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Author(s)	Arai, Asao
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Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: Canonical commutation relations, quantum algebras, and reduction to lattice quantum systems

Asao Arai^{a)}

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

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Some representation-theoretic aspects of a two-dimensional quantum system of a charged particle in a vector potential \mathbf{A} , which may be singular on an infinite discrete subset \mathbf{D} of \mathbf{R}^2 are investigated. For each vector \mathbf{v} in a set $\mathbf{V}(\mathbf{D}) \subset \mathbf{R}^2 \setminus \{\mathbf{0}\}$, the projection $P_{\mathbf{v}}$ of the physical momentum operator $\mathbf{P} := \mathbf{p} - \alpha \mathbf{A}$ to the direction of \mathbf{v} is defined by $P_{\mathbf{v}} := \mathbf{v} \cdot \mathbf{P}$ as an operator acting in $L^2(\mathbf{R}^2)$, where $\mathbf{p} = (-iD_x, -iD_y)[(x, y) \in \mathbf{R}^2]$ with D_x (resp., D_y) being the generalized partial differential operator in the variable x (resp., y) and $\alpha \in \mathbf{R}$ is a parameter denoting the charge of the particle. It is proven that $P_{\mathbf{v}}$ is essentially self-adjoint and an explicit formula is derived for the strongly continuous one-parameter unitary group $\{e^{it\bar{P}_{\mathbf{v}}}\}_{t \in \mathbf{R}}$ generated by the self-adjoint operator $\bar{P}_{\mathbf{v}}$ (the closure of $P_{\mathbf{v}}$), i.e., the magnetic translation to the direction of the vector \mathbf{v} . The magnetic translations along curves in $\mathbf{R}^2 \setminus \mathbf{D}$ are also considered. Conjugately to $P_{\mathbf{v}}$ and $P_{\mathbf{w}}$ [$\mathbf{w} \in \mathbf{V}(\mathbf{D})$], a self-adjoint multiplication operator $Q_{\mathbf{v}, \mathbf{w}}$ is introduced, which is a linear combination of the position operators x and y , such that, if \mathbf{A} is flat on $\mathbf{R}^2 \setminus \mathbf{D}$, then $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}} := \{Q_{\mathbf{v}, \mathbf{w}}, Q_{\mathbf{w}, \mathbf{v}}, P_{\mathbf{v}}, P_{\mathbf{w}}\}$ gives a representation of the canonical commutation relations (CCR) with two degrees of freedom. Properties of the representation $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ are analyzed. In particular, a necessary and sufficient condition for $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ to be unitarily equivalent (or inequivalent) to the Schrödinger representation of CCR is established. The case where $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ is inequivalent to the Schrödinger representation corresponds to the Aharonov–Bohm effect. Quantum algebraic structures [quantum plane and the quantum group $U_q(sl_2)$] associated with the pair $\{\bar{P}_{\mathbf{v}}, \bar{P}_{\mathbf{w}}\}$ are also discussed. Moreover, for every \mathbf{A} in a class of vector potentials having singularities on the infinite lattice $\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) := \{m\boldsymbol{\omega}_1 + n\boldsymbol{\omega}_2 | m, n \in \mathbf{Z}\}$ [the case $\mathbf{D} = \mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$], where $\boldsymbol{\omega}_1 \in \mathbf{R}^2$ and $\boldsymbol{\omega}_2 \in \mathbf{R}^2$ are linearly independent, it is shown that the magnetic translations $e^{i\bar{P}_{\mathbf{v}}\boldsymbol{\omega}_j}$, $j = 1, 2$, with \mathbf{A} replaced by a *modified* vector potential are reduced by the Hilbert space $l^2(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$ identified with a closed subspace of $L^2(\mathbf{R}^2)$. This result, which may be regarded as one of the most important novel results of the present paper, establishes a connection of continuous quantum systems in vector potentials to lattice ones. © 1998 American Institute of Physics.

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I. INTRODUCTION

This article is a continuation of the previous articles^{1–5} concerning gauge theory (quantum mechanics of a particle interacting with an external gauge potential) on a nonsimply connected space in two dimensions. In such a gauge theory, a representation of the canonical commutation relations (CCR) with two degrees of freedom is realized by the position and the physical momentum operators if the gauge potential is flat. The nonsimply connectedness of the base space is essential for this representation to be nontrivial, i.e., not necessarily be equivalent to the Schrödinger representation of CCR with two degrees of freedom. The unitary equivalence or inequivalence of the representation to the Schrödinger representation is completely characterized in terms

^{a)}Electronic mail: arai@math.sci.hokudai.ac.jp

of the local Wilson loops. An interesting feature to be noted here is that the celebrated Aharonov–Bohm effect⁶ may be mathematically well understood as the representation inequivalent to the Schrödinger one. Some physical implications of the inequivalent representation in the Abelian case (the case of quantum mechanics of a charged particle in an external vector potential) are discussed in terms of the Dirac–Weyl operator (Ref. 2). Moreover, it is shown that, in the case where the vector potential is given in terms of the Weierstrass Zeta function with singularities at $z = m\omega_1 + in\omega_2$, $m, n \in \mathbf{Z}$ ($\omega_j > 0$, $j = 1, 2$), the inequivalent representation induces representations of quantum planes (rotation algebras) and the quantum group $U_q(sl_2)$ with $|q| = 1$, $q \in \mathbf{C}$ (Ref. 5).

In this paper we further pursue representation-theoretic aspects of a two-dimensional Abelian gauge theory in which the vector potential may have singularities on an infinite set of points in \mathbf{R}^2 and show by bringing some new aspects to light that such a gauge theory may have richer and deeper structures. We first demonstrate by an explicit construction that, associated with the position and the physical momentum operators, there exists a wide class of inequivalent representations of CCR that includes those given in Refs. 1 and 5. The basic idea of the construction is to consider projections of the position and the physical momentum operators to directions of vectors in \mathbf{R}^2 . As in the previous simpler cases discussed in Refs. 1 and 5, these new discovered representations of CCR give new representations of quantum planes and $U_q(sl_2)$ if the singularities of the vector potential form an infinite lattice.

Another new aspect of the present article is concerned with magnetic translations. These objects can be defined in both continuous and lattice quantum systems in external magnetic fields. Magnetic translations in the former systems in *uniform magnetic fields* have been discussed in some detail (e.g., Refs. 7–10), but, it seems that investigations of magnetic translations in the case of *nonuniform magnetic fields* are missing in the literature. In this paper we define, in the continuous quantum system under consideration, magnetic (parallel) translations as the strongly continuous one-parameter unitary groups generated by the projected physical momentum operators and study their properties.

On the other hand, magnetic translations on lattice quantum systems have been extensively discussed in connection with models of the Hofstadter type (e.g., Refs. 11–13 and references therein). Lattice models are usually defined by *ad hoc* procedures from continuous quantum systems (e.g., “tight-binding approximation” or other analogies). From a unified point of view, this situation is obviously unsatisfactory. It would be natural and interesting to investigate if there exists any *internal* (non-*ad hoc*) reduction mechanism by which a lattice quantum system “dynamically” emerges from a continuous quantum system. In this paper we show that such a mechanism exists. We regard this result as one of the most important new results of the present paper.

We now describe the outline of the present article in more detail. As already mentioned above, we consider a continuous quantum system of a charged particle with charge $\alpha \in \mathbf{R} \setminus \{0\}$ moving in the Euclidean plane \mathbf{R}^2 under the influence of a perpendicular magnetic field B . We denote by

$$\mathbf{A}(\mathbf{r}) := (A_1(\mathbf{r}), A_2(\mathbf{r})), \quad \mathbf{r} = (x, y) \in \mathbf{R}^2, \tag{1.1}$$

the vector potential of the magnetic field (up to gauge transformations), so that

$$B = D_x A_2 - D_y A_1, \tag{1.2}$$

where D_x and D_y are the generalized (distributional) partial differential operators in the variables x and y , respectively.

Let

$$\mathbf{D} := \{\mathbf{a}_n = (a_{n1}, a_{n2})\}_{n \in \mathbf{N}} \tag{1.3}$$

be a set of points in \mathbf{R}^2 such that $\mathbf{a}_n \neq \mathbf{a}_m$ if $n \neq m$ and the set $\{a_{nj}\}_{n=1}^\infty$ ($j = 1, 2$) has no accumulation point in \mathbf{R} . Then

$$\mathbf{M} := \mathbf{R}^2 \setminus \mathbf{D} \tag{1.4}$$

is an open set of \mathbf{R}^2 . We assume that \mathbf{A} is continuous on \mathbf{M} . But \mathbf{A} may be singular on \mathbf{D} and B may be a distribution on \mathbf{R}^2 with support in \mathbf{D} . Except for some general aspects, it is essential for

the theory presented below to be nontrivial that \mathbf{A} has singularities in \mathbf{D} . The Hilbert space of state vectors of the quantum system under consideration can be taken to be $L^2(\mathbf{R}^2)$.

The operators,

$$p_1 := -iD_x, \quad p_2 := -iD_y, \tag{1.5}$$

acting in $L^2(\mathbf{R}^2)$ with domain $D(p_1) := \{\Psi \in L^2(\mathbf{R}^2) | D_x \Psi \in L^2(\mathbf{R}^2)\}$ and $D(p_2) := \{\Psi \in L^2(\mathbf{R}^2) | D_y \Psi \in L^2(\mathbf{R}^2)\}$ are self-adjoint, where, for an operator T , we denote by $D(T)$ its domain. The physical momentum operator (the velocity operator up to a constant multiple) $\mathbf{P} := (P_1, P_2)$ is given by

$$P_j := p_j - \alpha A_j, \quad j = 1, 2, \tag{1.6}$$

acting in $L^2(\mathbf{R}^2)$ with $D(P_j) = D(p_j) \cap D(A_j)$.

For each vector $\mathbf{v} = (v_1, v_2) \in \mathbf{R}^2 \setminus \{0\}$, we define

$$P_{\mathbf{v}} := \mathbf{v} \cdot \mathbf{P} := \sum_{j=1}^2 v_j P_j. \tag{1.7}$$

We call it the *projection of the physical momentum operator* (or simply *the projected physical momentum operator*) to the direction of \mathbf{v} . The operator P_j is a special case of $P_{\mathbf{v}}$:

$$P_j = P_{\mathbf{e}_j}, \quad j = 1, 2, \tag{1.8}$$

with

$$\mathbf{e}_1 := (1, 0), \quad \mathbf{e}_2 := (0, 1). \tag{1.9}$$

We introduce a subset of vectors \mathbf{R}^2 . Let

$$\mathbf{v} \wedge \mathbf{w} := v_1 w_2 - v_2 w_1, \quad \mathbf{v}, \mathbf{w} \in \mathbf{R}^2. \tag{1.10}$$

Definition 1.1: We say that $\mathbf{v} \in \mathbf{R}^2 \setminus \{0\}$ is in the set $\mathbf{V}(\mathbf{D})$ if the sequence $\{\mathbf{v} \wedge \mathbf{a}_n\}_{n=1}^{\infty}$ has no accumulation point.

By the assumption for \mathbf{D} stated above, $\mathbf{e}_j \in \mathbf{V}(\mathbf{D})$, $j = 1, 2$.

In Sec. II we prove the essential self-adjointness of $P_{\mathbf{v}}$ with $\mathbf{v} \in \mathbf{V}(\mathbf{D})$ and clarify the spectral properties of the self-adjoint operator $\bar{P}_{\mathbf{v}}$ (the closure of $P_{\mathbf{v}}$) (Theorems 2.4 and 2.5).

The essential self-adjointness of $P_{\mathbf{v}}$ allows one to define the *continuous magnetic translations* to the direction of \mathbf{v} as the elements of the strongly continuous one-parameter unitary group,

$$T_{\mathbf{v}}^{\mathbf{A}}(t) := e^{it\bar{P}_{\mathbf{v}}}, \quad t \in \mathbf{R}, \tag{1.11}$$

generated by $\bar{P}_{\mathbf{v}}$. We prove that there exist no nontrivial finite-dimensional subspaces of $L^2(\mathbf{R}^2)$ left invariant by $T_{\mathbf{v}}^{\mathbf{A}}(t)$ (Proposition 2.6).

Section III is devoted to a basic analysis of the continuous magnetic translations. We derive an explicit formula for $T_{\mathbf{v}}^{\mathbf{A}}(t)$ (Theorem 3.2) and, using it, we compute commutation relations of $T_{\mathbf{v}}^{\mathbf{A}}(s)$ and $T_{\mathbf{w}}^{\mathbf{A}}(t)$ [$s, t \in \mathbf{R}, \mathbf{w} \in \mathbf{V}(\mathbf{D})$] (Theorem 3.3).

In Sec. IV we define magnetic translations along curves in \mathbf{M} and investigate their properties.

Section V is devoted to analysis of representations of CCR appearing in the quantum system under consideration. We first introduce multiplication operators given as linear combinations of the position operators,

$$q_1 := x, \quad q_2 := y. \tag{1.12}$$

Namely, for vectors $\mathbf{v}, \mathbf{w} \in \mathbf{V}(\mathbf{D})$ linearly independent, we define

$$Q_{\mathbf{v}, \mathbf{w}} := \frac{\mathbf{q} \wedge \mathbf{w}}{\mathbf{v} \wedge \mathbf{w}}, \tag{1.13}$$

where $\mathbf{q} := (q_1, q_2)$.

We denote by $C^k(\mathbf{M})$ the set of k times continuously differentiable functions on \mathbf{M} and by $C_0^k(\mathbf{M})$ the set of functions in $C^k(\mathbf{M})$ with bounded support in \mathbf{M} .

We say that \mathbf{A} with $A_j \in C^1(\mathbf{M})$ ($j = 1, 2$) is *flat* on \mathbf{M} if $B(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathbf{M}$. The flatness of \mathbf{A} on \mathbf{M} physically means the magnetic field B is concentrated on the discrete set \mathbf{D} in the distribution sense.

We show that, if $A_j \in C^1(\mathbf{M})$, then

$$\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}} := \{L^2(\mathbf{R}^2), C_0^2(\mathbf{M}), Q_{\mathbf{v}, \mathbf{w}}, \bar{Q}_{\mathbf{v}, \mathbf{w}}, \bar{P}_{\mathbf{v}}, \bar{P}_{\mathbf{w}}\} \tag{1.14}$$

is a representation of the CCR with two degrees of freedom if and only if \mathbf{A} is flat on \mathbf{M} (for the terminology on the representation theory of CCR, we refer to Ref. 5, Sec. 1). We analyze properties of this representation. We see that results similar to those of simpler cases in Refs. 1 and 5 hold in the present case too, generalizing them.

In Sect. VI we give some remarks on quantum algebraic structures [representations of quantum planes and $U_q(sl_2)$] associated with the representation $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$.

In the last section we consider the problem of reduction of the continuous quantum system under consideration to a lattice one. For this purpose, we fix arbitrarily two linearly independent vectors,

$$\boldsymbol{\omega}_1 := (\omega_{11}, \omega_{12}), \quad \boldsymbol{\omega}_2 := (\omega_{21}, \omega_{22}) \in \mathbf{R}^2, \tag{1.15}$$

such that $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 > 0$ and take as \mathbf{D} an infinite lattice,

$$\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) := \{\boldsymbol{\Omega}_{m,n} | m, n \in \mathbf{Z}\}, \tag{1.16}$$

where

$$\boldsymbol{\Omega}_{m,n} := m\boldsymbol{\omega}_1 + n\boldsymbol{\omega}_2. \tag{1.17}$$

The Hilbert space of state vectors of a quantum system on the lattice $\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ is taken to be

$$l^2(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)) := \left\{ \psi = \{\psi(\boldsymbol{\Omega}_{m,n})\}_{m,n \in \mathbf{Z}}, \psi(\boldsymbol{\Omega}_{m,n}) \in \mathbf{C}, m, n \in \mathbf{Z}, \sum_{m,n \in \mathbf{Z}} |\psi(\boldsymbol{\Omega}_{m,n})|^2 < \infty \right\}. \tag{1.18}$$

This Hilbert space can be regarded as a closed subspace of $L^2(\mathbf{R}^2)$ in a natural way. Indeed, let $\mathbf{S}_{m,n}$ be the interior domain of the parallelogram determined by the four vectors $\boldsymbol{\Omega}_{m+1,n} - \boldsymbol{\Omega}_{m,n}$, $\boldsymbol{\Omega}_{m,n+1} - \boldsymbol{\Omega}_{m,n}$, $\boldsymbol{\Omega}_{m+1,n+1} - \boldsymbol{\Omega}_{m,n+1}$ and $\chi_{m,n}$ be the characteristic function of $\mathbf{S}_{m,n}$. Then each element $\psi \in l^2(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$ can be regarded as an element of $L^2(\mathbf{R}^2)$ by the correspondence

$$\psi \rightarrow \tilde{\psi} := \sum_{m,n \in \mathbf{Z}} \psi(\boldsymbol{\Omega}_{m,n}) \chi_{m,n} \in L^2(\mathbf{R}^2),$$

so that, under this correspondence, $l^2(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$ can be identified with the closed subspace,

$$L_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}^2(\mathbf{R}^2) := \left\{ \Psi \in L^2(\mathbf{R}^2) \mid \Psi = \sum_{m,n \in \mathbf{Z}} \Psi_{m,n} \chi_{m,n}, \Psi_{m,n} \in \mathbf{C}, \sum_{m,n \in \mathbf{Z}} |\Psi_{m,n}|^2 < \infty \right\}, \tag{1.19}$$

consisting of elements in $L^2(\mathbf{R}^2)$ being constant on each $\mathbf{S}_{m,n}$. It is easy to see that $\boldsymbol{\omega}_j \in \mathbf{V}(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$, $j = 1, 2$. If the unitary operators,

$$T_{\pm \boldsymbol{\omega}_j}^{\mathbf{A}} := T_{\pm \boldsymbol{\omega}_j}^{\mathbf{A}}(1), \quad j = 1, 2, \tag{1.20}$$

leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant, then they are reduced by $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ and their restriction to $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ gives a set of magnetic translations on a quantum system on the lattice $\mathbf{L}(\omega_1, \omega_2)$. In that case, models of the Hofstadter type on $\mathbf{L}(\omega_1, \omega_2)$ are obtained as internal or ‘‘dynamical’’ reductions of models on the continuous space $\mathbf{R}^2 \setminus \mathbf{L}(\omega_1, \omega_2)$. Thus, the problem is to determine the class of vector potentials \mathbf{A} such that $T^\mathbf{A}_{\pm \omega_j}$ leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant. At first sight, such vector potentials may seem not to exist, except for some physically trivial cases. This is certainly true as long as one considers *regular* vector potentials on \mathbf{R}^2 . But, if the vector potential is allowed to be *singular* on the lattice $\mathbf{L}(\omega_1, \omega_2)$, then the situation changes drastically. Indeed, in that case, we can show that there exists a method of constructing vector potentials \mathbf{A} for which $T^\mathbf{A}_{\pm \omega_j}$ leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant. Roughly speaking, the method is as follows. Let

$$\omega_j = \omega_{j1} + i\omega_{j2}, \quad j = 1, 2, \tag{1.21}$$

be the complex numbers corresponding to $\omega_j \in \mathbf{R}^2$, $j = 1, 2 (i := \sqrt{-1})$. We take a meromorphic function $f(z)$ on \mathbf{C} with poles at

$$\Omega_{m,n} := m\omega_1 + n\omega_2, \quad m, n \in \mathbf{Z}, \tag{1.22}$$

such that $df(z)/dz$ is an elliptic function with periods ω_j , $j = 1, 2$ and define a vector potential \mathbf{A} by

$$A_1(\mathbf{r}) := \Im f(z), \quad A_2(\mathbf{r}) := \Re f(z), \quad \mathbf{r} = (x, y) \in \mathbf{R}^2, \quad z = x + iy. \tag{1.23}$$

We show that $T^\mathbf{A}_{\pm \omega_j}$ themselves may not leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant, but there exists a correspondence $\mathbf{A} \rightarrow \tilde{\mathbf{A}}$ of the vector potential such that $T^{\tilde{\mathbf{A}}}_{\pm \omega_j}$ do leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant.

Unfortunately we have been unable to solve the problem of determining *all* the vector potentials \mathbf{A} such that $T^\mathbf{A}_{\pm \omega_j}$ ($j = 1, 2$) leave $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$ invariant. We leave this problem for future study.

We remark that, in the same way as in Refs. 3 and 4, the results presented in Secs. II–V can be extended in a natural way to the case of non-Abelian gauge theories. If any significant aspects are discovered in the non-Abelian case, then we shall report them in a separate paper.

II. BASIC PROPERTIES OF THE PROJECTED PHYSICAL MOMENTUM OPERATORS

A. The physical momentum operator

The mathematical analysis of the physical momentum operator \mathbf{P} in the present case can be made quite similarly to that in the case where \mathbf{D} is a finite discrete set (Ref. 1). But, for the reader’s convenience as well as for later reference, we briefly describe some basic properties of P_j , $j = 1, 2$. We introduce two sets:

$$\mathbf{M}_1 := \{(x, y) \in \mathbf{R}^2 \mid x \in \mathbf{R}, y \neq a_{n2}, n \in \mathbf{N}\}, \tag{2.1}$$

$$\mathbf{M}_2 := \{(x, y) \in \mathbf{R}^2 \mid x \neq a_{n1}, n \in \mathbf{N}, y \in \mathbf{R}\}, \tag{2.2}$$

which, by the assumed property of \mathbf{D} , are open sets of \mathbf{R}^2 . Let

$$U_1(x, y) := e^{-i\alpha \int_0^x A_1(x', y) dx'}, \quad U_2(x, y) := e^{-i\alpha \int_0^y A_2(x, y') dy'}. \tag{2.3}$$

Then $U_j \in C(\mathbf{M}_j)$, $j = 1, 2$. Since the Lebesgue measure of the set $\{(x, y) \mid x \in \mathbf{R}, y = a_{n2}, n \in \mathbf{N}\}$ (resp., $\{(x, y) \mid y \in \mathbf{R}, x = a_{n1}, n \in \mathbf{N}\}$) is zero, U_1 (resp., U_2) defines a unique multiplication unitary operator on $L^2(\mathbf{R}^2)$, which we denote by the same symbol U_1 (resp., U_2).

In what follows, we assume the following.

Hypothesis (A)_k: For a non-negative integer k , $A_j \in C^k(\mathbf{M})$, $j = 1, 2$.

Proposition 2.1: The operator $P_j (j = 1, 2)$ is essentially self-adjoint on $C^k_0(\mathbf{M}_j)$ and the operator equations,

$$\bar{P}_j = U_j^{-1} p_j U_j, \quad j=1,2, \tag{2.4}$$

hold.

Proof: The unitary operator U_j maps $C_0^k(\mathbf{M}_j)$ to itself bijectively and, for all $\Psi \in C_0^k(\mathbf{M}_j)$, $P_j \Psi = U_j^{-1} p_j U_j \Psi$, $j=1,2$. Since p_j is essentially self-adjoint on $C_0^k(\mathbf{M}_j)$, it follows that P_j is essentially self-adjoint on $C_0^k(\mathbf{M}_j)$ and (2.4) is obtained. ■

We denote by $\sigma(\bar{P}_j)$ [resp., $\sigma_{ac}(\bar{P}_j)$, $\sigma_p(\bar{P}_j)$, $\sigma_{sc}(\bar{P}_j)$] the spectrum (resp., absolutely continuous, point, singular continuous spectrum) of \bar{P}_j .

Proposition 2.1 (the unitary equivalence of \bar{P}_j to p_j) and the well-known spectral property of p_j imply the following.

Proposition 2.2: For $j=1,2$,

$$\sigma(\bar{P}_j) = \sigma_{ac}(\bar{P}_j) = \mathbf{R}, \quad \sigma_p(\bar{P}_j) = \sigma_{sc}(\bar{P}_j) = \emptyset. \tag{2.5}$$

B. The projected physical momentum operators

Let $\mathbf{V}(\mathbf{D})$ be as in Definition 1.1. For $\mathbf{v} \in \mathbf{V}(\mathbf{D})$, we define

$$L(\mathbf{a}_n; \mathbf{v}) = \{\mathbf{a}_n + s\mathbf{v} \mid s \in \mathbf{R}\}, \tag{2.6}$$

which is the straight line passing through the point \mathbf{a}_n with the direction of \mathbf{v} . Then

$$\mathbf{M}_v(\mathbf{D}) := \mathbf{R}^2 \setminus \bigcup_{n=1}^{\infty} L(\mathbf{a}_n; \mathbf{v}), \tag{2.7}$$

is an open set of \mathbf{R}^2 .

It is well known that the angular momentum operator,

$$L := q_1 p_2 - q_2 p_1, \tag{2.8}$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$ (e.g., Ref. 14, Sec. 3). We denote its closure by the same symbol L . Then, for all $\theta \in \mathbf{R}$ and $\Psi \in L^2(\mathbf{R}^2)$, we have

$$(e^{i\theta L} \Psi)(\mathbf{r}) = \Psi(R(\theta)\mathbf{r}), \quad \text{a.e } \mathbf{r} \in \mathbf{R}^2, \tag{2.9}$$

where

$$R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \tag{2.10}$$

is the rotation matrix in \mathbf{R}^2 .

We write $\mathbf{v} = (v_1, v_2) \in \mathbf{V}(\mathbf{D})$ in the polar coordinate as

$$v_1 = v \cos \theta_v, \quad v_2 = v \sin \theta_v, \quad v = |\mathbf{v}|, \quad 0 \leq \theta_v < 2\pi. \tag{2.11}$$

The function

$$A_v(\mathbf{r}) := \frac{\mathbf{v} \cdot \mathbf{A}(R(\theta_v)\mathbf{r})}{v} \tag{2.12}$$

is in $C^k(R(-\theta_v)\mathbf{M}_v(\mathbf{D}))$. We define

$$P_1(\mathbf{v}) := p_1 - \alpha A_v. \tag{2.13}$$

Proposition 2.3: The operator $P_1(\mathbf{v})$ is essentially self-adjoint on $C_0^k(R(-\theta_v)\mathbf{M}_v(\mathbf{D}))$.

Proof: We need only to apply Proposition 2.1 with A_1 and \mathbf{a}_n replaced by A_v and $R(-\theta_v)\mathbf{a}_n$, respectively. ■

The following theorem is a generalization of Proposition 2.1.

Theorem 2.4: Let $\mathbf{v} \in \mathbf{V}(\mathbf{D})$. Then P_v is essentially self-adjoint on $C_0^k(\mathbf{M}_v(\mathbf{D}))$ and the operator equation,

$$e^{i\theta_{\mathbf{v}}L}\bar{P}_{\mathbf{v}}e^{-i\theta_{\mathbf{v}}L} = \overline{vP_1(\mathbf{v})}, \tag{2.14}$$

holds.

Proof: Let

$$\mathbf{p} := (p_1, p_2). \tag{2.15}$$

Then it is easy to see that, for all $\theta \in \mathbf{R}$,

$$e^{i\theta L}\mathbf{p}e^{-i\theta L} = R(\theta)\mathbf{p}, \quad \text{on } C_0^\infty(\mathbf{R}^2). \tag{2.16}$$

Hence

$$e^{i\theta_{\mathbf{v}}L}\mathbf{v}\cdot\mathbf{p}e^{-i\theta_{\mathbf{v}}L} = vp_1, \quad \text{on } C_0^\infty(\mathbf{R}^2). \tag{2.17}$$

It is easy to see that

$$e^{-i\theta_{\mathbf{v}}L}C_0^k(R(-\theta_{\mathbf{v}})\mathbf{M}_{\mathbf{v}}(\mathbf{D})) = C_0^k(\mathbf{M}_{\mathbf{v}}(\mathbf{D})) \tag{2.18}$$

and

$$e^{i\theta_{\mathbf{v}}L}\mathbf{v}\cdot\mathbf{A}e^{-i\theta_{\mathbf{v}}L} = vA_{\mathbf{v}}, \quad \text{on } C_0^k(R(-\theta_{\mathbf{v}})\mathbf{M}_{\mathbf{v}}(\mathbf{D})).$$

Hence

$$e^{i\theta_{\mathbf{v}}L}P_{\mathbf{v}}e^{-i\theta_{\mathbf{v}}L} = vP_1(\mathbf{v}), \quad \text{on } C_0^k(R(-\theta_{\mathbf{v}})\mathbf{M}_{\mathbf{v}}(\mathbf{D})). \tag{2.19}$$

It follows from Proposition 2.3, (2.18) and (2.19), that $P_{\mathbf{v}}$ is essentially self-adjoint on $C_0^k(\mathbf{M}_{\mathbf{v}}(\mathbf{D}))$. This result implies that (2.19) can be extended to the operator equation (2.14). ■

Remark 2.1: If \mathbf{D} is a finite discrete set, then $P_{\mathbf{v}}$ is essentially self-adjoint for all $\mathbf{v} \in \mathbf{R}^2 \setminus \{0\}$.

By (2.14) and Proposition 2.2, we obtain the following result on the spectral property of $\bar{P}_{\mathbf{v}}$.

Theorem 2.5: For all $\mathbf{v} \in \mathbf{V}(\mathbf{D})$,

$$\sigma(\bar{P}_{\mathbf{v}}) = \sigma_{\text{ac}}(\bar{P}_{\mathbf{v}}) = \mathbf{R}, \quad \sigma_{\text{p}}(\bar{P}_{\mathbf{v}}) = \sigma_{\text{sc}}(\bar{P}_{\mathbf{v}}) = \emptyset. \tag{2.20}$$

C. Nonexistence of nontrivial finite-dimensional subspaces left invariant by continuous magnetic translations

Let $T_{\mathbf{v}}^{\mathbf{A}}(t)$ be defined as in (1.11). The following fact is important in considering the algebra generated by the magnetic translations $T_{\mathbf{v}_j}^{\mathbf{A}}(t)$, $j = 1, \dots, n$ ($n \in \mathbf{N}$, $\mathbf{v}_j \in \mathbf{V}(\mathbf{D})$).

Proposition 2.6: Let $t \in \mathbf{R} \setminus \{0\}$ and $\mathbf{v} \in \mathbf{V}(\mathbf{D})$. Then there exist no nontrivial finite dimensional subspaces of $L^2(\mathbf{R}^2)$ that are left invariant by $T_{\mathbf{v}}^{\mathbf{A}}(t)$.

Proof: Suppose that there exists a finite-dimensional subspace $\mathcal{H} \neq \{0\}$ of $L^2(\mathbf{R}^2)$ that is left invariant by $T_{\mathbf{v}}^{\mathbf{A}}(t)$. Then $T_{\mathbf{v}}^{\mathbf{A}}(t)$ is reduced by \mathcal{H} . Since $T_{\mathbf{v}}^{\mathbf{A}}(t)$ is unitary and \mathcal{H} is finite dimensional, the reduced part $T_{\mathbf{v}}^{\mathbf{A}}(t)|_{\mathcal{H}}$ has an eigenvalue λ_0 with $|\lambda_0| = 1$. This is also an eigenvalue of $T_{\mathbf{v}}^{\mathbf{A}}(t)$ on $L^2(\mathbf{R}^2)$. It follows from the spectral theorem of self-adjoint operators that $\bar{P}_{\mathbf{v}}$ has an eigenvalue of the form $(\arg \lambda_0 + 2\pi m_0)/t$ ($m_0 \in \mathbf{Z}$). But this contradicts the fact that $\sigma_{\text{p}}(\bar{P}_{\mathbf{v}}) = \emptyset$ (Theorem 2.5). Thus, we obtain the desired result. ■

III. CONTINUOUS MAGNETIC TRANSLATIONS GENERATED BY THE PROJECTED PHYSICAL MOMENTUM OPERATORS

A. Explicit representations

Let

$$p_{\mathbf{v}} = \mathbf{v} \cdot \mathbf{p}, \quad \mathbf{v} \in \mathbf{R}^2. \tag{3.1}$$

Then we have for all $t \in \mathbf{R}$ and $\Psi \in L^2(\mathbf{R}^2)$,

$$(e^{it\bar{P}_v}\Psi)(\mathbf{r}) = \Psi(\mathbf{v} + t\mathbf{v}), \quad \text{a.e. } \mathbf{r} \in \mathbf{R}^2. \tag{3.2}$$

Let $T_v^A(t)$ be defined by (1.11). Proposition 2.1 implies that, for all $t \in \mathbf{R}$,

$$T_{e_j}^A(t) = e^{it\bar{P}_j} = U_j^{-1} e^{itp_j} U_j = U_j^{-1} e^{it\bar{p}_{e_j}} U_j, \quad j = 1, 2. \tag{3.3}$$

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2$, we denote by $\int_a^b \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}$ the line integral of \mathbf{A} over the straight line from \mathbf{a} to \mathbf{b} .

Using (3.3) and (3.2), we can obtain an explicit formula for $T_{e_j}^A(t)$.

Theorem 3.1: For all $t \in \mathbf{R}$, and $\Psi \in L^2(\mathbf{R}^2)$,

$$(T_{e_j}^A(t)\Psi)(\mathbf{r}) = e^{-i\alpha \int_{\mathbf{r}}^{\mathbf{r}+te_j} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \Psi(\mathbf{r} + te_j) = e^{-i\alpha \int_{\mathbf{r}}^{\mathbf{r}+te_j} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} (e^{it\bar{p}_{e_j}}\Psi)(\mathbf{r}), \quad \text{a.e. } \mathbf{r}. \tag{3.4}$$

Theorem 3.1 can be extended to the case of the magnetic translation $T_v^A(t)$ with any vector $\mathbf{v} \in \mathbf{V}(\mathbf{D})$.

Theorem 3.2: Let $\mathbf{v} \in \mathbf{V}(\mathbf{D})$. Then, for all $t \in \mathbf{R}$ and $\Psi \in L^2(\mathbf{R}^2)$,

$$(T_v^A(t)\Psi)(\mathbf{r}) = e^{-i\alpha \int_{\mathbf{r}}^{\mathbf{r}+t\mathbf{v}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \Psi(\mathbf{r} + t\mathbf{v}) = e^{-i\alpha \int_{\mathbf{r}}^{\mathbf{r}+t\mathbf{v}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} (e^{it\bar{P}_v}\Psi)(\mathbf{r}), \quad \text{a.e. } \mathbf{r}. \tag{3.5}$$

Proof: By Theorem 2.4, we have, for all $t \in \mathbf{R}$,

$$T_v^A(t) = e^{-i\theta_v L} e^{itv\bar{P}_1(\mathbf{v})} e^{i\theta_v L}. \tag{3.6}$$

Let $\Psi \in L^2(\mathbf{R}^2)$ and set $R(-\theta_v)\mathbf{r} = (x(\mathbf{v}), y(\mathbf{v}))$. Then we have, by (3.4),

$$(T_v^A(t)\Psi)(\mathbf{r}) = e^{-i\alpha \int_{x(\mathbf{v})}^{x(\mathbf{v})+tv} A_v(x', y(\mathbf{v})) dx'} \Psi(\mathbf{r} + tvR(\theta_v)\mathbf{e}_1), \quad \text{a.e. } \mathbf{r}.$$

Noting that $vR(\theta_v)\mathbf{e}_1 = \mathbf{v}$ and

$$\int_{x(\mathbf{v})}^{x(\mathbf{v})+tv} A_v(x', y(\mathbf{v})) dx' = t \int_0^1 \mathbf{v} \cdot \mathbf{A}(\mathbf{r} + \lambda t\mathbf{v}) d\lambda = \int_{\mathbf{r}}^{\mathbf{r}+t\mathbf{v}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}',$$

we obtain (3.5). ■

Remark 3.1: We have, for all $t \in \mathbf{R} \setminus \{0\}$ and $\mathbf{v} \in \mathbf{V}(\mathbf{D})$,

$$t\bar{P}_v = \bar{P}_{t\mathbf{v}}, \tag{3.7}$$

which implies that

$$T_v^A(t) = T_{t\mathbf{v}}^A(1). \tag{3.8}$$

Thus, as for the magnetic translations generated by \bar{P}_v , it is sufficient to consider the unitary operator,

$$T_v^A := T_v^A(1). \tag{3.9}$$

B. Commutation relations

Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}(\mathbf{D})$ such that \mathbf{v} and \mathbf{w} are linearly independent, and set

$$C_{\mathbf{r}}(\mathbf{v}, \mathbf{w}) := \{\mathbf{r} + \lambda\mathbf{v} | 0 \leq \lambda \leq 1\} \circ \{\mathbf{r} + \mathbf{v} + \lambda\mathbf{w} | 0 \leq \lambda \leq 1\} \circ \{\mathbf{r} + (1-\lambda)\mathbf{v} + \mathbf{w} | 0 \leq \lambda \leq 1\} \circ \{\mathbf{r} + (1-\lambda)\mathbf{w} | 0 \leq \lambda \leq 1\}, \tag{3.10}$$

which is the closed curve starting and ending at the point \mathbf{r} , forming the circumference of the parallelogram with vertices $\mathbf{r}, \mathbf{r} + \mathbf{v}, \mathbf{r} + \mathbf{v} + \mathbf{w}, \mathbf{r} + \mathbf{w}$. We introduce

$$\mathbf{M}_{\mathbf{v}, \mathbf{w}}(\mathbf{D}) := \mathbf{R}^2 \setminus \bigcup_{n=1}^{\infty} [L(\mathbf{a}_n; \mathbf{v}) \cup (L(\mathbf{a}_n; \mathbf{v}) - \mathbf{w}) \cup L(\mathbf{a}_n; \mathbf{w}) \cup (L(\mathbf{a}_n; \mathbf{w}) - \mathbf{v})]. \tag{3.11}$$

Note that, if $\mathbf{r} \in \mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$, then $C_{\mathbf{r}}(\mathbf{v},\mathbf{w})$ does not intersect \mathbf{D} . We define

$$\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}(\mathbf{r}) = \int_{C_{\mathbf{r}}(\mathbf{v},\mathbf{w})} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}', \quad \mathbf{r} \in \mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D}), \tag{3.12}$$

which physically means the magnetic flux passing through the interior domain of $C_{\mathbf{r}}(\mathbf{v},\mathbf{w})$. Since the Lebesgue measure of $\mathbf{R}^2 \setminus \mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$ is zero, the function $\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ defines a unique multiplication self-adjoint operator on $L^2(\mathbf{R}^2)$. We denote it by the same symbol.

Theorem 3.3: *Let \mathbf{v} and \mathbf{w} be as above. Then*

$$T_{\mathbf{v}}^{\mathbf{A}} T_{\mathbf{w}}^{\mathbf{A}} = \exp(-i\alpha\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}) T_{\mathbf{w}}^{\mathbf{A}} T_{\mathbf{v}}^{\mathbf{A}}. \tag{3.13}$$

Proof: Using Theorem 3.2, we have, for all $\Psi \in L^2(\mathbf{R}^2)$ and a.e. \mathbf{r} ,

$$(T_{\mathbf{v}}^{\mathbf{A}} T_{\mathbf{w}}^{\mathbf{A}} \Psi)(\mathbf{r}) = e^{-i\alpha \int_{\mathbf{r}}^{\mathbf{r}+\mathbf{v}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} e^{-i\alpha \int_{\mathbf{r}+\mathbf{v}}^{\mathbf{r}+\mathbf{v}+\mathbf{w}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \Psi(\mathbf{r} + \mathbf{v} + \mathbf{w}), \tag{3.14}$$

from which (3.13) easily follows. ■

Remark 3.2: Formula (3.13) is a generalization of a formula established in Ref. 1 [see (2.1) in Ref. 1, which corresponds to the case where $\mathbf{v} = \mathbf{e}_1$, $\mathbf{w} = \mathbf{e}_2$, and \mathbf{A} may be singular on a finite discrete set of points].

IV. MAGNETIC TRANSLATIONS ALONG CURVES

In this section we consider magnetic translations along curves in \mathbf{M} . *Throughout this section we assume that*

$$\mathbf{V}(\mathbf{D}) = \mathbf{R}^2 \setminus \{0\}, \tag{4.1}$$

so that, for every vector $\mathbf{v} \in \mathbf{R}^2 \setminus \{0\}$, the projected physical momentum operator $\bar{P}_{\mathbf{v}}$ can be defined as a self-adjoint operator.

An example of such \mathbf{D} is given by $\mathbf{D} = \{\Omega_{m,n} \mid m \in \mathbf{Z}, n \in [-M, M'] \cap \mathbf{Z}\}$, where $\Omega_{m,n}$ is given by (1.17) and $M, M' \in \mathbf{N}$.

Let C be a continuous curve in \mathbf{M} with parametrization $C = \{\mathbf{u}(s) \mid s \in [a, b]\}$ ($-\infty < a < b < \infty$). Let $\{s_0, s_1, \dots, s_n\}$ be a partition of $[a, b]$ ($a = s_0 < s_1 < \dots < s_{n-1} < s_n = b$) such that $\max_{k=1, \dots, n} (s_k - s_{k-1}) \rightarrow 0 (n \rightarrow \infty)$ and $\Delta \mathbf{u}_k := \mathbf{u}(s_k) - \mathbf{u}(s_{k-1})$, $k = 1, \dots, n$. Then we define

$$U(C) := s\text{-}\lim_{n \rightarrow \infty} e^{-i\bar{P}_{\Delta \mathbf{u}_n}} e^{-i\bar{P}_{\Delta \mathbf{u}_{n-1}}} \dots e^{-i\bar{P}_{\Delta \mathbf{u}_1}}, \tag{4.2}$$

if the right-hand side (rhs) exists, where s-lim means strong limit. It follows that $U(C)$ is unitary. We call $U(C)$ the *magnetic translation along the curve C* .

We introduce

$$\Phi(C) := \int_C \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}, \tag{4.3}$$

the line integral of \mathbf{A} along the curve C . For $\mathbf{r} \in \mathbf{R}^2$ we define a curve $C_{\mathbf{r}}$ by

$$C_{\mathbf{r}} := C + \mathbf{r} - \mathbf{u}(b). \tag{4.4}$$

We denote by $F(C)$ the multiplication operator by the function: $\mathbf{r} \rightarrow e^{i\alpha\Phi(C_{\mathbf{r}})}$.

$$(F(C)\Psi)(\mathbf{r}) = e^{i\alpha\Phi(C_{\mathbf{r}})} \Psi(\mathbf{r}), \quad \Psi \in L^2(\mathbf{R}^2), \quad \text{a.e. } \mathbf{r} \in \mathbf{R}^2. \tag{4.5}$$

Remark 4.1: In the case where \mathbf{A} has singularities on \mathbf{D} , the function: $\mathbf{r} \rightarrow \Phi(C_{\mathbf{r}})$ is originally defined only on $\mathbf{R}^2 \setminus \cup_{n \in \mathbf{N}} \{\mathbf{a}_n + \mathbf{u}(b) - \mathbf{u}(s) \mid s \in [a, b]\}$. But the Lebesgue measure of the set $\cup_{n \in \mathbf{N}} \{\mathbf{a}_n + \mathbf{u}(b) - \mathbf{u}(s) \mid s \in [a, b]\}$ is zero, so that $e^{i\alpha\Phi(C_{\mathbf{r}})}$ defines a unique multiplication unitary operator on $L^2(\mathbf{R}^2)$.

Theorem 4.1: *The rhs of (4.2) exists and*

$$U(C) = F(C)e^{i\bar{P}\mathbf{u}(a) - \mathbf{u}(b)}, \tag{4.6}$$

Proof: Let

$$U_n(C) = e^{-i\bar{P}\Delta\mathbf{u}_n} e^{-i\bar{P}\Delta\mathbf{u}_{n-1}} \cdots e^{-i\bar{P}\Delta\mathbf{u}_1}.$$

By applying (3.5) repeatedly, we have, for all $\Psi \in L^2(\mathbf{R}^2)$,

$$(U_n(C)\Psi)(\mathbf{r}) = e^{i\alpha\Phi_n(\mathbf{r})} (e^{i\bar{P}\mathbf{u}(a) - \mathbf{u}(b)}\Psi)(\mathbf{r}), \quad \text{a.e. } \mathbf{r},$$

with

$$\Phi_n(\mathbf{r}) = \sum_{k=1}^{n-1} \int_{\mathbf{r} - \Delta\mathbf{u}_k - \Delta\mathbf{u}_{k+1} - \cdots - \Delta\mathbf{u}_n}^{\mathbf{r} - \Delta\mathbf{u}_{k+1} - \Delta\mathbf{u}_{k+2} - \cdots - \Delta\mathbf{u}_n} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' + \int_{\mathbf{r} - \Delta\mathbf{u}_n}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'.$$

Hence

$$\begin{aligned} \int_{\mathbf{R}^2} |(U_n(C)\Psi)(\mathbf{r}) - (F(C)e^{i\bar{P}\mathbf{u}(a) - \mathbf{u}(b)}\Psi)(\mathbf{r})|^2 d\mathbf{r} &= \int_{\mathbf{R}^2} |e^{i\alpha\Phi_n(\mathbf{r})} - e^{i\alpha\Phi(C_{\mathbf{r}})}|^2 |\Psi(\mathbf{r} - \mathbf{u}(b) \\ &\quad + \mathbf{u}(a))|^2 d\mathbf{r}. \end{aligned}$$

It is easy to see that, for all $\mathbf{r} \in \mathbf{R}^2 \setminus \cup_{n \in \mathbf{N}} \{\mathbf{a}_n + \mathbf{u}(b) - \mathbf{u}(s) \mid s \in [a, b]\}$, $\lim_{n \rightarrow \infty} \Phi_n(\mathbf{r}) = \Phi(C_{\mathbf{r}})$, and

$$|e^{i\alpha\Phi_n(\mathbf{r})} - e^{i\alpha\Phi(C_{\mathbf{r}})}|^2 |\Psi(\mathbf{r} - \mathbf{u}(b) + \mathbf{u}(a))|^2 \leq 4 |\Psi(\mathbf{r} - \mathbf{u}(b) + \mathbf{u}(a))|^2.$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^2} |e^{i\alpha\Phi_n(\mathbf{r})} - e^{i\alpha\Phi(C_{\mathbf{r}})}|^2 |\Psi(\mathbf{r} - \mathbf{u}(b) + \mathbf{u}(a))|^2 d\mathbf{r} = 0.$$

Hence (4.6) follows. ■

Remark 4.2: (1) Suppose that C is continuously differentiable. Then we can show that $U(C)$ is the product integral of the operator-valued function: $s \rightarrow -i\bar{P}_{d\mathbf{u}(s)/ds}$ on the interval $[a, b]$ (for the product integral, see Ref. 15).

(2) By (4.6), we have, for all $\Psi \in L^2(\mathbf{R}^2)$,

$$(U(C)\Psi)(\mathbf{r}) = e^{i\alpha\Phi(C_{\mathbf{r}})} \Psi(\mathbf{r} + \mathbf{u}(a) - \mathbf{u}(b)), \quad \text{a.e. } \mathbf{r}. \tag{4.7}$$

Let $E = \mathbf{M} \times \mathbf{C}$ be the trivial vector bundle with base space \mathbf{M} , fibre \mathbf{C} , and structure group $U(1)$ (the one-dimensional unitary group), and $\mathbf{V}_{\mathbf{r}} = \mathbf{C}$ be the fibre at the point $\mathbf{r} \in \mathbf{M}$. Then the rhs of (4.7) geometrically means the parallel transport of the vector $\Psi(\mathbf{r} + \mathbf{u}(a) - \mathbf{u}(b)) \in \mathbf{V}_{\mathbf{r} + \mathbf{u}(a) - \mathbf{u}(b)}$ to the point \mathbf{r} along the curve $C_{\mathbf{r}}$ with the connection one-form $-iA := (-i)(A_1 dx + A_2 dy)$ (see, e.g., Ref. 16).

As a corollary of Theorem 4.1, we obtain the following.

Theorem 4.2: (i) *We have*

$$U(C)^* = e^{-i\bar{P}\mathbf{u}(a) - \mathbf{u}(b)} F(C)^* \tag{4.8}$$

$$= \text{s-lim}_{n \rightarrow \infty} e^{i\bar{P}\Delta\mathbf{u}_1} e^{i\bar{P}\Delta\mathbf{u}_2} \cdots e^{i\bar{P}\Delta\mathbf{u}_n}. \tag{4.9}$$

(ii) *Let C_1 and C_2 be any continuous curves in \mathbf{M} such that the terminal point of C_1 coincides with the initial point of C_2 , so that the composition $C_1 \circ C_2$ is also a continuous curve whose initial (resp., terminal) point is that of C_1 (resp., that of C_2). Then*

$$U(C_1)U(C_2) = U(C_1 \circ C_2). \tag{4.10}$$

Proof: (i) Equation (4.8) follows from taking the adjoint of (4.6). In general, if unitary operators V_n, V on a Hilbert space satisfy $s\text{-}\lim_{n \rightarrow \infty} V_n = V$, then $s\text{-}\lim_{n \rightarrow \infty} V_n^* = V^*$. Applying this fact, we obtain (4.9).

(ii) This easily follows from applying formula (4.6). ■

Let $a \leq t \leq b$ and

$$C(t) = \{\mathbf{u}(s) | s \in [a, t]\}. \tag{4.11}$$

Then the correspondence $t \rightarrow U(C(t))$ gives an operator-valued function from $[a, b]$ into the set of unitary operators on $L^2(\mathbf{R}^2)$. We want to derive a differential equation for $U(C(t))$.

For a Hilbert space \mathcal{H} , we denote by $\mathbf{B}(\mathcal{H})$ the set of bounded linear operators on \mathcal{H} .

Lemma 4.3: Let \mathcal{H} be a Hilbert space, and V and W be $\mathbf{B}(\mathcal{H})$ -valued functions on $[a, b]$. Assume the following (i) and (ii): (i) V is strongly continuous on \mathcal{H} ; (ii) there exist subspaces \mathcal{D}_1 and \mathcal{D}_2 of \mathcal{H} such that V and W are strongly differentiable on \mathcal{D}_1 and \mathcal{D}_2 , respectively, and $W(t)\mathcal{D}_2 \subset \mathcal{D}_1$ for all $t \in [a, b]$. Then the $\mathbf{B}(\mathcal{H})$ -valued function: $t \rightarrow V(t)W(t)$ is strongly differentiable on \mathcal{D}_2 and, for all $\psi \in \mathcal{D}_2$,

$$\frac{d}{dt} V(t)W(t)\psi = \frac{dV(t)}{dt} W(t)\psi + V(t) \frac{dW(t)}{dt} \psi, \quad t \in [a, b], \tag{4.12}$$

where $dV(t)/dt$ [resp., $dW(t)/dt$] denotes the strong derivative of $V(t)$ [resp., $W(t)$] on \mathcal{D}_1 [resp., \mathcal{D}_2].

Proof: Let $X(t) = V(t)W(t)$ and $\psi \in \mathcal{D}_2$. Then we have for all $t \in (a, b)$ and $\epsilon \in \mathbf{R}$ with $|\epsilon|$ sufficiently small,

$$\begin{aligned} & \frac{X(t+\epsilon) - X(t)}{\epsilon} \psi - \frac{dV(t)}{dt} W(t)\psi - V(t) \frac{dW(t)}{dt} \psi \\ &= V(t+\epsilon) \left\{ \frac{W(t+\epsilon) - W(t)}{\epsilon} \psi - \frac{dW(t)}{dt} \psi \right\} + [V(t+\epsilon) - V(t)] \frac{dW(t)}{dt} \psi \\ & \quad + \left\{ \frac{V(t+\epsilon) - V(t)}{\epsilon} - \frac{dV(t)}{dt} \right\} W(t)\psi. \end{aligned}$$

By the strong continuity of V and the principle of uniform boundedness, we have $\sup_{t \in [a, b]} \|V(t)\| < \infty$. Using this fact and the assumed properties of V and W , we obtain the desired result. ■

Let

$$\mathbf{M}_C = \mathbf{R}^2 \setminus \bigcup_{n \in \mathbf{N}} \{\mathbf{a}_n + \mathbf{u}(t) - \mathbf{u}(s) | s, t \in [a, b]\}. \tag{4.13}$$

Theorem 4.4: Let C be continuously differentiable. Suppose that \mathbf{M}_C is an open set and Hypothesis (A)₁ holds. Then the operator-valued function $U(C(\cdot))^*$ on $[a, b]$ is strongly differentiable on $C_0^1(\mathbf{M}_C)$ with

$$\frac{d}{dt} U(C(t))^* \Psi = iU(C(t))^* P_{d\mathbf{u}(t)/dt} \Psi, \quad t \in [a, b], \quad \Psi \in C_0^1(\mathbf{M}_C). \tag{4.14}$$

Proof (Outline): Let $V(t) = e^{-i\bar{p}\mathbf{u}(a) - \mathbf{u}(t)}$ and $W(t) = F(C(t))^*$. Then we have $U(C(t))^* = V(t)W(t)$. It is not so difficult to show that V is strongly continuous on $L^2(\mathbf{R}^2)$ and strongly differentiable on $C_0^1(\mathbf{R}^2)$ with

$$\frac{dV(t)}{dt} \Psi = iV(t)P_{d\mathbf{u}(t)/dt} \Psi, \quad \Psi \in C_0^1(\mathbf{R}^2)$$

(cf. the proof of Theorem 4.1). Let $\Psi \in C_0^1(\mathbf{M}_C)$. Then

$$(W(t)\Psi)(\mathbf{r}) = \exp\left(-i\alpha \int_a^t \mathbf{A}(\mathbf{u}(s) - \mathbf{u}(t) + \mathbf{r}) \cdot \frac{d\mathbf{u}(s)}{ds} ds\right) \Psi(\mathbf{r}).$$

It follows from this formula and the present assumption for \mathbf{A} that $W(t)C_0^1(\mathbf{M}_C) \subset C_0^1(\mathbf{M}_C)$ and $W(t)$ is strongly differentiable on $C_0^1(\mathbf{M}_C)$ with

$$\begin{aligned} \left(\frac{dW(t)}{dt} \Psi\right)(\mathbf{r}) &= \exp\left(-i\alpha \int_a^t \mathbf{A}(\mathbf{u}(s) - \mathbf{u}(t) + \mathbf{r}) \cdot \frac{d\mathbf{u}(s)}{ds} ds\right) \left(-i\alpha \frac{d\mathbf{u}(t)}{dt} \cdot \mathbf{A}(\mathbf{r})\right. \\ &\quad \left.+ i\alpha \sum_{j,k=1}^2 \int_a^t \frac{du_j(t)}{dt} (\partial_j A_k)(\mathbf{u}(s) - \mathbf{u}(t) + \mathbf{r}) \frac{du_k(s)}{ds} ds\right) \Psi(\mathbf{r}), \end{aligned}$$

where $\partial_1 := \partial/\partial x$, $\partial_2 := \partial/\partial y$. On the other hand, we have

$$\begin{aligned} (iP_{d\mathbf{u}(t)/dt} W(t)\Psi)(\mathbf{r}) &= \exp\left(-i\alpha \int_a^t \mathbf{A}(\mathbf{u}(s) - \mathbf{u}(t) + \mathbf{r}) \cdot \frac{d\mathbf{u}(s)}{ds} ds\right) \\ &\quad \times \left(-i\alpha \sum_{j,k=1}^2 \int_a^t \frac{du_j(t)}{dt} (\partial_j A_k)(\mathbf{u}(s) - \mathbf{u}(t) + \mathbf{r}) \frac{du_k(s)}{ds} ds\right) \Psi(\mathbf{r}) \\ &\quad + (iW(t)P_{d\mathbf{u}(t)/dt}\Psi)(\mathbf{r}). \end{aligned}$$

By applying Lemma 4.3, we obtain (4.14). ■

Remark 4.3: (1) It follows from Theorem 4.4 that

$$\frac{d}{dt} (\Psi, U(C(t))\Phi)_2 = (iP_{d\mathbf{u}(t)/dt}\Psi, U(C(t))\Phi)_2, \quad \Psi \in C_0^1(\mathbf{M}_C), \quad \Phi \in L^2(\mathbf{R}^2), \quad (4.15)$$

where $(\cdot, \cdot)_2$ denotes the inner product of $L^2(\mathbf{R}^2)$. But, in the case where \mathbf{A} has singularities on \mathbf{D} , it seems difficult to find a dense subspace \mathcal{D} on which $U(C(t))$ is strongly differentiable and such that $U(C(t))\mathcal{D} \subset D(\bar{P}_{d\mathbf{u}(t)/dt})$.

(2) If \mathbf{A} is differentiable on the whole space \mathbf{R}^2 , then $U(C(t))$ is strongly differentiable on $C_0^1(\mathbf{R}^2)$, with

$$dU(C(t))\Psi/dt = iP_{d\mathbf{u}(t)/dt}U(C(t))\Psi, \quad \Psi \in C_0^1(\mathbf{R}^2).$$

V. REPRESENTATIONS OF CCR

In this section we show that, associated with the projected physical momentum operators, there exist representations of the CCR with two degrees of freedom and give a complete characterization on the unitary equivalence or inequivalence of the representations to the Schrödinger representation of the CCR with two degrees of freedom. The inequivalent representations correspond to the Aharonov–Bohm effect (Ref. 6). The results of this section include generalizations of those of the previous works (Refs. 1 and 5).

Let $Q_{\mathbf{v},\mathbf{w}}[\mathbf{v}, \mathbf{w} \in \mathbf{V}(\mathbf{D})]$ be the self-adjoint multiplication operator given by (1.13). It is easy to see that, if $A_j \in C^1(\mathbf{M})$, $j = 1, 2$, then the set $\{Q_{\mathbf{v},\mathbf{w}}, Q_{\mathbf{w},\mathbf{v}}, \bar{P}_{\mathbf{v}}, \bar{P}_{\mathbf{w}}\}$ of self-adjoint operators has the following commutation properties: for all $\Psi \in C_0^2(\mathbf{M})$,

$$[Q_{\mathbf{v},\mathbf{w}}, \bar{P}_{\mathbf{v}}]\Psi = i\Psi, \quad [Q_{\mathbf{w},\mathbf{v}}, \bar{P}_{\mathbf{w}}]\Psi = i\Psi, \quad (5.1)$$

$$[Q_{\mathbf{v},\mathbf{w}}, \bar{P}_{\mathbf{w}}]\Psi = 0, \quad [Q_{\mathbf{w},\mathbf{v}}, \bar{P}_{\mathbf{v}}]\Psi = 0, \quad (5.2)$$

$$[Q_{\mathbf{v},\mathbf{w}}, Q_{\mathbf{w},\mathbf{v}}]\Psi = 0, \quad (5.3)$$

$$[\bar{P}_{\mathbf{v}}, \bar{P}_{\mathbf{w}}]\Psi = i\alpha(\mathbf{v} \wedge \mathbf{w})B\Psi. \quad (5.4)$$

Remark 5.1: Let \mathbf{G}_d be the algebra generated by elements $Q_j, \Pi_j, F_{jk}, j=1, \dots, d$, with identity I , obeying the commutation relations

$$[Q_j, Q_k]=0, \quad [Q_j, \Pi_k]=i\delta_{jk}I, \quad [\Pi_j, \Pi_k]=iF_{jk}, \quad [F_{jk}, Q_l]=0, \quad j, k, l=1, \dots, d.$$

These relations have an origin in a quantum theory of a charged particle in an external electromagnetic field on \mathbf{R}^d , where Q_j and Π_j are realized as the position and the physical momentum operators, and may be regarded as an extension or a deformation of the CCR with d degrees of freedom. A special class of representations of \mathbf{G}_d is discussed in Ref. 14 in connection with exactly solvable models. Relations (5.1)–(5.4) show that, if \mathbf{A} is in $C^\infty(\mathbf{M})$, then the operators $Q_{\mathbf{v}, \mathbf{w}}, Q_{\mathbf{w}, \mathbf{v}}, \bar{P}_{\mathbf{v}}$, and $\bar{P}_{\mathbf{w}}$ restricted to the subspace $C_0^\infty(\mathbf{M})$ give a representation of \mathbf{G}_2 with the correspondence

$$Q_1 \rightarrow Q_{\mathbf{v}, \mathbf{w}}, \quad Q_2 \rightarrow Q_{\mathbf{w}, \mathbf{v}}, \quad \Pi_1 \rightarrow \bar{P}_{\mathbf{v}}, \quad \Pi_2 \rightarrow \bar{P}_{\mathbf{w}}, \quad F_{12} \rightarrow \alpha(\mathbf{v} \wedge \mathbf{w})B.$$

It would be interesting to classify Hilbert space representations of the algebra \mathbf{G}_d .

The following fact easily follows from (5.1)–(5.4).

Proposition 5.1: Assume Hypothesis $(A)_1$. Then the set $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ given by (1.14) is a representation of the CCR with two degrees of freedom if and only if \mathbf{A} is flat on \mathbf{M} .

We denote by $\mathcal{F}(\mathbf{D})$ the set of vector potentials \mathbf{A} that satisfy Hypothesis $(A)_1$ and flat on \mathbf{M} . The following example shows that the set $\mathcal{F}(\mathbf{D})$ is large to some extent.

Example 5.1: Let $a_n = a_{n1} + ia_{n2} \in \mathbf{C}$ ($n \in \mathbf{N}$) be the complex number corresponding to the vector \mathbf{a}_n and $D := \{a_n\}_{n=1}^\infty$. Without loss of generality, we can assume that $|a_1| \leq |a_2| \leq |a_3| \leq \dots$. By the assumption for \mathbf{D} , we have $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Let

$$P_n(z) = \frac{c_{n,1}}{z - a_n} + \frac{c_{n,2}}{(z - a_n)^2} + \dots + \frac{c_{n,k_n}}{(z - a_n)^{k_n}},$$

where $k_n \in \mathbf{N}$ and $c_{n,j} \in \mathbf{C}$ ($j=1, \dots, k_n$) are arbitrarily given constants. Then, by the Mittag–Leffler theorem, there exists a meromorphic function $f(z)$ on \mathbf{C} having the following properties: (i) f is holomorphic on $\mathbf{C} \setminus D$; (ii) the principal part of f at $z = a_n$ is given by $P_n(z)$. Let $\mathbf{A} = (A_1, A_2)$ with $A_j, j=1, 2$, given by (1.23). Then $A_j \in C^\infty(\mathbf{M})$. Moreover, the Cauchy–Riemann equation for f implies that \mathbf{A} is flat on \mathbf{M} and divergence-free:

$$\partial_x A_1 + \partial_y A_2 = 0, \quad \text{on } \mathbf{M}. \tag{5.5}$$

Hence $\mathbf{A} \in \mathcal{F}(\mathbf{D})$. ■

The special case,

$$\pi_{\mathbf{e}_1, \mathbf{e}_2}^{\mathbf{A}} = \{L^2(\mathbf{R}^2), C_0^2(\mathbf{M}), \{q_j, \bar{P}_j\}_{j=1}^2\} \quad [\mathbf{A} \in \mathcal{F}(\mathbf{D})],$$

of the above representation of CCR with \mathbf{D} being a finite discrete set has been analyzed in detail (Refs. 1, 2, and 17). Similar analyses can be made in the present general case.

We first consider irreducibility of the representation $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ with $\mathbf{A} \in \mathcal{F}(\mathbf{D})$. For this purpose, we introduce two kinds of commutants.

A weak commutant of the representation $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ may be defined by

$$(\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}})' := \{T \in \mathbf{B}(L^2(\mathbf{R}^2)) \mid (TS\Psi, \Phi)_2 = (T\Psi, S\Phi)_2, \quad \text{for all } \Psi, \Phi \in C_0^2(\mathbf{M}) \text{ and}$$

$$S = \{Q_{\mathbf{v}, \mathbf{w}}, Q_{\mathbf{w}, \mathbf{v}}, P_{\mathbf{v}}, P_{\mathbf{w}}\}. \tag{5.6}$$

We set

$$U_{\mathbf{v}, \mathbf{w}}(t) = e^{itQ_{\mathbf{v}, \mathbf{w}}}, \quad t \in \mathbf{R}. \tag{5.7}$$

Another commutant is associated with the operator algebra generated by

$$\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}} := \{U_{\mathbf{v},\mathbf{w}}(t), U_{\mathbf{w},\mathbf{v}}(t), T_{\mathbf{v}}^{\mathbf{A}}(t), T_{\mathbf{w}}^{\mathbf{A}}(t) | t \in \mathbf{R}\}, \tag{5.8}$$

i.e.,

$$(\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' := \{T \in \mathbf{B}(L^2(\mathbf{R}^2)) | TW = WT, \text{ for all } W \in \mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}\}. \tag{5.9}$$

It is easy to see that

$$(\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' \subset (\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})'. \tag{5.10}$$

Theorem 5.2: *Let $\mathbf{A} \in \mathscr{F}(\mathbf{D})$. Then $(\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' = (\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' = \mathbf{C}I$, where I denotes the identity operator on $L^2(\mathbf{R}^2)$.*

Proof: Since \mathbf{v} and \mathbf{w} are linearly independent, q_j (resp., P_j) can be written as a linear combination of $Q_{\mathbf{v},\mathbf{w}}$ and $Q_{\mathbf{w},\mathbf{v}}$ (resp., $P_{\mathbf{v}}$ and $P_{\mathbf{w}}$). Hence, we have $(\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' = (\pi_{\mathbf{e}_1, \mathbf{e}_2}^{\mathbf{A}})'$. On the other hand, in quite the same way as in the proof of Theorem 3.8 of Ref. 17, we can show that $(\pi_{\mathbf{e}_1, \mathbf{e}_2}^{\mathbf{A}})' = \mathbf{C}I$. This result and (5.10) imply $(\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}})' = \mathbf{C}I$. ■

We next consider commutation properties of the operators in $\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$. Using Theorem 3.2, we can show that, for all \mathbf{A} satisfying Hypothesis $(\mathbf{A})_0$,

$$U_{\mathbf{v},\mathbf{w}}(t)U_{\mathbf{w},\mathbf{v}}(s) = U_{\mathbf{w},\mathbf{v}}(s)U_{\mathbf{v},\mathbf{w}}(t), \tag{5.11}$$

$$U_{\mathbf{v},\mathbf{w}}(t)T_{\mathbf{w}}^{\mathbf{A}}(s) = T_{\mathbf{w}}^{\mathbf{A}}(s)U_{\mathbf{v},\mathbf{w}}(t), \quad U_{\mathbf{w},\mathbf{v}}(t)T_{\mathbf{v}}^{\mathbf{A}}(s) = T_{\mathbf{v}}^{\mathbf{A}}(s)U_{\mathbf{w},\mathbf{v}}(t), \tag{5.12}$$

$$U_{\mathbf{v},\mathbf{w}}(t)T_{\mathbf{v}}^{\mathbf{A}}(s) = e^{-its}T_{\mathbf{v}}^{\mathbf{A}}(s)U_{\mathbf{v},\mathbf{w}}(t), \quad U_{\mathbf{w},\mathbf{v}}(t)T_{\mathbf{w}}^{\mathbf{A}}(s) = e^{-its}T_{\mathbf{w}}^{\mathbf{A}}(s)U_{\mathbf{w},\mathbf{v}}(t). \tag{5.13}$$

A commutation relation between $T_{\mathbf{v}}^{\mathbf{A}}(s)$ and $T_{\mathbf{w}}^{\mathbf{A}}(t)$ is given by (3.13), with \mathbf{v} and \mathbf{w} replaced by $s\mathbf{v}$ and $t\mathbf{w}$, respectively.

Definition 5.3: We say that the magnetic flux is locally quantized (with respect to the pair $\{\mathbf{v}, \mathbf{w}\}$) if the function $\Phi_{s\mathbf{v}, t\mathbf{w}}^{\mathbf{A}}$ on $\mathbf{M}_{s\mathbf{v}, t\mathbf{w}}(\mathbf{D})$ is $2\pi\mathbf{Z}/\alpha$ valued for all $s, t \in \mathbf{R}$.

Remark 5.2: It is easy to see that, if $A_j \in C^1(\mathbf{M})$, $j = 1, 2$, and the magnetic flux is locally quantized, then \mathbf{A} is flat on \mathbf{M} .

Theorem 5.4: *For all $s, t \in \mathbf{R}$, the set $\mathscr{U}_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ of unitary operators satisfies the Weyl relations with two degrees of freedom if and only if the magnetic flux is locally quantized.*

Proof: It follows from Theorem 3.3 and (3.8) that $T_{\mathbf{v}}^{\mathbf{A}}(s)$ and $T_{\mathbf{w}}^{\mathbf{A}}(t)$ commute for all $s, t \in \mathbf{R}$ if and only if the magnetic flux is locally quantized. This fact and (5.11)–(5.13) imply the desired assertion. ■

In the rest of this section, we consider only the case $\mathbf{A} \in \mathscr{F}(\mathbf{D})$; hence $\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ is a representation of the CCR with two degrees of freedom.

Theorems 5.2 and 5.4 together with the von Neumann uniqueness theorem on the Weyl form of CCR^{18,19} give the following result.

Theorem 5.5: *The representation $\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ is unitarily equivalent to the Schrödinger representation $\{q_j, p_j\}_{j=1}^2$ if and only if the magnetic flux is locally quantized.*

Remark 5.3: As in the case of the special representation $\pi_{\mathbf{e}_1, \mathbf{e}_2}^{\mathbf{A}}$ discussed in the previous papers (Refs. 1, 2, and 5), the case where the representation $\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ is inequivalent to the Schrödinger representation, which, by Theorem 5.4, occurs if and only if the magnetic flux is not locally quantized, physically corresponds to the Aharonov–Bohm effect [see (3.13) and Remark 4.2(2)].

The function $\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ can be explicitly represented, as is shown below. By the assumption for \mathbf{D} , we have

$$\delta := \inf_{m \neq n} |\mathbf{a}_n - \mathbf{a}_m| > 0. \tag{5.14}$$

It follows from the flatness of \mathbf{A} and the Green’s theorem that, for all $\epsilon \in (0, \delta)$, the line integral,

$$\gamma_{\mathbf{A}}(\mathbf{a}_n) := \int_{|\mathbf{r} - \mathbf{a}_n| = \epsilon} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}, \tag{5.15}$$

along the circle $\{\epsilon(\cos \theta, \sin \theta) + \mathbf{a}_n \mid 0 \leq \theta \leq 2\pi\}$ is independent of ϵ . It is easy to see that

$$\Phi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}(\mathbf{r}) = \sum_{\mathbf{a}_n \in D_{\mathbf{r}}(\mathbf{v}, \mathbf{w})} \text{sgn}(\mathbf{v} \wedge \mathbf{w}) \gamma_{\mathbf{A}}(\mathbf{a}_n), \quad \mathbf{r} \in \mathbf{M}_{\mathbf{v}, \mathbf{w}}(\mathbf{D}), \tag{5.16}$$

where $D_{\mathbf{r}}(\mathbf{v}, \mathbf{w})$ is the interior domain of the curve $C_{\mathbf{r}}(\mathbf{v}, \mathbf{w})$ and $\text{sgn}(\lambda) := 1$ (resp., -1) for $\lambda > 0$ (resp., $\lambda < 0$). Thus we obtain the following result.

Proposition 5.6: The magnetic flux is locally quantized if and only if $\gamma_{\mathbf{A}}(\mathbf{a}_n) \in 2\pi\mathbf{Z}/\alpha$ for all $n \in \mathbf{N}$.

This proposition and Theorem 5.4 imply the following theorem.

Theorem 5.7: The representation $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ is unitarily equivalent to the Schrödinger representation $\{q_j, p_j\}_{j=1}^2$ if and only if $\gamma_{\mathbf{A}}(\mathbf{a}_n) \in 2\pi\mathbf{Z}/\alpha$ for all $n \in \mathbf{N}$.

Example 5.2: Let \mathbf{A} be the vector potential given in Example 5.1. Then

$$\gamma_{\mathbf{A}}(\mathbf{a}_n) = 2\pi \mathfrak{R}c_{n,1}.$$

Hence, if $\mathfrak{R}c_{n_0,1} \notin \mathbf{Z}/\alpha$ for some $n_0 \in \mathbf{N}$, then $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ is unitarily inequivalent to the Schrödinger representation $\{q_j, p_j\}_{j=1}^2$. Thus, there exist lots of inequivalent representations of CCR. ■

In concluding this section, we make a remark on the unitary inequivalence between the two representations $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ and $\pi_{\mathbf{v}', \mathbf{w}'}^{\mathbf{A}'}$ [$\mathbf{A}' \in \mathcal{F}(\mathbf{D})$ and $\mathbf{v}', \mathbf{w}' \in \mathbf{V}(\mathbf{D})$ are linearly independent]. We introduce a 2×2 matrix:

$$K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}') = \frac{1}{\mathbf{v}' \wedge \mathbf{w}'} \begin{pmatrix} v_1 w'_2 - w_1 v'_2 & w_1 v'_1 - v_1 w'_1 \\ v_2 w'_2 - w_2 v'_2 & w_2 v'_1 - v_2 w'_1 \end{pmatrix}. \tag{5.17}$$

It is easy to see that $K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}')$ is bijective, with

$$\det K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}') = \frac{\mathbf{v} \wedge \mathbf{w}}{\mathbf{v}' \wedge \mathbf{w}'}, \tag{5.18}$$

$$K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}') K(\mathbf{v}', \mathbf{w}', \mathbf{v}, \mathbf{w}) = I. \tag{5.19}$$

Proposition 5.8: Suppose that $\pi_{\mathbf{v}, \mathbf{w}}^{\mathbf{A}}$ is unitarily equivalent to $\pi_{\mathbf{v}', \mathbf{w}'}^{\mathbf{A}'}$. Then, for all $s, t \in \mathbf{R}$,

$$\exp(-i\alpha \Phi_{s\mathbf{v}, t\mathbf{w}}^{\mathbf{A}}(K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}')\mathbf{r})) = \exp(-i\alpha \Phi_{s\mathbf{v}', t\mathbf{w}'}^{\mathbf{A}'}(\mathbf{r})), \quad \text{a.e. } \mathbf{r} \in \mathbf{R}^2. \tag{5.20}$$

Proof: To make explicit the dependence of $P_{\mathbf{v}}$ on \mathbf{A} , we write $\bar{P}_{\mathbf{v}} = \bar{P}_{\mathbf{v}}(\mathbf{A})$. By the present assumption, there exists a unitary operator U on $L^2(\mathbf{R}^2)$, such that

$$U Q_{\mathbf{v}, \mathbf{w}} U^{-1} = Q_{\mathbf{v}', \mathbf{w}'}, \quad U Q_{\mathbf{w}, \mathbf{v}} U^{-1} = Q_{\mathbf{w}', \mathbf{v}'}, \tag{5.21}$$

$$U \bar{P}_{\mathbf{v}}(\mathbf{A}) U^{-1} = \bar{P}_{\mathbf{v}'}(\mathbf{A}'), \quad U \bar{P}_{\mathbf{w}}(\mathbf{A}) U^{-1} = \bar{P}_{\mathbf{w}'}(\mathbf{A}'). \tag{5.22}$$

By Theorem 3.3 and (5.22), together with functional calculus of self-adjoint operators, we have for all $s, t \in \mathbf{R}$,

$$\exp(-i\alpha U \Phi_{t\mathbf{v}, s\mathbf{w}}^{\mathbf{A}} U^{-1}) = \exp(-i\alpha \Phi_{t\mathbf{v}', s\mathbf{w}'}^{\mathbf{A}'}). \tag{5.23}$$

It is easy to show that

$$q_1 = v_1 Q_{\mathbf{v}, \mathbf{w}} + w_1 Q_{\mathbf{w}, \mathbf{v}}, \tag{5.24}$$

$$q_2 = v_2 Q_{\mathbf{v}, \mathbf{w}} + w_2 Q_{\mathbf{w}, \mathbf{v}}. \tag{5.25}$$

These relations and (5.21) imply that

$$U\mathbf{q}U^{-1} = K(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}')\mathbf{q}. \tag{5.26}$$

Hence, by functional calculus, the self-adjoint operator $U\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}U^{-1}$ is equal to the multiplication operator by the function $\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}(\mathbf{K}(\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}')\mathbf{r})$. This fact and (5.23) imply (5.20). ■

Remark 5.4: (1) There exist many triples $\{\mathbf{v}, \mathbf{w}, \mathbf{A}\}$ and $\{\mathbf{v}', \mathbf{w}', \mathbf{A}'\}$, where $\mathbf{A}, \mathbf{A}' \in \mathcal{F}(\mathbf{D})$, such that (5.20) does not hold. Hence, there exist many pairs $\{\pi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}, \pi_{\mathbf{v}',\mathbf{w}'}^{\mathbf{A}'}\}$ of representations that are unitarily inequivalent to each other.

(2) In the case $\mathbf{v}=\mathbf{v}'$, $\mathbf{w}=\mathbf{w}'$, we have

$$K(\mathbf{v}, \mathbf{w}, \mathbf{v}, \mathbf{w}) = I, \tag{5.27}$$

so that (5.20) becomes

$$\exp(-i\alpha\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}(\mathbf{r})) = \exp(-i\alpha\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}'}(\mathbf{r})), \quad \text{a.e. } \mathbf{r} \in \mathbf{R}^2. \tag{5.28}$$

VI. QUANTUM ALGEBRAIC STRUCTURES

In this section we discuss quantum algebraic structures associated with continuous magnetic translations $\{T_{\mathbf{v}}^{\mathbf{A}}\}_{\mathbf{v} \in \mathbf{V}(\mathbf{D})}$. We first introduce a special class of vector potentials.

Definition 6.1: We say that a vector potential \mathbf{A} is in $\mathcal{A}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$ if the function $\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ is equal to a constant on $\mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$.

Example 6.1: Constant magnetic fields. Let $B_0 \in \mathbf{R}$ be a constant and

$$A_1(\mathbf{r}) = -\frac{B_0 y}{2}, \quad A_2(\mathbf{r}) = \frac{B_0 x}{2}.$$

Then $B(\mathbf{r}) = B_0$, i.e., the magnetic field is uniformly constant. Hence, by the Green's theorem, we have

$$\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}(\mathbf{r}) = (\mathbf{v} \wedge \mathbf{w})B_0, \quad \mathbf{r} \in \mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D}).$$

Thus $\mathbf{A} \in \mathcal{A}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$. ■

Example 6.2: Vector potentials singular on an infinite lattice. Let $\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ be the infinite lattice given by (1.16) and consider the case $\mathbf{D} = \mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ and $\mathbf{v} = \boldsymbol{\omega}_1$, $\mathbf{w} = \boldsymbol{\omega}_2$. Let $\Omega_{m,n}$ be as in (1.22) and $f(z)$ be a meromorphic function on \mathbf{C} with the following properties: (i) f is holomorphic on $\mathbf{C} \setminus \{\Omega_{m,n}\}_{m,n \in \mathbf{Z}}$; (ii) the principal part of f at $z = \Omega_{m,n}$ is of the form

$$P_{m,n}(z) = \frac{c}{z - \Omega_{m,n}} + \frac{c_{m,n}^{(2)}}{(z - \Omega_{m,n})^2} + \dots + \frac{c_{m,n}^{(k_{m,n})}}{(z - \Omega_{m,n})^{k_{m,n}}},$$

where $k_{m,n} \in \mathbf{N}$, $c, c_{m,n}^{(j)}$ are constants (c is independent of m, n). The existence of such a function f is ensured by the Mittag-Leffler theorem (see Example 5.1). As in Example 5.1, we define a vector potential $\mathbf{A} = (A_1, A_2)$ by (1.23). Then $\mathbf{A} \in \mathcal{F}(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$ with A_j being infinitely many times differentiable on the open set,

$$\mathbf{M}_{\mathbf{L}} := \mathbf{R}^2 \setminus \mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2). \tag{6.1}$$

As in Example 5.2, we have

$$\gamma_{\mathbf{A}}(\Omega_{m,n}) = 2\pi \Re c, \quad m, n \in \mathbf{Z}. \tag{6.2}$$

For all $\mathbf{r} \in \mathbf{M}_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$, $D_{\mathbf{r}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ contains only one point in $\{\Omega_{m,n}\}_{m,n \in \mathbf{Z}}$. Hence, noting that $\text{sgn}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = 1$, we have, from (5.16) and (6.2),

$$\Phi_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}^{\mathbf{A}}(\mathbf{r}) = 2\pi \Re c. \tag{6.3}$$

Thus, $\mathbf{A} \in \mathcal{A}_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2))$. ■

We denote by $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}$ the algebra generated by $T_{\mathbf{v}}^{\mathbf{A}}$ and $T_{\mathbf{w}}^{\mathbf{A}}$.

Let $\mathbf{A} \in \mathcal{A}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$. Then, by definition, $\Phi_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ is a constant on $\mathbf{M}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$. We denote the constant by $C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$. By Theorem 3.3, we have

$$T_{\mathbf{v}}^{\mathbf{A}} T_{\mathbf{w}}^{\mathbf{A}} = e^{-i\alpha C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}} T_{\mathbf{w}}^{\mathbf{A}} T_{\mathbf{v}}^{\mathbf{A}}. \tag{6.4}$$

Hence $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}$ is a representation on $L^2(\mathbf{R}^2)$ of a rotation algebra (e.g., Ref. 20, Chap. VI) or the quantum plane with the deformation parameter $q = e^{-i\alpha C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}}$ (Ref. 21).

Proposition 6.2: *There exist no nontrivial finite-dimensional subspaces of $L^2(\mathbf{R}^2)$ left invariant by $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}$.*

Proof: This follows from Proposition 2.6. ■

Proposition 6.3: Let $\mathbf{A} \in \mathcal{A}_{\mathbf{v},\mathbf{w}}(\mathbf{D})$ and $\mathbf{A}' \in \mathcal{A}_{\mathbf{v}',\mathbf{w}'}(\mathbf{D})$. Suppose that $C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}} - C_{\mathbf{v}',\mathbf{w}'}^{\mathbf{A}'} \notin 2\pi\mathbf{Z}/\alpha$. Then $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}$ and $\mathcal{R}_{\mathbf{v}',\mathbf{w}',\mathbf{A}'}$ are unitarily inequivalent.

Proof: Suppose that there exists a unitary operator U on $L^2(\mathbf{R}^2)$ such that $UT_{\mathbf{v}}^{\mathbf{A}}U^{-1} = T_{\mathbf{v}'}^{\mathbf{A}'}$, $UT_{\mathbf{w}}^{\mathbf{A}}U^{-1} = T_{\mathbf{w}'}^{\mathbf{A}'}$. By (6.4) and the fact that $C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}$ and $C_{\mathbf{v}',\mathbf{w}'}^{\mathbf{A}'}$ are constants, we have $\exp(-i\alpha C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}) = \exp(-i\alpha C_{\mathbf{v}',\mathbf{w}'}^{\mathbf{A}'})$, which is equivalent to that $C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}} - C_{\mathbf{v}',\mathbf{w}'}^{\mathbf{A}'} \in 2\pi\mathbf{Z}/\alpha$. Thus, the desired assertion follows. ■

Example 6.3: Let \mathbf{A} be the vector potential given in Example 6.2. Then, by (6.3), we have

$$T_{\omega_1}^{\mathbf{A}} T_{\omega_2}^{\mathbf{A}} = e^{-2\pi i \alpha \Re c} T_{\omega_2}^{\mathbf{A}} T_{\omega_1}^{\mathbf{A}}. \tag{6.5}$$

Hence we have a one-parameter family $\mathcal{R}_c := \mathcal{R}_{\omega_1, \omega_2, \mathbf{A}}$ of representations [on $L^2(\mathbf{R}^2)$] of rotation algebras (quantum planes). It follows from Proposition 6.3 that, if $\Re(c - c') \notin \mathbf{Z}/\alpha$, then the two representations \mathcal{R}_c and $\mathcal{R}_{c'}$ are unitarily inequivalent to each other. Thus, associated with singular magnetic fields concentrated on $\mathbf{L}(\omega_1, \omega_2)$, there exist infinitely many representations on $L^2(\mathbf{R}^2)$ of rotation algebras inequivalent to each other. ■

Remark 6.1: The same consideration as above applies to the algebra $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}^*$ generated by $(T_{\mathbf{v}}^{\mathbf{A}})^*$ and $(T_{\mathbf{w}}^{\mathbf{A}})^*$, which is a representation of a rotation algebra. We can also consider the algebra generated by $T_{\mathbf{v}}^{\mathbf{A}}$, $T_{\mathbf{w}}^{\mathbf{A}}$, $(T_{\mathbf{v}}^{\mathbf{A}})^*$ and $(T_{\mathbf{w}}^{\mathbf{A}})^*$, which is a $*$ -subalgebra of $\mathbf{B}(L^2(\mathbf{R}^2))$.

As is shown in Ref. 5, for any representation of a quantum plane where the generators are represented as bijections, one can construct a representation of the quantum group $U_q(sl_2)$. We can apply this method to the present case to construct, from $\mathcal{R}_{\mathbf{v},\mathbf{w},\mathbf{A}}$, representations of $U_q(sl_2)$ on $L^2(\mathbf{R}^2)$ with $q = e^{-i\alpha C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}}$ or $q = e^{-i\alpha C_{\mathbf{v},\mathbf{w}}^{\mathbf{A}}/2}$ and analyze them in quite the same way as in Ref. 5. Here we only mention a basic feature of those representations: they have no weight vectors and hence no nontrivial finite-dimensional reductions (cf. Proposition 6.2). Note that this makes a big difference from the case of representations of $U_q(sl_2)$ constructed in terms of discrete magnetic translations on a lattice (Ref. 12), where finite-dimensional representations of $U_q(sl_2)$ appear (cf. also Ref. 13).

VII. REDUCTION TO LATTICE QUANTUM SYSTEMS

In this section we focus our attention on the case $\mathbf{D} = \mathbf{L}(\omega_1, \omega_2)$ and consider the problem of reduction of the continuous magnetic translations,

$$T_j := T_{\omega_j}^{\mathbf{A}}, \quad j = 1, 2, \tag{7.1}$$

to the closed subspace $L^2_{\omega_1, \omega_2}(\mathbf{R}^2)$, which is given by (1.19) and naturally identified with the Hilbert space $l^2(\mathbf{L}(\omega_1, \omega_2))$.

A. General aspects

We denote by $\mathbf{S}_n^{(1)}$ (resp., $\mathbf{S}_m^{(2)}$) the open set between the straight lines $\{n\omega_2 + s\omega_1 | s \in \mathbf{R}\}$ (resp., $\{m\omega_1 + s\omega_2 | s \in \mathbf{R}\}$) and $\{(n+1)\omega_2 + s\omega_1 | s \in \mathbf{R}\}$ (resp., $\{(m+1)\omega_1 + s\omega_2 | s \in \mathbf{R}\}$).

Let

$$\phi_j^{\mathbf{A}}(\mathbf{r}) := \int_{\mathbf{r}}^{\mathbf{r} + \omega_j} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}', \quad j = 1, 2. \tag{7.2}$$

Definition 7.1: We say that \mathbf{A} is in the class \mathcal{E}_j ($j=1,2$) if, for each $n \in \mathbf{Z}$, $\phi_j^{\mathbf{A}}$ is constant on $\mathbf{S}_n^{(j)}$, i.e., there exists a constant $c_j(n) \in \mathbf{R}$, such that

$$\phi_j^{\mathbf{A}}(\mathbf{r}) = c_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}, \quad n \in \mathbf{Z}. \tag{7.3}$$

Remark 7.1: It is easy to see that, if $\mathbf{A} \in \mathcal{E}_j$, then the function: $\mathbf{r} \rightarrow \mathbf{A}(\mathbf{r}) \cdot \boldsymbol{\omega}_j$ is periodic on $\cup_{n \in \mathbf{Z}} \mathbf{S}_n^{(j)}$ with period $\boldsymbol{\omega}_j$. But the converse is not true.

Proposition 7.2: For each $j=1,2$, T_j and T_j^{-1} leave $L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)$ invariant if and only if $\mathbf{A} \in \mathcal{E}_j$.

Proof: By Theorem 3.2, we have, for all $\Psi \in L^2(\mathbf{R}^2)$,

$$(T_j \Psi)(\mathbf{r}) = e^{-i\alpha \phi_j^{\mathbf{A}}(\mathbf{r})} (e^{i\bar{p}} \boldsymbol{\omega}_j \Psi)(\mathbf{r}), \quad \text{a.e. } \mathbf{r} \in \mathbf{S}_n^{(j)}.$$

Note that $e^{\pm i\bar{p}\boldsymbol{\omega}_j}$ leave $L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)$ invariant. Hence, T_j leaves $L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)$ invariant if and only if $\phi_j^{\mathbf{A}}(r)$ is constant on $\mathbf{S}_{m,n} = \mathbf{S}_n^{(1)} \cap \mathbf{S}_m^{(2)}$ for all $m, n \in \mathbf{Z}$. Since $\mathbf{S}_n^{(j)}$ is connected and $\phi_j^{\mathbf{A}}$ is continuous on $\mathbf{S}_n^{(j)}$, it follows that $\phi_j^{\mathbf{A}}(\mathbf{r})$ is constant on $\mathbf{S}_{m,n}$ for all $m, n \in \mathbf{Z}$ if and only if it is constant on $\mathbf{S}_n^{(j)}$ for all $n \in \mathbf{Z}$. Thus, we obtain the desired result. ■

We set

$$\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2. \tag{7.4}$$

As a corollary of Proposition 7.2, we have the following.

Theorem 7.3: The four unitary operators T_1, T_1^{-1}, T_2 and T_2^{-1} leave $L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)$ invariant if and only if $\mathbf{A} \in \mathcal{E}$.

By this theorem, if $\mathbf{A} \in \mathcal{E}$, then T_1, T_1^{-1}, T_2 and T_2^{-1} are reduced by $L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)$, and the restrictions

$$\hat{T}_j := T_j|_{L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)}, \quad \widehat{T}_j^{-1} := T_j^{-1}|_{L^2_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}(\mathbf{R}^2)}, \quad j=1,2, \tag{7.5}$$

induce magnetic translations on the lattice $\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ with $(\hat{T}_j)^{-1} = \widehat{T}_j^{-1}$. But, if these reduced magnetic translations have the trivial holonomy, i.e., $\widehat{T}_1 \widehat{T}_2 \widehat{T}_1^{-1} \widehat{T}_2^{-1} = I$, then they are uninteresting. Thus, we need to find conditions for the magnetic translations to have a nontrivial holonomy. The following proposition gives a necessary condition for that.

Proposition 7.4: Let $\mathbf{A} \in \mathcal{E}$. Suppose that A_1 and A_2 are continuous on \mathbf{R}^2 . Then $T_1 T_2 = T_2 T_1$.

Proof: By the condition $\mathbf{A} \in \mathcal{E}$, we have, for all $\mathbf{r} \in \mathbf{S}_{m,n}$,

$$\Phi_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}^{\mathbf{A}}(\mathbf{r}) = \phi_1^{\mathbf{A}}(\mathbf{r}) + \phi_2^{\mathbf{A}}(\mathbf{r} + \boldsymbol{\omega}_1) - \phi_1^{\mathbf{A}}(\mathbf{r} + \boldsymbol{\omega}_2) - \phi_2^{\mathbf{A}}(\mathbf{r}) = c_1(n) + c_2(m+1) - c_1(n+1) - c_2(m).$$

Since A_j is continuous on \mathbf{R}^2 , so is $\phi_j^{\mathbf{A}}$. This implies that $c_j(n) = c_j(n+1)$ for all $n \in \mathbf{Z}$ and $j=1,2$. Hence $\Phi_{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2}^{\mathbf{A}} = 0$. By this fact and Theorem 3.3 we obtain the desired result. ■

Proposition 7.4 shows that, in the case $\mathbf{A} \in \mathcal{E}$, for the magnetic translations $\hat{T}_j, \widehat{T}_j^{-1}, j=1,2$, to have a nontrivial holonomy, it is necessary for \mathbf{A} to have singularities in $\mathbf{L}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$.

We next show that, for a given vector potential \mathbf{A} satisfying certain properties, we can construct an element in \mathcal{E} .

Definition 7.5: We say that \mathbf{A} is in $\mathcal{D}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ if there exist real sequences $\{v_j(n)\}_{n \in \mathbf{Z}}, j=1,2$, and real-valued continuous functions F_j on $\cup_{n \in \mathbf{Z}} \mathbf{S}_n^{(j)}$, such that

$$\phi_j^{\mathbf{A}}(\mathbf{r}) + \int_0^1 F_j(\mathbf{r} + s\boldsymbol{\omega}_j) ds = v_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}, \quad j=1,2. \tag{7.6}$$

It is obvious that $\mathcal{E} \subset \mathcal{D}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ [note that every element $\mathbf{A} \in \mathcal{E}$ satisfies (7.6) with $F_j=0, j=1,2$].

The matrix

$$W := \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \tag{7.7}$$

is regular, since ω_1 and ω_2 are linearly independent.

Proposition 7.6: Suppose that $\mathbf{A} \in \mathcal{D}(\omega_1, \omega_2)$ and let $\nu_j(n), F_j$ be as in Definition 7.5. Let

$$\mathbf{F} := (F_1, F_2), \tag{7.8}$$

$$\tilde{\mathbf{A}} := \mathbf{A} + W^{-1}\mathbf{F}. \tag{7.9}$$

Then $\tilde{\mathbf{A}} \in \mathcal{C}$ with

$$\phi_j^{\tilde{\mathbf{A}}}(\mathbf{r}) = \nu_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}, \quad j = 1, 2. \tag{7.10}$$

Proof: We have, for all $\mathbf{r} \in \mathbf{S}_n^{(j)}$,

$$\phi_j^{\tilde{\mathbf{A}}}(\mathbf{r}) = \phi_j^{\mathbf{A}}(\mathbf{r}) + \int_0^1 W^{-1}\mathbf{F}(\mathbf{r} + s\omega_j) \cdot \omega_j ds.$$

Since $(W^{-1})\omega_j = \mathbf{e}_j$, we have $W^{-1}\mathbf{F}(\mathbf{r} + s\omega_j) \cdot \omega_j = F_j(\mathbf{r} + s\omega_j)$. Hence, (7.10) follows. ■

Remark 7.2: The modified vector potential $\tilde{\mathbf{A}}$ is not necessarily flat on $\mathbf{M}_{\mathbf{L}}$, even if \mathbf{A} is flat on $\mathbf{M}_{\mathbf{L}}$. See the next section.

B. Construction of a subset of \mathcal{C} from quasiperiodic functions

Definition 7.7: Let $f(z)$ be a holomorphic function on $\mathbf{C} \setminus \{\Omega_{m,n}\}_{m,n \in \mathbf{Z}}$ with possible poles at $z = \Omega_{m,n}, m, n \in \mathbf{Z}$. We say that f is in the class \mathcal{C} if there exist constants $\xi_j := \xi_{j1} + i\xi_{j2} \in \mathbf{C}, j = 1, 2$, such that

$$f(z + \omega_j) = f(z) + \xi_j, \quad j = 1, 2, \quad z \in \mathbf{C} \setminus \{\Omega_{m,n}\}_{m,n \in \mathbf{Z}}. \tag{7.11}$$

Remark 7.3: Every function $f \in \mathcal{C}$ is a primitive function of an elliptic function with periods $\omega_j, j = 1, 2$. The constant ξ_j on the rhs of (7.11) is given by, e.g.,

$$\xi_j = f\left(\frac{\omega_j}{2}\right) - f\left(-\frac{\omega_j}{2}\right).$$

Let $f \in \mathcal{C}$ and $A_j, j = 1, 2$, be defined from f by (1.23). Set $\mathbf{A} = (A_1, A_2)$. Then the Cauchy–Riemann equation implies that \mathbf{A} is flat on $\mathbf{M}_{\mathbf{L}}$ and

$$\partial_x A_1 + \partial_y A_2 = 0, \quad \text{on } \mathbf{M}_{\mathbf{L}}. \tag{7.12}$$

Let ξ_j be as in (7.11) and

$$\xi_j := (\xi_{j1}, \xi_{j2}), \quad \xi'_j := (\xi_{j2}, \xi_{j1}), \quad j = 1, 2. \tag{7.13}$$

Proposition 7.8: For all $n \in \mathbf{Z}$, there exist constants $c_j(n), j = 1, 2$, such that

$$\phi_j^{\mathbf{A}}(\mathbf{r}) = \mathbf{r} \cdot \xi'_j + c_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}. \tag{7.14}$$

Proof: Let ω_j be as in (1.21) and $\theta_j := \arg \omega_j$. Let $R(\theta)$ be defined by (2.10) and let

$$g_j(\mathbf{r}) := A(R(\theta_j)\mathbf{r}) \cdot \omega_j, \quad \phi'_j(\mathbf{r}) := \phi_j^{\mathbf{A}}(R(\theta_j)\mathbf{r}).$$

Then we have

$$\phi'_j(\mathbf{r}) = \int_0^1 g_j(\mathbf{r} + s\omega'_j) ds,$$

where $\boldsymbol{\omega}'_j = |\omega_j| \mathbf{e}_1$. On the connected domain $R(-\theta_j)\mathbf{S}_n^{(j)}$, $g_j(\cdot + s\boldsymbol{\omega}'_j)$, and ϕ'_j are infinitely many times differentiable with

$$\partial_x \phi'_j(\mathbf{r}) = \int_0^1 \partial_x g_j(\mathbf{r} + s\boldsymbol{\omega}'_j) ds, \quad \mathbf{r} \in R(-\theta_j)\mathbf{S}_n^{(j)}.$$

It is easy to see that

$$\partial_x g_j(\mathbf{r} + s\boldsymbol{\omega}'_j) = |\omega_j|^{-1} \frac{d}{ds} g_j(\mathbf{r} + s\boldsymbol{\omega}'_j).$$

Hence

$$\partial_x \phi'_j(\mathbf{r}) = \frac{g_j(\mathbf{r} + \boldsymbol{\omega}'_j) - g_j(\mathbf{r})}{|\omega_j|}.$$

On the other hand, we have, by (7.11) and (1.23),

$$\mathbf{A}(\mathbf{r} + \boldsymbol{\omega}_j) - \mathbf{A}(\mathbf{r}) = \boldsymbol{\xi}'_j, \quad \mathbf{r} \in \mathbf{M}_L. \tag{7.15}$$

Thus, we obtain

$$\partial_x \phi'_j(\mathbf{r}) = \frac{\boldsymbol{\omega}_j \cdot \boldsymbol{\xi}'_j}{|\omega_j|}, \quad \mathbf{r} \in U_{n \in \mathbf{Z}} R(-\theta_j)\mathbf{S}_n^{(j)}. \tag{7.16}$$

By the flatness of \mathbf{A} on \mathbf{M}_L and (7.12), we can show that

$$\partial_y g_j(\mathbf{r}) = \partial_x \boldsymbol{\omega}_j \wedge \mathbf{A}(\mathbf{R}(\theta_j)\mathbf{r}).$$

Hence, in the same way as above, we obtain

$$\partial_y \phi'_j(\mathbf{r}) = \frac{\boldsymbol{\omega}_j \wedge \boldsymbol{\xi}'_j}{|\omega_j|}, \quad \mathbf{r} \in U_{n \in \mathbf{Z}} R(-\theta_j)\mathbf{S}_n^{(j)}. \tag{7.17}$$

It follows from (7.16) and (7.17) that there exist constants $c_j(n)$, $j = 1, 2$, $n \in \mathbf{Z}$, such that

$$\phi_j^{\mathbf{A}}(\mathbf{r}) = \left[R(\theta_j) \left(\frac{\boldsymbol{\omega}_j \cdot \boldsymbol{\xi}'_j}{|\omega_j|}, \frac{\boldsymbol{\omega}_j \wedge \boldsymbol{\xi}'_j}{|\omega_j|} \right) \right] \cdot \mathbf{r} + c_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}.$$

The rhs of this equation is equal to the rhs of (7.14). ■

Theorem 7.9: *Let*

$$h_j(\mathbf{r}) := \left(\frac{\boldsymbol{\omega}_j}{2} - \mathbf{r} \right) \cdot \boldsymbol{\xi}'_j, \quad \mathbf{r} \in \mathbf{R}^2, \quad j = 1, 2, \quad \mathbf{h} = (h_1, h_2), \tag{7.18}$$

and

$$\tilde{\mathbf{A}} := \mathbf{A} + W^{-1}\mathbf{h}. \tag{7.19}$$

Then $\tilde{\mathbf{A}}$ is in \mathcal{C} with

$$\phi_j^{\tilde{\mathbf{A}}}(\mathbf{r}) = c_j(n), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}, \quad j = 1, 2, \quad n \in \mathbf{Z}. \tag{7.20}$$

Moreover,

$$(T_{\tilde{\mathbf{A}}}^{\tilde{\mathbf{A}}})(\mathbf{r}) = e^{-iac_j(n)} \Psi(\mathbf{r} + \boldsymbol{\omega}_j), \quad \mathbf{r} \in \mathbf{S}_n^{(j)}, \quad j = 1, 2, \quad n \in \mathbf{Z}. \tag{7.21}$$

Proof: By Proposition 7.8, (7.6) holds with $F_j = h_j$ and $v_j(n) = c_j(n)$. Hence, by Proposition 7.6, the desired assertion follows. Formula (7.21) follows from (3.5) and (7.20). ■

Theorem 7.9 establishes the existence of a mapping from \mathcal{E} to a subset of \mathcal{E} by the correspondence $f \rightarrow \tilde{\mathbf{A}}$ defined by (7.19).

It is interesting to see when $\tilde{\mathbf{A}}$ is flat on $\mathbf{M}_{\mathbf{L}}$.

Lemma 7.10: The vector potential $\tilde{\mathbf{A}}$ given by (7.19) is flat on $\mathbf{M}_{\mathbf{L}}$ if and only if

$$\xi'_1 \cdot \omega_2 = \xi'_2 \cdot \omega_1. \tag{7.22}$$

Proof: The magnetic field,

$$\tilde{B}(\mathbf{r}) := D_x \tilde{A}_2 - D_y \tilde{A}_1, \tag{7.23}$$

of the vector potential $\tilde{\mathbf{A}}$ is constant, with

$$\tilde{B}(\mathbf{r}) = \frac{\xi'_1 \cdot \omega_2 - \xi'_2 \cdot \omega_1}{\det W}, \quad \mathbf{r} \in \mathbf{M}_{\mathbf{L}}. \tag{7.24}$$

Thus, the desired assertion follows. ■

We note the following fact.

Lemma 7.11: Let r_f be the residue of f at $z = \Omega_{m,n}$. Then r_f is independent of $m, n \in \mathbf{Z}$ and obeys the relation

$$\xi_1 \omega_2 - \xi_2 \omega_1 = 2\pi i r_f. \tag{7.25}$$

Proof: Let $a = (m - \frac{1}{2})\omega_1 + (n - \frac{1}{2})\omega_2$ and $C_{m,n}$ be the contour formed by the edges of the cell whose corners are $a, a + \omega_1, a + \omega_1 + \omega_2, a + \omega_2$, where the orientation of $C_{m,n}$ is taken to be anticlockwise. Then we have

$$\begin{aligned} 2\pi i r_f &= \int_{C_{m,n}} f(z) dz = \int_a^{a+\omega_1} \{f(z) - f(z + \omega_2)\} dz - \int_a^{a+\omega_2} \{f(z) - f(z + \omega_1)\} dz \\ &= -\xi_2 \omega_1 + \xi_1 \omega_2, \end{aligned}$$

where we have used the quasiperiodicity (7.11) of f . Thus (7.25) is obtained. ■

The following proposition gives a complete characterization on the flatness of $\tilde{\mathbf{A}}$ in terms of only f .

Proposition 7.12: The vector potential $\tilde{\mathbf{A}}$ given by (7.19) is flat on $\mathbf{M}_{\mathbf{L}}$ if and only if $\Re r_f = 0$.

Proof: We note that

$$\xi'_1 \cdot \omega_2 - \xi'_2 \cdot \omega_1 = \Im(\xi_1 \omega_2 - \xi_2 \omega_1).$$

By Lemma 7.11, the rhs is equal to $2\pi \Re r_f$. From this fact and Lemma 7.10, the desired result follows. ■

Remark 7.4: Let $f \in \mathcal{E}$ and $\Re r_f = 0$. Then, by Proposition 7.12, $\tilde{\mathbf{A}}$ is flat on $\mathbf{M}_{\mathbf{L}}$. We have $\Phi_{\omega_1, \omega_2}^{\tilde{\mathbf{A}}} = 2\pi \Re r_f = 0$ and

$$\Phi_{\omega_1, \omega_2}^{\tilde{\mathbf{A}}} = \Phi_{\omega_1, \omega_2}^{\mathbf{A}} + \Phi_{\omega_1, \omega_2}^{W^{-1}\mathbf{h}} = 0.$$

Hence, in this case, the representations $\pi_{\omega_1, \omega_2}^{\mathbf{A}}$ and $\pi_{\omega_1, \omega_2}^{\tilde{\mathbf{A}}}$ of CCR are unitarily equivalent to the Schrödinger representation and the algebras $\mathcal{R}_{\omega_1, \omega_2, \mathbf{A}}$ and $\mathcal{R}_{\omega_1, \omega_2, \tilde{\mathbf{A}}}$ are commutative. Thus, for $f \in \mathcal{E}$, the case $\Re r_f = 0$ is uninteresting and only the case where $\tilde{\mathbf{A}}$ is not flat on $\mathbf{M}_{\mathbf{L}}$ may be interesting.

Example 7.1: Consider the case where $f(z)$ is the Weierstrass Zeta function:

$$f(z) = \zeta(z) := \frac{1}{z} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \left\{ \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right\}. \tag{7.26}$$

It is well known (e.g., Ref. 22, Chap. XX, pp. 20–41) that

$$\zeta(z + \omega_j) = \zeta(z) + 2\eta_j, \quad j = 1, 2, \tag{7.27}$$

where

$$\eta_j = \zeta(\omega_j/2). \tag{7.28}$$

Hence, $\zeta \in \mathcal{E}$. Since $r_\zeta = 1$, Lemma 7.11 implies that, in the present case, $\tilde{\mathbf{A}}$ is not flat on \mathbf{M}_L . By Lemma 7.11 we have

$$\eta_1 \omega_2 - \eta_2 \omega_1 = i\pi.$$

(This is a well-known formula: e.g., Ref. 22, Chap. XX, 20·411, pp. 446–447.) Hence

$$\eta'_1 \cdot \omega_2 - \eta'_2 \cdot \omega_1 = \pi.$$

Thus, in the present example,

$$\tilde{B}(\mathbf{r}) = \frac{2\pi}{\det W} \neq 0, \quad \mathbf{r} \in \mathbf{M}_L. \quad \blacksquare$$

C. Derivation of Hamiltonians of the Hofstadter type from continuous systems

It is well known that some transport phenomena in two-dimensional solids can be modeled in part by Hamiltonians of the Hofstadter type, which are usually defined on two-dimensional infinite lattices (e.g., Ref. 13). It may be interesting to investigate if Hamiltonians of the Hofstadter type can be obtained as reductions of self-adjoint Hamiltonians of continuous quantum systems whose Hilbert spaces of state vectors are equal to $L^2(\mathbf{R}^2)$. For that purpose, we introduce a family $\{H^{\mathbf{A}}(t)\}_{t \in \mathbf{R}}$ of bounded self-adjoint operators,

$$\begin{aligned} H^{\mathbf{A}}(t) := & T_{\omega_1}^{\mathbf{A}}(t) + \mu T_{\omega_2}^{\mathbf{A}}(t) + \epsilon [T_{\omega_1}^{\mathbf{A}}(t)^2 + T_{\omega_2}^{\mathbf{A}}(t)^2] + \lambda T_{\omega_1}^{\mathbf{A}}(t) T_{\omega_2}^{\mathbf{A}}(t) \\ & + \nu T_{\omega_1}^{\mathbf{A}}(t) T_{\omega_2}^{\mathbf{A}}(t)^* + \gamma T_{\omega_1}^{\mathbf{A}}(t)^* T_{\omega_2}^{\mathbf{A}}(t) + \text{h.c.}, \end{aligned} \tag{7.29}$$

where $\mu, \epsilon, \lambda, \nu, \gamma$ are complex parameters and h.c. means the Hermitian conjugate. The operator $H^{\mathbf{A}}(t)$ is a continuous version of Hamiltonians of the Hofstadter type. By Theorem 7.3, if $\mathbf{A} \in \mathcal{E}$, then

$$H^{\mathbf{A}} := H^{\mathbf{A}}(1) = T_1 + \mu T_2 + \epsilon(T_1^2 + T_2^2) + \lambda T_1 T_2 + \nu T_1 T_2^* + \gamma T_1^* T_2 + \text{h.c.}, \tag{7.30}$$

is reduced by $L^2_{\omega_1, \omega_2}(\mathbf{R}^2) \cong l^2(\mathbf{L}(\omega_1, \omega_2))$ and its reduced part yields a Hamiltonian of the Hofstadter type on the lattice $\mathbf{L}(\omega_1, \omega_2)$. This shows that, in the case $\mathbf{A} \in \mathcal{E}$, it is possible to obtain Hamiltonians of the Hofstadter type on $\mathbf{L}(\omega_1, \omega_2)$ as reductions of Hamiltonians of the type $H^{\mathbf{A}}$. By Theorem 7.9, there exists a wide class of vector potentials \mathbf{A} that give this kind of reduction for the Hamiltonian $H^{\mathbf{A}}$. We also remark that the reduction of $H^{\mathbf{A}}$ with $\mathbf{A} \in \mathcal{E}$ to the subspace $l^2(\mathbf{L}(\omega_1, \omega_2))$ makes it possible to identify the spectrum of $H^{\mathbf{A}}$ in part by analyzing the spectra of Hamiltonians of the Hofstadter type on the lattice $\mathbf{L}(\omega_1, \omega_2)$ that may have interesting structures such as fractal ones (Ref. 13).

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