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NON-POLYNOMIAL CONSERVED QUANTITIES FOR ODE SYSTEMS AND ITS APPLICATION TO THE LONG-TIME BEHAVIOR OF SOLUTIONS TO CUBIC NLS SYSTEMS

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ABSTRACT. In this paper, we investigate the asymptotic behavior of small solutions to the initial value problem for a system of cubic nonlinear Schrödinger equations (NLS) in one spatial dimension. We identify a new class of NLS systems for which the global boundedness and asymptotics of small solutions can be established, even in the absence of any effective conserved quantity. The key to this analysis lies in utilizing conserved quantities for the reduced ordinary differential equation (ODE) systems derived from the original NLS systems. In [27], the first author investigated conserved quantities expressed as quartic polynomials. In contrast, the conserved quantities considered in the present paper are of a different type and are not necessarily polynomial.

1. INTRODUCTION

In this paper, we consider the large time behavior of solutions to the initial value problem of the system of cubic nonlinear Schrödinger equations in one space dimension:

$$(1.1) \quad \begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 = F_1(u_1, u_2), & t \in \mathbb{R}, x \in \mathbb{R}, \\ i\partial_t u_2 + \frac{1}{2}\partial_x^2 u_2 = F_2(u_1, u_2), & t \in \mathbb{R}, x \in \mathbb{R}, \\ u_1(0, x) = u_{1,0}(x), \quad u_2(0, x) = u_{2,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $u_j : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ ($j = 1, 2$) are unknown functions and $u_{j,0} : \mathbb{R} \rightarrow \mathbb{C}$ ($j = 1, 2$) are given functions. The nonlinear terms F_1 and F_2 are given by

$$\begin{aligned} F_1(u_1, u_2) &= c_1|u_1|^2 u_1 + c_2|u_1|^2 u_2 + c_3 u_1^2 \bar{u}_2 + c_4 u_1 |u_2|^2 + c_5 \bar{u}_1 u_2^2 + c_6 |u_2|^2 u_2, \\ F_2(u_1, u_2) &= c_7 |u_1|^2 u_1 + c_8 |u_1|^2 u_2 + c_9 u_1^2 \bar{u}_2 + c_{10} u_1 |u_2|^2 + c_{11} \bar{u}_1 u_2^2 + c_{12} |u_2|^2 u_2, \end{aligned}$$

and c_j ($j = 1, \dots, 12$) are real constants. Our aim is to show the global existence of the solution to (1.1) for small initial data and to derive their asymptotic behavior under suitable conditions on the parameters of the nonlinearities. It is well-known that the cubic nonlinearities are critical in determining the asymptotic behavior of solutions in one dimension, making this an important and intriguing problem. We provide two new types of systems of the form (1.1). Typical examples in our scope are

$$(1.2) \quad \begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 = (\zeta + 1)|u_1|^2 u_2 + \zeta |u_2|^2 u_2, & t \in \mathbb{R}, x \in \mathbb{R}, \\ i\partial_t u_2 + \frac{1}{2}\partial_x^2 u_2 = 2\zeta |u_1|^2 u_1 + 2\zeta u_1 |u_2|^2 + \bar{u}_1 u_2^2, & t \in \mathbb{R}, x \in \mathbb{R} \end{cases}$$

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with $\zeta \in \mathbb{R} \setminus \{0\}$ and

$$(1.3) \quad \begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 = 6|u_1|^2 u_1 + 3|u_1|^2 u_2 + u_1^2 \overline{u_2}, & t \in \mathbb{R}, x \in \mathbb{R}, \\ i\partial_t u_2 + \frac{1}{2}\partial_x^2 u_2 = -u_1|u_2|^2 - 4|u_2|^2 u_2, & t \in \mathbb{R}, x \in \mathbb{R}. \end{cases}$$

Considering the single gauge invariant cubic nonlinear Schrödinger equation:

$$(1.4) \quad i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda|u|^2 u, \quad t \in \mathbb{R}, x \in \mathbb{R}$$

with $\lambda \in \mathbb{C}$, two typical asymptotic behavior of the solution are known, depending on whether $\lambda \in \mathbb{C}$ is a real number or a pure imaginary number. When λ is a real number, Ozawa [31] showed the modified scattering of the small solution to (1.4), namely, for a suitable function u_+ depending only on the spatial variable,

$$(1.5) \quad u(t, x) = (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+} \left(\frac{x}{t} \right) \exp(-i\lambda |\widehat{u_+}(\frac{x}{t})| \log t) + o(t^{-\frac{1}{2}})$$

in L^∞ as $t \rightarrow \infty$ (see also [4, 6, 7, 9, 11, 23]). When λ is a pure imaginary number, Shimomura [32] showed that the equation has the dissipative property in one time direction (See also [8, 17]). Kita [15] established a counterpart of the above result, namely, small-data blow-up occurs in the reverse time direction. For related blow-up results, we refer the reader to [2], [3], [12], and [13].

In studying the general system of cubic nonlinear Schrödinger equations (1.1), we encounter the various situations depending on the coefficients of the nonlinearity [10, 14, 18–22, 30, 33]. In [28, 29], we provided the classification of the system (1.1). First, we review it in detail. In [28], they showed that the system (1.1) can be identified with a pair of a matrix $\mathcal{A} \in M_3(\mathbb{R}) \simeq \mathbb{R}^9$ and a vector $\mathcal{V} \in \mathbb{R}^3$ as follows: Given $(c_1, \dots, c_{12}) \in \mathbb{R}^{12}$, we define

$$(1.6) \quad \mathcal{A} := \begin{bmatrix} c_2 - c_3 & -c_1 + c_8 - c_9 & -c_7 \\ c_5 & -c_3 + c_{11} & -c_9 \\ c_6 & -c_4 + c_5 + c_{12} & -c_{10} + c_{11} \end{bmatrix}$$

and

$$(1.7) \quad \mathcal{V} := \begin{bmatrix} c_8 - 2c_9 \\ \frac{1}{2}(-c_2 + 2c_3 - c_{10} + 2c_{11}) \\ c_4 - 2c_5 \end{bmatrix}.$$

Conversely, for a given pair $(\mathcal{A} = (a_{ij})_{1 \leq i, j \leq 3}, (q_k)_{1 \leq k \leq 3}) \in M_3(\mathbb{R}) \times \mathbb{R}^3$, the system of the form (1.1) is given with the following nonlinearity (see [28]);

$$(1.8) \quad \begin{cases} F_1(u_1, u_2) = -(a_{12} + a_{23})|u_1|^2 u_1 + a_{11}(2|u_1|^2 u_2 + u_1^2 \overline{u_2}) + a_{21}(2u_1|u_2|^2 + \overline{u_1} u_2^2) \\ \quad + a_{31}|u_2|^2 u_2 - (\text{tr } \mathcal{A}) \text{Re}(\overline{u_1} u_2) u_1 + \mathcal{V}(u_1, u_2) u_1, \\ F_2(u_1, u_2) = -a_{13}|u_1|^2 u_1 - a_{23}(2|u_1|^2 u_2 + u_1^2 \overline{u_2}) - a_{33}(2u_1|u_2|^2 + \overline{u_1} u_2^2) \\ \quad + (a_{21} + a_{32})|u_2|^2 u_2 + (\text{tr } \mathcal{A}) \text{Re}(\overline{u_1} u_2) u_2 + \mathcal{V}(u_1, u_2) u_2, \end{cases}$$

where $\mathcal{V}(u_1, u_2) = q_1|u_1|^2 + 2q_2 \text{Re}(\overline{u_1} u_2) + q_3|u_2|^2$ is a real-valued quadratic potential. The matrix $\mathcal{A}$ naturally appears in the analysis of the following system of ordinary differential equations, which is obtained by removing the term $\frac{1}{2}\partial_x^2 u_j$ from the system (1.1):

$$(1.9) \quad \begin{cases} i\partial_t \varphi_1 = F_1(\varphi_1, \varphi_2), \\ i\partial_t \varphi_2 = F_2(\varphi_1, \varphi_2). \end{cases}$$

The system (1.9) is called the *reduced ODE system*. We introduce the four quadratic quantities

$$(1.10) \quad \rho_1 = |\varphi_1|^2, \quad \rho_2 = |\varphi_2|^2, \quad \mathcal{R} = 2 \operatorname{Re}(\overline{\varphi_1} \varphi_2), \quad \mathcal{I} = 2 \operatorname{Im}(\overline{\varphi_1} \varphi_2)$$

for a solution (φ_1, φ_2) to (1.9). Further, for a vector $\vec{a} = {}^t(a_1, a_2, a_3) \in \mathbb{R}^3$ and a solution (φ_1, φ_2) to (1.9), we define a quadratic form

$$(1.11) \quad \mathcal{Q}(\vec{a}) := \begin{bmatrix} \rho_1 & \mathcal{R} & \rho_2 \end{bmatrix} \vec{a} = \begin{bmatrix} \overline{\varphi_1} & \overline{\varphi_2} \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}.$$

Following proposition provides the time evolution of the quadratic form $\mathcal{Q}(\vec{a})$ along the flow given by (1.9).

Proposition 1.1 ([29]). *For any $\vec{a} \in \mathbb{R}^3$, it holds that*

$$(1.12) \quad \frac{d}{dt} \mathcal{Q}(\vec{a}) = 2\mathcal{I} \mathcal{Q}(\vec{a}).$$

In particular, if $\vec{a} \in \ker \mathcal{A}$ then $\mathcal{Q}(\vec{a})$ is independent of time.

We introduce the following sets:

$$(1.13) \quad \begin{aligned} \mathcal{P}_+ &:= \{{}^t(a, b, c) \in \mathbb{R}^3 \mid ac > b^2\} \subset \mathbb{R}^3, \\ \mathcal{P}_0 &:= \{{}^t(a, b, c) \in \mathbb{R}^3 \mid ac = b^2\} \subset \mathbb{R}^3, \\ \mathcal{P}_- &:= \{{}^t(a, b, c) \in \mathbb{R}^3 \mid ac < b^2\} \subset \mathbb{R}^3. \end{aligned}$$

Note that if $\vec{a} \in \mathcal{P}_+$ then the quadratic form $\mathcal{Q}(\vec{a})$ is sign-definite, i.e., $|\mathcal{Q}(\vec{a})| \sim_{\vec{a}} |\varphi_1|^2 + |\varphi_2|^2$.

From Proposition 1.1, we see that $\operatorname{rank} \mathcal{A}$ indicates the complexity of the system since the number of the conserved quadratic quantities of the reduced ODE system (1.9) is $3 - \operatorname{rank} \mathcal{A}$. In [25, 29], the system (1.1) satisfying $\operatorname{rank} \mathcal{A} \leq 2$ are classified. Roughly speaking, there are the following seven types;

- (1) $\operatorname{rank} \mathcal{A} = 0$;
- (2) $\operatorname{rank} \mathcal{A} = 1$ and $\ker \mathcal{A} \cap \mathcal{P}_+ \neq \emptyset$;
- (3) $\operatorname{rank} \mathcal{A} = 1$, $\ker \mathcal{A} \cap \mathcal{P}_+ = \emptyset$, and $\ker \mathcal{A} \cap \mathcal{P}_0 \neq \{0\}$;
- (4) $\operatorname{rank} \mathcal{A} = 1$, $\ker \mathcal{A} \cap \mathcal{P}_+ = \emptyset$, and $\ker \mathcal{A} \cap \mathcal{P}_0 = \{0\}$;
- (5) $\operatorname{rank} \mathcal{A} = 2$ and $\ker \mathcal{A} \cap \mathcal{P}_+ \neq \emptyset$;
- (6) $\operatorname{rank} \mathcal{A} = 2$ and $\ker \mathcal{A} \cap \mathcal{P}_0 \neq \{0\}$;
- (7) $\operatorname{rank} \mathcal{A} = 2$ and $\ker \mathcal{A} \cap \mathcal{P}_- \neq \emptyset$.

The case (1) includes the Manakov system [24] which is an integrable system arising in the nonlinear optics.

In the cases (1), (2) and (5), a structural condition called the *weak null gauge condition* is satisfied. As for (1.1), this condition is expressed as

$$(1.14) \quad \ker \mathcal{A} \cap \mathcal{P}_+ \neq \emptyset$$

(see [26]). If this condition is satisfied, then one can find a conserved quadratic form which is equivalent to $|\varphi_1(t)|^2 + |\varphi_2(t)|^2$ in view of Proposition 1.1. Hence, solutions to the reduced ODE system (1.9) are global and bounded. Kim [14] studied the global existence and time decay of solution to (1.1) under the weak null gauge condition (see also Li-Sunagawa [21, 22] for (1.1) with derivative cubic nonlinearity). Furthermore, Katayama-Sakoda [10] proved that, under the weak null gauge condition, the asymptotic behavior of the solution of the system (1.1) is given by

$$(1.15) \quad u_j(t, x) = (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \varphi_j \left(\log t, \frac{x}{t} \right) + O(t^{-\delta}),$$

for $\delta > 1/2$, where φ_1 and φ_2 are the solution to (1.9) (see also [5] for the two dimensional case). We note that they considered not only (1.1) but also more general N -coupled systems of cubic nonlinear Schrödinger equations, including derivatives in the nonlinearity.

In our previous studies [16, 29], we showed a similar asymptotic results for several examples belonging to the case (3) or (6). These examples have the special structure; one equation of the system is the single cubic equation (1.4) and the other equation is linear with time dependent potential given by the first component.

Our main motivation here is to study the case $\text{rank } \mathcal{A} = 3$. We remark that the condition (1.14) holds only in the case $\text{rank } \mathcal{A} \leq 2$ because $\ker \mathcal{A} = \{0\}$ if $\text{rank } \mathcal{A} = 3$. Recently, the first author obtained the global existence of the solution for small initial data and its asymptotic behavior including the case of $\text{rank } \mathcal{A} = 3$ in [27]. He showed that if $\mathcal{A}$ has two pure imaginary eigenvalues, say $\pm ik$ ($k > 0$), and if $\ker(\mathcal{A}^2 + k^2 E_3) \cap \mathcal{P}_+ \neq \emptyset$, where E_3 is the identity matrix of size 3, then the reduced ODE system has the conserved quantity which is given as a *quartic quantity* equivalent to

$$(1.16) \quad (|\varphi_1(t)|^2 + |\varphi_2(t)|^2)^2.$$

Moreover, the precise asymptotic behavior of the solution is given by explicitly solving the reduced ODE for a typical case (see also [26] for other “solvable” cases).

In this paper, we find more systems (1.1) which satisfy $\text{rank } \mathcal{A} = 3$ and the behavior of small solutions is specified. More precisely, we prove that (1.9) has conserved quantities which are *not necessarily a polynomial* and described by eigenvalues and eigenvectors of $\mathcal{A}$. We further find a class of systems of (1.1) for which the global boundedness of solutions to the corresponding reduced ODE systems (1.9) is obtained by utilizing the conservation of these quantities.

1.1. A conserved quantity of (1.9). Before stating our result on (1.1), we state a simple result on the reduced ODE system (1.9). This is an immediate consequence of Proposition 1.1.

Theorem 1.2. *Let $\mathcal{A}$ be the matrix part of the system (1.9). If $\vec{a}_1 \in \mathbb{R}^3$ and $\vec{a}_2 \in \mathbb{R}^3$ are eigenvectors of $\mathcal{A}$ associated with eigenvalues λ_1 and λ_2 , respectively, then*

$$|\mathcal{Q}(\vec{a}_1)|^{\lambda_2} |\mathcal{Q}(\vec{a}_2)|^{-\lambda_1}$$

is conserved when it is well-defined, where $\mathcal{Q}(\cdot)$ is defined as in (1.11).

By utilizing this conserved quantity, one can prove a global existence and boundedness result for a class of reduced ODE system (1.9). Because it is a main ingredient for our main result, let us introduce here. To specify the class, let us now introduce assumptions on the system. To this end, we make a notation. For a complex number λ , we let

$$W(\lambda) = W(\lambda, \mathcal{A}) = \ker(\mathcal{A} - \lambda E_3) \subset \mathbb{R}^3$$

be the eigenspace, where E_3 is the identity matrix of size 3. Note that W contains nonzero element if and only if λ is an eigenvalue of $\mathcal{A}$. Our new class of systems is those that satisfy one of the following two assumptions:

Assumption 1. *The matrix $\mathcal{A}$ has two distinct nonzero real eigenvalues λ_1 and λ_2 such that $W(\lambda_j) \cap \mathcal{P}_+ \neq \emptyset$ for $j = 1, 2$.*

Assumption 2. *The matrix $\mathcal{A}$ has three distinct real eigenvalues λ_j ($j = 1, 2, 3$) such that $\lambda_3 \neq 0$, $\frac{\lambda_1}{\lambda_3} < 1$, and $\frac{\lambda_2}{\lambda_3} < 1$. Further, $W(\lambda_1) \cap \mathcal{P}_+ = W(\lambda_2) \cap \mathcal{P}_+ = \emptyset$, $(W(\lambda_1) \oplus W(\lambda_2)) \cap \mathcal{P}_+ \neq \emptyset$, and $W(\lambda_3) \cap \mathcal{P}_+ \neq \emptyset$.*

Remark 1.3. (1) In Assumptions 1 and 2, we assumed λ_1 and λ_2 are nonzero. This is because if one of them is zero then the weak null gauge condition (1.14) is satisfied.

- (2) We assumed $W(\lambda_1) \cap \mathcal{P}_+ = W(\lambda_2) \cap \mathcal{P}_+ = \emptyset$ in Assumption 2 because otherwise Assumption 1 or the weak null gauge condition (1.14) is fulfilled. Note that $\lambda_1 \lambda_2 = 0$ is allowed in Assumption 2.
- (3) We do not need to make any assumptions about $\mathcal{V}$.

The following result holds.

Theorem 1.4. *We assume Assumptions 1 or 2. Then any solution to the reduced ODE system (1.9) exists globally in time and satisfies*

$$\sup_{t \in \mathbb{R}} (|\varphi_1(t)|^2 + |\varphi_2(t)|^2) \lesssim \inf_{t \in \mathbb{R}} (|\varphi_1(t)|^2 + |\varphi_2(t)|^2).$$

1.2. Main result. Now we turn to our main topic, i.e., the asymptotic behavior of small solutions to the NLS system (1.1). We define the Sobolev space H^1 by

$$H^1 = H^1(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) ; \|f\|_{H^1} = \|f\|_{L^2} + \|\partial_x f\|_{L^2} < \infty\},$$

and the weighted L^2 space by

$$H^{0,1} = H^{0,1}(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) ; \|f\|_{H^{0,1}} = \|f\|_{L^2} + \|xf\|_{L^2} < \infty\}.$$

Theorem 1.5. *We assume either Assumption 1 or Assumption 2. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon := \|u_{1,0}\|_{H^{0,1}} + \|u_{2,0}\|_{H^{0,1}}$ satisfies $\varepsilon \leq \varepsilon_0$, then there exists a unique solution $(u_1, u_2) \in (C(\mathbb{R}; L^2(\mathbb{R})))^2$ to (1.1) satisfying $(U(-t)u_1, U(-t)u_2) \in (C(\mathbb{R}; H^{0,1}(\mathbb{R})))^2$. Furthermore, there exist $\delta > 0$ and $(\psi_1^\pm, \psi_2^\pm) \in C(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|\psi_j^\pm\|_{L^2 \cap L^\infty} \lesssim \varepsilon$ ($j = 1, 2$) such that*

$$(1.17) \quad u_j(t, x) = (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} A_j^\pm(\text{sign}(t) \log |t|, \frac{x}{t}) + O(\varepsilon |t|^{-\frac{3}{4} + \frac{1}{2p} + \delta \varepsilon^2})$$

in $L^p(\mathbb{R})$ as $|t| \rightarrow \infty$ for $p \in [2, \infty]$, where $(A_1^\pm(t, \xi), A_2^\pm(t, \xi))$ is a solution to (1.9) subject to the initial condition

$$(1.18) \quad (A_1^\pm(0, \xi), A_2^\pm(0, \xi)) = (\psi_1^\pm(\xi), \psi_2^\pm(\xi)).$$

Since our assumption is given in terms of eigenvalue and eigenvector of matrix $\mathcal{A}$, it is easy to check. Further, it is also easy to construct examples of $\mathcal{A}$ satisfying the assumption.

1.3. Examples. Let us check that the systems (1.2) and (1.3) satisfy our assumptions.

The matrix-vector representation of (1.2) is

$$\mathcal{A}_1 = \begin{bmatrix} \zeta + 1 & 0 & -2\zeta \\ 0 & 1 & 0 \\ \zeta & 0 & -2\zeta + 1 \end{bmatrix}, \quad \mathcal{V}_1 = \begin{bmatrix} 0 \\ -\frac{3\zeta+1}{2} \\ 0 \end{bmatrix}.$$

Since $\zeta \neq 0$, it follows that

$$W(1; \mathcal{A}_1) = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad W(1 - \zeta; \mathcal{A}_1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore, it satisfies Assumption 1. One sees from Theorem 1.2 that the reduced ODE system

$$\begin{cases} i\partial_t \varphi_1 = (\zeta + 1)|\varphi_1|^2 \varphi_2 + \zeta|\varphi_2|^2 \varphi_2, \\ i\partial_t \varphi_2 = 2\zeta|\varphi_1|^2 \varphi_1 + 2\zeta\varphi_1|\varphi_2|^2 + \overline{\varphi_1} \varphi_2^2 \end{cases}$$

has a conserved quantity

$$(2|\varphi_1|^2 + |\varphi_2|^2)^{\zeta-1} (|\varphi_1|^2 + |\varphi_2|^2),$$

which is *not a power of a polynomial* if $\zeta \notin \mathbb{Q}$ or $\zeta < 1$. Notice that the quantity is equivalent to $(|\varphi_1|^2 + |\varphi_2|^2)^\zeta$. Hence, under the assumption $\zeta \neq 0$, the conservation of this quantity implies

that the solution is global and uniformly bounded. Assumption 1 implies the existence of a conserved quantity which is equivalent to a power of $|\varphi_1|^2 + |\varphi_2|^2$. Note that there are two more conserved quantities¹

$$|2 \operatorname{Re}(\overline{\varphi_1} \varphi_2)|^{\zeta-1} (|\varphi_1|^2 + |\varphi_2|^2), \quad |2 \operatorname{Re}(\overline{\varphi_1} \varphi_2)| (2|\varphi_1|^2 + |\varphi_2|^2)^{-1}.$$

However, they do not yield a global bound of the solution directly since they are not equivalent to a power of $|\varphi_1|^2 + |\varphi_2|^2$.

The matrix-vector representation of (1.3) is

$$\mathcal{A}_2 = \begin{bmatrix} 2 & -6 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix}, \quad \mathcal{V}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to see that $\mathcal{A}_2$ has three distinct eigenvalues 2, 1, and -1 and

$$W(2; \mathcal{A}_2) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad W(1; \mathcal{A}_2) = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad W(-1; \mathcal{A}_2) = \operatorname{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

By Theorem 1.2, one sees that

$$\begin{cases} i\partial_t \varphi_1 = 6|\varphi_1|^2 \varphi_1 + 3|\varphi_1|^2 \varphi_2 + \varphi_1^2 \overline{\varphi_2}, \\ i\partial_t \varphi_2 = -\varphi_1 |\varphi_2|^2 - 4|\varphi_2|^2 \varphi_2 \end{cases}$$

has two conserved quantities

$$h_1 := (2|\varphi_1|^2 + 2 \operatorname{Re}(\overline{\varphi_1} \varphi_2) + 2|\varphi_2|^2)^2 (|\varphi_1|^2), \quad h_2 := (2|\varphi_1|^2 + 2 \operatorname{Re}(\overline{\varphi_1} \varphi_2) + 2|\varphi_2|^2) (|\varphi_2|^2).$$

Each of them does not give us the global bound of a solution. However, the desired global bound follows from a combination of these two quantities. Indeed, we see that

$$h_1^2 + h_2^3 \sim (|\varphi_1|^2 + |\varphi_2|^2)^6$$

is also a conserved quantity. Assumption 2 implies that the existence of such a pair of conserved quantities.

1.4. Standard form of systems satisfying Assumptions 1 or 2. We next give a standard form of systems satisfying Assumptions 1 and 2. The standard forms are explicit examples of the systems. We let

$$\widetilde{W}(\lambda) = \widetilde{W}(\lambda, \mathcal{A}) = \bigcup_{k \geq 1} \ker((\mathcal{A} - \lambda E_3)^k) \subset \mathbb{R}^3$$

be the generalized eigenspace. Here, we state our result as a transformation of the NLS system (1.1). However, the transformation of (1.9) is completely the same.

Theorem 1.6. *Suppose that a system (1.1) satisfies Assumption 1. Let (u_1, u_2) be the unknown for the system. Then, there exists $\mathcal{M} \in GL_2(\mathbb{R})$ such that the matrix part $\mathcal{A}$ of the system which the new variable*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

becomes one of $\mathcal{A}_{1,j}$ for $j = 1, 2, 3$ given as follows:

¹It would be more appropriate to say “one more conserved quantity” here since the latter quantity is given in terms of the former quantity and the above quantity, and vice versa.

- $\mathcal{A}_{1,1}$ is given as

$$\begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 - \lambda_2) & \eta_2(1 - \lambda_1) + (1 + \eta_1)(\eta_2 - \eta_3)(\lambda_1 - \lambda_2) & (1 + \eta_1)(\lambda_1 - \lambda_2) \\ 0 & 1 & 0 \\ -\eta_1(\lambda_1 - \lambda_2) & \eta_3(1 - \lambda_1) + \eta_1(\eta_2 - \eta_3)(\lambda_1 - \lambda_2) & \lambda_1 + \eta_1(\lambda_1 - \lambda_2) \end{bmatrix}$$

with $\lambda_1, \lambda_2 \notin \{0, 1\}$, $\lambda_1 > \lambda_2$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in \mathbb{R}$. $\mathcal{A}_{1,1}$ has three distinct eigenvalues λ_1, λ_2 , and 1 and it holds that

$$\begin{aligned} W(\lambda_1; \mathcal{A}_{1,1}) &= \text{span}\{^t(1, 0, 1)\}, \\ W(\lambda_2; \mathcal{A}_{1,1}) &= \text{span}\{^t(1 + \eta_1, 0, \eta_1)\}, \\ W(1; \mathcal{A}_{1,1}) &= \text{span}\{^t(\eta_2, 1, \eta_3)\}. \end{aligned}$$

- $\mathcal{A}_{1,2}$ is given as

$$\begin{bmatrix} 1 - (1 + \eta_1)(1 - \lambda) & (1 + \eta_1)(\eta_2 - \eta_3)(1 - \lambda) + \eta_4 & (1 + \eta_1)(1 - \lambda) \\ 0 & 1 & 0 \\ -\eta_1(1 - \lambda) & \eta_1(\eta_2 - \eta_3)(1 - \lambda) + \eta_4 & 1 + \eta_1(1 - \lambda) \end{bmatrix}$$

with $\lambda \notin \{0, 1\}$, $\eta_1 > 0$, and $\eta_2, \eta_3, \eta_4 \in \mathbb{R}$. $\mathcal{A}_{1,2}$ has exactly two distinct eigenvalues 1 and λ and it holds that

$$\begin{aligned} W(1; \mathcal{A}_{1,2}) &= \begin{cases} \text{span}\{^t(1, 0, 1)\}, & \eta_4 \neq 0, \\ \text{span}\{^t(1, 0, 1), ^t(\eta_2, 1, \eta_3)\}, & \eta_4 = 0, \end{cases} \\ \widetilde{W}(1; \mathcal{A}_{1,2}) &= \text{span}\{^t(1, 0, 1), ^t(\eta_2, 1, \eta_3)\}, \\ W(\lambda; \mathcal{A}_{1,2}) &= \text{span}\{^t(1 + \eta_1, 0, \eta_1)\}. \end{aligned}$$

- $\mathcal{A}_{1,3}$ is given as

$$\begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 + 1) & -\eta_2\lambda_1 + (1 + \eta_1)(\eta_2 - \eta_3)(\lambda_1 + 1) & (1 + \eta_1)(\lambda_1 + 1) \\ 0 & 0 & 0 \\ -\eta_1(\lambda_1 + 1) & -\eta_3\lambda_1 + \eta_1(\eta_2 - \eta_3)(\lambda_1 + 1) & \lambda_1 + \eta_1(\lambda_1 + 1) \end{bmatrix}$$

with $\lambda_1 \in (-1, 1] \setminus \{0\}$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in \mathbb{R}$. $\mathcal{A}_{1,3}$ has three distinct eigenvalues λ_1 , 0, and -1 and it holds that

$$\begin{aligned} W(\lambda_1; \mathcal{A}_{1,3}) &= \text{span}\{^t(1, 0, 1)\}, \\ W(0; \mathcal{A}_{1,3}) &= \text{span}\{^t(\eta_2, 1, \eta_3)\}, \\ W(-1; \mathcal{A}_{1,3}) &= \text{span}\{^t(1 + \eta_1, 0, \eta_1)\}. \end{aligned}$$

Let us next consider the standard form of systems satisfying Assumption 2.

Theorem 1.7. Suppose that a system (1.1) satisfies Assumption 2. Let (u_1, u_2) be the unknown for the system. Then, there exists $\mathcal{M} \in GL_2(\mathbb{R})$ such that the matrix part $\mathcal{A}$ of the system which the new variable

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

solves becomes one of $\mathcal{A}_{2,j}$ for $j = 1, 2$ given as follows:

- $\mathcal{A}_{2,1}$ is given as

$$\begin{bmatrix} \lambda_1 & -\eta(\lambda_1 + 1) & 0 \\ 0 & -1 & 0 \\ 0 & -\eta(\lambda_2 + 1) & \lambda_2 \end{bmatrix}$$

with $\lambda_1 > \lambda_2 > -1$ and $\eta \in \mathbb{R}$ with $|\eta| > 1$. $\mathcal{A}_{2,1}$ has three distinct eigenvalues λ_1 , λ_2 , and -1 and it holds that

$$\begin{aligned} W(\lambda_1; \mathcal{A}_{2,1}) &= \text{span}\{^t(1, 0, 0)\}, \\ W(\lambda_2; \mathcal{A}_{2,1}) &= \text{span}\{^t(0, 0, 1)\}, \\ W(-1; \mathcal{A}_{2,1}) &= \text{span}\{^t(\eta, 1, \eta)\}. \end{aligned}$$

• $\mathcal{A}_{2,2}$ is given as

$$\begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 - \lambda_2) & (1 + \eta_1)(\eta_2 + \eta_3)(\lambda_1 - \lambda_2) - (\lambda_1 + 1)\eta_2 & -(1 + \eta_1)(\lambda_1 - \lambda_2) \\ 0 & -1 & 0 \\ \eta_1(\lambda_1 - \lambda_2) & -\eta_1(\eta_2 + \eta_3)(\lambda_1 - \lambda_2) - (\lambda_1 + 1)\eta_3 & \eta_1(\lambda_1 - \lambda_2) + \lambda_1 \end{bmatrix}$$

with $\lambda_1, \lambda_2 \in (-1, \infty)$, $\lambda_1 \neq \lambda_2$, $\eta_1 \geq 0$, $\eta_2, \eta_3 \in \mathbb{R}$, and $\eta_2\eta_3 > 1$. Further, if $\eta_1 \neq 0$ then $\lambda_1 > \lambda_2$. $\mathcal{A}_{2,2}$ has three distinct eigenvalues λ_1 , λ_2 , and -1 and it holds that

$$\begin{aligned} W(\lambda_1; \mathcal{A}_{1,3}) &= \text{span}\{^t(1, 0, -1)\}, \\ W(\lambda_2; \mathcal{A}_{1,3}) &= \text{span}\{^t(1 + \eta_1, 0, -\eta_1)\}, \\ W(-1; \mathcal{A}_{1,3}) &= \text{span}\{^t(\eta_2, 1, \eta_3)\}. \end{aligned}$$

The rest of the paper is organized as follows. In Sections 2 and 3, we prove Theorem 1.5. In particular, in Section 2, we show the global existence of solution to (1.1) and in Section 3, we derive the asymptotic formula (1.17). Finally in Section 4, we prove Theorems 1.6 and 1.7.

2. GLOBAL EXISTENCE OF THE SOLUTION

In this section we prove the global existence of the solution to (1.1). We will denote by $U(t)$ the free Schrödinger group, namely,

$$(2.1) \quad (U(t)f)(x) = (e^{\frac{it}{2}\partial_x^2}f)(x) = \left(\frac{1}{2\pi it}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{\frac{i|x-y|^2}{2t}} f(y) dy.$$

The following factorization formula is well-known:

$$(2.2) \quad U(t) = M(t)D(t)\mathcal{F}M(t),$$

where

$$\begin{aligned} (M(t)f)(x) &= e^{\frac{i|x|^2}{2t}} f(x), & (\text{Multiplication}) \\ (D(t)f)(x) &= (it)^{-\frac{1}{2}} f\left(\frac{x}{t}\right), & (\text{Dilation}) \end{aligned}$$

and $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. The generator of the Galilean transformation is given as $J(t) = x + it\partial_x$. Note that the operators $J(t)$ and $i\partial_t + (1/2)\partial_x^2$ commute. The following identities are useful:

$$J(t) = U(t)xU(-t) = M(t)it\partial_x M(t)^{-1},$$

where the second identity is valid for $t \neq 0$.

By the standard theory of local well-posedness for the nonlinear Schrödinger equation (see [1] for instance), we have the following.

Proposition 2.1. *There exists $\tilde{\varepsilon}_0 > 0$ such that if $\varepsilon := \|u_{1,0}\|_{H^{0,1}} + \|u_{2,0}\|_{H^{0,1}}$ satisfies $\varepsilon \leq \tilde{\varepsilon}_0$, then there exists a unique solution $(u_1, u_2) \in (C([-1, 1]; L^2(\mathbb{R})))^2$ to (1.1) such that*

$$(U(-t)u_1, U(-t)u_2) \in (C([-1, 1]; H^{0,1}(\mathbb{R})))^2$$

and

$$(2.3) \quad \max_{t \in [-1, 1]} (\|u_1(t)\|_{L^2} + \|J(t)u_1(t)\|_{L^2} + \|u_2(t)\|_{L^2} + \|J(t)u_2(t)\|_{L^2}) \leq 2\varepsilon.$$

In what follows, we only treat the forward time direction $t > 0$, because the other direction can be handled in the same way. We shall show the existence of global solution by proving that $\|U(-t)u_1\|_{H^{0,1}} + \|U(-t)u_2\|_{H^{0,1}}$ is finite for all $t > 0$. To this end, we introduce a new variable $w_j(t, \xi) := \mathcal{F}[U(-t)u_j(t, \cdot)](\xi)$. Note that $U(-t)u_j(t) \in H^{0,1}$ is equivalent to $w_j(t) \in H^1$. Applying $\mathcal{F}U(-t)$ to the both sides of (1.1), we have

$$(2.4) \quad i\partial_t w_j(t) = \frac{1}{t} F_j(w_1, w_2) + r_j$$

for $j = 1, 2$ and $t \neq 0$, where

$$(2.5) \quad r_j := \mathbf{I}_j + \mathbf{II}_j$$

with

$$(2.6) \quad \mathbf{I}_j := \left(U\left(\frac{1}{t}\right) - 1 \right) D(t)^{-1} M(t)^{-1} F_j(u_1(t), u_2(t))$$

and

$$(2.7) \quad \mathbf{II}_j := \frac{1}{t} \left\{ \left(F_j\left(U\left(-\frac{1}{t}\right)w_1(t), U\left(-\frac{1}{t}\right)w_2(t)\right) - F_j(w_1(t), w_2(t)) \right) \right\}.$$

From the unitarity of the Fourier transform and $U(t)$ in $L^2(\mathbb{R})$ and $J(t)$, we have

$$(2.8) \quad \|w_j(t)\|_{L^2} = \|u_j(t)\|_{L^2}, \quad \|\partial_\xi w_j(t)\|_{L^2} = \|J(t)u_j(t)\|_{L^2}.$$

Since $u_j(t) = M(t)D(t)U(-\frac{1}{t})w_j(t)$, it holds that for any $t \neq 0$,

$$(2.9) \quad \|U(-\frac{1}{t})w_j(t)\|_{L^\infty} = t^{\frac{1}{2}} \|u_j(t)\|_{L^\infty}.$$

For $\delta > 0$ and $T > 1$, we define

$$(2.10) \quad X_T := \sup_{t \in [1, T]} t^{-\delta\varepsilon^2} \sum_{j=1}^2 (\|u_j(t)\|_{L^2} + \|J(t)u_j(t)\|_{L^2})$$

and

$$(2.11) \quad Y_T := \sup_{t \in [1, T]} t^{\frac{1}{2}} \sum_{j=1}^2 \|u_j(t)\|_{L^\infty}.$$

Note that by the Gagliardo-Nirenberg inequality and (2.8), we have for any $t \in [1, T]$,

$$(2.12) \quad \|w_j(t)\|_{L^\infty} \lesssim \|w_j(t)\|_{L^2}^{\frac{1}{2}} \|\partial_\xi w_j(t)\|_{L^2}^{\frac{1}{2}} \lesssim t^{\delta\varepsilon^2} X_T.$$

The following gives us the global existence of a solution to (1.1).

Proposition 2.2. *Let $\tilde{\varepsilon}_0$ be given by Proposition 2.1. Then there exist $\delta > 0$, $C > 0$, and $\varepsilon_0 \in (0, \tilde{\varepsilon}_0]$ such that if $\varepsilon = \|u_{1,0}\|_{H^{0,1}} + \|u_{2,0}\|_{H^{0,1}}$ satisfies $\varepsilon \leq \varepsilon_0$, then the unique solution to (1.1) given in Proposition 2.1 exists globally in time for positive time direction and obeys the bound*

$$(2.13) \quad \sup_{T \geq 1} (X_T + Y_T) \leq C\varepsilon.$$

Proof. Since X_T and Y_T are continuous in T on $[1, \infty)$, it suffices to show that there exist $\delta > 0$, $C_0 > 0$, $\tilde{C}_0 > 0$, and $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$ and

$$(2.14) \quad X_T \leq C_0 \varepsilon, \quad Y_T \leq \tilde{C}_0 \varepsilon$$

for some $T \geq 1$ then it holds that

$$(2.15) \quad X_T \leq \frac{1}{2} C_0 \varepsilon, \quad Y_T \leq \frac{1}{2} \tilde{C}_0 \varepsilon.$$

We divide the proof into four steps.

Step 1. We estimate X_T . Let $\delta > 0$, $C_0 > 0$, $\tilde{C}_0 > 0$, and $\varepsilon_0 > 0$ to be chosen later. Suppose that (2.14) is true for some $T \geq 1$. By the Duhamel formula for (1.1), we see that

$$\begin{aligned} \sum_{j=1}^2 \|u_j(t)\|_{L^2} &\leq C_1 \varepsilon + C_1 \int_1^t \left(\sum_{j=1}^2 \|u_j(\tau)\|_{L^\infty} \right)^2 \sum_{j=1}^2 \|u_j(\tau)\|_{L^2} d\tau \\ &\leq C_1 \varepsilon + C_1 X_T Y_T^2 \int_1^t \tau^{\delta \varepsilon^2 - 1} d\tau \\ &\leq C_1 \varepsilon + \delta^{-1} \tilde{C}_0^2 C_0 C_1 \varepsilon t^{\delta \varepsilon^2} \end{aligned}$$

for $t \in [1, T]$. In the same manner, we obtain the estimate for $J(t)u_j(t)$:

$$(2.16) \quad \sum_{j=1}^2 \|J(t)u_j(t)\|_{L^2} \leq C_1 \varepsilon + \delta^{-1} \tilde{C}_0^2 C_0 C_1 \varepsilon t^{\delta \varepsilon^2}$$

for $t \in [1, T]$. Hence, it holds that

$$(2.17) \quad X_T \leq C_1 (1 + \delta^{-1} C_0 \tilde{C}_0^2) \varepsilon.$$

Step 2. We estimate Y_T . By using the Gagliardo-Nirenberg inequality and (2.8), we have

$$\begin{aligned} (2.18) \quad \|(U(-\tfrac{1}{t}) - 1)w_j(t)\|_{L^\infty} &\lesssim \|(U(-\tfrac{1}{t}) - 1)w_j(t)\|_{L^2}^{\frac{1}{2}} \|\partial_x (U(-\tfrac{1}{t}) - 1)w_j(t)\|_{L^2}^{\frac{1}{2}} \\ &\lesssim t^{-\frac{1}{4}} \|\partial_x w_j(t)\|_{L^2} \\ &= t^{-\frac{1}{4}} \|J(t)u_j(t)\|_{L^2}. \end{aligned}$$

Hence, for $t \in [1, T]$, it holds that

$$\begin{aligned} \|u_j(t)\|_{L^\infty} &= \|U(t)U(-t)u_j(t)\|_{L^\infty} \\ &\leq \|M(t)D(t)w_j(t)\|_{L^\infty} + \|M(t)D(t)\mathcal{F}(M(t) - 1)\mathcal{F}^{-1}w_j(t)\|_{L^\infty} \\ &\leq Ct^{-\frac{1}{2}} \|w_j(t)\|_{L^\infty} + Ct^{-\frac{1}{2}} \|(U(-\tfrac{1}{t}) - 1)w_j(t)\|_{L^\infty} \\ &\leq Ct^{-\frac{1}{2}} \|w_j(t)\|_{L^\infty} + Ct^{-\frac{3}{4}} \|J(t)u_j(t)\|_{L^2}. \end{aligned}$$

This yields

$$(2.19) \quad Y_T \leq C \sum_{j=1}^2 \|w_j(t)\|_{L^\infty} + CX_T.$$

In view of (2.19), to estimate Y_T , it suffices to estimate w_j in L^∞ . We shall do so in the next two steps by using the integral form of (2.4). In this step, we estimate the remainder term r_j given in (2.5)-(2.7). For the estimate of I_j given in (2.6), it follows from the Gagliardo-Nirenberg inequality that

$$\|I_j\|_{L^\infty} \lesssim \|I_j\|_{L^2}^{\frac{1}{2}} \|\partial_x I_j\|_{L^2}^{\frac{1}{2}}$$

$$\begin{aligned}
&\lesssim t^{-\frac{1}{4}} \|\partial_x D(t)^{-1} M(t)^{-1} F_j(u_1(t), u_1(t))\|_{L^2} \\
&\lesssim t^{-\frac{1}{4}} \|J(t) F_j(u_1(t), u_2(t))\|_{L^2} \\
&\lesssim t^{-\frac{5}{4}} \left(\sum_{j=1}^2 t^{\frac{1}{2}} \|u_j(t)\|_{L^\infty} \right)^2 \sum_{j=1}^2 \|J(t) u_j(t)\|_{L^2} \\
&\lesssim t^{-\frac{5}{4} + \delta \varepsilon^2} X_T Y_T^2.
\end{aligned}$$

As for Π_j given in (2.7), the estimates (2.9), (2.12) and (2.18) yield

$$\begin{aligned}
\|\Pi_j\|_{L^\infty} &\lesssim t^{-1} \left(\sum_{j=1}^2 \|U(-\frac{1}{t}) w_j(t)\|_{L^\infty}^2 + \sum_{j=1}^2 \|w_j(t)\|_{L^\infty}^2 \right) \sum_{j=1}^2 \|(U(-\frac{1}{t}) - 1) w_j(t)\|_{L^\infty} \\
&\lesssim t^{-1} \left(\sum_{j=1}^2 (t^{\frac{1}{2}} \|u_j(t)\|_{L^\infty})^2 + \sum_{j=1}^2 \|w_j(t)\|_{L^\infty}^2 \right) \sum_{j=1}^2 t^{-\frac{1}{4}} \|\partial_x w_j(t)\|_{L^2} \\
&\lesssim t^{-\frac{5}{4} + 3\delta \varepsilon^2} (X_T^2 + Y_T^2) X_T.
\end{aligned}$$

Thus, we end up with

$$(2.20) \quad \|r_j(t)\|_{L^\infty} \lesssim t^{-\frac{5}{4} + 3\delta \varepsilon^2} (X_T^2 + Y_T^2) X_T$$

in view of (2.5).

Step 3. We complete the estimates of Y_T under Assumption 1. One may suppose that $\lambda_1 > \lambda_2$ without loss of generality. Let

$$\rho_1 = |w_1|^2, \quad \rho_2 = |w_2|^2, \quad \mathcal{R} = 2 \operatorname{Re}(\overline{w_1} w_2), \quad \mathcal{I} = 2 \operatorname{Im}(\overline{w_1} w_2)$$

and for $\vec{a} = {}^t(a_1, a_2, a_3) \in \mathbb{R}^3$,

$$(2.21) \quad \mathcal{Q}(\vec{a}) := \begin{bmatrix} \rho_1 & \mathcal{R} & \rho_2 \end{bmatrix} \vec{a} = \begin{bmatrix} \overline{w_1} & \overline{w_2} \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Then, we see from (2.4) and Proposition 1.1 that

$$(2.22) \quad \frac{d}{dt} \mathcal{Q}(\vec{a}) = 2t^{-1} \mathcal{I} \mathcal{Q}(\mathcal{A} \vec{a}) + \vec{r} \cdot \vec{a}$$

for any $\vec{a} \in \mathbb{R}^3$, where $\vec{r} = \vec{r}(t)$ is given by

$$(2.23) \quad \vec{r} = {}^t \begin{bmatrix} 2 \operatorname{Im}(\overline{w_1} r_1) & 2 \operatorname{Im}(\overline{w_1} r_2) + 2 \operatorname{Im}(\overline{w_2} r_1) & 2 \operatorname{Im}(\overline{w_2} r_2) \end{bmatrix}.$$

Pick $\vec{p}_j = {}^t(p_{j,1}, p_{j,2}, p_{j,3}) \in W(\lambda_j) \cap \mathcal{P}_+$ and define

$$Q_j := \mathcal{Q}(\vec{p}_j), \quad R_j = \vec{r} \cdot \vec{p}_j$$

for $j = 1, 2$. Without loss of generality, we may suppose that $p_{j,1} > 0$ so that

$$(2.24) \quad Q_1 \sim |w_1|^2 + |w_2|^2 \sim Q_2.$$

Then, one sees from (2.22) and the fact that $\vec{p}_j$ is eigenvector of $\mathcal{A}$ associated with λ_j for $j = 1, 2$, respectively, that

$$(2.25) \quad \frac{d}{dt} Q_j = 2\lambda_j t^{-1} \mathcal{I} Q_j + R_j.$$

Let m be a real number satisfying $m > 1/(\lambda_1 - \lambda_2)$. Then, by (2.25), we have

$$\partial_t(Q_1^{-m\lambda_2} Q_2^{m\lambda_1}) = -m\lambda_2 Q_1^{-m\lambda_2-1} Q_2^{m\lambda_1} \partial_t Q_1 + m\lambda_1 Q_1^{-m\lambda_2} Q_2^{m\lambda_1-1} \partial_t Q_2$$

$$\begin{aligned}
&= -m\lambda_2 Q_1^{-m\lambda_2-1} Q_2^{m\lambda_1} (2t^{-1} \mathcal{I} \lambda_1 Q_1 + R_1) \\
&\quad + m\lambda_1 Q_1^{-m\lambda_2} Q_2^{m\lambda_1-1} (2t^{-1} \mathcal{I} \lambda_2 Q_2 + R_2) \\
&= -m\lambda_2 Q_1^{-m\lambda_2-1} Q_2^{m\lambda_1} R_1 + m\lambda_1 Q_1^{-m\lambda_2} Q_2^{m\lambda_1-1} R_2
\end{aligned}$$

if $|w_1|^2 + |w_2|^2 > 0$. This identity is valid also when $|w_1|^2 + |w_2|^2 = 0$ in view of (2.24) and $m(\lambda_1 - \lambda_2) > 1$. Integrating this equation on $[1, t]$, we see that

$$\begin{aligned}
(2.26) \quad Q_1(t)^{-m\lambda_2} Q_2(t)^{m\lambda_1} &= Q_1(1)^{-m\lambda_2} Q_2(1)^{m\lambda_1} \\
&\quad + m \int_1^t (Q_1^{-m\lambda_2-1} Q_2^{m\lambda_1-1} (-\lambda_2 Q_2 R_1 + \lambda_1 Q_1 R_2))(\tau) d\tau.
\end{aligned}$$

Substituting (2.12), (2.20), and (2.24) into (2.26), we obtain

$$\begin{aligned}
&(|w_1(t)|^2 + |w_2(t)|^2)^{m(\lambda_1-\lambda_2)} \\
&\lesssim (|w_1(1)|^2 + |w_2(1)|^2)^{m(\lambda_1-\lambda_2)} \\
&\quad + \int_1^t (\|w_1(\tau)\|_{L^\infty}^2 + \|w_2(\tau)\|_{L^\infty}^2)^{m(\lambda_1-\lambda_2)-\frac{1}{2}} (\|r_1(\tau)\|_{L^\infty} + \|r_2(\tau)\|_{L^\infty}) d\tau \\
&\lesssim \varepsilon^{2m(\lambda_1-\lambda_2)} + (X_T^2 + Y_T^2)^{m(\lambda_1-\lambda_2)+1} \int_1^t \tau^{-\frac{5}{4}+2\{m(\lambda_1-\lambda_2)+1\}\delta\varepsilon_0^2} d\tau \\
&\lesssim \varepsilon^{2m(\lambda_1-\lambda_2)} + (X_T^2 + Y_T^2)^{m(\lambda_1-\lambda_2)+1}
\end{aligned}$$

as long as $-\frac{5}{4} + 2\{m(\lambda_1 - \lambda_2) + 1\}\delta\varepsilon_0^2 \leq -\frac{9}{8}$. Here, we note that

$$\|w_j(1)\|_{L^\infty} \lesssim \|u_j(1)\|_{L^2}^{\frac{1}{2}} \|J(1)u_j(1)\|_{L^2}^{\frac{1}{2}} \lesssim \varepsilon$$

follows from Proposition 2.1. This yields that

$$\sum_{j=1}^2 \|w_j(t)\|_{L^\infty} \lesssim \varepsilon + (X_T^2 + Y_T^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1-\lambda_2)}}.$$

Plugging this estimate with (2.19), we obtain

$$Y_T \leq C_2 \varepsilon + C_2 (X_T^2 + Y_T^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1-\lambda_2)}} + C_2 X_T$$

with some constant $C_2 > 0$. Substituting (2.14) into the above inequality, we have

$$\begin{aligned}
(2.27) \quad Y_T &\leq C_2 \varepsilon + C_2 (C_0^2 + \tilde{C}_0^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1-\lambda_2)}} \varepsilon^{1 + \frac{1}{m(\lambda_1-\lambda_2)}} + C_2 C_0 \varepsilon \\
&\leq C_2 (1 + C_0) \varepsilon + C_2 (C_0^2 + \tilde{C}_0^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1-\lambda_2)}} \varepsilon_0^{\frac{1}{m(\lambda_1-\lambda_2)}} \varepsilon.
\end{aligned}$$

So far, we obtained (2.17) and (2.27), i.e.,

$$\begin{cases} X_T \leq C_1 (1 + \delta^{-1} C_0 \tilde{C}_0^2) \varepsilon, \\ Y_T \leq C_2 (1 + C_0) \varepsilon + C_2 (C_0^2 + \tilde{C}_0^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1-\lambda_2)}} \varepsilon_0^{\frac{1}{m(\lambda_1-\lambda_2)}} \varepsilon. \end{cases}$$

To conclude (2.15), we choose constants $\delta > 0$, C_0 , $\tilde{C}_0$, and $\varepsilon_0 > 0$ suitable. Firstly, we take the constant C_0 such that

$$4C_1 \leq C_0.$$

Secondly we choose the constant $\tilde{C}_0$ such that

$$4C_2 (1 + C_0) \leq \tilde{C}_0.$$

Thirdly, we take the constant δ such that

$$4C_1\tilde{C}_0^2 \leq \delta,$$

and finally, we take the constant ε_0 such that

$$4C_2(C_0^2 + \tilde{C}_0^2)^{\frac{1}{2} + \frac{1}{2m(\lambda_1 - \lambda_2)}} \varepsilon_0^{\frac{1}{m(\lambda_1 - \lambda_2)}} \leq \tilde{C}_0.$$

Then it holds that

$$X_T \leq \frac{1}{4}C_0\varepsilon + \frac{1}{4}C_0\varepsilon = \frac{1}{2}C_0\varepsilon$$

and

$$Y_T \leq \frac{1}{4}\tilde{C}_0\varepsilon + \frac{1}{4}\tilde{C}_0\varepsilon = \frac{1}{2}\tilde{C}_0\varepsilon.$$

Hence, we obtain (2.15). In the above argument, by choosing ε_0 even smaller if necessary, we also assume that $-\frac{5}{4} + 2\{m(\lambda_1 - \lambda_2) + 1\}\delta\varepsilon_0^2 \leq -\frac{9}{8}$.

Step 4. We next derive the L^∞ estimate of the solution to (1.1) under Assumption 2. Let $\mathcal{Q}(\vec{a})$ be given by (2.21). Pick

$$\vec{p}_j = {}^t(p_{j,1}, p_{j,2}, p_{j,3}) \in W(\lambda_j)$$

so that $\vec{p}_1 + \vec{p}_2 \in \mathcal{P}_+$, $\vec{p}_1 \neq 0$, $\vec{p}_2 \neq 0$, $\vec{p}_3 \in \mathcal{P}_+$, and $p_{3,1} > 0$. It is possible by assumption. We introduce

$$Q_{+,j} := \mathcal{Q}(\vec{p}_j), \quad R_{+,j} := \vec{r} \cdot \vec{p}_j$$

for $j = 1, 2$ and

$$Q_- := \mathcal{Q}(\vec{p}_3), \quad R_- := \vec{r} \cdot \vec{p}_3,$$

where $\vec{r}$ is as in (2.23). Then, one has

$$(2.28) \quad \frac{d}{dt}Q_{+,j} = 2t^{-1}\mathcal{I}(\lambda_j Q_{+,j}) + R_{+,j}$$

for $j = 1, 2$ and

$$(2.29) \quad \frac{d}{dt}Q_- = 2t^{-1}\mathcal{I}(\lambda_3 Q_-) + R_-.$$

As in the previous step, since $\vec{p}_3 \in \mathcal{P}_+$ and $p_{3,1} > 0$, one sees that

$$(2.30) \quad D_-^{-1}(|w_1|^2 + |w_2|^2) \leq Q_- \leq D_- (|w_1|^2 + |w_2|^2)$$

for some constant $D_- \geq 1$. Further, by assumption $\vec{p}_1 + \vec{p}_2 \in \mathcal{P}_+$, one has

$$|Q_{+,1} + Q_{+,2}| \gtrsim |w_1|^2 + |w_2|^2.$$

Further, by an elementary computation,

$$|Q_{+,1} + Q_{+,2}| \leq |Q_{+,1}| + |Q_{+,2}| \lesssim |w_1|^2 + |w_2|^2.$$

Hence, there exists a constant $D_+ \geq 1$ such that

$$(2.31) \quad D_+^{-1}(|w_1|^2 + |w_2|^2) \leq |Q_{+,1}| + |Q_{+,2}| \leq D_+ (|w_1|^2 + |w_2|^2).$$

In particular, we see from (2.30) and (2.31) that

$$(2.32) \quad \max(|Q_{+,1}|, |Q_{+,2}|) \leq D_+ D_- Q_-.$$

For $j = 1, 2$, we define

$$\theta_j = (1 - \frac{\lambda_j}{\lambda_3})^{-1} > 0.$$

We may suppose that $\theta_1 > \theta_2$ without loss of generality. We note that the elementary relation

$$(2.33) \quad \lambda_j \theta_j + \lambda_3 (1 - \theta_j) = 0$$

holds for $j = 1, 2$. Fix $m > 1$ so that $m\theta_2 > 1$. For $j = 1, 2$, by (2.28), (2.29), we have

$$\begin{aligned}
& \partial_t(|Q_{+,j}|^{m\theta_j} Q_-^{m(1-\theta_j)}) \\
&= m\theta_j |Q_{+,j}|^{m\theta_j-2} Q_{+,j} \partial_t Q_{+,j} Q_-^{m(1-\theta_j)} + m(1-\theta_j) |Q_{+,j}|^{m\theta_j} Q_-^{m(1-\theta_j)-1} \partial_t Q_- \\
&= m\theta_j |Q_{+,j}|^{m\theta_j-2} Q_{+,j} (2\lambda_j t^{-1} \mathcal{I} Q_{+,j} + R_{+,j}) Q_-^{m(1-\theta_j)} \\
&\quad + m(1-\theta_j) |Q_{+,j}|^{m\theta_j} Q_-^{m(1-\theta_j)-1} (2\lambda_3 t^{-1} \mathcal{I} Q_- + R_-) \\
&= m\theta_j |Q_{+,j}|^{m\theta_j-2} Q_{+,j} Q_-^{m(1-\theta_j)} R_{+,j} + m(1-\theta_j) |Q_{+,j}|^{m\theta_j} Q_-^{m(1-\theta_j)-1} R_-,
\end{aligned}$$

where we have used (2.33) to obtain the final line. Note that this identity is valid also when $w_1 = w_2 = 0$ since the right hand side becomes zero in view of (2.32), $m\theta_j > 1$, and $m > 1$. Integrating this equation on $[1, t]$ and using the equivalence relations (2.30) and (2.32), and the estimates (2.12) and (2.20), one obtains

$$\begin{aligned}
(2.34) \quad & |Q_{+,j}(t)|^{m\theta_j} Q_-(t)^{m(1-\theta_j)} \\
& \lesssim |Q_{+,j}(1)|^{m\theta_j} Q_-^{m(1-\theta_j)}(1) \\
& + \int_1^t (\|w_1(\tau)\|_{L^\infty}^2 + \|w_2(\tau)\|_{L^\infty}^2)^{m-\frac{1}{2}} (\|r_1(\tau)\|_{L^\infty} + \|r_2(\tau)\|_{L^\infty}) d\tau \\
& \lesssim \varepsilon^{2m} + (X_T^2 + Y_T^2)^{m+1} \int_1^t \tau^{-\frac{5}{4}+2(m+1)\delta\varepsilon_0^2} d\tau \\
& \lesssim \varepsilon^{2m} + C(X_T^2 + Y_T^2)^{m+1}
\end{aligned}$$

as long as $-\frac{5}{4} + 2(m+1)\delta\varepsilon_0^2 \leq -\frac{9}{8}$. Further, recalling $m\theta_1 > m\theta_2 > 1$, we see from (2.32) and (2.31) that

$$\begin{aligned}
& |Q_{+,1}(t)|^{m\theta_1} Q_-(t)^{m(1-\theta_1)} + |Q_{+,2}(t)|^{m\theta_2} Q_-(t)^{m(1-\theta_2)} \\
& \geq |Q_{+,1}(t)|^{m\theta_1} Q_-(t)^{m(1-\theta_1)} + |Q_{+,2}(t)|^{m\theta_2} ((D_+ D_-)^{-1} |Q_{+,2}(t)|)^{m(\theta_1-\theta_2)} Q_-(t)^{m(1-\theta_1)} \\
& \geq (D_+ D_-)^{-m(\theta_1-\theta_2)} (|Q_{+,1}(t)|^{m\theta_1} + |Q_{+,2}(t)|^{m\theta_1}) Q_-(t)^{m(1-\theta_1)} \\
& \geq 2^{1-m\theta_1} (D_+ D_-)^{-m(\theta_1-\theta_2)} (|Q_{+,1}(t)| + |Q_{+,2}(t)|)^{m\theta_1} Q_-(t)^{m(1-\theta_1)} \\
& \geq 2^{1-m\theta_1} (D_+ D_-)^{-m(\theta_1-\theta_2)} D_+^{-m\theta_1} D_-^{-m(1-\theta_1)} (|w_1(t)|^2 + |w_2(t)|^2)^m.
\end{aligned}$$

Thus, we obtain

$$(|w_1(t)|^2 + |w_2(t)|^2)^m \lesssim \varepsilon^{2m} + C(X_T^2 + Y_T^2)^{m+1},$$

which shows

$$(2.35) \quad Y_T \leq C_3 \varepsilon + C_3 (X_T^2 + Y_T^2)^{\frac{1}{2} + \frac{1}{2m}}.$$

Arguing as in Step 3, we are able to choose constants C_0 and $\tilde{C}_0$ suitably and obtain (2.15). $\square$

Remark 2.3. The global existence result for the ODE system (Theorem 1.4) follows from the arguments used in Steps 3 and 4 of the proof.

We conclude this section with a global-in-time estimate on the error terms r_j given in (2.5).

Corollary 2.4. *Let r_j be as in (2.5). Let δ and ε_0 be as in Proposition 2.2. If $\varepsilon = \|u_{0,1}\|_{H^{0,1}} + \|u_{0,2}\|_{H^{0,1}} \leq \varepsilon_0$, then it holds that*

$$(2.36) \quad \|r_j(t)\|_{L^2} \lesssim \varepsilon^3 t^{-\frac{3}{2}+3\delta\varepsilon^2}, \quad \|r_j(t)\|_{L^\infty} \lesssim \varepsilon^3 t^{-\frac{5}{4}+3\delta\varepsilon^2}$$

for $t > 1$.

Proof. By Proposition 2.2, (2.14) is true for all $T > 0$. Hence, the L^∞ -estimate for r_j follows from (2.20). Let us prove L^2 -estimate. Arguing as in Step 2 of the proof of Proposition 2.2, one has

$$\begin{aligned}\|\mathbb{I}_j\|_{L^2} &\lesssim t^{-\frac{1}{2}} \|\partial_x D(t)^{-1} M(t)^{-1} F_j(u_1(t), u_1(t))\|_{L^2} \\ &\lesssim t^{-\frac{3}{2} + \delta \varepsilon^2} X_T Y_T^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathbb{II}_j\|_{L^2} &\lesssim t^{-1} \left(\sum_{j=1}^2 \|U(-\tfrac{1}{t}) w_j(t)\|_{L^\infty}^2 + \sum_{j=1}^2 \|w_j(t)\|_{L^\infty}^2 \right) \sum_{j=1}^2 \|(U(-\tfrac{1}{t}) - 1) w_j(t)\|_{L^2} \\ &\lesssim t^{-1} \left(\sum_{j=1}^2 (t^{\frac{1}{2}} \|u_j(t)\|_{L^\infty})^2 + \sum_{j=1}^2 \|w_j(t)\|_{L^\infty}^2 \right) \sum_{j=1}^2 t^{-\frac{1}{2}} \|\partial_x w_j(t)\|_{L^2} \\ &\lesssim t^{-\frac{3}{2} + 3\delta \varepsilon^2} (X_T^2 + Y_T^2) X_T.\end{aligned}$$

Thus we have (2.36) for L^2 norm of r_j . This completes the proof. $\square$

3. APPROXIMATION WITH A SOLUTION TO THE ODE SYSTEM

In this section, we prove the asymptotic formula (1.17) in Theorem 1.5. We introduce $(\alpha_1(t, \xi), \alpha_2(t, \xi))$ by

$$(3.1) \quad \alpha_j(\log t, \xi) = w_j(t, \xi)$$

for $t > 0$, where $w_j(t, \xi) = \mathcal{F}[U(-t)u_j(t, \cdot)](\xi)$. We see that (α_1, α_2) solves

$$(3.2) \quad i\partial_t \alpha_j = F_j(\alpha_1, \alpha_2) + \tilde{r}_j$$

for $j = 1, 2$, where

$$(3.3) \quad \tilde{r}_j(t, \xi) = e^t r_j(e^t, \xi).$$

We first give the error estimate in Corollary 2.4 in terms of $\tilde{r}_j$.

Proposition 3.1. *Let $\tilde{r}_j$ be as in (3.3). Let δ and ε_0 be as in Proposition 2.2. If $\varepsilon < \varepsilon_0$, then it holds that*

$$(3.4) \quad \|\tilde{r}_j(t)\|_{L^2} \lesssim \varepsilon^3 e^{-(\frac{1}{2} - 3\delta \varepsilon^2)t}, \quad \|\tilde{r}_j(t)\|_{L^\infty} \lesssim \varepsilon^3 e^{-(\frac{1}{4} - 3\delta \varepsilon^2)t}$$

for $t > 0$.

Proof of Theorem 1.5. We now prove (1.17) in Theorem 1.5. We construct a one-parameter family $\{(\varphi_1(\cdot, \xi), \varphi_2(\cdot, \xi))\}_{\xi \in \mathbb{R}}$ of solutions to the ODE system (1.9) satisfying

$$\|\varphi_j(t) - \alpha_j(t)\|_{L^2} \lesssim \varepsilon^3 e^{-(\frac{1}{2} - 3\delta \varepsilon^2)t}, \quad \|\varphi_j(t) - \alpha_j(t)\|_{L^\infty} \lesssim \varepsilon^3 e^{-(\frac{1}{4} - 3\delta \varepsilon^2)t}$$

for $t \geq 1$. Employing the idea of Hayashi-Li-Naumkin [5], we construct a sequence of approximate solution $\{(\varphi_1^{(n)}, \varphi_2^{(n)})\}_{n \in \mathbb{N}_0}$. We first define $\varphi_j^{(0)}(t, \xi) := \alpha_j(t, \xi)$, namely $\varphi_j^{(0)}$ satisfy

$$(3.5) \quad i \frac{d}{dt} \varphi_j^{(0)} = F_j(\varphi_1^{(0)}, \varphi_2^{(0)}) + \tilde{r}_j.$$

For $n \geq 1$, we inductively define $(\varphi_1^{(n)}, \varphi_2^{(n)})$ by

$$(3.6) \quad \begin{cases} i \frac{d}{dt} \varphi_1^{(n)}(t, \xi) = F_1(\varphi_1^{(n-1)}(t, \xi), \varphi_2^{(n-1)}(t, \xi)), & t \geq 1, \\ i \frac{d}{dt} \varphi_2^{(n)}(t, \xi) = F_2(\varphi_1^{(n-1)}(t, \xi), \varphi_2^{(n-1)}(t, \xi)), & t \geq 1, \end{cases}$$

with the final state condition

$$(3.7) \quad \lim_{t \rightarrow \infty} \|\varphi_j^{(n)}(t) - \alpha_j(t)\|_{L^2 \cap L^\infty} = 0.$$

Taking the difference between $\varphi_j^{(0)}$ and $\varphi_j^{(1)}$ and integrating on $[t, \infty)$, we have the integral form

$$(3.8) \quad \varphi_j^{(1)}(t, \xi) = \alpha_j(t, \xi) + i \int_t^\infty \tilde{r}_j(s, \xi) ds.$$

Taking the difference between $\varphi_j^{(n)}$ and $\varphi_j^{(n-1)}$, we also have the integral form

$$(3.9) \quad \begin{aligned} & \varphi_j^{(n)}(t, \xi) - \varphi_j^{(n-1)}(t, \xi) \\ &= -i \int_t^\infty \left(F_j(\varphi_1^{(n-1)}(s, \xi), \varphi_2^{(n-1)}(s, \xi)) - F_j(\varphi_1^{(n-2)}(s, \xi), \varphi_2^{(n-2)}(s, \xi)) \right) ds \end{aligned}$$

for $n \geq 2$. Hereafter, we consider the integral form (3.8) and (3.9). We claim that $\{(\varphi_1^{(n)}, \varphi_2^{(n)})\}_{n \in \mathbb{N}_0}$ belongs to

$$(3.10) \quad Z := \left\{ (a_1, a_2) \in (C([1, \infty) \times \mathbb{R}))^2 ; \sum_{j=1}^2 \|a_j(t) - \alpha_j(t)\|_{L^p} \leq C_1 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)t} \text{ for } p = 2, \infty \right\}$$

for suitable $C_1 > 0$. Since Proposition 2.2 implies that there exists C_0 such that

$$(3.11) \quad \sum_{j=1}^2 \|\alpha_j(\cdot)\|_{L^\infty([1, \infty) \times \mathbb{R})} \leq C_0 \varepsilon,$$

we see that for any choice of $C_1 > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$

$$(3.12) \quad \sum_{j=1}^2 \|a_j(\cdot)\|_{L^\infty([1, \infty) \times \mathbb{R})} \leq 2C_0 \varepsilon$$

holds for any $(a_1, a_2) \in Z$. By the definition of the space Z , we easily see $(\varphi_1^{(0)}, \varphi_2^{(0)}) \in Z$. Further, thanks to (3.4), there exists $C_2 > 0$ such that

$$(3.13) \quad \sum_{j=1}^2 \|\varphi_j^{(1)}(t) - \varphi_j^{(0)}(t)\|_{L^p} = \sum_{j=1}^2 \left\| \int_t^\infty \tilde{r}_j(s) ds \right\|_{L^p} \leq C_2 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)t}$$

for $p = 2, \infty$ and $t \geq 1$. We now choose $C_1 = 2C_2$. Then $(\varphi_1^{(1)}, \varphi_1^{(2)}) \in Z$. Now, suppose for some $k \geq 1$ that $(\varphi_1^{(n)}, \varphi_1^{(n)}) \in Z$ and

$$(3.14) \quad \sum_{j=1}^2 \|\varphi_j^{(n)}(t) - \varphi_j^{(n-1)}(t)\|_{L^p} \leq 2^{-n+1} C_2 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)t}$$

hold for $p = 2, \infty$, $t \geq 1$, and $n \in [1, k]$. Then, since

$$|F_j(\varphi_1^{(k)}, \varphi_2^{(k)}) - F_j(\varphi_1^{(k-1)}, \varphi_2^{(k-1)})| \leq C \left(\sum_{j=1}^2 |\varphi_j^{(k)}| + \sum_{j=1}^2 |\varphi_j^{(k-1)}| \right)^2 \sum_{j=1}^2 |\varphi_j^{(k)} - \varphi_j^{(k-1)}|,$$

we see from the assumption of the induction that $(\varphi_j^{(k+1)}, \varphi_j^{(k+1)})$ is well-defined as an element of $C([0, \infty) \times \mathbb{R})$ and the bound

$$\begin{aligned} \sum_{j=1}^2 \|\varphi_j^{(k+1)}(t) - \varphi_j^{(k)}(t)\|_{L^p} &\leq 16C_0^2 C\varepsilon_0^2 \int_t^\infty 2^{-k+1} C_2 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)s} ds \\ &\leq (128C_0^2 C\varepsilon_0^2) 2^{-k} C_2 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)t} \end{aligned}$$

holds for $p = 2, \infty$, and $t \geq 1$, which implies (3.14) for $n = k + 1$ if ε_0 is chosen so that $128C_0^2 C\varepsilon_0^2 \leq 1$. Moreover,

$$\sum_{j=1}^2 \|\varphi_j^{(k+1)}(t) - \alpha_j(t)\|_{L^p} \leq \sum_{\ell=1}^{k+1} \sum_{j=1}^2 \|\varphi_j^{(\ell)}(t) - \varphi_j^{(\ell-1)}(t)\|_{L^p} \leq 2C_2 \varepsilon^3 e^{-(\frac{1}{4} + \frac{1}{2p} - 3\delta\varepsilon^2)t}$$

holds for $p = 2, \infty$, and $t \geq 1$, and hence $(\varphi_1^{(k+1)}, \varphi_2^{(k+1)}) \in Z$. Thus an induction argument shows that $\{(\varphi_1^{(n)}, \varphi_2^{(n)})\}_{n \in \mathbb{N}_0} \subset Z$ is well-defined and (3.14) holds true for all $n \in \mathbb{N}$. One easily sees that Z is complete with respect to the metric on $(L^\infty([1, \infty); L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})))^2$. Note that (3.14) implies that $\{(\varphi_1^{(n)}, \varphi_2^{(n)})\}_{n \in \mathbb{N}_0} \subset Z$ is a Cauchy sequence with respect to this metric. Hence, we find the limit $(\varphi_1, \varphi_2) \in Z$. Let us confirm that the limit (φ_1, φ_2) is a solution to (1.9). To this end, we note that (3.14) implies that $\{(\varphi_1^{(n)}, \varphi_2^{(n)})\}_{n \in \mathbb{N}_0}$ converges to (φ_1, φ_2) also in $(L^1(1, \infty; L^\infty(\mathbb{R})))^2$. Hence,

$$\int_t^\infty F_j(\varphi_1^{(n)}, \varphi_2^{(n)})(s, \xi) ds \rightarrow \int_t^\infty F_j(\varphi_1, \varphi_2)(s, \xi) ds$$

in $L^\infty([1, \infty) \times \mathbb{R})$ as $n \rightarrow \infty$. Then, by using (3.8) and the definition of the sequence, it holds for each $t_1, t_2 \in [1, \infty)$ that

$$\begin{aligned} \varphi_j(t_2) - \varphi_j(t_1) &= \lim_{n \rightarrow \infty} (\varphi_j^{(n)}(t_2) - \varphi_j^{(n)}(t_1)) \\ &= -i \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} F_j(\varphi_1^{(n-1)}, \varphi_2^{(n-1)}) ds \\ &= -i \int_{t_1}^{t_2} F_j(\varphi_1, \varphi_2) ds. \end{aligned}$$

This shows that (φ_1, φ_2) is a solution to (1.9).

Finally, we show the asymptotic estimate (1.17). We see from Proposition 3.1 and the fact that $(\varphi_1, \varphi_2) \in Z$ that

$$\begin{aligned} u_j(t) &= (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} w_j(t, \frac{x}{t}) + O(\varepsilon t^{-\frac{3}{4} + \frac{1}{2p} + \delta\varepsilon^2}) \\ &= (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \alpha_j(\log t, \frac{x}{t}) + O(\varepsilon t^{-\frac{3}{4} + \frac{1}{2p} + \delta\varepsilon^2}) \\ &= (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \varphi_j(\log t, \frac{x}{t}) + O(\varepsilon^3 t^{-\frac{3}{4} - \frac{1}{2p} + 3\delta\varepsilon^2}) + O(\varepsilon t^{-\frac{3}{4} + \frac{1}{2p} + \delta\varepsilon^2}) \\ &= (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \varphi_j(\log t, \frac{x}{t}) + O(\varepsilon t^{-\frac{3}{4} + \frac{1}{2p} + \delta\varepsilon^2}) \end{aligned}$$

in $L^p(\mathbb{R})$ as $t \rightarrow \infty$ for $p = 2, \infty$. The non-endpoint case $p \in (2, \infty)$ follows by interpolation. Thus we obtain (1.17). This completes the proof of Theorem 1.5. $\square$

4. NORMALIZATION OF (1.1)

In this section, we prove Theorems 1.6 and 1.7. To this end, we note that the system (1.1) and (1.9) are closed under the linear transform of the unknowns. The change of the system is described as a matrix manipulation.

Proposition 4.1 ([29]). *Let $(\mathcal{A}, \mathcal{V})$ be a matrix-vector representation of a system of the form (1.1) or (1.9) with unknown (u_1, u_2) . Define a new unknown (v_1, v_2) via*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}).$$

Let $(\mathcal{A}', \mathcal{V}')$ be the matrix-vector representation of the system for (v_1, v_2) . Then,

$$(4.1) \quad \mathcal{A}' = \frac{1}{\det \mathcal{M}} \mathcal{D}(\mathcal{M}) \mathcal{A} \mathcal{D}(\mathcal{M})^{-1}$$

and

$$(4.2) \quad \mathcal{V}' = \frac{1}{\det \mathcal{M}} \mathcal{D}(\mathcal{M}) \mathcal{V}$$

hold, where

$$\mathcal{D}(\mathcal{M}) := \frac{1}{\det \mathcal{M}} \begin{bmatrix} d^2 & -2cd & c^2 \\ -bd & ad+bc & -ac \\ b^2 & -2ab & a^2 \end{bmatrix} \in SL_3(\mathbb{R}),$$

$$\mathcal{D}(\mathcal{M})^{-1} = \frac{1}{\det \mathcal{M}} \begin{bmatrix} a^2 & 2ac & c^2 \\ ab & ad+bc & cd \\ b^2 & 2bd & d^2 \end{bmatrix} \in SL_3(\mathbb{R}).$$

Moreover, $\vec{p} \in \mathbb{R}^3$ is an eigenvector of $\mathcal{A}$ associated with an eigenvalue $\lambda \in \mathbb{C}$ if and only if $\mathcal{D}(\mathcal{M})\vec{p} \in \mathbb{R}^3$ is an eigenvector of $\mathcal{A}'$ associated with an eigenvalue $(\det \mathcal{M})^{-1}\lambda \in \mathbb{C}$. Furthermore, for any $\mathcal{M} \in GL_2(\mathbb{R})$, $\mathcal{D}(\mathcal{M})\vec{p} \in \mathcal{P}_+$ if and only if $\vec{p} \in \mathcal{P}_+$.

Proof of Theorem 1.6. We consider ODE system (1.9) instead of (1.1). It is sufficient because the change of parameters of the nonlinearity does not depend on the linear part of the equation.

Pick an ODE system (1.9) satisfying Assumption 1 and let $\mathcal{A}^{(0)}$ be the matrix part of the system. Let $(u_1^{(0)}, u_2^{(0)})$ be the unknown for the system. Let $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ be two eigenvalues such that $W(\lambda_j^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_+ \neq \emptyset$. We apply several changes of variables to make the system is of the desired standard form.

Case 1. We first consider the case where $\mathcal{A}^{(0)}$ has three distinct eigenvalues. Let $\lambda_0^{(0)}$ be the third eigenvalue. By changing $(u_1^{(0)}, u_2^{(0)}) \mapsto (u_1^{(0)}, -u_2^{(0)})$ if necessary and by relabeling the eigenvalues if necessary, we may suppose without loss of generality that either

- (a) $\lambda_0^{(0)} > 0$ and $\lambda_1^{(0)} > \lambda_2^{(0)}$;
- (b) $\lambda_0^{(0)} = 0$, $\lambda_2^{(0)} < 0$, and $\lambda_2^{(0)} < \lambda_1^{(0)} \leq |\lambda_2^{(0)}|$.

(Note that the former manipulation changes the sign of all the eigenvalues.)

Subcase 1-(a). We first consider the subcase (a). Pick

$$\vec{p} = {}^t(p_1, p_2, p_3) \in W(\lambda_1^{(0)}, \mathcal{A}^{(0)}) \cap \mathcal{P}_+$$

with $p_1 > 0$ and introduce a new variable $(u_1^{(1)}, u_2^{(1)})$ by

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \mathcal{M}^{(1)} \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}, \quad \mathcal{M}^{(1)} = \begin{bmatrix} p_1^{1/2} & p_1^{-1/2} p_2 \\ 0 & (p_1 p_3 - p_2^2)^{1/2} / p_1 \end{bmatrix}.$$

Let $\mathcal{A}^{(1)}$ be the matrix corresponding to the system for $(u_1^{(1)}, u_2^{(1)})$. A computation shows $\mathcal{D}(\mathcal{M}^{(1)})\vec{p} = \sqrt{p_1 p_3 - p_2^2} \, {}^t(1, 0, 1)$, which is a rephrase of the square completion

$$p_1 |u_1^{(0)}|^2 + 2p_2 \operatorname{Re}(\overline{u_1^{(0)}} u_2^{(0)}) + p_3 |u_2^{(0)}|^2 = |u_1^{(1)}|^2 + |u_2^{(1)}|^2.$$

One sees from Proposition 4.1 that ${}^t(1, 0, 1) \in W(\lambda_1^{(1)}; \mathcal{A}^{(1)})$, where $\lambda_1^{(1)} := (\det \mathcal{M}^{(1)})^{-1} \lambda_1^{(0)}$. Note that $\lambda_2^{(1)} := (\det \mathcal{M}^{(1)})^{-1} \lambda_2^{(0)}$ satisfies $W(\lambda_2^{(1)}; \mathcal{A}^{(1)}) \cap \mathcal{P}_+ \neq \emptyset$. The third eigenvalue is $\lambda_0^{(1)} := (\det \mathcal{M}^{(1)})^{-1} \lambda_0^{(0)} > 0$.

We next adjust the magnitude of the matrix so that the third eigenvalue becomes 1. Define $(u_1^{(2)}, u_2^{(2)})$ by $u_j^{(2)} = (\lambda_0^{(1)})^{\frac{1}{2}} u_j^{(1)}$. Let $\mathcal{A}^{(2)}$ be the matrix corresponding to the system for $(u_1^{(2)}, u_2^{(2)})$. We remark that the positive and negative eigenvalues are changed into $\lambda_1 := \lambda_1^{(1)} (\lambda_0^{(1)})^{-1} \notin \{0, 1\}$ and $\lambda_2 := \lambda_2^{(1)} (\lambda_0^{(1)})^{-1} \notin \{0, 1\}$, respectively. Then, one has $\lambda_1 > \lambda_2$ and

$${}^t(1, 0, 1) \in W(\lambda_1; \mathcal{A}^{(2)}), \quad W(\lambda_2; \mathcal{A}^{(2)}) \cap \mathcal{P}_+ \neq \emptyset, \quad W(1; \mathcal{A}^{(2)}) \neq \{0\}$$

as desired.

We then apply further change of variable to simplify the eigenspace associated with eigenvalue λ_2 . Pick

$$\vec{q} = {}^t(q_1, q_2, q_3) \in W(\lambda_2; \mathcal{A}^{(2)}) \cap \mathcal{P}_+.$$

Define $(u_1^{(3)}, u_2^{(3)})$ by

$$\begin{bmatrix} u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} = \mathcal{M}^{(3)} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}, \quad \mathcal{M}^{(3)} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

for $\theta \in \mathbb{R}$ and let $\mathcal{A}^{(3)}$ be the matrix corresponding to the system for $(u_1^{(3)}, u_2^{(3)})$. This does not change eigenvalues since $\det \mathcal{M}^{(3)} = 1$. One has $\mathcal{D}(\mathcal{M}^{(3)}) {}^t(1, 0, 1) = {}^t(1, 0, 1)$ for any θ , which implies ${}^t(1, 0, 1) \in W(\lambda_1; \mathcal{A}^{(3)})$ for any θ . We choose θ so that $\mathcal{D}(\mathcal{M}^{(3)})\vec{q}$ is of the form ${}^t(\alpha, 0, \beta)$. If $q_2 = 0$, we let $\theta = 0$. Otherwise, since the second component of the vector $\mathcal{D}(\mathcal{M}^{(3)})\vec{q}$ is explicitly given as

$$\frac{q_3 - q_1}{2} \sin 2\theta + q_2 \cos 2\theta,$$

we choose θ so that $\cot 2\theta = \frac{q_1 - q_3}{2q_2}$. Then, one sees that

$$W(\lambda_2; \mathcal{A}^{(3)}) = \operatorname{span}\{{}^t(1, 0, k)\}$$

for some positive $k \neq 1$. Note that k is positive since $W(\lambda_2; \mathcal{A}^{(3)}) \cap \mathcal{P}_+ \neq \emptyset$ and that $k \neq 1$ follows from $W(\lambda_1; \mathcal{A}^{(3)}) \cap W(\lambda_2; \mathcal{A}^{(3)}) = \{0\}$.

If $k > 1$ then we further apply

$$\begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} = \mathcal{M}^{(4)} \begin{bmatrix} u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}, \quad \mathcal{M}^{(4)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $\mathcal{A}^{(4)}$ be the matrix corresponding the system for $(u_1^{(4)}, u_2^{(4)})$. Since $\det \mathcal{M}^{(4)} = 1$, the eigenvalues do not change. Further, since

$$\mathcal{D}(\mathcal{M}^{(4)}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

we see from Proposition 4.1 that $W(\lambda_1; \mathcal{A}^{(4)}) = \operatorname{span}\{{}^t(1, 0, 1)\}$ and

$$W(\lambda_2; \mathcal{A}^{(4)}) = \operatorname{span}\{{}^t(k, 0, 1)\} = \operatorname{span}\{{}^t(1, 0, k^{-1})\}.$$

If $0 < k < 1$ then we simply let $\mathcal{A}^{(4)} = \mathcal{A}^{(3)}$. In either case, one has

$$W(1; \mathcal{A}^{(4)}) = \text{span}\{{}^t(\eta_2, 1, \eta_3)\}$$

with $\eta_2, \eta_3 \in \mathbb{R}$. Note that the second component of the eigenvector does not vanish since otherwise we have $W(\lambda_1; \mathcal{A}^{(4)}) \oplus W(\lambda_2; \mathcal{A}^{(4)}) \oplus W(1; \mathcal{A}^{(4)}) \subsetneq \mathbb{R}^3$, a contradiction.

Thus, by slightly changing the notation, we have

$$\begin{aligned} W(\lambda_1; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1, 0, 1)\}, \\ W(\lambda_2; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1 + \eta_1, 0, \eta_1)\}, \\ W(1; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(\eta_2, 1, \eta_3)\} \end{aligned}$$

with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}$, $\lambda_1 > \lambda_2$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} \mathcal{A}^{(4)} &= \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 - \lambda_2) & \eta_2(1 - \lambda_1) + (1 + \eta_1)(\eta_2 - \eta_3)(\lambda_1 - \lambda_2) & (1 + \eta_1)(\lambda_1 - \lambda_2) \\ 0 & 1 & 0 \\ -\eta_1(\lambda_1 - \lambda_2) & \eta_3(1 - \lambda_1) + \eta_1(\eta_2 - \eta_3)(\lambda_1 - \lambda_2) & \lambda_1 + \eta_1(\lambda_1 - \lambda_2) \end{bmatrix}, \end{aligned}$$

which is the desired standard form.

Subcase 1-(b). We then move onto the subcase (b). Define $\mathcal{A}^{(1)}$ as in the subcase (a). We modified the definition of $\mathcal{A}^{(2)}$. Set the magnitude of the matrix so that λ_2 becomes -1 . Define $(u_1^{(2)}, u_2^{(2)})$ by $u_j^{(2)} = |\lambda_2^{(1)}|^{\frac{1}{2}} u_j^{(1)}$. Let $\mathcal{A}^{(2)}$ be the matrix corresponding to the system for $(u_1^{(2)}, u_2^{(2)})$. Set $\lambda_1 := \lambda_1^{(1)} |\lambda_2^{(1)}|^{-1} \in (-1, 0) \cup (0, 1]$. One has

$${}^t(1, 0, 1) \in W(\lambda_1; \mathcal{A}^{(2)}), \quad W(-1; \mathcal{A}^{(2)}) \cap \mathcal{P}_+ \neq \emptyset, \quad \text{rank } \mathcal{A}^{(2)} = 2.$$

The rest is the same as in subcase (a). By defining $\mathcal{A}^{(3)}$ and $\mathcal{A}^{(4)}$ via exactly the same way, one obtains

$$\begin{aligned} W(\lambda_1; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1, 0, 1)\}, \\ W(-1; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1 + \eta_1, 0, \eta_1)\}, \\ W(0; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(\eta_2, 1, \eta_3)\} \end{aligned}$$

with $\lambda_1 \in (-1, 0) \cup (0, 1]$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} \mathcal{A}^{(4)} &= \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 + 1) & -\eta_2\lambda_1 + (1 + \eta_1)(\eta_2 - \eta_3)(\lambda_1 + 1) & (1 + \eta_1)(\lambda_1 + 1) \\ 0 & 0 & 0 \\ -\eta_1(\lambda_1 + 1) & -\eta_3\lambda_1 + \eta_1(\eta_2 - \eta_3)(\lambda_1 + 1) & \lambda_1 + \eta_1(\lambda_1 + 1) \end{bmatrix}, \end{aligned}$$

which is of the desired form.

Case 2. Let us consider the exceptional case, i.e., $\lambda_0^{(0)}$ is equal to $\lambda_1^{(0)}$ or $\lambda_2^{(0)}$. By changing $(u_1^{(0)}, u_2^{(0)}) \mapsto (u_1^{(0)}, -u_2^{(0)})$ if necessary and by relabeling the eigenvalues if necessary, we may suppose without loss of generality that $\mathcal{A}^{(0)}$ has exactly two distinct eigenvalues $\lambda_0^{(0)} > 0$ and $\lambda^{(0)} \notin \{0, \lambda_0^{(0)}\}$. By assumption, we have $W(\lambda_0^{(0)}) \cap \mathcal{P}_+ \neq \emptyset$ and $W(\lambda^{(0)}) \cap \mathcal{P}_+ \neq \emptyset$. There exist two subcases;

- (a) $\dim W(\lambda_0^{(0)}) = 2$;
- (b) $\dim W(\lambda_0^{(0)}) = 1$,

In the both cases, the procedure in Subcase 1-(a) works. The only difference is that one of λ_1 and λ_2 is equal to λ_0 . Further, in the Subcase 2-(b), $\mathcal{A}^{(0)}$ is not diagonalized and the Jordan block of size two appears. One has

$$\begin{aligned} W(1; \mathcal{A}^{(4)}) &\supset \text{span}\{{}^t(1, 0, 1)\}, \\ \widetilde{W}(1; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1, 0, 1), {}^t(\eta_2, 1, \eta_3)\}, \\ W(\lambda; \mathcal{A}^{(4)}) &= \text{span}\{{}^t(1 + \eta_1, 0, \eta_1)\} \end{aligned}$$

with $\lambda \in \mathbb{R} \setminus \{0, 1\}$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} \mathcal{A}^{(4)} &= \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \eta_4 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ 1 & \eta_1 & \eta_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 - (1 + \eta_1)(1 - \lambda) & (1 + \eta_1)(\eta_2 - \eta_3)(1 - \lambda) + \eta_4 & (1 + \eta_1)(1 - \lambda) \\ 0 & 1 & 0 \\ -\eta_1(1 - \lambda) & \eta_1(\eta_2 - \eta_3)(1 - \lambda) + \eta_4 & 1 + \eta_1(1 - \lambda) \end{bmatrix} \end{aligned}$$

with $\lambda \notin \{0, 1\}$, $\eta_1 > 0$, and $\eta_2, \eta_3, \eta_4 \in \mathbb{R}$. Note that $\eta_4 = 0$ and $\eta_4 \neq 0$ correspond to the Subcases 2-(a) and 2-(b), respectively. This is the desired standard forms. $\square$

Proof of Theorem 1.7. Pick an ODE system (1.9) satisfying Assumption 2 and let $\mathcal{A}^{(0)}$ be the matrix part of the system. Let $(u_1^{(0)}, u_2^{(0)})$ be the unknown for the system. Let $\lambda_j^{(0)}$ be the three distinct eigenvalues such that $\lambda_1^{(0)}/\lambda_3^{(0)} < \lambda_2^{(0)}/\lambda_3^{(0)} < 1$, $(W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \oplus W(\lambda_2^{(0)}; \mathcal{A}^{(0)})) \cap \mathcal{P}_+ \neq \emptyset$, $W(\lambda_3^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_+ \neq \emptyset$, and $W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_+ = W(\lambda_2^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_+ = \emptyset$. By changing $(u_1^{(0)}, u_2^{(0)}) \mapsto (u_1^{(0)}, -u_2^{(0)})$ if necessary, we may suppose that $\lambda_3^{(0)} < 0$ without loss of generality. One has $\lambda_1^{(0)} > \lambda_2^{(0)} > \lambda_3^{(0)}$. There are four subcases:

- (a) $W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \subset \mathcal{P}_0$ and $W(\lambda_2^{(0)}; \mathcal{A}^{(0)}) \subset \mathcal{P}_0$;
- (b) $W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \subset \mathcal{P}_0$ and $W(\lambda_2^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_0 = \{0\}$;
- (c) $W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_0 = \{0\}$ and $W(\lambda_2^{(0)}; \mathcal{A}^{(0)}) \subset \mathcal{P}_0$;
- (d) $W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_0 = \{0\}$ and $W(\lambda_2^{(0)}; \mathcal{A}^{(0)}) \cap \mathcal{P}_0 = \{0\}$.

Subcase (a). By assumption, one can choose $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ with $\theta_1 \neq \theta_2$ such that

$$\vec{p}_j = {}^t(1 + \sin \theta_j, \cos \theta_j, 1 - \sin \theta_j) \in W(\lambda_j^{(0)}; \mathcal{A}^{(0)})$$

for $j = 1, 2$. With this θ_j and $\sigma_0 \in \{\pm 1\}$, we introduce

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \mathcal{M}^{(1)} \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}, \quad \mathcal{M}^{(1)} = \begin{bmatrix} \sigma_0 \sqrt{1 + \sin \theta_1} & \sigma_0 \frac{\cos \theta_1}{\sqrt{1 + \sin \theta_1}} \\ \sqrt{1 + \sin \theta_2} & \frac{\cos \theta_2}{\sqrt{1 + \sin \theta_2}} \end{bmatrix}$$

if $\theta_1, \theta_2 \neq 3\pi/2$. If $\theta_1 = 3\pi/2 \neq \theta_2$ or $\theta_1 \neq 3\pi/2 = \theta_2$, we define

$$\begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{1 + \sin \theta_2} & \frac{\cos \theta_2}{\sqrt{1 + \sin \theta_2}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sqrt{1 + \sin \theta_1} & \frac{\cos \theta_1}{\sqrt{1 + \sin \theta_1}} \\ 0 & \sqrt{2} \end{bmatrix},$$

respectively. We choose $\sigma_0 \in \{\pm 1\}$ so that $\det \mathcal{M}^{(1)} > 0$. Let $\mathcal{A}^{(1)}$ be the matrix corresponds to the system for $(u_1^{(1)}, u_2^{(1)})$. One sees by a direct computation that

$$\mathcal{D}(\mathcal{M}^{(1)})\vec{p}_1 = {}^t(c, 0, 0), \quad \mathcal{D}(\mathcal{M}^{(1)})\vec{p}_2 = {}^t(0, 0, c)$$

with the constant $c = (\det \mathcal{M}^{(1)})^{-1}2(1 - \cos(\theta_1 - \theta_2)) \neq 0$. Hence,

$$W(\lambda_1^{(1)}; \mathcal{A}^{(1)}) = \text{span}\{{}^t(1, 0, 0)\}, \quad W(\lambda_2^{(1)}; \mathcal{A}^{(1)}) = \text{span}\{{}^t(0, 0, 1)\},$$

where $\lambda_j^{(1)} := (\det \mathcal{M}^{(1)})^{-1}\lambda_j^{(0)}$. One has

$$W(\lambda_3^{(1)}; \mathcal{A}^{(1)}) = \text{span}\{{}^t(\eta_2, 1, \eta_3)\}$$

for some $\eta_2, \eta_3 \in \mathbb{R}$ with $\eta_2\eta_3 > 1$. The condition comes from $(\eta_2, 1, \eta_3) \in \mathcal{P}_+$.

We next apply

$$\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \mathcal{M}^{(2)} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}, \quad \mathcal{M}^{(2)} = \begin{bmatrix} (\eta_2/\eta_3)^{1/4}|\lambda_3^{(1)}|^{1/2} & 0 \\ 0 & (\eta_2/\eta_3)^{-1/4}|\lambda_3^{(1)}|^{1/2} \end{bmatrix}.$$

Let $\mathcal{A}^{(2)}$ be the matrix corresponds to the system for $(u_1^{(2)}, u_2^{(2)})$. Let $\lambda_j := \lambda_j^{(1)}|\lambda_3^{(1)}|^{-1}$ for $j = 1, 2$. One has $\lambda_1 > \lambda_2 > \lambda_3 = -1$ and

$$W(\lambda_1; \mathcal{A}^{(2)}) = \text{span}\{{}^t(1, 0, 0)\}, \quad W(\lambda_2; \mathcal{A}^{(2)}) = \text{span}\{{}^t(0, 0, 1)\}, \quad W(-1; \mathcal{A}^{(2)}) = \text{span}\{{}^t(\eta, 1, \eta)\}$$

with a number $\eta \in (-\infty, -1) \cup (1, \infty)$. Thus,

$$\mathcal{A}^{(2)} = \begin{bmatrix} 1 & 0 & \eta \\ 0 & 0 & 1 \\ 0 & 1 & \eta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \eta \\ 0 & 0 & 1 \\ 0 & 1 & \eta \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1 & -\eta(\lambda_1 + 1) & 0 \\ 0 & -1 & 0 \\ 0 & -\eta(\lambda_2 + 1) & \lambda_2 \end{bmatrix}.$$

This is of the desired form.

Subcases (b) and (c). Let us consider Subcase (b). We generalize the assumption $\lambda_1^{(0)} > \lambda_2^{(0)}$ into $\lambda_1^{(0)} \neq \lambda_2^{(0)}$ in this subcase so that Subcase (c) is handled in a unified way. As in Subcase (a), one can find $\theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\vec{p}_2 = {}^t(1 + \sin \theta_2, \cos \theta_2, 1 - \sin \theta_2) \in W(\lambda_2^{(0)}; \mathcal{A}^{(0)})$$

With this θ_2 , we introduce

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \mathcal{M}^{(1)} \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}, \quad \mathcal{M}^{(1)} = \begin{bmatrix} \sqrt{1 + \sin \theta_2} & \frac{\cos \theta_2}{\sqrt{1 + \sin \theta_2}} \\ 0 & 1/\sqrt{1 + \sin \theta_2} \end{bmatrix} \in SL_2(\mathbb{R})$$

if $\theta_2 \neq 3\pi/2$. If $\theta_2 = 3\pi/2$, we define

$$\mathcal{M}^{(1)} = \begin{bmatrix} 0 & \sqrt{2} \\ -1/\sqrt{2} & 0 \end{bmatrix} \in SL_2(\mathbb{R}).$$

Let $\mathcal{A}^{(1)}$ be the new matrix part. One sees that the eigenvalues do not change and

$$W(\lambda_2^{(0)}; \mathcal{A}^{(1)}) = \text{span}\{{}^t(1, 0, 0)\}.$$

Note that

$$W(\lambda_1^{(0)}; \mathcal{A}^{(1)}) = \text{span}\{{}^t(a, b, 1)\}$$

with some numbers $a, b \in \mathbb{R}$ such that $a < b^2$. Note that the third component of a nonzero element of $W(\lambda_1^{(0)}; \mathcal{A}^{(1)})$ do not vanish since otherwise $(W(\lambda_1^{(0)}; \mathcal{A}^{(1)}) \oplus W(\lambda_2^{(0)}; \mathcal{A}^{(1)})) \cap \mathcal{P}_+ = \emptyset$, a contradiction.

We next apply the following change of variables:

$$\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \mathcal{M}^{(2)} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}, \quad \mathcal{M}^{(2)} = |\lambda_3|^{\frac{1}{2}}(1+b)^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ \sqrt{b^2-a} & 1+b \end{bmatrix}.$$

Let $\mathcal{A}^{(2)}$ be the new matrix. Then, since $\det \mathcal{M}^{(2)} = |\lambda_3| > 0$, one has

$W(\lambda_1; \mathcal{A}^{(2)}) = \text{span}\{^t(1, 0, -1)\}$, $W(\lambda_2; \mathcal{A}^{(2)}) = \text{span}\{^t(1, 0, 0)\}$, $W(-1; \mathcal{A}^{(2)}) \cap \mathcal{P}_+ \neq \emptyset$, where $\lambda_j := \lambda_j^{(0)} |\lambda_3^{(0)}|^{-1}$ for $j = 1, 2$. Then, $\lambda_1 > -1$ and $\lambda_2 > -1$. Thus, we obtain

$$\begin{aligned} \mathcal{A}^{(2)} &= \begin{bmatrix} 1 & 1 & \eta_2 \\ 0 & 0 & 1 \\ -1 & 0 & \eta_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \eta_2 \\ 0 & 0 & 1 \\ -1 & 0 & \eta_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \lambda_2 & -\eta_2(\lambda_2 + 1) + \eta_3(\lambda_1 - \lambda_2) & -(\lambda_1 - \lambda_2) \\ 0 & -1 & 0 \\ 0 & -\eta_3(\lambda_1 + 1) & \lambda_1 \end{bmatrix} \end{aligned}$$

with $\eta_2\eta_3 > 1$. This is the special case $\eta_1 = 0$ of $\mathcal{A}_{2,2}$.

Subcase (d). One writes

$$W(\lambda_1^{(0)}; \mathcal{A}^{(0)}) = \text{span}\{^t(p_1, p_2, p_3)\}$$

with $p_1 p_3 < p_2^2$. We introduce

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \mathcal{M}^{(1)} \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}$$

with

$$\mathcal{M}^{(1)} = \begin{bmatrix} \sigma_0 p_1 & \sigma_0 p_2 \\ 0 & (p_2^2 - p_1 p_3)^{1/2} \end{bmatrix}$$

if $p_1 \neq 0$,

$$\mathcal{M}^{(1)} = \begin{bmatrix} \sigma_0 p_2 & 0 \\ p_2 & p_3 \end{bmatrix}$$

if $p_1 = 0$ and $p_3 \neq 0$, and

$$\mathcal{M}^{(1)} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

if $p_1 = p_3 = 0$, where $\sigma_0 \in \{\pm 1\}$ is chosen so that $\det \mathcal{M}^{(1)} > 0$. Let $\mathcal{A}^{(1)}$ be the new matrix part and let $\lambda_j^{(1)} := (\det \mathcal{M}^{(1)}) \lambda_j^{(0)}$ be the eigenvalues of the new matrix. Then, one sees that

$$W(\lambda_1^{(1)}; \mathcal{A}^{(1)}) = \text{span}\{(1, 0, -1)\}.$$

Let us next simplify the characteristic space associated with eigenvalue $\lambda_2^{(1)}$. Pick

$$\vec{q} = {}^t(q_1, q_2, q_3) \in W(\lambda_2^{(1)}; \mathcal{A}^{(1)}) \setminus \{0\}.$$

Note that $q_1 q_3 - q_2^2 < 0$. Since $W(\lambda_2^{(1)}; \mathcal{A}^{(1)}) \cap \mathcal{P}_+ = \emptyset$ and $(W(\lambda_1^{(1)}; \mathcal{A}^{(1)}) \oplus W(\lambda_2^{(1)}; \mathcal{A}^{(1)})) \cap \mathcal{P}_+ \neq \emptyset$, one has $\max_{t \in \mathbb{R}} ((q_3 + t)(q_1 - t) - q_2^2) > 0$, which reads as $2|q_2| < |q_1 + q_3|$. In particular, $q_1 + q_3 \neq 0$. We define

$$\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \mathcal{M}^{(2)} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}$$

with

$$\mathcal{M}^{(2)} = \begin{bmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{bmatrix} \in SL_2(\mathbb{R}), \quad \tau := \frac{1}{2} \tanh^{-1} \left(\frac{2q_2}{q_1 + q_3} \right) \in \mathbb{R}.$$

Let $\mathcal{A}^{(2)}$ be the new matrix. A computation shows that $\mathcal{D}(\mathcal{M}^{(2)})^t(1, 0, -1) = {}^t(1, 0, -1)$ and that the second component of $\mathcal{D}(\mathcal{M}^{(2)})\vec{q}$ vanishes. Hence,

$$W(\lambda_2^{(1)}; \mathcal{A}^{(2)}) = \text{span}\{{}^t(1, 0, -1)\}, \quad W(\lambda_2^{(1)}; \mathcal{A}^{(2)}) = \text{span}\{{}^t(1, 0, -\tilde{q}_3)\}$$

with some $\tilde{q}_3 \in (0, 1) \cup (1, \infty)$. We may suppose $\tilde{q}_3 \in (0, 1)$ by applying $(u_1^{(2)}, u_2^{(2)}) \mapsto (u_2^{(2)}, -u_1^{(2)})$ if necessary, as in Subcase 1-(a) of the proof of Theorem 1.6. We define $\eta_1 = \tilde{q}_3/(1 - \tilde{q}_3) \in (0, \infty)$ so that $W(\lambda_2^{(1)}; \mathcal{A}^{(2)}) = \text{span}\{{}^t(1 + \eta_1, 0, -\eta_1)\}$.

We finally adjust the magnitude of the matrix by $u_j^{(3)} := |\lambda_3^{(1)}|^{1/2} u_j^{(2)}$ so that the minimum eigenvalue becomes -1 . Let $\mathcal{A}^{(3)}$ be the new matrix. Let $\lambda_j := \lambda_j^{(1)}/|\lambda_3^{(1)}|$ ($j = 1, 2$) be the positive eigenvalues of $\mathcal{A}^{(3)}$. We have

$$W(\lambda_1; \mathcal{A}^{(3)}) = \text{span}\{{}^t(1, 0, -1)\},$$

$$W(\lambda_2; \mathcal{A}^{(3)}) = \text{span}\{{}^t(1 + \eta_1, 0, -\eta_1)\},$$

$$W(-1; \mathcal{A}^{(3)}) = \text{span}\{{}^t(\eta_2, 1, \eta_3)\}$$

with $\eta_1 > 0$ and $\eta_2\eta_3 > 1$. Thus, we obtain

$$\begin{aligned} \mathcal{A}^{(3)} &= \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ -1 & -\eta_1 & \eta_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 + \eta_1 & \eta_2 \\ 0 & 0 & 1 \\ -1 & -\eta_1 & \eta_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \lambda_1 - (1 + \eta_1)(\lambda_1 - \lambda_2) & (1 + \eta_1)(\eta_2 + \eta_3)(\lambda_1 - \lambda_2) - (\lambda_1 + 1)\eta_2 & -(1 + \eta_1)(\lambda_1 - \lambda_2) \\ 0 & -1 & 0 \\ \eta_1(\lambda_1 - \lambda_2) & -\eta_1(\eta_2 + \eta_3)(\lambda_1 - \lambda_2) - (\lambda_1 + 1)\eta_3 & \eta_1(\lambda_1 - \lambda_2) + \lambda_1 \end{bmatrix} \end{aligned}$$

with $\eta_1 > 1$ and $\eta_2\eta_3 > 1$. This corresponds to the case $\eta_1 > 0$ of $\mathcal{A}_{2,2}$. $\square$

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