An asymptotic analysis and its application to the nonrelativistic limit of the Pauli–Fierz and a spin-boson model

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An abstract asymptotic theory of a family of self-adjoint operators $\{H_\kappa\}_{\kappa>0}$ acting in the tensor product of two Hilbert spaces is presented and it is applied to the nonrelativistic limit of the Pauli–Fierz model in quantum electrodynamics and of a spin-boson model. It is proven that the resolvent of H_κ converges strongly as $\kappa\to\infty$ and the limit is a pseudoresolvent, which defines an "effective operator" of H_κ at $\kappa\gtrsim\infty$. As corollaries of this result, some limit theorems for H_κ are obtained, including a theorem on spectral concentration. An asymptotic estimate of the infimum of the spectrum (the ground state energy) of H_κ is also given. The application of the abstract theory to the above models yields some new rigorous results for them.

I. INTRODUCTION

This paper consists of two parts: one is concerned with an abstract asymptotic theory of a family of self-adjoint operators and the other presents its application to the nonrelativistic limit of the Pauli–Fierz[1–9] and a spin-boson model.[10–15]

The Pauli–Fierz model is a model in quantum electrodynamics and describes a nonrelativistic one-electron atom coupled to a quantized radiation field. It is known that the model is a realistic one in the sense that in a nonrelativistic region, it explains well some physical phenomena such as the Lamb shift, although the explanations are usually done by using formal perturbation calculations to which rigorous mathematical basis has not yet been given. Only a few mathematically rigorous results have been obtained for the model.[4,5,7,9] The spin-boson model we consider describes a two-level atom coupled to a quantized Bose field and can be regarded as a simplified version of the Pauli–Fierz model.[8] The nonrelativistic limit we study on these models is a scaling limit of the speed of light at the same time as the coupling constant of the models gets a scale transformation, which, as far as we know, has not been discussed in the literature.

To treat the problem of the nonrelativistic limit of the Pauli–Fierz and the spin-boson model in a unified way, we first present in Sec. II an abstract asymptotic theory of a family of self-adjoint operators $\{H_\kappa\}_{\kappa>0}$ acting in the tensor product of two Hilbert spaces. The self-adjoint operator H_κ is an abstract version of operators *unitarily equivalent* to Hamiltonians of some models of an atom coupled to a quantized radiation field, including the Pauli–Fierz and the spin-boson model. We prove that the resolvent of H_κ converges strongly as $\kappa\to\infty$ and the limit is a pseudoresolvent, which defines an "effective" operator of H_κ at $\kappa\gtrsim\infty$. Introducing a concept of "partial expectation" of operators, we represent the effective operator more explicitly. In applications, partial expectations can be used also to describe "fluctuations" caused by a quantized radiation field on an atom (see Sec. III). Further, we obtain an asymptotic estimate of the infimum of the spectrum (the ground state energy) of H_κ. The abstract theory presented here is closely related to asymptotic theories given in Refs. 16–18. But our class of H_κ is different from the operators considered there in the scaling order with respect to κ. There may be different asymptotic theories depending on the scaling order of κ and the form of the relevant operators. We also discuss the spectral concentration of H_κ.

In Sec. III we discuss the Pauli-Fierz model, which, as mentioned above, describes a one-electron atom coupled to a quantized radiation field. For a mathematical generality, we consider the case where the one-electron atom is placed in the d-dimensional space ($d\geqslant 2$). The total Hamiltonian of the model is given by a self-adjoint operator $H(c,e)$ with parameters $c>0$ and $e\in\mathbb{R}\setminus\{0\}$ denoting the speed of light and the elementary charge (the coupling constant in this model), respectively. The scaled Hamiltonian is defined by $H(\kappa)=H(c(\kappa),e(\kappa))$ with $c(\kappa)=\kappa c$ and $e(\kappa)=\kappa^{3/2}e$. The nonrelativistic limit we study is taken in the sense of the scaling limit $\kappa\to\infty$. Since $|e(\kappa)|\to\infty$ as $\kappa\to\infty$, the nonrelativistic limit is a scaling limit of the speed of light at the same time as the magnitude of the coupling constant becomes infinite. We show that $H(\kappa)$ is unitarily equivalent to an operator $\widetilde{H}(\kappa)$, which is of the form of H_κ discussed in Sec. II. Applying the abstract theory in Sec. II to $\widetilde{H}(\kappa)$, we find that the effective operator of $\widetilde{H}(\kappa)$ is a Schrödinger operator $H_{A,\text{eff}}$. In the case $d=3$, the potential operator of $H_{A,\text{eff}}$ coincides with the effective potential that Welton[3] proposed to calculate some observable effects of the quantized radiation field such as the Lamb shift. In Ref. 3 the effective potential was derived by physical arguments. We derive it as a scaling limit in the sense described above, starting from the total Hamiltonian $H(\kappa)$. This does not only justify rigorously the effective potential of Welton but also clarifies a mathematical meaning of it, in other words, in what sense the effective potential is "effective." Further, we show that the ground state energy of the model is nondecreasing as a function of κ and obtain an estimate of the ground state energy, which, to our knowledge, has not been given so far in the literature. We also prove that the spectrum of $H(\kappa)$ is asymptotically concentrated on the spectrum of $H_{A,\text{eff}}$ "locally" as $\kappa\to\infty$.

In Sec. IV we consider the spin-boson model. The nonrelativistic limit of this model is also a scaling limit of the

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speed of light at the same time as the magnitude of the coupling constant becomes infinite, but the scaling order of the coupling constant is different from that of the Pauli–Fierz model. We show that the total Hamiltonian of the model is unitarily equivalent to an operator $\widetilde{H}(\kappa)$ of the form of H_κ in Sec. II. We derive the effective operator of $\widetilde{H}(\kappa)$. Moreover, we show that the ground state energy is nondecreasing in the scaling parameter κ and obtain an estimate of the ground state energy, which slightly improves that given by Davies.[10] We also give a meaning to the transition probability between the two degenerate ground states of the model without the atom part (cf. Ref. 11). Finally, we prove the existence of a "local" spectral concentration of the total Hamiltonian.

In the last section some remarks are given. We conclude the present paper with an Appendix, where we prove some limit theorems related to the strong convergence of resolvents in which the limiting operator is a pseudoresolvent.

II. AN ABSTRACT ASYMPTOTIC THEORY

In this section we present an abstract asymptotic theory for a class of self-adjoint operators. The theory developed below may be formulated in a more abstract setting using a Banach space as in Ref. 16 and for a more general class of operators. In the present paper, however, we take a Hilbert space formulation and restrict our consideration to a class of self-adjoint operators, which allows us to obtain more concrete and stronger results in some respects.

In what follows, we use the following notation: $(\cdot,\cdot)_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$ denote the inner product and the norm of the Hilbert space $\mathscr{H}$, respectively. If there is no danger of confusion, then we omit the subscript $\mathscr{H}$ of them. The domain (resp. range) of an operator T is denoted by $D(T)$ (resp. Ran T). For bounded operators T, we denote by $\|T\|$ the operator norm. By I we denote identity.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces and

$$\mathscr{F} = \mathscr{H} \otimes \mathscr{K}. \tag{2.1}$$

Let A and B be non-negative self-adjoint operators in $\mathscr{H}$ and $\mathscr{K}$, respectively. We assume that

$$\text{Ker}\, B \neq \{0\}. \tag{2.2}$$

Let $\{C_\kappa\}_{\kappa>0}$ be a family of symmetric operators in $\mathscr{F}$ satisfying the following conditions.

(i) For all $\kappa>0$, $D(A\otimes I)\subset D(C_\kappa)$ and $C_\kappa(A\otimes I+\lambda)^{-1}$ is bounded for all $\lambda>0$ with

$$\lim_{\lambda\to\infty} \|C_\kappa(A\otimes I+\lambda)^{-1}\| = 0,$$

where the convergence is uniform in $\kappa\geqslant\kappa_0$ for some $\kappa_0>0$.

(ii) For all $\lambda>0$, $C_\kappa(A\otimes I+\lambda)^{-1}$ is strongly continuous in $\kappa>0$.

(iii) There exists a symmetric operator C in $\mathscr{F}$ such that $D(A\otimes I)\subset D(C)$ and for all $\lambda>0$,

$$\text{s-}\lim_{\kappa\to\infty} C_\kappa(A\otimes I+\lambda)^{-1} = C(A\otimes I+\lambda)^{-1},$$

where s-lim means strong limit.

For each $\kappa>0$, we define

$$H_{0,\kappa} = A\otimes I + \kappa I\otimes B. \tag{2.3}$$

The above property (i) of C_κ implies that for every $\epsilon>0$, there exists a constant $\lambda_0=\lambda_0(\epsilon,\kappa_0)>0$, independent of $\kappa\geqslant\kappa_0$, such that for all $\lambda\geqslant\lambda_0$,

$$\|C_\kappa\Psi\| < \epsilon\|(A\otimes I+\lambda)\Psi\|, \quad \Psi\in D(A\otimes I). \tag{2.4}$$

Since $I\otimes B$ is non-negative and commutes with $A\otimes I$, it follows that for $\lambda\geqslant\lambda_0$,

$$\|C_\kappa\Psi\| < \epsilon\|(H_{0,\kappa}+\lambda)\Psi\|,$$
$$\Psi\in D(H_{0,\kappa}) = D(A\otimes I)\cap D(I\otimes B) \equiv D_{A,B}. \tag{2.5}$$

Hence, C_κ with $\kappa\geqslant\kappa_0$ is infinitesimally small with respect to $H_{0,\kappa}$. Therefore, by the Kato–Rellich theorem (e.g., Refs. 19 and 20), the operator

$$H_\kappa = H_{0,\kappa} + C_\kappa \tag{2.6}$$

with $\kappa\geqslant\kappa_0$ is self-adjoint on $D_{A,B}$ and essentially self-adjoint on every core of $H_{0,\kappa}$. Further, H_κ is bounded from below with

$$H_\kappa \geqslant -\epsilon\lambda/(1-\epsilon), \tag{2.7}$$

where $0<\epsilon<1$ and $\lambda\geqslant\lambda_0(\epsilon,\kappa_0)$. The ground state energy E_κ of H_κ is defined by

$$E_\kappa = \inf\sigma(H_\kappa), \tag{2.8}$$

where $\sigma(T)$ denotes the spectrum of operator T. By (2.7), E_κ is bounded from below uniformly in $\kappa\geqslant\kappa_0$:

$$\epsilon_0 \equiv \inf_{\kappa\geqslant\kappa_0} E_\kappa > -\infty. \tag{2.9}$$

Our aim is to consider the limit $\kappa\to\infty$ of H_κ and to give an asymptotic estimate of E_κ for large κ.

Let P_0 be the orthogonal projection from $\mathscr{K}$ onto Ker B. Then it follows from property (iii) of C_κ and (2.4) that $(I\otimes P_0)C(I\otimes P_0)$ is infinitesimally small with respect to $A\otimes I$. Hence, by the Kato–Rellich theorem again, the operator

$$H_\infty = A\otimes I + (I\otimes P_0)C(I\otimes P_0) \tag{2.10}$$

is self-adjoint on $D(A\otimes I)$ and bounded from below. It is easy to see that the resolvent of H_∞ commutes with $I\otimes P_0$. Hence, H_∞ is reduced by Ran $I\otimes P_0 = \mathscr{H}\otimes\text{Ker}\,B$. We define

$$E_\infty = \inf\sigma(H_\infty \restriction \mathscr{H}\otimes\text{Ker}\,B). \tag{2.11}$$

The first of the main results in this section is the following.

Theorem 2.1: For all $z\in\mathbb{C}$ with Im $z\neq0$ or for $z<0$ with $|z|$ sufficiently large, $(H_\kappa-z)^{-1}$ is strongly continuous in $\kappa\geqslant\kappa_0$ and

$$\text{s-}\lim_{\kappa\to\infty}(H_\kappa-z)^{-1} = (H_\infty-z)^{-1}(I\otimes P_0). \tag{2.12}$$

Further,

$$\overline{\lim_{\kappa\to\infty}} E_\kappa \leqslant E_\infty. \tag{2.13}$$

Proof: Let $\lambda>0$ be sufficiently large so that $-\lambda\in\rho(H_\kappa)\cap\rho(H_\infty)\cap\rho(H_{0,\kappa})$ for all $\kappa\geqslant\kappa_0$, where $\rho(T)$ denotes the resolvent set of T. Iterating the second resolvent formula with respect to the pair $(H_\kappa,H_{0,\kappa})$, we have

$$(H_\kappa+\lambda)^{-1} = \sum_{n=0}^{N}(-1)^n(H_{0,\kappa}+\lambda)^{-1}T_\kappa^n + R_N(\kappa),$$

where

$$T_\kappa = C_\kappa (H_{0,\kappa} + \lambda)^{-1},$$

and

$$R_N(\kappa) = (-1)^{N+1}(H_\kappa + \lambda)^{-1} T_\kappa^{N+1}.$$

It follows from (2.5) that

$$\|R_N(\kappa)\| < (\epsilon_0 + \lambda)^{-1}\epsilon^{N+1}.$$

Hence, taking $\epsilon < 1$, we see that for $\lambda > 0$ sufficiently large

$$(H_\kappa + \lambda)^{-1} = \sum_{n=0}^{\infty} (-1)^n (H_{0,\kappa} + \lambda)^{-1} T_\kappa^n \qquad (2.14)$$

is norm convergent uniformly in $\kappa \geqslant \kappa_0$. It is easy to see that

$$\text{s-}\lim_{\kappa\to\infty} (H_{0,\kappa} + \lambda)^{-1} = (A \otimes I + \lambda)^{-1} I \otimes P_0.$$

Further, by property (iii) of C_κ, we have

$$\text{s-}\lim_{\kappa\to\infty} T_\kappa = C(A\otimes I + \lambda)^{-1}(I \otimes P_0).$$

By the uniform convergence of the series on the right-hand side (rhs) of (2.14), we can interchange the limit $\kappa \to \infty$ and the summation Σ_n to obtain

$$\text{s-}\lim_{\kappa\to\infty} (H_\kappa + \lambda)^{-1} = \sum_{n=0}^{\infty} (-1)^n (A \otimes I + \lambda)^{-1} \times \{\widetilde{C}(A\otimes I + \lambda)^{-1}\}^n (I\otimes P_0), \qquad (2.15)$$

where

$$\widetilde{C} = (I \otimes P_0) C (I \otimes P_0),$$

and we have used the fact that $I \otimes P_0$ is a projection. The rhs of (2.15) is equal to

$$(H_\infty + \lambda)^{-1}(I \otimes P_0).$$

Thus (2.12) with $z = -\lambda$ follows. Once (2.12) is proved for some $z = -\lambda \in \mathbb{R} \cap \rho(H_\kappa) \cap \rho(H_\infty)$, it can be extended to the case $\operatorname{Im} z \neq 0$ by mimicking a standard argument for resolvents (e.g., the proof of Theorem VIII.19 in Ref. 21). The strong continuity of $(H_\kappa - z)^{-1}$ in κ follows similarly. Inequality (2.13) follows from an application of Theorem A.1 in the Appendix. ∎

Theorems 2.1 can be generalized by the following theorem.

Theorem 2.2: Denote by $C_\infty(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ vanishing at ∞. Then, for all $F \in C_\infty(\mathbb{R})$,

$$\text{s-}\lim_{\kappa\to\infty} F(H_\kappa) = F(H_\infty) I \otimes P_0.$$

Proof: This follows from Theorem 2.1 and an application of Theorem A.1 in the Appendix. ∎

We next consider the asymptotic behavior of the ground state energy E_κ. Concerning this problem, we have been able to obtain a result only in the case where C_κ and C are bounded. Let

$$H_\infty(\kappa) = A \otimes I + (I \otimes P_0) C_\kappa (I \otimes P_0) \qquad (2.16)$$

and

$$E_\infty(\kappa) = \inf \sigma(H_\infty(\kappa) \upharpoonright \mathscr{H} \otimes \operatorname{Ker} B). \qquad (2.17)$$

Lemma 2.3: Let C_κ and C be bounded. Then, for all $\kappa > 0$,

$$|E_\infty - E_\infty(\kappa)| \leqslant \|C_\kappa - C\|. \qquad (2.18)$$

In particular, if $\|C_\kappa - C\| \to 0$ as $\kappa \to \infty$, then

$$\lim_{\kappa\to\infty} E_\infty(\kappa) = E_\infty.$$

Proof: For $\Psi \in D(A \otimes P_0)$ with $\|\Psi\| = 1$, we have

$$|(\Psi, H_\infty(\kappa)\Psi) - (\Psi, H_\infty \Psi)| \leqslant \|C_\kappa - C\|,$$

which, combined with the variational principle, gives (2.18). ∎

An estimate of the ground state energy E_κ is given by the following theorem.

Theorem 2.4: Let C_κ and C be bounded. Suppose that $B \upharpoonright (\operatorname{Ker} B)^\perp \geqslant b$ with some constant $b > 0$. Then, for all $\kappa > (E_\infty(\kappa) + \|C_\kappa\|)/b$,

$$E_\infty(\kappa) - \|C_\kappa\| \eta_\kappa (1 + \sqrt{1 + \eta_\kappa^2})^{-1} \leqslant E_\kappa \leqslant E_\infty(\kappa), \qquad (2.19)$$

where

$$\eta_\kappa = 2\|C_\kappa\| / [b\kappa - E_\infty(\kappa) - \|C_\kappa\|].$$

In particular, if $\|C_\kappa - C\| \to 0$ as $\kappa \to \infty$, then,

$$\lim_{\kappa\to\infty} E_\kappa = E_\infty. \qquad (2.20)$$

Proof: We have

$$\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2,$$

where

$$\mathscr{X}_1 = \mathscr{H} \otimes \operatorname{Ker} B, \quad \mathscr{X}_2 = \mathscr{H} \otimes (\operatorname{Ker} B)^\perp.$$

Let P_j ($j = 1, 2$) be the orthogonal projection from $\mathscr{X}$ onto $\mathscr{X}_j$. It is easy to see that

$$P_1 = I \otimes P_0, \quad P_2 = I \otimes (I - P_0).$$

We can write

$$H_\kappa = H_\infty(\kappa) + \kappa I \otimes B (I - P_0) + P_1 C_\kappa P_2 + P_2 C_\kappa P_1 + P_2 C_\kappa P_2.$$

For all $\Psi \in \mathscr{X}_1 \cap D_{A,B}$ with $\|\Psi\| = 1$, we have

$$(\Psi, H_\kappa \Psi) = (\Psi, H_\infty(\kappa)\Psi).$$

Hence, it follows from the variational principle that

$$E_\kappa \leqslant (\Psi, H_\infty(\kappa)\Psi),$$

which implies the second inequality of (2.19). [Note that $D_{A,B}$ is a core of $H_\infty(\kappa)$.]

To prove the first inequality of (2.19), we write $\Psi \in D_{A,B}$ with $\|\Psi\| = 1$ as

$$\Psi = \Psi_1 + \Psi_2,$$

with $\Psi_j \in \mathscr{X}_j$ ($j = 1,2$). Then, using the Schwarz inequality and the fact that A is non-negative and $B \upharpoonright (\operatorname{Ker} B)^\perp \geqslant b > 0$, we have

$$(\Psi, H_\kappa \Psi) \geqslant E_\infty(\kappa)\|\Psi_1\|^2 + (b\kappa - \|C_\kappa\|)\|\Psi_2\|^2 - 2\|C_\kappa\|\,\|\Psi_1\|\,\|\Psi_2\|.$$

Since

$$\|\Psi_2\|^2 = 1 - \|\Psi_1\|^2,$$

it follows that

$$E_\kappa \geqslant E_\infty(\kappa) - \sup_{0<x<1} \lambda(x),$$

where

$$\lambda(x) = (b\kappa - E_\infty(\kappa) - \|C_\kappa\|)x^2 + 2\|C_\kappa\|x\sqrt{1-x^2} - (b\kappa - \|C_\kappa\| - E_\infty(\kappa)).$$

It is easy to show that the inequality

$$\alpha x^2 + \beta x\sqrt{1-x^2} - \alpha \leqslant \tfrac{1}{2}(-\alpha + \sqrt{\alpha^2+\beta^2}) = \frac{\beta}{2}\left(\frac{\beta}{\alpha}\right)\left(1 + \sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\right)^{-1}$$

holds for all $\alpha > 0$, $\beta \geqslant 0$ and $0 \leqslant x \leqslant 1$. Applying this inequality with $\alpha = b\kappa - E_\infty(\kappa) - \|C_\kappa\| > 0$ and $\beta = 2\|C_\kappa\|$, we obtain the first inequality of (2.19). Formula (2.20) follows from (2.19), Lemma 2.3, and the fact that $\eta_\kappa \to 0$ $(\kappa \to \infty)$. ■

Remark: As the above proof shows, the second inequality of (2.19) holds also for the case where C_κ and C are not bounded.

In order to write H_∞ in a more explicit way, we introduce a concept of "partial expectation" for linear operators. For $S \in \mathbf{B}(\mathscr{F})$ (the space of all bounded linear operators on $\mathscr{F}$) and $f,g \in \mathscr{K}$, we define the sesquilinear form $q_{f,g}(\cdot,\cdot)$ on $\mathscr{H} \times \mathscr{H}$ by

$$q_{f,g}(u,v) = (u \otimes f, S(v \otimes g))_{\mathscr{F}}, \quad u,v \in \mathscr{H},$$

which is bounded with

$$|q_{f,g}(u,v)| \leqslant \|S\|\,\|f\|\,\|g\|\,\|u\|\,\|v\|.$$

Therefore, by the Riesz lemma, there exists a unique $E_{f,g}(S) \in \mathbf{B}(\mathscr{H})$ such that

$$(u \otimes f, S(v \otimes g))_{\mathscr{F}} = (u, E_{f,g}(S)v)_{\mathscr{H}}$$

and

$$\|E_{f,g}(S)\| \leqslant \|f\|\,\|g\|\,\|S\|.$$

We also define $E_f(S) \in \mathbf{B}(\mathscr{H})$ by

$$E_f(S) = E_{f,f}(S).$$

We call the operator $E_{f,g}(S)$ [resp. $E_f(S)$] the *partial expectation* of S with respect to $\{f,g\}$ (resp. f). Note that, in the case $S = L \otimes M$ with $L \in \mathbf{B}(\mathscr{H})$ and $M \in \mathbf{B}(\mathscr{K})$, we have

$$E_{f,g}(L \otimes M) = (f, Mg)_{\mathscr{K}} L.$$

Some elementary facts of $E_{f,g}(S)$ are summarized in the following proposition, whose proof is left to the reader.

Proposition 2.5: (i) For all $f,g,h \in \mathscr{K}$, $\alpha,\beta \in \mathbb{C}$, and $S \in \mathbf{B}(\mathscr{F})$,

$$E_{h,\alpha f+\beta g}(S) = \alpha E_{h,f}(S) + \beta E_{h,g}(S).$$

(ii) For all $f,g \in \mathscr{K}$, $\alpha,\beta \in \mathbb{C}$, and $S,T \in \mathbf{B}(\mathscr{F})$,

$$E_{f,g}(\alpha S + \beta T) = \alpha E_{f,g}(S) + \beta E_{f,g}(T).$$

(iii) For all $f,g \in \mathscr{K}$ and $S \in \mathbf{B}(\mathscr{F})$,

$$E_{f,g}(S)^* = E_{g,f}(S^*).$$

The following continuity properties of the map $:S \to E_{f,g}(S)$ can also be easily proved.

Proposition 2.6: Let $S, S_n \in \mathbf{B}(\mathscr{F})$, $n \geqslant 1$, and $f,g \in \mathscr{K}$. Then:

(i) If $S_n \to S(n \to \infty)$ in operator norm, then $E_{f,g}(S_n) \to E_{f,g}(S)(n \to \infty)$ in operator norm.

(ii) If $S_n \to S(n \to \infty)$ strongly, then $E_{f,g}(S_n) \to E_{f,g}(S)(n \to \infty)$ strongly.

(iii) If $S_n \to S(n \to \infty)$ weakly, then $E_{f,g}(S_n) \to E_{f,g}(S)(n \to \infty)$ weakly.

Lemma 2.7: Let $P \in \mathbf{B}(\mathscr{K})$ be an orthogonal projection with dim Ran $P = n < \infty$. Let $\{f_j\}_{j=1}^n$ be an orthonormal basis of Ran P. Then, for all $S \in \mathbf{B}(\mathscr{F})$,

$$(I \otimes P)S(I \otimes P) = \sum_{j,k=1}^{n} E_{f_j,f_k}(S) \otimes P_{kj}, \tag{2.21}$$

where $P_{kj} \in \mathbf{B}(\mathscr{K})$ is defined by

$$P_{kj} f = (f_k, f)_{\mathscr{K}} f_j. \tag{2.22}$$

In particular, if $n = 1$ and Ran $P = \{\alpha f_0 | \alpha \in \mathbb{C}\}$ with $\|f_0\| = 1$, then

$$(I \otimes P)S(I \otimes P) = E_{f_0}(S) \otimes P. \tag{2.23}$$

Proof: Let $u,v \in \mathscr{H}$ and $f,g \in \mathscr{K}$. Then we have

$$\begin{aligned}(u \otimes f, (I \otimes P)S(I \otimes P)v \otimes g)_{\mathscr{F}} &= \sum_{j,k=1}^{n} (f,f_j)_{\mathscr{K}} (f_k,g)_{\mathscr{K}} (u \otimes f_j, S(v \otimes f_k))_{\mathscr{F}} \\ &= \sum_{j,k=1}^{n} (f, P_{kj} g)_{\mathscr{K}} (u, E_{f_j,f_k}(S)v)_{\mathscr{H}} \\ &= \left(u \otimes f, \left\{\sum_{j,k=1}^{n} E_{f_j,f_k}(S) \otimes P_{kj}\right\} v \otimes g\right)_{\mathscr{F}}.\end{aligned}$$

Thus (2.21) follows. ■

We next define the partial expectation for unbounded operators. For this purpose, we introduce a class of linear operators in $\mathscr{F}$.

Definition 2.8: We say that a densely defined linear operator S in $\mathscr{F}$ is in $\mathbf{E}(\mathscr{F})$ if and only if there exist subspaces $D_{\mathscr{H}}(S)$ and $D_{\mathscr{K}}(S)$ dense in $\mathscr{H}$ and $\mathscr{K}$, respectively, such that

$$D_{\mathscr{H}}(S) \hat{\otimes} D_{\mathscr{K}}(S) \subset D(S),$$

where $\hat{\otimes}$ denotes algebraic tensor product.

Let $S \in \mathbf{E}(\mathscr{F})$. Then, for all $f \in \mathscr{K}$, $g \in D_{\mathscr{K}}(S)$, and $v \in D_{\mathscr{H}}(S)$, the conjugate linear functional

$$L(u) = (u \otimes f, S(v \otimes g))_{\mathscr{F}}, \quad u \in \mathscr{H},$$

on $\mathscr{H}$ is bounded with

$$|L(u)| \leqslant \|f\|\,\|S(v \otimes g)\|\,\|u\|.$$

Therefore, by the Riesz lemma, there exists a unique vector $E_{f,g}(S)v \in \mathscr{H}$ such that

$$L(u) = (u, E_{f,g}(S)v)_{\mathscr{H}}$$

and

$$\|E_{f,g}(S)v\| \leqslant \|f\|\,\|S(v \otimes g)\|.$$

The map $:v \to E_{f,g}(S)v \in \mathscr{H}$ is linear. Hence, $E_{f,g}(S)$ gives a densely defined linear operator in $\mathscr{H}$ with $D(E_{f,g}(S)) = D_{\mathscr{H}}(S)$. We remark that $E_{f,g}(S)$ may depend on the choice of the pair of the subspaces $D_{\mathscr{H}}(S)$ and $D_{\mathscr{K}}(S)$. A criterion for the closability of $E_{f,g}(S)$ is given by the following proposition.

Proposition 2.9: Let $S \in \mathbf{E}(\mathscr{F})$. Suppose that S^* is in

$\mathbb{E}(\mathscr{K})$. Then, for all $f\in D_{\mathscr{K}}(S^*)$, $g\in D_{\mathscr{K}}(S)$, $E_{f,g}(S)$ is closable and

$$E_{g,f}(S^*)\subset E_{f,g}(S)^*.$$

Proof: It is straightforward to see that for all $u\in D_{\mathscr{K}}(S^*)$, $v\in D_{\mathscr{K}}(S)$, $f\in D_{\mathscr{K}}(S^*)$, and $g\in D_{\mathscr{K}}(S)$,

$$(u,E_{f,g}(S)v)_{\mathscr{K}}=(E_{g,f}(S^*)u,v)_{\mathscr{K}},$$

which implies the desired result. ■

Lemma 2.7 is translated into the present case as follows.

Lemma 2.10: Let P and $\{f_j\}_{j=1}^n$ be as in Lemma 2.7. Suppose that $S\in\mathbb{E}(\mathscr{K})$ with $\operatorname{Ran} P\subset D_{\mathscr{K}}(S)$. Then, the same conclusion as in Lemma 2.7 holds for S.

The above lemma and (2.10) immediately give the following result.

Proposition 2.11: Suppose that $\dim \operatorname{Ker} B=n<\infty$ and C is in $\mathbb{E}(\mathscr{K})$ with $D_{\mathscr{K}}(C)\supset \operatorname{Ker} B$. Let $\{f_j\}_{j=1}^n$ be an orthonormal basis of Ker B. Then,

$$H_\infty=A\otimes I+\sum_{j,k=1}^{n}E_{f_j f_k}(C)\otimes(P_0)_{kj}.$$

In particular, if $\operatorname{Ker} B=\{\alpha f_0|\alpha\in\mathbb{C}\}$ with $\|f_0\|=1$, then

$$H_\infty=H_{\rm eff}\otimes P_0+A\otimes(I-P_0),$$

where

$$H_{\rm eff}=A+E_{f_0}(C). \tag{2.24}$$

The following fact easily follows from Theorems 2.1, 2.2, and (2.24).

Theorem 2.12: Let C be as in Proposition 2.11 and $\operatorname{Ker} B=\{\alpha f_0|\alpha\in\mathbb{C}\}$ with $\|f_0\|=1$. Then: (i) Let $z\in\mathbb{C}$ be as in Theorem 2.1. Then

$$\operatorname*{s-lim}_{\kappa\to\infty}(H_\kappa-z)^{-1}=(H_{\rm eff}-z)^{-1}\otimes P_0.$$

(ii) For all $F\in C_\infty(\mathbb{R})$,

$$\operatorname*{s-lim}_{\kappa\to\infty}F(H_\kappa)=F(H_{\rm eff})\otimes P_0.$$

Under the assumption of Theorem 2.12, the self-adjoint operator $H_{\rm eff}$ may be regarded as an "effective" operator, in the asymptotic region $\kappa\gtrsim\infty$, of H_κ restricted to the subspace $\mathscr{K}\otimes\operatorname{Ker} B$.

We next consider the relation between the spectrum of H_κ and of $H_{\rm eff}$.

Theorem 2.13: Under the assumption of Theorem 2.12, we have:

(i) If $a,b\in\mathbb{R}$, $a<b$, and $(a,b)\cap\sigma(H_\kappa)=0$ for all large κ, then $(a,b)\cap\sigma(H_{\rm eff})=0$.

(ii) Let $\{E_\lambda(H_\kappa)\}$ and $\{E_\lambda(H_{\rm eff})\}$ be the spectral family of H_κ and of $H_{\rm eff}$, respectively. Let $a,b\in\mathbb{R}$, $a<b$, and $a,b\notin\sigma_{\rm pp}(H_{\rm eff})$, where $\sigma_{\rm pp}(H_{\rm eff})$ denotes the pure point spectrum of $H_{\rm eff}$. Then,

$$\operatorname*{s-lim}_{\kappa\to\infty}E_{(a,b)}(H_\kappa)=E_{(a,b)}(H_{\rm eff})\otimes P_0.$$

Proof: This follows from Theorem 2.12 and an application of Theorem A.2 in the Appendix. ■

Let A_1 be a symmetric operator in $\mathscr{K}$ such that $A+A_1$ has a discrete spectrum. We may write H_κ as

$$H_\kappa=\widetilde{H}_{0,\kappa}+H_I(\kappa),$$

where

$$\widetilde{H}_{0,\kappa}=(A+A_1)\otimes I+\kappa I\otimes B$$

and

$$H_I(\kappa)=C_\kappa-A_1\otimes I.$$

If the spectrum of B is of the form $[0,\infty)$, then all the eigenvalues of $\widetilde{H}_{0,\kappa}$ are embedded in the continuous spectrum of $\widetilde{H}_{0,\kappa}$ and hence H_κ gives an example for perturbation problem of embedded eigenvalues. In general, embedded eigenvalues may be unstable under perturbations, i.e., they may disappear under perturbations (e.g., Refs. 6, 9, and 22); H_κ may have no eigenvalues more than the ground state energy. On the other hand, the effective operator $H_{\rm eff}$ may be regarded as the unperturbed operator of H_κ in the sense of Theorem 2.12 and its eigenvalues may be discrete (see Secs. III and IV). It is well known that one of the concepts to handle such a situation in perturbation problems is *spectral concentration* (e.g., Chap. VIII, Sec. 5 in Ref. 20 and Sec. XII.5 in Ref. 22). We recall the following definition.

Definition: Let T_n be a family of self-adjoint operators and $E_\lambda(T_n)$ be the spectral family of T_n. Let $\{\Lambda_n\}_{n=1}^\infty$ and Λ be subsets of $\mathbb{R}$. We say that *the part of the spectrum of T_n in Λ is asymptotically concentrated on Λ_n as $n\to\infty$* if and only if

$$\operatorname*{s-lim}_{n\to\infty}E_{\Lambda\cap\Lambda_n^c}(T_n)=0,$$

where $\Lambda_n^c=\mathbb{R}-\Lambda_n$.

Theorem 2.14: Let C and Ker B be as in Theorem 2.12. Let $R>0$ and Λ be the union of a finite number of mutually disjoint, bounded open intervals of $\mathbb{R}$ such that $[-R,R]\cap\sigma(H_{\rm eff})\subset\Lambda$. Then, the part of the spectrum of H_κ in $[-R,R]$ is asymptotically concentrated on Λ as $\kappa\to\infty$.

Proof: We write

$$\Lambda=\cup_{j=1}^n(a_j,b_j).$$

It suffices to consider the case where $a_1<-R<b_1<a_2<b_2\cdots<a_n<R<b_n$. Then we have

$$\Lambda^c(R)\equiv[-R,R]\cap\Lambda^c=\cup_{j=1}^{n-1}[b_j,a_{j+1}].$$

For all $j=1,\dots,n-1$, the interval $[b_j,a_{j+1}]$ is included in the resolvent set $\rho(H_{\rm eff})$. Hence, for each $j-1,\dots,n-1$, there exist constants a_j' and b_j' such that $[b_j,a_{j+1}]\subset(b_j',a_{j+1}')\subset\rho(H_{\rm eff})$ and $b_j',a_{j+1}'\notin\sigma(H_{\rm eff})$. Hence, by Theorem 2.13(ii), we have

$$E_{(b_j',a_{j+1}')}(H_\kappa)\to E_{(b_j',a_{j+1}')}(H_{\rm eff})\otimes P_0=0,$$

strongly as $\kappa\to\infty$. Since

$$\Lambda^c(R)\subset\cup_{j=1}^{n-1}(b_j',a_{j+1}')$$

and hence

$$E_{\Lambda^c(R)}(H_\kappa)\leqslant\sum_{j=1}^{n-1}E_{(b_j',a_{j+1}')}(H_\kappa),$$

we obtain

$$E_{\Lambda^c(R)}(H_\kappa)\to 0,$$

strongly as $\kappa\to\infty$. Thus the desired result follows. ■

Remark: The above result is weaker than the standard result on spectral concentration (e.g., Chap. VIII, Sec. 5, Theorem 5.1 in Ref. 20). This is due to the fact that the strong resolvent convergence of H_κ is different from the usual strong resolvent convergence where the limiting operator is also a resolvent. Theorem 2.14 may be interpreted as a *local* spectral concentration of H_κ on the spectrum of $H_{\rm eff}$.

III. THE PAULI–FIERZ MODEL

In this section we apply the abstract theory in the last section to the Pauli–Fierz model to study its nonrelativistic limit.

A. Definition of the model and some fundamental facts

The model describes a quantum system of a one-electron atom coupled to a quantized radiation field[1-4] (cf. also Refs. 5–9). For a mathematical generality, we assume that the one-electron atom is placed in the d-dimensional space $\mathbb{R}^d$ ($d\geqslant 2$). We shall denote by $\hbar$ (resp. m, c) the Planck constant divided by 2π (resp. the electronic mass, the speed of light), regarding them as positive parameters. In what follows, the differential operators $\partial/\partial x_j$, $j=1,\ldots,d$, $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, are taken in the generalized sense. We set

$$p=\left(-i\hbar\frac{\partial}{\partial x_1},\ldots,-i\hbar\frac{\partial}{\partial x_d}\right). \tag{3.1}$$

We take the potential $V(x)$ of the atom to be a real-valued measurable function on $\mathbb{R}^d$ which satisfies:

(V-1) $D(p^2)\subset D(V)$ and for all $\lambda>0$, $V(p^2+\lambda)^{-1}$ is bounded with

$$\lim_{\lambda\to\infty}\|V(p^2+\lambda)^{-1}\|=0.$$

(V-2) For all $t>0$ and $x\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}e^{-t|x-y|^2}|V(y)|dy<\infty.$$

Condition (V-1) implies that V is infinitesimally small with respect to p^2 and hence the Hamiltonian of the atom

$$H_A=(1/2m)p^2+V \tag{3.2}$$

is self-adjoint on $D(p^2)$ and bounded from below.

Remark: If V is a Phillips perturbation of p^2, then V satisfies (V-1) (see Refs. 23 and 24). It was proved in Ref. 23 that if

$$V\in L^q(\mathbb{R}^d)+L^\infty(\mathbb{R}^d),$$

with $q>d/2$ and $q\geqslant 2$, then V is a Phillips perturbation of p^2. In particular, it follows that the Coulomb potential in the case $d=3$ satisfies (V-1) and (V-2).

We use the Coulomb gauge in quantizing the radiation field. The Hilbert space of state vectors for the quantized radiation field is then defined by the boson Fock space:

$$\mathscr{F}_{\rm EM}=\oplus_{n=0}^\infty\otimes_s^n W \tag{3.3}$$

over the Hilbert space

$$W=\underbrace{L^2(\mathbb{R}^d)\oplus\cdots\oplus L^2(\mathbb{R}^d)}_{d-1\ \text{times}}, \tag{3.4}$$

where $\otimes_s^n W$ denotes the n-fold symmetric tensor product of W with convention $\otimes_s^{n=0}W=\mathbb{C}$. We denote by $a(F)$, $F\in W$, the annihilation operator in $\mathscr{F}_{\rm EM}$. For $r=1,\ldots,d-1$ and $f\in L^2(\mathbb{R}^d)$, we define $f_r\in W$ by $f_r=(0,\ldots,f,\ldots,0)$ (the rth component is equal to f and the other components are zero). The map $:f\to a(f_r)$ defines an operator-valued distribution on $\mathbb{R}^d$. We denote the distribution kernel by $a_r(k)$, $r=1,\ldots,d-1$, $k\in\mathbb{R}^d$. Then the following canonical commutation relations hold in the sense of operator-valued distribution:

$$[a_r(k),a_q(k')]=[a_r(k)^*,a_q(k')^*]=0,$$
$$[a_r(k),a_q(k')^*]=\delta_{rq}\delta(k-k'),\quad r,q=1,\ldots,d-1.$$

Let $e_r(k)$ be an $\mathbb{R}^d$-valued measurable function on $\mathbb{R}^d$ such that

$$k\cdot e_r(k)=0,\quad e_r(k)\cdot e_q(k)=\delta_{rq},$$
$$a.e.\,k\in\mathbb{R}^d,\quad r,q=1,\ldots,d-1.$$

The vectors $e_r(k)$, $r=1,\ldots,d-1$, serve as polarization vectors of "photon."

The free Hamiltonian of the quantized radiation field is defined by

$$H_F=\hbar c\sum_{r=1}^{d-1}\int dk\,\omega(k)a_r(k)^*a_r(k). \tag{3.5}$$

Here, $\omega(k)$ is a non-negative measurable function on $\mathbb{R}^d$ with $\omega\in L^2_{\rm loc}(\mathbb{R}^d)$ which depends only on $|k|$. The physical choice for $\omega(k)$ is given by $\omega(k)=|k|$.

The Hilbert space $\mathscr{F}$ of state vectors for the interacting system of the atom and the radiation field is taken to be the tensor product of $L^2(\mathbb{R}^d)$ and $\mathscr{F}_{\rm EM}$:

$$\mathscr{F}=L^2(\mathbb{R}^d)\otimes\mathscr{F}_{\rm EM}. \tag{3.6}$$

To define the interaction between the atom and the radiation field as an operator in $\mathscr{F}$, we have to introduce a cutoff for photon momenta: Let $\rho(x)$ be a real distribution on $\mathbb{R}^d$ such that its Fourier transform

$$\hat\rho(k)=\frac{1}{(2\pi)^{d/2}}\int dx\,\rho(x)e^{-ikx} \tag{3.7}$$

is a measurable function and depends only on $|k|$ with

$$\int dk\frac{|\hat\rho(k)|^2}{\omega(k)^3}<\infty,\quad\int dk\frac{|\hat\rho(k)|^2}{\omega(k)}<\infty. \tag{3.8}$$

Then we define the time-zero radiation field with cutoff ρ by

$$A(x;\rho)=\sum_{r=1}^{d-1}\int dk\frac{\sqrt{\hbar c}}{\sqrt{2\omega(k)}}e_r(k)\times\{\hat\rho(k)^*a_r(k)^*e^{-ikx}+\hat\rho(k)a_r(k)e^{ikx}\}. \tag{3.9}$$

The total Hamiltonian of the coupled system of the atom and the radiation field with the full minimal interaction reads:

$$H=(1/2m)(p-(e/c)A(x;\rho))^2+H_F+V, \tag{3.10}$$

where $e\in\mathbb{R}\setminus\{0\}$ is a coupling parameter denoting the elementary charge. In the present paper, however, we take as the total Hamiltonian of the coupled system a version of H simplified in the following way: (i) We use the dipole approximation, i.e., we replace $A(x;\rho)$ by $A(0;\rho)$; (ii) We neglect the term $A(x;\rho)^2$.

Further, we take the mass renormalization of the electron into account, i.e., we introduce the "bare mass" m_0 of the electron by

$$\frac{1}{m_0}=\frac{1}{m}+\frac{(d-1)}{d}\left(\frac{e}{mc}\right)^2\int dk\,\frac{|\hat\rho(k)|^2}{\omega(k)^2}, \qquad (3.11)$$

and define the "renormalized" atom Hamiltonian $H_A^{\rm ren}$ by

$$H_A^{\rm ren}=(1/2m_0)p^2+V. \qquad (3.12)$$

Thus the total Hamiltonian of our model is defined by

$$H(c,e)=H_A^{\rm ren}\otimes I+I\otimes H_F+H_I, \qquad (3.13)$$

where

$$H_I=-(e/mc)p\otimes A(0;\rho).$$

For $\kappa>0$, we introduce

$$c(\kappa)=\kappa c,\quad e(\kappa)=\kappa^{3/2}e,$$

which are regarded as a scaled speed of light and a scaled elementary charge, respectively. Then we define the scaled Hamiltonian $H(\kappa)$ by

$$H(\kappa)\equiv H(c(\kappa),e(\kappa))$$
$$=\left(\frac{1}{2m(\kappa)}p^2+V\right)\otimes I+\kappa I\otimes H_F+\kappa H_I, \qquad (3.14)$$

where $m(\kappa)$ is defined by

$$\frac{1}{m(\kappa)}=\frac{1}{m}+\kappa\frac{(d-1)}{d}\left(\frac{e}{mc}\right)^2\int dk\,\frac{|\hat\rho(k)|^2}{\omega(k)^2}.$$

We want to consider the scaling limit $\kappa\to\infty$ of the model in terms of $H(\kappa)$. Obviously $c(\kappa),|e(\kappa)|\to\infty$ as $\kappa\to\infty$. In this sense, the scaling limit $\kappa\to\infty$ in $H(\kappa)$ corresponds to the nonrelativistic limit at the same time as the magnitude of the coupling charge becomes infinite. Note also that the "scaled bare mass" $m(\kappa)\to 0$ as $\kappa\to\infty$.

Before stating the main results on the scaling limit, we give some known facts. We denote by Ω the Fock vacuum in $\mathscr{F}_{\rm EM}$:

$$\Omega=\{1,0,0,...\}. \qquad (3.15)$$

Let $\mathscr{F}_{\rm EM,0}$ be the dense subspace in $\mathscr{F}_{\rm EM}$ spanned by vectors of the form

$$a(F_1)^*\cdots a(F_n)^*\Omega,\ \Omega,\ \ F_j\in W,\ j=1,...,n,\ n\geqslant 1$$

and

$$\mathscr{S}_0(\mathbb{R}^d)=\{f\in\mathscr{S}(\mathbb{R}^d)|\hat f\in C_0^\infty(\mathbb{R}^d)\}, \qquad (3.16)$$

where $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz test function space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ and $C_0^\infty(\mathbb{R}^d)$ denotes the space of C^∞-functions on $\mathbb{R}^d$ with compact support. Then the subspace

$$\mathscr{F}_0=\mathscr{S}_0(\mathbb{R}^d)\,\hat\otimes\,\mathscr{F}_{\rm EM,0} \qquad (3.17)$$

is dense in $\mathscr{F}$.

Let

$$T=i\frac{e}{mc}\sum_{r=1}^{d-1}\int dk\,\frac{1}{\omega(k)\sqrt{2\hbar c\omega(k)}}p$$
$$\cdot e_r(k)\{\hat\rho(k)^*a_r(k)^*-\hat\rho(k)a_r(k)\}. \qquad (3.18)$$

Then we can show that $\mathscr{F}_0$ is a set of analytic vectors of T and hence T is essentially self-adjoint on it.[5] We denote the unique self-adjoint extension of $T\upharpoonright\mathscr{F}_0$ by the same symbol.

Let

$$C(V)=e^{iT}(V\otimes I)e^{-iT}. \qquad (3.19)$$

Since $\exp(-tp^2\otimes I)$ commutes with $\exp(i\lambda T)(\lambda\in\mathbb{R})$ and $\exp(i\lambda T)$ is unitary, it follows from (V-1) that $C(V)$ is infinitesimally small with respect to $p^2\otimes I$ with

$$\lim_{\lambda\to\infty}\|C(V)(p^2\otimes I+\lambda)^{-1}\|=0, \qquad (3.20)$$

which implies that $C(V)$ is infinitesimally small with respect to $(p^2\otimes I)/2m+\kappa I\otimes H_F$ with

$$\lim_{\lambda\to\infty}\|C(V)[(1/2m)p^2\otimes I+\kappa I\otimes H_F+\lambda]^{-1}\|=0 \qquad (3.21)$$

uniformly in κ. Therefore, the operator

$$\widetilde H(\kappa)=(1/2m)p^2\otimes I+\kappa I\otimes H_F+C(V) \qquad (3.22)$$

is self-adjoint on

$$D_0=D(p^2\otimes I)\cap D(I\otimes H_F) \qquad (3.23)$$

and bounded from below. We have

$$\widetilde H(\kappa)\geqslant\inf\sigma(H_A). \qquad (3.24)$$

This follows from the non-negativity of H_F and the fact that $p^2\otimes I$ commutes with $\exp(\pm iT)$.

A fundamental fact concerning our model $H(\kappa)$ is the following lemma.

Lemma 3.1: The unitary operator e^{iT} maps D_0 onto D_0 and for all $\kappa>0$,

$$e^{iT}H(\kappa)e^{-iT}=\widetilde H(\kappa)$$

on D_0. In particular, $H(\kappa)$ is self-adjoint on D_0 and bounded from below with

$$H(\kappa)\geqslant\inf\sigma(H_A).$$

Proof: See Ref. 5 (cf. also Ref. 4). ■

Proposition 3.2: Let

$$E(\kappa)=\inf\sigma(H(\kappa)). \qquad (3.25)$$

Then, $E(\kappa)$ is nondecreasing in κ.

Proof: Since H_F is non-negative, we have from (3.22)

$$\widetilde H(\kappa)\geqslant\widetilde H(\kappa'),$$

for all $\kappa>\kappa'>0$. By Lemma 3.1, we have

$$E(\kappa)=\inf\sigma(\widetilde H(\kappa)).$$

Hence, $E(\kappa)\geqslant E(\kappa')$, for $\kappa>\kappa'>0$. ■

B. Convergence of the Hamiltonian, effective potential, and an estimate of the ground state energy

Lemma 3.3: The operator $C(V)$ is in $\mathbb{E}(\mathscr{F})$ (see Definition 2.8) with $D_{L^2(\mathbb{R}^d)}(C(V))=D(p^2)$ and $D_{\mathscr{F}_{\rm EM}}(C(V))=D(H_F)$. Further, the partial expectation $E_\Omega(C(V))$ of $C(V)$ with respect to Ω is given by

$$E_\Omega(C(V))=V_{\rm eff}\ \text{on}\ D(p^2), \qquad (3.26)$$

where $V_{\rm eff}$ is the multiplication operator associaed with the function

$$V_{\rm eff}(x)=(2\pi C(\rho))^{-d/2}\int dy\,e^{-|x-y|^2/2C(\rho)}V(y), \qquad (3.27)$$

with

$$C(\rho)=\frac{(d-1)}{2d}\left(\frac{\hbar}{mc}\right)^2\frac{e^2}{\hbar c}\int dk\,\frac{|\hat\rho(k)|^2}{\omega(k)^3}.$$

Proof: By Lemma 3.1 and condition (V-1), $D(p^2)\hat\otimes D(H_F)\subset D_0\subset D(C(V))$. Hence, $C(V)$ is in $\mathbb{E}(\mathscr{F})$. To prove (3.26), we first consider the case where $V\in\mathscr{S}(\mathbb{R}^d)$. Then it follows that for all $f,g\in L^2(\mathbb{R}^d)$,

$$(f,E_\Omega(C(V))g)_{L^2}=\frac{1}{(2\pi)^{d/2}}\int d\xi\,\hat V(\xi)(f\otimes\Omega,e^{i\xi x(T)}g\otimes\Omega)_{\mathscr{F}},$$

where

$$x(T)=e^{iT}x\otimes Ie^{-iT}. \tag{3.28}$$

Let

$$X=i\frac{e}{mc}\sqrt{\frac{\hbar}{2c}}\times\sum_{r=1}^{d-1}\int\frac{e_r(k)}{\omega(k)^{3/2}}\{\hat\rho(k)^*a_r(k)^*-\hat\rho(k)a_r(k)\}.$$

Then, it is not so difficult to see that

$$e^{iT}D(x_\mu\otimes I)=D(x_\mu\otimes I+I\otimes X_\mu),$$

$$x(T)_\mu=x_\mu\otimes I+I\otimes X_\mu,\quad \mu=1,...,d.$$

Hence, we have

$$(f\otimes\Omega,e^{i\xi x(T)}g\otimes\Omega)_{\mathscr{F}}=(f,e^{i\xi x}g)_{L^2}(\Omega,e^{i\xi X}\Omega)_{\mathscr{F}_{\rm EM}}.$$

By the standard Fock space calculus, we find

$$(\Omega,e^{i\xi X}\Omega)_{\mathscr{F}_{\rm EM}}=e^{-|\xi|^2C(\rho)/2}.$$

Thus (3.26) follows.

We next consider the case where V is bounded, but, not in $\mathscr{S}(\mathbb{R}^d)$. In this case, we approximate V by a sequence $\{V_n\}_n\subset\mathscr{S}(\mathbb{R}^d)$ in the sense of strong convergence in $L^2(\mathbb{R}^d)$. Then, by Proposition 2.6 (ii), we have

$$E_\Omega(C(V_n))\to E_\Omega(C(V))(n\to\infty)$$

strongly. On the other hand, we have

$$(V_n)_{\rm eff}(x)\to V_{\rm eff}(x)(n\to\infty),$$

for all $x\in\mathbb{R}^d$. Thus we obtain (3.26).

Finally, let V satisfy (V-1) and (V-2). Denoting by χ_n the characteristic function of $[0,n]$, $n\in\mathbb{N}$, we define

$$V_n(x)=\chi_n(|x|)V(x).$$

Then V_n is bounded and hence (3.26) holds with V replaced by V_n. It is easy to see that for all $\Psi\in D_0$,

$$C(V_n)\Psi\to C(V)\Psi$$

strongly. Hence, for all $f\in C_0^\infty(\mathbb{R}^d)$ and g in $D(p^2)$,

$$(f,E_\Omega(C(V_n))g)\to(f,E_\Omega(C(V))g). \tag{3.29}$$

It follows from condition (V-2) that $|V|_{\rm eff}$ is a continuous function on $\mathbb{R}^d$. Hence, by using the dominated convergence theorem, we have

$$(f,(V_n)_{\rm eff}g)\to(f,V_{\rm eff}g),$$

which, combined with (3.29), gives

$$(f,E_\Omega(C(V))g)=(f,V_{\rm eff}g).$$

Thus (3.26) follows. ■

By (3.26) and the fact that $C(V)$ is infinitesimally small with respect to $p^2\otimes I$, it follows that $V_{\rm eff}$ is infinitesimally small with respect to $p^2/2m$. Hence,

$$H_{A,\rm eff}=(1/2m)p^2+V_{\rm eff} \tag{3.30}$$

is self-adjoint on $D(p^2)$ and bounded from below. Let

$$E_{A,\rm eff}=\inf\sigma(H_{A,\rm eff}). \tag{3.31}$$

We denote by P_0 the orthogonal projection from $\mathscr{F}_{\rm EM}$ onto the Fock vacuum sector $\{\alpha\Omega|\alpha\in\mathbb{C}\}$.

Theorem 3.4: For all $z\in\mathbb{C}$ with $\operatorname{Im}z\neq0$ or $z\in\mathbb{R}$ with $z<\min\{\inf\sigma(H_A),\inf\sigma(H_{A,\rm eff})\}$,

$$\text{s-}\lim_{\kappa\to\infty}(H(\kappa)-z)^{-1}=e^{-iT}\{(H_{A,\rm eff}-z)^{-1}\otimes P_0\}e^{iT}. \tag{3.32}$$

Proof: By Lemma 3.1, we need only to consider the scaling limit $\kappa\to\infty$ of $(\tilde H(\kappa)-z)^{-1}$. Note that $\tilde H(\kappa)$ is just of the form of the operator H_κ considered in Theorem 2.1 with the following identifications:

$$\mathscr{H}=L^2(\mathbb{R}^d),\quad \mathscr{K}=\mathscr{F}_{\rm EM},$$

$$A=p^2/2m,\quad B=H_F,\quad C_\kappa=C=C(V). \tag{3.33}$$

We have $\operatorname{Ker}H_F=\{\alpha\Omega|\alpha\in\mathbb{C}\}$. Hence we obtain from Theorem 2.12(i)

$$\text{s-}\lim_{\kappa\to\infty}(\tilde H(\kappa)-z)^{-1}=(p^2/2m+E_\Omega(C(V))-z)^{-1}\otimes P_0,$$

which, together with (3.26), gives (3.32). ■

Remarks: (i) In the case $d=3$, $V_{\rm eff}$ coincides with the effective potential given by Welton,[3] who derived it by physical arguments to calculate some observable effects of the quantized radiation field such as the Lamb shift. Theorem 3.4 shows that the effective potential can be derived as a scaling limit of the total Hamiltonian $H(\kappa)$. This does not only justify rigorously the effective potential but also clarifies a mathematical meaning of it.

(ii) As (3.27) shows, the effective potential $V_{\rm eff}$ is a Gaussian transformation of the original potential V. The functional $C(\rho)$ of ρ, which characterizes the Gaussian transformation, has a mathematical meaning: let $x(T)$ be defined by (3.28). Then it follows from the proof of Lemma 3.3 that

$$E_\Omega[(x(T)-x\otimes I)^2]=C(\rho)I. \tag{3.34}$$

Hence, $C(\rho)$ can be identified with the partial expectation of the square of $\Delta x\equiv x(T)-x\otimes I$ with respect to the Fock vacuum Ω. In Ref. 3, Δx was regarded as a fluctuation in position of a free electron and $C(\rho)$ was interpreted as the mean-square fluctuation in position of a free electron. In this sense, (3.34) suggests that mean fluctuations of observables under the action of a quantized radiation field may be formulated in terms of the notion of partial expectation.

Theorem 3.5: Suppose that V is bounded and

$$\inf_{k\in\mathbb{R}^d}\omega(k)\geqslant\omega_0, \tag{3.35}$$

with a constant $\omega_0>0$. Then, for all $\kappa>(E_{A,\rm eff}+\|V\|)/\hbar c\omega_0$,

$$E_{A,\rm eff}-\|V\|\nu_\kappa(1+\sqrt{1+\nu_\kappa^2})^{-1}\leqslant E(\kappa)\leqslant E_{A,\rm eff},$$

where

$$\nu_\kappa = 2\|V\|/(\hbar c\omega_0\kappa - E_{A,\text{eff}} - \|V\|).$$

In particular,

$$\lim_{\kappa\to\infty} E(\kappa) = E_{A,\text{eff}}.$$

Proof: Since V is bounded in the present case, we have $\|C(V)\| = \|V\|$.

Under condition (3.35), $H_F \upharpoonright (\text{Ker}\, H_F)^\perp \geqslant \hbar c\omega_0$. Hence, an application of Theorem 2.4 to the present case with the identifications (3.33) and $b = \hbar c\omega_0$ yields the desired result. ■

C. Spectral concentration

In the physical case $\omega(k) = |k|$, all the eigenvalues of the unperturbed Hamiltonian $H_A \otimes I + \kappa I \otimes H_F$ are embedded in its continuous spectrum. Hence, as already remarked in Sec. II, they may be unstable under the perturbation $\kappa H_I +$ (the self-energy term), i.e., they may disappear under the perturbation (for an example, see Ref. 6). On the other hand, eigenvalues of $H_{A,\text{eff}}$ may be discrete. The concept of spectral concentration may be helpful to handle such a situation in perturbation problems. We have the following result on the spectral concentration of $H(\kappa)$.

Theorem 3.6: Let $R > 0$ and Λ be the union of a finite number of mutually disjoint, bounded open intervals of $\mathbb{R}$ such that $[-R,R] \cap \sigma(H_{A,\text{eff}}) \subset \Lambda$. Then, the part of the spectrum of $H(\kappa)$ in $[-R,R]$ is asymptotically concentrated on Λ as $\kappa\to\infty$.

Proof: Let $E_\lambda(H(\kappa))$ be the spectral family of $H(\kappa)$. Then, by Lemma 3.1,

$$\exp(iT)E_\lambda(H(\kappa))\exp(-iT) \equiv E_\lambda(\widetilde{H}(\kappa))$$

is the spectral family of $\widetilde{H}(\kappa)$ By Theorems 3.4 and 2.14, we have

$$E_{\Lambda^c(R)}(\widetilde{H}(\kappa)) \to 0$$

strongly as $\kappa\to\infty$. Hence,

$$E_{\Lambda^c(R)}(H(\kappa)) \to 0$$

strongly as $\kappa\to\infty$. ■

IV. THE SPIN-BOSON MODEL

The spin-boson model we are going to discuss describes a two-level atom coupled to a quantized Bose field (a simplified version of a quantized radiation field) (e.g., Ref. 13 for a review and Refs. 10–12, 14, and 15 for some rigorous results). We denote by $\mu > 0$ the half of the gap of the two energy levels of the unperturbed atom. The total Hamiltonian of the model is given as follows:

$$\begin{aligned} H &= H(c,\lambda) \\ &= I\otimes H_b + \mu\sigma_1 \otimes I + \sigma_3 \\ &\quad \otimes \int dk\{\lambda(k)^* a(k)^* + \lambda(k)a(k)\} - E_0(c,\lambda). \end{aligned} \tag{4.1}$$

Here, the Hilbert space in which H acts is

$$\mathscr{F} = \mathbb{C}^2 \otimes \mathscr{F}_B(L^2(\mathbb{R}^d)) = \mathscr{F}_B(L^2(\mathbb{R}^d)) \oplus \mathscr{F}_B(L^2(\mathbb{R}^d)), \tag{4.2}$$

where $\mathscr{F}_B(L^2(\mathbb{R}^d))$ denotes the boson Fock space over $L^2(\mathbb{R}^d)$ $(d\geqslant 1)$. The 2×2 matrices σ_1 and σ_3 are the standard Pauli matrices and $a(k)$ is the operator-valued distribution kernel of the boson annihilation operator acting in $\mathscr{F}_B(L^2(\mathbb{R}^d))$. The operator H_b is the free boson Hamiltonian:

$$H_b = c\int dk\, \omega(k)a(k)^* a(k), \tag{4.3}$$

where $\omega(k)$ is a non-negative measurable function on $\mathbb{R}^d$ with $\omega \in L^2_{\text{loc}}(\mathbb{R}^d)$. We assume that $\lambda(k)$ is a measurable function on $\mathbb{R}^d$ satisfying the following conditions:

$$\int dk\, |\lambda(k)|^2 < \infty, \quad \int dk \frac{|\lambda(k)|^2}{\omega(k)^2} < \infty. \tag{4.4}$$

The functional $E_0(c,\lambda): \lambda \to \mathbb{C}$ is defined by

$$E_0(c,\lambda) = -\frac{1}{c}\int dk \frac{|\lambda(k)|^2}{\omega(k)}, \tag{4.5}$$

which physically means the ground state energy of the bosonic Hamiltonian

$$H_b(c,\lambda) = I\otimes H_b + \sigma_3 \otimes \int dk\{\lambda(k)^* a(k)^* + \lambda(k)a(k)\}.$$

We have set $\hbar = 1$.

It is known (or easy to see by applying the Kato–Rellich theorem) that H is self-adjoint with $D(H) = D(I\otimes H_b)$ and bounded from below.

For $\kappa > 0$, we set

$$H(s) = H(\kappa c, \kappa\lambda), \tag{4.6}$$

which is the scaled Hamiltonian we are going to study in the asymptotic region $\kappa \approx \infty$. The operators

$$T_\pm = \pm i\frac{1}{c}\int dk \frac{1}{\omega(k)}\{\lambda(k)^* a(k)^* - \lambda(k)a(k)\} \tag{4.7}$$

are self-adjoint in $\mathscr{F}_B$. Hence, we can define the unitary operators

$$U_\pm = e^{iT_\pm} \tag{4.8}$$

on $\mathscr{F}_B$. Set

$$U = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \tag{4.9}$$

which is unitary on $\mathscr{F}$. The following fact is easily proved.

Lemma 4.1: For all $\kappa > 0$,

$$\widetilde{H}(\kappa) \equiv U^{-1}H(\kappa)U = \kappa I\otimes H_b + W, \tag{4.10}$$

where

$$W = \mu\begin{pmatrix} 0 & U_-^2 \\ U_+^2 & 0 \end{pmatrix}. \tag{4.11}$$

Let

$$E(\kappa) = \inf \sigma(H(\kappa)). \tag{4.12}$$

Proposition 4.2: The ground state energy $E(\kappa)$ is nondecreasing in $\kappa > 0$ and

$$\inf_{\kappa>0} E(\kappa) \geqslant -\mu. \tag{4.13}$$

Proof: By (4.10), we have

$$\widetilde{H}(\kappa) \geqslant \widetilde{H}(\kappa'),$$

for all $\kappa > \kappa' > 0$. Since we have

$$E(\kappa) = \inf \sigma(\widetilde{H}(\kappa)),$$

the nondecreasingness of $E(\kappa)$ follows. Note that $W \geqslant -\mu$, which, combined with (4.10) and the non-negativity of H_b, gives (4.13). ■

Lemma 4.1 shows that $H(\kappa)$ is unitarily equivalent to an operator of the form of H_κ discussed in Sec. II. Hence, we can apply the main theorems in Sec. II to the present case. Let us compute the partial expectation of W with respect to the vectors in Ker H_b first. We have Ker $H_b = \{\alpha\Omega | \alpha \in \mathbb{C}\}$, where Ω is the Fock vacuum in $\mathscr{F}_B(L^2(\mathbb{R}^d))$.

Lemma 4.3: Let

$$F(c,\lambda) = \exp\left(-\frac{2}{c^2}\int dk \frac{|\lambda(k)|^2}{\omega(k)^2}\right). \tag{4.14}$$

Then,

$$E_\Omega(W) = \mu F(c,\lambda)\sigma_1. \tag{4.15}$$

Proof: By computing the inner product $(u \otimes \Omega, W(v \otimes \Omega))$ for

$$u = (z_1, z_2), v = (w_1, w_2) \in \mathbb{C}^2,$$

we see that

$$E_\Omega(W) = \mu\begin{pmatrix} 0 & (\Omega, U_-^2 \Omega) \\ (\Omega, U_+^2 \Omega) & 0 \end{pmatrix}.$$

It is straightforward to show that

$$(\Omega, U_\pm^2 \Omega) = F(c,\lambda).$$

Thus (4.15) follows. ■

Theorem 4.4: For all $z \in \mathbb{C} \setminus [-\mu, \infty)$,

$$\operatorname*{s-lim}_{\kappa\to\infty} (H(\kappa) - z)^{-1} = U(\mu F(c,\lambda)\sigma_1 - z)^{-1} \otimes P_0 U^{-1}. \tag{4.16}$$

Proof: We need only to use (4.10) and apply Theorem 2.12 with the following identifications:

$$\mathscr{H} = \mathbb{C}^2, \ \mathscr{K} = \mathscr{F}_B(L^2(\mathbb{R}^d)),$$

$$A = 0, \quad B = H_b, \quad C_\kappa = C = W. \tag{4.17}$$

[Note that $\mu F(c,\lambda)\sigma_1 \geqslant -\mu F(c,\lambda) \geqslant -\mu$.] ■

The estimate (4.13) of the ground state energy $E(\kappa)$ is improved as follows.

Theorem 4.5: Suppose that $\omega(k)$ satisfies (3.35). Then, for all $\kappa > \mu(1 - F(c,\lambda))/c\omega_0$, we have

$$-\mu F(c,\lambda) - \mu d_\kappa (1 + \sqrt{1 + d_\kappa^2})^{-1} \leqslant E(\kappa) \leqslant -\mu F(c,\lambda), \tag{4.18}$$

where

$$d_\kappa = 2\mu/[c\omega_0\kappa - \mu(1 - F(c,\lambda))].$$

Proof: Under condition (3.35), $H_b \restriction (\text{Ker } H_b)^\perp \geqslant c\omega_0$. Further, we have

$$\|W\| = \mu, \quad \inf \sigma(\mu F(c,\lambda)\sigma_1) = -\mu F(c,\lambda).$$

Hence, applying Theorem 2.4 with the identifications (4.17), we obtain (4.18). ■

Remarks: (i) The ground state of the model $H(\kappa)$ with $\mu = 0$ is twofold degenerate. We note that $F(c,\lambda)^2$ is equal to the transition probability between the two ground states at zero temperature and in the case $\kappa = 1$ (cf. Ref. 11).

(ii) Inequality (4.18) gives a nonperturbative estimate of the ground state energy of $H(\kappa)$ with respect to both parameters μ and λ and slightly improves the estimate given by Davies.[10]

As for the spectral concentration of $H(\kappa)$, we have the following result.

Theorem 4.6: Let $R > 0$ and $\epsilon > 0$. Then, the part of the spectrum of $H(\kappa)$ in $[-R,R]$ is asymptotically concentrated on

$$(-\mu F(c,\lambda) - \epsilon, -\mu F(c,\lambda) + \epsilon)$$
$$\cup(\mu F(c,\lambda) - \epsilon, \mu F(c,\lambda) + \epsilon)$$

as $\kappa \to \infty$.

Proof: We first note that the spectrum of the effective operator $\mu F(c,\lambda)\sigma_1$ is equal to $\{\pm\mu F(c,\lambda)\}$. Then, in the same way as in the proof of Theorem 3.6, we obtain the desired result. ■

V. CONCLUDING REMARKS

In this paper we have developed an abstract asymptotic theory of a family of self-adjoint operators, which allows us to study in a unified way the nonrelativistic limit of the Pauli–Fierz and a spin-boson model. We have obtained some new rigorous results for the models, including an asymptotic estimate of their ground state energy and the existence of "local" spectral concentration.

Our method can be applied to other quantum field models whose Hamiltonians can be transformed by unitary transformations ("dressing transformations") to operators of the form of H_κ discussed in Sec. II.

In the present paper we have considered only the case where the quantum system under consideration is at zero temperature. In the case of finite temperature, we have to reformulate the asymptotic theory in Sec. II in terms of correlation functions of a KMS state associated with H_κ.

The following topics may be worth being studied as a continuation of the present work.

(i) Extension of the abstract theory in Sec. II to the case where the operator C_κ is more singular and/or the scaling order of C_κ in κ is different.

(ii) The nonrelativistic limit of the Pauli–Fierz model (3.13) without the dipole approximation. In this case, a formal perturbation calculation suggests that we should have an effective potential different from $V_{\rm eff}$ given by (3.27) (cf. Ref. 3).

(iii) The nonrelativistic limit of the model whose Hamiltonian is given by (3.10). In the case of the dipole approximation, we may use the results in Ref. 7.

APPENDIX: SOME LIMIT THEOREMS

In Sec. II we encounter a strong convergence of resolvent that is different from the usual strong resolvent convergence in that the limiting operator is not the resolvent of an operator. In this Appendix we present some limit theorems related to such a strong resolvent convergence, which are variants of the standard limit theorems of the usual strong resolvent convergence (e.g., Refs. 20 and 21). We first consider a general case.

Theorem A.1: Let T_n, $n\in\mathbb{N}$, and T be self-adjoint operators in a Hilbert space $\mathscr{H}$, and Q be an orthogonal projection on $\mathscr{H}$. Suppose that Q commutes with the resolvent of T and for all $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\text{s-}\lim_{s\to\infty}(T_n - z)^{-1} = (T-z)^{-1}Q. \tag{A1}$$

Then, for all $F\in C_\infty(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$ vanishing at ∞), $F(T)$ commutes with Q and

$$\text{s-}\lim_{n\to\infty} F(T_n) = F(T)Q. \tag{A2}$$

Further, if T_n is bounded from below uniformly in n, then T is bounded from below and

$$\overline{\lim_{n\to\infty}}\, E_n \leqslant E, \tag{A3}$$

where

$$E_n = \inf\sigma(T_n), \quad E = \inf\sigma(T\restriction \text{Ran } Q).$$

Remark: The commutativity of the resolvent of T with Q implies that T is reduced by Ran Q, so that $T\restriction$ Ran Q is also self-adjoint.

Proof: The proof of (A2) can be done in the same way as in the proof of Theorem VIII.20(a) in Ref. 21. The point that we are careful about in the present case is that the limiting operator $(T-z)^{-1}Q$ is not a resolvent. We can show that for all polynomials $P = P(x,y)$ in two variables x, y:

$$P((T_n+i)^{-1},(T_n-i)^{-1}) \to P((T+i)^{-1},(T-i)^{-1})Q$$

strongly as $n\to\infty$, where we have used the assumption that Q is an orthogonal projection commuting with the resolvent of T. We then see that (A2) follows from a slight modification of the proof of Theorem VIII.20(a) in Ref. 21.

If T_n is bounded from below uniformly in n, then a standard method shows that T is bounded from below and (A1) holds for $z<z_0\equiv\min\{\inf_n E_n, E\}$. We fix a real number $\lambda<z_0$. Then, a standard formula on the strong convergence of bounded linear operators gives

$$\|(T-\lambda)^{-1}Q\| \leqslant \underline{\lim}_{n\to\infty}\|(T_n-\lambda)^{-1}\|.$$

Note that

$$\|(T-\lambda)^{-1}Q\| = \|(T\restriction \text{Ran } Q-\lambda)^{-1}\|.$$

Thus (A3) follows. ■

We next consider operators in the Hilbert space $\mathscr{L}$ given by (2.1).

Theorem A.2: Let T_n, $n\in\mathbb{N}$, be self-adjoint operators in $\mathscr{L}$ and S be a self-adjoint operator in $\mathscr{H}$. Suppose that for all $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\text{s-}\lim_{n\to\infty}(T_n-z)^{-1} = (S-z)^{-1}\otimes P$$

with an orthogonal projection P on $\mathscr{K}$. Denote by $E_\lambda(T_n)$ and $E_\lambda(S)$ the spectral family of T_n and S, respectively. Then: (i) If $a,b\in\mathbb{R}$, $a<b$, and $(a,b)\cap\sigma(T_n)=0$ for all n, then $(a,b)\cap\sigma(S)=0$. (ii) If $a,b\in\mathbb{R}$, $a<b$, and $a,b\notin\sigma_{\rm pp}(S)$ (the pure point spectrum of S), then $E_{(a,b)}(T_n)\to E_{(a,b)}(S)\otimes P$ strongly as $n\to\infty$.

Proof: The proof of part (i) is similar to that of Theorem VIII.24(a) in Ref. 21; we need only to note that

$$\|(S-z)^{-1}\| = \|(S-z)^{-1}\otimes P\|.$$

To prove part (ii), we note that Theorem A.1 applied to the present case gives

$$F(T_n)\to F(S)\otimes P$$

strongly for all $F\in C_\infty(\mathbb{R})$. Then the method of the proof of Theorem VIII.24(b) in Ref. 21 works and the desired result follows. ■

Remark: Under the assumption of Theorem A.2 and the condition that $P\neq I$, $\{\sigma(T_n)\}_{n\in\mathbb{N}}$ cannot be bounded in n.

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