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Author(s)	Wu, Xiaomin; Lu, Zhenzhou; Ding, Xu
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An improved single-loop importance sampling method based on progressive hierarchical strategy for generalized failure probability function under high-dimensional and high-reliability applications

XiaominWu[1], ZhenzhouLu[1,]*, Xu Ding
[1]*School of Aeronautics, Northwestern Polytechnical University, 127 West Youyi Road, Beilin District, Xi'an Shaanxi, 710072,China.*
E-mail: xiaominwu@mail.nwpu.edu.cn
[2] *School of Aeronautics, Northwestern Polytechnical University, 127 West Youyi Road, Beilin District, Xi'an Shaanxi, 710072,China.*
E-mail: zhenzhoulu@nwpu.edu.cn
[3] *School of Aeronautics, Northwestern Polytechnical University, 127 West Youyi Road, Beilin District, Xi'an Shaanxi, 710072,China.*
E-mail: xuding@mail.nwpu.edu.cn
**Corresponding Author*

For high-dimensional and high-reliability aerospace applications, accurately estimating how design-parameter variations influence structural reliability levels under probabilistic inputs and fuzzy states assumption is of paramount importance, specifically, the generalized failure-probability function. Therefore, this paper proposes an improved single-loop importance-sampling method based on a progressive hierarchical technique for efficiently estimating the generalized failure probability function. The key novelties include: first, establishing an analytical expression for the unified importance-sampling density that enables rapid sample generation; second, designing a progressive hierarchical technique that adaptively updates the importance-sampling center using samples with high contributions to the generalized failure probability estimation, ensuring accurate targeting of critical high-contribution regions. By circumventing computationally expensive optimization procedures through analytical construction and progressive stratification, the proposed approach substantially improves computational efficiency in high-dimensional and high-reliability scenarios while guaranteeing estimation accuracy. Numerical example demonstrate that the proposed method outperforms existing single-loop importance-sampling methods in sampling efficiency, estimation stability, and high-dimensional adaptability.

Keywords: Generalized Failure Probability Function; Fuzzy State; High-Dimensional; Importance Sampling Method

1 Introduction

Reliability analysis models can objectively evaluate the structural safety level by considering various uncertainties that affect structural output performance. Among these, reliability models based on probabilistic inputs and the binary state assumption, known as binary state reliability analysis models, have been widely studied. The probabilistic input assumption quantifies the randomness of input variables by specifying probability density function (PDF). The binary state assumption characterizes the structural state using an indicator function of the failure domain, representing complete failure and complete safety. However, for many engineering structures, failure occurs as a gradual degradation process without a clear boundary between safe and failed states. To address this limitation, a fuzzy failure domain can be introduced and described using a membership function, enabling a more refined representation of structural safety. This leads to fuzzy state reliability models (Ling 202) under probabilistic inputs, in which safety is quantified by the generalized failure probability (G-FP). In the generalized reliability optimization design process with G-FP as a constraint, the distribution parameters (DPs) of input variables are typically treated as design variables. By establishing the relationship between the G-FP and DPs within the design domain, i.e., the generalized failure probability function (G-FPF), the coupling between parameter optimization and constraint analysis in generalized reliability optimization can be decoupled.

The G-FPF can be directly evaluated using a double-loop analysis framework combined with Monte Carlo simulation (DL-MCS), where the outer loop discretizes the DPs and the inner loop estimates the G-FP for each parameter realization (Yang 2022). Although straightforward, this approach becomes computationally prohibitive in high-reliability problems due to the large number of required samples. To improve efficiency, advanced simulation techniques such as line sampling, subset simulation, and importance sampling have been introduced. Nevertheless, the intrinsic double-loop structure still limits efficiency due to repeated reliability evaluations. Given the limited research on G-FPF under fuzzy state models and its relation to failure probability functions in binary state models, this paper reviews existing methods for estimating such functions in high-dimensional, high-reliability settings.

To overcome the inefficiency of the double-loop framework, numerous single-loop analysis frameworks and corresponding solution methods have been proposed for binary state reliability models, broadly classified into Bayesian-based and unified density-based methods. The Bayesian-based single-loop framework, first proposed by Au

(Au 2005), decomposes the failure probability function into three components: the augmented failure probability, the conditional probability density of DPs given failure, and the prior distribution of the parameters. Since the prior distribution is predefined and the conditional probability is obtained as a by-product of augmented failure probability estimation, the complete failure probability function can be obtained through a single simulation, thereby eliminating the need for a double-loop structure. To further reduce computational cost, surrogate models, variance reduction techniques, and subset simulation have been incorporated. However, this approach suffers from high computational complexity in approximating the conditional probability density under failure conditions, particularly due to challenges in accurately capturing tail behavior. To address this issue, alternative methods based on differential approximation theory (Feng 2019) and iterative application of Bayesian formulas (Yuan 2021) have been proposed. Nevertheless, these methods still face issues of instability and high computational cost, especially in high-dimensional and extremely low failure probability scenarios.

In addition to the Bayesian-based single-loop analysis framework, a unified sampling density independent of DPs can be constructed to replace the original sample density and achieve sample information sharing. This is known as the unified density-based single-loop analysis framework. Several single-loop solution methods for failure probability functions have been derived based on various forms of unified density construction. Yuan et al. (Yuan 2013) constructed a local unified importance sampling density (ISD) to achieve single-loop failure probability function estimation. Later, a global solution method for failure probability functions (Yuan 2023) was developed using combinatorial optimization algorithms. Based on the metamodel importance sampling method (Dubourg 2013), Chen et al. (Chen 2024), Lu et al. (Lu 2023), and Wu et al. (Wu 2025) respectively proposed single-loop solutions for multi-failure modes, time-dependent failure probability functions, and fuzzy state G-FPFs. However, the unified importance sampling densities constructed in these methods are irregular and implicitly expressed, making sample extraction difficult. Although samples can be extracted using Markov Chain Monte Carlo (Papaioannou 2015) or acceptance-rejection methods (Straub 2015), these sampling methods are time-consuming in high-dimensional and small failure probability scenarios, and the accuracy and efficiency of Markov Chain Monte Carlo are still not well resolved, while the acceptance-rejection method is prohibitively costly.

From the above review, it is evident that existing single-loop methods for estimating G-FPF still face significant challenges in high-dimensional, high-reliability scenarios. Therefore, this paper constructs an explicit and easy-to-sample unified ISD and proposes an asymptotic hierarchical strategy to achieve efficient and robust estimation of the G-FPF in high-dimensional, high-reliability scenarios.

The remainder of this paper is organized as follows. Section 2 reviews existing double-loop and single-loop frameworks for G-FPF estimation. Section 3 presents the proposed asymptotic hierarchical importance sampling method. Section 4 provides numerical example to demonstrate the efficiency and accuracy of the proposed approach. Finally, Section 5 concludes the paper.

2 Background

2.1 *Generalized failure probability function*

Let the PDF of performance function $g(\boldsymbol{x})$ with respect to n-dimensional input variable $\boldsymbol{X}=\{X_1,\ X_2,\cdots,X_n\}$ be defined as $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ represents the DPs. The structural safety level can be quantified by the G-FP $P_{\tilde{F}}(\boldsymbol{\theta})$ defined in Eq. (1).

$$P_{\tilde{F}}(\boldsymbol{\theta})=\int \rho_{\tilde{F}}\left[g(\boldsymbol{x})\right] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})\mathrm{d}\boldsymbol{x} \tag{1}$$

where $\rho_{\tilde{F}}\left[g(\boldsymbol{x})\right]$ is the membership function of the performance function $g(\boldsymbol{x})$ to the fuzzy failure domain $\tilde{F}$。

When the DPs vary within their concerned design domain $[\boldsymbol{\theta}^L,\boldsymbol{\theta}^U]$ the relationship between the G-FP $P_{\tilde{F}}(\boldsymbol{\theta})$ and the DPs is referred to as G-FPF $P_{\tilde{F}}(\boldsymbol{\theta})(\boldsymbol{\theta}\in[\boldsymbol{\theta}^L,\boldsymbol{\theta}^U])$, which is defined as:

$$\begin{aligned} P_{\tilde{F}}(\boldsymbol{\theta}) &= \int \rho_{\tilde{F}}[g(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})\mathrm{d}\boldsymbol{x} \\ &= E_{f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}\{\rho_{\tilde{F}}[g(\boldsymbol{x})]\} \end{aligned} \quad \boldsymbol{\theta}\in[\boldsymbol{\theta}^L,\boldsymbol{\theta}^U] \tag{2}$$

where $E_{f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}\{\cdot\}$ is the expectation operator with $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ acting as the weighting function.

2.2 *Single-Loop Analysis Framework Combined with Importance Sampling*

The constructed unified optimal ISD $h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x})$ (Wu 2025), which is independent of the DPs, is introduced into G-FPF in Eq. (2), thereby enabling a single-loop estimation of the G-FPF.

$$P_{\tilde{F}}(\boldsymbol{\theta})=\int \frac{\rho_{\tilde{F}}\left[g(\boldsymbol{x})\right] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}{h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x})} h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \quad \boldsymbol{\theta}\in[\boldsymbol{\theta}^L,\boldsymbol{\theta}^U] \tag{3}$$

where,

$$\begin{aligned} h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x}) &= \int h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \\ &= \int \frac{\rho_{\tilde{F}}[g(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}{P_{\tilde{F}}(\boldsymbol{\theta})} f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \end{aligned} \tag{4}$$

After obtaining $\boldsymbol{\theta}_i (i = 1, 2, \cdots, N_{\boldsymbol{x}}^h)$ drawn from $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$, the corresponding samples $\boldsymbol{x}_i^h$ drawn from $h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x}|\boldsymbol{\theta}_i)$ are generated to form an $N_{\boldsymbol{x}}^h$ sample set $\boldsymbol{S}_{\boldsymbol{x}}^h = \left\{ \boldsymbol{x}_1^h, \boldsymbol{x}_2^h, \cdots, \boldsymbol{x}_{N_{\boldsymbol{x}}^h}^h \right\}$. Then, the generalized failure probabilities $P_{\tilde{F}}(\boldsymbol{\theta})$ corresponding to different realizations $\boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U]$ of the DPs can be simultaneously estimated using Eq. (5).

$$P_{\tilde{F}}(\boldsymbol{\theta}) = \frac{1}{N_{\boldsymbol{x}}^h} \sum_{i=1}^{N_{\boldsymbol{x}}^h} \frac{\rho_{\tilde{F}}\left[g(\boldsymbol{x}_i)\right] f_{\boldsymbol{X}}(\boldsymbol{x}_i|\boldsymbol{\theta})}{h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{x}_i)} \quad \boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U] \tag{5}$$

3 Hierarchical Importance Sampling Method for G-FPF Estimation

3.1 Unified Importance Sampling Density

Assume that the n-dimensional independent random variable $\boldsymbol{X}$ follows a normal distribution characterized by mean μ_{X_i} and standard deviation $\sigma_{X_i} (i = 1, 2, \cdots, n)$, i.e., $\boldsymbol{\theta} = [\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\sigma}_{\boldsymbol{X}}]$. Then, $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ can be expressed as follows,

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) = f_{\boldsymbol{X}}(\boldsymbol{x} \,|\, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\sigma}_{\boldsymbol{X}}) \tag{6}$$

Based on Eq. (4), the average contribution $\omega(\boldsymbol{x})$ of sample $\boldsymbol{x}$ to G-FPF over DPs is defined as follows,

$$\begin{aligned} \omega(\boldsymbol{x}) &= \int \omega(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \\ &= \int \rho_{\tilde{F}}[g(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \end{aligned} \tag{7}$$

By performing weighted clustering analysis with the average contribution $\omega(\boldsymbol{x})$ as weights, a set of clustering centers $\boldsymbol{S}_{\boldsymbol{x}}^c = \left\{ \boldsymbol{x}_k^c, k = 1, \cdots, N_c \right\}$ is obtained. Taking these clustering centers $\boldsymbol{S}_{\boldsymbol{x}}^c$ as density centers, the explicit expression $h_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}_j)$ for estimating G-FP $P_{\tilde{F}}(\boldsymbol{\theta}_j)$ under a given $\boldsymbol{\theta}_j = [\boldsymbol{\mu}_{\boldsymbol{X}}^{(j)}, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(j)}]$ can be constructed accordingly.

$$h_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}_j) = \frac{1}{N_c} \sum\nolimits_{k=1}^{N_c} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{x}_k^c, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(j)}) \tag{8}$$

By further integrating $h_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}_j)$ with respect to $\boldsymbol{\theta}$ over the design domain $[\boldsymbol{\theta}^L, \boldsymbol{\theta}^U]$, an explicit expression of the unified ISD $h_{\boldsymbol{X}}(\boldsymbol{x})$ independent of the DPs can be derived.

$$h_{\boldsymbol{X}}(\boldsymbol{x}) = \int h_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{9}$$

3.2 *Hierarchical Strategy for Unified Importance Sampling Density*

3.2.1 *Initialization*

Since no prior information about the unified density is available at this stage, the first-level density $h_{\boldsymbol{X}}^{(1)}(\boldsymbol{x})$ is constructed based on the original probability density $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$ of the random inputs, expressed as follows,

$$h_{\boldsymbol{X}}^{(1)}(\boldsymbol{x}) = \int f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} \tag{10}$$

3.2.2 *Adaptive Hierarchical Iteration Rule*

At the m-th level, since the expression of $h_{\boldsymbol{X}}^{(m)}(\boldsymbol{x})$ is explicit and regular, a sample set $\boldsymbol{S}_{\boldsymbol{x}}^{hm} = \{\boldsymbol{x}_1^{hm}, \boldsymbol{x}_2^{hm}, \cdots, \boldsymbol{x}_{N_{\boldsymbol{x}}^h}^{hm}\}$ can be directly drawn. The average contributions of these samples to the G-FPF are then evaluated, forming a contribution set $\boldsymbol{S}_{\omega}^{hm}$ as defined in Eq. (11). The elements of this set are subsequently sorted in ascending order to obtain an ordered contribution set $\boldsymbol{S}_{[\omega]}^{hm}$ as in Eq. (12).

$$\boldsymbol{S}_{\omega}^{hm} = \{\omega(\boldsymbol{x}_1^{hm}), \omega(\boldsymbol{x}_2^{hm}), \cdots, \omega(\boldsymbol{x}_{N_{\boldsymbol{x}}^h}^{hm})\} \tag{11}$$

$$\boldsymbol{S}_{[\omega]}^{hm} = \{\omega(\boldsymbol{x}_{[1]}^{hm}) < \omega(\boldsymbol{x}_{[2]}^{hm}) < \cdots < \omega(\boldsymbol{x}_{[N_{\boldsymbol{x}}^h]}^{hm})\} \tag{12}$$

A predefined proportion $\alpha N_{\boldsymbol{x}}^h$ of samples with the largest average contributions $\boldsymbol{S}_{[\omega]}^{hm}$ is selected to form a subset Ω_m, as defined in Eq. (13). Typically, α is chosen within the range of 0.1–0.4, which is adopted as 0.1.

$$\Omega_m = \{\boldsymbol{x} \,|\, \omega(\boldsymbol{x}) \geq \omega(\boldsymbol{x}_{[\alpha N_{\boldsymbol{x}}^h]}^h), \omega(\boldsymbol{x}) \in \boldsymbol{S}_{\omega}^{hm}\} \tag{13}$$

Weighted clustering analysis is then performed on this subset $\boldsymbol{x} \in \Omega_m$ using the average contributions $\omega(\boldsymbol{x})$ as weights, resulting in a set of clustering centers $\boldsymbol{S}_{\boldsymbol{x}}^{cm}$, as given in Eq. (14).

$$\boldsymbol{S}_{\boldsymbol{x}}^{cm} = \{\boldsymbol{x}_1^{cm}, \boldsymbol{x}_2^{cm}, \cdots, \boldsymbol{x}_{N_c}^{cm}\} \tag{14}$$

Based on these clustering centers, the unified ISD for the $(m+1)$-th level $h_{\boldsymbol{X}}^{(m+1)}(\boldsymbol{x})$ can be constructed.

3.2.3 *Stopping Criterion*

The hierarchical iteration is terminated when the minimum average contribution $\omega(\boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{S}_{\boldsymbol{x}}^{c(m+1)})$ of the clustering centers at the next level is smaller than that of $\omega(\boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{S}_{\boldsymbol{x}}^{cm})$ the current level, i.e., $\min\{\omega(\boldsymbol{x})|\boldsymbol{x} \in \boldsymbol{S}_{\boldsymbol{x}}^{cm}\} \geq \min\{\omega(\boldsymbol{x})|\boldsymbol{x} \in \boldsymbol{S}_{\boldsymbol{x}}^{c(m+1)}\}$. At this point, the hierarchical exploration is stopped, and the unified ISD constructed is taken as $h_{\boldsymbol{X}}(\boldsymbol{x}) = h_{\boldsymbol{X}}^{(m)}(\boldsymbol{x})$ as defined in Eq. (15).

$$h_{\boldsymbol{X}}(\boldsymbol{x}) = \int [\frac{1}{N_c} \sum\nolimits_{i=1}^{N_c} \varphi_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{x}_i^{cm}, \boldsymbol{\sigma}_{\boldsymbol{X}})] \cdot f_{\boldsymbol{\sigma}_{\boldsymbol{X}}}(\boldsymbol{\sigma}_{\boldsymbol{X}}) \mathrm{d}\boldsymbol{\sigma}_{\boldsymbol{X}} \tag{15}$$

3.3 Single-Loop Estimation of the Generalized Failure Probability Function

By introducing the constructed unified ISD $h_{\boldsymbol{X}}(\boldsymbol{x})$ into Eq. (2), the G-FPF can be rewritten as follows,

$$P_{\tilde{F}}(\boldsymbol{\theta}) = \int \frac{\rho_{\tilde{F}}[g(\boldsymbol{x})] f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})}{h_{\boldsymbol{X}}(\boldsymbol{x})} h_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \quad \boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U] \tag{16}$$

Since the constructed $h_{\boldsymbol{X}}(\boldsymbol{x})$ is explicit and regular, its samples can be efficiently generated via direct sampling. Specifically, samples of the DPs $\boldsymbol{\theta}_i (i = 1, 2, \cdots, N_{\boldsymbol{x}}^h / N_c)$ are first drawn from $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$. Then, for each $f_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{x}_j^c, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(i)})(j = 1, 2, \cdots, N_c)$, the corresponding random input samples $\boldsymbol{x}_i^h (i = 1, 2, \cdots, N_{\boldsymbol{x}}^h / N_c)$ are generated. This process yields a complete sample set $\boldsymbol{S}_{\boldsymbol{x}}^h = \{\boldsymbol{x}_1^h, \boldsymbol{x}_2^h, \cdots, \boldsymbol{x}_{N_{\boldsymbol{x}}^h}^h\}$.

After obtaining $\boldsymbol{S}_{\boldsymbol{x}}^h = \{\boldsymbol{x}_1^h, \boldsymbol{x}_2^h, \cdots, \boldsymbol{x}_{N_{\boldsymbol{x}}^h}^h\}$, $\hat{P}_{\tilde{F}}(\boldsymbol{\theta})$ $(\boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U])$, variance estimate $\hat{Var}[P_{\tilde{F}}(\boldsymbol{\theta})]$, and coefficient of variation $\hat{Cov}[\hat{P}_{\tilde{F}}(\boldsymbol{\theta})]$ can be evaluated using Eqs. (17)-(19), respectively.

$$\hat{P}_{\tilde{F}}(\boldsymbol{\theta}) = \frac{1}{N_{\boldsymbol{x}}^h} \sum_{i=1}^{N_{\boldsymbol{x}}^h} \frac{\rho_{\tilde{F}}[g(\boldsymbol{x}_i)] f_{\boldsymbol{X}}(\boldsymbol{x}_i|\boldsymbol{\theta})}{\hat{h}_{\boldsymbol{X}}(\boldsymbol{x}_i)} \quad \boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U] \tag{17}$$

$$\hat{Var}[\hat{P}_{\tilde{F}}(\boldsymbol{\theta})] = \frac{1}{N_{\boldsymbol{x}}^h - 1} \left\{ \frac{1}{N_{\boldsymbol{x}}^h} \sum_{i=1}^{N_{\boldsymbol{x}}^h} \left[\frac{\rho_{\tilde{F}}[g(\boldsymbol{x}_i)] f_{\boldsymbol{X}}(\boldsymbol{x}_i|\boldsymbol{\theta})}{\hat{h}_{\boldsymbol{X}}(\boldsymbol{x}_i)} \right]^2 - \hat{P}_{\tilde{F}}^2 \right\} \tag{18}$$

$$\hat{Cov}[\hat{P}_{\tilde{F}}(\boldsymbol{\theta})] = \sqrt{\hat{Var}[\hat{P}_{\tilde{F}}(\boldsymbol{\theta})]} / \hat{P}_{\tilde{F}}(\boldsymbol{\theta}) \quad \boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U] \tag{19}$$

where $\hat{h}_{\boldsymbol{X}}(\boldsymbol{x}_i)$ can be estimated from a set of samples $\boldsymbol{S}_{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_{\boldsymbol{X}}^{(1)}, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(2)}, \cdots, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(N_\sigma)}\}$ drawn from $f_{\boldsymbol{\sigma}_{\boldsymbol{X}}}(\boldsymbol{\sigma}_{\boldsymbol{X}})$,

$$\hat{h}_{\boldsymbol{X}}(\boldsymbol{x}_i) = \frac{1}{N_{\boldsymbol{\sigma}}} \sum_{j=1}^{N_{\boldsymbol{\sigma}}} \left[\frac{1}{N_c} \sum_{k=1}^{N_c} f_{\boldsymbol{X}}(\boldsymbol{x}_i|\boldsymbol{x}_k^c, \boldsymbol{\sigma}_{\boldsymbol{X}}^{(j)}) \right] \tag{20}$$

4 Numerical Example

In this section, the compared methods include DL-MCS, SL-MCS (Wu 2025), Single loop based Global importance sampling method (SL-GIS) (Yuan 2023), and the proposed method.

For accuracy comparison, the G-FPF estimated by DL-MCS is taken as the reference solution. The mean relative error (MRE), defined in Eq. (21), is employed to evaluate the accuracy of different methods. Specifically, the MRE is calculated as the average of the relative errors between the estimated G-FP $\hat{P}_{\tilde{F}}(\boldsymbol{\theta}_i)$ and the DL-MCS reference solution $\hat{P}_{\tilde{F}}^{\text{DL-MCS}}(\boldsymbol{\theta}_i)$ over $N_{\boldsymbol{\theta}}$ realizations $\boldsymbol{\theta}_i (i = 1, \cdots, N_{\boldsymbol{\theta}})$. In this study, $N_{\boldsymbol{\theta}}$ is set to 10.

$$\text{MRE} = \frac{1}{N_{\boldsymbol{\theta}}} \sum_{i=1}^{N_{\boldsymbol{\theta}}} \frac{\left| \hat{P}_{\tilde{F}}(\boldsymbol{\theta}_i) - \hat{P}_{\tilde{F}}^{\text{DL-MCS}}(\boldsymbol{\theta}_i) \right|}{\hat{P}_{\tilde{F}}^{\text{DL-MCS}}(\boldsymbol{\theta}_i)} \tag{21}$$

4.1 *Cantilever Wing Structure*

The proposed method is applied to a cantilever wing structure, as shown in Fig. 1, to further validate its performance. The structure consists of 17 ribs, three groups of stringers, and one spar. The wingspan is 4.02 m, and the chord length is 1.0 m. The ribs are made of aluminum alloy 2A12, while the spar and stringers are made of aluminum alloy 2A16, and the skin is made of aluminum alloy 7075.

For the cantilever wing structure, the performance function $g(\boldsymbol{x})$ is defined as the difference between the maximum stress $\sigma_{\max}(\boldsymbol{x})$ and the structural yield strength σ_s, as given in Eq. (22). The membership function of the performance function with respect to the fuzzy failure domain is defined in Eq. (23).

$$g(\boldsymbol{x}) = \sigma_s - \sigma_{\max}(\boldsymbol{x}) \tag{22}$$

$$\rho_{\tilde{F}}[g(\boldsymbol{x})] = \begin{cases} 1 & g(\boldsymbol{x}) \le -0.1 \\ \exp[-\dfrac{(g(\boldsymbol{x})+0.1)^2}{0.001}] & -0.1 < g(\boldsymbol{x}) \end{cases} \tag{23}$$

Figure 1. Finite element model

The notations, distribution types, and corresponding parameters of all random input variables are listed in Table 1. The structural loads and boundary conditions are illustrated in Fig. 2. Fig. 3 presents the stress distribution obtained from finite element analysis when all random variables are set to their mean values. Since the computational cost of DL-MCS and SL-MCS becomes prohibitive for examples requiring repeated calls to the finite element model, only SL-GIS and the proposed method are employed in this study. The resulting G-FPF surfaces are shown in Fig. 4, and the corresponding computational costs are reported in Table 2.

Table 1. Distribution type and corresponding parameter

Variable	Meaning	Notation	Distribution Type	μ_{X_i}	σ_{X_i}
E_1 (MPa)	2A12 Elastic modulus	X_1	Normal	70000	1400
E_2 (MPa)	2A16 Elastic modulus	X_2	Normal	67700	1354
E_3 (MPa)	7075 Elastic modulus	X_3	Normal	74000	1480
λ_1	2A12 Poisson ratio	X_4	Normal	0.33	0.0099
λ_2	2A16 Poisson ratio	X_5	Normal	0.33	0.0099
λ_3 S	7075 Poisson ratio	X_6	Normal	0.33	0.0099
P (MPa)	External pressure	X_7	Normal	3.3×10^{-3}	1.5×10^{-4}
σ_s (MPa)	Yield strength	X_8	Normal	440	20

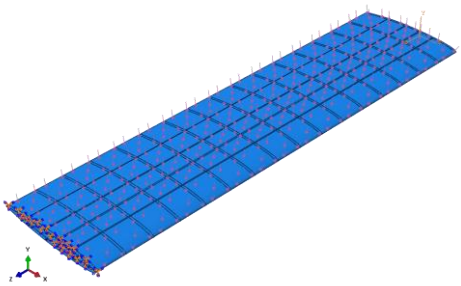

Figure 2. Structural loads and boundary conditions

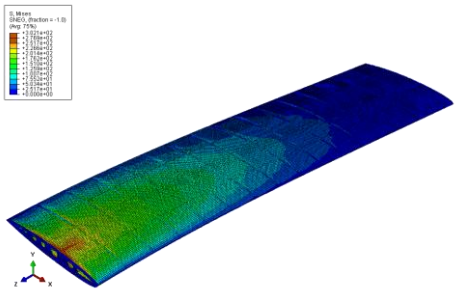

Figure 3. Finite element analysis

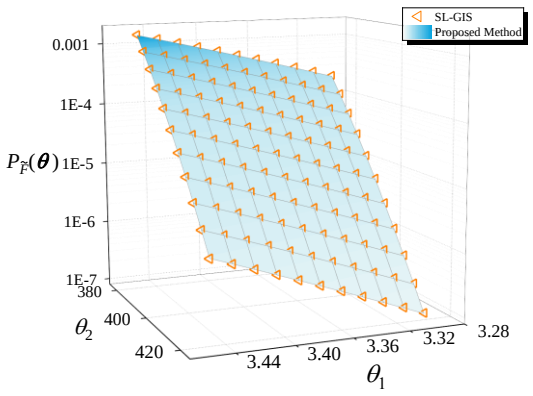

Figure 4. G-FPF curve

Table 2. Computational cost

Method	Covm	Ncall	Time (day)
SL-GIS	0.024	24367	13.3
Proposed Method	0.040	$4\times4\times10^3+5\times2$	8.1

Note: a. Covm denotes the mean value of the standard deviation of the G-FPF; b. Ncalldenotes the total number of performance function evaluations; c. Time denotes the total computational time of each method.

The results demonstrate that G-FPFs estimated by SL-GIS and the proposed method under the single-loop framework are in close agreement, while the proposed method achieves convergence with the fewest finite element evaluations. These findings indicate that the unified ISD constructed in this study possesses an explicit, regular, and easily sampleable form, which significantly reduces additional computational cost and enables efficient and robust estimation of G-FPF.

5 Conclusion

To efficiently estimate the generalized failure probability function (G-FPF) under probabilistic inputs and fuzzy state assumptions in high-dimensional, high-reliability scenarios, this paper proposes an asymptotic hierarchical importance sampling method.

By using weighted clustering centers—where the weights are defined as the average contributions of samples to the G-FPF estimation—as density centers, the proposed method constructs an explicit expression of a unified importance sampling density that is independent of the distribution parameters. This enables sample information sharing when evaluating the generalized failure probability at different realizations of the distribution parameters. Moreover, the explicit and regular form of the constructed density avoids the difficulties associated with sampling from implicit densities and significantly reduces computational cost.

The constructed unified importance sampling density has a regular form that facilitates efficient sampling. Furthermore, through an additional asymptotic hierarchical clustering strategy with low computational overhead, the resulting density approximates the important regions of the optimal unified importance sampling density. In the proposed asymptotic hierarchical approximation process, the centers of the importance sampling density at the next level are determined via clustering, where the weights are defined by the average contributions of samples from the previous level to the G-FPF estimation. As a result, the converged unified importance sampling density effectively covers regions with high contributions to the G-FPF, thereby achieving high computational efficiency. Although the constructed density is not strictly optimal, it progressively approximates the key regions of the optimal unified importance sampling density with limited additional computational cost, leading to superior efficiency. Numerical example presented in subsequent sections demonstrate that the proposed method achieves a favorable balance between accuracy and efficiency, significantly outperforming existing methods.

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