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ON THE SOLUTION SET OF A SYSTEM OF LINEAR INTERVAL EQUATIONS

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Abstract.

In this paper, a necessary and sufficient condition for the solution set of a system of linear interval equations to be convex is given. It is also discussed the relation between the convexity and feasibility of an optimization problem.

Key words: linear interval equations, convexity.

1. Introduction.

We are concerned with the system

$$(1) \qquad \tilde{A}\tilde{x} = \tilde{b} \qquad (\tilde{A} \in A,\ \tilde{b} \in b),$$

where A is a regular interval matrix (i.e., $\forall \tilde{A} \in A$ are nonsingular) and b is an interval vector, which originated from a practical situation where the coefficients and the right-hand vector cannot be sharply defined. However, a large part of researchers in this field aimed at evaluating the hull of the solution set of (1) efficiently [2], [3], so there are few literatures deal with the

solution set. Among them, Oettli [5] showed the intersection of the solution set with each orthant is a convex polytope. Recently Rohn [9] proposed a necessary and sufficient condition for the solution set to be nonconvex by utilizing solutions of the auxiliary equations.

In this paper, we proposed a necessary and sufficient condition for the solution set to be convex under the condition where the coefficient interval matrix is inverse positive, which is usually assumed in considering solution methods to (1) [3].

The terminology follows Neumaier [3]; we will summarize it briefly for convenience sake. Let $\mathbf{IR}, \mathbf{IR}^n$, and $\mathbf{IR}^{n \times n}$ be the set of real closed intervals, n-dimensional interval vectors and $n \times n$ interval matrices, respectively. For each $A \in \mathbf{IR}^{n \times n}$, we write $A_c := (\overline{A} + \underline{A})/2$ for the midpoint (matrix), $\Delta := (\overline{A} - \underline{A})/2$ for the radius (matrix), and $|A| := \sup\{|\tilde{A}| \mid \tilde{A} \in A\} = |\check{A}| + \rho(A)$ for the absolutely value (matrix). The *Ostrowski operator* $\langle \cdot \rangle$ is defined for $A \in \mathbf{IR}^{n \times n}$ as $\langle A \rangle = A'$, where $A'_{ii} := \min\{|\tilde{a}| \mid \tilde{a} \in A_{ii}\}$ and $A'_{ij} := -|A_{ij}|$ for $i \neq j$. The *hull* of a bounded subset $\Sigma \subseteq \mathbf{R}^n$ is defined as $\square\Sigma := [\inf \Sigma, \sup \Sigma]$ (componentwise), the smallest interval vector containing Σ. Note that sum, difference and inverse for $A, B \in \mathbf{IR}^{n \times n}$ are defined by

$$A \pm B := \square\{\tilde{A} \pm \tilde{B} \mid \tilde{A} \in A, \ \tilde{B} \in B\},$$

$$A^{-1} := \square\{\tilde{A}^{-1} \mid \tilde{A} \in A\}.$$

We introduce the norm for $A \in \mathbf{IR}^{n \times n}$ by $\|A\| := \|\sup_{\|x\|=1} \||A|x|\|$.

2. Results for inverse positive matrices.

Oettli and Prager [6] gave the following explicit expression for the solution set X of (1):

$$(2) \qquad X = \{x \mid |A_c x - b_c| \le \Delta |x| + \delta\},$$

where $|x|_i = |x_i|$, $i = 1, ..., n$, from which it is shown that the intersection of X with each orthant is a convex polytope [5].

Let Y be the set of 2^n vectors such that $Y = \{y \in \mathbf{R}^n \mid |y_j| = 1, \ j = 1, ..., n\}$ and T_y be diag(y). Rohn [10] considers the following equation associated with (1):

$$(3) \qquad A_c x - b_c = T_y(\Delta |x| + \delta),$$

which has a unique solution $x_y \in X$ for each $y \in Y$ constructing conv X in total. The main theorem in [9] is as follows.

THEOREM 1[9]: *Let $A \in \mathbf{IR}^{n \times n}$ be regular. Then the solution set X of (1) is nonconvex if and only if there exist $y, z \in Y$, $i, j \in \{1, ..., n\}$ such that $y_i = z_i$, $(x_y)_j (x_z)_j < 0$, $\Delta_{ij} > 0$.* □

By substituting $x = x^+ - x^-$, $|x| = x^+ + x^-$ into (3) and setting $A_c - T_y \Delta = A_{ye}$, $A_c + T_y \Delta = A_{yf}$, $b_y = b_c + T_y \delta$, we get

$$(4) \qquad x^+ = A_{ye}^{-1} A_{yf} x^- + A_{ye}^{-1} b_y.$$

Each x_y exists and can be obtained by solving the above linear complementary problem (see [10]). Rohn's results are, however, rather conceptual because it is troublesome to solve (3) for all $y \in Y$.

Let us consider the case where the coefficient matrix A is inverse positive. This assumption is representatively satisfied when the coefficient matrix $A \in \mathbf{IR}^{n \times n}$ is an M-*matrix* in the sence of [3].

We begin with the following facts.

FACT 1: Let $A \in \mathbf{IR}^{n \times n}$ be such that $A^{-1} \geq 0$. Then the solution set X of (1) is convex if one of the following conditions holds.

(a) $b \in \mathbf{IR}^n$ satisfies $b \geq \mathbf{0}$ or $b \leq \mathbf{0}$.

(b) b is thin (i.e., $\delta = \mathbf{0}$).

PROOF: (a): From the properties of inverse positive matrices, the solution set $X = \{x \mid x = \tilde{A}^{-1}\tilde{b},\ \tilde{A} \in A,\ \tilde{b} \in b\} \geq \mathbf{0}$ or $X \leq \mathbf{0}$. Since X is part of a single orthant, it is convex from $[5]$.

(b): For $x^1, x^2 \in X$, there exist $\tilde{A}^1, \tilde{A}^2 \in A$ such that $\tilde{A}^1 x^1 = b$, $\tilde{A}^2 x^2 = b$. Then, since for any $0 \leq \lambda \leq 1$ it holds

$$
\begin{aligned}
x^c &= (1-\lambda)x^1 + \lambda x^2 \\
&= (1-\lambda)x^1 + \lambda(\tilde{A}^2)^{-1}\tilde{A}^1 x^1 = (I - \lambda(I - (\tilde{A}^2)^{-1}\tilde{A}^1))x^1,
\end{aligned}
$$

the matrix $\tilde{A}^c$ satisfying $\tilde{A}^c x^c = b$ can be written as $\tilde{A}^c = \tilde{A}^1(I - \lambda(I - (\tilde{A}^2)^{-1}\tilde{A}^1))^{-1}$.

Let D be the diagonal matrix such that $d_{ii} = 1$ for $b_i \geq 0$ and $d_{jj} = -1$ for $b_j < 0$, and consider the problem

$$
(5) \qquad\qquad \tilde{A}u = D\tilde{A}x = Db.
$$

Since the solution set of (4) is convex by (a), for $u^c = (1-\lambda)u^1 + \lambda u^2$ where $\tilde{A}^1 u^1 = \tilde{A}^2 u^2 = Db$, there exists the matrix $B \in A$ satisfying $Bu^c = Db$. In fact $B = \tilde{A}^1(I - \lambda(I - (\tilde{A}^2)^{-1}\tilde{A}^1))^{-1} = \tilde{A}^c$, which completes the proof.

$\square$

EXAMPLE **1:** Consider (1) with

$$A = \begin{pmatrix} 1 & [-\frac{1}{2}, 0] \\ [-\frac{1}{2}, 0] & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Since

$$A^{-1} = \begin{pmatrix} [1, \frac{4}{3}] & [0, \frac{2}{3}] \\ [0, \frac{2}{3}] & [1, \frac{4}{3}] \end{pmatrix},$$

X is shown to be convex from Theorem 2(b). In fact, X is the triangle whose vertexes are $(0, -2)$, $(1, -2)$, and $(1, -\frac{3}{2})$.

On the other hand, the next fact holds with a slightly restrictive condition imposed on A.

FACT **2:** Let $A \in \mathbf{IR}^{n \times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq O$, and $\Delta > O$. Then the solution set X is nonconvex if $0 \in int\ b$.

PROOF: It follows from [10, Theorem 4.7] that $\overline{x} = x_y$, $\underline{x} = x_{-y}$ for $y = e = (1, ..., 1)^T$. Since $\overline{x} \geq \tilde{A}^{-1}e > 0$, $\underline{x} \leq -\tilde{A}^{-1}e < 0$ for all $\tilde{A} \in A$, we have only to show $x_z \neq 0$ for $z \neq e, -e$ due to Theorem 1. If $x_z = 0$, then $A_{ye}^{-1}b_y = 0$ by (4), which contradicts the regularness of A_{ye}. □

For the sake of simplicity, we suppose the following assumption.

ASSUMPTION **(N):** For each $y \in Y$, the solution of (4) satisfies $x_i^+ > 0$ or $x_i^- > 0$.

The assumption asserts that the solution of (4) does not degenerate and is usually assumed.

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LEMMA 2:   Let $A \in \mathbf{IR}^{n \times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq O$, and $\Delta > O$. Suppose also that Assumption (N) holds. Then the solution set X is convex if and only if $x_i^1 x_i^2 \geq 0$, $i = 1, ..., n$, for any $x_i^1, x_i^2 \in X$.

PROOF: Since the intersection of X with each orthant is a convex polytope (see [6]), sufficiency of the condition is trivial.

To prove necessity, suppose that there exist $x^1, x^2 \in X$ such that $x_j^1 x_j^2 < 0$. Then $\overline{x}_j \underline{x}_j < 0$ holds since $\overline{x}_j \geq \max\{x_j^1, x_j^2\}$ and $\underline{x}_j \leq \min\{x_j^1, x_j^2\}$. On the other hand, it follows $\overline{x} = x_y$, $\underline{x} = x_{-y}$ for $y = e$. Hence we obtain either $(x_z)_j (x_y)_j < 0$ or $(x_z)_j (x_{-y})_j < 0$, which implies nonconvexity of X by Theorem 1.   □

Note that under the hypothesis of the lemma, $A^H b$, the exact hull of X can be determined by x_y and x_{-y}.

Now we can establish the following theorem.

THEOREM 3:   Let $A \in \mathbf{IR}^{n \times n}$ $(n \geq 2)$ be such that A is regular, $A^{-1} \geq O$, and $\Delta > O$. Suppose also that Assumption (N) holds. Then the solution set X is convex if and only if $(x_y)_i (x_{-y})_i \geq 0$, $i = 1, ..., n$, for $y = e$.

PROOF: Directly from the proof of Lemma 2.   □

3. Discussion.

Rohn [8] considers the problem

$$\text{(6)} \qquad \max\{\tilde{c}^T x \mid \tilde{A}x = \tilde{b}, x \geq 0\},$$

where $\tilde{A} \in A$, $\tilde{b} \in b$, $\tilde{c} \in c$, and gives necessary and sufficient conditions for strong feasibility of it. Recall that we call an interval linear system $Ax = b$ *strongly feasible* when each its subsystem is feasible.

THEOREM 4: *Let $A \in \mathbf{IR}^{n \times n}$ ($n \geq 2$) be such that A is regular, $A^{-1} \geq O$, and $\Delta > O$. Suppose also that Assumption (N) holds and there exists an $x > 0$ in X. Then the interval linear system $Ax = b$ is strongly feasible if and only if the solution set X is convex.*

PROOF: Necessity of the condition is obvious.

Sufficiency is proved by Lemma 2 since X is confined in the orthant $\{x \mid x > 0\}$ under the hypothesis. $\square$

To solve the linear complementary problem (4), since $A_{ye}^{-1} A_{yf}$ is a *P*-matrix [10], Lemke's method may not be valid. Further research is needed to solve each x_y efficiently.

Finally, it is noted that convexity of X is not a necessary or sufficient condition for $A^H b = A^G b$, where $A^G b$ denotes the hull by Gauss elimination [1], [3].

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4. Conclusion.

We have proposed an efficient condition for the solution set of linear interval equations to be convex under mild assumptions. In our framework, convexity of the solution set has been shown to be necessary and sufficient for feasibility of the optimization problem.

References

[1] W. Barth and E. Nuding, *Optimale lösung von intervallgleichungssystemen*, Computing 12 (1974), 117–125.

[2] D.M. Gay, *Solving interval linear equations*, SIAM J. Numer. Anal. 19 (1982), 858–870.

[3] A. Neumaier, *New techniques for the analysis of linear interval equations*, Linear Algebra Appl. 58 (1984), 273–325.

[4] A. Neumaier, *Further results on linear interval equations*, Linear Algebra Appl. 87 (1987), 155–179.

[5] W. Oettli, *On the solution set of a linear system with inaccurate coefficients*, SIAM J. Numer. Anal. 2 (1965), 115–118.

[6] W. Oettli and W. Prager, *Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides*, Numerische Mathematik 6 (1964), 405–409.

[7] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, 1970.

[8] J. Rohn, *Solvability of interval linear programming problems*, Computing 26 (1981), 79–82.

[9] J. Rohn, *On nonconvexity of the solution set of a system of linear interval equations*, BIT 30 (1989), 161–165.

[10] J. Rohn, *Systems of linear interval systems*, Linear Algebra Appl. 126 (1989), 39–78.