K. Tsugawa, Well-posedness and parabolic smoothing effect for higher order Schrödinger type equations with constant coefficients, *Osaka J. Math.*, **59** (2022), 465–480.

[Ta] T. Tao, Global well-posedness of Benjamin-Ono equation in $\mathbb{H}^1(\mathbf{R})$, *J. Hyperbolic Differ. Equ.*, **1** (2004), 27–49.

[Tsug] K. Tsugawa, Local well-posedness and parabolic smoothing effect of fifth order dispersive equations on the torus, *RIMS Kôkyûroku Bessatsu*, **B60** (Series of Lecture Notes from RIMS) (2016), 177-193.

STABILITY OF TRAVELING WAVES IN NON-COOPERATIVE SYSTEMS WITH NONLOCAL DISPERSAL VIA AN ENTROPY FRAMEWORK

ARNAUD DUCROT, JONG-SHENQ GUO, AND MASAHIKO SHIMOJO

1. Introduction

Propagation phenomena described by reaction-diffusion equations have been a central topic in mathematical biology and ecology since the pioneering work on the Fisher–KPP equation. Traveling wave solutions are crucial as they represent invasion fronts.

The stability of these waves remains a core challenge. Historically, three methods have dominated the analytical landscape. The first category is the comparison principles, which is applicable to cooperative systems, or monotone flows. For this technique, see [25, 24] and references therein. In the paper [31], the weighted energy method, together with a squeezing argument, is employed to show convergence toward traveling wave fronts. The second category is related to the spectral analysis started by [21, 25], which has a relation with the Evans function to track point spectrum (see, e.g.,[27, 1, 16, 17, 32]). The third one is the variational methods, using energy functionals for gradient-type systems ([20]).

However, when modeling predator-prey dynamics or multi-species competition, cooperativity is often lost. In these non-monotone settings, traditional comparison-based trapping arguments fail. This survey focuses on the emerging entropy framework, which provides a robust alternative for non-cooperative systems.

This lecture provides a stability theory for traveling wave solutions in reaction–diffusion systems, specifically focusing on non-cooperative structures. While classical stability results rely heavily on comparison principles and order-preserving properties, many biological and ecological models lack such structures. We introduce a unified framework based on relative entropy on weighted spaces. This approach allows for the derivation of nonlinear stability for a wide class of systems, including those with unequal diffusivities and nonlocal dispersal kernels.

2. The relative entropy framework–local diffusion

We investigate the stability of traveling wave solutions for the following reaction-diffusion system:

$$\partial_t u_i(x,t) = d_i \partial_x^2 u_i(x,t) + u_i(x,t) g_i(\mathbf{u}(x,t)), \qquad x \in \mathbb{R},\ t > 0, \quad i = 1, \ldots, m, \tag{2.1}$$

where $\mathbf{u} := (u_1, \ldots, u_m)$, $d_i > 0$, and $g_i : \mathbb{R}^m \to \mathbb{R}$ are smooth functions on an appropriate subset Σ. A traveling wave solution $\{c, \mathbf{\Phi}\}$ connecting distinct constant equilibria $E^{\pm}$ is defined as a solution of the form $u_i(x,t) = \phi_i(x - ct)$ such that $\mathbf{\Phi}(-\infty) = E^-$ and $\mathbf{\Phi}(\infty) = E^+$ for $\mathbf{\Phi} = (\phi_1, \ldots, \phi_m)$.Traveling waves play a crucial role in describing the spatio-temporal dynamics of invasion phenomena.

In non-cooperative cases, existence of traveling wave is typically proven using the method of generalized upper-lower solutions and Schauder's fixed point theorem. Recent works such as [15,

This work was supported in part by by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) (No. 24K06817) and partially supported by JSJSBP120233210.

19, 3, 14, 2] have successfully constructed these waves for two-species or three-species predator-prey models. While their existence has been extensively studied, their stability analysis for non-cooperative systems—such as predator-prey or epidemic models—remains a significant challenge due to the lack of order-preserving structures. In this study, we establish a general framework to derive the global asymptotic stability of traveling waves in systems without comparison principles. Our main theorem is as follows:

Theorem 2.1 (**Stability theorem** [8]). *Assume that the system* (2.1) *admits a bounded, rectangular positively invariant set* $\Sigma \subset (0,\infty)^m$ *in a neighborhood of the traveling wave. That is, we assume that if the initial data* $\mathbf{u}^0 = (u_1^0, \ldots, u_m^0) \in BUC(\mathbb{R})$ *satisfies* $\mathbf{u}^0(x) \in \Sigma$ *for all* $x \in \mathbb{R}$, *then the corresponding solution* $\mathbf{u}$ *satisfies* $\mathbf{u}(x,t) \in \Sigma$ *for all* $x \in \mathbb{R}$ *and* $t > 0$.

Suppose there exist constants $\{\sigma_i\}_{i=1,\ldots,m} \subset (0,\infty)^m$ *such that*

$$\sum_{i=1}^{m} \sigma_i \left(u_i - v_i\right) \{g_i(\mathbf{u}) - g_i(\mathbf{v})\} \leq 0, \qquad \mathbf{u}, \mathbf{v} \in \Sigma. \tag{2.2}$$

For $R := \max_{1\leq i\leq m}\{\|g_i(\mathbf{\Phi})\|_{L^\infty(\mathbb{R})}\}$ *and* $d_{\max} := \max_{1\leq i\leq m}\{d_i\}$, *assume that*

$$c > 2\sqrt{d_{\max}R}, \quad \text{and} \quad \mu \in \left(\frac{c - \sqrt{c^2 - 4d_{\max}R}}{2d_{\max}}, \frac{c + \sqrt{c^2 - 4d_{\max}R}}{2d_{\max}}\right). \tag{2.3}$$

Suppose that $\phi_i(x) \geq Ce^{-\alpha_0|x|}$ *for some* $C, \alpha_0 > 0$, *and that there exist constants* $\alpha^+ > \mu + \alpha_0$ *and* $\alpha^- < -\alpha_0 - \mu$ *such that the initial perturbation satisfies*

$$x \mapsto \left(u_i^0(x) - \phi_i(x)\right) e^{-\alpha^\pm x} \in H^1(\mathbb{R}).$$

Then, the solution $\mathbf{u}(\cdot + ct, t)$ *converges to* $\mathbf{\Phi}(\cdot)$ *as* $t \to \infty$ *locally uniformly in* $\mathbb{R}$.

By applying this theorem, the asymptotic stability of traveling waves can be established for various predator- prey and epidemic models, including those presented in [3, 14, 2, 10, 22]. Notably, in standard predator-prey or epidemic models where the predator (or infected) species diffuses faster than the prey (or susceptible) species, the threshold $\sqrt{2d_{\max}R}$ often coincides with the minimum wave speed for the existence of a traveling wave solution.

The main novelty of this theorem lies in removing the artificial requirement that all species must share the same diffusion rate—a long-standing limitation in the entropy-based analysis of non-cooperative systems. Indeed, while the previous work [11, 10, 22] proved stability for $c \geq \sqrt{2d_{\max}R}$ under the restrictive and often biologically unrealistic assumption of identical diffusion coefficients $d_i = d$ for all $1 \leq i \leq m$.

Remark 2.2. We impose the dissipativity condition (2.2). This condition is satisfied by many predator-prey type models or some type of SIR model. Note that the above condition simply means that the corresponding kinetic system to (2.1) admits a suitable Lyapunov function. To see this, note that this kinetic problem is written as

$$\frac{du_i}{dt}(t) = u_i(t)g_i(\mathbf{u}(t)), \quad i = 1, \ldots, m. \tag{2.4}$$

Let $\{\sigma_i \mid 1 \leq i \leq m\}$ be the set of positive constants provided by (2.2), and let $\mathbf{u}(t) = (u_1(t), \cdots, u_m(t))$ be a positive solution of (2.4), and let $\mathbf{u}^* = (u_1^*, \cdots, u_m^*)$ be a positive equilibrium of (2.4). By calculating the time derivative of

$$W[\mathbf{u}(t)] := \sum_{i=1}^{m} \sigma_i \Big(u_i(t) - u_i^* - u_i^* \log \frac{u_i(t)}{u_i^*}\Big), \tag{2.5}$$

we obtain

$$\frac{d}{dt}W[\mathbf{u}(t)] = \sum_{i=1}^{m}\sigma_i\Big(1-\frac{u_i^*}{u_i(t)}\Big)\frac{du_i(t)}{dt} = \sum_{i=1}^{m}\sigma_i\,(u_i(t)-u_i^*)\,\{g_i(\mathbf{u}(t))-g_i(\mathbf{u}^*)\}. \tag{2.6}$$

Here we used the relation $g_i(\mathbf{u}^*)=0$ for all $i=1,\ldots,m$. Note that the right-hand side of (2.6) is non-positive if (2.2) is satisfied. This means that (2.5) is a Lyapunov functional to the ordinary differential system (2.4).

Outline of the proof of Theorem 2.1. Let $\mathbf{u}(x,t)$ be a positive solution of (2.1). Using the moving coordinate $z=x-ct$, we define $\mathbf{u}(z+ct,t)=\hat{\mathbf{u}}(z,t)$. To shorten the notation, we denote $\hat{\mathbf{u}}(z,t)$ by $\mathbf{u}(z,t)$ and $\hat{u}_i(z,t)$ by $u_i(z,t)$ for $1\le i\le m$. Then, the function $u_i=u_i(z,t)$ satisfies

$$\partial_t u_i(z,t)=\partial_z^2 u_i(z,t)+c\partial_z u_i(z,t)+u_i(z,t)g_i(\mathbf{u}(z,t)),\ z\in\mathbb{R},\ t>0,\ i=1,\ldots,m. \tag{2.7}$$

Note that traveling wave profile $\mathbf{\Phi}(z)$ is a stationary solution.

The core idea is to measure the "distance" from the wave profile using a convex entropy functional rather than a simple L^2-norm. We define the individual entropy for each component i as:

$$\mathcal{K}_i[U_i]=U_i-\phi_i-\phi_i\ln\frac{U_i}{\phi_i},\qquad 1\le i\le m,$$

for positive functions $U_i=U_i(z)$, and define the total entropy with weights $\sigma_i>0$ by

$$\mathcal{K}[\mathbf{U}]=\sum_{i=1}^{m}\sigma_i\mathcal{K}_i[U_i],\qquad \mathbf{U}:=(U_1,U_2\ldots,U_m).$$

After calculations, the functions

$$W_i(z,t)=\sigma_i\mathcal{K}_i[u_i(\cdot,t)](z),\quad i=1,2,\ldots,m$$

satisfy

$$\partial_t W_i=d_i\partial_z^2 W_i+c\partial_z W_i+\sigma_i(u_i-\phi_i)(g_i(\mathbf{u})-g_i(\mathbf{\Phi}))+g_i(\mathbf{\Phi})W_i-\phi_i\Big[\Big(\log\frac{u_i}{\phi_i}\Big)_z\Big]^2.$$

We define the weighted entropy

$$\int_{\mathbb{R}} W(z,t)e^{\mu z}\,dz,\qquad W(z,t):=\sum_{i=1}^{m}W_i(z,t).$$

and show an entropy dissipation property. Thus we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}W(z,t)e^{\mu z}\,dz &\le \sum_{i=1}^{m}\int_{\mathbb{R}}\{d_i\partial_z^2W_i(z,t)+c\partial_zW_i(z,t)+g_i(\mathbf{\Phi}(z))W_i(z,t)\}e^{\mu z}\,dz\\ &\le \sum_{i=1}^{m}\int_{\mathbb{R}}\{d_i\partial_z^2W_i(z,t)+c\partial_zW_i(z,t)+RW_i(z,t)\}e^{\mu z}\,dz.\end{aligned}$$

A formal calculation of the integration by parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}}W(z,t)e^{\mu z}\,dz\le\sum_{i=1}^{m}\int_{\mathbb{R}}(d_i\mu^2-c\mu+R\mu)W_i(z,t)e^{\mu z}\,dz. \tag{2.8}$$

This formal calculation is rigorously justified by the approach originally developed by Arnaud Ducrot and described in [8]. By condition (2.3) and the inequality (2.8), there exists $\varepsilon>0$ such that

$$\frac{d}{dt}\int_{\mathbb{R}}W(z,t)e^{\mu z}\,dz\le-\varepsilon\int_{\mathbb{R}}W(z,t)e^{\mu z}\,dz.$$

Therefore, we conclude that

$$\int_{\mathbb{R}} W(z,t)e^{\mu z}\,dz \le e^{-\varepsilon t}\int_{\mathbb{R}} W(z,0)e^{\mu z}\,dz \to 0 \text{ as } t\to\infty.$$

This and the parabolic estimates give us the desired result. □

Remark 2.3. When all $d_i = d$, which corresponds the equal diffusion coefficients case, the entropy function evolves according to a heat-like inequality:

$$W_t \le dW_{zz} + cW_z + RW, \tag{2.9}$$

where $W(z,t) = \mathcal{K}[\mathbf{u}(\cdot,t)](z)$ for $z\in\mathbb{R}$ and $t\ge 0$. This yields the convergence $\mathbf{u}(z,t)\to\mathbf{\Phi}(z)$ locally uniformly as $t\to\infty$ when $c\ge 2\sqrt{dR}$. See [11] for the detail calculation. Recent developments [8] utilize a weighted space $L^1(\mathbb{R}, e^{\mu z}dz)$. By carefully choosing the weight μ as above, and obtain the exponential stability in the weighted norm, even when the diffusion coefficients are different.

3. The relative entropy framework–nonlocal dispersal

In models where dispersal is represented by an integral kernel $J(x-y)$, the diffusion operator is replaced by $\mathcal{N}[u] = J*u-u$. The entropy method is particularly advantageous here because it does not require the compactness often missing in nonlocal spectral analysis.

Table 1. Comparison of Stability Frameworks

Feature	Comparison Principle	Evans Function	Entropy Framework
Cooperativity	Required	Not required	Not required
Linearization	Not needed	Central	Not needed
Non-local Kernel	Limited	Complex	Robust

Now we examine the stability of traveling wave solutions for a class of reaction–diffusion systems incorporating nonlocal dispersal:

$$\partial_t u_i(x,t) = d_i\mathcal{N}_i[u_i(\cdot,t)](x) + u_i(x,t)g_i(\mathbf{u}(x,t)),\quad (x,t)\in\mathbb{R}\times(0,\infty),\quad 1\le i\le m, \tag{3.1}$$

where $u(x,t) := (u_1(x,t),\dots,u_m(x,t))$, $d_i>0$, and $g_i\in C^1([0,\infty)^m)$. The operator $\mathcal{N}_i$ represents nonlocal dispersal, defined as:

$$\mathcal{N}_i[u_i(\cdot,t)](x) := \int_{\mathbb{R}} J_i(y)u_i(x-y,t)dy - u_i(x,t) = (J_i*u_i-u_i)(x,t),$$

with the kernel J_i satisfying the following properties:

(J1) J_i is a non-negative, smooth, and even function;

(J2) J_i is normalized such that $\int_{\mathbb{R}} J_i(y)dy = 1$;

(J3) The moment-generating function $\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy$ is finite for any $\lambda\in(0,\hat\lambda_i)$ and diverges as $\lambda\uparrow\hat\lambda_i$ for some $\hat\lambda_i\in(0,\infty]$.

A traveling wave solution $\{c,\mathbf{\Phi}\}$ of (3.1) connecting two distinct equilibria $E^\pm$ is a solution of the form $u_i(x,t)=\phi_i(x-ct)$, where the profile $\mathbf{\Phi} := (\phi_1,\dots,\phi_m)$ satisfies the boundary conditions $\mathbf{\Phi}(-\infty)=E^-$ and $\mathbf{\Phi}(\infty)=E^+$.

The existence of traveling waves in nonlocal reaction–diffusion systems has been a subject of extensive research. For scalar equations [4] and cooperative systems possessing an order structure, characterizing the minimal wave speed is well-established [18]. Recently, traveling waves have also been successfully constructed for non-monotone systems, such as predator–prey and epidemic models, where the comparison principle is inapplicable ([29, 28, 6, 30, 26, 33, 23]).

The objective of this study is to provide a unified framework for establishing the stability of these traveling wave solutions. While stability in nonlocal cooperative systems can be addressed using general order-preserving methods [31], results for non-cooperative systems remain scarce due to the breakdown of the comparison principle. By utilizing Theorem 3.1 presented in this talk, we demonstrate the asymptotic stability of traveling waves in a weighted space for various predator–prey and epidemic models, including those investigated in [6, 30, 26, 33, 23, 29, 28].

Our main result is stated as follows.

Theorem 3.1. (Stability theorem [7]**)** *Assume that the following assumptions (A1)-(A3) are satisfied.*

(A1) *The system* (3.1) *possesses a bounded rectangular invariant set* $\Sigma \subset [0,\infty)^m$. *On this set, there exist positive constants* $\{\sigma_i\}_{i=1}^m$ *such that the reaction terms satisfy the following dissipative condition:*

$$\sum_{i=1}^m \sigma_i \left(u_i - v_i\right)\left\{g_i(\mathbf{u}) - g_i(\mathbf{v})\right\} \le 0 \quad \text{for all } \mathbf{u}, \mathbf{v} \in \Sigma.$$

(A2) *Let* $\{c, \mathbf{\Phi}\}$ *be a positive traveling wave solution in* $\mathbf{\Phi}(x) \in \Sigma$ *for all* $x \in \mathbb{R}$ *with* $c > c^*$ *for a suitable constant* c^*. *We assume the profile* ϕ_i *satisfies a lower exponential bound* $\phi_i(x) \ge Ce^{-\alpha_0|x|}$ *for some* $C, \alpha_0 > 0$.

(A3) *Let* R *be a constant such that* $\max_i \|g_i(\mathbf{\Phi})\|_{L^\infty} \le R$. *There exists a decay rate* $\mu \in (0, \min \hat{\lambda}_i)$ *such that the following dispersal-dominated inequality holds for all* $1 \le i \le m$:

$$d_i \left(\int_{\mathbb{R}} J_i(y) e^{\mu y} dy - 1 \right) - c\mu + R < 0.$$

Additionally, the kernels satisfy $x \mapsto J_i(x)e^{\alpha^\pm x} \in L^1(\mathbb{R})$ *for appropriate constants* $\alpha^\pm$ *such that* $\alpha^+ > \alpha_0$ *and* $\alpha^- < -\max\{\alpha_0, (\alpha_0+\mu)/2\}$.

Consider the solution $\mathbf{u}(x,t)$ *with initial data* $\mathbf{u}^0 \in \Sigma \cap BUC(\mathbb{R})^m$. *If the initial perturbation* $u_i^0 - \phi_i$ *belongs to the weighted Sobolev spaces* $H^1(\mathbb{R})$ *with weights* $e^{-\alpha^\pm x}$, *then* $\mathbf{u}(\cdot + ct, t)$ *converges to the traveling wave profile* $\mathbf{\Phi}(\cdot)$ *in* $L^1_{\mathrm{loc}}(\mathbb{R})$ *as* $t \to \infty$.

Outline of the proof of Theorem 3.1. To analyze the stability, we adopt the moving coordinate system $z := x - ct$. Let $u_i(z,t)$ denote the solution in this frame. We define the weighted relative entropy functional as

$$\int_{\mathbb{R}} W(z,t) e^{\mu z} dz = \sum_{i=1}^m \sigma_i \int_{\mathbb{R}} \left(u_i - \phi_i - \phi_i \log \frac{u_i}{\phi_i} \right) e^{\mu z} dz,$$

where the entropy density W_i is consistent with the local diffusion case.

Time differentiation of W_i yields a decomposition into dispersal ($I_{D,i}$) and reaction ($I_{R,i}$) contributions. A direct differentiation and integration of the function $W_i(z,t) = \mathcal{K}_i[u_i(\cdot,t)](z)$ yields the decomposition:

$$\partial_t W_i - d_i \mathcal{N}_i[W_i] - c\,\partial_z W_i - g_i(\mathbf{\Phi}) W_i = I_{D,i} + I_{R,i},$$

where

$$I_{D,i} := d_i \int_{\mathbb{R}} J_i(z-y) \left\{ \phi_i(y) - \frac{u_i(y,t)}{u_i(z,t)} \phi_i(z) + \phi_i(y) \ln \left[\frac{u_i(y,t)\phi_i(z)}{u_i(z,t)\phi_i(y)} \right] \right\} dy$$

and

$$I_{R,i} := (u_i - \phi_i)(g_i(\mathbf{u}) - g_i(\mathbf{\Phi})).$$

By the standard entropy inequality ($X - 1 \geq \log X$), the dispersal term $I_{D,i}$ is shown to be non-positive, acting as a critical dissipative mechanism.

Assumption (A3) ensures that the weight function $\rho(z) = e^{\mu z}$ satisfies

$$d_i \mathcal{N}_i[\rho] - c\partial_z \rho + R\rho \leq -\epsilon\rho, \quad \epsilon > 0.$$

This inequality guarantees that the drift and dispersal effects dominate any potential growth from the reaction terms. Combining these estimates, we arrive at the fundamental functional inequality:

$$\frac{d}{dt}\int_{\mathbb{R}} W(z,t)e^{\mu z}dz \leq -\epsilon \int_{\mathbb{R}} W(z,t)e^{\mu z}dz. \tag{3.2}$$

Here we apply the assumption (A1). This inequality proves that the entropy decays exponentially, which yields the convergence after some technical calculations. □

Remark 3.2 (Critical speed stability)**.** The entropy framework bridges the gap between simple scalar equations and complex non-cooperative systems. However, several questions remain. For example, proving stability at the minimal speed $c = 2\sqrt{d_{\max}R}$ for local diffusion problem, and $c = c^*$ for nonlocal dispersal one, in the setting of unequal diffusivities remains delicate. If the diffusion operators are completely the same for all components, the convergence holds when c satisfies the equality, for both local and nonlocal diffusion ([11, 12]).

Remark 3.3 (Other extensions)**.** There are other problems to which we can apply the entropy framework. One example is the stability of forced waves in media with moving climate change effect [9]. Although this paper focuses on the monostable stability of traveling waves for a single equation, which in itself was an open problem. It is interesting to generalize this result to multi-component reaction-diffusion systems with shifting environment.

Another example is the problem in the graph-based domains. Recently [13], we provide a solution to the problem posed as an open question by the authors in [5] by extending these entropy dissipative structures to reaction-diffusion systems on networks.

By organizing recent advances around the entropy functional, we have shown that stability is often a consequence of the underlying nonlinear geometry and dissipativity of the system rather than purely spectral properties. This provides a powerful tool for researchers dealing with multi-species ecological models where traditional monotonicity is unavailable.

References

[1] J. Alexander, R. Gardner, C. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math. 410 (1990), 167-212.

[2] Y.-S. Chen, J.-S. Guo, M. Shimojo, *Recent developments on a singular predator-prey model*, Discrete Contin. Dyn. Syst. Ser. B, 26 (2021), 1797–1809.

[3] Y.-Y. Chen, J.-S. Guo, C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator-prey model*, J. Math. Anal. Appl. 445 (2017), 212–239.

[4] J. Coville, J. Dávila, S. Martinez, *Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity*, SIAM J. Math. Anal., 39 (2008), 1693-1709.

[5] Y. Du, B. Lou, R. Peng, M. Zhou, *The Fisher-KPP equation over simple graphs: varied persistence states in river networks*, J. Math. Biology, 80 (2020), 1559-1616.

[6] A. Ducrot, J.-S. Guo, G. Lin, S. Pan, *The spreading speed and the minimal wave speed of a predator-prey system with nonlocal dispersal*, Z. Angew. Math. Phys., 70 (2019), Art. 146.

[7] A. Ducrot, J.-S. Guo, M. Shimojo, *Stability of traveling waves in non-cooperative systems with nonlocal dispersal of general diffusivities*, Discrete and Continuous Dynamical Systems - Series B, 37 (2026), 43-57.

[8] A. Ducrot, M Shimojo, *Convergence to traveling waves for reaction-diffusion systems using Lyapunov type arguments*, J. Dynamics and Differential Equations, DOI 10.1007/s10884-026-10489-z.

[9] J.-S. Guo, K. Guo, M. Shimojo, *Uniqueness and stability of forced waves for the Fisher–KPP equation in a shifting environment*, Nonlinear Analysis, 247 (2024), Art 113607.

[10] J.-S. Guo, K. Guo, M. Shimojo, *Stability of monostable traveling waves in diffusive three-species competition systems*, Applied Mathematics Letters, 148 (2024), 108891

[11] J.-S. Guo, M. Shimojo, *Convergence to traveling waves in reaction-diffusion systems with equal diffusivities*, J. Differential Equations, 375 (2023), 156–171.

[12] J.-S. Guo and M. Shimojo, *Stability of traveling waves in noncooperative systems with nonlocal dispersal of equal diffusivities*, Asymptotic Analysis, 142 (2025), 582–591.

[13] J.-S. Guo, M. Shimojo, C.-C. Wu, *Convergence to the stationary solutions in a river network*, submitted.

[14] J.-S. Guo, K.-I. Nakamura, T. Ogiwara and C.-C. Wu, *Traveling wave solutions for a predator-prey system with two predators and one prey*, Nonlinear Analysis: Real World Applications 54 (2020), Art. 103111.

[15] Y.L. Huang, G. Lin, *Traveling wave solutions in a diffusive system with two preys and one predator*, J. Math. Anal. Appl., 418 (2014), 163-184.

[16] Y. Li, Y. Wu, *Stability of traveling front solutions with algebraic spatial decay for some autocatalytic chemical reaction systems*, SIAM J. Math. Anal., 44 (2012), 1474-1521.

[17] Y. Li, Y.Wu, *Existence and stability of travelling front solutions for general auto-catalytic chemical reaction systems*, Math. Model. Nat. Phenom., 8 (2013), 104-132.

[18] X. Liang, X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math., **60** (2007) 1-40.

[19] J.-J. Lin, W. Wang, C. Zhao, T.-H. Yang, *Global dynamics and traveling wave solutions of two-predators-one-prey models*, Discrete Contin. Dynam. Syst. Ser. B, 20 (2015), 1135-1154.

[20] C. B. Muratov and M. Novaga, *Global exponential convergence to variational traveling waves in cylinders*, SIAM J. Math. Anal., 44 (2012), 293–317.

[21] D.H. Sattinger, *Weighted norms for the stability of travelling waves*, J. Differential Equations, 25 (1977), 179-201.

[22] M. Shimojo, Y. Tanaka, *Global asymptotic stability of endemic equilibria and stability of traveling waves for a diffusive SIR epidemic model with logistic growth*, J. Elliptic and Parabolic Equations, 11 (2025) 1917–1928.

[23] W.-Y. Tang, F.-Y. Yang, B.-E. Jiang, *Minimal wave speed for a predator-prey system with nonlocal dispersal*, International Journal of BiomathematicsVol. 18, No. 06, (2025) 2450017.

[24] M. Taniguchi, *Traveling Front Solutions in Reaction-Diffusion Equations*, MSJ Memoirs, Vol. 39, Mathematical Society of Japan, Tokyo, 2021.

[25] V. Volpert, A. Volpert, V. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, Vol. 140, American Mathematical Society, Providence, RI, 1994.

[26] C.-C. Wu, *On the stable tail limit of traveling wave for a predator-prey system with nonlocal dispersal*, Appl. Math. Lett., 113 (2021), 5 pp.

[27] E. Yanagida, K. Maginu, *Stability of double-pulse solutions in nerve axon equations*, SIAM J. Appl.Math. 49 (1989),1158–1173.

[28] F.-Y. Yang, W.-T. Li, *Traveling waves in a nonlocal dispersal SIR model with critical wave speed*, J. Math. Anal. Appl. 458 (2018), 1131–1146.

[29] F.-Y. Yang, Y. Li, W.-T. Li, Z.-C. Wang, *Traveling waves in a nonlocal dispersal Kermack-McKendrick epidemic model*, Discr. Contin. Dyn. Syst. Ser. B, 18 (2013), 1969–1993.

[30] F.-Y. Yang, W.-T. Li, R.-H. Wang, *Invasion waves for a nonlocal dispersal predator-prey model with two predators and one prey*, Commun. Pure. Appl. Anal., 20 (2021), 4083–4105.

[31] Z. Yu, J. Pei, *Stability of traveling wave fronts for a cooperative system with nonlocal dispersal*, Japan J. Industrial Appl. Math., 35 (2018), 817-834.

[32] H. Zhang, H. Izuhara, Y. Wu, *Asymptotic stability of two types of traveling waves for some predator-prey models*, Discrete Contin. Dynam. Syst. Ser. B, 26 (2021), 2323-2342.

[33] X.-D. Zhao, F.-Y. Yang, W.-T. Li, *Traveling waves for a nonlocal dispersal predator-prey model with two preys and one predator*, Z. Angew. Math. Phys., 73 (2022), work No. 124, 29 pp.

(A. Ducrot) Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France

(J.-S. Guo) Department of Applied Mathematics and Data Science, Tamkang University, Tamsui, New Taipei City 251301, Taiwan

(M. Shimojo) Department of Mathematical Sciences, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan

Vanishing viscosity and quasi-steady state limit for a chemo-repulsion system in one-dimensional space

Markus Schmidtchen[1] and Yoshitaro Tanaka*[2]

[1] Institute of Scientific Computing, Faculty of Mathematics, TU Dresden, Zellescher Weg 25, 01069 Dresden, Deutschland
[2] Department of Complex and Intelligent Systems, School of Systems Information Science, Future University Hakodate, 116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan, E-mail: y-tanaka@fun.ac.jp

1 Introduction

Cells communicate with each other through a variety of interactions. Through these interactions, cells control differentiation, migration, and tissue development, and the resulting tissues perform various functions. For example, in neural firing, neurons transmit membrane potentials to other cells via dendrites and axons, e.g., [9, Chapter 7.5], [1, Chapter 8]. During cell movement, cells transmit biological signals to other cells and sense the surrounding cell density by contacting other cells with parts of their bodies. Furthermore, in quorum sensing and chemotaxis, cells secrete diffusible chemical signals and communicate with other cells through the concentration or gradient of those signals, see, e.g., [8, Chapter 5], as well as [2, 3, 4, 13].

Many mathematical models describing the evolution of a **cell density**, $n = n(x,t)$ have been proposed to analyse the aforementioned biological phenomena, in which interactions are often modelled using convolutions with suitable **integral kernels**, $K(x)$, and are therefore referred to as *nonlocal* or *long-range*. Nonlocal interactions can represent various biological functions by appropriately choosing the shape of the integral kernel and where they enter the equation. Nonlocal interaction equations and local systems of partial differential equations share many qualitative properties in the behavior of their solutions. Motivated by this observation, considerable effort has been devoted to investigating the relationship between nonlocal interactions and PDE systems.

The connection between nonlocal interaction operators and differential operators can be established through Taylor expansion [9]. Furthermore, a methodology has been proposed to approximate nonlocal interactions by superposing Green's functions associated with the steady-state problems of reaction–diffusion systems [5, 6, 7, 10, 11]. This approach is justified by combining the theory of approximating a given interaction kernel with Green's functions and the singular limit arising from the quasi-steady-state approximation of the corresponding PDE system.

In the analysis of such singular limits, the diffusion terms typically provide sufficient regularity of solutions, and this regularity plays a crucial role in establishing the convergence. However, in realistic models of cell migration and cell growth, diffusion may be extremely weak. It is therefore of both mathematical and phenomenological interest to determine whether the above approximation remains valid in regimes where diffusion is negligible and the regularity of solutions is consequently limited.

In this report, we investigate whether the above approximation remains valid under such low-regularity conditions. More precisely, we study a repulsive Keller–Segel system and analyze whether the quasi-steady-state limit can be justified after first taking the vanishing viscosity limit. We first derive entropy estimates and establish uniform bounds with respect to the relevant parameters. Using these uniform estimates together with compactness arguments, we prove the convergence of weak solutions in the vanishing viscosity limit. We then employ the method of compensated compactness to justify the quasi-steady-state limit and establish the convergence of weak solutions. As a consequence, we show that the repulsive chemotaxis model converges to a nonlocal advection equation.

1.1 Organisation of this paper

This paper aims to address the gap of local approximation of aggregation-type equations in the absence of diffusion and when regularity is low. The starting point of our study is the parabolic-parabolic Keller-Segel-type chemorepulsion system

$$\frac{\partial n}{\partial t} = \alpha \frac{\partial^2 n}{\partial x^2} + \frac{\partial}{\partial x}\left(n \sum_{j=1}^{N} \beta_j \frac{\partial v_j}{\partial x}\right),$$
$$\epsilon \frac{\partial v_j}{\partial t} = d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n,$$

in $Q_T := \Pi \times (0, T)$. Notably, the energy dissipation identity usually employed, see for instance [12], does not yield estimates uniform in $\alpha > 0$, which is why we propose a different strategy to derive uniform estimates. Then, we prove that solutions converge to solutions to the hyperbolic-parabolic system

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(n \sum_{j=1}^{N} \beta_j \frac{\partial v_j}{\partial x}\right), \tag{2a}$$
$$\epsilon \frac{\partial v_j}{\partial t} = d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n, \tag{2b}$$

in Q_T. Continuing in this low-regularity setting, we finally establish the quasi-steady-state approximation, $\varepsilon \to 0$, and we obtain a hyperbolic-elliptic system of the form

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(n \sum_{j=1}^{N} \beta_j \frac{\partial v_j}{\partial x}\right), \tag{3a}$$
$$0 = d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n, \tag{3b}$$

in Q_T.

2 Mathematical setting and main results

2.1 The starting point

We begin by revisiting the chemorepulsion system, (4), for some density, $n = n(x,t)$ coupled to the $N \in \mathbb{N}$ chemical species, $v_j = v_j(x,t)$:

$$\frac{\partial n}{\partial t} = \alpha \frac{\partial^2 n}{\partial x^2} + \frac{\partial}{\partial x}\left(n \sum_{j=1}^{N} \beta_j \frac{\partial v_j}{\partial x} \right), \tag{4a}$$

$$\epsilon \frac{\partial v_j}{\partial t} = d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n. \tag{4b}$$

The system is posed on $Q_T = \Pi \times (0,T)$, where Π denotes the one-dimensional torus, and $j = 1,\dots,N$, and it is equipped with periodic boundary conditions

$$n(L,t) = n(-L,t), \quad n_x(L,t) = n_x(-L,t), \quad v_j(L,t) = v_j(-L,t), \quad v_{j,x}(L,t) = v_{j,x}(-L,t),$$

for any $t \geq 0$ and $j = 1,\dots,N$. Furthermore, we impose the following initial conditions

$$n(x,0) := n_0(x), \quad v_j(x,0) := (v_j)_0(x), \quad (j = 1,\dots,N). \tag{5}$$

We note that all solutions depend on α and ε. For the subsequent analysis, we will make the following assumptions on the initial data and the constants

(H1) $n_0 \in H^1(\Pi)$ with $n_0 \geq 0$ and $\|n_0\|_{L^1(\Pi)} = 1$,

(H2) $(v_j)_0 \in C^2(\Pi)$, and $(v_j)_0 \geq 0$, $\|(v_j)_0\|_{L^1(\Pi)} = 1$, $\quad (j = 1,\dots,N)$,

(H3) $0 \leq \alpha \leq 1$, as well as $\beta_j, d_j > 0$, for all $j = 1,\dots,N$.

To define the mild solution to the system (4a) and (4b), we introduce the following fundamental solutions:

$$G(x,t) = G(x,t;\alpha) := \frac{1}{2L} \sum_{k\in\mathbb{Z}} e^{-\alpha\sigma_k^2 t} e^{i\sigma_k x},$$

$$G_j^{\varepsilon}(x,t) := e^{-\frac{t}{\epsilon}} G(x,t;d_j/\varepsilon) = \frac{1}{2L} \sum_{k\in\mathbb{Z}} e^{-\frac{d_j\sigma_k^2+1}{\varepsilon}t} e^{i\sigma_k x},$$

where i denotes the imaginary unit, and

$$\sigma_k := \frac{k\pi}{L}, \quad (k \in \mathbb{Z})$$

is the wave number. For any $\tau > 0$, we define the maps

$$\Psi_j : C([0,\tau], H^1(\Pi)) \to C([0,\tau], H^1(\Pi)),$$

for $j = 1,\dots,N$, and

$$\Phi : C([0,\tau], H^1(\Pi)) \to C([0,\tau], H^1(\Pi)),$$

by

$$\Psi_j[n](x,t) := (G_j^\varepsilon \star (v_j)_0)(x,t) + \frac{1}{\varepsilon}\iint_{Q_T} G_j^\varepsilon(x-y,t-s)n(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$\Phi[n](x,t) := (G\star n_0)(x,t) - \iint_{Q_t} G(x-y,t-s)\Big(\frac{\partial}{\partial x}\Big(n\frac{\partial}{\partial x}\sum_{j=1}^N \beta_j\Psi_j[n]\Big)\Big)(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

for $n \in C([0,\tau], H^1(\Pi))$, respectively.

Definition 2.1. [Mild solution] We say that

$$(n,(v_j)_{j=1}^N) \in C([0,T]; H^1(\Pi)) \times C([0,T]; C^2(\Pi))^N,$$

for any $T>0$, is a mild solution to (4a) and (4b) with (5), provided $n = \Phi[n]$ and $v_j = \Psi_j[n]$ for $j = 1,\dots,N$.

Remark 2.2. [Existence of mild solutions] We refer to the result on the existence of the unique mild solution by Murakawa and Tanaka, cf. [7]. Using this result, we see that this mild solution satisfies system (4a) and (4b) in $L^2(Q_T)$ for any time $T>0$, and has the following regularity

$$n \in H^1(0,T;L^2(\Pi)) \cap L^2(0,T;H^2(\Pi)) \cap C([0,T];H^1(\Pi)),$$

and

$$v_j \in L^2(0,T;H^3(\Pi)), \quad \varepsilon\frac{\partial v_j}{\partial t} \in L^2(0,T;H^1(\Pi)),$$

for all j.

2.2 Notions of weak solutions

In the singular limits we later consider, the regularity of mild solutions will be lost. We shall therefore introduce our notions of weak solutions to the hyperbolic-parabolic system, System (2), and the hyperbolic-elliptic quasi-steady-state system, System (3).

We begin with the notion of weak solutions to the inviscid system.

Definition 2.3. [Weak solutions of System (2)] We say that $(n,(v_j)_{j=1}^N)$ is a weak solution of (2a) and (2b) with the initial condition (5) if

$$n \in L^2(Q_T) \cap L^\infty(0,T;\mathcal{M}(\Pi)), \quad \text{and} \quad \frac{\partial n}{\partial t} \in \mathcal{M}(0,T;H^2(\Pi)^*),$$

as well as

$$v_j \in L^2(0,T;H^2(\Pi)) \cap H^1(0,T;L^2(\Pi)),$$

for $j=1,\dots,N$, and, such that

$$\iint_{Q_T} n\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Pi n_0\varphi(0)\,\mathrm{d}x = \iint_{Q_T} n\sum_{j=1}^N \beta_j\frac{\partial v_j}{\partial x}\frac{\partial\varphi}{\partial x}\,\mathrm{d}x\,\mathrm{d}t,$$

for all $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(\cdot, T) = 0$, and

$$\epsilon \frac{\partial v_j}{\partial t} = d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n,$$

almost everywhere in Q_T, for $j = 1, \ldots, N$.

Finally, introduce our notion of weak solutions to the inviscid quasi-steady-state system.

Definition 2.4. [Weak solutions of System (3)] We say that $(n, (v_j)_{j=1}^N)$ is a weak solution of (3a) and (3b) with the initial condition (5) if

$$n \in L^2(Q_T) \cap L^\infty(0, T; \mathcal{M}(\Pi)), \quad \text{and} \quad \frac{\partial n}{\partial t} \in \mathcal{M}(0, T; H^2(\Pi)^*),$$

as well as

$$v_j \in L^2(0, T; H^2(\Pi)),$$

for $j = 1, \ldots, N$ and, such that

$$\iint_{Q^T} n \frac{\partial \varphi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t + \int_\Pi n_0 \varphi(0) \, \mathrm{d}x = \iint_{Q^T} n \sum_{j=1}^N \beta_j \frac{\partial v_j}{\partial x} \frac{\partial \varphi}{\partial x} \, \mathrm{d}x \, \mathrm{d}t,$$

for all $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(\cdot, T) = 0$, and

$$d_j \frac{\partial^2 v_j}{\partial x^2} - v_j + n = 0,$$

almost everywhere in Q_T, for $j = 1, \ldots, N$.

2.3 Main results

We are now ready to state the main results of our work.

Theorem 2.5 (Vanishing viscosity limit)**.** *Assume that* (H1)-(H3) *hold. Let* $(n^{\alpha,\epsilon}, (v_j^{\alpha,\epsilon})_{j=1}^N)_{\alpha>0}$ *be a family of solutions to System* (4)*. Then, there exists a subsequences in* α *(not relabelled) and functions* $n^\epsilon \in L^2(Q_T) \cap L^\infty(0, T; \mathcal{M}(\Pi))$ *and*

$$v_j^\epsilon \in L^2(0, T; H^2(\Pi)), \quad \textit{with} \quad \frac{\partial v_j^\epsilon}{\partial t} \in L^2(Q_T),$$

for $j = 1, \ldots, N$*, such that*

$$v_j^{\alpha,\epsilon} \to v_j^\epsilon, \qquad \textit{strongly in } L^2(Q_T),$$

and

$$\frac{\partial v_j^{\alpha,\epsilon}}{\partial x} \to \frac{\partial v_j^\epsilon}{\partial x}, \qquad \textit{strongly in } L^2(Q_T),$$

as well as

$$n^{\alpha,\epsilon} \rightharpoonup n^\epsilon,$$

weakly in $L^2(Q_T)$*, as* $\alpha \to 0$*. The limits are weak solutions of System* (2) *in the sense of Definition 2.3.*

Theorem 2.6 (Quasi-steady-state limit)**.** *Assume that* (H1)-(H3) *hold. Let* $(n^\epsilon, (v_j^\epsilon)_{j=1}^N)_{\epsilon>0}$ *be a family of weak solutions to System* (2)*.Then, there exists a subsequences in* ϵ *(not relabelled) and functions* $n \in L^2(Q_T) \cap L^\infty(0,T;\mathcal{M}(\Pi))$ *and*

$$v_j \in L^2(0,T;H^2(\Pi)),$$

for $j = 1, \ldots, N$*, such that*

$$v_j^\epsilon \rightharpoonup v_j, \qquad \textit{weakly in } L^2(Q_T),$$

and

$$\frac{\partial v_j^\epsilon}{\partial x} \rightharpoonup \frac{\partial v_j}{\partial x}, \qquad \textit{weakly in } L^2(Q_T),$$

as well as

$$n^\epsilon \rightharpoonup n,$$

weakly in $L^2(Q_T)$*, as* $\varepsilon \to 0$*. The limits are weak solutions of System* (3) *in the sense of Definition 2.4.*

Acknowledgments

YT was partially supported by DAAD Research Stays for University Academics and Scientists program Grant Number 57698956, by JSPS KAKENHI Grant Numbers 23K21704, 24H00188, and 24K06848, and by Special Research Expenses of Future University Hakodate.

References

[1] L. Edelstein-Keshet. *Mathematical models in biology.*, volume 46 of *Class. Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2005.

[2] T. Hillen and K. J. Painter. A user ' s guide to pde models for chemotaxis. *Journal of mathematical biology*, 58(1):183–217, 2009.

[3] D. Horstmann. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Dtsch. Math.-Ver.*, 105(3):103–165, 2003.

[4] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Dtsch. Math.-Ver.*, 106(2):51–69, 2004.

[5] H. Ishii, H. Murakawa, and Y. Tanaka. Relationship between haptotaxis and chemotaxis in cell dynamics, 2025.

[6] H. Ishii and Y. Tanaka. Reaction, diffusion, and nonlocal interaction in high-dimensional space. *Journal of Mathematical Biology*, 92(5):67, 2026.

[7] H. Murakawa and Y. Tanaka. Keller–segel type approximation for nonlocal fokker–planck equations in one-dimensional bounded domain. *European Journal of Applied Mathematics*, pages 1–37, 2025.

[8] J. D. Murray. *Mathematical biology II: Spatial models and biomedical applications*, volume 3. Springer New York, 2001.

[9] J. D. Murray. *Mathematical biology: I. An introduction.* Springer, 2002.

[10] H. Ninomiya, Y. Tanaka, and H. Yamamoto. Reaction, diffusion and non-local interaction. *J. Math. Biol.*, 75(5):1203–1233, 2017.

[11] H. Ninomiya, Y. Tanaka, and H. Yamamoto. Reaction-diffusion approximation of non-local interactions using Jacobi polynomials. *Jpn. J. Ind. Appl. Math.*, 35(2):613–651, 2018.

[12] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Analysis: Theory, Methods & Applications*, 51(1):119–144, 2002.

[13] K. J. Painter and T. Hillen. Volume-filling and quorum-sensing in models for chemosensitive movement. *Can. Appl. Math. Q.*, 10(4):501–543, 2002.