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Multiplier cocycles of a flow on a C^*-algebra

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Abstract

We show that a multiplier cocycle of a flow on a non-unital C^*-algebra can be approximated by a norm-continuous cocycle in the strict topology. As an application among others we show that a flow is approximately inner if and only if the restriction of the flow to a full invariant hereditary C*-subalgebra is approximately inner.

1 Introduction

Let A be a C^*-algebra and let α be a flow on A, i.e., a strongly continuous one-parameter automorphism group of A. If α is uniformly continuous, we can say that we understand it very well as well as its generator [12, 13]; e.g., the generator of a uniformly continuous flow is a derivation, i.e., a linear map $\delta : A \to A$ with $\delta(xy) = \delta(x)y + x\delta(y),\ \delta(x)^* = \delta(x^*)$ for $x, y \in A$, and a derivation generates such a flow. And we know that if A is simple, then a derivation is of the form $\mathrm{ad}\, ih$, where h is a self-adjoint multiplier. If α is universally weakly inner (or extendible to the second dual), we can also say that we understand it fairly well [7, 5] though we do not seem to know how to characterize its generator among (unbounded) derivations. As a next class of flows to study, it looks that approximately inner flows were proposed in 70's mainly because of relevance to physical models [2, 3, 13]; so the study at that time was rather restricted to the situations which bore direct relations with such models, a rather different approach from the uniformly continuous or universally weakly inner flows. When we try to explore approximately inner flows per se, we encounter a wide range of problems which are still not answered; specifically we have in mind a problem with respect to invariant hereditary C*-subalgebras and a problem concerning extensions. We will answer the former by studying multiplier cocycles (see 1.4) and consider the latter in a very special case (see 1.7).

Suppose that the C^*-algebra A does not have an identity. We denote by $M(A)$ the multiplier algebra of A and by $\overline{\alpha}$ the flow on $M(A)$ induced by α. Note that $t \mapsto \overline{\alpha}_t(x)$ is continuous in the strict topology for any $x \in M(A)$.

Let u be an α-cocycle in $M(A)$, or a multiplier α-cocycle; i.e., u is a continuous function on $\mathbf{R}$ into $\mathcal{U}(M(A))$, equipped with the strict topology, such that $u_s\overline{\alpha}_s(u_t) = u_{s+t},\ s, t \in \mathbf{R}$. Note that $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ defines another flow on A, which is called a cocycle perturbation of α and written as $\mathrm{Ad}u\alpha$.

When we say that u is an α-cocycle in A, u is a continuous function of $\mathbf{R}$ into $\mathcal{U}(A+\mathbf{C}1)$ (with the norm topology) such that $u_s\alpha_s(u_t) = u_{s+t},\ s,t \in \mathbf{R}$. For an $h \in A_{sa}$, the equation $du_t/dt = u_t\alpha_t(ih)$ with $u_0 = 1$ defines a differentiable α-cocycle in A, which we denote by $u^{(h)}$. In general, an α-cocycle in A is given by $t \mapsto wu_t^{(h)}\alpha_t(w^*)$ with $h \in A_{sa}$ and $w \in \mathcal{U}(A + \mathbf{C}1)$. We shall prove:

Theorem 1.1 *Let u be an α-cocycle in $M(A)$, $p \in A$, and $\epsilon > 0$. Then there is an α-cocycle v in A such that $\|(u_t - v_t)p\| < \epsilon,\ t \in [-1, 1]$.*

We denote by δ_α the generator of α; the domain $D(\delta_\alpha)$ is a dense $*$-subalgebra of A and δ_α is a derivation of $D(\delta_\alpha)$ into A. Note that $D(\delta_\alpha)$ is a Banach $*$-algebra; the norm may be defined by embedding $D(\delta_\alpha)$ into $M_2(A)$:

$$x \mapsto \begin{pmatrix} x & \delta_\alpha(x) \\ 0 & x \end{pmatrix}.$$

When $h \in A_{sa}$ we denote by $\alpha^{(h)}$ the flow generated by $\delta_\alpha + \operatorname{ad} ih$, i.e., $\alpha^{(h)} = \operatorname{Ad} u^{(h)}\alpha$, which is called an inner perturbation of α. See [2, 3, 13] for details.

Let β be another flow on A. If for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$, there is an $h \in A_{sa}$ such that

$$\|\beta_t(x) - \alpha_t^{(h)}(x)\| < \epsilon, \quad x \in \mathcal{F}, \quad t \in [-1, 1],$$

we say that β is an approximately inner perturbation of α. We get the following as a corollary to the above theorem:

Corollary 1.2 *Any cocycle perturbation of a flow is an approximately inner perturbation.*

Note that if A is unital, this follows easily.

We have introduced the notion of asymptotic innerness for flows, which we do not know is really stronger than the approximate innerness or not but is useful for a lifting problem for flows [11]. Let h be a continuous function on $\mathbf{R}$ into A_{sa} such that $\alpha^{(h(s))}$ converges as $s\to\infty$, say to a flow α' in the sense that $\lim_{s\to\infty} \max_{t\in[-1,1]} \|\alpha'_t(x) - \alpha_t^{(h(s))}(x)\| = 0$ for $x \in A$. We then call α' as an asymptotically inner perturbation of α. If $\alpha = \mathrm{id}$, then we say that α' is asymptotically inner. The same proof of the above results will show:

Corollary 1.3 *Suppose that the C^*-algebra A has a strictly positive element p. Then for any α-cocycle u in $M(A)$ there is a continuous function $h : [0,\infty)\to A_{sa}$ such that $\max_{t\in[-1,1]} \|(u_t - u_t^{(h(s))})p\|\to 0$ as $s\to\infty$. Hence any cocycle perturbation of the flow is an asymptotically inner perturbation.*

Another consequence, which we have referred to above in connection with approximately inner flows, is

Corollary 1.4 *Let α be a flow on a C*-algebra A and B an α-invariant hereditary C*-subalgebra of A. Then if α is approximately inner, then the restriction $\alpha|B$ is approximately inner. Moreover if B is full in the sense that the closed linear span of ABA is A, the converse is also true; i.e., in this case α is approximately inner if and only if $\alpha|B$ is approximately inner. If each of A and B has a strictly positive element, then the same statements hold with* asymptotic innerness *in place of approximate innerness.*

A flow α on an AF algebra is called an AF flow if there is an increasing sequence of α-invariant finite-dimensional C*-subalgebras with dense union. We have left open a problem of whether a multiplier cocycle perturbation of an AF flow on a non-unital AF algebra is approximately AF or not in [10].

Corollary 1.5 *Let A be a non-unital AF algebra and α an AF flow on A. Let u be an α-cocycle in $M(A)$. Then* $\mathrm{Ad}\, u\alpha$ *is approximately AF in the sense that there is an increasing sequence (A_n) of finite-dimensional C*-subalgebras of A with dense union such that* $\max_{t\in[-1,1]} \mathrm{dist}(\alpha_t(A_n), A_n)\to 0$.

We note that the distance on C*-subalgebras is defined as follow: For two C*-subalgebras B and C of A, $\mathrm{dist}(B, C)$ is defined as the infimum of $r > 0$ such that $\forall x \in B\ \exists y \in C\ \|x - y\| \leq r\|x\|$ and vice versa (see [6]).

Another consequence on AF flows is

Corollary 1.6 *Let α be a flow on an AF algebra A and B an α-invariant hereditary C*-subalgebra of A. (Then B is an AF algebra.) If α is approximately AF, then $\alpha|B$ is also approximately AF. Moreover if B is full, then the converse also holds.*

In section 2 we prove one of the lemmas needed for proving Theorem 1.1. In particular, if α is approximately inner on A and $E \in M(A)$ is an α-invariant projection with $0 \lneqq E \lneqq 1$, then we show that the restriction $\alpha|EAE$ is also approximately inner. We can also show that the extension $\alpha|A+\mathbf{C}E+\mathbf{C}(1-E)$ is approximately inner. This is strengthened in Corollary 2.10 and applied to yield:

Corollary 1.7 *Let α be a flow on an AF algebra A and I be an α-invariant ideal of A. Suppose that $\alpha|I$ is approximately inner and the flow $\dot{\alpha}$ on $B = A/I$ induced by α is approximately inner. Moreover suppose that there is an increasing sequence (B_n) of finite-dimensional C*-subalgebras of B such that $\bigcup_n B_n \subset D(\dot{\delta}_\alpha)$ and $\bigcup_n B_n$ is a core for $\dot{\delta}_\alpha$, where $\dot{\delta}_\alpha$ is the generator of $\dot{\alpha}$. Then α is approximately inner.*

Note that the condition on $\dot{\delta}_\alpha$ implies that $\dot{\alpha}$ is approximately inner [13], but not vice versa [8]. We will prove this corollary essentially by reducing it to the case $B = A/I$ is finite-dimensional. The converse holds without any assumptions on A or $\dot{\alpha}$ (see [11]).

In section 3 we prove the main theorem and Corollary 1.3. In section 4 we prove Corollary 1.4 and in section 5 we prove Corollaries 1.5 and 1.6.

2 Invariant corners

If α is an approximately inner flow on A and E is an α-invariant projection of A, then it is easy to see that the restriction $\alpha|EAE$ is also approximately inner. The following shows that this remains true if E is just an invariant multiplier projection.

Lemma 2.1 *Let A be a non-unital C*-algebra and let α, β be flows on A. Let $E \in M(A) \setminus A$ be a non-zero projection such that $\alpha_t(E) = E = \beta_t(E)$ for $t \in \mathbf{R}$. If α is an approximately inner perturbation of β on A, then the same is true for the restrictions of α and β to EAE.*

Before going to the proof, we prepare a few lemmas.

In the situation of the above lemma, let A_1 be an arbitrary separable C*-subalgebra of A. Then there is a sequence (h_n) in A_{sa} such that $\alpha_t(x) = \lim_n \beta_t^{(h_n)}(x)$ for $x \in A_1$ (uniformly in t on every compact subset of $\mathbf{R}$) as $n \to \infty$. Let B_2 be the C*-algebra generated by E and $\beta_t(x)$ with $x \in A_2 \cup \{h_1, h_2, \ldots\}$ and $t \in \mathbf{R}$ and let $A_2 = B_2 \cap A$. Then A_2 is a β-invariant separable C*-subalgebra of A such that $A_1 \subset A_2$, $h_n \in A_2$ for all n, and $EA_2, A_2E \subset A_2$. Let (k_n) be a sequence in A_{sa} such that $\alpha_t(x) = \lim_n \beta_t^{(k_n)}(x)$ for $x \in A_2$ and let A_3 be an α-invariant C*-subalgebra of A such that $A_2 \subset A_3$, $k_n \in A_3$, and $EA_3, A_3E \subset A_3$. We will repeat this process indefinitely to get an increasing sequence (A_n) of separable C*-subalgebras of A. Then the closure A_0 of the union $\bigcup_n A_n$ is a separable α-invariant and β-invariant C*-subalgebra of A such that A_0 contains A_1, E acts on A_0 as a multiplier, and $\alpha|A_0$ is an approximately inner perturbation of $\beta|A_0$. Thus we may suppose that A is separable and that there is a sequence (h_n) in A_{sa} such that

$$\max_{|t| \le 1} \|\alpha_t(x) - \beta_t^{(h_n)}(x)\| \to 0$$

for all $x \in A$.

We let $\beta_t^{(n)} = \beta_t^{(h_n)}$ and denote by δ_n the generator $\delta_\beta + \operatorname{ad} ih_n$ of $\beta^{(n)} = \beta^{(h_n)}$.

For $\varphi \in C_c^\infty(\mathbf{R})$ and $x \in A$ we let

$$\alpha_\varphi(x) = \int \varphi(t)\alpha_t(x)dt.$$

Note that $\alpha_\varphi(x) \in D(\delta_\alpha)$ and

$$\delta_\alpha(\alpha_\varphi(x)) = -\int \varphi'(t)\alpha_t(x)dt.$$

Lemma 2.2 *Let $\varphi \in C_c^\infty(\mathbf{R})$ and $x \in D(\delta_\alpha)$. Then $\|\beta_\varphi^{(n)}(x) - \alpha_\varphi(x)\| \to 0$ and*

$$\|\delta_n(\beta_\varphi^{(n)}(x)) - \delta_\alpha(\alpha_\varphi(x))\| \to 0$$

as $n \to \infty$.

Proof. This is immediate because $\beta_\varphi^{(n)}(x) = \int \varphi(t)\beta_t^{(n)}(x)dt \to \int \varphi(t)\alpha_t(x)dt$ and

$$\delta_n(\beta_\varphi^{(n)}(x)) = -\int \varphi'(t)\beta_t^{(n)}(x)dt \to -\int \varphi'(t)\alpha_t(x)dt.$$

We call $e \in A$ a pseudo-projection if $0 \le e \le 1$ and $ae = a$ for some non-zero $a \in A$.

Let $\varphi, \psi \in C_c^\infty(\mathbf{R})$ such that $0 \le \varphi \le \psi \le 1$, $\varphi(0) = 0 = \psi(0)$, $\varphi(1) = 1 = \psi(1)$, and $\varphi\psi = \varphi$. We fix these φ, ψ in the following arguments.

Lemma 2.3 *For any $\epsilon > 0$ there is a $\mu > 0$ satisfying the following condition: If δ is a generator and $e \in D(\delta)$ is a pseudo-projection such that $\|\delta(e)\| < \mu$, then $\varphi(e) \in D(\delta)$ and*

$$\|\delta(\varphi(e)\| < \epsilon.$$

Proof. Express $\varphi(t)$ as $\int \hat{\varphi}(p)e^{ipt}dp$, where $\hat{\varphi}$ is a rapidly decreasing function; then

$$\varphi(e) = \int \hat{\varphi}(p)e^{ipe}dp.$$

Since $e^{ipe} \in D(\delta)$ and $\|\delta(e^{ipe})\| \le |p|\|\delta(e)\|$, we get the estimate

$$\|\delta(\varphi(e))\| \le \int |\hat{\varphi}(p)p|dp \cdot \mu.$$

This concludes the proof (see [2, 13] for details).

Lemma 2.4 *For any $\epsilon > 0$ there is a $\mu > 0$ satisfying the following condition: If e, x are pseudo-projections in A such that $\|ex - e\| < \mu$, then $\|\psi(e)x - \psi(e)\| < \epsilon$.*

Proof. Since $\psi(0) = 0$, we approximate ψ by a polynomial $f(t) = \sum_{k=1}^N a_k t^k$ such that $\max_{t\in[0,1]} |\psi(t) - f(t))| < \epsilon/3$. If $\|ex - e\| < \mu$, then $\|f(e)x - f(e)\| < C\mu$ with $C = \sum_{k=1}^N k|a_k|$, which implies that $\|\psi(e)x - \psi(e)\| < 2\epsilon/3 + C\mu$. Hence if $\mu \le \epsilon/3C$, then we get the conclusion.

Lemma 2.5 *There exists a sequence (e_n) of pseudo-projections in EAE such that $\|xe_n - x\| \to 0$ for all $x \in EAE$, $e_n \in D(\delta_\alpha)$, and $\|\delta_\alpha(e_n)\| \to 0$.*

Proof. Let $\chi \in C_c^\infty(\mathbf{R})$ such that $\chi \ge 0$ and $\int \chi(t)dt = 1$ and set $\chi_k(t) = k^{-1}\chi(k^{-1}t)$ for $k \in \mathbf{N}$. Since EAE is separable, we have a sequence (p_n) of pseudo-projections in EAE which forms an approximate identity. Then for each $k \in \mathbf{N}$ the sequence $(\alpha_{\chi_k}(e_n))$ also forms an approximate identity. Since $\|\delta_\alpha(\alpha_{\chi_k}(e_n))\| < k^{-1}\int |\chi'(t)|dt$, we can get such a sequence as a subsequence of $(\psi(\chi_k(e_n)))_{k,n}$.

Lemma 2.6 *For any pseudo-projection $x_0 \in EAE$, any finite subset of $B = \{a \in A \mid x_0 a x_0 = a\}$, and $\epsilon > 0$, there is an $n \in \mathbf{N}$ and a pseudo-projection $e \in D(\delta_n)$ satisfying the following conditions: Let $p = \varphi(e)$ and $q = \psi(e)$.*

$$\begin{aligned} pq &= p, \\ \|\delta_n(p)\| &< \epsilon, \\ \|x_0 p - x_0\| &< \epsilon, \\ \|qE - q\| &< \epsilon, \\ \max_{|t|\le 1} \|\alpha_t(a) - \beta_t^{(n)}(a)\| &< \epsilon, \quad a \in \mathcal{F}. \end{aligned}$$

Proof. Let (e_n) be a sequence in EAE as in Lemma 2.5.

We fix a $\chi \in C_c^\infty(\mathbf{R})$ such that $\chi \ge 0$ and $\int \chi(t)dt = 1$. Suppose that $\mathrm{supp}(\chi) \subset [-1, 1]$. We will define e as $\beta_\chi^{(n)}(e_m)$ for sufficiently large m, n.

For any $\epsilon_1 > 0$ we can find m such that

$$\begin{aligned} \|x_0 \alpha_t(e_m) - x_0\| &< \epsilon_1, \quad t \in [-1, 1], \\ \|\delta_\alpha(e_m)\| &< \epsilon_1, \end{aligned}$$

where the latter condition implies that $\|\delta_\alpha(\alpha_\chi(e_m))\| < \epsilon_1$.

By Lemma 2.2 and the fact $\beta^{(n)} \to \alpha$, we choose n so large that $\|\delta_n(\beta_\chi^{(n)}(e_m))\| < \epsilon_1$ and

$$\max_{|t|\le 1} \|\alpha_t(a) - \beta_t^{(n)}(a)\| < \epsilon_1, \quad a \in \mathcal{F} \cup \{e_m\}.$$

We set $e = \beta_\chi^{(n)}(e_m)$.

The first condition of the lemma follows from $\varphi\psi = \varphi$ and the last is given just above. If ϵ_1 is sufficiently small, from $\|\delta(e)\| < \epsilon_1$ we get, by Lemma 2.3, the second condition for $p = \varphi(e)$. From $\|x_0 e - x_0\| < 2\epsilon_1$, we get, by Lemma 2.4, the third condition. From $\|E\beta_t^{(n)}(e_m) - \beta_t^{(n)}(e_m)\| \le \|E\beta_t^{(n)}(e_m) - E\alpha_t(e_m)\| + \|\alpha_t(e_m) - \beta_t^{(n)}(e_m)\| < 2\epsilon_1$, $t \in [-1, 1]$, we get the fourth condition by Lemma 2.4.

Proof of Lemma 2.1

Let x_0 be a pseudo-projection in EAE, $\mathcal{F}$ a finite subset of $B = \{a \in A \mid x_0 a x_0 = a\}$, and $\epsilon > 0$. We will show that there is a $b \in (EAE)_{sa}$ such that $\max_{|t|\le 1} \|\alpha_t(a) - \beta_t^{(b)}(a)\| < \epsilon$, $a \in \mathcal{F}$.

We take p, q and $h = h_n$ as in Lemma 2.6. We regard $\beta^{(n)} = \beta^{(h)}$ as a flow on $A + \mathbf{C}E + \mathbf{C}1$ with δ as its generator. Let

$$E' = 2q - q^2 + (1 - q)E(1 - q) \in A + \mathbf{C}E.$$

Since $q, E \in D(\delta) = D(\delta_\beta)$, we get that $E' \in D(\delta_\beta)$. Since $pq = p$, we get that $pE' = p = E'p$. Since $qE \approx q$, we have that $(1 - q)E(1 - q) \approx E - 2q + q^2$, i.e., $E' \approx E$. Since $0 \le E' \le 1$, we have that $\mathrm{Sp}(E') \subset [0, \|E - E'\|] \cup [1 - \|E - E'\|, 1]$. By C^∞ functional

calculus on E' we get a projection $F \in A + \mathbf{C}E$ such that $\|F - E'\| \leq \|E - E'\|$ (so $\|F - E\| \leq 2\|E - E'\| \approx 0$). We should note that $F \in D(\delta_\beta)$ (since $E' \in D(\delta_\beta)$; see the proof of Lemma 2.3). Since $pE' = p$, we get that $pF = p$, or $p \leq F$. By the polar decomposition of $FE + (1-F)(1-E)$, which is an element in $D(\delta_\beta)$ close to 1, we get a unitary $Z \in A + \mathbf{C}E + \mathbf{C}1$ such that $Z \approx 1$, $ZEZ^* = F$, and $Z \in D(\delta_\beta)$.

Let

$$k = -iF\delta(F) + i\delta(F)F.$$

Then $i[k, F] = -\delta(F)$ and $k^* = k \in A$ (as the range of $\delta|A + \mathbf{C}E + \mathbf{C}1$ is contained in A). Hence, since $\delta = \delta_\beta + \operatorname{ad} ih$, we get that

$$\operatorname{Ad} Z^* \beta_t^{(h+k)} \operatorname{Ad} Z(E) = E.$$

Note that the generator of the flow $t \mapsto \operatorname{Ad} Z^* \beta_t^{(h+k)} \operatorname{Ad} Z$ is $\delta_\beta + \operatorname{ad} i(-iZ^*\delta_\beta(Z) + Z^*(h+k)Z)$. Since $\delta_\beta(E) = 0$, we get that $\operatorname{ad} iH(E) = E$, where $H = -iZ^*\delta_\beta(Z) + Z^*(h+k)Z \in A$.

What remains to show is that

$$\operatorname{Ad} Z^* \beta_t^{(h+k)} \operatorname{Ad} Z(a) \approx \alpha_t(a), \quad a \in \mathcal{F}, \quad t \in [-1, 1].$$

Using $Z \approx 1$, $\mathcal{F} \subset B$, and $\alpha_t(a) \approx \beta_t^{(h)}(a),\ a \in \mathcal{F}$, it suffices to show that

$$\beta_t^{(h+k)}(x) \approx \beta_t^{(h)}(x), \quad x \in B, \quad t \in [-1, 1].$$

Define a $\beta^{(h)}$-cocycle $u = u^{(k)}$ by $du_t/dt = u_t \beta_t^{(h)}(ik)$ with $u_0 = 1$. Then $\beta_t^{(h+k)} = \operatorname{Ad} u_t^{(k)} \beta_t^{(h)}$. We shall show that $\|(u_t^{(k)} - 1)\beta_t^{(h)}(p)\| \approx 0$ for $t \in [-1, 1]$. Since $\|px - x\| \leq \epsilon\|x\|$ for $x \in B$, this suffices to show the conclusion.

Let $G(t) = u_t^{(k)} \beta_t^{(h)}(p)$. Then $G'(t) = u_t^{(k)} \beta_t^{(h)}(ikp + \delta(p))$, which implies that $\|G'(t)\| \leq \|kp\| + \|\delta(p)\|$.

Since $pF = p$ and $\delta(F)p = \delta(Fp) - F\delta(p) = (1-F)\delta(p)$, we get that

$$ikp = (F\delta(F) - \delta(F)F)p = F(1-F)\delta(p) - (1-F)\delta(p) = -(1-F)\delta(p).$$

Hence $\|kp\| \leq \|\delta(p)\|$, which is very small. In this way we get that $\|G'(t)\| \leq 2\|\delta(p)\| \approx 0$. Since $G(0) = p$, we get that $\|u_t^{(k)} \beta_t^{(h)}(p) - p\| \leq 2\|\delta(p)\||t|$. Since $\|\beta_t^{(h)}(p) - p\| \leq \|\delta(p)\||t|$, this implies that $\|(u_t^{(k)} - 1)\beta_t^{(h)}(p)\| \leq 3\|\delta(p)\||t|$, completing the proof of Lemma 2.1.

In the above proof, if $E \in A$, we could take $x_0 = E$ and $e_n = E$ and assume that $p = q = F \approx E$; the proof would be much simpler.

Lemma 2.7 *Let A be a non-unital C^*-algebra and let β be a flow on A. Let $E \in M(A) \backslash A$ be a non-zero projection such that $\beta_t(E) = E$, $t \in \mathbf{R}$. Let u be a β-cocycle in $M(A)$ (continuous in the strict topology) such that $u_t E = E u_t$, $t \in \mathbf{R}$. If u can be approximated, in the strict topology, by β-cocycles in A (continuous in the norm topology), then the same is true for the $\beta|EAE$-cocycle $t \mapsto u_t E$.*

Proof. This is just what is proved in the proof of 2.1.

Let x_0 be a pseudo-projection of EAE and $\epsilon > 0$. By replacing x_0 by a larger one, we may suppose that $\|\beta_t(x_0) - x_0\| < \epsilon,\ t \in [-1, 1]$. Then, as in the proof of Lemma 2.6, we have pseudo-projections $p, q \in A$ and an $h \in A_{sa}$ such that $pq = p$, $\|(\delta_\beta + \operatorname{ad} ih)(p)\| < \epsilon$, $\|x_0 p - x_0\| < \epsilon$, $\|qE - q\| < \epsilon$, and $\|(u_t - v_t^{(h)})x_0\| < \epsilon$ for $t \in [-1, 1]$, where $v^{(h)}$ denotes the β-cocycle determined by $dv_t^{(h)}/dt = v_t^{(h)}\beta_t(ih)$. Then, as in the proof of 2.1, we find a $Z \in \mathcal{U}(A + \mathbf{C}E + \mathbf{C}1) \cap D(\delta_\beta)$ and $k \in A_{sa}$ such that $\operatorname{Ad} Z^* \beta_t^{(h+k)} \operatorname{Ad} Z(E) = E$, $\|Z - 1\| < \epsilon$, and $\|kp\| \le \|(\delta_\beta + \operatorname{ad} ih)(p)\| < \epsilon$. Then we show that

$$\|\frac{d}{dt} v_t^{(h+k)} \beta_t(p)(v_t^{(h)})^*\| = \|v_t^{(h+k)} \beta_t(i(h+k)p + \delta_\beta(p) - iph) v_t^{(h)}\| \le \|kp\| + \|(\delta_\beta + \operatorname{ad} ih)(p)\| < 2\epsilon.$$

Since $\beta_t^{(h)}(p) v_t^{(h)} = v_t^{(h)} \beta_t(p)$ and $\|\beta_t^{(h)}(p) - p\| \le \epsilon|t|$, we can conclude that

$$\|v_t^{(h+k)} \beta_t(p) - v_t^{(h)} \beta_t(p)\| \le 3\epsilon|t|.$$

Since $\|x_0 \beta_t(p) - x_0\| = \|\beta_{-t}(x_0)(1 - p)\| < \|x_0(1 - p)\| + \epsilon < 2\epsilon$, we get that for $t \in [-1, 1]$,

$$\|u_t x_0 - v_t^{(h+k)} x_0\| < \|v_t^{(h)} x_0 - v_t^{(h+k)} x_0\| + \epsilon < \|v_t^{(h)} \beta_t(p) - v_t^{(h+k)} \beta_t(p)\| + 5\epsilon \le 8\epsilon.$$

Set $H = -iZ^* \delta_\beta(Z) + Z^*(h + k)Z \in A$ and note that $HE = EH$. Then $t \mapsto v_t^{(H)} E$ is the $\beta|EAE$-cocycle determined by $HE \in (EHE)_{sa}$ and $v_t^{(H)} E = Z v_t^{(h+k)} \beta_t(Z^*) E$. Since $\|Z - 1\| < \epsilon$, we can conclude that $\|(u_t E - v_t^{(H)} E)x_0\| < 10\epsilon,\ t \in [-1, 1]$.

Lemma 2.8 *Suppose that A has a strictly positive element p with a flow β. Then there is a continuous function $e : [0, \infty) \to A_{sa}$ such that $0 \le e(s) \le 1$, $\|pe(s) - p\| \to 0$ as $s \to \infty$, $e(s) \in D(\delta_\beta)$, and $\|\delta_\beta(e(s))\| \to 0$ as $s \to \infty$.*

Proof. Let $f \in C_c^\infty(\mathbf{R})$ be such that $0 \le f \le 1$, $f(0) = 0$, and $f(t) = 1$ for $t \ge 1$. For $s \in [0, \infty)$ let $p(s) = f(sp)$. Then it follows that $0 \le p(s) \le 1$ and that $\|p \cdot p(s) - p\| \to 0$ as $s \to \infty$.

Let $\chi \in C_c^\infty(\mathbf{R})$ be such that $0 \le \chi$ and $\int \chi(t)dt = 1$. For $\mu \in (0, 1]$ let $p_\mu(s) = \int \mu\chi(\mu t)\beta_t(p(s))dt$. Then $0 \le p_\mu(s) \le 1$, $\|p \cdot p_\mu(s) - p\| \to 0$, uniformly in μ in a compact subset of $(0, 1]$, as $s \to \infty$, $p_\mu(s) \in D(\delta_\beta)$, and $\|\delta_\beta(p_\mu(s))\| < C\mu$ with some constant C. Hence for any $n \in \mathbf{N}$ there is an $s_n \in (0, \infty)$ such that $\|pp_\mu(s) - p\| < 1/n$ for $\mu \in [1/n, 1]$ and $s \ge s_n$. We may suppose that $s_{n+1} > s_n$ for all n. We define a continuous function $\mu : [0, \infty) \to [1, 0)$ by $\mu(s) = 1$ for $s < s_2$ and by, for $s \in [s_n, s_{n+1})$ with $n \ge 2$,

$$\mu(s) = \frac{s_{n+1} - s}{s_{n+1} - s_n}(n - 1)^{-1} + \frac{s - s_n}{s_{n+1} - s_n} n^{-1} \in [1/n, 1/(n - 1)).$$

We define $e(s)$ by $e(s) = p_{\mu(s)}(s)$, which is continuous in s. If $s_n \le s < s_{n+1}$, we have that $\|pe(s) - p\| < 1/n$ and $\|\delta_\beta(e(s))\| < C/n$. This concludes the proof.

Lemma 2.9 *Suppose that A has a strictly positive element p and let β be a flow on A. Let $E \in M(A)\setminus A$ be a non-zero projection such that $\beta_t(E) = E$. Let u be a β-cocycle in $M(A)$ such that $u_tE = Eu_t$. If u can be approximated by differentiable β-cocycles in A asymptotically, i.e., there is a continuous $h : [0,\infty)\to A_{sa}$ such that $\max_{t\in[-1,1]} \|(u_t - v_t^{(h(s))})p\|\to 0$ as $s\to\infty$, where $v_t^{(h(s))}$ is the β-cocycle determined by $dv_t^{(h(s))}/dt = v_t^{(h(s))}\beta_t(ih(s))$, then the same is true for $\beta|EAE$ and for the $\beta|EAE$-cocycle $t \mapsto u_tE$.*

Proof. By inspecting the proof of Lemmas 2.6 and 2.7, we know that the unitary Z and $k \in A_{sa}$ there depend on the choice of e_m and h_n. Now we have $e(\sigma),\ \sigma \in [0,\infty)$ as given in Lemma 2.8 instead of (e_m) and $h(s),\ s \in [0,\infty)$ instead of (h_n) by the assumption of this lemma. Thus we can find $Z(\sigma,s) \in \mathcal{U}(A + \mathbf{C}E + \mathbf{C}1) \cap D(\delta_\beta)$ and $k(\sigma,s) \in A_{sa}$ for suitable pair (σ,s) such that

$$Z(\sigma,s)v_t^{(h(s)+k(\sigma,s))}\beta_t(Z(\sigma,s)^*)$$

exactly commutes with E and is close to u_t on the strictly positive element p. Here the closeness is limited by only the choice of σ if s is sufficiently large. This statement remains true for any given compact subset of σ if s is sufficiently large. Just as in the proof of the previous lemma, we can find a continuous function $\sigma(s),\ s \in [0,\infty)$ such that $\sigma(s)\to\infty$ as $s\to\infty$ and $Z(s) = Z(\sigma(s),s)$ and $k(s) = k(\sigma(s),s)$ give the desired family of $\beta|EAE$-cocycles.

In the proof of Lemma 2.1 we can adopt the same argument also for $1 - E$ as for E, with a common $h = h_n$ by making n even larger. Thus for any pseudo-projection $y_0 \in (1-E)A(1-E)$, we get $Z_1 \in \mathcal{U}(A + \mathbf{C}E + \mathbf{C}1) \cap D(\delta_\beta)$ and $k_1 \in A_{sa}$ such that $Z_1 \approx 1$, $\mathrm{Ad}\, Z_1^*\beta_t^{(h+k_1)}\mathrm{Ad}\, Z_1(1-E) = 1-E$. We also have:

$$\|(Z_1^*v_t^{(h+k_1)}\beta_t(Z_1) - v_t^{(h)})y_0\| \approx 0$$

as well as $\|(Z^*v_t^{(h+k)}\beta_t(Z) - v_t^{(h)})x_0\| \approx 0$, where $v_t^{(h+k_1)}$ and $v_t^{(h+k)}$ are the β-cocycles defined for $h + k_1$ and $h + k$ as in Lemma 2.7. Thus, since $\beta_t(E) = E$, we get

$$Z^*v_t^{(h+k)}\beta_t(Z)E + Z_1^*v_t^{(h+k_1)}\beta_t(Z_1)(1-E)$$

is a β-cocycle commuting with E and is close to $v_t^{(h)}$ on $x_0 + y_0$. Since $x_0 + y_0$ is an arbitrary pseudo-projection in $EAE+(1-E)A(1-E)$ (which generates A as a hereditary C*-subalgebra), we get that α on $A + \mathbf{C}E$ is an approximate inner perturbation of β on $A + \mathbf{C}E$. This result can be generalized as follows:

Corollary 2.10 *Let A be a non-unital C^*-algebra and let α, β be flows on A. Let B be a C^*-subalgebra of $M(A)$ such that such that $t \mapsto \overline{\alpha}_t(x)$ and $t \mapsto \overline{\beta}_t(x)$ are norm-continuous and $\overline{\alpha}_t(x) - x \in A$ and $\overline{\beta}_t(x) - x \in A$ for any $x \in B$. Let D be the C^*-subalgebra of $M(A)$ generated by A and B, i.e., $D = A + B$; then $\overline{\alpha}$ and $\overline{\beta}$ act on D as flows. Suppose that B is an AF algebra. If α is an approximately inner perturbation of β on A, then $\overline{\alpha}|D$ is an approximately inner perturbation of $\overline{\beta}|D$.*

Proof. Note that D has A as an ideal and the quotient D/A is an AF algebra as being a quotient of B. Note also that the flows $\overline{\alpha}$ and $\overline{\beta}$ induce the trivial flow on D/A.

We regard $\overline{\alpha}$ and $\overline{\beta}$ as flows on D. We have to show that for any finite subset $\mathcal{F}$ of the unit ball of D and $\epsilon > 0$ there is an $h \in D_{sa}$ such that

$$\|\overline{\alpha}_t(x) - \overline{\beta}_t^{(h)}(x)\| < \epsilon, \quad x \in \mathcal{F},\ t \in [-1, 1].$$

Since B is an AF algebra, there is an increasing sequence (B_n) of finite-dimensional C*-subalgebras of B with dense union. Hence we may suppose that $\mathcal{F}$ is a finite subset of the C^*-algebra $D_1 = A + B_n$ for some n. Since D_1 is left invariant under $\overline{\alpha}$ and $\overline{\beta}$, we may suppose that B is finite-dimensional.

By modifying B slightly if necessary we may suppose that $B \subset D(\overline{\delta}_\alpha)$, where $\overline{\delta}_\alpha$ is the generator of $\overline{\alpha}$. Hence there is an $h_1 \in D_{sa}$ such that $\overline{\alpha}_t^{(h_1)}$ is trivial on B. There is also an $h_2 \in D_{sa}$ and $w \in \mathcal{U}(D)$ such that $\gamma_t = \mathrm{Ad}\, w^* \overline{\beta}_t^{(h_2)} \mathrm{Ad}\, w$ is trivial on B. Then it suffices to show that for any finite subset $\mathcal{F}$ of the unit ball of A and $\epsilon > 0$, there is an $H \in (D \cap B')_{sa}$ such that

$$\|\overline{\alpha}_t^{(h_1)}(x) - \gamma_t^{(H)}(x)\| < \epsilon, \quad x \in \mathcal{F},\ t \in [-1, 1].$$

The reasoning is as follows: Define an $\overline{\alpha}^{(h_1)}$-cocycle U by $dU_t/dt = U_t \overline{\alpha}_t^{(h_1)}(-ih_1)$ and a $\gamma^{(H)}$-cocycle V by $dV_t/dt = V_t \gamma_t^{(H)}(-ih_1)$. Then $\overline{\alpha}_t = \mathrm{Ad}\, U_t \overline{\alpha}_t^{(h_1)}$ and $\gamma_t^{(H-h_1)} = \mathrm{Ad}\, V_t \gamma_t^{(H)}$. To get the assertion we have to show that $\|U_t - V_t\|$ is small for $t \in [-1, 1]$. Since

$$\frac{d}{dt} U_t V_t^* = U_t (\gamma_t^{(H)}(ih_1) - \overline{\alpha}_t(ih_1)) V_t^*,$$

it then suffices to make $\max_{t \in [-1,1]} \|\overline{\alpha}_t^{(h_1)}(h_1) - \gamma_t^{(H)}(h_1)\|$ as small as we wish. Since h_1 is a finite linear combination of xb with $x \in A$ and $b \in B$, we can attain this by assuming that $\mathcal{F}$ contains all those x and ϵ is sufficiently small. Since

$$\mathrm{Ad}\, V_t \gamma_t^{(H)} = \gamma_t^{(H-h_1)} = \mathrm{Ad}\, w^* \overline{\beta}_t^{(h_2 + w(H-h_1)w^*)} \mathrm{Ad}\, w,$$

we get the assertion by approximating w by a unitary in $D(\delta_\beta)$.

Hence we may assume that $\overline{\alpha}_t | B = \mathrm{id}$ and $\overline{\beta}_t | B = \mathrm{id}$. We may assume that B has the unit of $M(A)$. Let $(e_{ij}^{(k)})$ be a generating family of matrix units for B. Since $\sum_k \sum_i e_{i1}^{(k)} e_{1i}^{(k)} = 1$ and $e_{1i}^{(k)} A e_{j1}^{(\ell)} \subset A$, the C^*-algebra generated by A and B is the linear span of $e_{ij}^{(k)}$ and $e_{i1}^{(k)} x e_{1j}^{(\ell)}$ with $x \in e_{11}^{(k)} A e_{11}^{(\ell)}$. Let $E = \sum_k e_{11}^{(k)}$. Then for the family of $e_{11}^{(k)}$ with all k, any finite subset $\mathcal{F}$ of EAE, and $\epsilon > 0$, we apply 2.1, as discussed before this corollary, to get an $h_k \in e_{11}^{(k)} A e_{11}^{(k)}$ such that $\overline{\alpha}_t \approx \overline{\beta}_t^{(H)}$ on $\mathcal{F}$ within ϵ (and $\overline{\beta}_t^{(H)}(e_{11}^{(k)}) = e_{11}^{(k)}$), where $H = \sum_k h_k$. We set $K = \sum_{i,k} e_{i1}^{(k)} H e_{1i}^{(k)} = \sum_{i,k} e_{i1}^{(k)} h_k e_{1i}^{(k)}$. Since $K \in A \cap B'$, it follows that $\overline{\beta}_t^{(K)}(e_{ij}^{(k)}) = e_{ij}^{(k)}$ and also that $\overline{\beta}_t^{(K)}(e_{i1}^{(k)} x e_{1j}^{(\ell)}) = e_{i1}^{(k)} \overline{\beta}_t^{(H)}(x) e_{1j}^{(\ell)} \approx e_{i1}^{(k)} \overline{\alpha}_t(x) e_{1j}^{(\ell)} = \overline{\alpha}_t(e_{i1}^{(k)} x e_{1j}^{(\ell)})$ for $x \in \mathcal{F}$. This completes the proof.

Now we shall prove Corollary 1.7: Let I be an ideal of A and let α be a flow on A such that $\alpha_t(I) = I$, $\alpha|I$ is approximately inner and the induced flow $\dot{\alpha}$ on the quotient $B = A/I$ is approximately inner. We then wanted to conclude that α is approximately inner, but for that we have to impose a technical assumption: A is AF (and so is B) and there is an increasing sequence (B_n) of finite-dimensional C*-subalgebras of B with dense union such that $\bigcup_m B_m \subset D(\dot{\delta}_\alpha)$ and the closure of the restriction $\dot{\delta}_\alpha|\bigcup_n B_n$ is $\dot{\delta}_\alpha$, which in particular implies that $\dot{\alpha}$ is approximately inner [13] (and even asymptotically inner).

From a theory of derivations in AF algebras [13], there is a sequence (b_n) in B_{sa} such that $\dot{\delta}_\alpha + \operatorname{ad} ib_n$ leaves B_n invariant and $(\dot{\delta}_\alpha + \operatorname{ad} ib_n)(x) \to \dot{\delta}_\alpha(x)$ for $x \in \bigcup_n B_n$, i.e., $\operatorname{ad} ib_n(x) \to 0$ for $x \in \bigcup_n B_n$. To see this let $h_n \in A_{sa}$ be such that $\dot{\delta}_\alpha|B_n = \operatorname{ad} ih_n|B_n$ and let $\mathcal{E}_n$ be a projection of norm one of B onto B_n. We set $b_n = \mathcal{E}_n(h_n) - h_n$. Then we have that $(\dot{\delta}_\alpha + \operatorname{ad} ib_n)|B_n = \operatorname{ad} i\mathcal{E}_n(h_n)|B_n$, which has range in B_n and that for $x \in B_m$ and $n \geq m$, $(\dot{\delta}_\alpha + \operatorname{ad} ib_n)(x) = \operatorname{ad}\mathcal{E}_n(ih_n)(x) = \mathcal{E}_n(\dot{\delta}_\alpha(x)) \to \dot{\delta}_\alpha(x)$.

Let (e_n) be a sequence in I_{sa} such that $0 \leq e_n \leq 1$, $e_n \in D(\delta_\alpha)$, $\|xe_n - x\| \to 0$ for $x \in I$, $\|[e_n, y]\| \to 0$ for $y \in A$, and $\|\delta_\alpha(e_n)\| \to 0$. We may further suppose that $e_n e_{n+1} = e_n$ (see e.g., Lemma 5.1 below). We will choose a $w_n \in A_{sa}$ with $Q(w_n) = b_n$ in a suitable way, where Q is the quotient map of A onto B. Note that $\alpha^{(w_n)}$ leaves the C*-subalgebra $A_n = Q^{-1}(B_n)$ invariant. We shall show that $\alpha^{(w_n)}|A_n$ is approximately inner and $\alpha^{(w_n)}$ converges to α as $n \to \infty$, which implies the conclusion.

The first assertion follows from the previous corollary. Since A is AF, we find a copy D_n of B_n in A_n such that $D_n \subset D(\delta_\alpha)$ and $Q|D_n$ is an isomorphism onto B_n; then we have $A_n = I + D_n$. We find an $h = h^* \in A_n$ such that $\alpha^{(w_n+h)}|A_n$ restricts to the trivial flow on D_n. Thus $\alpha^{(w_n+h)}|A_n$ is approximately inner because $\alpha^{(w_n+h)}|I$ is approximately inner by Corollary 1.2, which will be shown later without depending on this, since this is a perturbation of $\alpha|I$ by a cocycle in $M(I)$. Hence $\alpha^{(w_n)}|A_n$ is approximately inner (since it is an inner perturbation of $\alpha^{(w_n+h)}|A_n$).

We shall show the the second assertion; during the course of the proof, we will choose w_n more carefully. Let δ_n be the generator of $\alpha^{(w_n)}$ and let Δ be the graph limit of (δ_n). We have to show that $\Delta = \delta_\alpha$, which follows from the weaker condition $\Delta \supset \delta_\alpha$ since $\pm\Delta$ is dissipative.

First note that $D = \{x \in D(\delta_\alpha) \mid Q(x) \in \bigcup_m B_m\}$ is a core for δ_α. Because if $(\mathrm{id} - \delta_\alpha)D$ is not dense in A, there is a non-zero $f \in A^*$ such that $f(x - \delta_\alpha(x)) = 0$, $x \in D$. Since $(\mathrm{id} - \delta_\alpha)D \supset I$ (as $D \supset D(\delta_\alpha) \cap I$), we can regard f as a linear functional on $B = A/I$ and have that $f(x - \dot{\delta}_\alpha(x)) = 0$, $x \in \bigcup_m B_m$. By the assumption on $\dot{\delta}_\alpha$, we get that $f = 0$, which is a contradiction.

Since $\|\operatorname{ad} ib_n|B_m\| \to 0$ as $n \to \infty$, for any $m \in \mathbf{N}$ there is an $n_m \in \mathbf{N}$ such that $\|\operatorname{ad} ib_n\|B_m\| < 1/m$ for all $n \geq n_m$. We may suppose that (n_m) is an increasing sequence.

We choose an increasing sequence (V_m) of finite-dimensional subspaces of $D(\delta_\alpha)$ such that $A_m = I + V_m$. Let

$$T_m = \{[w_k, x] \mid n_m \leq k < n_{m+1},\ x \in V_m,\ \|x\| \leq 1\},$$

which is a compact subset of A such that $\|Q(y)\| < 1/m$ for $y \in T_m$. Let $C_m = \max\{\|w_k\| \mid n_m \le k < n_{m+1}\} + 1$. Then we have an $\ell = \ell_m \in \mathbf{N}$ such that $\ell \ge n_{m+1}$, $\|[x, e_\ell]\| \le (mC_m)^{-1}\|x\|$, $x \in V_m$ and

$$\|(1 - e_\ell)y(1 - e_\ell)\| < 1/m, \quad y \in T_m.$$

Then we obtain that for any x of the unit ball of V_m and $n_m \le k < n_{m+1}$,

$$\|[(1 - e_\ell)w_k(1 - e_\ell), x]\| \le 2\|w_k\|\|[e_\ell, x]\| + \|(1 - e_\ell)[w_k, x](1 - e_\ell)\| < 3/m.$$

Now we replace w_k by $(1-e_{\ell_m})w_k(1-e_{\ell_m})$ for $n_m \le k < n_{m+1}$, which satisfies the original conditions $w_k^* = w_k$ and $Q(w_k) = b_k$. Moreover we have that $w_k e_k = 0$ (as $e_{\ell_m} e_k = e_k$ with $n_m \le k < n_{m+1}$) and that $\|[w_k, x]\| \to 0$, $x \in \bigcup_m V_m$, i.e.,

$$\delta_n(x) = \delta_\alpha(x) + \operatorname{ad} i w_n(x) \to \delta_\alpha(x), \quad x \in \bigcup_m V_m.$$

Let $x \in D(\delta_\alpha)$ with $Q(x) \in B_m$ for some m. Then we have $b \in D(\delta_\alpha) \cap I$ and $c \in \bigcup_m V_m$ such that $x = b + c$. Let $x_n = e_n b e_n + c \in D(\delta_\alpha)$. Then $\|x - x_n\| \to 0$ and

$$\delta_n(x_n) = \delta_\alpha(e_n b e_n) + \delta_n(c) \to \delta_\alpha(b) + \delta_\alpha(c) = \delta_\alpha(x)$$

since $\delta_\alpha(e_n) \to 0$ and (e_n) is an approximate identity for I. This implies that $D(\Delta) \supset D = D(\delta_\alpha) \cap I + \bigcup_m V_m$ and $\Delta = \delta_\alpha$ on D. Since D is a core for δ_α, this completes the proof.

3 Multiplier cocycles

We will prove Theorem 1.1 in this section.

Let α be a flow on a non-unital C^*-algebra A and u an α-cocycle in $M(A)$.

When u is an α-cocycle, we define a flow T of isometries on A by

$$T_t(x) = u_t \alpha_t(x), \quad x \in A.$$

Note that $t \mapsto T_t(x)$ is continuous for $x \in A$. We denote by L the generator of T. Let $\alpha' = \mathrm{Ad}u\alpha$ and δ'_α the generator of α'. We note that $\alpha'_t(x)T_t(y)\alpha_t(z) = T_t(xyz)$, $T_t(x)T_t(y)^* = \alpha'_t(xy^*)$, and $T_t(x)^*T_t(y) = \alpha_t(x^*y)$ for $x, y, z \in A$. We then obtain the following: $D(\delta'_\alpha)D(L)D(\delta_\alpha) \subset D(L)$ and $D(L)D(L)^* \subset D(\delta'_\alpha)$ and $D(L)^*D(L) \subset D(\delta_\alpha)$.

Since $D(L)$ is dense in A, we get that the linear span of $D(L)D(L)^*$ is dense in A. Since it is invariant under α', it follows that the linear span of $D(L)D(L)^*$ is dense in the Banach $*$-algebra $D(\delta'_\alpha)$. Moreover we have the following:

Lemma 3.1 *The closure of the convex hull of xx^*, $x \in D(L)$ is $D(\delta'_\alpha) \cap A_+$ in the Banach $*$-algebra $D(\delta'_\alpha)$.*

Proof. Let P denote the closure of the convex hull of $xx^*,\ x \in D(L)$ in $D(\delta'_\alpha)$. Then obviously $P \subset D(\delta'_\alpha) \cap A_+$ and P is α'-invariant.

For $\varphi \in C_c^\infty(\mathbf{R})$ and $x \in A$ we denote

$$T_\varphi(x) = \int \varphi(t) T_t(x) dt.$$

We then know that $T_\varphi(x) \in D(L)$. By choosing a sequence (φ_n) which converges to the Dirac function at $0 \in \mathbf{R}$, we get that $\|T_{\varphi_n}(x) - x\| \to 0$. Thus P is dense in A_+.

Suppose that $P \neq D(\delta'_\alpha) \cap A_+$ and let $x_0 \in D(\delta'_\alpha) \cap A_+ \setminus P$. Since $(1-\delta'_\alpha)P$ is a closed convex cone of A_{sa} and $(1-\delta'_\alpha)(x_0) \notin (1-\delta'_\alpha)P$, there is a $f \in A^*$ such that $f^* = f$, $f(x_0 - \delta'_\alpha(x_0)) < 0$, and $f(y - \delta'_\alpha(y)) \geq 0,\ y \in P$. Then we can find a function $\varphi \in C_c^\infty(\mathbf{R})$ such that $\varphi \geq 0$ and

$$\int \varphi(t) f \circ \alpha'_t(x_0 - \delta'_\alpha(x_0)) dt < 0.$$

Define $f_0 \in A^*$ by

$$f_0 = \int \varphi(t) f \circ \alpha'_t dt$$

and note that $f_0(x_0) < f_0\delta'_\alpha(x_0)$ and that $x \in D(\delta'_\alpha) \mapsto f_0\delta'_\alpha(x) = -\int \varphi'(t) f \circ \alpha'_t(x) dt$ is continuous as a functional on A. Since $f_0\delta'_\alpha(y) \leq f_0(y),\ y \in P$ and x_0 is in the closure of P in A, we get the inequality that $f_0\delta'_\alpha(x_0) \leq f_0(x_0)$, which is a contradiction.

We recall an element $x \in A$ is called a pseudo-projection if $0 \leq x \leq 1$ and there is a non-zero $a \in A$ with $xa = a$.

Lemma 3.2 *Let $e, f, p \in A$ be a pseudo-projection such that $e, f \in D(\delta_\alpha)$ and $ef = e$. For any $\epsilon > 0$ there are pseudo-projections $E, F \in A$ such that $E, F \in D(\delta_\alpha)$, $EF = E$, $\|\delta_\alpha(E)\| < \epsilon$, $eE = e$, and $\|pE - p\| < \epsilon$.*

Proof. By (the proof of) Lemma 2.5 we get a sequence (E_n) in $D(\delta_\alpha)$ such that $0 \leq E_n \leq 1$, $\|\delta_\alpha(E_n)\| \to 0$, $\|yE_n - y\| \to 0$ for $y = f, \delta_\alpha(f), p$. We set

$$x_n = 2f - f^2 + (1-f)E_n(1-f).$$

Note that $x_n \in D(\delta_\alpha)$, $0 \leq x_n \leq 1$, $ex_n = e$, $\|x_n - E_n\| \to 0$ (so $\|px_n - p\| \to 0$), and $\|\delta_\alpha(x_n)\| \to 0$ (as $\delta_\alpha(x_n) = \delta_\alpha(f)(1-E_n)(1-f) + (1-f)(1-E_n)\delta_\alpha(f) + (1-f)\delta_\alpha(E_n)(1-f)$). We prepare two functions $\varphi, \psi \in C_c^\infty(\mathbf{R})$ such that $0 \leq \varphi \leq \psi \leq 1$, $\varphi(0) = 0 = \psi(0)$, $\varphi(1) = 1 = \psi(1)$, and $\varphi\psi = \varphi$. We also assume that $\varphi(t) = 1$ for $t \in [1-\epsilon, 1]$ for some $\epsilon > 0$. We can then set $E = \varphi(x_n)$ and $F = \psi(x_n)$ for a sufficiently large n. Since $\|px_n - p\| \to 0$, we may assume that $\|pE - p\| < \epsilon$. Since $ex_n = e$, it follows that $eE = e$. The other properties of E, F follow from those of φ, ψ and Lemmas 2.2 and 2.4.

We now consider $A \otimes K, \alpha \otimes \mathrm{id}, u \otimes 1$ etc. instead of A, α, u etc., where K is the compact operators on an infinite-dimensional separable Hilbert space. We will use the same notation α, u, etc. for $\alpha \otimes \mathrm{id}, u \otimes 1$, etc. as before in this new situation, except for $A \otimes K$, which is kept as a reminder.

Lemma 3.3 *For any $\epsilon > 0$ and any pseudo-projection $p \in A \otimes K$ there are pseudo-projections $e, f \in D(\delta'_\alpha)$ and $z \in \mathcal{U}(A \otimes K + 1)$ such that*

$$\begin{aligned} ef &= e, \\ \|\delta'_\alpha(e)\| &< \epsilon, \\ \|z - 1\| &< \epsilon, \\ \|pe - p\| &< \epsilon, \\ fz &\in D(L), \\ e, f, z - 1 &\in A \otimes M_n, \end{aligned}$$

for some n, where $A \otimes M_n$ is regarded as a C-subalgebra of $A \otimes K$.*

Proof. We may suppose that $p \in A \otimes M_n \subset A \otimes K$. Since $M_n \otimes K \cong K$, we may suppose, by regarding $A \otimes M_n$ as A, that $p \in A = A \otimes e_{11}$, where (e_{ij}) is a generating family of matrix units of K.

We choose four functions $\varphi_1, \varphi_2, \psi, \chi \in C_c^\infty(\mathbf{R})$ such that $0 \le \varphi_1 \le \varphi_2 \le \psi \le 1$, $\varphi_i(0) = 0 = \psi(0)$, $\varphi_i(1) = 1 = \psi(1)$, $\varphi_1\varphi_2 = \varphi_1$, $\varphi_2\psi = \varphi_2$, and $\psi(t) = t\chi(t)^2$.

By Lemma 2.5, for any $\epsilon_1 > 0$, we choose a pseudo-projection $b \in A$ such that $\|pb - p\| < \epsilon_1$ and $\|\delta'_\alpha(b)\| < \epsilon_1$.

By 3.1 there is a finite sequence $(x_1, \dots, x_n)$ in $D(L) \cap A \otimes e_{11}$ such that $c = \sum_{i=1}^n x_i x_i^*$ is close to b in $D(\delta'_\alpha)$; more specifically, $\|b - c\| < \epsilon_1$ and $\|\delta'_\alpha(b) - \delta'_\alpha(c)\| < \epsilon_1$ (and so $\|\delta'_\alpha(c)\| < 2\epsilon_1$). This entails that $\|pc - p\| < \epsilon_1 + \|pb - p\| < 2\epsilon_1$. We set $e = \varphi_1(c)$ and $f = \varphi_2(c)$.

If ϵ_1 is sufficiently small, then the condition $\|\delta'_\alpha(c)\| < 2\epsilon_1$ implies that $\|\delta'_\alpha(e)\| < \epsilon$. Also the condition $\|pc - p\| < 2\epsilon_1$ implies that $\|pe - p\| < \epsilon$. That $ef = e$ follows from the property of φ_i.

We set $y = \sum_{i=1}^n \chi(c) x_i e_{1,n+i} \in A \otimes K$. Since $\chi(c) \in D(\delta'_\alpha)$ and $e_{1,n+i} \in D(\delta_\alpha)$, we have that $y \in D(L)$. It also follows that $yy^* = \chi(c)c\chi(c) = \psi(c)$. We define a unitary $z \in A \otimes K + 1$ by

$$z = (E_{1n} - yy^*)^{1/2} + y - y^* + (E_{n+1,2n} - y^*y)^{1/2} + (1 - E_{1,2n}),$$

where $E_{1,n} = \sum_{i=1}^n 1 \otimes e_{ii}$ etc. Note that z is indeed a unitary because $y = E_{1,n} y E_{n+1,2n}$ and z is essentially of the form

$$\begin{pmatrix} (1 - yy^*)^{1/2} & y \\ -y^* & (1 - y^*y)^{1/2} \end{pmatrix}.$$

Since $f = \varphi_2(c) \in A \otimes e_{11}$ and $f \cdot yy^* = f$, we get that $fz = fy \in D(L)$. Since $D(\delta_\alpha) \cap \mathcal{U}(A \otimes K + \mathbf{C}1)$ is dense in $\mathcal{U}(A \otimes K + \mathbf{C}1)$ and $z(1 - E_{1,2n}) = 1 - E_{1,2n} \in D(\delta_\alpha)$, we can find a unitary $v \in D(\delta_\alpha) \cap \mathcal{U}(A \otimes K + 1)$ such that $v(1 - E_{1,2n}) = 1 - E_{1,2n}$ and $\|z - v\| < \epsilon$. Since $fzv^* \in D(L)$, we can take zv^* as z in the statement.

By using the unitary $z \approx 1$ above, we define a flow β on $A \otimes K$ by $\beta_t = \mathrm{Ad}\, z\alpha_t \mathrm{Ad}\, z^* = \mathrm{Ad}(z\alpha_t(z)^*)\alpha_t$. Let $v_t = u_t\alpha_t(z)z^*$, which is a β-cocycle in $M(A \otimes K)$. We have that $\mathrm{Ad}\, v_t\beta_t = \mathrm{Ad}\, u_t\alpha_t \equiv \alpha'_t$. Note that

$$t \mapsto v_t\beta_t(f) = u_t\alpha_t(z)z^*\mathrm{Ad}(z\alpha_t(z)^*)\alpha_t(f) = u_t\alpha_t(fz)z^*$$

is differentiable in t as $fz \in D(L)$, where $f \in D(\delta'_\alpha)$ is given in the previous lemma.

Note that $t \mapsto v_t^*$ is α'-cocycle and that $\mathrm{Ad}\, v_t^*\alpha'_t = \beta_t$. Define a flow T' of isometries on $A \otimes K$ by

$$T'_t(x) = v_t^*\alpha'_t(x)$$

and let L' denote the generator of T'. Since $v_t^*\alpha'_t(f) = \beta_t(f)v_t^* = (v_t\beta_t(f))^*$, we have that $f \in D(L')$ and $L'(f) = zL(fz)^*$. Hence $f^2 \in D(L')D(L')^* \subset D(\delta_\beta)$ besides $f \in D(\delta'_\alpha) \cap D(L')$. On the other hand we have that $e \in D(\delta'_\alpha) \cap D(L') \cap D(\delta_\beta)$; to see that $e \in D(\delta_\beta)$ we express $e = fe \in D(L')D(L')^* \subset D(\delta_\beta)$. Obviously we might assume that $f \in D(\delta_\beta)$ as well.

Now we apply a similar argument as in the proof of Lemma 3.3 to the pair α' and v^* (with $\mathrm{Ad}\, v^*\alpha' = \beta$). (Actually the argument will be essentially the same; but we retain the previous lemma because then the proof of Theorem 1.1 below will be written down explicitly.)

Lemma 3.4 *Let $\alpha, \alpha', u, e, f, z$ be as in Lemma 3.3 above and let $\beta_t = \mathrm{Ad}(z\alpha_t(z^*))\alpha_t, v_t = u_t\alpha_t(z)z^*, T', L'$ be as above. For any $\epsilon > 0$ and a pseudo-projection $P \in A \otimes K$ there exist pseudo-projections $E, F \in D(\delta_\beta)$ and a unitary $Z \in \mathcal{U}(A \otimes K + 1)$ such that*

$$\begin{aligned} eE &= e, \\ EF &= E, \\ \|\delta_\beta(E)\| &< \epsilon, \\ \|Z - 1\| &< \epsilon, \\ \|PE - P\| &< \epsilon, \\ eZ &= e, \\ FZ &\in D(L'). \end{aligned}$$

Moreover E, F, and $Z - 1$ can be chosen from $A \otimes M_n$ for some n.

Proof. We may suppose that $P, e \in A \otimes e_{11}$, as in the proof of the previous lemma. By doing so the new (e_{ij}) generates a C*-subalgebra of the original K on which α' acts trivially.

We choose four functions $\varphi_1, \varphi_2, \psi, \chi \in C_c^\infty(\mathbf{R})$ such that $0 \le \varphi_1 \le \varphi_2 \le \psi \le 1$, $\varphi_i(0) = 0 = \psi(0)$, $\chi(0) = 0$, $\varphi_i(1) = 1 = \psi(1)$, $\varphi_1\varphi_2 = \varphi_1$, $\varphi_2\psi = \varphi_2$, and $\psi(t) = t\chi(t)^2$.

For any $\epsilon_1 > 0$ there is a pseudo-projection $b \in D(\delta_\beta)$ such that $\|xb - x\| < \epsilon_1$ for $x = P, f^2, \delta_\beta(f^2)$, and $\|\delta_\beta(b)\| < \epsilon_1$.

By 3.2 there is a finite sequence $(x_1, \dots, x_{n-1})$ in $D(L') \cap A \otimes e_{11}$ such that $c = \sum_i x_i x_i^*$ is close to b in $D(\delta_\beta)$; more precisely $\|b - c\| < \epsilon_1$ and $\|\delta_\beta(c)\| < \epsilon_1$. Let

$$C = 2f^2 - f^4 + \sum_{i=1}^{n-1}(1-f^2)x_i x_i^*(1-f^2) = 2f^2 - f^4 + (1-f^2)c(1-f^2),$$

which belongs to $D(\delta_\beta) \cap A \otimes e_{11}$. We define $E = \varphi_1(C)$, $F = \varphi_2(C)$, and

$$y = \chi(C)(2-f^2)^{1/2} f e_{1,1} + \sum_{i=1}^{n-1} \chi(C)(1-f^2)x_i e_{1,1+i}.$$

If ϵ_1 is sufficiently small, then C is close to c and b and we may assume that $\|PE - P\| < \epsilon$ and $\|\delta_\beta(E)\| < \epsilon$. Since $\varphi_1\varphi_2 = \varphi_1$, we have that $EF = E$. Since $f^2, (2-f^2)^{1/2} \in D(\delta_\beta)$ (as $\mathrm{Sp}(2-f^2) \subset [1,2]$), $\chi(C) \in D(\delta_\beta)$, $f \in D(L')$, and $e_{1,i} \in D(\delta'_\alpha)$, we have that $y \in D(L')$. We also have that $yy^* = \psi(C)$ and so $Fyy^* = F$. Since $eC = e = Ce$ and $ef = e$, we get that $eE = e$ and $ey = e$.

Since $y \in A \otimes M_n$, we may assume that there are pseudo-projection $F_1, F_2 \in A \otimes M_n$ such that $F_1 y = y$ and $y F_2 = y$. (By assuming that there are other $\psi_1, \chi_1 \in C_c^\infty(\mathbf{R})$ such that $\psi\psi_1 = \psi$ and $\psi_1(t) = t\chi_1(t)^2$ etc. we may construct y_1 using χ_1 just we did y using χ above and set $F_1 = y_1 y_1^*$ and $F_2 = y_1^* y_1$.)

Let $Y = y + (E_{1,n} - F_1)(E_{1,n} - F_2) \in A \otimes M_n$ and check that $YY^* = yy^* + (E_{1,n} - F_1)(E_{1,n} - F_2)^2(E_{1,n} - F_1) \le E_{1,n}$. Let $W = \sum_{i=1}^n e_{i,n+i}$ and define a unitary $Z \in A \otimes K + 1$ by

$$Z = Y + (E_{1,n} - YY^*)^{1/2} W - W^*(E_{1,n} - Y^*Y)^{1/2} + W^* Y^* W + 1 - E_{1,2n}.$$

Since $Y \equiv E_{1,n} \mod A \otimes K$, we can check that this is indeed a unitary belonging to $A \otimes K + 1$. Since $FYY^* = F$ we get that $FZ = FY = Fy \in D(L')$, which implies that $eZ = ey = e$.

Let $V \in (A \otimes K + 1) \cap D(\delta'_\alpha)$ be a unitary such that $V(1 - E_{1,2n}) = 1 - E_{1,2n}$ and $\|Z - V\| \approx 0$. Let $f_1 \in A \otimes e_{11} \cap D(\delta'_\alpha)$ be a pseudo-projection such that $ef_1 = e$ and $\|f_1 f - f_1\| \approx 0$. Since $f_1 y \approx f_1$ (as $f_1 C \approx f_1$), we get that $f_1 Z \approx f_1$ and so $f_1 V \approx f_1$. Then $V_1 = 2f_1 - f_1^2 + (1 - f_1)V(1 - f_1)$ is close to V and satisfies that $V_1 \in D(\delta'_\alpha)$ and $V_1 e = e = e V_1$. Thus by the polar decomposition of V_1 we get a unitary $V_2 \in (A \otimes K + 1) \cap D(\delta'_\alpha)$ such that $Z \approx V_2$, $V_2(1 - E_{1,2n}) = 1 - E_{1,2n}$, and $eV_2 = e$. Note that $FZV_2^* \in D(L')$ and $eZV_2^* = eV_2^* = e$. We take ZV_2^* for Z, completing the proof.

Proof of Theorem 1.1

By using $e \in D(L')$ above, we know that $v_t^* \alpha'_t(e) = \beta_t(e) v_t^*$ is differentiable in t. We define

$$x = \lim_{t \to 0} \frac{1}{t}(v_t \beta_t(e) - e) = L'(e)^*.$$

We choose a pseudo-projection P such that $Px \approx x$. We then choose E and Z as in the previous lemma. Note that $xE \approx xPE \approx xP \approx x$ and that $T'_t(EZ) = v_t^* \alpha'_t(EZ) =$

$\beta_t(EZ)v_t^*$, which is differentiable in t. Since $E \in D(\delta_\beta)$, the following limits exists:

$$\begin{aligned} y &= \lim_{t\longrightarrow 0} \frac{1}{t}(EZv_t\beta_t(Z^*)E - E^2) \\ &= \lim_{t\longrightarrow 0} \frac{1}{t}EZ(v_t\beta_t(Z^*E) - Z^*E) + \lim_{t\longrightarrow 0} \frac{1}{t}EZv_t\beta_t(Z^*)(E - \beta_t(E)) \\ &= EZL'(EZ)^* - E\delta_\beta(E). \end{aligned}$$

We assert that $y^* = -y$. Since $v_t^* = \beta_t(v_{-t})$, we have that

$$\begin{aligned} y^* &= \lim_{t\longrightarrow 0} t^{-1}(E\beta_t(Z)v_t^*Z^*E - E^2) \\ &= \lim_{t\longrightarrow 0} t^{-1}(E\beta_t(Zv_{-t}\beta_{-t}(Z^*))E - E^2) \\ &= \lim_{t\longrightarrow 0} t^{-1}(\beta_{-t}(E)Zv_{-t}\beta_{-t}(Z^*)\beta_{-t}(E) - \beta_{-t}(E^2)). \end{aligned}$$

The last limit is equal to $-\delta_\beta(E)E - E\delta_\beta(E) - y + \delta_\beta(E^2) = -y$.

Since $eE = e$ and $eZ = e$, we have that

$$\begin{aligned} ey &= \lim_{t\longrightarrow 0} t^{-1}(ev_t\beta_t(Z^*)E - e) \\ &= \lim_{t\longrightarrow 0} t^{-1}(e - v_t\beta_t(e)v_t^*)v_t\beta_t(Z^*)E + \lim_{t\longrightarrow 0} t^{-1}(v_t\beta_t(e)E - e) \\ &= -\delta'_\alpha(e)Z^*E + xE, \end{aligned}$$

which is close to x (as $\|\delta'_\alpha(e)\| \approx 0$ and $xE \approx x$).

We define a β-cocycle w in $A \otimes K + \mathbf{C}1$ by

$$\frac{dw_t}{dt} = w_t\beta_t(y).$$

Let $G(t) = v_t\beta_t(e)w_t^*$. Since $(d/dt)v_t\beta_t(e) = v_t\beta_t(x)$, we get that

$$\frac{dG(t)}{dt} = v_t(\beta_t(x) - \beta_t(ey))w_t^*,$$

which implies that $\|G(t) - G(0)\| \leq |t|\|x - ey\|$. Since $G(0) = e$, we get that

$$\|v_t\beta_t(e) - ew_t\| \leq |t|\|x - ey\|.$$

Since $v_t\beta_t(e) = \alpha'_t(e)v_t = \alpha'_t(e)u_t\alpha_t(z)z^*$ and $\|\alpha'_t(e) - e\| \leq |t|\|\delta'_\alpha(e)\|$, we get that

$$\|e(u_t - w_tz\alpha_t(z^*))\| \leq \epsilon|t|,$$

where $\epsilon = \|\delta'_\alpha(e)\| + \|x - ey\|$, which can be made arbitrarily small. Note that $t \mapsto w_tz\alpha_t(z^*)$ is an α-cocycle in $A \otimes K + \mathbf{C}1$.

This shows that for any α-cocycle u in $M(A)$, any pseudo projection $p \in A \otimes K$, and $\epsilon > 0$, there is an $\alpha \otimes \mathrm{id}$-cocycle v in $A \otimes K$ such that

$$\|p(u_t \otimes 1 - v_t)\| < \epsilon, \quad t \in [-1, 1].$$

Let $E \in K$ be a minimal projection. Since $1 \otimes E \in M(A \otimes K)$, $(\alpha_t \otimes \mathrm{id})(1 \otimes E) = 1 \otimes E$, and $\mathrm{Ad}(u_t \otimes 1)\alpha_t \otimes \mathrm{id}(1 \otimes E) = 1 \otimes E$, we can apply Lemma 2.7 to conclude that the α-cocycle u can be approximated by α-cocycles in A in the strict topology.

Before going to the proof of Corollary 1.3, we give:

Lemma 3.5 *Suppose that a non-unital C^*-algebra A has a strictly positive element. Let α be a flow on A and let u be an α-cocycle in $M(A)$. Then there are unitaries $y, Y \in A \otimes K + 1$, sequences (e_n) and (E_n) of pseudo-projections in $A \otimes K$ satisfying the following conditions: Let $\beta_t = \mathrm{Ad}(y\alpha_t(y^*))\alpha_t$ and $w_t = Yu_t\alpha_t(Y^*)\alpha_t(y)y^*$,and $\gamma_t = \mathrm{Ad}\, w_t\beta_t = \mathrm{Ad}(Yu_t\alpha_t(Y^*))\alpha_t$. Let $T_t(x) = w_t\beta_t(x)$ on $A \otimes K$ and let L be the generator of T. Then all e_n, E_n belong to $D(\delta_\beta) \cap D(\delta_\gamma) \cap D(L)$ and satisfy that $\|\delta_\gamma(e_n)\| \to 0$ and $\|\delta_\beta(E_n)\| \to 0$.*

Proof. We apply the previous lemma inductively. First, by Lemma 3.3, we construct e_1, z_1 based on $\beta^{(0)} = \alpha$, $w_t^{(1)} = u_t$, and $\gamma^{(1)} = \mathrm{Ad}\, w^{(1)}\beta^{(0)}$ such that $e_1 \in D(\delta_\gamma^{(1)})$, $\delta_\gamma^{(1)}(e_1) \approx 0$, $z_1 \approx 1$, and $e_1z_1 \in D(\frac{d}{dt}w_t^{(1)}\beta_t^{(0)})$, where $\delta_\gamma^{(1)}$ is the generator of the flow $\gamma^{(1)}$ and the last condition means that $t \mapsto w_t^{(1)}\beta_t^{(0)}(e_1z_1)$ is differentiable. Second, by Lemma 3.4, we construct E_1, Z_1 based on $\gamma^{(1)}$, $(v_t^{(1)})^* = (w_t^{(1)}\beta_t^{(0)}(z_1)z_1^*)^*$, and $\beta^{(1)} = \mathrm{Ad}\,(v^{(1)})^*\gamma^{(1)} = \mathrm{Ad}(z_1\alpha_t(z_1^*))\alpha$ such that $e_1E_1 = e_1$, $e_1Z_1 = e_1$, $E_1 \in D(\delta_\beta^{(1)})$, $\delta_\beta^{(1)}(E_1) \approx 0$, $Z_1 \approx 1$, and $E_1Z_1 \in D(\frac{d}{dt}(v_t^{(1)})^*\gamma_t^{(1)})$. Next, by the same lemma, we construct e_2, z_2 based on $\beta^{(1)}$, $w_t^{(2)} = Z_1v_t^{(1)}\beta_t^{(1)}(Z_1^*)$, and $\gamma^{(2)} = \mathrm{Ad}\, w^{(2)}\beta^{(1)}$ such that $E_1e_2 = E_1$, $E_1z_2 = E_1$, $e_2 \in D(\delta_\gamma^{(2)})$, $\delta_\gamma^{(2)}(e_2) \approx 0$, $z_2 \approx 1$, and $e_2z_2 \in D(\frac{d}{dt}w_t^{(2)}\beta_t^{(1)})$. Here we want to make clear the following point in the proof Lemma 3.4: When we construct a suitable element in $D(L^{(1)})$, which will be used to define z_2, from a bunch of elements in $D(L^{(1)})$, we use a family of matrix units in K, which should be in $D(\delta_\beta^{(1)})$, where $L^{(1)}$ is the generator of $t \mapsto w_t^{(2)}\beta_t^{(1)}$. This argument is based on the given E_1, Z_1 and e_1, z_1. We have supposed that $E_1, Z_1 - 1$ and $e_1, z_1 - 1$ all belong to $A \otimes M_n$ for some n and identified $M_n \otimes K$ with K. Let us denote by K_1 the image of $\mathbf{C} \otimes K$ in K. Thus what we use is a generating family of matrix units $(e_{ij}^{(1)})$ in K_1 such that $E_1, Z_1 - 1, e_1, z_1 - 1 \in e_{11}^{(1)}(A \otimes K)e_{11}^{(1)}$. Since K_1 need not be contained in $D(\delta_\beta^{(1)})$, we actually use $\mathrm{Ad}\, z_1(e_{ij}^{(1)})$, to define y there, on which $\beta^{(1)}$ acts trivially (as α acts trivially on K_1). Since $\mathrm{Ad}\, z_1(e_{ii}^{(1)}) = e_{ii}^{(1)}$, this causes no problem. (In this explanation we omitted a *companion* pseudo-projection, denoted by using letter f or F, for each of $e_1, E_1, e_2, \ldots$, which plays an important role in the proof but does not show up in the outcome.)

Repeating this way we obtain sequences (e_n) and (E_n) of pseudo-projections in $A \otimes K$ and sequences (z_n) and (Z_n) in $\mathcal{U}(A \otimes K + 1)$ as follows:

We define p to be a strictly positive element of $A \otimes K$ and

$$\begin{aligned}
\beta_t^{(0)} &= \alpha_t, \\
w_t^{(1)} &= u_t, \\
\beta_t^{(n)} &= \mathrm{Ad}(z_n \beta_t^{(n-1)}(z_n^*))\beta_t^{(n-1)}, \\
v_t^{(n)} &= w_t^{(n)} \beta_t^{(n-1)}(z_n) z_n^*, \\
w_t^{(n)} &= Z_{n-1} v_t^{(n-1)} \beta_t^{(n-1)}(Z_{n-1})^*, \quad n \geq 2, \\
\gamma_t^{(n)} &= \mathrm{Ad}\, w_t^{(n)} \beta_t^{(n-1)},
\end{aligned}$$

for $n \geq 1$ if unspecified, where $t \mapsto v_t^{(n)}$ and $t \mapsto w_t^{(n+1)}$ are $\beta^{(n)}$-cocycles as shown by the implications $w^{(n)} : \beta^{(n-1)}$-cocycle $\Rightarrow$ $v^{(n)} : \beta^{(n)}$-cocycle $\Rightarrow$ $w^{(n+1)} : \beta^{(n)}$-cocycle.

Then we have the following:

$$\begin{aligned}
\|pe_n - p\| &< 1/n, \\
e_n E_n &= e_n, \\
E_n e_{n+1} &= E_n, \\
e_n &\in D(\delta_\gamma^{(n)}) \cap D(\delta_\beta^{(n)}), \\
E_n &\in D(\delta_\beta^{(n)}) \cap D(\delta_\gamma^{(n+1)}), \\
e_n Z_n &= e_n, \\
E_n z_{n+1} &= E_n, \\
e_n z_n &\in D(\frac{d}{dt} w_t^{(n)} \beta_t^{(n-1)}), \\
E_n Z_n &\in D(\frac{d}{dt} (v_t^{(n)})^* \gamma_t^{(n)}), \\
\|\delta_\gamma^{(n)}(e_n)\| &< 1/n, \\
\|\delta_\beta^{(n)}(E_n)\| &< 1/n, \\
\|z_n - 1\| &< 2^{-n}, \\
\|Z_n - 1\| &< 2^{-n}.
\end{aligned}$$

Let $y_n = z_n z_{n-1} \cdots z_1$ and $Y_n = Z_n Z_{n-1} \cdots Z_1$. Then, by the last two conditions above, (y_n) converges, say to $y \in \mathcal{U}(A \otimes K + 1)$, and (Y_n) converges, say to $Y \in \mathcal{U}(A \otimes K + 1)$. Moreover we have the following:

$$\begin{aligned}
e_n Y &= e_n Y_{n-1}, \\
E_n y &= E_n y_n, \\
\beta_t^{(n)} &= \mathrm{Ad}(y_n \alpha_t(y_n^*))\alpha_t, \\
\gamma_t^{(n)} &= \mathrm{Ad}(Y_{n-1} u_t \alpha_t(Y_{n-1}^*))\alpha_t, \\
w_t^{(n)} &= Y_{n-1} u_t \alpha_t(Y_{n-1}^*) \alpha_t(y_{n-1}) y_{n-1}^*, \\
v_t^{(n)} &= Y_{n-1} u_t \alpha_t(Y_{n-1}^*) \alpha_t(y_n) y_n^*.
\end{aligned}$$

The first two equalities are obvious. The third equality follows from $\beta_t^{(n)} = \mathrm{Ad}\, z_n \circ \beta_t^{(n-1)} \circ \mathrm{Ad}\, z_n^*$. The fourth equality is obtained by computing:

$$\gamma_t^{(n)} = \mathrm{Ad}\, Z_{n-1}^* \circ \mathrm{Ad}\, v_t^{(n-1)} \beta_t^{(n-1)} \circ \mathrm{Ad}\, Z_{n-1}^*$$

and

$$\mathrm{Ad}\, v_t^{(n-1)} \beta_t^{(n-1)} = \mathrm{Ad}\, w_t^{(n-1)} \circ \mathrm{Ad}(\beta_t^{(n-2)}(z_{n-1}) z_{n-1}^*) \beta_t^{(n-1)} = \mathrm{Ad}\, w_t^{(n-1)} \mathrm{Ad}\, \beta_t^{(n-2)} = \gamma_t^{(n-1)}$$

from which we get that

$$\gamma_t^{(n)} = \mathrm{Ad}\, Z_{n-1} \circ \gamma_t^{(n-1)} \circ \mathrm{Ad}\, Z_{n-1}^*.$$

The other equalities are obtained by induction: $w_t^{(1)} = u_t$, $w_t^{(n)} = Z_{n-1} \gamma_t^{(n-1)}(Z_{n-1}^*) v_t^{(n-1)} = Y_{n-1} u_t \alpha_t(Y_{n-1}^*) \cdot \alpha_t(Y_{n-2}) u_t^* Y_{n-2}^* v_t^{(n-1)}$, and $v_t^{(n)} = w_t^{(n)} y_{n-1} \alpha_t(y_{n-1}^*) \cdot \alpha_t(y_n) y_n^*$.

We define

$$\begin{aligned} \beta_t &= \mathrm{Ad}(y \alpha_t(y^*)) \alpha_t, \\ \gamma_t &= \mathrm{Ad}(Y u_t \alpha_t(Y^*)) \alpha_t, \\ w_t &= Y u_t \alpha_t(Y^*) \alpha_t(y) y^*, \end{aligned}$$

where $t \mapsto w_t$ is a β-cocycle and satisfies that $\gamma_t = \mathrm{Ad}\, w_t \beta_t$. We also define T by $T_t(x) = w_t \beta_t(x)$, $x \in A \otimes K$ and L to be the generator of T as in the statement.

We assert that all e_n, E_n belong to $D(\delta_\beta) \cap D(\delta_\gamma) \cap D(L)$ and that $\|\delta_\gamma(e_n)\| < 1/n$ and $\|\delta_\beta(E_n)\| < 1/n$.

By computation we have that

$$w_t \beta_t(e_n) = Y u_t \alpha_t(Y^*) \alpha_t(e_n) \alpha_t(y) y^* = Y u_t \alpha_t(Y_{n-1}^* e_n y_n) y^*,$$

where we have used that $Y^* e_n y = Y^* e_n E_n y = Y_{n-1}^* e_n y_n$, and that

$$w_t^{(n)} \beta_t^{(n-1)}(e_n z_n) = Y_{n-1} u_t \alpha_t(Y_{n-1}^*) \alpha_t(e_n z_n) \alpha_t(y_{n-1}) y_{n-1}^* = Y_{n-1} u_t \alpha_t(Y_{n-1}^* e_n y_n) y_{n-1}^*.$$

Hence we get that $e_n \in D(L)$. Since $\gamma_t(e_n) = w_t \beta_t(e_n) w_t^* = Y u_t \alpha_t(Y_{n-1}^* e_n Y_{n-1}) u_t^* Y^*$ and $\gamma_t^{(n)}(e_n) = Y_{n-1} u_t \alpha_t(Y_{n-1}^* e_n Y_{n-1}) u_t^* Y_{n-1}^*$, we get that $e_n \in D(\delta_\gamma)$ and $\|\delta_\gamma^{(n)}(e_n)\| = \|\delta_\gamma(e_n)\|$. Similarly we have that $\beta_t(e_n) = y \alpha_t(y_n^* e_n y_n) y^*$ and $\beta_t^{(n)}(e_n) = y_n \alpha_t(y_n^* e_n y_n^*) y_n^*$, which implies that $e_n \in D(\delta_\beta)$.

For E_n we have that

$$w_t \beta_t(E_n) = Y u_t \alpha_t(Y^*) \alpha_t(E_n) \alpha_t(y) y^* = Y u_t \alpha_t(Y_n^* E_n y_n) y^*$$

and

$$(v_t^{(n)})^* \gamma_t^{(n)}(E_n Z_n) = y_n \alpha_t(y_n^* E_n Y_n) u_t^* Y_{n-1},$$

we get that $E_n \in D(L)$. Since $\beta_t(E_n) = y\alpha_t(y^*E_ny)y^* = y\alpha_t(y_n^*E_ny_n)y^*$ and $\beta_t^{(n)}(E_n) = y_n\alpha_t(y_n^*E_ny_n)y_n^*$, we get that $E_n \in D(\delta_\beta)$ and $\|\delta_\beta(E_n)\| = \|\delta_\beta^{(n)}(E_n)\|$. Similarly we have that $\gamma_t(E_n) = Yu_t\alpha_t(Y_n^*E_nY_n)u_t^*Y^*$ and $\gamma_t^{(n+1)}(E_n) = Y_nu_t\alpha_t(Y_n^*E_nY_n)u_t^*Y_n^*$, which implies that $E_n \in D(\gamma)$. This concludes the proof.

Proof of Corollary 1.3

We define

$$a_n = \lim_{t\to 0} t^{-1}(w_t\beta_t(e_n) - e_n) = \lim_{t\to 0} t^{-1}(Yu_t\alpha_t(Y^*e_ny)y^* - e_n)$$

as an element of $A \otimes K$. We also define

$$b_n = \lim_{t\to 0} t^{-1}E_n(w_t - 1)E_n = \lim_{t\to 0} t^{-1}(E_nYu_t\alpha_t(Y^*y)y^*E_n - E_n^2),$$

where the limit exists because both $t \mapsto w_t\beta_t(E_n)$ and $t \mapsto \beta_t(E_n)$ are differentiable. We can also conclude that $b_n^* = -b_n$. We compute: if $m \geq n$,

$$e_nb_m = \lim_{t\to 0} t^{-1}(e_nYu_t\alpha_t(Y^*)\alpha_t(y)y^*E_m - e_n) = -\delta_\gamma(e_n)E_m + \lim_{t\to 0} t^{-1}(Yu_t\alpha_t(Y^*e_ny)y^* - e_n)E_m,$$

where we have used that $\gamma_t(e_n)Yu_t\alpha_t(Y^*) = Yu_u\alpha_t(Y^*e_n)$. Thus we get that for $m \geq n$,

$$e_nb_m = a_nE_m - \delta_\gamma(e_n)E_m,$$

which implies that $\limsup_m \|e_nb_m - a_n\| \leq \|\delta_\gamma(e_n)\|$.

We define a continuous map $h : [0,\infty) \to (A \otimes K)_{sa}$ by: For $t \in [n-1, n]$,

$$-ih(t) = (n-t)b_{n-1} + (t-n+1)b_n,$$

with $b_0 = 0$. Then, just as in the proof of Theorem 1.1, we can conclude that $\|w_t\beta_t(e_n) - e_nw_t^{(h(s))}\| \leq \|e_nh(s) - a_n\||t|$, where $w_t^{(h(s))}$ is the β-cocycle determined by $(d/dt)w_t^{(h(s))} = w_t^{(h(s))}\beta_t(ih(s))$. Thus we get that $\|e_n(w_t - w_t^{(h(s))})\| \leq \epsilon_n(s)|t|$ with $\epsilon_n(s) = \|e_nh(s) - a_n\| + \|\delta_\gamma(e_n)\|$. Since $\lim_n \limsup_s \epsilon_n(s) = 0$ and (e_n) forms an approximate identity for $A \otimes K$, we can conclude that for a strictly positive element p of $A \otimes K$, $\max_{t\in[-1,1]} \|p(w_t - w_t^{(h(s))})\| \to 0$ as $s \to \infty$. Let $z_t^{(s)} = Y^*w_t^{(h(s))}y\alpha_t(y^*)\alpha_t(Y)$, which is an α-cocycle in $A \otimes K$ and satisfies that $\max_{t\in[-1,1]} \|p(u_t \otimes 1 - z_t^{(s)})\| \to 0$ as $s \to \infty$. Then Lemma 2.9 implies that the α-cocycle u (in $M(A)$) can be approximated by α-cocycles in A asymptotically.

4 Invariant hereditary C*-subalgebras

We will prove Corollary 1.4 in this section. We note that Theorem 1.1 implies:

Lemma 4.1 *Any cocycle perturbation of an approximately inner flow is approximately inner.*

The main part is to show:

Theorem 4.2 *Let A be a C^*-algebra and α a flow on A. Let B be a full α-invariant hereditary C^*-subalgebra of A. Then α is approximately inner if and only if $\alpha|B$ is approximately inner.*

Proof. We follow the proof of Theorem 2.8 of [4] and then apply Theorem 1.1.

We equip the C^*-algebra $A \otimes M_2$ with the flow $\overline{\alpha} = \alpha \otimes \mathrm{id}$. Let C be the hereditary C*-subalgebra of $A \otimes M_2$ generated by $A \otimes e_{11}$ and $B \otimes e_{22}$, which is $\overline{\alpha}$-invariant.

We will write $e_1 = 1 \otimes e_{11} \in M(C)$ and $e_2 = 1 \otimes e_{22} \in M(C)$ and denote $\overline{\alpha}$ by α.

If A is separable, then, by [4], there is a partial isometry $v \in M(C \otimes K)$ such that $v^*v = e_1 \otimes 1$ and $vv^* = e_2 \otimes 1$. Hence we can define an isomorphism Φ of $A \otimes K$ onto $B \otimes K$ by $\Phi(x) = vxv^*$, where $A \otimes K$ and $B \otimes K$ are regarded as corners of $C \otimes K$. Then it follows that $\Phi(\alpha_t(x)) = \mathrm{Ad}\, u_t \alpha_t(\Phi(x)), \quad x \in A \otimes K$, where $u_t = v\alpha_t(v^*)$, which is a cocycle in the multiplier algebra $M(B \otimes K)$. Hence, by Corollary 1.3, $\alpha|A \otimes K$ is approximately inner if and only if $\alpha|B \otimes K$ is approximately inner. Thus, by Lemma 2.1, this completes the proof in the separable case.

In general, for any finite subset $\mathcal{F}$ of A, we can find an α-invariant separable C*-subalgebra A_0 of A such that $A_0 \supset \mathcal{F}$ and $B_0 = B \cap A_0$ is a full hereditary C*-subalgebra of A_0. Furthermore, we can suppose that $\alpha|A_0$ is approximately inner if α is approximately inner, and that $\alpha|B \cap A_0$ is approximately inner if $\alpha|B$ is approximately inner.

To see this suppose that α is approximately inner. Let A_1 be the C*-subalgebra generated by $\alpha_t(x),\ x \in \mathcal{F},\ t \in \mathbf{R}$, which is separable and α-invariant. Let (x_n) be a dense sequence in $\{x \in A_1 \mid 0 \le x \le 1\}$; for each x_n we choose a finite sequence $(y_1^{(n)}, \dots, y_{k_n}^{(n)})$ in AB such that $\|x_n - \sum_i y_i^{(n)}(y_i^{(n)})^*\| < 1/n$ and let A_2 be the C*-subalgebra generated by A_1 and all $y_i^{(n)}$. Then it follows that A_2 is separable and the closed linear span of $A_2B_2A_2$ contains A_1, where $B_2 = B \cap A_2$ is a hereditary C*-subalgebra of A_2. Let (h_n) be a sequence in A_{sa} such that $\alpha_t(x) = \lim_{n\to\infty} \mathrm{Ad}\, e^{ith_n}(x),\ x \in A_2$. Let A_3 be the C*-subalgebra generated by A_2 and all h_n. Let A_4 be the C*-subalgebra generated by $\alpha_t(A_3),\ x \in \mathbf{R}$, which is separable and α-invariant. Then regarding A_4 as A_1 we repeat this process; the closure of the union of the increasing sequence (A_n) of C*-subalgebras of A so obtained is the desired C*-subalgebra A_0. Then by the previous argument we get that $\alpha|B \cap A_0$ is approximately inner, or in particular, for any $\epsilon > 0$ there is an $h = h^* \in B \cap A_0$ such that $\|\alpha_t(x) - \mathrm{Ad}\, e^{iht}(x)\| < \epsilon,\ x \in B \cap \mathcal{F},\ t \in [-1, 1]$. Since $B \cap \mathcal{F}$ is an arbitrary subset of B, this implies that $\alpha|B$ is approximately inner. The other case can be treated similarly.

Proof of Corollary 1.4

In the case the hereditary C*-subalgebra B is full this is shown in the above theorem.

In the case B is not full, let I be the ideal generated by B. Then if α is approximately inner, then the restriction $\alpha|I$ is approximately inner (this is easy; see [11]). Since B is a full hereditary C*-subalgebra of I, we can apply the above theorem to conclude that $\alpha|B$

is approximately inner in this case. To get the statement with asymptotic innerness, we just apply Corollary 1.3 and Lemma 2.8.

5 AF flows

Let A be an AF algebra. We call a flow α on A an AF flow if there is an increasing sequence (A_n) of finite-dimensional C*-subalgebras of A such that $\alpha_t(A_n) = A_n$ for all $t \in \mathbf{R}$ and $n \in \mathbf{N}$ and A is the closure of $\bigcup_n A_n$. We assume that A has no unit and let u be an α-cocycle in $M(A)$ and $\alpha'_t = \operatorname{Ad} u_t \alpha_t$. Our purpose is to prove that a cocycle perturbation of an AF flow is approximately AF (see [10]).

We first note the following simple lemma:

Lemma 5.1 *Let α be a flow on a non-unital C^*-algebra A with a strict positive element and let u be an α-cocycle in $M(A)$. Let $\alpha' = \operatorname{Ad} u\alpha$. Then there is an approximate identity (e_n) consisting of pseudo projections of A such that $e_n \in D(\delta'_\alpha)$, $e_n e_{n+1} = e_n$, $\|\delta'_\alpha(e_n)\| < 1/n$, and $\max_{t\in[-1,1]} \|\alpha_t(e_n) - e_n\| < 1/n$.*

Proof. We define a flow γ on $A \otimes M_2$ by $\gamma_t(x \otimes e_{12}) = \alpha_t(x)u_t^* \otimes e_{12},\ x \in A$. Then we find a sequence (E_n) in $A \otimes M_2$ such that $0 \le E_n \le 1$, $\|xE_n - x\| \to 0$ for $x \in A$, $\|[1 \otimes e_{ij}, E_n]\| \to 0$, $E_n \in D(\delta_\gamma)$, and $\|\delta_\gamma(E_n)\| \to 0$. We write $E_n = \sum_{ij}(E_n)_{ij} \otimes e_{ij}$ and let $e_n = (E_n)_{22}$. Then $0 \le e_n \le 1$, $\|xe_n - x\| \to 0$ for $x \in A$, $e_n \in D(\delta'_\alpha)$, $\|\delta'_\alpha(e_n)\| \to 0$, and $\max_{t\in[-1,1]} \|\alpha_t(e_n) - e_n\| \to 0$ (as $\|(E_n)_{11} - (E_n)_{22}\| \to 0$). We have to modify (e_n) to make it satisfy $e_n e_{n+1} = e_n$.

Let $\varphi, \psi \in C_c^\infty(\mathbf{R})$ be such that $0 \le \varphi \le \psi \le 1$, $\varphi(0) = 0 = \psi(0)$ and $\varphi(1) = 1 = \psi(1)$.

We denote the above (e_n) by (p_n). We define $e_1 = \varphi(p_n)$ and $f_1 = \psi(p_n)$ for a large n so that $\|\delta'_\alpha(e_1)\| < 1$ and $\max_{t\in[-1,1]} \|\alpha_t(e_1) - e_1\| < 1$. Let $x_{1n} = 2f_1 - f_1^2 + (1 - f_1)p_n(1 - f_1)$, which satisfies that $e_1 x_n = e_1$ and is close to p_n (as elements of $D(\delta'_\alpha)$) for large n. We choose n so large that $\|\delta'_\alpha(e_2)\| < 1/2$ and $\max_{t\in[-1,1]} \|\alpha_t(e_2) - e_2\| < 1/2$ follow for $e_2 = \varphi(x_{1n})$. Note that $e_1 e_2 = e_1$. Next we set $f_2 = \psi(x_{1n})$ and $x_{2n} = 2f_2 - f_2^2 + (1 - f_2)p_n(1 - f_2)$, and we repeat this process.

Lemma 5.2 *Let α be an AF flow on an AF algebra A and let $u, \alpha' = \operatorname{Ad} u\alpha$ etc. be as given before Lemma 5.1. Let $\mathcal{F}$ be a finite subset of the unit ball of A and $\epsilon > 0$. Then there is a finite-dimensional C^*-subalgebra B of A and an α'-cocycle w in A such that each $x \in \mathcal{F}$ has $x' \in B$ with $\|x - x'\| < \epsilon$, $\|w_t - 1\| < \epsilon,\ t \in [-1, 1]$, and $\operatorname{Ad} w_t \alpha'_t(B) = B$.*

Proof. For any small $\epsilon_1 > 0$ we choose pseudo-projections $p, q \in A$ such that $\|xp - x\| < \epsilon_1$ for $x \in \mathcal{F}$, $pq = p$, $\|\alpha_t(q) - q\| < \epsilon_1,\ t \in [-1, 1]$, $q \in D(\delta'_\alpha)$, and $\|\delta'_\alpha(q)\| < \epsilon_1$ (by applying the previous lemma). Then, by Theorem 1.1, we choose $H \in A_{sa}$ such that $\|(u_t - u_t^{(H)})q\| < \epsilon_1,\ t \in [-1, 1]$, which entails that

$$\|(u_t - u_t^{(H)})\alpha_t(q)\| < 3\epsilon_1.$$

Then it follows that $\|u_t\alpha_t(q)u_t^* - u_t^{(H)}\alpha_t(q)(u_t^{(H)})^*\| < 6\epsilon_1,\ t \in [-1,1]$. Thus we get that $\|q - \alpha_t^{(H)}(q)\| < 7\epsilon_1,\ t \in [-1,1]$. Note that then, by taking $q' = \alpha_\varphi^{(H)}(q) \approx q$ with an appropriate $\varphi \in C_c^\infty(\mathbf{R})$, we get that $(\delta_\alpha + \operatorname{ad} iH)(q') \approx 0$. Then, for any $\epsilon_2 \in (0, 1/2)$, by Theorem 3.6 of [1] for a sufficiently small ϵ_1, we get $q_1 \in D(\delta_\alpha)$ and $b \in A_{sa}$ such that

$$\|q - q_1\| < \epsilon_2, \quad \|b\| < \epsilon_2, \quad (\delta_\alpha + \operatorname{ad}(iH + ib))(q_1) = 0,$$

and the spectrum of q_1 is finite. Let e be the spectral projection of q_1 corresponding to $[1/2, 3/2]$. Since $\|pq_1 - p\| < \epsilon_2 + \epsilon_1$, we may suppose, for any $\epsilon > 0$, that $\|pe - p\| < \epsilon$. (For $\varphi \in C_c^\infty$ with $0 \le \varphi \le 1$, $\operatorname{supp}(\varphi) \subset [1/2, 3/2]$, and $\varphi(1) = 1$, we could assume that $\|p\varphi(q_1) - p\| < \epsilon$, which implies that $\|p(1-e)\| \le \|p(1 - \varphi(q_1))\| < \epsilon$.) Since $\|(u_t - u_t^{(H)})\alpha_t(q_1)\| < 3\epsilon_1 + 2\epsilon_2,\ t \in [-1,1]$ and $e \le 2q_1$, we get that $\|(u_t - u_t^{(H)})\alpha_t(e)\| < 6\epsilon_1 + 4\epsilon_2$, which we may suppose is smaller than ϵ:

$$\|(u_t - u_t^{(H)})\alpha_t(e)\| < \epsilon, \quad t \in [-1,1].$$

Since $\|pe - p\| < \epsilon$ and $\|x(1-p)\| < \epsilon_1,\ x \in \mathcal{F}$, it follows that $\|xe - x\| < \epsilon_1 + \epsilon < 2\epsilon,\ x \in \mathcal{F}$. Since $(\delta_\alpha + \operatorname{Ad}(iH + ib))(e) = 0$, the restriction $\alpha^{(H+b)}|eAe$ is approximately AF, or a small inner perturbation of an AF flow. Thus we get a unital finite-dimensional C*-subalgebra B of eAe and $b' \in (eAe)_{sa}$ such that $\alpha_t^{(H+b+b')}(B) = B$, $\|b'\| < \epsilon$, and each $x \in \mathcal{F}$ has $y \in B$ with $\|exe - y\| < \epsilon$ (which implies that $\|x - y\| < 5\epsilon$). Since $\|(u_t - u_t^{(H+b+b')})\alpha_t(e)\| < \epsilon + \|b + b'\| < \epsilon + \epsilon_2 + \epsilon < 3\epsilon$ for $t \in [-1,1]$, we get that

$$\|\alpha_t'(e) - e\| < 6\epsilon.$$

Hence there is an α'-cocycle v in $A + 1$ such that $\max_{t\in[-1,1]} \|v_t - 1\| \approx 0$ depending on ϵ and $\operatorname{Ad} v_t\alpha_t'(e) = e$. Then we set

$$w_t = (u_t^{(H+b+b')}u_t^*v_t^*e + 1 - e)v_t,$$

which is an α'-cocycle in A. Since $\|u_t^{(H+b+b')}u_t^*v_t^*e - e\| = \|e(v_tu_t - u_t^{(H+b+b')})\| = \|(v_tu_t - u^{(H+b+b')})\alpha_t(e)\|$, we get that $\|w_t - 1\| \approx 0$ for $t \in [-1,1]$. Since $\operatorname{Ad} w_t\alpha_t'(B) = \operatorname{Ad} u_t^{(H+b+b')}\alpha_t(B) = B$, this completes the proof.

Corollary 1.5 follows from:

Theorem 5.3 *Let α be a flow on a (non-unital) AF algebra. Then the following conditions are equivalent:*

1. *There is an α-cocycle u in $M(A)$ such that $\operatorname{Ad} u\alpha$ is an AF flow.*
2. *For any $\epsilon > 0$ there is an α-cocycle in A such that $\operatorname{Ad} u\alpha$ is an AF flow and $\|u_t - 1\| < \epsilon,\ t \in [-1,1]$.*

3. *α is an approximate AF flow, i.e., there is an increasing sequence (A_n) of finite-dimensional C*-subalgebras of A such that $\bigcup_n A_n$ is dense in A and*

$$\lim_{n\longrightarrow\infty}\max_{t\in[-1,1]}\mathrm{dist}(\alpha_t(A_n),A_n)=0.$$

Proof. The equivalence of the last two conditions is shown in [10] (see Remark 1.4 there). The first condition is apparently weaker than the second. A repeated application of the above lemma shows the converse implication.

We shall prove Corollary 1.6. Specifically we state:

Theorem 5.4 *Let A be an AF algebra and α a flow on A. Let B be a full α-invariant hereditary C*-subalgebra of A. Then α is an approximate AF flow if and only if $\alpha|B$ is an approximate AF flow.*

Proof. Just as in the proof of Theorem 4.2, we have an isomorphism Φ of $A\otimes K$ onto $B\otimes K$ and a $\alpha\otimes\mathrm{id}|B\otimes K$-cocycle u in $M(A\otimes K)$ such that $\Phi(\alpha\otimes\mathrm{id}(x))=\mathrm{Ad}\,u_t(\alpha_t\otimes\mathrm{id})(\Phi(x)),\ x\in A\otimes K$. Thus, by the previous theorem, $\alpha\otimes\mathrm{id}|A\otimes K$ is approximately AF if and only if $\alpha\otimes\mathrm{id}|B\otimes K$. It is easy to see that $\alpha\otimes\mathrm{id}|A\otimes K$ is approximately AF if and only if $\alpha|A$ is approximately AF. This concludes the proof.

To prove the rest of Corollary 1.6, let B be an α-invariant hereditary C*-subalgebra of A and let I be the ideal generated by B. We have to show that if α is approximately AF, then $\alpha|I$ is approximately AF, which implies, by the previous theorem, that $\alpha|B$ is approximately AF. If α is an approximately AF, then there is an $h\in A_{sa}$ such that $\alpha^{(h)}$ is an AF flow, i.e., there is an increasing sequence (A_n) of $\alpha^{(h)}$-invariant finite-dimensional C*-subalgebras of A with dense union. Then, since I is the closure $\bigcup_n(I\cap A_n)$, we have that $\alpha^{(h)}|I$ is an AF flow. Hence $\alpha|I$ is an approximate AF flow by Theorem 5.3.

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