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Reliability function analysis of a linear oscillator subjected to non-Gaussian random excitation using the equivalent non-Gaussian excitation method and the Hermite moment model

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This study develops an approximate analytical technique for evaluating the reliability function of a single-degree-of-freedom linear system subjected to non-Gaussian random excitation. The excitation is modeled as a stationary stochastic process characterized by an arbitrary probability density function (PDF) and a power spectral density (PSD) with a bandwidth parameter, and is described by an Itô stochastic differential equation. In the proposed approach, the equivalent non-Gaussian excitation method is first employed to derive moment equations of the system response, enabling the evaluation of response moments up to the fourth order. Based on these response moments, the Hermite moment model is then used to estimate the mean upcrossing rate of the displacement response under two approximations: statistical independence between displacement and velocity responses, and Gaussianity of the velocity response. The reliability function associated with first-passage failure is subsequently determined from the mean upcrossing rate by assuming that threshold crossings follow a Poisson process. Numerical examples demonstrate that the proposed method accurately estimates the reliability function for various excitation PDFs with different tail heaviness. Furthermore, it is shown that heavier-tailed excitation PDFs lead to lower system reliability.

Keywords: Stochastic dynamics, dynamic reliability analysis, probabilistic method, non-Gaussian random excitation, first passage failure, Tail heaviness.

1 Introduction

Dynamic reliability analysis under random excitation is essential for evaluating the safety and durability of building structures and mechanical systems. In conventional probabilistic structural dynamics, stochastic external forces acting on dynamic systems are generally assumed to be Gaussian. However, it has been widely reported that real-world random excitations often deviate from Gaussianity. Typical examples of non-Gaussian random excitations include pressure fluctuations induced by strong winds and turbulence (Zhang 2023), earthquake ground motions (Grigoriu 1995), and shallow-water waves (Ochi 1986). Importantly, when non-Gaussian stochastic excitation follows a heavy-tailed probability density function (PDF), large-amplitude events that rarely appear in Gaussian excitation can occur intermittently.

Through previous studies on dynamic systems exposed to non-Gaussian random excitation, it has been clarified that the non-Gaussianity of excitation is generally transmitted to the system response, and consequently, the probabilistic characteristics of the response under non-Gaussian excitation can differ significantly from those under Gaussian excitation (Tsuchida and Kimura 2022, Tsuchida and Kimura 2023). Therefore, neglecting excitation non-Gaussianity and performing reliability analysis under the Gaussian assumption may lead to inaccurate evaluation of system safety and risk. In particular, when the excitation PDF has heavy tails and produces large-amplitude events not observed in Gaussian excitation, its influence on system reliability is expected to be substantial. For this reason, it is crucial to establish reliability evaluation methods that appropriately account for excitation non-Gaussianity, especially the tail heaviness of the PDF.

Motivated by this background, this study aims to develop an approximate analytical method for evaluating the reliability function of a single-degree-of-freedom (SDOF) linear system subjected to non-Gaussian random excitation. In this study, first-passage failure is considered, where system failure occurs when the response exceeds a specified threshold (tolerance limit) for the first time. This failure mode is one of the most fundamental and important mechanisms associated with dynamically induced failure. The non-Gaussian random excitation is modeled as a stationary stochastic process characterized by an arbitrary PDF and a low-pass-type power spectral density (PSD), and is described using an Itô stochastic differential equation. In the proposed method, the mean upcrossing rate of the system response (i.e., the expected number of times per unit time that the response exceeds a specified threshold) is first evaluated following the analytical framework developed in the author's previous study (Tsuchida and Kimura 2023). By assuming that threshold-crossing events of the response follow a Poisson process, an approximate solution for the system reliability function associated with first-passage failure can then be obtained using the mean upcrossing rate. The validity of the proposed method is verified by comparing the analytical reliability functions with the corresponding results obtained from Monte Carlo simulations (MCSs). Furthermore, by comparing reliability functions

for several excitation PDFs with different tail heaviness, it is demonstrated that properly accounting for excitation non-Gaussianity is critically important for accurate reliability evaluation of randomly excited dynamic systems.

2 System Equation of Motion and Non-Gaussian Random Excitation

2.1 *Equation of Motion*

We consider an SDOF linear system described by the following nondimensional equation of motion:

$$\ddot{X}(t) + 2\zeta\dot{X}(t) + X(t) = U(t) \tag{1}$$

where $X(t)$ is the displacement response of the system, ζ is the damping ratio, $U(t)$ is the non-Gaussian random excitation, and t denotes time nondimensionalized by the undamped natural angular frequency of the system. In this study, the damping ratio is set to $\zeta = 0.05$. The present reliability analysis considers stationary system responses.

2.2 *Non-Gaussian Random Excitation*

The non-Gaussian excitation $U(t)$ is modeled as a stationary stochastic process with zero mean, characterized by a PDF $p_U(u)$ and a PSD $S_U(\omega)$. In this study, the PSD is assumed to take the following low-pass form:

$$S_U(\omega) = \frac{\alpha \mathrm{E}[U^2]}{\pi(\omega^2 + \alpha^2)} \tag{2}$$

where $\mathrm{E}[U^2]$ is the second-order moment of the excitation, and $\alpha > 0$ is the excitation bandwidth parameter. The PSD has a peak at $\omega = 0$. When α is small, $S_U(\omega)$ becomes strongly concentrated around zero frequency, whereas larger values of α result in broader spectral bandwidth.

To investigate the influence of excitation PDF tail heaviness on system reliability, the excitation PDF $p_U(u)$ is given by the generalized Gaussian distribution. The zero-mean generalized Gaussian distribution is defined as (Varanasi and Aazhang 1989, Sharifi and Leon-Garcia 1995)

$$p_U(u) = \frac{K}{2\theta\Gamma\left(\frac{1}{K}\right)} \exp\left(-\frac{|u|^K}{\theta^K}\right) \tag{3}$$

where $\Gamma(\cdot)$ denotes the gamma function, $\theta > 0$ is the scale parameter, and $K > 0$ is the shape parameter. When $K = 2$, the distribution reduces to the well-known Gaussian distribution. In this study, the parameter θ is chosen such that the variance of $p_U(u)$ becomes unity. By varying K, PDFs with different tail heaviness can be represented. Specifically, $K < 2$ corresponds to heavier-tailed distributions than the Gaussian distribution, whereas $K > 2$ corresponds to lighter-tailed distributions.

A zero-mean stationary process $U(t)$ with an arbitrary non-Gaussian PDF $p_U(u)$ and PSD $S_U(\omega)$ given by Eq. (2) can be represented by the following Itô stochastic differential equation (Cai and Lin 1996):

$$dU(t) = -\alpha U(t)dt + D\big(U(t)\big)dB(t) \tag{4}$$

where α is the excitation bandwidth parameter and $B(t)$ is standard Brownian motion. The diffusion coefficient $D^2(\cdot)$ is determined by

$$D^2(u) = -\frac{2\alpha}{p_U(u)} \int_{-\infty}^{u} s p_U(s) ds \tag{5}$$

For the generalized Gaussian distribution, the diffusion coefficient $D^2(\cdot)$ is given by

$$D^2(u) = \frac{2\alpha\theta^2}{K} \Gamma\left(\frac{2}{K}, \frac{|u|^K}{\theta^K}\right) \exp\left(\frac{|u|^K}{\theta^K}\right) \tag{6}$$

where $\Gamma(\cdot,\cdot)$ denotes the incomplete gamma function of the second kind.

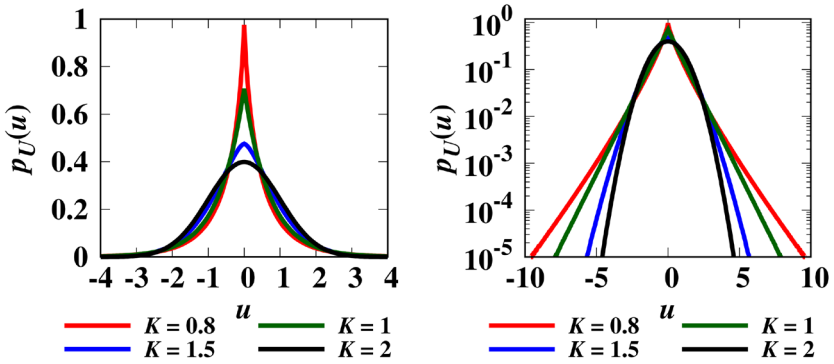

Figure 1. Generalized Gaussian distribution with zero mean and unit variance (left: linear scale; right: semi log scale).

3 Proposed Method

3.1 *Calculation of Response Moments Using the Equivalent Non-Gaussian Excitation Method*

Let $X_1 = X$, $X_2 = \dot{X}$, and $X_3 = U$. By applying the equivalent non-Gaussian excitation method to Eqs. (1) and (4), the stationary moment equations of order $(m_1 + m_2 + m_3)$ can be derived as (Tsuchida and Kimura 2023)

$$\begin{aligned} &m_1 \mathrm{E}\left[X_1^{m_1-1} X_2^{m_2+1} X_3^{m_3}\right] - m_2\left(\mathrm{E}\left[X_1^{m_1+1} X_2^{m_2-1} X_3^{m_3}\right] + 2\zeta \mathrm{E}[X_1^{m_1} X_2^{m_2} X_3^{m_3}] - \mathrm{E}\left[X_1^{m_1} X_2^{m_2} X_3^{m_3+1}\right]\right) \\ &\quad -m_3 \alpha \mathrm{E}[X_1^{m_1} X_2^{m_2} X_3^{m_3}] + \frac{1}{2} m_3(m_3-1)\left(A_{eq}\mathrm{E}[X_1^{m_1} X_2^{m_2} X_3^{m_3}] + B_{eq}\mathrm{E}\left[X_1^{m_1} X_2^{m_2} X_3^{m_3-1}\right]\right) \\ &\quad +\frac{1}{2} m_3(m_3-1) C_{eq}\mathrm{E}\left[X_1^{m_1} X_2^{m_2} X_3^{m_3-2}\right] = 0 \end{aligned} \tag{7}$$

where

$$A_{eq} = \frac{\alpha}{M}\left[2\mathrm{E}[X_3^2]\left\{\frac{1}{3}\mathrm{E}[X_3^4] - (\mathrm{E}[X_3^2])^2\right\} - (\mathrm{E}[X_3^3])^2\right], B_{eq} = \frac{\alpha \mathrm{E}[X_3^3]}{M}\left\{\frac{1}{3}\mathrm{E}[X_3^4] + (\mathrm{E}[X_3^2])^2\right\},$$
$$C_{eq} = (2\alpha - A_{eq})\mathrm{E}[X_3^2], M = \mathrm{E}[X_3^2]\{\mathrm{E}[X_3^4] - (\mathrm{E}[X_3^2])^2\} - (\mathrm{E}[X_3^3])^2$$

The excitation moments $\mathrm{E}[X_3^n]$ $(n = 2,3,4)$ required to determine the coefficients A_{eq}, B_{eq} and C_{eq} can be directly calculated from the excitation PDF $p_U(u)$. By solving Eq. (7), exact response moments up to the fourth order can be obtained for arbitrary excitation PDF $p_U(u)$ and excitation bandwidth α.

3.2 *Calculation of Displacement Response PDF Using the Hermite Moment Model*

Using the response moments up to the fourth order obtained in the previous subsection, the stationary displacement response PDF $p_{X_1}(x_1)$ is evaluated using the Hermite moment model (Winterstein 1988), which is expressed as

$$p_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}k^2\left(\frac{x_1}{\sqrt{\mathrm{E}[X_1^2]}}\right)\right\} \frac{d}{dx_1} k\left(\frac{x_1}{\sqrt{\mathrm{E}[X_1^2]}}\right) \tag{8}$$

where the function $k(\cdot)$ depends on the sign of the displacement kurtosis $\gamma_{4X_1}(= \mathrm{E}[X_1^4]/(\mathrm{E}[X_1^2])^2 - 3)$.

(i) Case of positive kurtosis

$$k(y) = \left(\sqrt{\xi^2(y) + c} + \xi(y)\right)^{\frac{1}{3}} - \left(\sqrt{\xi^2(y) + c} - \xi(y)\right)^{\frac{1}{3}} - \frac{h_3}{3h_4} \tag{9}$$

where

$$\xi(y) = \frac{1}{2h_4}\left(\frac{h_3}{3h_4} + \sqrt{1 + 2h_3^2 + 6h_4^2 y}\right) - \left(\frac{h_3}{3h_4}\right)^3, c(t) = \left\{\frac{1}{3h_4} - 1 - \left(\frac{h_3}{3h_4}\right)^2\right\}^3 \tag{10}$$

The coefficients h_3 and h_4 are determined from the skewness $\gamma_{3X_1}(= \mathrm{E}[{X_1}^3]/(\mathrm{E}[{X_1}^2])^{1.5})$ and kurtosis γ_{4X_1} of the displacement response X_1 according to the following equations (Yang 2013):

$$\begin{aligned} h_3 = {} & 0.1967\gamma_{3X_1} - 0.01646\gamma_{3X_1}\left(\gamma_{4X_1}+3\right) + 0.01809\gamma_{3X_1}^3(t) + 7.438\times10^{-4}\gamma_{3X_1}\left(\gamma_{4X_1}+3\right)^2 \\ & -9.209\times10^{-4}\gamma_{3X_1}^3\left(\gamma_{4X_1}+3\right) - 1.366\times10^{-5}\gamma_{3X_1}\left(\gamma_{4X_1}+3\right)^3 + 1.527\times10^{-4}\gamma_{3X_1}^5 \\ & +1.07\times10^{-5}\gamma_{3X_1}^3\left(\gamma_{4X_1}+3\right)^2 + 8.823\times10^{-8}\gamma_{3X_1}\left(\gamma_{4X_1}+3\right)^4 \end{aligned} \tag{11}$$

$$\begin{aligned} h_4 = {} & -0.0721 + 0.03176\left(\gamma_{4X_1}+3\right) - 0.02942\gamma_{3X_1}^2 - 0.00179\left(\gamma_{4X_1}+3\right)^2 + 0.002348\gamma_{3X_1}^2\left(\gamma_{4X_1}+3\right) \\ & +5.965\times10^{-5}\left(\gamma_{4X_1}+3\right)^3 - 6.282\times10^{-4}\gamma_{3X_1}^4 - 6.355\times10^{-5}\gamma_{3X_1}^2\left(\gamma_{4X_1}+3\right)^2 \\ & -9.692\times10^{-7}\left(\gamma_{4X_1}+3\right)^4 + 1.497\times10^{-5}\gamma_{3X_1}^4(t)\left(\gamma_{4X_1}+3\right) + 5.457\times10^{-7}\gamma_{3X_1}^2\left(\gamma_{4X_1}+3\right)^3 \\ & +6.049\times10^{-9}\left(\gamma_{4X_1}+3\right)^5 \end{aligned} \tag{12}$$

(ii) Case of negative kurtosis

$$k(y) = \; y - \frac{\gamma_{3X_1}}{6}(y^2-1) - \frac{\gamma_{4X_1}}{24}(y^3-3y) \tag{13}$$

In addition, following Ref. (Tsuchida and Kimura 2023), the velocity response PDF $p_{X_2}(x_2)$ is approximated as a zero-mean Gaussian distribution having the second-order moment $\mathrm{E}[X_2^2]$ obtained from the moment equations.

3.3 *Calculation of Mean Upcrossing Rate*

The mean upcrossing rate $\nu_{X_1}^+(\lambda)$ means the expected number of times per unit time that the displacement response X_1 crosses a specified threshold $\lambda > 0$ with positive slope, and is mathematically defined as (Lin 1967):

$$\nu_{X_1}^+(\lambda) = \int_0^\infty s p_{X_1X_2}(\lambda, s) ds \tag{14}$$

where $p_{X_1X_2}(x_1, x_2)$ is the joint PDF of stationary displacement and velocity responses. To construct this joint PDF, the following two approximations are introduced (Tsuchida and Kimura 2023):

1. The velocity response X_2 follows a Gaussian distribution.
2. The displacement and velocity responses are statistically independent.

The second approximation is supported by the fact that displacement and velocity responses are uncorrelated. Under these assumptions, the mean upcrossing rate in Eq. (14) becomes

$$\nu_{X_1}^+(\lambda) = \sqrt{\frac{\mathrm{E}[X_2^2]}{2\pi}}\, p_{X_1}(\lambda) \tag{15}$$

3.4 *Derivation of Reliability Function for First-passage Failure*

In this study, first-passage failure is considered, where failure occurs when the system response exceeds a specified threshold for the first time. It is assumed that at the initial time $t = 0$, the response is already stationary and has not exceeded the threshold.

Assuming that threshold-crossing events follow a Poisson process, the probability $P(\lambda, n, T)$ that the displacement response $X_1(t)$ crosses the threshold λ, n times during time interval T is expressed as (Lin1967)

$$P(\lambda, n, T) = \frac{1}{n!}\left[\int_0^T \nu_{X_1}^+(\lambda)\, dt\right]^n \exp\left\{-\int_0^T \nu_{X_1}^+(\lambda)\, dt\right\} \tag{16}$$

By setting $n = 0$, the reliability function $R(\lambda, T)$ for first-passage failure is obtained as

$$R(\lambda, T) = \exp\left\{-\int_0^T \nu_{X_1}^+(\lambda)\, dt\right\} = \exp\left[-\nu_{X_1}^+(\lambda)T\right] \tag{17}$$

The proposed method can be applied to arbitrary excitation PDFs $p_U(u)$ as long as the combination of displacement skewness γ_{3X_1} and kurtosis γ_{4X_1} falls within the applicable range of the Hermite moment model, which is known to cover a wide range of skewness–kurtosis combinations.

4 Numerical Examples

The proposed method is applied while varying the shape parameter K of the generalized Gaussian distribution. Specifically, the following values of K are considered:

- $K = 2$ (kurtosis $\gamma_{4U} = 0$, Gaussian distribution)
- $K = 1.5$ ($\gamma_{4U} = 0.7620$)
- $K = 1$ ($\gamma_{4U} = 3$)
- $K = 0.8$ ($\gamma_{4U} = 5.565$)

The bandwidth parameter α of the excitation PSD $S_U(\omega)$ is set to $\alpha = 0.1$. The failure threshold λ is defined, based on the standard deviation σ_{X_1} of the stationary displacement X_1, as $\lambda = 3\sigma_{X_1}$. For $\alpha = 0.1$, the standard deviation is $\sigma_{X_1} = 1.4003$.

Figure 2 shows the calculated reliability functions $R(\lambda, T)$. In the figure, solid lines represent the analytical results obtained using the proposed method, whereas circular markers represent the corresponding MCS results computed from 100,000 sample paths.

It can be seen from Fig. 2 that the proposed method provides accurate reliability estimates for all considered values of the shape parameter K. In general, the proposed method slightly underestimates system reliability compared with the MCS results (conservative reliability estimation). This tendency originates from the Poisson process assumption for threshold crossing used in deriving the reliability function (Roberts 1978).

Furthermore, it is observed that smaller values of K, corresponding to heavier-tailed excitation PDFs, result in lower system reliability. This result indicates that if non-Gaussian excitation following a heavy-tailed PDF is approximated as Gaussian by neglecting its non-Gaussian characteristics, system reliability may be overestimated, leading to non-conservative safety evaluation. These results clearly demonstrate the importance of explicitly accounting for excitation non-Gaussianity, especially excitation PDF tail heaviness, when evaluating dynamic system reliability.

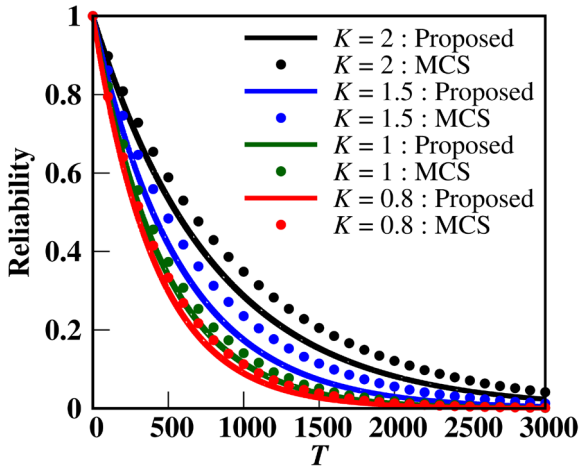

Figure 2. Comparison of reliability functions for four different excitation PDFs (threshold $\lambda = 3\sigma_{X_1}$).

5 Conclusions

In this study, an approximate analytical technique was developed to evaluate the reliability function associated with first-passage failure of an SDOF linear system under non-Gaussian random excitation. The random excitation was modeled as a stationary stochastic process characterized by its PDF and PSD, and was described using a stochastic differential equation. To investigate the influence of excitation PDF tail heaviness on system reliability, the generalized Gaussian distribution was adopted, which enables systematic variation of tail heaviness.

In the proposed method, the mean upcrossing rate of the displacement response was first evaluated using the equivalent non-Gaussian excitation method together with the Hermite moment model. By assuming that threshold-crossing events follow a Poisson process, the reliability function of the system was then determined using the mean upcrossing rate.

Through numerical examples, it was confirmed that the proposed method can estimate system reliability with good accuracy for various excitation PDFs with different tail heaviness. Furthermore, it was demonstrated that system reliability decreases as the excitation PDF becomes heavier-tailed.

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