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A weighted importance sampling method for estimating local reliability sensitivity

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Local reliability sensitivity (LRS) quantifies the influence of distribution parameters of random inputs on the failure probability. Existing numerical simulation methods can simultaneously estimate both the failure probability and LRS using a shared set of samples. However, achieving convergence for all LRS estimates typically requires substantially more samples than needed for the failure probability alone, leading to high computational costs. To address this challenge, this paper proposes a weighted importance sampling method. First, a Gaussian mixture model (GMM) is constructed to approximate the optimal importance sampling density for failure probability estimation. This GMM is then used to compute the coefficient of variation for each estimated LRS. Subsequently, these coefficients are employed as weights to formulate an enhanced importance sampling density, which targets to reduce the variance of LRS with large coefficients of variation. The enhanced importance sampling strategy in the proposed method improves the efficiency of local sensitivity estimation. This improvement is clearly demonstrated by the presented examples.

Keywords: Reliability analysis, Local sensitivity analysis, Cross-entropy, Importance sampling method

1 Introduction

Local reliability sensitivity (LRS) quantifies how input distribution parameters affect structural failure probability (Chiron 2023), playing a crucial role in parameter identification and gradient-based optimization (Wang 2023, Dubourg 2011). Existing approaches include moment methods, point estimation, and digital simulation. Moment methods (Bjægerager 1989) often neglect nonlinearities, while point estimation (Rosenblueth 1981, Seo 2002) suffers from double approximation errors and poor high-dimensional scalability. Although dimension-reduced integration with stochastic collocation improves efficiency (Ding 2025), digital simulation methods (Lu 2008, Yuan 2020, Au 2005, Proppe 2021) remain the most general. However, ensuring convergence for all sensitivity parameters typically requires sample sizes far exceeding those for failure probability alone, significantly increasing computational costs.

To address this, we propose a Weighted Importance Sampling (WIS) method. First, a Gaussian Mixture Model (GMM) approximates the optimal sampling density to estimate the coefficient of variation (COV) for each sensitivity parameter. Then, an enhanced sampling density is constructed using these COVs as weights and approximated by a second GMM. This strategy specifically reduces the variance of high-variability sensitivity estimates. Numerical examples validate the improved efficiency of the proposed method.

2 The Definition of LRS

Assuming that $Y = g(\mathbf{X})$ is the performance function of the structure and $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is the n -dimensional independent random input vector. $f_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta}_{\mathbf{X}})$ is the joint probability density function (PDF) of $\mathbf{X}$, where $\boldsymbol{\theta}_{\mathbf{X}}$ is the distribution parameter vector of $\mathbf{X}$. For n -dimensional independent input vector, its PDF can be rewritten as

$f_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta}_{\mathbf{X}}) = \prod_{i=1}^n f_{X_i}(x_i | \boldsymbol{\theta}_{X_i})$, where $f_{X_i}(x_i | \boldsymbol{\theta}_{X_i})$ is the marginal PDF of the i -dimensional input variable X_i , $\boldsymbol{\theta}_{X_i}$ is the distribution parameter vector of X_i .

According to the failure domain $F = \{g(\mathbf{x}) \leq 0\}$ defined by the performance function, the failure probability P_f of the structure can be expressed as an integral shown in Eq. (1).

$$P_f = \int_{R^n} f_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta}_{\mathbf{X}}) d\mathbf{x} = \int_{R^n} I_F(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta}_{\mathbf{X}}) d\mathbf{x} \quad (1)$$

where R^n is the n -dimensional real number space, $I_F(\mathbf{x})$ is the indicator function of the failure domain F , if $\mathbf{x} \in F$, $I_F(\mathbf{x}) = 1$, otherwise, $I_F(\mathbf{x}) = 0$.

Based on Eq. (1), the LRS can be defined by the partial derivative of failure probability P_f with respect to the distribution parameters as follows,

$$\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}} = \int_{R^n} I_F(\mathbf{x}) \frac{\partial f_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta}_{\mathbf{X}})}{\partial \theta_{X_i}^{(k)}} d\mathbf{x} \quad (2)$$

where $\theta_{X_i}^{(k)} (i=1,2,\dots,n+1; k=1,2,\dots,m_i)$ is the k -th distribution parameter of an m_i -dimensional distribution parameter vector θ_{X_i} of X_i ($i=1,2,\dots,n$).

Eq. (2) can be rewritten as Eq. (3),

$$\begin{aligned} \frac{\partial P_f}{\partial \theta_{X_i}^{(k)}} &= \int_{R^n} I_F(\mathbf{x}) \frac{\partial f_X(\mathbf{x}|\theta_X)}{\partial \theta_{X_i}^{(k)}} \cdot \frac{1}{f_X(\mathbf{x}|\theta_X)} f_X(\mathbf{x}|\theta_X) d\mathbf{x} \\ &= \int_{R^n} I_F(\mathbf{x}) \cdot \eta_{X_i}^{(k)}(\mathbf{x}) f_X(\mathbf{x}|\theta_X) d\mathbf{x} \end{aligned} \quad (3)$$

where $\eta_{X_i}^{(k)}(\mathbf{x}) = \frac{\partial f_X(\mathbf{x}|\theta_X)}{\partial \theta_{X_i}^{(k)}} \cdot \frac{1}{f_X(\mathbf{x}|\theta_X)}$ is the score function of $\theta_{X_i}^{(k)}$.

3 The Weighted Importance Sampling Method for Estimating LRS

3.1 The Importance Sampling Method for Estimating P_f and LRS

Direct Monte Carlo simulation for P_f in Eq. (1) is computationally inefficient. By introducing an Importance Sampling (IS) PDF $h(\mathbf{x})$, Eq. (2) transforms into Eq. (4).

$$P_f = \int_{R^n} \frac{I_F(\mathbf{x}) f_X(\mathbf{x}|\theta_X)}{h(\mathbf{x})} h(\mathbf{x}) d\mathbf{x} \quad (4)$$

While the theoretical optimal IS PDF is given by Eq. (5).

$$h^{opt}(\mathbf{x}) = \frac{I_F(\mathbf{x}) f_X(\mathbf{x}|\theta_X)}{P_f} \quad (5)$$

The theoretical optimal IS PDF $h^{opt}(\mathbf{x})$ is analytically inaccessible since P_f is unknown and the limit state is implicit. To address this problem, a Gaussian Mixture Model (GMM) $h(\mathbf{x}, \mathbf{v})$ minimizing the Cross-Entropy (CE) is constructed to approximate $h^{opt}(\mathbf{x})$.

Considering the general case, the GMM $h(\mathbf{x}, \mathbf{v})$ is expressed as:

$$h(\mathbf{x}, \mathbf{v}) = \sum_{k=1}^K \pi_k \cdot \varphi(\mathbf{x} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \quad (6)$$

where K is the number of Gaussian components, π_k is weight of each Gaussian model. The PDF $\varphi(\mathbf{x} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ is an n -dimensional joint Gaussian PDF with the mean vector $\boldsymbol{\mu}_k$ and covariance matrix $\boldsymbol{\Sigma}_k$. Denote $\mathbf{v}$ as the distribution parameter set of $h(\mathbf{x}, \mathbf{v})$, i.e., $\mathbf{v} = \{\pi_1, \dots, \pi_K, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_K, \boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_K\}$. Then, $\mathbf{v}$ has $(3 \times K)$ categories.

To approximate the optimal IS density $h^{opt}(\mathbf{x})$, we minimize the Kullback-Leibler (KL) divergence of $h^{opt}(\mathbf{x})$ and $h^{opt}(\mathbf{x})$ as shown in Eq. (7),

$$\min D(h^{opt}(\mathbf{x}), h(\mathbf{x}, \mathbf{v})) = \min \left[\int h^{opt}(\mathbf{x}) \ln h^{opt}(\mathbf{x}) d\mathbf{x} - \int h^{opt}(\mathbf{x}) \ln h(\mathbf{x}, \mathbf{v}) d\mathbf{x} \right] \quad (7)$$

Since the first term in Eq. (7) is independent of $\mathbf{v}$, minimizing $D(h^{opt}(\mathbf{x}), h(\mathbf{x}, \mathbf{v}))$ is equivalent to maximizing the second term. Eq. (7) can be transformed into the expectation form shown in Eq. (8).

$$\begin{aligned} \max_{\mathbf{v}} \int h^{opt}(\mathbf{x}) \ln h(\mathbf{x}, \mathbf{v}) d\mathbf{x} &= \max_{\mathbf{v}} \int I_F(\mathbf{x}) \ln h(\mathbf{x}, \mathbf{v}) f_X(\mathbf{x}|\theta_X) d\mathbf{x} \\ &= \max_{\mathbf{v}} E_{f_X} [I_F(\mathbf{x}) \cdot \ln h(\mathbf{x}, \mathbf{v})] \end{aligned} \quad (8)$$

where $E_{f_X}[\cdot]$ denotes the expectation with respect to the PDF $f_X(\mathbf{x}|\theta_X)$.

Typically, Eq (8) is convex and differentiable with respect to $\mathbf{v}$, therefore, the corresponding problems can be solved by setting the gradient of the objective function to zero. The sequential iterative algorithm can significantly reduce the difficulty of solving the aforementioned Eq (8), wherein the distribution parameter set $\mathbf{v}$ updated in the previous step can serve as the distribution parameters for the current sampling step. The specific update rules are as follows,

$$\gamma_{t,k}^{(l)} = \frac{\pi_k^{(l)} \varphi(\mathbf{x}_t | \boldsymbol{\mu}_k^{(l)}, \boldsymbol{\Sigma}_k^{(l)})}{\sum_{j=1}^K \pi_k^{(l)} \varphi(\mathbf{x}_t | \boldsymbol{\mu}_k^{(l)}, \boldsymbol{\Sigma}_k^{(l)})}, \quad t=1, \dots, N, k=1, \dots, K \quad (9)$$

where, $\gamma_{t,k}^{(l)}$ denotes proportion of the k -th Gaussian density in the GMM $h(\mathbf{x}, \mathbf{v})$ at sample $\mathbf{x}_t$ ($t=1,2,\dots,N$) in the l -th iteration,

the update rules for the other parameters are as follows,

$$\mu_k^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \gamma_{t,k}^{(l)} \mathbf{x}_t f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})} \quad (10)$$

$$\Sigma_k^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \gamma_{t,k}^{(l)} (\mathbf{x}_t - \mu_k^{(l)}) (\mathbf{x}_t - \mu_k^{(l)})^T f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})} \quad (11)$$

$$\pi_k^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}^{(l)})} \quad (12)$$

After the algorithm converges, $\hat{P}_f$ and $\text{COV}(\hat{P}_f)$ can be derived from Eqs. (13) and (14)

$$\hat{P}_f = \frac{1}{N} \sum_{t=1}^N I_F(\mathbf{x}_t) \cdot f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}) \quad (13)$$

$$\text{COV}(\hat{P}_f) = \sqrt{\frac{1}{N-1} \left(\frac{1}{N} \sum_{t=1}^N \left[\frac{I_F(\mathbf{x}_t) f_X(\mathbf{x}_t | \boldsymbol{\theta}_X)}{h(\mathbf{x}_t, \mathbf{v})} \right]^2 - \hat{P}_f^2 \right)} \quad (14)$$

Using the current IS PDF $h(\mathbf{x}, \mathbf{v})$, the LRS $\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}$ and $\text{COV}(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}})$ can be estimated by Eqs. (15) and (16).

$$\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}} = \frac{1}{N} \sum_{t=1}^N \eta_{X_i}^{(k)}(\mathbf{x}_t) I_F(\mathbf{x}_t) f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h(\mathbf{x}_t, \mathbf{v}) \quad (15)$$

$$\text{COV}(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}) = \sqrt{\frac{1}{N-1} \left(\frac{1}{N} \sum_{t=1}^N \left[\eta_{X_i}^{(k)}(\mathbf{x}_t) I_F(\mathbf{x}_t) \cdot \frac{f_X(\mathbf{x}_t | \boldsymbol{\theta}_X)}{h(\mathbf{x}_t, \mathbf{v})} \right]^2 - \frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}^2 \right)} \quad (16)$$

3.2 The Weighted Importance Sampling Method for Estimating LRS

Eq (3) implies that the optimal IS PDF for estimating different LRS are not identical. Consequently, using the IS PDF of P_f to compute multiple LRS may obtain a converged estimate of P_f , while certain LRS terms still exhibit high variability.

To address this issue, section 3.2 reconstructs the weighted IS PDF as in Eq. (17), leveraging $\text{COV}(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}})$

derived from the importance density used for P_f estimation in section 3.1,

$$h_\eta^r(\mathbf{x}) = \frac{\sum_{i=1}^n \sum_{k=1}^{m_i} \omega_i^{(k)} I_F(\mathbf{x}) \eta_{X_i}^{(k)}(\mathbf{x}) f_X(\mathbf{x} | \boldsymbol{\theta}_X)}{\sum_{i=1}^n \sum_{k=1}^{m_i} \omega_i^{(k)} \left| \partial P_f / \partial \theta_{X_i}^{(k)} \right|} \quad (17)$$

where the weight $\omega_i^{(k)}$ is defined shown as Eq. (18),

$$\omega_i^{(k)} = [\text{COV}(\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}})]^2 / \sum_{i=1}^n \sum_{k=1}^{m_i} [\text{COV}(\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}})]^2 \quad (18)$$

Eqs. (18) and (19) indicate that LRS with larger COVs are assigned higher weights, thereby steering the reconstructed density $h_\eta^r(\mathbf{x})$ closer to the theoretical optimal IS PDF for each corresponding LRS.

To approximate the weighted importance sampling density in Eq. (18), we minimize the KL divergence between the GMM $h_\eta(\mathbf{x}, \mathbf{v}_\eta)$ and the target density $h_\eta^r(\mathbf{x})$. Denote $\mathbf{v}_\eta$ as the distribution parameter set of $h_\eta(\mathbf{x}, \mathbf{v}_\eta)$, i.e., $\mathbf{v}_\eta = \{\pi_{\eta 1}, \dots, \pi_{\eta K}, \mu_{\eta 1}, \dots, \mu_{\eta K}, \Sigma_{\eta 1}, \dots, \Sigma_{\eta K}\}$. The update procedure are as follows,

$$\mu_{\eta k}^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) \gamma_{t,k}^{(l)} \mathbf{x}_t f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})} \quad (19)$$

$$\Sigma_{\eta k}^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) \gamma_{t,k}^{(l)} (\mathbf{x}_t - \mu_k^{(l)}) (\mathbf{x}_t - \mu_k^{(l)})^T f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})} \quad (20)$$

$$\pi_{\eta k}^{(l+1)} = \frac{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) \gamma_{t,k}^{(l)} f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})}{\sum_{t=1}^N I_F(\mathbf{x}_t) \bar{\eta}_{X_i}^{(k)}(\mathbf{x}_t) f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}^{(l)})} \quad (21)$$

where, $\bar{\eta}_{X_i}^{(k)}$ is the abbreviated form of $\sum_{i=1}^n \sum_{k=1}^{m_i} \omega_i^{(k)} |\eta_{X_i}^{(k)}(\mathbf{x})|$.

After the algorithm converges, $\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}$ and $\text{COV}(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}})$ can be estimated by Eqs. (22) and (23).

$$\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}} = \frac{1}{N} \sum_{t=1}^N \eta_{X_i}^{(k)}(\mathbf{x}_t) I_F(\mathbf{x}_t) f_X(\mathbf{x}_t | \boldsymbol{\theta}_X) / h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta}) \quad (22)$$

$$\text{COV}(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}) = \sqrt{\frac{1}{N-1} \left(\frac{1}{N} \sum_{t=1}^N [\eta_{X_i}^{(k)}(\mathbf{x}_t) I_F(\mathbf{x}_t) \cdot \frac{f_X(\mathbf{x}_t | \boldsymbol{\theta}_X)}{h_{\eta}(\mathbf{x}_t, \mathbf{v}_{\eta})}]^2 - \frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}} \right)^2} \quad (23)$$

4 Examples

4.1 Example 1: Concrete Beam Structure

This example is taken from reference (Rashki 2012), with the performance function defined by Eq. (53). All random inputs follow normal distributions, with their specific parameters detailed in Table 1. The LRS estimation results for this example are presented in Table 2.

$$Y = g(\mathbf{X}) = X_2 \cdot X_1 \cdot X_4 - 0.59 \frac{(X_2 \cdot X_1)^2}{X_3 \cdot X_5} - X_6 \quad (24)$$

Table 1. The information of random input for example 1

X_i	μ_{X_i}	σ_{X_i}	X_i	μ_{X_i}	σ_{X_i}	X_i	μ_{X_i}	σ_{X_i}
X_1	45	4.5	X_3	4	0.6	X_5	12	1.2
X_2	3	0.24	X_4	20	2	X_6	800	240

Table 2. LRS results for example 1

	$\partial \hat{P}_f / \partial \mu_{X_1}$	$\partial \hat{P}_f / \partial \mu_{X_2}$	$\partial \hat{P}_f / \partial \mu_{X_3}$	$\partial \hat{P}_f / \partial \mu_{X_4}$	$\partial \hat{P}_f / \partial \mu_{X_5}$	$\partial \hat{P}_f / \partial \mu_{X_6}$	Ncall
MCS	-1.231×10^{-5}	-1.699×10^{-4}	-1.392×10^{-5}	-3.599×10^{-5}	-4.503×10^{-6}	3.568×10^{-7}	10^8
Cov.	1.241%	1.143%	3.433%	1.112%	4.508%	1.441%	
WIS	-1.233×10^{-5}	-1.685×10^{-4}	-1.397×10^{-5}	-3.575×10^{-5}	-4.301×10^{-6}	3.536×10^{-7}	1.10×10^5
Cov.	0.308%	0.372%	1.249%	0.267%	1.878%	0.217%	
IS	-1.228×10^{-5}	-1.695×10^{-4}	-1.405×10^{-5}	-3.583×10^{-5}	-4.339×10^{-6}	3.575×10^{-7}	1.27×10^5
Cov.	0.181%	0.282%	1.531%	0.187%	2.106%	0.301%	
	$\partial \hat{P}_f / \partial \sigma_{X_1}$	$\partial \hat{P}_f / \partial \sigma_{X_2}$	$\partial \hat{P}_f / \partial \sigma_{X_3}$	$\partial \hat{P}_f / \partial \sigma_{X_4}$	$\partial \hat{P}_f / \partial \sigma_{X_5}$	$\partial \hat{P}_f / \partial \sigma_{X_6}$	
MCS	2.225×10^{-5}	2.272×10^{-4}	1.007×10^{-5}	8.161×10^{-5}	1.752×10^{-6}	9.153×10^{-7}	
Cov.	1.180%	0.970%	5.356%	0.774%	4.593%	1.628%	
WIS	2.241×10^{-5}	2.258×10^{-4}	1.012×10^{-5}	8.070×10^{-5}	1.814×10^{-6}	9.005×10^{-7}	
Cov.	0.496%	0.705%	2.829%	0.412%	4.169%	0.281%	
IS	2.218×10^{-5}	2.265×10^{-4}	1.123×10^{-5}	8.113×10^{-5}	2.012×10^{-6}	9.201×10^{-7}	
Cov.	0.367%	0.581%	6.487%	0.327%	7.271%	0.439%	

Table 2 shows that WIS outperforms IS with fewer function evaluations, keeping all sensitivity COVs under 5% for greater stability. Although low-variance components show slightly higher COVs due to the strategy prioritizing high-COV terms, WIS overall enhances estimation stability at a lower computational cost.

4.2 Example 2: Automobile Front Axle Structure

This example is taken from reference (Rahman 2004), with the performance function defined by Eq. (53). All random inputs follow normal distributions, with their specific parameters detailed in Table 1. The LRS estimation results for this example are presented in Table 2.

$$y = g(\mathbf{X}) = X_6 - \sqrt{\sigma_{\max}^2 + 3\tau_{\max}^2} \quad (25)$$

Table 3. The information of random input for example 2

Input	X_i	μ_{X_i}	σ_{X_i}	Input	X_i	μ_{X_i}	σ_{X_i}
a	X_1	12	1.8	h	X_4	85	12.75
b	X_2	65	3.25	T	X_5	3.1×10^6	3.1×10^5
t	X_3	14	0.7	σ_s	X_6	600	120

Table 4. LRS results for example 2

	$\partial \hat{P}_f / \partial \mu_{X_1}$	$\partial \hat{P}_f / \partial \mu_{X_2}$	$\partial \hat{P}_f / \partial \mu_{X_3}$	$\partial \hat{P}_f / \partial \mu_{X_4}$	$\partial \hat{P}_f / \partial \mu_{X_5}$	$\partial \hat{P}_f / \partial \mu_{X_6}$	Ncall
MCS	-1.231×10^{-5}	-1.699×10^{-4}	-1.392×10^{-5}	-3.599×10^{-5}	-4.503×10^{-6}	3.568×10^{-7}	10^8
Cov.	1.241%	1.143%	3.433%	1.112%	4.508%	1.441%	
WIS	-1.233×10^{-5}	-1.685×10^{-4}	-1.397×10^{-5}	-3.575×10^{-5}	-4.301×10^{-6}	3.536×10^{-7}	1.10×10^5
Cov.	0.308%	0.372%	1.249%	0.267%	1.878%	0.217%	
IS	-1.228×10^{-5}	-1.695×10^{-4}	-1.405×10^{-5}	-3.583×10^{-5}	-4.339×10^{-6}	3.575×10^{-7}	1.27×10^5
Cov.	0.181%	0.282%	1.531%	0.187%	2.106%	0.301%	
	$\partial \hat{P}_f / \partial \sigma_{X_1}$	$\partial \hat{P}_f / \partial \sigma_{X_2}$	$\partial \hat{P}_f / \partial \sigma_{X_3}$	$\partial \hat{P}_f / \partial \sigma_{X_4}$	$\partial \hat{P}_f / \partial \sigma_{X_5}$	$\partial \hat{P}_f / \partial \sigma_{X_6}$	
MCS	2.225×10^{-5}	2.272×10^{-4}	1.007×10^{-5}	8.161×10^{-5}	1.752×10^{-6}	9.153×10^{-7}	
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Cov.	0.496%	0.705%	2.829%	0.412%	4.169%	0.281%	
IS	2.218×10^{-5}	2.265×10^{-4}	1.123×10^{-5}	8.113×10^{-5}	2.012×10^{-6}	9.201×10^{-7}	
Cov.	0.367%	0.581%	6.487%	0.327%	7.271%	0.439%	

Table 4 shows that WIS outperforms IS with fewer function evaluations, keeping all sensitivity COVs under 5% for greater stability. Although low-variance components show slightly higher COVs due to the strategy prioritizing high-COV terms, WIS overall enhances estimation stability at a lower computational cost.

5 Conclusion

To efficiently estimate reliability local sensitivities, this paper proposes a weighted importance sampling. The main conclusions are summarized as follows:

(1) A Gaussian Mixture Model (GMM) is constructed to approximate the optimal importance sampling density for failure probability. By deriving specific parameter update rules, this model enables the computation of the coefficients of variation (COV) for local sensitivity estimates.

(2) Leveraging these preliminary COV estimates, an enhanced importance sampling density is reconstructed and subsequently approximated by GMM. This step effectively mitigates the variance associated with highly variable local sensitivity estimates.

(3) The proposed method synergizes standard importance sampling variance reduction with a targeted enhanced sampling strategy, significantly enhancing the overall efficiency of local sensitivity estimation.

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References

Chiron, M., Morio, J. and Dubreuil, S., A review on local failure probability sensitivity analysis, Appl. Sci., 13(21), 12021, 2023.

- Wang, B., Huang, X. and Chang, M., Regional reliability sensitivity analysis based on dimension reduction technique, *Probab. Eng. Mech.*, 74, 103533, 2023.
- Dubourg, V., Sudret, B. and Bourinet, J.-M., Reliability-based design optimization using Kriging surrogates and subset simulation, *Struct. Multidiscip. Optim.*, 44(5), 673-690, 2011.
- Bjerager, P. and Krenk, S., Parametric sensitivity in first order reliability theory, *J. Eng. Mech.*, 115(7), 1577-1582, 1989.
- Rosenblueth, E., Two-point estimates in probabilities, *Appl. Math. Model.*, 5(5), 329-335, 1981.
- Seo, H.S. and Kwak, B.M., Efficient statistical tolerance analysis for general distributions using three-point information, *Int. J. Prod. Res.*, 40(4), 931-944, 2002.
- Ding, X. and Lu, Z.Z., A dimensional reduction integration based stochastic collocation algorithm for estimating local reliability sensitivity, *Structures*, 82, 110655, 2025.
- Wu, Y.T., Computational methods for efficient structural reliability and reliability sensitivity analysis, *AIAA J.*, 32(8), 1717-1723, 1994.
- Lu, Z.Z., Song, S.F., Yue, Z.F. et al., Reliability sensitivity method by line sampling, *Struct. Saf.*, 30(6), 517-532, 2008.
- Yuan, X.K., Zheng, Z.X. and Zhang, B.Q., Augmented line sampling for approximation of failure probability function in reliability-based analysis, *Appl. Math. Model.*, 80, 895-910, 2020.
- N, J. and Ellingwood, B.R., Directional methods for structural reliability analysis, *Struct. Saf.*, 22(3), 233-249, 2000.
- Au, S.K., Reliability-based design sensitivity by efficient simulation, *Comput. Struct.*, 83(14), 1048-1061, 2005.
- Song, S.F., Lu, Z.Z. and Qiao, H.W., Subset simulation for structural reliability sensitivity analysis, *Reliab. Eng. Syst. Saf.*, 94(3), 658-665, 2009.
- Proppe, C., Local reliability based sensitivity analysis with the moving particles method, *Reliab. Eng. Syst. Saf.*, 207, 107269, 2021.
- Rashki, M., Miri, M. and Azhdary, M.M., A new efficient simulation method to approximate the probability of failure and most probable point, *Struct. Saf.*, 39, 22-29, 2012.
- Rahman, S. and Xu, H., A univariate dimension-reduction method for multidimensional integration in stochastic mechanics, *Probab. Eng. Mech.*, 19(4), 393-408, 2004.