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Title	Adaptive radial sampling combined with stratified mixture importance sampling for efficiently estimating extremely small failure probability under random-interval uncertainty
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Relation	Proceedings of the 10th International Symposium on Reliability Engineering and Risk Management (ISRERM 2026), June 28 - July 1, 2026, Sapporo, Japan
Citation	Proceedings of the 10th International Symposium on Reliability Engineering and Risk Management (ISRERM 2026)
Issue Date	2026-07-01
Doc URL	https://hdl.handle.net/2115/100480
Type	conference paper
File Information	C000273_1.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

10th International Symposium on Reliability Engineering and Risk Management (ISRERM 2026),
28 June-01 July 2026, Sapporo, Japan
Kitahara, T., Miyamori, Y. & Beer, M. (editors)

Adaptive radial sampling combined with stratified mixture importance sampling for efficiently estimating extremely small failure probability under random-interval uncertainty

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To efficiently estimate the extremely small failure probability under random-interval uncertainty, this paper proposes an adaptive radial sampling combined with stratified mixture importance sampling method. The proposed method first constructs a hypersphere with relatively large radius in the standard normal space, outside which the samples are generated by radial sampling to explore the region far away from the coordinate origin. Then the rare failure domain can be quickly approached by adaptively stratifying. By clustering analysis for the failure samples of each layer, an explicit and regular mixture importance sampling probability density function (MIS-PDF) can be gradually constructed. Since the constructed explicit and regular MIS-PDF can efficiently explore the rare failure domain and greatly reduce the variance of estimating the extremely small failure probability with relatively small additional computational cost, it shows higher efficiency compared to the most update-to-date competing methods. To further reduce the performance function evaluations, the adaptive Kriging model of performance function is employed, thereby efficiently constructing the MIS-PDF and identifying the state of candidate sample pool. Several examples are investigated to validate the superiority of the proposed method.

Keywords: Random-interval uncertainty, Failure probability, Radial sampling, Stratified mixture importance sampling, Adaptive Kriging.

1 Introduction

Denote $G(\boldsymbol{X},\boldsymbol{Y})$ as the performance function under random-interval uncertainty (RIU), where $\boldsymbol{Y}=\{Y_1,Y_2,\cdots,Y_m\}^{\mathrm{T}}$ is an m-dimensional interval input vector with the value-taking region $\boldsymbol{y}_I^u=\{[y_1^l,y_1^u],[y_2^l,y_2^u],\cdots,[y_m^l,y_m^u]\}$, where $\boldsymbol{X}=\{X_1,X_2,\cdots,X_n\}^{\mathrm{T}}$ is n-dimensional random input vector with probability density function (PDF) $f_{\boldsymbol{X}}(\boldsymbol{x})$. Then the failure probability P_f under RIU is defined as follows (Wang et al. 2025),

$$P_f=\Pr\{F\}=\int_F f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\int_{R^n} I_F(\boldsymbol{x})\,f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1}$$

where $\Pr\{\cdot\}$ denotes the probability operator. $F=\{G(\boldsymbol{x},\boldsymbol{y})\le 0,\exists \boldsymbol{y}\in\boldsymbol{y}_I^u\}=\{G_{\min}(\boldsymbol{x})=\min\limits_{\boldsymbol{y}\in\boldsymbol{y}_I^u}G(\boldsymbol{x},\boldsymbol{y})\le 0\}$ is most conserve failure domain in $\boldsymbol{X}$ space. $G_{\min}(\boldsymbol{x})=\min\limits_{\boldsymbol{y}\in\boldsymbol{y}_I^u}G(\boldsymbol{x},\boldsymbol{y})$ denotes the minimum performance function of $\boldsymbol{x}$. $I_F(\boldsymbol{x})$ denotes the indicator function of failure domain F. If $\boldsymbol{x}\in F$, $I_F(\boldsymbol{x})=1$, otherwise $I_F(\boldsymbol{x})=0$.

The universal method for estimating failure probability under RIU is Monte Carlo simulation (MCS) method, which offers the benefits of dimension-independent convergence and ease of implementation. The convergent result of MCS is typically regarded as the reference solution for validating other new methods. However, for common small failure probability problems in engineering, MCS demands a large sample size to achieve convergence, which limits its applicability to problems involving time-consuming implicit performance functions (Lee et al. 2022). To reduce the number of calling performance functions, Kriging model is adaptively embedded into MCS (Wang and Lu 2020). However, for the extremely small failure probability problems, this method will consume considerable time in identifying the state of the large size candidate sample pool (Lu et al. 2024).

To further reduce candidate sample pool size required by gaining convergent failure probability estimation, the importance sampling (IS) variance reduction strategy is proposed. The basic idea of IS strategy is to construct an IS-PDF that enables the generation of more samples in the failure domain, thereby enhancing sampling efficiency and reducing the variance of failure probability estimation (Tabandeh et al. 2022). To approach the theoretically optimal IS-PDF, the Gaussian mixture model combined with cross-entropy based IS method (Kurtz and Song 2013) and metamodel based IS method have been proposed (Dubourg et al. 2013). Although the Gaussian mixture model combined with cross-entropy based IS-PDF is regular and easy sampling, solving undetermined parameter set becomes challenging for problems involving complex failure domains. In the metamodel based IS method, the IS-PDF

constructed using the metamodel of the performance function is irregular, which brings difficulty in generating IS samples accurately and efficiently.

To address the challenge of accurately and efficiently estimating extremely small failure probability under RIU, this paper proposes an adaptive radial sampling combined with stratified mixture importance sampling method (RS-SMIS). The proposed RS-SMIS first establishes a hypersphere with relatively large radius is in the standard normal space. Then the samples outside the hypersphere are generated by radial sampling to explore the region far away from the coordinate origin. Subsequently, the rare failure domain is quickly approached by the efficient adaptively stratifying strategy. In the stratifying strategy, an explicit and regular mixture importance sampling probability density function is gradually established through clustering analysis for the failure samples of each layer. To further reduce the number of calling performance function when establishing the mixture importance sampling density and identifying the states of candidate samples, the adaptive Kriging model is inserted to RS-SMIS (AK-RS-SMIS).

The rest of the work is organized as follows. Section 2 reviews the basic principle of IS. Section 3 establishes the AK-RS-SMIS method for efficiently estimating extremely small failure probability under RIU. Section 4 verifies the superiority of the proposed AK-RS-SMIS by two examples. Section 5 concludes the whole paper.

2 Basic principle of IS

By the isoprobabilistic transformation $\boldsymbol{U}=T(\boldsymbol{X})$, $\boldsymbol{X}$ can be converted into the standard normal vector $\boldsymbol{U}=\{U_1,U_2,\cdots,U_n\}^{\mathrm{T}}$. The corresponding performance function in $\boldsymbol{U}$ space can be obtained by $g(\boldsymbol{u},\boldsymbol{y})=G[T^{-1}(\boldsymbol{u}),\boldsymbol{y}]$ and the corresponding minimum performance function can be obtained by $g_{\min}(\boldsymbol{u})=G_{\min}[T^{-1}(\boldsymbol{u})]$. Then failure probability P_f under RIU can be rewritten as follows,

$$P_f=\int I_F(\boldsymbol{u})f_U(\boldsymbol{u})\mathrm{d}\boldsymbol{u} \tag{2}$$

After introducing the IS-PDF $h_U(\boldsymbol{u})$ into Eq. (2), the failure probability P_f under RIU can be reformulated as follows,

$$P_f=\int I_F(\boldsymbol{u})f_U(\boldsymbol{u})\mathrm{d}\boldsymbol{u}=\int \frac{I_F(\boldsymbol{u})f_U(\boldsymbol{u})}{h_U(\boldsymbol{u})}h_U(\boldsymbol{u})\mathrm{d}\boldsymbol{u}=E_h\left[\frac{I_F(\boldsymbol{u})f_U(\boldsymbol{u})}{h_U(\boldsymbol{u})}\right] \tag{3}$$

where $E_h[\cdot]$ denotes the expectation operator with weight density function $h_U(\boldsymbol{u})$.

According to the law of large numbers, after generating an N_h -size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^h=\{\boldsymbol{u}_1^h,\boldsymbol{u}_2^h,\cdots,\boldsymbol{u}_{N_h}^h\}$ by $h_U(\boldsymbol{u})$, P_f in Eq. (3) can be evaluated by the sample mean $\hat{P}_f$ in the following Eq. (4),

$$\hat{P}_f=\frac{1}{N_h}\sum_{i=1}^{N_h}\frac{I_F(\boldsymbol{u}_i^h)f_U(\boldsymbol{u}_i^h)}{h_U(\boldsymbol{u}_i^h)} \tag{4}$$

According to the classical statistical theory, if $h_U(\boldsymbol{u})=h_U^{\mathrm{opt}}(\boldsymbol{u})=I_F(\boldsymbol{u})f_U(\boldsymbol{u})/P_f$, and then the variance of $\hat{P}_f$ will become zero. Thus $h_U^{\mathrm{opt}}(\boldsymbol{u})$ is known as the optimal IS-PDF. Since that $h_U^{\mathrm{opt}}(\boldsymbol{u})$ is irregular and related to the unsolved failure probability P_f as well as the failure domain indicator function $I_F(\boldsymbol{u})$, it is time-consuming to extract samples of $h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{u})$. Fortunately, it can be observed from the form of $h_{\boldsymbol{X}}^{\mathrm{opt}}(\boldsymbol{u})$ that if the most of samples with large $f_U(\boldsymbol{u})$ values of the established $h_U(\boldsymbol{u})$ fall into the failure domain, and then the variance of $\hat{P}_f$ can be reduced. To efficiently explore the rare failure domain and construct $h_U(\boldsymbol{u})$, Section 3 proposes an adaptive Kriging-assisted radial sampling combined with stratified mixture importance sampling method.

3 Adaptive Kriging-assisted radial sampling combined with stratified mixture importance sampling

In this section, the proposed adaptive Kriging-assisted radial sampling combined with stratified mixture importance sampling method (AK-RS-SMIS) will be introduced by three subsections. Subsection 3.1 constructs an adaptive radial sampling combined with stratified mixture importance sampling. Subsection 3.2 illustrates the MIS-PDF construction and sample state identification assisted by adaptive Kriging model. Subsection 3.3 provides the detailed steps.

3.1 *Adaptive radial sampling combined with stratified mixture importance sampling*

3.1.1 *Radial sampling outside a hypersphere with preset radius*

Preset the initial failure probability level P_{f0} ($P_{f0}=1\times10^{-5}$ is suggested in this paper), and then the preset radius of the hypersphere can be determined by the following equation,

$$r_0=\sqrt{\chi_n^{-2}(1-P_{f0})} \tag{5}$$

where $\chi_n^{-2}(\cdot)$ denotes the of inverse cumulative distribution function of chi-square distribution with *n* degrees of freedom. An N_1 -size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{(1)}=\{\boldsymbol{u}_1^{(1)},\boldsymbol{u}_2^{(1)},\cdots,\boldsymbol{u}_{N_1}^{(1)}\}$ outside the hypersphere with radius r_0 can be generated by the following two steps (Hong et al. 2025).

Step (1). Generate N_1 samples $\boldsymbol{z}_i$ $(i=1,2,\cdots,N_1)$ following the standard normal distribution and N_1 samples v_i $(i=1,2,\cdots,N_1)$ following the uniform distribution in the interval $[0,1]$.

Step (2). Obtain $\boldsymbol{u}_i^{(1)}$ $(i=1,2,\cdots,N_1)$ by the following equation.

$$\boldsymbol{u}_i^{(1)}=\frac{\boldsymbol{z}_i}{\|\boldsymbol{z}_i\|}\cdot\sqrt{\chi_n^{-2}[(1-\chi_n^2(r_0^2))\cdot v_i+\chi_n^2(r_0^2)]} \tag{6}$$

3.1.2 *Stratified mixture importance sampling*

To construct the MIS-PDF $h_U(\boldsymbol{u})$ for efficiently exploring the rare target failure domain and estimating extremely small failure probability, this subsection establishes an adaptively stratifying strategy on the basis of the radial sampling in subsection 3.1.1.

First, randomly interval input vector candidate sample pool $\boldsymbol{S}_y=\{\boldsymbol{y}_1,\boldsymbol{y}_2,\cdots,\boldsymbol{y}_{N_y}\}^{\mathrm{T}}$ by uniformly discretizing $\boldsymbol{y}_l^u$. Preset the failure samples ratio ρ ($\rho=0.3$ is suggested in this paper) and initialize the MIS-PDF $h_U^{(1)}(\boldsymbol{u})=f_U(\boldsymbol{u};0,1)$. Evaluate $g_{\min}(\boldsymbol{u}_i^{(1)})\approx\min\limits_{\boldsymbol{y}\in\boldsymbol{S}_y}g(\boldsymbol{u}_i^{(1)},\boldsymbol{y})$ $(i=1,2,\cdots,N_1)$ and gain $g_{\min}(\boldsymbol{u}_{[1]}^{(1)})<g_{\min}(\boldsymbol{u}_{[2]}^{(1)})<\cdots<g_{\min}(\boldsymbol{u}_{[N_1]}^{(1)})$ sorted ascendingly. Set $b_1=g_{\min}(\boldsymbol{u}_{[\rho N_1]}^{(1)})$ and the first intermediate failure domain $F_1=\{g_{\min}(\boldsymbol{u})\le b_1\}$. Then the stratified mixture importance sampling process for estimating failure probability can be divided into two cases according to the relation between b_1 and 0.

Case I. $b_1\ge 0$

In this case, the failure probability P_f under RIU is smaller than P_{f0}. Therefore, the subsequent thresholds for stratifying failure domains satisfy $b_1<b_2<b_3<\cdots<b_m=0$. Screen an N_{F_1}-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{F_1}=\{\boldsymbol{u}_1^{F_1},\boldsymbol{u}_2^{F_1},\cdots,\boldsymbol{u}_{N_{F_1}}^{F_1}\}$ in F_1 from $\boldsymbol{S}_{\boldsymbol{u}}^{(1)}$. Perform clustering analysis in $\boldsymbol{S}_{\boldsymbol{u}}^{F_1}$ to gain an $N_c^{(1)}$-size centroid set $\boldsymbol{S}_{\boldsymbol{u}}^{c(1)}=\{\boldsymbol{u}_1^{c(1)},\boldsymbol{u}_2^{c(1)},\cdots,\boldsymbol{u}_{N_c^{(1)}}^{c(1)}\}$ and establish MIS-PDF $h_U^{(2)}(\boldsymbol{u})$ by the following equation,

$$h_U^{(2)}(\boldsymbol{u})=\frac{1}{N_c^{(1)}}\sum_{i=1}^{N_c^{(1)}}f_U(\boldsymbol{u};\boldsymbol{u}_i^{c(1)},1) \tag{7}$$

Then generate an N_2-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{(2)}=\{\boldsymbol{u}_1^{(2)},\boldsymbol{u}_2^{(2)},\cdots,\boldsymbol{u}_{N_2}^{(2)}\}$ by $h_U^{(2)}(\boldsymbol{u})$. Evaluate $g_{\min}(\boldsymbol{u}_i^{(2)})\approx\min\limits_{\boldsymbol{y}\in\boldsymbol{S}_y}g(\boldsymbol{u}_i^{(2)},\boldsymbol{y})$ $(i=1,2,\cdots,N_2)$ and gain $g_{\min}(\boldsymbol{u}_{[1]}^{(2)})<g_{\min}(\boldsymbol{u}_{[2]}^{(2)})<\cdots<g_{\min}(\boldsymbol{u}_{[N_2]}^{(2)})$ sorted ascendingly. Set $b_2=g_{\min}(\boldsymbol{u}_{[\rho N_2]}^{(2)})$ and the second intermediate failure domain $F_2=\{g_{\min}(\boldsymbol{u})\le b_2\}$. Screen an N_{F_2}-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{F_2}=\{\boldsymbol{u}_1^{F_2},\boldsymbol{u}_2^{F_2},\cdots,\boldsymbol{u}_{N_{F_2}}^{F_2}\}$ in F_2 from $\boldsymbol{S}_{\boldsymbol{u}}^{(2)}$. Perform clustering analysis in $\boldsymbol{S}_{\boldsymbol{u}}^{F_2}$ to gain an $N_c^{(2)}$-size centroid set $\boldsymbol{S}_{\boldsymbol{u}}^{c(2)}=\{\boldsymbol{u}_1^{c(2)},\boldsymbol{u}_2^{c(2)},\cdots,\boldsymbol{u}_{N_c^{(2)}}^{c(2)}\}$ and establish MIS-PDF $h_U^{(2)}(\boldsymbol{u})$ by the following equation,

$$h_U^{(3)}(\boldsymbol{u})=\frac{1}{N_c^{(2)}}\sum_{i=1}^{N_c^{(2)}}f_U(\boldsymbol{u};\boldsymbol{u}_i^{c(2)},1) \tag{8}$$

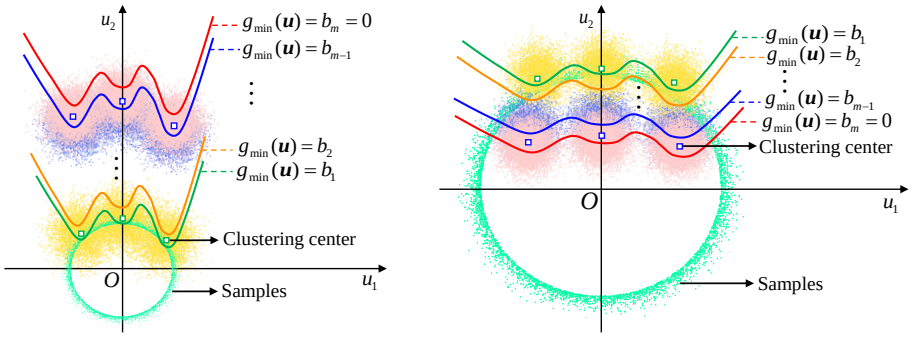
Continue the above strategy for establishing the MIS-PDF in each layer until the rare target failure domain $F_m=\{g_{\min}(\boldsymbol{u})\le b_m\}=\{g(\boldsymbol{u})_{\min}\le 0\}$ is approached. Let $h_U(\boldsymbol{u})=h_U^{(m)}(\boldsymbol{u})$ and generate an N_h-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{h}=\{\boldsymbol{u}_1^{h},\boldsymbol{u}_2^{h},\cdots,\boldsymbol{u}_{N_h}^{h}\}$ by $h_U(\boldsymbol{u})$, and then estimate P_f by Eq. (4).

Case II. $b_1<0$

In this case, the failure probability P_f under RIU is larger than P_{f0}. Therefore, the subsequent thresholds for stratifying failure domains satisfy $b_1>b_2>b_3>\cdots>b_m=0$. Screen an N_{F_1}-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{F_1}=\{\boldsymbol{u}_1^{F_1},\boldsymbol{u}_2^{F_1},\cdots,\boldsymbol{u}_{N_{F_1}}^{F_1}\}$ in F_1 from $\boldsymbol{S}_{\boldsymbol{u}}^{(1)}$. Perform clustering analysis in $\boldsymbol{S}_{\boldsymbol{u}}^{F_1}$ to gain an $N_c^{(1)}$-size centroid set $\boldsymbol{S}_{\boldsymbol{u}}^{c(1)}=\{\boldsymbol{u}_1^{c(1)},\boldsymbol{u}_2^{c(1)},\cdots,\boldsymbol{u}_{N_c^{(1)}}^{c(1)}\}$ and establish MIS-PDF $h_U^{(2)}(\boldsymbol{u})$ by Eq. (7). Then generate an N_2-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{(2)}=\{\boldsymbol{u}_1^{(2)},\boldsymbol{u}_2^{(2)},\cdots,\boldsymbol{u}_{N_2}^{(2)}\}$ by $h_U^{(2)}(\boldsymbol{u})$. Evaluate $g_{\min}(\boldsymbol{u}_i^{(2)})\approx\min\limits_{\boldsymbol{y}\in\boldsymbol{S}_y}g(\boldsymbol{u}_i^{(2)},\boldsymbol{y})$ $(i=1,2,\cdots,N_2)$ and gain $g_{\min}(\boldsymbol{u}_{[1]}^{(2)})<g_{\min}(\boldsymbol{u}_{[2]}^{(2)})<\cdots<g_{\min}(\boldsymbol{u}_{[N_2]}^{(2)})$ sorted ascendingly. Set $b_2=g_{\min}(\boldsymbol{u}_{[(1-\rho)N_2]}^{(2)})$ and the second intermediate failure domain $F_2=\{g_{\min}(\boldsymbol{u})\le b_2\}$. Screen an N_{F_2}-size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^{F_2}=\{\boldsymbol{u}_1^{F_2},\boldsymbol{u}_2^{F_2},\cdots,\boldsymbol{u}_{N_{F_2}}^{F_2}\}$ in F_2 from $\boldsymbol{S}_{\boldsymbol{u}}^{(2)}$. Perform clustering analysis in $\boldsymbol{S}_{\boldsymbol{u}}^{F_2}$ to gain an $N_c^{(2)}$-size centroid set $\boldsymbol{S}_{\boldsymbol{u}}^{c(2)}=\{\boldsymbol{u}_1^{c(2)},\boldsymbol{u}_2^{c(2)},\cdots,\boldsymbol{u}_{N_c^{(2)}}^{c(2)}\}$ and establish MIS-PDF $h_U^{(2)}(\boldsymbol{u})$ by Eq. (8). Continue the above strategy for establishing the MIS-PDF in each layer until the rare target failure domain $F_m=\{g_{\min}(\boldsymbol{u})\le b_m\}=\{g(\boldsymbol{u})_{\min}\le 0\}$ is

approached. Let $h_U(\boldsymbol{u}) = h_U^{(m)}(\boldsymbol{u})$ and generate an N_h -size sample pool $\boldsymbol{S}_{\boldsymbol{u}}^h = \{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h, \cdots, \boldsymbol{u}_{N_h}^h\}$ by $h_U(\boldsymbol{u})$, and then estimate P_f by Eq. (4).

The process of RS-SMIS for exploring the rare target failure domain is illustrated in Figure 1.



(a) Case I: $b_1 \geq 0$ (b) Case II: $b_1 < 0$

Figure 1. The process of RS-SMIS for exploring the rare target failure domain

3.2 *MIS-PDF construction and sample state identification assisted by adaptive Kriging model*

In this subsection, the double-loop adaptive Kriging model (DL-AK) of performance function $g(\boldsymbol{u}, \boldsymbol{y})$ is embedded into RS-SMIS to efficiently estimate failure probability under RIU. In the outer loop of DL-AK, the Kriging model $\hat{g}_{\min}(\boldsymbol{u})$ of the minimum performance function $g_{\min}(\boldsymbol{u})$ is adaptively trained to be convergent for establishing $h_U(\boldsymbol{u})$. In the inner loop of DL-AK, the Kriging model $\hat{g}(\boldsymbol{u}^*, \boldsymbol{y})$ of $g(\boldsymbol{u}^*, \boldsymbol{y})$ is established at the given random input vector realization $\boldsymbol{u}^*$ from the outer loop. The purpose of establishing $\hat{g}(\boldsymbol{u}^*, \boldsymbol{y})$ is to gain the minimum value of the performance function $g(\boldsymbol{u}^*, \boldsymbol{y})$ over interval input vector $\boldsymbol{y}$ using the convergent $\hat{g}(\boldsymbol{u}^*, \boldsymbol{y})$. The detailed construction of the convergent inner loop Kriging model $\hat{g}(\boldsymbol{u}^*, \boldsymbol{y})$ can refer to reference (Wang and Lu 2020).

In the process of establishing $h_U(\boldsymbol{u})$, the convergence of approximating $g_{\min}(\boldsymbol{u})$ by $\hat{g}_{\min}(\boldsymbol{u})$ can be measured by the difference between "1" and leave-one-out cross-validation value α_{LOO} (Lu et al. 2025).

After generating the $\boldsymbol{S}_{\boldsymbol{u}}^h = \{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h, ..., \boldsymbol{u}_{N_h}^h\}$ by $h_U(\boldsymbol{u})$ and convergent $\hat{g}_{\min}(\boldsymbol{u})$, $I_F(\boldsymbol{u}_i^h)$ $(\boldsymbol{u}_i^h \in \boldsymbol{S}_{\boldsymbol{u}}^h)$ still needs to estimate accurately to estimate failure probability by RS-SMIS. Thus, this paper adopts the training strategy in reference (Yan et al. 2024) to further update $\hat{g}_{\min}(\boldsymbol{u})$ until $\hat{g}_{\min}(\boldsymbol{u})$ can identify the state of candidate samples in $\boldsymbol{S}_{\boldsymbol{u}}^h$.

Then $I_F(\boldsymbol{u}_i^h)$ $(\boldsymbol{u}_i^h \in \boldsymbol{S}_{\boldsymbol{u}}^h)$ can evaluated by the prediction mean $\mu_{\hat{g}_{\min}}(\boldsymbol{u})$ of the convergent $\hat{g}_{\min}(\boldsymbol{u})$, i.e., If $\mu_{\hat{g}_{\min}}(\boldsymbol{u}_i^h) \leq 0$, $I_F(\boldsymbol{u}_i^h) \approx 1$, otherwise $I_F(\boldsymbol{u}_i^h) \approx 0$. Substitute $I_F(\boldsymbol{u}_i^h)$ $(\boldsymbol{u}_i^h \in \boldsymbol{S}_{\boldsymbol{u}}^h)$ into Eq. (4) and estimate P_f.

3.3 *Detailed steps of AK-RS-SMIS*

According to the illustrations in subsections 3.1 and 3.2, the detailed steps of the proposed AK-RS-SMIS are shown as follows.

Step (1). Form the initial training set and construct the Kriging model $\hat{g}_{\min}(\boldsymbol{u})$.

Step (2). Update $\hat{g}_{\min}(\boldsymbol{u})$ until it can approximate $g_{\min}(\boldsymbol{u})$.

Step (3). Construct MIS-PDF $h_U(\boldsymbol{u})$ by $g_{\min}(\boldsymbol{u}) \approx \hat{g}_{\min}(\boldsymbol{u})$ using the method in subsection 3.1.2.

Step (4). Generate an N_h -size candidate sample pool $\boldsymbol{S}_{\boldsymbol{u}}^h = \{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h, \cdots, \boldsymbol{u}_{N_h}^h\}$ by the convergent $h_U(\boldsymbol{u})$.

Step (5). By the strategy in subsection 3.2, update $\hat{g}_{\min}(\boldsymbol{u})$ in $\boldsymbol{S}_{\boldsymbol{u}}^h$ until it can evaluate $I_F(\boldsymbol{u}_i)$ $(\boldsymbol{u}_i \in \boldsymbol{S}_{\boldsymbol{u}}^h)$ accurately.

Step (6). Evaluate $I_F(\boldsymbol{u}_i)$ $(\boldsymbol{u}_i \in \boldsymbol{S}_{\boldsymbol{u}}^h)$ by the prediction mean $\mu_{\hat{g}_{\min}}(\boldsymbol{u})$ of the convergent $\hat{g}_{\min}(\boldsymbol{u})$.

Step (7). Estimate failure probability P_f under RIU by Eq. (4).

4 Examples

In this section, the accuracy and efficiency of the proposed method are verified by two examples including a numerical example and a 23-bar truss structure. And the proposed AK-RS-SMIS is compared with other three computational methods, i.e., Monte Carlo simulation (MCS), double-loop adaptive Kriging model combined with Monte Carlo simulation (AK-MCS) (Wang and Lu 2020) and double-loop adaptive Kriging model based meta-model importance sampling (AK-Meta-IS) (Lu et al. 2024). For each method used in these examples, 20 independent runs are repeated to gain 20 average estimations of failure probability under RIU. Moreover, to check the robustness of these comparative methods, the variation coefficient of the 20 failure probability estimations is calculated.

4.1 *Numerical example*

A numerical example with two failure modes is utilized to demonstrate the advantage of the proposed method. The performance function is given as follows,

$$G(\boldsymbol{X},\boldsymbol{Y}) = \min\begin{cases} 0.1(X_1 - X_2)^2 - (X_1 + X_2)/\sqrt{2} + Y_1 \\ Y_1 - \sin\left(\dfrac{\pi(Y_1 - 1)}{2}\right) X_1 X_2 \end{cases} \tag{9}$$

where the random input vector $\boldsymbol{X} = \{X_1, X_2\}^{\mathrm{T}}$ is independently and normally distributed. $\boldsymbol{Y} = Y_1$ is the interval input vector and the corresponding distributed range is $[y_1^l,\ y_1^l + 1]$.

When y_1^l takes different values (6 and 10), the proposed AK-RS-SMIS is compared with other three methods (MCS, AK-MCS, AK-Meta-IS) and the comparative results are listed in Table 1, where N_{call} denotes the average number of calling performance functions. Relative error measures the accuracy of AK-MCS, AK-Meta-IS and the proposed AK-RS-SMIS when taking the failure probability estimation result of MCS as the reference. From Table 1, it can be observed that the four comparative methods gain the similar failure probability estimations and variation coefficients when the constant y_1^l is preset. Taking the situation when $y_1^l = 10$ as an example, MCS requires the most performance function callings (1×10^{10}), while the proposed AK-RS-SMIS requires the least one (96.5). Moreover, the computational time of the proposed AK-RS-SMIS (1.51 s) is shorter than that of AK-MCS (7677.53 s) and AK-Meta-IS (1177.72 s), which indicates the proposed AK-RS-SMIS has higher efficiency than the other two adaptive Kriging model based methods.

Table 1. The comparative results for estimating the failure probability in example 4.1

y_1^l	Method	$\hat{P}_f$	Cov($\hat{P}_f$)	Relative error	N_{call}	Time (s)
6	MCS	3.687×10^{-4}	0.0400	-	$1.7\times10^{6}\times10^{2}$	0.062
	AK-MCS	3.725×10^{-4}	0.0399	1.03%	144.58	152.42
	AK-Meta-IS	3.685×10^{-4}	0.0404	0.05%	100	73.88
	AK-RS-SMIS	3.720×10^{-4}	0.0401	0.89%	95.7	1.48
10	MCS	5.395×10^{-6}	0.0392	-	$1\times10^{8}\times10^{2}$	8.13
	AK-MCS	5.375×10^{-6}	0.0408	0.37%	148.3	7677.53
	AK-Meta-IS	5.373×10^{-6}	0.0405	0.41%	110.0	1177.72
	AK-RS-SMIS	5.408×10^{-6}	0.0396	0.24%	96.5	1.51

4.2 *A 23-bar truss structure*

A 23-bar truss structure shown in Figure 2 is considered in this example to show the practicability of the proposed method in the engineering structures with implicit performance function and multiple random inputs (Yan and Lu 2025). As illustrated in Figure 2, the structure contains 11 horizontal bars and 12 sloping bars. A_1, E_1 denote horizontal bars' cross-section area and Young's modulus respectively, while A_2, E_2 denote sloping bars' cross-section area and Young's modulus respectively. Six loads, from P_1 to P_6, are applied on the nodes of horizontal bars. $\boldsymbol{X} = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}\}^{\mathrm{T}} = \{P_1, P_2, P_3, P_4, P_5, P_6, A_1, A_2, E_1, E_2\}^{\mathrm{T}}$ is the independent and normal random input vector and $\boldsymbol{Y} = \{Y_1, Y_2\}^{\mathrm{T}} = \{H, L\}^{\mathrm{T}}$ is the interval input vector. Table 2 presents the concrete types and parameters of inputs of the truss structure. In this example, the displacement of node D is denoted by $u_D(\boldsymbol{X},\boldsymbol{Y})$ and should not exceed the threshold $u^{\mathrm{cr}} = 0.12\mathrm{m}$. The maximum stress of horizontal bars is denoted by $\sigma_1(\boldsymbol{X},\boldsymbol{Y})$ and should not exceed the threshold $\sigma_1^{\mathrm{cr}} = 325\mathrm{MPa}$. The maximum stress of horizontal bars is denoted by $\sigma_2(\boldsymbol{X},\boldsymbol{Y})$ and should not exceed the threshold $\sigma_2^{\mathrm{cr}} = 285\mathrm{MPa}$. Thus, the performance function $G(\boldsymbol{X},\boldsymbol{Y})$ of the structure is expressed as,

$$G(\boldsymbol{X},\boldsymbol{Y}) = \min\begin{cases} u^{\mathrm{cr}} - u_D(\boldsymbol{X},\boldsymbol{Y}) \\ \sigma_1^{\mathrm{cr}} - \sigma_1(\boldsymbol{X},\boldsymbol{Y}) \\ \sigma_2^{\mathrm{cr}} - \sigma_2(\boldsymbol{X},\boldsymbol{Y}) \end{cases} \tag{10}$$

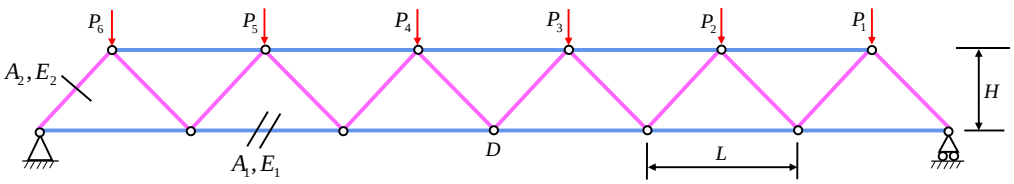


Figure 2. The 23-bar truss structure

Table 2. Types and parameters of inputs in example 4.2

Gumbel input (unit)	Notation	Mean	Standard deviation
P_1 (N)	X_1	8100	405
P_2 (N)	X_2	8200	410
P_3 (N)	X_3	8300	415
P_4 (N)	X_4	8400	420
P_5 (N)	X_5	8500	425
P_6 (N)	X_6	8600	430

Lognormal input (unit)	Notation	Mean	Standard deviation
A_1 (m^2)	X_7	7.5×10^{-4}	3.75×10^{-5}
A_2 (m^2)	X_8	7.0×10^{-4}	3.5×10^{-5}
E_1 (GPa)	X_9	74	3.7
E_2 (GPa)	X_{10}	66	3.3

Interval input (unit)	Notation	Lower bound	Upper bound
H (m)	Y_1	1.95	2
L (m)	Y_2	3.9	4

Since a large number of finite element analyses of the structure are required, MCS is too time-consuming to handle this problem. Thus, three adaptive Kriging model based methods including AK-MCS, AK-Meta-IS and the proposed AK-RS-SMIS are employed to evaluate the failure probability for cross-validation, respectively. The comparative results of the above three methods are listed in Table 3. It can be found from Table 3 that the three methods gain the similar failure probability estimations under the close variation level. Compared with AK-MCS, the proposed AK-RS-SMIS can significantly reduce the number of calling and the computational time since it has much smaller candidate sample pool. Compared with AK-Meta-IS, the proposed AK-RS-SMIS can reduce the computational time due to its more efficient sampling strategy.

Table 3. The comparative results for estimating the failure probability of example 4.2

Method	$\hat{P}_f$	Cov($\hat{P}_f$)	N_{call}	Time (s)
AK-MCS	6.082×10^{-5}	0.0407	408.7	17654.27
AK-Meta-IS	6.057×10^{-5}	0.0406	285.8	3163.25
AK-RS-SMIS	6.028×10^{-5}	0.0404	265.2	332.84

5 Conclusions

To efficiently estimate the extremely small failure probability under random-interval uncertainty, this paper proposes an adaptive method by combining radial sampling combined with stratified mixture importance sampling. To more efficiently construct the mixture importance sampling density and identify the states of candidate samples, the adaptive Kriging model is inserted to further reduce the performance function evaluations. The efficiency and accuracy of the proposed method are verified by a numerical example and an engineering example. Compared with adaptive Kriging model combined with Monte Carlo simulation method, the proposed method can improve the efficiency due to its much smaller candidate sample pool. Compared with adaptive Kriging model based meta-model importance sampling, the proposed method can reduce the runtime benefitting from its more efficient sampling strategy. Overall, the proposed method performs stronger applicability for engineering problems with small failure probability.

Acknowledgments

This research received financial support from the National Natural Science Foundation of China (NSFC Grant Nos. 12572141 and 12272300).

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