Proceedings of the 31st Sapporo Symposium on Partial Differential Equations

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T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,
Y. Tonegawa, K. Tsutaya, and T. Sakajo

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 2 through August 4 in 2006 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broard spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 30 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,
Y. Tonegawa, K. Tsutaya, and T. Sakajo

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（第３１回偏微分方程式論札幌シンポジウム）

代表者: 小澤 徹, 儀我 美一, 神保 秀一, 中村 玄, 利根川 吉廣, 津田谷 公利, 坂上 貴之
Organizers: T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura, Y. Tonegawa, K. Tsutaya, T. Sakajo

1. Period (期間) August 2 , 2006 - August 4 , 2006
2. Venue (場所) **Room 203, Faculty of Science Building #5, Hokkaido University**
   北海道大学 理学部 5 号館大講義室 (203 号室)
3. Program（プログラム）

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Low order quadrilateral finite elements for Reissner-Mindlin plate model

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Abstract

Reissner-Mindlin plate model is one of the most commonly used models of a moderately thick to thin plate. However, a direct and seemingly reasonable finite element discretization usually yields very poor results which is referred to LOCKING phenomenon. In the past two decades, many efforts have been devoted to the design of locking free finite elements to resolve this model, most of these work focus on triangular and rectangular elements, the latter may be extended to parallelograms, but very few on quadrilaterals.

In this talk we will give an overview of the recent development of low order quadrilateral elements for Reissner-Mindlin plate model and present our new results.

Numerical computation of solitons in nonlinear optics

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Abstract.— We present some numerical methods in order to compute spatially localized stationnary states of nonlinear scalar equations or coupled systems, referred in the literature as *solitons*. We first make use of the well-known shooting method in order to find k^{th} excited states for the classical nonlinear Schrödinger equation. It is then possible to derive some asymptotics power laws of the L^∞ norm of theses states in terms of k or σ. We finally compute solitons for a nonlinear system governing the propagation of two coupled waves in a quadratic media, starting from one-dimensional states obtained with the above shooting method and considering the dimension as a continuation parameter.

1 Introduction

Solitons have been intensively studied both theoretically and experimentally, as spatial structures propagating in a nonlinear media with shape invariance. As a possible model investigated for the description of wave propagation relevant in nonlinear optics as well as plasmas physics or quantum mechanics, we first consider the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + \alpha|\psi|^{2\sigma}\psi = 0, \;\; \psi = \psi(t,x) \tag{1}$$

where $x \in \mathbb{R}^d$ is the transverse coordinate, $t \in \mathbb{R}^+$ is the propagation distance normalized to the diffraction length, $\alpha \in \mathbb{R}$, $\sigma > 0$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ (see [11]). Seeking solutions that express as $\psi(t,x) = u(||x||)\exp(i\omega t)$, where $\omega \in \mathbb{R}$ and u is assumed to be spatially localized (say, u goes fastly to zero as $r := ||x||$ goes to infinity), (1) simply reduces to

$$-\omega u + \Delta u + \alpha|u|^{2\sigma}u = 0, \;\; x \in \mathbb{R}^d. \tag{2}$$

In the one-dimensional case $d = 1$, such a nonlinear elliptic problem can be explicitely integrated for $\omega > 0$ and $\alpha > 0$ to find sech-profiles and only one nonegative localized state (the *ground state*) can be obtained. For higher space dimensions, the situation becomes completely different: for the same assumptions made on ω and α, equation (2) admits lots of localized radial solutions for reasonable values of σ (namely, $\sigma \in \mathbb{R}^+$ if $d \leq 2$ and $\sigma < \sigma^* = 2/(d-2)$ if $d \geq 3$) that can be parametrized by the number k of *nodes* often referred as *bound states* or *excited states* (for the sake of correspondence with quantum mechanics context; the solution obtained for $k = 0$ is still referred as the ground state); see for example [7]. Since these solutions are not explicitly known, a numerical procedure should be used for their computation.

This question still makes sense for more general models have been investigated for the concern of propagation in media where the susceptibility tensor is quadratic in tems of input fields. A typical example is given by the so-called type I polarization, governed by the system

$$\left\{ \begin{array}{l} i\dfrac{\partial u}{\partial z} + \Delta u - u + \bar{u}v = 0 \\ i\sigma\dfrac{\partial v}{\partial z} + \Delta v - \rho v + \dfrac{1}{2}u^2 = 0, \end{array} \right. \tag{3}$$

up to various rescalings in longitudinal and transverse domains (see [1] for a review of quadratic models in nonlinear optics). This model describes the propagation of the fundamental wave u of frequency ω and the first harmonic wave v characterized by the frequency 2ω. The numerical determination of solitons is also presented for any spatial dimensional case.

2 Solitons for NLS equation

2.1 Computation of solitons

We first seek solutions of (1) that write $\psi(t,x) = e^{i\omega t}u(\|x\|)$ (with $\omega > 0$), leading us to the elliptic equation (2) that will be seen later as a differential second-order equation. This problem has been intensively studied and it has been proved that in the case $\alpha > 0$ and $\omega > 0$, H^1 solutions exist only when $\sigma < \sigma^*$ (which means for every $\sigma > 0$ if $d \leq 2$). First, a simple rescaling argument shows that it is always possible to assume that $\omega = 1$: if u denotes a solution of (2), then for each $\lambda > 0$, the new function $\tilde{u}(x) = \lambda^{1/\sigma}u(\lambda x)$ solves (2) with $\tilde{\omega} = \omega\lambda^{-2/\sigma}$. Taking $\lambda = \omega^{\sigma/2}$ thus leads to the correctly rescaled equation. Furthermore, looking for a C^2 solution at $r = 0$ requires to set $u'(0) = 0$ in order to avoid an angular point at the origin. Setting now $\beta := u(0)$, we thus investigate the differential system which expresses as

$$\left\{ \begin{array}{l} u''(r) + \dfrac{d-1}{r}u'(r) - u(r) + \alpha|u(r)|^{2\sigma}u(r) = 0, \quad r > 0 \\ u(0) = u_0, \quad u'(0) = 0 \end{array} \right. \tag{4}$$

and we look for globally defined solution of (4) that tends to zero as $r \to \infty$. First, this system can be rewritten as a general first-order system of the form $X'(r) = F(r, X(r))$ with $X(r) = (u(r), u'(r)) \in \mathbb{R}^2$. Here, the function F may be delicate to evaluate at $r = 0$ because of the term $u'(r)/r$. However, a Taylor expansion around $r = 0$ of the derivative leads us to $u'(r)/r \sim u''(0)$ as $r \to 0$. Consequently, we have $u''(r) + (d-1)u'(r)/r \sim d\,u''(0)$ and the second derivative of u at zero can be expressed using (4).

We now present the numerical algorithm in order to find the ground state noted u_0 of (1), that is with no node. This algorithm is based on the fact that (2) admits a unique positive radial solution that tends to zero at infinity. At each step, say n, the exact value u_0 is seeked in an interval $[a_n, b_n]$. We compute $c_n = (a_n + b_n)/2$ and we solve (4) for $u_0 = c_n$. If the approximate solution becomes negative at some r_0, then $u_0 < c_n$. We thus set $a_{n+1} = a_n$ and $b_{n+1} = c_n$. If the solution is always larger some some small prescribed value $\varepsilon > 0$, then we $a_{n+1} = c_n$ and $b_{n+1} = b_n$. The initial interval is chosen in such a way that $a_0 < u_0 < b_0$: a_0 is set to the value leading to the constant solution of (4) $u_0^* = (\alpha)^{1/2\sigma}$ and b_0 is such that the solution of the corresponding Cauchy problem changes its sign. The algorithm for the determination of the k^{th} excited state (noted u_k) is very similar to the one previously described. Here, at step n, if the solution computed with c_n changes its sign more than k times, we fix $a_{n+1} = a_n$ and $b_{n+1} = c_n$. In the opposite case, we prescribe $a_{n+1} = c_n$ and $b_{n+1} = b_n$. The differential system is solved numerically with a standart fourth-order Runge-Kutta method on a bounded interval $[0, R]$ with R is assumed to be larger than all the nodes of the seeked solution (we take R sufficiently large and check that the computed solution with this algorithm does not depend on R). The approximate solution of the soliton is then computed on prescribed discrete points $r_j = j\delta r$ $(0 \leq j \leq N)$ where N is given and $\delta r = R/N$. In all that follows, the value of u_k at the origin is noted $u_{0,k}$. In Figure 1, we plot the results obtained in the two-dimensional case $d = 2$ for $\sigma = 1$ (that is the cubic case), when looking for the ground state u_0 and the first two excited states u_1 and u_2.

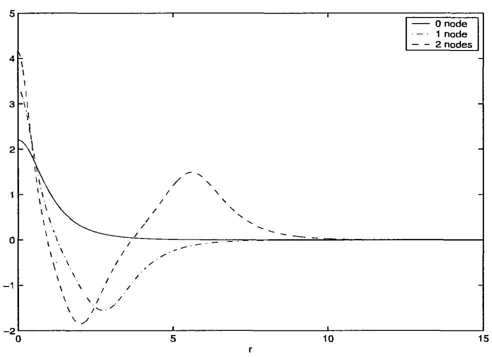

Figure 1: The first three solitons of (1), $d = 2$, $\alpha = 1$, $\sigma = 1$: ground state and the first two excited states.

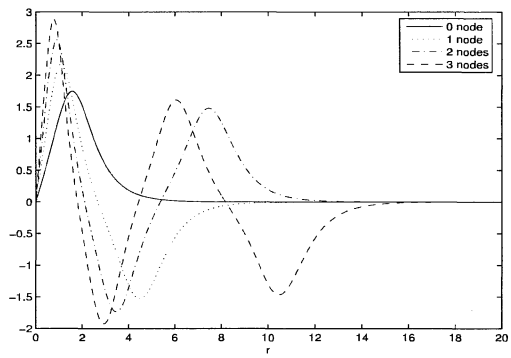

Figure 2: Plots of the first four 2d vortex solutions computed with $m = 1$ $(\alpha = 1, \sigma = 1)$.

In the particular case $d = 2$, it is also possible to look for vortex stationnary states that express as $\psi(t, x) = e^{i\omega t} e^{im\theta} u(r)$, where m is an integer (sometimes known as the vortex charge or the winding number) and (r, θ) stand for the polar coordinates of x (that is $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ with $\theta \in [0, 2\pi[$). That means that the spatial solution is no more radial, even if the function u still depends on r. Plugging this expression into (1), we now find the new ordinary differential equation

$$u''(r) + \frac{1}{r}u'(r) - \frac{m^2}{r^2}u(r) - u(r) + \alpha|u(r)|^{2\sigma}u(r) = 0, \ r > 0. \tag{5}$$

As opposed to the previous situation, it is no more possible to assume that $u(0) \neq 0$ and we have to find another way of expressing this problem. We then set $u(r) = r^m U(r)$ in order to get rid of the term $m^2 u(r)/r^2$. It is easy to check that the new function U solves the modified equation

$$U''(r) + \frac{2m+1}{r}U'(r) - U(r) + \alpha r^{2m\sigma}|U(r)|^{2\sigma}U(r) = 0, \ r > 0$$

which closely looks like the one that had to be solved in the radial case, except that the nonlinearity now depends on r. We also use a shooting strategy in this new context in order to find stationnary states denoted by $u_{k,m}$ (that is with k nodes and corresponding to the winding number m). These vortex states have been studied in [9] in the case $k = 0$ and in [4] for any k. In Figure 2 are shown the results obtained in the two-dimensional case $d = 2$ for $\sigma = 1$ and $m = 1$.

2.2 Asymptotics

We now intend to analyze the asymptotic behaviour the excited states as the number of nodes increases. Indeed, it has been found in [5] that the L^2 and H^1 norms of u_k asymptotically behave as $k^{d/2}$ under suitable assumptions made on the nonlinearity. In [8], the k^{th}-energy level for the Schrödinger-Newton equation, defined as the frequency ω_k associated to the L^2 normalized solution of the nonlinear eigenvalue problem has been numerically found to evolve as k^{-2}. Our goal is to study the influence of d, σ and m (for $d = 2$) on the amplitude of the stationnary excited state with large numbers of nodes.

2.2.1 Bright solitons

First, we have observed that the value $u_k(0)$ of the k^{th} node seems to increase with k. The nonlinear system (4) has been solved on a spatial domain that increases with the number of nodes (we have observed in our computations that a suitable choice for R in the computation of u_k previously described is $R_k = R_0 + k\,\delta$, where δ is a constant depending on σ). In order to study more accurately the dependence of these quantities regarded as functions of k, we also view log $u_{0,k}$ versus log k and plot the derivative versus k. It provides us a simple way of determining an exponential rate of some sequence f_k with respect to k: if the slope tends to some γ, then f_k is governed by the asymptotics $f_k \sim k^\gamma$. The values obtained in Figures 3 seem to converge to the limit value 0.5 for $u_{0,k}$ and suggests that if $|u_{0,k}| = ||u_k||_{L^\infty}$ (which is obvious for $k = 0$ and is numerically satisfied for all our experiments), then the asymptotic behaviour $||u_k||_{L^\infty} \sim \sqrt{k}$ holds. The same computations have been made for higher values of d. Surprisingly, they showed that for $d \geq 3$, the asymptotic growth now depends on σ (see Figure 4 for $d = 3$). These experiments suggest us that for $d \geq 2$, the asymptotics of the L^∞ norm of the solutions is governed by $||u_k||_{L^\infty} \sim k^{\gamma(d,\sigma)}$ where $\gamma(2, \sigma) = 1/2$, $\forall \sigma > 0$ and where $\gamma(d, .)$ is an increasing function of σ if $d \geq 3$. Figures 4 suggests that γ is nonlinear with respect to σ.

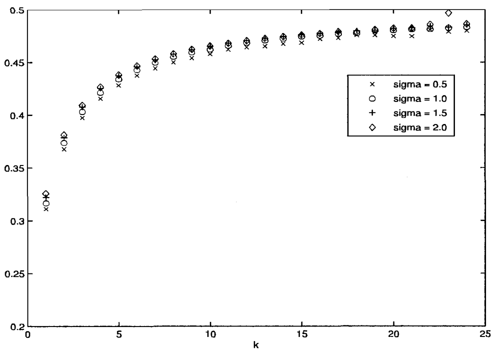

Figure 3: Derivative of log $u_{0,k}$ as a function of log k for $0 \leq k \leq 25$, $\sigma = 0.5,\ 1.0,\ 1.5,\ 2.0\ (d = 2)$.

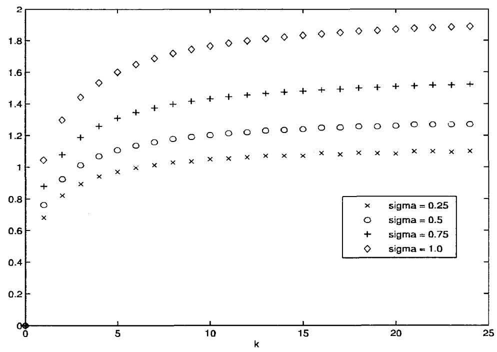

Figure 4: Derivative of log $u_{0,k}$ as a function of log k for $0 \leq k \leq 25$, $\sigma = 0.25,\ 0.5,\ 0.75,\ 1.0\ (d = 3)$.

In order to find the profile of γ, we compute the limit value of the derivative when σ continuously varies from 0 to σ^* and it is found that γ diverges when σ tends to σ^*. We then study $\log \gamma(d, \sigma)$ as a function of $\log(\sigma^* - \sigma)$ and observe a slope very close to -1 for every d; we thus set $\gamma(d, \sigma) = C/(\sigma^* - \sigma)$, $\sigma \in [0, \sigma^*[$. The constant C is then numerically found by comparing the numerical profiles of $\gamma(d, \sigma)$ with the desired expression. It has been found in all our tests that C closely looks like the constant $(d-1)/(d-2) = 1 + \sigma^*/2$. Examinating the results shown in Figure 5, we thus conjecture that

(6) $$||u_k||_{L^\infty} \sim k^{\frac{1+\sigma^*/2}{\sigma^*-\sigma}}, \ k \to \infty.$$

This expression still makes sense for $d = 2$: we have $\sigma^* = \infty$ and this implies $\gamma(2, \sigma) = 1/2$ that had been observed below. Figure 5 displays a remarkably good agreement between the computed γ and the expression given in (6), recalling the each value of σ is obtained as a limit numerically calculated when k is large enough.

The same tests have been made in the case of negative index $-1/2 < \sigma < 0$ (and for $\alpha = \omega = -1$) in order to look again for asymptotic rates in various norms. The situation now differs from what was found above: the limit value of the derivatives does not depend on σ and the profiles obtained seem to indicate that $||u_k||_{L^\infty} \sim k^{\frac{d-1}{2}}$. Finally, asymptotics have been studied for vortex solitons in terms of σ and m. We notice that in this case, the rate that is numerically found seems to fit the explicit one given by

$$U_{0,k,m} \sim k^{\frac{1+m\sigma}{2}}, \ k \to \infty.$$

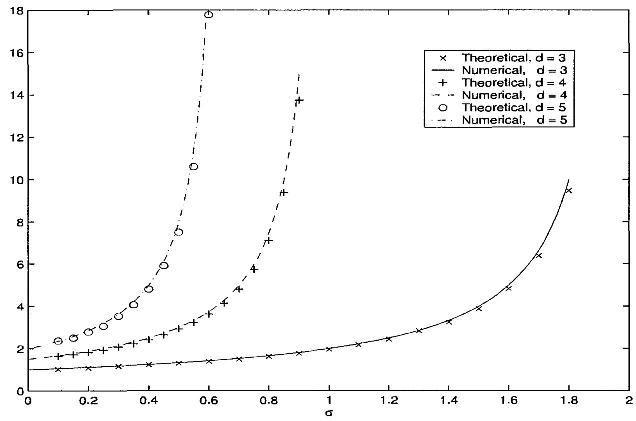

Figure 5: Comparison between numerical and possible theoretical values of $\gamma(d,\sigma)$, $\sigma \in [0, \sigma^*[$, $d = 3,\ 4,\ 5$.

2.2.2 Large nonlinear exponent

As recalled in the beginning of Paragraph 2.1, solitons can be computed when $\alpha > 0$ and $\omega > 0$ only for $\sigma < \sigma^*$ and we now focus on the asymptotics $\sigma \to \sigma^*$: we compute stationnary states for large values of σ in order to derive a scaling law for $\|u_k\|_{L^\infty}$. Our experiments first show that for $d = 2$, the value of the soliton at the origin seems to converge to some limit depending on the number of nodes. Furthermore, a spatial plot of the fundamental solution u_0 for various σ suggests the occurrence of a shrink around the origin when $\sigma \to \infty$. We now consider the case $d \geq 3$ for which $\sigma^* < \infty$ and observe that $u_{0,k}$ tends to infinity as $\sigma \to \sigma^*$. We intend to investigate the rate of blow-up of this quantity in terms of $\sigma^* - \sigma$: as before, we view the derivative of log $u_{0,k}$ as depending on log $(\sigma^* - \sigma)$ (see Figure 6 for $d = 3$ and Figure 7 for $d = 4$) for the first four excited states u_0, u_1, u_2 and u_3. We conjecture that $\|u_k\|_{L^\infty} \sim (\sigma^* - \sigma)^{-\frac{1}{2}-k}$ as $\sigma \to \sigma^*$.

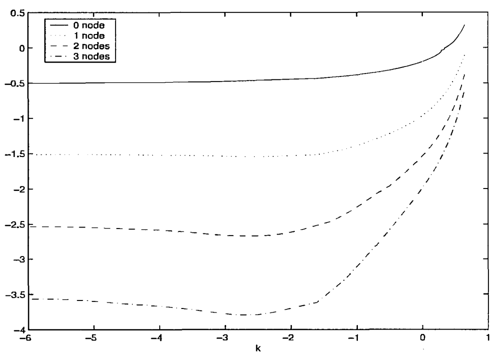

Figure 6: Derivative of log $\|u_k\|_{L^\infty}$ as a function of log $(\sigma^* - \sigma)$ versus σ ($d = 3$).

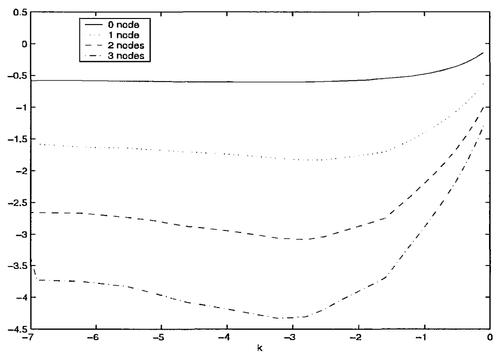

Figure 7: Derivative of log $\|u_k\|_{L^\infty}$ as a function of log $(\sigma^* - \sigma)$ versus σ ($d = 4$).

3 Solitons for quadratic media

We are now concerned with the computation of solitons for system (3) that governs the propagation of two electromagnetic waves in a quadratic media. Seeking stationnary states of the form $u(z,x) = e^{i\omega z}u(x)$ and $v(z,x) = e^{2i\omega z}v(x)$, the system now turns into a nonlinear elliptic system in $\mathbb{R}^d$

$$(7) \qquad \begin{cases} -(1+\omega)u + \Delta u + \bar{u}v = 0 \\ -(\rho + 2\sigma)v + \Delta v + \dfrac{1}{2}u^2 = 0. \end{cases}$$

In all that follows, we will assume for the sake of convenience that $\omega = 0$ without loss of generality. Looking for radial solutions, this system will consist in two differential second-order equations where the unknows u and v have to be computed for $r \in [0, \infty]$. In this case, the shooting method cannot be used directly in any transverse dimension, since two independent parameters $u(0)$ and $v(0)$ have to be adjusted in order to enforce a nice decay property at infinity. We first focus on the case $d = 1$.

3.1 The one-dimensional case

If we assume that u and v stand for even functions of x, smoothness requires that the two derivatives have to vanish at $x = 0$. System (7) thus reduces to

$$(8) \qquad \begin{cases} u''(x) - u(x) + u(x)v(x) = 0 & x > 0 \\ v''(x) - \rho v(x) + \dfrac{1}{2}u^2(x) = 0 \\ u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = 0, \quad v'(0) = 0. \end{cases}$$

In this case, if (u_0, v_0) the initial values of a spatially localized solution (u, v) of (8) at $x = 0$, then one has the identity $u_0^2 + \rho v_0^2 - u_0^2 v_0 = 0$. As a consequence, it is possible to express one of the two parameters in terms of the other one, which enables us to define a shooting strategy as for the case of single equations. We have equivalently

$$u_0 = \pm\gamma\sqrt{\frac{\rho}{1 - v_0}} \quad \text{or} \quad v_0 = \frac{u_0^2 \pm \sqrt{u_0^2 - 4\rho}}{2\rho},$$

provided that $v_0 > 1$ and $u_0 > 2\sqrt{\rho}$. Note that if (u, v) solves (7), so does $(-u, v)$ and it is possible to make the further assumption $u_0 > 0$. We have always chosen to express v_0 with use of u_0 since the second equation implies by virtue of the maximum principle that v remains strictly positive: u_0 will be adjusted in order to obtain a solution which can vanish k times and the k^{th} excited solutions will be noted (u_k, v_k).

We plot in Figure 8 the solution u obtained with the shooting technique for different values of ρ on the domain $[0, 10]$ using the Runge-Kutta method with $N = 1{,}000$. As opposed to NLS equation (1), it can be observed that the values of the k^{th} node solution at $x = 0$ seem to converge to a limit value (u^*, v^*) that is conjectured to be the Cauchy data of the periodic solution of (8).

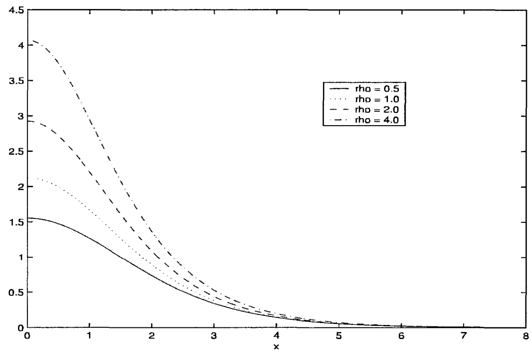

Figure 8: Plot of u, $\rho = 0.5$, $\rho = 1$, $\rho = 2$ and $\rho = 4$.

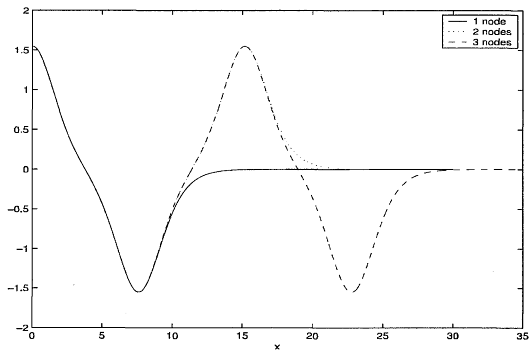

Figure 9: Plot of u, $\rho = 0.5$, $k = 0$, $k = 1$, $k = 2$ and $k = 3$.

3.2 Higher-dimensional bright solitons

We now investigate the determination of radial stationnary solution of (7) (referred as *simultons* in [3]) in any space dimension. The new system writes

$$(9)\qquad \begin{cases} u''(r) + \dfrac{d-1}{r}u'(r) - u(r) + u(r)v(r) = 0 & r > 0 \\ v''(r) + \dfrac{d-1}{r}v'(r) - \rho v(r) + \dfrac{1}{2}u^2(r) = 0 & r > 0 \\ u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = 0, \quad v'(0) = 0. \end{cases}$$

Unlike (8), it is not possible to find for localized solutions of (9) a relation between β and γ, because of the frictional terms u'/r and v'/r that come into play if $d \geq 2$. We then use a continuation method that enables to follow a path between one-dimensional solitons and d-dimensional solitons. Let (u_s, v_s) a spatially localized solution of (9) where s is regarded as a continuous parameter that varies between 1 and d. Since it is possible to evaluate numerically any excited one-dimensional soliton (see Subsection 3.1), we assume that the starting point, say (u_1, v_1) is known. We now follow the path between (u_1, v_1) and the desired solution (u_d, v_d), expressing that for each s, (u_s, v_s) satisfies the system (9): it implies that the derivative $(\dot{u}_s, \dot{v}_s)$ with respect to s identically vanishes. We thus get for each $r > 0$ the identities

$$\dot{u}_s''(r) + \frac{s-1}{r}\dot{u}_s'(r) - \dot{u}_s(r) + \dot{v}_s(r)u_s(r) + \dot{u}_s\ v_s(r) = -\frac{1}{r}u_s'(r)$$

and

$$(10)\qquad \dot{v}_s''(r) + \frac{d-1}{r}\dot{v}_s'(r) - \rho\dot{v}_s(r) + u_s(r)\dot{u}_s(r) = -\frac{1}{r}v_s'(r).$$

This means that for each s, $(\dot{u}_s, \dot{v}_s)$ can be obtained from (u_s, v_s) solving a linear nonhomogeneous elliptic system that is noted $(\dot{u}_s, \dot{v}_s) = F((u_s, v_s))$. In order to compute (u_d, v_d) starting from the initial data (u_1, v_1), we have to solve the differential problem

$$(11) \qquad \begin{cases} (\dot{u}_s, \dot{v}_s) = F((u_s, v_s)), & s \in [1, d] \\ (u_s, v_s)|_{s=1} = (u_1, v_1), \end{cases}$$

which can be easily done with a standart Runge-Kutta routine. Since radial exponentially decaying states are seeked, we consider the numerical domain $[0, r_{max}]$ where this elliptic system is discretized on a uniform grid with mesh h. Here, we have to deal with boundary condition at the right extremity of the segment in order to preserve the decreasing rate at infinity. Simple arguments based on the solution of the linear system lead us to

$$(12) \quad \frac{\partial \dot{u}_s}{\partial r} + \Big(1 + \frac{s-1}{2r}\Big)\dot{u}_s = -\frac{1}{2r}u_s \quad \text{and} \quad \frac{\partial \dot{v}_s}{\partial r} + \Big(\sqrt{\rho} + \frac{s-1}{2r}\Big)\dot{v}_s = -\frac{1}{2r}v_s.$$

These boundary conditions guarantee that the computed solutions keep a correct asymptotics as $r \to \infty$. We first plot the first two stationnary states comparing them with the profiles obtained in the one-dimensional case taken as starting points. We have used the following values of parameters : $\rho = 0.2$, $N_d = 100$ discretization points for the numerical resolution of (11) and $r_{max} = 20$. As seen in Figure 10 and 11, the one-dimensional solitons have travelled through the continuous path and have given radial excited two-dimensional profiles. Note that in [6], a continuation method has been used starting from one-dimensional explicit solutions with no nodes that can be calculated for $\rho = 1$; in [10], a variational approach has been used in order to find Gaussian ansatz of fundamental states.

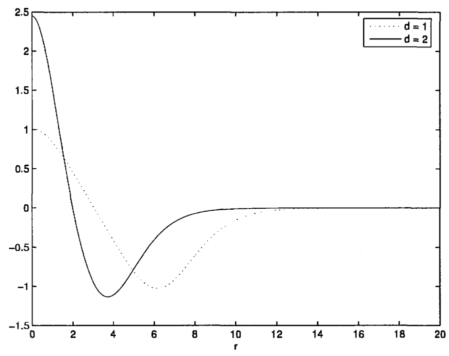

Figure 10: Plot of u, one-dimensional starting point and two-dimensional excited state, $\rho = 0.2$.

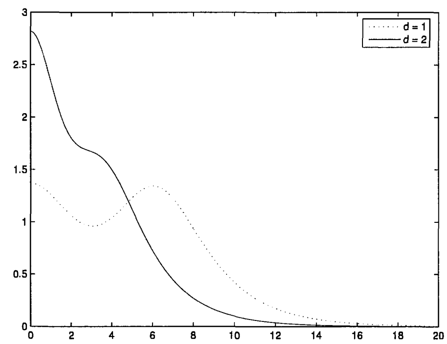

Figure 11: Plot of v, one-dimensional starting point and two-dimensional excited state, $\rho = 0.2$.

3.3 Vortex solitons in two-dimensional case

As for (NLS) equation, it is possible to look for solutions with dependence with respect to angular parameter θ. We thus look for solutions of the form $u(r)\exp(im\theta)$ and $u(r)\exp(2im\theta)$; the system to be solved now reads (see [2])

$$
(13) \qquad \begin{cases} u''(r) + \dfrac{1}{r}u'(r) - \dfrac{m^2}{r^2}u(r) - u(r) + \bar{u}(r)v(r) = 0 \\ v''(r) + \dfrac{1}{r}v'(r) - \dfrac{4m^2}{r^2}v(r) - \rho v(r) + \dfrac{1}{2}u^2(r) = 0. \end{cases}
$$

Such solutions vanish at $r = 0$ and we set $u(r) = r^m U(r)$ and $v(r) = r^{2m} V(r)$. We decide to use the continuation procedure described above, taking m as a continuation parameter that is noted s: we define $(U_s(r), V_s(r))$ as the solutions of the new system derived form (13). Differentiating with respect to $s \in [0, m]$, we get

$$
(14) \qquad \begin{cases} \dot{U}_s''(r) + \dfrac{1+2s}{r}\dot{U}_s'(r) - \dot{U}_s(r) + r^{2s}V_s(r)\dot{U}_s(r) + r^{2s}U_s(r)\dot{V}_s(r) = \\ \qquad\qquad -\dfrac{2}{r}U_s(r) - 2\log r \; r^{2s}U_s(r)V_s(r) \\ \dot{V}_s''(r) + \dfrac{1+4s}{r}\dot{V}_s'(r) - \rho\dot{V}_d(r) + U_s(r)\dot{U}_s(r) = -\dfrac{4}{r}V_s(r). \end{cases}
$$

If (U_s, V_s) is known, this gives us $(\dot{U}_s, \dot{V}_s)$. Here again, a differential solver enables us to compute (U_m, V_m) from Cauchy data (U_0, V_0) at $s = 0$ which appears to be the bright solitons in the two-dimensional case, because of the the friction terms $U'(r)/r$ and $V'(r)/r$ in the system. The boundary conditions set at the left side of the numerical domain are the same as for bright solitons. We show in Figures 12 and 13 the profiles of excited states obtained in the case $m = 1$.

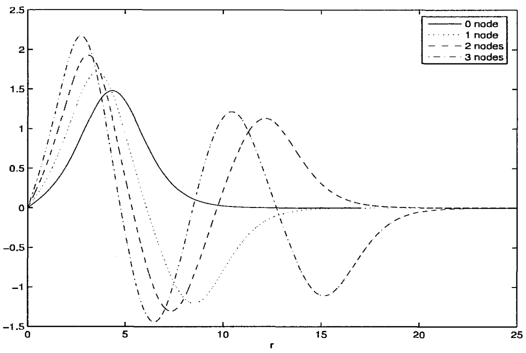

Figure 12: Plot of u, $m = 1$, $\rho = 0.2$, $k = 0$, $k = 1$, $k = 2$ and $k = 3$.

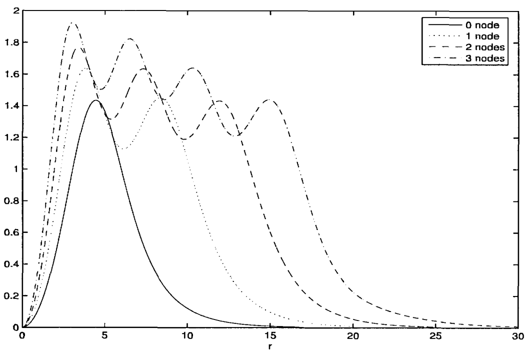

Figure 13: Plot of v, $m = 1$, $\rho = 0.2$, $k = 0$, $k = 1$, $k = 2$ and $k = 3$.

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A reaction-diffusion system approximation to nonlinear diffusion problems

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We consider the following degenerate parabolic systems:

$$
\text{(P)}\begin{cases}
\dfrac{\partial z}{\partial t} = \Delta\beta(z) + f(\beta(z)) & \text{in} \quad Q := \Omega \times (0,T), \\
\dfrac{\partial \beta(z)}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\
z(x,0) = a(x) & \text{for} \quad x \in \Omega,
\end{cases}
$$

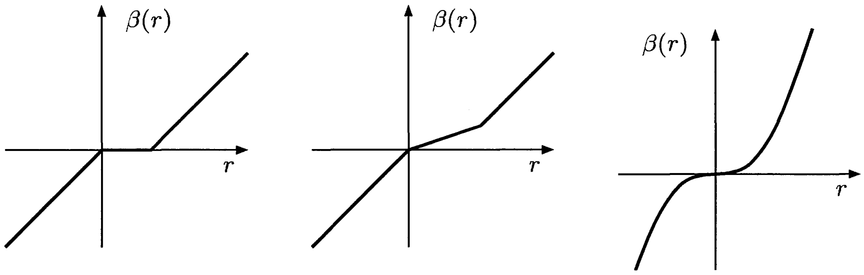
where Ω is a bounded domain in $\mathbf{R}^N$ ($N \in \mathbf{N}$) with smooth boundary $\partial\Omega$, T is a positive constant, ν is the outward normal unit vector to the boundary $\partial\Omega$, z is the n-dimensional vectors $(z_1, \ldots, z_n)$ ($n \in \mathbf{N}$), and $a : \Omega \to \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ and $\beta : \mathbf{R}^n \to \mathbf{R}^n$ are given functions.

Problem (P) includes a large number of important problems, for instance, the two-phase Stefan problem, which is the typical model of the solid-liquid phase transition problem, if $n = 1$ and

$$\beta(r) = d_1 \max(r - \lambda, 0) + d_2 \min(r, 0), \tag{1}$$

where d_1, d_2 and λ are non-negative constants (see Figure 1 (a)). The functions z and $\beta(z)$ represent physically the enthalpy and the temperature, respectively, and the function f denotes the intensity of a distributed heat source or sink. The diffusion vanishes in the above problem, where $z \in (0, \lambda)$. Vanishing diffusion characterizes the presence of a free boundary, and the solution exhibits a lack of regularity. Consequently, regularizations of the

Figure 1: Constitutive relations for (a) two-phase Stefan problem (b) a regularized Stefan problem (c) porous medium equation.



enthalpy-temperature constitutive relation β are sometimes used for the Stefan problem. Some regularized Stefan problems are also included by (P) (for example Figure 1 (b)).

The isentropic flow through porous media is also described by (P), the so-called porous medium equation, with $n = 1$ and

$$\beta(r) = r^m, \tag{2}$$

where $m > 1$ is a constant. In this problem, the function z denotes the density of fluid. It is known that these problems differ considerably from parabolic ones, since they allow solutions with compact support.

Another relevant problem with $n = 2$ is a nonlinear cross diffusion population model, namely:

$$\begin{aligned} \frac{\partial z_1}{\partial t} &= \Delta[(a_1 + b_1 z_2 + c_1 z_1) z_1] + g_1(z_1, z_2), \\ \frac{\partial z_2}{\partial t} &= \Delta[(a_2 + b_2 z_1 + c_2 z_2) z_2] + g_2(z_1, z_2), \end{aligned} \tag{3}$$

where a_i, b_i, c_i $(i = 1, 2)$ are non-negative constants and g_i $(i = 1, 2)$ are given functions. Here, z_1 and z_2 represent the population density of two competing species.

In the present note a reaction-diffusion system approximation to Problem (P) is proposed. There are a number of publications dealing with reaction-diffusion system approximations, for example, to the classical Stefan problem without latent heat, that is $\lambda = 0$ in (1), by Dancer et al. [1], to the classical Stefan problem by Hilhorst et al. [2] and Murakawa [5, 6], to the porous medium equation by Knabner [4], to the cross-diffusion system (3) with $c_1 = c_2 = 0$ by Iida, Mimura and Ninomiya [3]. Since these problems can be represented in the form (P), we would like to deal with the general problem (P) and propose a reaction-diffusion system approximations to (P).

Now, we illustrate our ideas and construct a reaction-diffusion system in the case of single equations ($n = 1$) with $f = 0$ for simplicity. The presence of nonlinear diffusion makes it difficult to analyze. In order to remove the nonlinearity of the diffusion, let us define $u := \beta(z)$ and $v := z - u$. Then, we have

$$\frac{\partial u}{\partial t} = \Delta u - \frac{\partial v}{\partial t}.$$

Although there are many choices for the time derivative of v, we propose for a sufficiently small parameter ε

$$\frac{\partial v}{\partial t} = \frac{1}{\varepsilon}(u - \beta(u + v)).$$

Thus, we consider a solution $(u^\varepsilon, v^\varepsilon)$ of the following reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \dfrac{1}{\varepsilon}(u - \beta(u+v)) & \text{in } \ Q, \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{\varepsilon}(u - \beta(u+v)) & \text{in } \ Q. \end{cases} \tag{4}$$

Let us define $z^\varepsilon := u^\varepsilon + v^\varepsilon$. We expect that z^ε and u^ε approximate the solution z of (P) and $\beta(z)$, respectively. Now, let us see the reaction part of (4). If u^ε is greater than $\beta(z^\varepsilon)$, then u^ε decreases. Conversely, u^ε is less than $\beta(z^\varepsilon)$, then u^ε increases. Thus, u^ε and $\beta(z^\varepsilon)$ approach each other:

$$u^\varepsilon \approx \beta(z^\varepsilon). \tag{5}$$

On the other hand, it follows from (4) that

$$\frac{\partial z^\varepsilon}{\partial t} = \Delta u^\varepsilon.$$

Therefore, (5) implies that

$$\frac{\partial z^\varepsilon}{\partial t} \approx \Delta \beta(z^\varepsilon).$$

Thus, it can formally be expected from (5) that z^ε and u^ε approximate z and $\beta(z)$, respectively.

We explain about the idea of construction of the system (4) from another point of view. A similar idea can be found in [3]. Suppose that there are two types of states: active (with linear diffusion) and inactive (without diffusion), that is, we divide z into active u and inactive v. The diffusion rate of z is decided by the proportion of u to v, specifically, it depends only on the value of u. The transition rates between the two states play an important role. These are taken so that u may approach $\beta(z)$, since it is expected that the diffusion rates of z and u agree. In this work, we take $(u - \beta(z))/\varepsilon$. Since it is required that the transition is much faster than the diffusion, ε is taken sufficiently small. These idea leads to the system (4).

Our ideas are easily generalized to Problem (P). Thus, we propose the following reaction-diffusion system approximation to Problem (P):

$$
\text{(RD)}\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u + f(u) - \dfrac{1}{\varepsilon}(u - \beta(u+v)) & \text{in} \quad Q, \\
\dfrac{\partial v}{\partial t} = \dfrac{1}{\varepsilon}(u - \beta(u+v)) & \text{in} \quad Q, \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\
u(x,0) = u_0^\varepsilon(x), \quad v(x,0) = v_0^\varepsilon(x) & \text{for} \quad x \in \Omega,
\end{cases}
$$

where u_0^ε and v_0^ε are given functions.

Before stating our results we define a weak solution of (P).

Definition 1. A function $z \in (L^\infty(Q))^n$ is a weak solution of (P) with an initial datum $a \in (L^\infty(\Omega))^n$ if it satisfies $\beta(z) \in L^2(0,T;H^1(\Omega))$ and

$$
\int_0^T \langle z_i, \frac{\partial \zeta_i}{\partial t}\rangle + \langle a_i, \zeta_i(\cdot,0)\rangle = \int_0^T \langle \nabla\beta_i(z), \nabla\zeta_i\rangle - \int_0^T \langle f_i(\beta(z)), \zeta_i\rangle \tag{6}
$$

for all functions $\zeta \in \mathcal{K} := \{\zeta \in (H^1(Q))^n \mid \zeta(\cdot,T) = 0\}$ and $i \in \{1,2,\cdots,n\}$. Here, $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\Omega)$.

The following assumptions are imposed on β, f and the initial functions:

(H_β) The function β is continuous and monotone, that is,

$$
(\beta(r) - \beta(s)) \cdot (r - s) \geq 0 \quad \text{for all } r,\ s \in \mathbf{R}^n.
$$

(H_f) The function f is continuous.

(H_0) The initial functions u_0^ε and v_0^ε satisfy that $a_i^\varepsilon \rightharpoonup a_i$ weakly in $L^2(Q)$ as ε tends to zero for each $i \in \{1,2,\cdots,n\}$, where $a^\varepsilon := u_0^\varepsilon + v_0^\varepsilon$.

Theorem 1. *Let $(u^\varepsilon,\ v^\varepsilon)$ be a solution of (RD) with an initial datum $(u_0^\varepsilon,\ v_0^\varepsilon)$. Suppose ($H_\beta$), ($H_f$) and ($H_0$) are satisfied and that u^ε and v^ε are uniformly bounded with respect to ε in $(L^\infty(Q))^n$ and in $(L^\infty(Q))^n \cap (W^{1,1}(0,T;L^1(\Omega)))^n$,* *respectively. Then, there exist a weak solution of (P) with the initial datum a and subsequences* $\{u^{\varepsilon_k}\}$ *and* $\{z^{\varepsilon_k}\}$ *of* $\{u^\varepsilon\}$ *and* $\{z^\varepsilon\}$*, respectively, such that*

$$\begin{array}{ll} u^{\varepsilon_k} \to \beta(z) & \text{strongly in } (L^2(Q))^n \text{ and weakly in } (L^2(0,T;H^1(\Omega)))^n, \\ z^{\varepsilon_k} \rightharpoonup z & \text{weakly in } (L^2(Q))^n \end{array}$$

as k tends to infinity. If the weak solution of (P) is unique, then the original sequences $\{u^\varepsilon\}$ *and* $\{z^\varepsilon\}$ *converge to u and z, respectively.*

We can estimate the rates of convergence under the stronger assumption:

$(\mathrm{H}_\beta)'$ The functions β_i depend only on the i-th component and are nondecreasing and Lipschitz continuous for each $i \in \{1, 2, \cdots, n\}$.

$(\mathrm{H}_f)'$ The function f is Lipschitz continuous.

Theorem 2. *Let* $z \in (L^\infty(Q))^n$ *be a weak solution of (P) with an initial datum a and* $(u^\varepsilon, v^\varepsilon)$ *be a solution of (RD) with an initial datum* $(u_0^\varepsilon, v_0^\varepsilon)$*. Suppose that* $(H_\beta)'$ *and* $(H_f)'$ *hold and that* u^ε *and* v^ε *are uniformly bounded with respect to* ε *in* $(L^\infty(Q))^n$ *and in* $(L^\infty(Q))^n \cap (W^{1,1}(0,T;L^1(\Omega)))^n$*, respectively. Then, there exists a positive constant* C_1 *independent of* ε *such that*

$$\|u-u^\varepsilon\|_{(L^2(Q))^n} + \left\| \int_0^t (u-u^\varepsilon) \right\|_{(L^\infty(0,T;H^1(\Omega)))^n} + \|z-z^\varepsilon\|_{(L^\infty(0,T;(H^1(\Omega))^*))^n} \le C_1(\varepsilon + \sigma(\varepsilon))^{1/2},$$

where $u = \beta(z)$ *and* $\sigma(\varepsilon) := \|a - (u_0^\varepsilon + v_0^\varepsilon)\|^2_{(L^2(\Omega))^n}$.

Moreover, if β_i *is strongly increasing function for* $i \in \{1, 2, \cdots, n\}$*, then there exists a positive constant* C_2 *independent of* ε *such that*

$$\|z_i - z_i^\varepsilon\|_{L^2(Q)} \le C_2(\varepsilon + \sigma(\varepsilon))^{1/2}.$$

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Cell Membranes, Lipid Bilayers, and the Elastica Functional

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Abstract: We study an energy functional that arises in a simplified two-dimensional model for lipid bilayer membranes. We demonstrate that this functional, defined on a class of spatial mass densities, favors concentrations on 'thin structures'. Stretching, fracture and bending of such structures all carry an energy penalty. In this sense we show that the models captures essential features of lipid bilayers, namely *partial localization* and a *solid-like behavior.*

Our findings are made precise in a Gamma-convergence result. We prove that a rescaled version of the energy functional converges in the 'zero thickness limit' to a functional that is defined on a class of planar curves. Finiteness of the limit value enforces both optimal thickness and non-fracture; if these conditions are met, then the value of this functional is given by the classical Elastica (bending) energy.

1 Introduction

In this presentation we discuss a new model for cell membranes. For more detailed informations, a precise statement of the full result and a complete proof we refer to our forthcoming paper [7].

1.1 Lipid Bilayers

Cell membranes shield the interior of the cell from the outside and are typically formed by lipid bilayers. Their main component is a lipid molecule which consists of a head and two tails. The head is hydrophilic, while the tails are hydrophobic. This difference in water affinity causes lipids to aggregate in structures as shown in Fig. 1, reducing energetically unfavorable tail-water interactions. Despite the structured appearance of the bilayer, there is no covalent bonding between any two lipids; the bilayer structure is entirely the result of the hydrophobic effect.

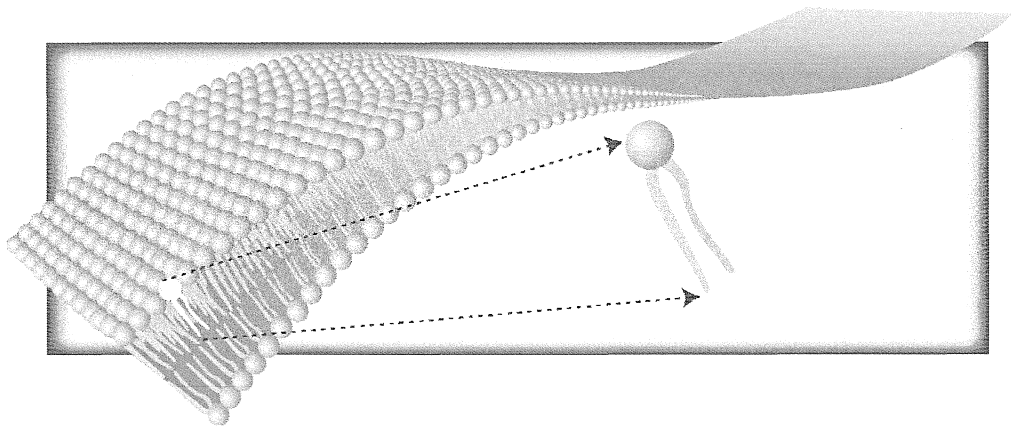

Figure 1: Lipid molecules aggregate into macroscopically surface-like structures

1.2 Partial localization and solid-like behavior

Lipid bilayers are *thin structures*, in the sense that there is a separation of length scales: the thickness of a lipid bilayer is fixed to approximately two lipid lengths, while the in-plane spatial extent is only limited by the surroundings. Structures that are thin in one or more directions and 'large' in others we call *partially localized.*

In a cell membrane, the lipid molecules behave fluid-like with respect to in-plane rearrangements. However, as planar structures lipid bilayers show a remarkable *solid-like behavior*: they resist various types of deformation, such as extension, bending, and fracture, much in the way a sheet of rubber does.

2 Descriptions on different scales

Several descriptions of lipid bilayers can be found in the literature. Microscopic models describe positions of molecules or, in 'coarse grained' versions, of groups of molecules (*e.g.* heads and tails). On the macroscale cell membranes are considered as smooth closed hypersurfaces. We will investigate a *mesoscale* model that introduces an energy functional defined on a class of density functions of head and tail particles.

2.1 Macroscale energy: The Helfrich Hamiltonian and the Elastica functional

Canham, Helfrich, and Evans pioneered the modelling of lipid bilayer vesicles by energy methods [3, 6, 4]. The name of Helfrich is now associated with a surface energy for closed vesicles, represented by a smooth boundaryless surface S, of the form

$$E_{\text{Helfrich}}(S) = \int_S \left[k(H - H_0)^2 + \overline{k}K\right] d\mathcal{H}^2.$$

Here $k > 0$ and $\overline{k}, H_0 \in \mathbb{R}$ are constants, H and K are the (scalar) total and Gaussian curvature, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. This energy functional, and many generalizations in the same vein, have been remarkably successful in describing the wide variety of vesicle shapes [8].

A natural two-dimensional reduction of the Helfrich curvature energy is given by the classical bending energy of the curve, the *Elastica functional*

$$\mathcal{W}(\gamma) = \frac{1}{4}\int_\gamma \kappa^2 \, d\mathcal{H}^1.$$

Here γ is a smooth closed curve in $\mathbb{R}^2$ and κ equals the scalar curvature of γ. This functional has a long history going back at least to Jakob Bernoulli; critical points of this energy are known as *Euler Elastica.*

2.2 Mesoscale energy: The functional $\mathcal{F}_\varepsilon$

Whereas microscale descriptions often are too complex, the macroscale models are only phenomenological and sometimes too simple. We consider here an energy functional on a scale in between. The derivation is inspired by well-known models of block copolymers, and uses a number of sometimes radical simplifications. Nevertheless we demonstrate that the model captures enough of the essence of lipid bilayers to address the issues of partial localization and solid-like behavior.

In a rescaled version the mesoscale energy is given by a class of admissible functions

$$\mathcal{K}_\varepsilon := \left\{(u,v) \in \mathrm{BV}(\mathbb{R}^2;\{0,\varepsilon^{-1}\}) \times L^1(\mathbb{R}^2;\{0,\varepsilon^{-1}\}) : \int u = \int v = M,\ uv = 0 \text{ a.e.}\right\} \tag{1}$$

and functionals $\mathcal{F}_\varepsilon$,

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \varepsilon \displaystyle\int |\nabla u| + \frac{1}{\varepsilon} d_1(u,v) & \text{if } (u,v) \in \mathcal{K}_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

Here $\varepsilon > 0$ is a small parameter and $M > 0$ is fixed. The term $d_1(u,v)$ denotes the Monge-Kantorovich distance between u and v, which can be characterized as the minimal cost for transporting the mass density u to the mass distribution v [1, 2, 5].

Despite of the simplifications in the derivation the physical origin of the various elements of $\mathcal{F}_\varepsilon$ remains identifiable:

- The functions u and v represent densities of tail and head particles;
- The term $\varepsilon \int |\nabla u|$, coupled with the restriction to functions u and v that take only two values, and have disjoint support, represents an interfacial energy;

- The Monge-Kantorovich distance $d_1(u, v)$ between u and v is a weak remnant of the covalent bonding between head and tail particles.

By (1) the small parameter $\varepsilon > 0$ introduces a *large density* scaling. We prove that $\mathcal{F}_\varepsilon$ favors partially localized structures and that the parameter ε in fact corresponds to 'the thickness' of the support of u, v. We will also show that $\mathcal{F}_\varepsilon$ displays solid-like behavior in the sense of a penalization of stretching, fracture, and bending.

3 Examples

To gain some insight into the properties of $\mathcal{F}_\varepsilon$ one can study particular choices of u, v which allow to compute the value of $\mathcal{F}_\varepsilon(u, v)$.

1. A **disc:** concentrate all the mass of u into a disc in $\mathbb{R}^2$, of radius $R \sim \varepsilon^{1/2}M^{1/2}$, surrounded by the mass of v in an annulus. Since $d_1(u, v)$ is approximated by an 'average transport distance' multiplied by the total mass M this gives
$$\mathcal{F}_\varepsilon(u, v) \sim \varepsilon^{-1/2}M^{3/2} + \varepsilon^{1/2}M^{1/2}$$
2. A **strip**: take the support of u to be a rectangle of width $t\varepsilon$ and length M/t, flanked by two strips of half this width for $\operatorname{supp} v$. Then
$$\mathcal{F}_\varepsilon(u, v) = \frac{t}{2}M + \frac{2}{t}M + 2t\varepsilon.$$

Comparing the two one observes that the strip structure has lower energy, for small ε, than a spherical one. For the strip structure we find that there is a preferred thickness given by 2ε and that the energy is at least $2M$ plus 'higher order terms'. This is a first indication that $\mathcal{F}_\varepsilon$ may favor partial localization.

Explicit calculations can also be done for *ring structures.* There one observes that the curvature of the structures create an energy penalty on the order $O(\varepsilon^2)$. This gives a hint how $\mathcal{F}_\varepsilon$ is related to a bending energy.

4 Results

We study the behavior of the mesoscale energy $\mathcal{F}_\varepsilon$ in the singular limit $\varepsilon \to 0$ and consider the rescaled functionals
$$\mathcal{G}_\varepsilon = \frac{\mathcal{F}_\varepsilon - 2M}{\varepsilon^2}.$$
Our result can be described as follows:

Theorem 4.1. *Take any sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$ with small energy, in the sense that $\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$ remains bounded. Then the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges as measures to a finite collection of closed curves of class $W^{2,2}$. Moreover, each curve in this collections appears an even number of times and the curves do not transversally intersect.*

Theorem 4.2. *The functionals $\mathcal{G}_\varepsilon$ Gamma-converge to the curve bending energy $\mathcal{W}$, generalized to systems of curves: For a system of curves $\Gamma = (\gamma_1, ..., \gamma_N), N \in \mathbb{N}$, which satisfies the conditions that $\gamma_1, ..., \gamma_N$ are closed, of class $W^{2,2}$, have no transversal crossings and that $\sum_{i=1}^N \operatorname{length}(\gamma_i) = M$, the value of the limit functional is given by*
$$\mathcal{W}(\Gamma) := \sum_{i=1}^N \mathcal{W}(\gamma_i).$$
If any of the conditions above is not satisfied the value of the limit functional is infinity.

Our results in fact show that the essence of lipid bilayer behavior is captured by the mesoscale model:

- *Partial localization:* Boundedness of $\mathcal{G}_\varepsilon$ along a sequence $u_\varepsilon, v_\varepsilon$ implies that the support of u_ε resembles a tubular ε-neighborhood of a curve.

- *No Stretching:* All curves which belong to a system of curves with finite limit energy carry a constant density function (the constant being one).
- *No Fracture:* A curve which belongs to a system of curves with finite energy is necessarily closed.
- *Bending stiffness:* The Gamma-limit of the functional $\mathcal{G}_\varepsilon$ equals the generalized Elastica functional.

5 Ingredients of our analysis

At first glance it is not obvious that the functionals $\mathcal{F}_\varepsilon$ encode bending stiffness effects. Curvature terms in fact appear first on an order ε^2 of an asymptotic development, which asks for a careful analysis. Most information is hidden in the nonlocal Monge-Kantorovich distance term. The special property that makes our analysis work is that this distance decouples into one-dimensional problems. This reduction is one major step in our analysis and the connection to the *optimal mass transport problem* is the most important tool here.

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Semi-classical analysis for Hartree equations in some supercritical cases

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1 An introduction to semi-classical analysis for a class of nonlinear Schrödinger equations

We consider the asymptotic behavior of the solution to the following semi-classical nonlinear Schrödinger equation:

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \varepsilon^\alpha F(u^\varepsilon), & (t,x)\in\mathbb{R}^{1+N} \\ u^\varepsilon_{|t=0} = u_0^\varepsilon \end{cases} \tag{NLS$^\varepsilon$}$$

as positive parameter $\varepsilon \to 0$. The parameter ε corresponds to the Planck constant and the limit $\varepsilon \to 0$ is known as the semi-classical limit. It is relevant when coupling quantum models to classical models.

1.1 Focusing at a point

We assume the initial datum of (NLS$^\varepsilon$) is of the following form:

$$u_0^\varepsilon(x) = e^{-i\frac{|x|^2}{2\varepsilon}} f(x). \tag{1}$$

Then, the caustic occurs at origin at $t = 1$. We shall observe this.

Let us consider the approximate solution of the following form

$$u^\varepsilon(t,x) \sim a_0(t,x) e^{i\frac{\phi(t,x)}{\varepsilon}} \quad (\varepsilon \to 0). \tag{2}$$

If $\alpha > 0$ then, substituting this to (NLS$^\varepsilon$), we obtain the eikonal equation (characteristic equation)

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{3}$$

Then, its bicharacteristics are the system

$$\begin{cases} \dot{x} = \xi, \\ \dot{t} = 1, \\ \dot{\xi} = 0, \\ \dot{\tau} = 0, \end{cases} \tag{4}$$

where $\xi = \nabla\phi$ and $\tau = \partial_t \phi$. We regard t as the parameter, since $\dot{t} = 1$. With the initial phase $\phi(0,x) = -|x|^2/2$ (see (1)), the rays are the line

$$x = \xi t + x_0 = x_0(1-t).$$

Therefore, all rays pass through $(t,x) = (1,0)$. Because of the quadratic oscillation of the initial datum, the caustic occurs, where the approximate of the geometrical optics (2) ceases to be valid.

At the caustic, rays focus, and therefore the modulus of the solution may become bigger. Indeed, let w^ε be the solution to free semi-classical Schrödinger equation $(i\varepsilon\partial_t + (1/2)\varepsilon^2\Delta)w^\varepsilon = 0$ and $w^\varepsilon(0,x) = e^{-i|x|^2/2\varepsilon} f(x)$. Then, it holds that

$$|w^\varepsilon(1,x)| = \frac{1}{\varepsilon^{n/2}} |\mathcal{F} f(x/\varepsilon)|.$$

$\mathcal{F}$ denotes the Fourier transform. Now, let us turn our attention to nonlinear equations. In general, if the modulus of solution becomes big, then the nonlinearity becomes more bigger than its linear part (for example, $F(u) = |u|^\beta u$, $\beta > 0$). As stated above, at the caustic, the modulus of the solution becomes big. Hence, we deduce that the nonlinearity may become big at the caustic. One of our aim here is to investigate this strong nonlinear effect around the caustic.

Remark 1.1. Adding harmonic potential $V(x) = |x|^2/2$ to the equation (NLS$^\varepsilon$), we consider the equation

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{1}{2}\varepsilon^2 \Delta u^\varepsilon - \dfrac{|x|^2}{2} u^\varepsilon = \varepsilon^\alpha F(u^\varepsilon), & (t,x) \in \mathbb{R}^{1+N} \\ u^\varepsilon_{|t=0} = f(x). \end{cases}$$

Then, the caustic occurs at origin periodically in time $t = \pi/2 + k\pi$ ($k = 0, \pm 1, \pm 2, \dots$). In this case, we do not need the quadratic oscillation in the initial datum ([5]).

1.2 Difficulty

The difficulty of this problem is caused by the existence of two individual critical indices on the size of the nonlinearity, that is, on the constant α. The problem (NLS$^\varepsilon$) has micro structure and macro structure, simultaneously. And, each structure provides one critical index. In Subsections 1.3 and 1.4, let us observe these critical indices. The argument there is formal.

1.3 WKB analysis and the first critical index

Let us again consider the approximate solution of the solution to (NLS$^\varepsilon$) which has the following form:

$$u^\varepsilon(t,x) \sim a_0(t,x) e^{i\frac{\phi(t,x)}{\varepsilon}} \quad (\varepsilon \to 0).$$

Then, we obtain equation (3) from ε^0 terms. We can solve the (3) and $\phi(0,x) = -|x|^2/2$, and then obtain

$$\phi(t,x) = \frac{|x|^2}{2(t-1)}. \tag{5}$$

We shall consider the next order. Considering the ε^1 terms, we obtain the following transport equation:

$$\partial_t a_0 + \nabla\phi \cdot \nabla a_0 + \frac{1}{2} a_0 \Delta\phi = \begin{cases} 0, & \text{if } \alpha > 1, \\ \varepsilon^{\alpha-1} F(a_0 e^{i\frac{\phi}{\varepsilon}}) e^{-i\frac{\phi}{\varepsilon}}, & \text{if } \alpha \leqslant 1. \end{cases} \tag{6}$$

Therefore, we see that the nonlinear effect on propagation depends on whether α is bigger than one or not.

$\alpha > 1$: linear propagation (or Linea WKB)

$\alpha = 1$: nonlinear propagation (or nonlinearly WKB)

$\alpha < 1$: supercritical propagation (or supercritical WKB)

This $\alpha = 1$ is the first critical index. In the linear propagation case (the case $\alpha > 1$), the transport equation (6) is the same one as that for free semi-classical Schrödinger equation $(i\varepsilon\partial_t + (1/2)\varepsilon^2\Delta)w^\varepsilon = 0$. Therefore, we can deduce that the true solution can be estimated by the free solution.

Remark 1.2. We can solve the above transport equation (6) explicitly. Now, we assume the right hand side of (6) is equal to 0, that is, we restrict our attention to the case $\alpha > 1$. Then, with the initial datum $a_0(0,x) = f(x)$, we obtain $a_0(t,x) = (1-t)^{-n/2} f(x/(1-t))$. Therefore, we obtain the approximate solution of the geometrical optics form (2):

$$v^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} e^{i\frac{|x|^2}{2\varepsilon(t-1)}} f\left(\frac{x}{1-t}\right).$$

This solution approximates the free solution away from the caustic. Indeed, if $f \in L^2$ then there exists a real-valued continuous function h with $h(0) = 0$ such that

$$\|w^\varepsilon(t) - v^\varepsilon(t)\|_{L^2} \leqslant h\left(\frac{\varepsilon t}{1-t}\right).$$

Remark 1.3. If the nonlinearity F is Gauge invaliant, that is, if F satisfies

$$F(\omega u) = \omega F(u), \quad \forall \omega \in \mathbb{C} \text{ s.t. } |\omega| = 1,$$

then we can solve the transport equation (6) also for $\alpha \leqslant 1$. By the method of variation of constants, we obtain

$$a_0(t,x) = \frac{1}{(1-t)^{n/2}} f\left(\frac{x}{1-t}\right) \times e^{ig^\varepsilon\left(t, \frac{x}{1-t}\right)},$$

where

$$g^\varepsilon(t,y) = -i\varepsilon^{\alpha-1} \int_0^t \frac{F((1-s)^{-n/2} f(y))}{(1-s)^{-n/2} f(y)} ds.$$

1.4 Scaling argument and the second critical index

Next, we consider the following scaling:

$$u^{\varepsilon}(t,x) = \varepsilon^{-\frac{n}{2}} \phi^{\varepsilon}\left(\frac{t-1}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{7}$$

This scaling means the closeup with the center $(t,x) = (1,0)$. With this scaling, we can observe the micro structure of the equation (NLS$^{\varepsilon}$)–(1) around caustic. The equation for ϕ^{ε} becomes

$$\begin{cases} i\partial_t \phi^{\varepsilon} + \dfrac{1}{2}\Delta \phi^{\varepsilon} = \varepsilon^{\alpha} F^{\varepsilon}(\phi^{\varepsilon}), & (t,x) \in \mathbb{R}^{1+N}, \\ \phi^{\varepsilon}_{|t=-1/\varepsilon} = \varepsilon^{\frac{n}{2}} e^{-i\frac{\varepsilon|x|^2}{2}} f(\varepsilon x), \end{cases} \tag{SNLS$^{\varepsilon}$}$$

where $F^{\varepsilon}(\phi)(t,x) = \varepsilon^{\frac{N}{2}} F\big(\varepsilon^{-\frac{N}{2}}\phi(\frac{\cdot-1}{\varepsilon}, \frac{\cdot}{\varepsilon})\big)(\varepsilon t + 1, \varepsilon x)$. We note

$$\begin{aligned} F(u) = |u|^{\beta} u &\Longrightarrow F^{\varepsilon}(\phi) = \varepsilon^{-\frac{N\beta}{2}} |\phi|^{\beta}\phi, \\ F(u) = (|x|^{-\gamma} * |u|^2)u &\Longrightarrow F^{\varepsilon}(\phi) = \varepsilon^{-\gamma}(|x|^{-\gamma} * |\phi|^2)\phi. \end{aligned}$$

Then, the second critical index is $\alpha = N\beta/2$ or $\alpha = \gamma$. This critical index depends on the nonlinearity, and one sees that this index is completely deferent from the first critical index $\alpha = 1$. If $F(u) = |u|^{\beta}u$ (resp. $F(u) = (|x|^{-\gamma} * |u|^2)u$) then we have the following distinguish:

$$\begin{aligned} \alpha &> \frac{N\beta}{2} \text{ (resp. } \alpha > \gamma \text{) : linear caustic} \\ \alpha &= \frac{N\beta}{2} \text{ (resp. } \alpha = \gamma \text{) : nonlinear caustic} \\ \alpha &< \frac{N\beta}{2} \text{ (resp. } \alpha < \gamma \text{) : supercritical caustic} \end{aligned}$$

Notice that the equation (SNLS$^{\varepsilon}$) has no parameter in its linear part (the left hand side). If α is critical value ($N\beta/2$ or γ), then the parameter does not appear in the equation (it appears in the initial condition). For example, if $F(u) = |u|^{\beta}u$ then the equation (SNLS$^{\varepsilon}$) becomes

$$\begin{cases} i\partial_t \phi^{\varepsilon} + \dfrac{1}{2}\Delta \phi^{\varepsilon} = \varepsilon^{\alpha - \frac{N\beta}{2}} |\phi|^{\beta}\phi, \\ \phi^{\varepsilon}_{|t=-1/\varepsilon} = \varepsilon^{\frac{n}{2}} e^{-i\frac{\varepsilon|x|^2}{2}} f(\varepsilon x). \end{cases}$$

Therefore, in the critical case $\alpha = N\beta/2$, this is the (ordinary) nonlinear Schrödinger equation with parameter-dependent initial condition. We next consider this initial condition, in the following Subsection 1.5.

1.5 Relation to the scattering theory

In this subsection, we assume $F(u) = |u|^\beta u$, $f \in \Sigma = H^1 \cap \mathcal{F}(H^1)$, and $\alpha = N\beta/2$ unless otherwise stated. We consider the micro-scale equation (SNLS$^\varepsilon$) with these conditions:

$$\begin{cases} i\partial_t \phi^\varepsilon + \dfrac{1}{2}\Delta \phi^\varepsilon = |\phi^\varepsilon|^\beta \phi^\varepsilon, \\ \phi^\varepsilon_{|t=-1/\varepsilon} = \varepsilon^{\frac{N}{2}} e^{-i\frac{\varepsilon|x|^2}{2}} f(\varepsilon x). \end{cases} \tag{8}$$

We put $U_0(t) = e^{i\frac{t}{2}\Delta}$, which solves the free Schrödinger equation. Then we have

$$\begin{aligned} U_0\left(\frac{1}{\varepsilon}\right)\phi^\varepsilon\left(-\frac{1}{\varepsilon}\right) =& \varepsilon^{\frac{N}{2}} U\left(\frac{1}{\varepsilon}\right) e^{-i\frac{\varepsilon|x|^2}{2}} f(\varepsilon x) \\ =& i^{-\frac{N}{2}} e^{i\frac{\varepsilon|x|^2}{2}} \mathcal{F}f \to i^{-\frac{N}{2}}\mathcal{F}f \quad (\varepsilon \to 0) \end{aligned}$$

in Σ. Therefore, we can deduce that the equation (8) is almost equivalent to the following problem:

$$\begin{cases} i\partial_t \phi + \dfrac{1}{2}\Delta \phi = |\phi|^\beta \phi, \\ \lim\limits_{t\to-\infty} U_0(-t)\phi(t) = i^{-\frac{N}{2}}\mathcal{F}f. \end{cases} \tag{9}$$

In fact, the scattering theory and this nonlinear caustic case is closely related. The nonlinear effect near the caustic can be estimated by the use of scattering operator in this nonlinear caustic case ([1, 4]). On the other hand, the semi-classical analysis in nonlinear caustic case is applicable to the scattering theory ([2]).

Now, we shall consider general α. We rescale (SNLS$^\varepsilon$) by the scaling $\phi^\varepsilon = \varepsilon^{-(\alpha-N\beta/2)/2}\psi^\varepsilon$, then we obtain

$$\begin{cases} i\partial_t \psi^\varepsilon + \dfrac{1}{2}\Delta \psi^\varepsilon = |\psi^\varepsilon|^\beta \psi^\varepsilon, \\ \psi^\varepsilon_{|t=-1/\varepsilon} = \varepsilon^{(\alpha-N\beta/2)/2} \times (\varepsilon^{\frac{N}{2}} e^{-i\frac{\varepsilon|x|^2}{2}} f(\varepsilon x)). \end{cases}$$

From the argument above, we see that this problem is almost equivalent to the problem (9) with initial datum of size $O(\varepsilon^{(\alpha-N\beta/2)/2})$. Therefore, this problem in linear caustic case $\alpha > N\beta/2$ is almost scattering problem with small data, and so is in supercritical caustic case with divergent data.

2 Main result

In this talk, we consider the following semi-classical Hartree equation:

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{1}{2}\varepsilon^2\Delta u^\varepsilon = \lambda\varepsilon^\alpha(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon, & (t,x) \in \mathbb{R}^{1+N} \\ u^\varepsilon_{|t=0} = e^{-i\frac{|x|^2}{2\varepsilon}} f(x), \end{cases} \tag{HE$^\varepsilon$}$$

where space dimensions $N \geqslant 2$, $\alpha > 0$, $1 < \gamma < \min(4, N)$, $\lambda > 0$, $f \in \Sigma = H^1 \cap \mathcal{F}(H^1)$. With the Hartree type nonlinearity $F(u) = \lambda(|x|^{-\gamma} * |u|^2)u$, the two critical indices are $\alpha = 1$ and $\alpha = \gamma$ (See Subsections 1.3 and 1.4).

	$\alpha > 1$	$\alpha = 1$	$\alpha < 1$
$\alpha > \gamma$	linear propagation linear caustic	nonlinear propagation linear caustic	supercritical propagation linear caustic
$\alpha = \gamma$	linear propagation nonlinear caustic	nonlinear propagation nonlinear caustic	supercritical propagation nonlinear caustic
$\alpha < \gamma$	linear propagation supercritical caustic	nonlinear propagation supercritical caustic	supercritical propagation supercritical caustic

Concerning this problem, Carles and Lannes study the cases $\alpha > \gamma > 1$ (linear propagation and linear caustic case) and $\alpha = \gamma > 1$ (linear propagation and nonlinear caustic case) in [4], and the author treats the case $\alpha > 1 \geqslant \gamma$ (linear propagation and linear caustic) in [7]. If $\alpha > \gamma > 1$ then the nonlinear effect is negligible everywhere, and the solution therefore asymptotically behaves as a free solution even near the caustic. The case $\alpha > 1 \geqslant \gamma$ is essentially the same. The difference is that the solution does not asymptotically behave as a free solution on $\mathbb{R}$, but on any bounded time interval. Since the nonlinearity becomes long range type, the above asymptotics fails near time $t = \pm\infty$. On the other hand, if $\alpha = \gamma > 1$ then the nonlinear effect appears only around caustic. The solution asymptotically behaves as a free solution away from the focal point. Moreover, the asymptotic behavior changes beyond the caustic and this change can be described by the scattering operator (See Subsection 1.5).

In this talk, we consider the linear propagation and super critical caustic case $1 < \alpha < \gamma$. In [7], the author shows that the nonlinear effect is negligible before the focus even in the supercritical case $\gamma > \alpha > \max(1, \gamma/2)$.

2.1 Definitions

Before stating the details, let us make some definitions. A pair of numbers (q, r) is called to be admissible if

$$\frac{2}{q} = \delta(r) := N\left(\frac{1}{2} - \frac{1}{r}\right)$$

and $2 \leqslant r \leqslant 2N/(N-2)$ ($2 \leqslant r \leqslant \infty$ if $N = 1$, $2 \leqslant r < \infty$ if $N = 2$). The operator $J^\varepsilon(t)$ is defined by

$$J^\varepsilon(t) = \frac{x}{\varepsilon} + i(t-1)\nabla,$$

which is a scaled version of the Galilean operator $x + it\nabla$. For an interval $I \subset \mathbb{R}$, we first define

$$\|\cdot\|_{M(I)} = \sup_{(q,r):\text{admissible}} \|\cdot\|_{L^q(I;L^r)}.$$

Then, we define the spaces $X^\varepsilon(I)$ and $Y^\varepsilon(I)$ as follows.

$$X^\varepsilon(I) = \left\{\phi \in C(I, L^2) : \|\phi\|_{X^\varepsilon(I)} < \infty\right\},$$
$$Y^\varepsilon(I) = \left\{\phi \in C(I, L^2) : \|\phi\|_{Y^\varepsilon(I)} < \infty\right\},$$

where

$$\|\cdot\|_{X^\varepsilon(I)} = \|\cdot\|_{M(I)} + \|\varepsilon\nabla\cdot\|_{M(I)} + \|J^\varepsilon(t)\cdot\|_{M(I)}$$

and

$$\|\cdot\|_{Y^\varepsilon(I)} = \|\cdot\|_{M(I)} + \|\varepsilon\nabla\cdot\|_{M(I)}.$$

Remark 2.1. If $N = 2$ then the pair $(2, \infty)$ is not admissible. Therefore, we understand that the above supremum is took over all admissible pairs (q, r) which satisfy $2 \leqslant r \leqslant r_0$ with fixed sufficiently large r_0.

2.2 Main theorem 1 – Existence and Boundedness

Let C_0 be a positive constant and define

$$I_0(\mu) = (-\infty, 1 - C_0\varepsilon^\mu], \tag{10}$$
$$I_1(\mu) = [1 + C_0\varepsilon^\mu, \infty). \tag{11}$$

Then we have the following theorem.

Theorem 2.1 (existence and boundedness). *Let $N \geqslant 2$, $\sqrt{2} < \gamma < \min(4, N)$, and $\lambda > 0$. Assume that*

$$\gamma \geqslant \alpha > \max\left(1, \frac{\gamma}{2}\right).$$

Define

$$\mu_0 = \frac{\alpha - \max(1, \gamma/2)}{\gamma - \max(1, \gamma/2)} - \zeta_0 \tag{12}$$

and

$$\mu_1 = \begin{cases} 1 - (\gamma - \alpha)\dfrac{\gamma^2 + 2\gamma - 2}{(\gamma - 1)(\gamma^2 - 2)} - \zeta_1 & \text{if } \gamma < 2, \\ 1 - (\gamma - \alpha)\dfrac{2(\gamma + 4)}{\gamma^2} - \zeta_1 & \text{if } \gamma \geqslant 2, \end{cases} \tag{13}$$

where ζ_0 and ζ_1 are sufficiently small constants. Set

$$P = \begin{cases} 1 - \dfrac{\gamma}{2}\mu_0 - \left(1 - \dfrac{\gamma}{2}\right)\mu_1 & \text{if } \gamma < 2, \\ 1 - \mu_0 & \text{if } \gamma \geqslant 2. \end{cases} \tag{14}$$

Then, for all $f \in \Sigma$, there exists $\varepsilon^* = \varepsilon^*(N, \|f : \Sigma\|, \lambda, \alpha, \gamma, C_0)$ such that (HE$^\varepsilon$) *has a unique solution in $X^\varepsilon(\mathbb{R})$ for all $0 < \varepsilon < \varepsilon^*$. Moreover the solution has the following upper bounds:*

$$\|u^\varepsilon\|_{Y^\varepsilon(I_0(\mu_0))} \leqslant C_1, \quad \|u^\varepsilon\|_{X^\varepsilon(I_0(\mu_0))} \leqslant C_2, \quad \|u^\varepsilon\|_{X^\varepsilon(\mathbb{R}\setminus I_0(\mu_0))} \leqslant C_3\varepsilon^{-P},$$

where C_1 and C_2 depend on $\|f; \Sigma\|$, and C_3 on γ, C_0, and $\|f; \Sigma\|$.

2.3 Main theorem 2 – Asymptotic expansion

For functions u_1, u_2, and u_3 of $(t,x) \in \mathbb{R}^{1+n}$, we define F_{t_0} as follows:

$$(F_{t_0}(u_1,u_2,u_3))(t,x) = -i\lambda\varepsilon^{\alpha-1}\int_{t_0}^{t} U^\varepsilon(t-s)(|x|^{-\gamma} * \mathrm{Re}(u_1\overline{u_2}))(s,x)u_3(s,x)ds, \tag{15}$$

where $U^\varepsilon(t) = e^{i(\varepsilon t/2)\Delta}$. Then, we define the functions w_n and $\widetilde{w_n}$ as follows:

$$w_1 = U^\varepsilon(t)u^\varepsilon(0), \tag{16}$$
$$\widetilde{w_1} = U^\varepsilon(t-2)u^\varepsilon(2), \tag{17}$$

and

$$w_n = \sum_{\substack{i,j,k\in\mathbb{N}\\ i+j+k=n+1}} F_0(w_i,w_j,w_k), \tag{18}$$
$$\widetilde{w_n} = \sum_{\substack{i,j,k\in\mathbb{N}\\ i+j+k=n+1}} F_2(\widetilde{w_i},\widetilde{w_j},\widetilde{w_k}) \tag{19}$$

for $n \geqslant 2$. Moreover, we use the following notation:

$$p(\alpha,\gamma,\mu) = \begin{cases} \alpha - 1 - \mu(\gamma-1) & \text{if } \gamma < 2, \\ \alpha - \dfrac{\gamma}{2} - \eta - \mu\left(\dfrac{\gamma}{2} - \eta\right) & \text{if } \gamma \geqslant 2 \end{cases} \tag{20}$$

with sufficiently small $\eta > 0$. Then, we have the following asymptotic result.

Theorem 2.2 (asymptotic expansions). *Let $N \geqslant 2$, $\sqrt{2} < \gamma < \min(4,N)$, and $\lambda > 0$. Let $f \in \Sigma$. Assume that*

$$\gamma \geqslant \alpha > \begin{cases} \gamma - \dfrac{(\gamma-1)(\gamma^2-2)}{\gamma^2+2\gamma-2} & \text{if } \gamma < 2, \\ \gamma - \dfrac{\gamma^2}{2(\gamma+4)} & \text{if } \gamma \geqslant 2. \end{cases}$$

Let μ_0, μ_1, and P be as in Theorem 2.1, and let u^ε be the unique solution to the equation (HE$^\varepsilon$) *given by Theorem 2.1.*

- *If $0 \leqslant \mu \leqslant \mu_0$ then it holds for any positive integer n that*

$$\|w_n\|_{X^\varepsilon(I_0(\mu))} = O(\varepsilon^{(n-1)p(\alpha,\gamma,\mu)}) \tag{21}$$

and

$$\left\| u^\varepsilon - \sum_{m=1}^{n} w_m \right\|_{X^\varepsilon(I_0(\mu))} = O(\varepsilon^{np(\alpha,\gamma,\mu)}) \tag{22}$$

as $\varepsilon \to 0$.

- *If* $0 \leqslant \mu \leqslant \mu_1$ *then it holds for any positive integer* n *that*

$$\|\widetilde{w_n}\|_{X^\varepsilon(I_1(\mu))} = O(\varepsilon^{(n-1)Q(\mu)-P}), \tag{23}$$
$$\|\widetilde{w_n}\|_{Y^\varepsilon(I_1(\mu))} = O(\varepsilon^{(n-1)Q(\mu)}), \tag{24}$$

and that

$$\left\| u^\varepsilon - \sum_{m=1}^{n} \widetilde{w_m} \right\|_{X^\varepsilon(I_1(\mu))} = O(\varepsilon^{nQ(\mu)-P}), \tag{25}$$
$$\left\| u^\varepsilon - \sum_{m=1}^{n} \widetilde{w_m} \right\|_{Y^\varepsilon(I_1(\mu))} = O(\varepsilon^{nQ(\mu)}) \tag{26}$$

as $\varepsilon \to 0$*, where* $Q(\mu) = p(\alpha, \gamma, \mu) - \max(2, \gamma + \nu)P$ *with arbitrary small* $\nu > 0$.

Remark 2.2. Figure 1 below shows the range

$$\gamma \geqslant \alpha > \begin{cases} \gamma - \dfrac{(\gamma-1)(\gamma^2-2)}{\gamma^2+2\gamma-2} & \text{if } \gamma < 2, \\ \gamma - \dfrac{\gamma^2}{2(\gamma+4)} & \text{if } \gamma \geqslant 2. \end{cases}$$

in Theorem 2.2, where μ_1 is positive.

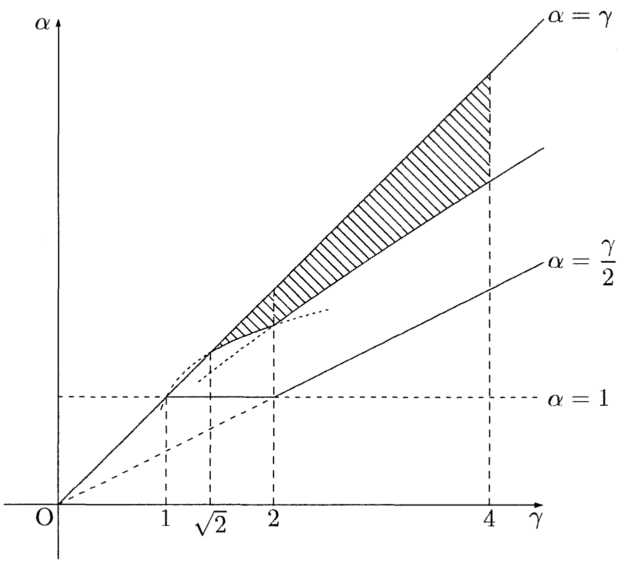

Fig.1

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On the Keller Segel system in higher dimensions

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1 Introduction

We consider the following reaction-diffusion equation:

$$\text{(KS)} \quad \begin{cases} u_t = \nabla \cdot \left(\nabla u^m - u^{q-1}\nabla v\right), & x \in \mathbb{R}^N,\ 0 < t < \infty, \\ 0 = \Delta v - v + u, & x \in \mathbb{R}^N,\ 0 < t < \infty, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 1,\ m \geq 1$ and $q > \{m + \frac{2}{N}, \frac{3}{2}\}$. The initial data u_0 is a non-negative function in $L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$. This equation was proposed by Keller-Segel [9] to describe the motion of the chemotaxis molds, and nowadays it is called Keller-Segel model.

The first equation of (KS) without the perturbation term is written as follows:

$$\text{(PM)} \qquad \psi_t(x,t) \ = \ \Delta \psi^m(x,t).$$

It is known that (PM) has the exact solution $V(x,t;M)$ with self-similarity, called *Barenblatt solution.*

For (KS), for $q > 2$ in [12], for $q = 2$ in [13], and for $\frac{3}{2} < q < 2$ in [14], it was shown that the exponent $q = m + \frac{2}{N}$ represents so called Fujita's one which divides the situation between the global existence and finite time blow-up to a solution of (KS). Specifically, it was proved in [12]–[14] that under the assumption $q > \frac{3}{2}$;
(i) when $q < m + \frac{2}{N}$, (KS) is globally solvable without the any restriction on the size of the initial data, and
(ii) when $m \geq 1$ and $q \geq m + \frac{2}{N}$, (KS) is globally solvable for small $L^{\frac{N(q-m)}{2}}$- initial data.

Furthermore, the decay of solution in $L^p(\mathbb{R}^N)(1 < p < \infty)$ was shown.

In the present article, we shall consider the above case (ii) and obtain the asymptotic profile of the solution $u(t)$ with a definite convergence rate in $L^p(\mathbb{R}^N)$. More precisely, we shall show that

(I) for (KS) with $m > 1$, we obtain the optimal convergence rate such as

$$\lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})} \left\| u(\cdot,t) - V(\cdot,t;\|u_0\|_{L^1(\mathbb{R}^N)}) \right\|_{L^p(B_{t,R})} = 0 \qquad \text{for } 1 < p < \infty$$

with $B_{t,R} := \{x \in \mathbb{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}$, where $V(x,t;M)$ is the well-known *Barenblatt solution* of (PM) such that $\displaystyle\int_{\mathbb{R}^N} V(x,t;M)\ dx = M$. For detail, see (2.4) in the next section.

We also discuss the semi-linear case: $m = 1$ of (KS) and

(II) for (KS) with $m = 1$, we prove that

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG_t(\cdot)\|_{L^p(B_{t,R})} = 0 \qquad \text{for } 1 < p < \infty,$$

where $G_t(x)$ is the heat kernel and $M = \|u_0\|_{L^1(\mathbb{R}^N)}$.

We thus propose the method to prove the asymptotic profile with the optimal convergence rate without "comparison principles and the representation formula of solutions." In many systems, it is difficult to show that a comparison principle holds. Our method could be applied to other nonlinear systems which do not make comparison principles ensure.

2 Results

Throughout this article, we deal with the weak solution of (KS). Our definition of the weak solution now reads:

Definition 1 *Let $m \geq 1$, $q > 1$ and let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$ and $u_0 \geq 0$. A pair (u,v) of non-negative functions defined in $\mathbb{R}^N \times [0,T)$ is called a weak solution of* (KS) *on $[0,T)$ if*

i) $u \in L^\infty(0,T;L^1 \cap L^\infty(\mathbb{R}^N))$, $\nabla u^m \in L^2(0,T;L^2(\mathbb{R}^N))$,

ii) $v \in L^\infty(0,T;H^1(\mathbb{R}^N))$,

iii) (u,v) *satisfies the equations in the sense of distribution, i.e., that*

$$\int_0^\infty \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla\varphi - u^{q-1}\nabla v \cdot \nabla\varphi - u\varphi_t\right)\ dxdt = \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\ dx,$$

$$\int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla\psi + v\psi - u\psi\right)(t)\ dx = 0 \qquad \textit{for a.a. } t \in (0,T)$$

for all functions $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,T))$ *and* $\psi \in C_0^\infty(\mathbb{R}^N)$.

We introduce the existence and decay property of a weak solution (u, v). The following proposition is a direct consequence of [10],[12]–[14].

Proposition 2.1 ([10],[12]–[14]) *Let* $1 \le p < \infty$, $N \ge 1$, $m \ge 1$, $q > \frac{3}{2}$ *and* $q \ge m + \frac{2}{N}$, $\ell \ge \frac{N(q-m)}{2}$ (≥ 1). *Suppose that the initial data* u_0 *is non-negative everywhere. Then, there exist an absolute constant* M *and a positive number* ε *depending only on* M, p, N, m, ℓ *such that if* $u_0 \in L^1 \cap L^\ell(\mathbb{R}^N)$ *satisfies that*

$$\|u_0\|_{L^1(\mathbb{R}^N)} = M, \quad \|u_0\|_{L^\ell(\mathbb{R}^N)} \le \varepsilon, \tag{2.1}$$

then (KS) *has a weak solution* (u, v) *on* $[0, \infty)$ *with the following decay property: there exists a constant* C_p *depending only on* p, $\|u_0\|_{L^p(\mathbb{R}^N)}$ *together with* N, m, q, M, $\|u_0\|_{L^{(N+2)q}(\mathbb{R}^N)}$ *such that*

$$\|u(t)\|_{L^p(\mathbb{R}^N)} + \|v(t)\|_{L^p(\mathbb{R}^N)} \le C_p(1+t)^{-d} \qquad \textit{for all } 0 < t < \infty, \tag{2.2}$$

where

$$d = \sigma_m \left(1 - \frac{1}{p}\right), \quad \sigma_m = \frac{N}{N(m-1)+2}. \tag{2.3}$$

Remark 1 (i) The decay rate d depends on m, N but not on q.
(ii) The above convergence rate d seems to be optimal. In fact, for $m = 1$, we find that $\sigma_m = \frac{N}{2}$ whose decay rate d coincides with the L^1-L^p estimate for the linear heat equation.

We introduce the self-similar solution $V(x, t; M)$ by Barenblatt [1]:

$$V(x,t;M) := \frac{1}{t^{\sigma_m}} \left(\beta^2 M^{\frac{2\sigma_m(m-1)}{N}} - \frac{\sigma_m(m-1)}{2mN} \cdot \frac{|x|^2}{t^{\frac{2\sigma_m}{N}}}\right)_+^{\frac{1}{m-1}}, \tag{2.4}$$

where β is the parameter. In this article, we take β in such a way that $V(x, t; M)$ satisfies $\int_{\mathbb{R}^N} V(x,t;M)\,dx = M$ for all $t > 0$. We call the above function $V(x, t; M)$ *the Barenblatt solution.* Moreover, it is known that $V(x, t; M)$ is the weak solution for the Cauchy problem of (PM) corresponding to the initial data δM, where δ is the Dirac mass at the origin.

We denote the heat kernel $G_t(x)$ by $G_t(x) := \dfrac{1}{(4\pi t)^{\frac{N}{2}}} \exp\Big(-\dfrac{|x|^2}{4t}\Big)$.

We now give two main theorems. The first one is for the quasilinear case of $m > 1$.

Theorem 2.2 (asymptotic profile: Barenblatt solution) *Let the same assumption as that in Proposition 2.1 hold. In addition, let $m > 1$ and $q > m + \frac{2}{N}$. Then, the weak solution u obtained in Proposition 2.1 satisfies that*

$$(2.5) \qquad \lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})} \left\| u(\cdot,t) - V(\cdot,t; \|u_0\|_{L^1(\mathbf{R}^N)}) \right\|_{L^p(B_{t,R})} = 0, \qquad 1 < p < \infty$$

for all $R > 0$, where σ_m is the exponent defined in (2.3) *and $B_{t,R}$ is the ball defined by*

$$(2.6) \qquad B_{t,R} := \{x \in \mathbf{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}.$$

Remark 2 (i) The solution of (PM) has the similar property as Theorem 2.2. Indeed, for the solution ψ of (PM), it holds

$$(2.7) \qquad \lim_{t\to\infty} t^{\sigma_m(1-\frac{1}{p})} \|\psi(\cdot,t) - V(\cdot,t; \|\psi(0)\|_{L^1(\mathbf{R}^N)})\|_{L^p(\mathbf{R}^N)} = 0$$

for any $1 \le p \le \infty$. (we refer to Bénilan [2], Friedman-Kamin [5], Kamin [7], Kamin-Vazquez [8], Véron [15].) Hence, Theorem 2.2 implies that Δu^m is dominant to $\nabla(u^{q-1}\nabla v)$ in the case of "$q > m + \frac{2}{N}$ and small initial data",

(ii) Proposition 2.1 includes the case of $q = m + \frac{2}{N}$. On the other hand, Theorem 2.2 excludes the case of $q = m + \frac{2}{N}$.

The next theorem is for the semi-linear case of $m = 1$.

Theorem 2.3 (asymptotic profile: heat kernel) *Let the same assumption as that in Proposition 2.1 hold. In addition, let $m = 1$ and $q > 1 + \frac{2}{N}$. Then, the weak solution u obtained in Proposition 2.1 satisfies that*

$$(2.8) \qquad \lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})} \|u(\cdot,t) - \|u_0\|_{L^1(\mathbf{R}^N)} G_t(\cdot)\|_{L^p(B_{t,R})} = 0, \qquad 1 < p < \infty$$

for all $R > 0$, where $B_{t,R}$ is the ball defined in (2.6).

Remark 3 The asymptotic profile as (2.8) (in the whole domain) was firstly obtained by Nagai-Syukuinn-Umesako [11] for Keller-Segel model of parabolic-parabolic type. Their argument is based on the representation formula of solutions. On the other hand, we study the Keller-Segel model of parabolic-elliptic type and give another proof without using any representation formula of solutions.

To prove our main theorems, we make fully use of the scaling argument. Let us introduce rescaled functions w_k and z_k defined by

$$w_k(x,t) = k^N u(kx, k^{N(m-1)+2}t) \quad \text{and} \quad z_k(x,t) = k^N v(kx, k^{N(m-1)+2}t) \quad \text{for } k \ge 1.$$

Then we see that (KS) can be rewritten as

$$
{}^{w}(\mathrm{KS}) \begin{cases} w_{kt} &= \nabla\cdot\Big(\nabla(w_k)^m - k^{-N(q-m)}(w_k)^{q-1}\nabla z_k\Big), \quad (x,t)\in\mathbb{R}^N\times(0,\infty), \quad \cdots(1)_w \\ 0 &= k^{-2}\Delta z_k - z_k + w_k, \qquad (x,t)\in\mathbb{R}^N\times(0,\infty), \quad \cdots(2)_w \\ w_k(x,0) &= k^N u_0(kx), \qquad x\in\mathbb{R}^N \end{cases}
$$

where $N\geq 1,\ m>1,\ q>\frac{3}{2},\ q\geq m+\frac{2}{N}$.

It should be noted that ^{w}(KS) does not have any invariance under change of scaling more. However, under the hypothesis $q>m+\frac{2}{N}$, it has an advantage since we can gain the negative power $-N(q-m)$ to k of the coefficient $w_k^{q-1}\nabla z_k$ which may be regarded as the small perturbation term. Hence, for $q>m+\frac{2}{N}$ we [10] proved that the sequence $\{w_k\}_{k=1}^{\infty}$ is bounded in $L^{\infty}(\mathbb{R}^N\times(\delta,T))$ together with the fact that $\{w_k^m\}_{k=1}^{\infty}$ is also bounded in $H^1(\delta,T;L^2(\mathbb{R}^N))\cap L^{\infty}(\delta,T;H^1(\mathbb{R}^N))$ for all $\delta>0$. These bounds and the standard compactness argument yield a subsequence of $\{w_k\}_{k=1}^{\infty}$, which we denote by $\{w_k\}_{k=1}^{\infty}$ itself for simplicity, and a function $U(x,t)$ such that

$$
(2.9) \qquad \|w_k(\cdot,t)-U(\cdot,t)\|_{L^p(B_R)}\to 0 \qquad \text{for all } 1<p<\infty \quad \text{as } k\to\infty
$$

with the ball $B_R:=\{x\in\mathbb{R}^N;|x|<R\}$. Here, we may take arbitrary $R>0$. On account of the negative power $-N(q-m)$ to k in ^{w}(KS) as is described above, we see that U is, in fact, a weak solution of (PM) with the property that $\|U(\cdot,t)\|_{L^1(\mathbb{R}^N)}=M=:\|u_0\|_{L^1(\mathbb{R}^N)}$. Furthermore, it turns out that both $U(\cdot,t)$ and $V(\cdot,t;M)$ converge to $M\delta$ in the sense of distributions as $t\downarrow 0$, which yields with the aid of uniqueness result due to Dahlberg-Kenig [4] that $U(x,t)\equiv V(x,t;M)$. Now, taking $k=t^{\sigma_m/N}$ in (2.9) and then returning to our original solution u from the rescaled sequence $\{w_k\}_{k=1}^{\infty}$, we obtain the desired asymptotic profile such as (2.5).

We will use the simplified notations:

1) $\partial_t=\frac{\partial}{\partial t},\quad \partial_i=\frac{\partial}{\partial x_i},\ \partial_{ij}^2=\partial_i\partial_j,\ \nabla u=\Big(\partial_1,\partial_2,\cdots\Big),\ \nabla^2 u=\Big(\partial_{11}^2,\partial_{12}^2,\cdots\Big)$,

2) $\|\cdot\|_{L^r}=\|\cdot\|_{L^r(\mathbb{R}^N)},(1\leq r\leq\infty),\ \int\cdot\,dx:=\int_{\mathbb{R}^N}\cdot\,dx$.

3) $Q_T:=\mathbb{R}^N\times(0,T),\ B_R:=\{x\in\mathbb{R}^N;|x|<R\}$.

4) When the weak derivatives $\nabla u,\nabla^2 u$ and $\partial_t u$ are in $L^p(Q_T)$ for some $p\geq 1$, we say that $u\in W_p^{2,1}(Q_T)$, *i.e.*,

$$
\begin{aligned} W_p^{2,1}(Q_T) &:= \Big\{u\in L^p(0,T;W^{2,p}(\mathbb{R}^N))\cap W^{1,p}(0,T;L^p(\mathbb{R}^N)); \\ &\qquad \|u\|_{W_p^{2,1}(Q_T)}:=\|u\|_{L^p(Q_T)}+\|\nabla u\|_{L^p(Q_T)}+\|\nabla^2 u\|_{L^p(Q_T)}+\|\partial_t u\|_{L^p(Q_T)}<\infty\Big\}. \end{aligned}
$$

3 Outline of proof

Let us recall ^{w}(KS) introduced in Section 2. The problem ^{w}(KS) does not have any invariance under change of scaling. However, we can show that the sequence $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded in $\mathbb{R}^N \times (\delta, T)$ together with the fact that

$$\text{(3.1)} \qquad \{(w_k)^m\}_{k=1}^{\infty} \text{ is also bounded in } H^1(\delta, T; L^2(\mathbb{R}^N)) \cap L^{\infty}(\delta, T; H^1(\mathbb{R}^N))$$

for all $0 < \delta < T < \infty$. By (3.1) and the standard compactness theorem, we find that there exist a subsequence, still denoted by $\{w_k\}$, and a function U on $\mathbb{R}^N \times (0, \infty)$ such that

$$\text{(3.2)} \qquad \|w_k(t) - U(t)\|_{L^p(B_R)} \to 0 \qquad \text{with } 1 < p < \infty, \text{ as } k \to \infty$$

for all $0 < t < \infty$ and all $R > 0$, where $B_R := \{x \in \mathbb{R}^N;\ |x| < R\}$.
On account of the negative power $-N(q-m)$ to k of the coefficient $w_k^{q-1}\nabla z_k$, we may treat $k^{-N(q-m)}\nabla(w_k^{q-1}\nabla z_k)$ as the small perturbation term. As a result, we find that this function U satisfies (PM) in the following weak sense:

$$\text{(3.3)} \qquad \int_0^{\tau}\int_{\mathbb{R}^N} (U\varphi_t + U^m \Delta\varphi)\ dxdt \ = \ \int_{\mathbb{R}^N} U(x,\tau)\varphi(\cdot,\tau)\ dx - \|u_0\|_{L^1(\mathbb{R}^N)}\varphi(0,0)$$

for all C^{∞} functions $\varphi(x,t)$ with compact support in $\mathbb{R}^N \times (0, T]$, and all $0 < \tau < T$. It should be noted that the Barenblatt solution $V(x, t; M)$ also satisfies (3.3).

Furthermore, it turns out that

$$\text{(H1)} \qquad U(t) \in L^1(\mathbb{R}^N) \quad \text{and} \quad \|U(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)} \qquad \text{for all } 0 < t < \infty$$

with the property that

$$\text{(H2)} \qquad \lim_{t \downarrow 0} \int_{\mathbb{R}^N} U(x,t)\psi(x)\ dx \ = \ \|u_0\|_{L^1(\mathbb{R}^N)}\psi(0).$$

On the other hand, it is easy to show that

$$\text{(H3)} \qquad \lim_{t \downarrow 0} \int_{\mathbb{R}^N} V(x,t; \|u_0\|_{L^1(\mathbb{R}^N)})\psi(x)\ dx \ = \ \|u_0\|_{L^1(\mathbb{R}^N)}\psi(0)$$

for all $\varphi \in C_0^{\infty}(\mathbb{R}^N)$. Then, by the uniqueness theorem given by Dahlberg-Kenig [4], we conclude that

$$\text{(3.4)} \qquad U(x,t) = V(x,t; \|u_0\|_{L^1(\mathbb{R}^N)}) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0, T].$$

Combining (3.2) with (3.4), we have

$$\text{(3.5)} \qquad \|w_k(\cdot, 1) - V(\cdot, 1; \|u_0\|_{L^1})\|_{L^p(B_R)} \to 0, \qquad 1 < p < \infty$$

as $k \to \infty$ for all $R > 0$, where $B_R := \{x \in \mathbb{R}^N;\ |x| < R\}$. Now taking k as $k = t^{\frac{\sigma_m}{N}}$ in (3.5), we conclude that

$$t^{\sigma_m(1-\frac{1}{p})} \left\| u(\cdot,t) - V(\cdot,t; \|u_0\|_{L^1(\mathbb{R}^N)}) \right\|_{L^p(B_{t,R})} \to 0 \quad \text{with } 1 < p < \infty, \text{ as } t \to \infty$$

for all $R > 0$, where $B_{t,R} := \{x \in \mathbb{R}^N;\ |x| < Rt^{\frac{1}{N(m-1)+2}}\}$. Thus, we obtain the optimal convergence rate.

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Stability of stationary solutions for surface diffusion flow with boundary conditions*

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1 Introduction

The geometrical evolution law

$$V = -\Delta\kappa$$

was derived by Mullins [6] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces (for simplicity we set the diffusion constant to 1). Here V is the normal velocity of the evolving interface, Δ is the Laplace-Beltrami operator and κ is the mean curvature of the interface where we use the sign convention that a sphere with the normal pointing to the inside has positive curvature.

In this talk we study the following problem. Given an open bounded domain $\Omega \subset \mathbb{R}^2$ we look for evolving curves $\Gamma = \{\Gamma_t\}_{t>0}$ (for a definition, see Gurtin [5]), which lies in Ω and satisfies $\partial\Gamma_t \subset \partial\Omega$, with the properties for $t > 0$:

$$\begin{cases} V = -\kappa_{ss} & \text{on} \quad \Gamma_t, \\ \Gamma_t \perp \partial\Omega & \text{at} \quad \Gamma_t \cap \partial\Omega, \\ \kappa_s = 0 & \text{at} \quad \Gamma_t \cap \partial\Omega, \end{cases} \tag{1.1}$$

where a subscript s denotes the differentiation with respect to the arc-length parameter.

For the readers convenience, we show some differences between the curvature flow and the surface diffusion flow.

- The curvature flow: $V = \kappa$
  - The gradient flow of the length with respect to the L^2-inner product.
  - Not area-preserving.
  - Stationary solutions are the line segments.
  - A singular limit of Allen-Cahn type equations.
- The surface diffusion flow: $V = -\kappa_{ss}$
  - The gradient flow of the length with respect to the H^{-1}-inner product (see [7]).
  - Area-preserving.

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– Stationary solutions are the line segments and the part of circle.

– A singular limit of Cahn-Hilliard equation.

Our goal in this talk is to derive the global existence and the nonlinear stability of stationary solutions for (1.1) when stationary solutions are linear stable.

2 Local existence

Let Ω be a domain in $\mathbb{R}^2$ such that there exists a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $\psi(x) = 0$ and that

$$\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) < 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) = 0\}.$$

In order to derive the local existence for (1.1), we use the parameterization considered in [4]. For the readers convenience, we state such parameterization below.

Let Γ^* be a stationary solution and σ be the arc-length parameter of Γ^*. Then we denote an arc-length parameterization of Γ^* as

$$\Gamma^* = \{\Phi^*(\sigma) \mid \sigma \in [\ell_-, \ell_+]\}.$$

Note that we can extend Γ^* naturally either to the full circle when Γ^* is a part of circle or to the straight line when Γ^* is a line segment. Also note that the curvature κ^* of Γ^* is a constant. We denote

$$\bar{\ell} := \begin{cases} 2\pi/|\kappa^*|, & \kappa^* \neq 0, \\ +\infty, & \kappa^* = 0. \end{cases}$$

i.e. $\bar{\ell}$ is the length of the extension of Γ_* to a full circle (if $\kappa_* \neq 0$). Define

$$\begin{cases} \xi_+(q) = \max\{\sigma \in (-\bar{\ell}, \bar{\ell}) \mid \Phi^*(\sigma) + qN^*(\sigma) \in \Omega\}, \\ \xi_-(q) = \min\{\sigma \in (-\bar{\ell}, \bar{\ell}) \mid \Phi^*(\sigma) + qN^*(\sigma) \in \Omega\}. \end{cases}$$

where $q \in [-d, d]$ for a small $d > 0$, and $N^*(\sigma)$ is a unit normal vector of Γ^* at σ and is obtained by rotating the unit tangent vector $T^*(\sigma)$ of Γ^* with $\pi/2$. Then it holds $\psi(\Phi^*(\xi_\pm(q)) + qN^*(\xi_\pm(q))) = 0$. In addition, we have $\xi_\pm(0) = \ell_\pm$. Using the implicit function theorem, we see that $\xi_+(q)$ and $\xi_-(q)$ are smooth. Let

$$\Psi(\sigma, q) := \Phi^*(\xi(\sigma, q)) + qN^*(\xi(\sigma, q))$$

with

$$\xi(\sigma, q) := \xi_-(q) + \frac{\sigma - \ell_-}{\ell_+ - \ell_-}(\xi_+(q) - \xi_-(q)).$$

Note that $\xi(\ell_\pm, q) = \xi_\pm(q)$ and $\xi(\sigma, 0) = \sigma$.

Let Γ be curves in the neighbourhood of Γ^*, which touch the boundary $\partial\Omega$ and are contained in Ω. For some functions $\rho : [\ell_-, \ell_+] \to [-d, d]$, we define $\Phi(\sigma) := \Psi(\sigma, \rho(\sigma))$ for $\sigma \in [\ell_-, \ell_+]$, which denotes a parameterization of such curves Γ. Thus we set

$$\Gamma_t := \{\Phi(\sigma, t) \mid \sigma \in [\ell_-, \ell_+]\} \tag{2.1}$$

with $\Phi(\sigma,t) := \Psi(\sigma,\rho(\sigma,t))$ for a function ρ depending on σ and t. We remark that $\rho\equiv 0$ means that curves Γ coincide with a stationary curve Γ^*.

Let us derive the representation of (1.1) to the parameterization (2.1). For the arc-length parameter s of Γ, we have

$$\frac{ds}{d\sigma} = |\Phi_\sigma| = \sqrt{|\Psi_\sigma|^2 + 2(\Psi_\sigma,\Psi_q)_{\mathbb{R}^2}\rho_\sigma + |\Psi_q|^2\rho_\sigma^2} \ (=: J(\rho)). \tag{2.2}$$

Here and hereafter $(\cdot,\cdot)_{\mathbb{R}^2}$ denotes the inner product in $\mathbb{R}^2$. Then we find

$$T = \frac{1}{J(\rho)}\Phi_\sigma, \quad N = \frac{1}{J(\rho)}R\Phi_\sigma,$$

where T and N are the unit tangent and normal vector of Γ respectively, and R is the rotation matrix with $\pi/2$. The normal velocity V of Γ_t is denoted by

$$V = (\Phi_t, N)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Phi_t, R\Phi_\sigma)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_t.$$

Furthermore, since (2.2) gives

$$\partial_s^2 = \frac{1}{J(\rho)}\partial_\sigma\left(\frac{1}{J(\rho)}\partial_\sigma\right) = \frac{1}{(J(\rho))^2}\partial_\sigma^2 + \frac{1}{J(\rho)}\left(\partial_\sigma\frac{1}{J(\rho)}\right)\partial_\sigma \ (=: \Delta(\rho)), \tag{2.3}$$

the curvature κ of Γ_t is written by

$$\begin{aligned}\kappa(\rho) &= (\Delta(\rho)\Phi, N)_{\mathbb{R}^2}\\ &= \frac{1}{(J(\rho))^3}(\Phi_{\sigma\sigma}, R\Phi_\sigma)_{\mathbb{R}^2}\\ &= \frac{1}{(J(\rho))^3}\Big[(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_{\sigma\sigma} + \big\{2(\Psi_{\sigma q}, R\Psi_\sigma)_{\mathbb{R}^2} + (\Psi_{\sigma\sigma}, R\Psi_q)_{\mathbb{R}^2}\big\}\rho_\sigma\\ &\quad +\big\{(\Psi_{qq}, R\Psi_\sigma)_{\mathbb{R}^2} + 2(\Psi_{\sigma q}, R\Psi_q)_{\mathbb{R}^2} + (\Psi_{qq}, R\Psi_q)_{\mathbb{R}^2}\rho_\sigma\big\}\rho_\sigma^2\\ &\quad +(\Psi_{\sigma\sigma}, R\Psi_\sigma)_{\mathbb{R}^2}\Big].\end{aligned} \tag{2.4}$$

If we note that the Neumann boundary condition $(\Phi_\sigma, T_{\partial\Omega})_{\mathbb{R}^2} = 0$ on $\partial\Omega$ is equivalent to $(R\Phi_\sigma, \nabla\psi(\Phi))_{\mathbb{R}^2} = 0$ on $\partial\Omega$, (1.1) are represented by

$$\begin{cases} \rho_t = -a(\rho)\Delta(\rho)\kappa(\rho) & \text{for } \ \sigma\in(\ell_-,\ell_+),\ t>0,\\ (R\Psi_\sigma + R\Psi_q\rho_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} = 0 & \text{at } \ \sigma = \ell_\pm,\\ \partial_\sigma\kappa(\rho) = 0 & \text{at } \ \sigma = \ell_\pm, \end{cases} \tag{2.5}$$

where $a(\rho) := J(\rho)/(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}$, and $\Delta(\rho)$ and $\kappa(\rho)$ are defined by (2.3) and (2.4), respectively. Now we are ready to state the local existence theorem.

Theorem 2.1 (Local existence) *Let $\alpha\in(0,1)$ and let us assume that $\rho_0\in C^3[\ell_-,\ell_+]$ fulfills the compatibility conditions. Then there exists a $T_0 = T_0(1/\|\rho_0\|_{C^{2+\alpha}[\ell_-,\ell_+]}) > 0$ such that the problem (2.5) with $\rho(\cdot,0) = \rho_0$ has a unique solution.*

3 Linearized stability

Define the bilinear form

$$I^*[w_1, w_2] = \int_{\ell_-}^{\ell_+} \{\partial_\sigma w_1 \partial_\sigma w_2 - (\kappa^*)^2 w_1 w_2\}\, d\sigma + h_+^* w_1 w_2\big|_{\sigma=\ell_+} + h_-^* w_1 w_2\big|_{\sigma=\ell_-}$$

for $w_i \in H^1(\Gamma^*)$ $(i = 1, 2)$ with

$$\int_{\ell_-}^{\ell_+} w_i \, d\sigma = 0.$$

Here κ^* is the curvature of Γ^* and $h_\pm^*$ are the curvature of $\partial\Omega$ at $\Gamma^* \cap \partial\Omega$. According to [4], the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ for the linearized problem derived from (2.5) are real and characterized by

$$\lambda_n = -\inf_{W\in\Sigma_n} \sup_{\rho\in W\setminus\{0\}} \frac{I^*[\rho,\rho]}{(\rho,\rho)_{-1}}, \quad \lambda_n = -\sup_{W\in\Sigma_{n-1}} \inf_{\rho\in W^\perp\setminus\{0\}} \frac{I^*[\rho,\rho]}{(\rho,\rho)_{-1}}$$

where Σ_n is the collection of n-dimensional subspaces of

$$\mathcal{E} = \{\rho \in H^1(\ell_-, \ell_+) \,|\, \int_{\ell_-}^{\ell_+} \rho \, d\sigma = 0\}.$$

and $W^\perp$ is the orthogonal complement with respect to the inner product $(\cdot\,,\cdot)_{-1}$. In particular, the maximal eigenvalue λ_1 is represented by

$$\lambda_1 = -\inf_{\rho\in\mathcal{E}\setminus\{0\}} \frac{I^*[\rho,\rho]}{(\rho,\rho)_{-1}}.$$

Then we see that $I^*[\rho, \rho]$ is positive for $\rho \in \mathcal{E} \setminus \{0\}$ if $\lambda_1 < 0$, which implies the linearized stability of Γ^*. In [4], we obtain the following theorem.

Theorem 3.1 *Let N_U be the number of positive eigenvalues and N_N be the number of zero eigenvalues. In addition, set*

$$D(h_+^*, h_-^*) = h_+^* h_-^* + b\,(h_+^* + h_-^*) + c$$

where b and c are constants depending on κ^* and $L^*\,(= \ell_+ - \ell_-)$. Then it holds

Case A: If $D(h_-^*, h_+^*) > 0$ and $h_-^* > -b$, then

$$N_U = N_N = 0. \quad \textit{i.e. } \Gamma^* \textit{ is linear stable.}$$

Case B: If $D(h_-^*, h_+^*) = 0$ and $h_-^* > -b$, then

$$N_U = 0, \;\; N_N = 1.$$

Case C: If $D(h_-^*, h_+^*) < 0$, then

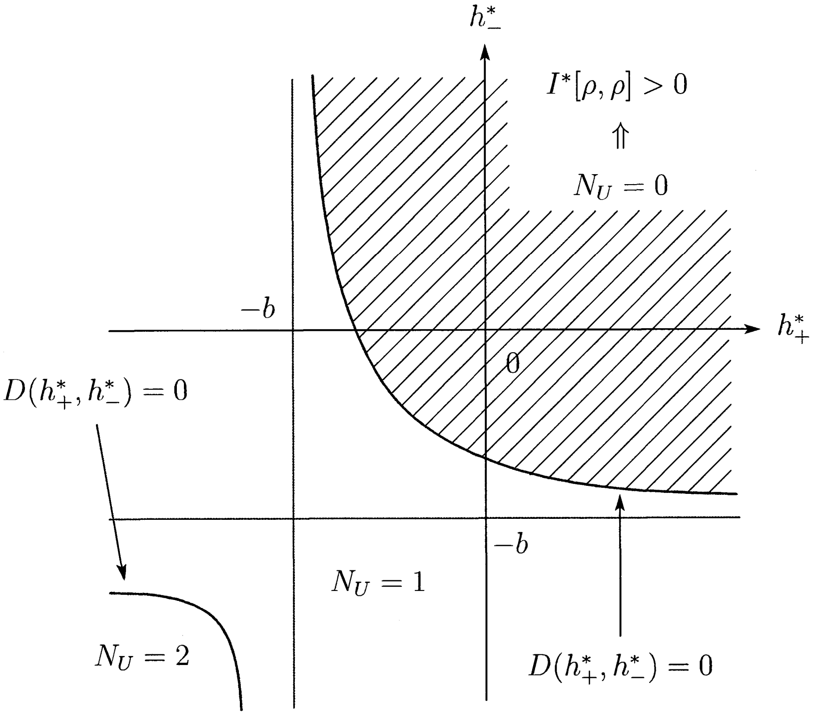
$$N_U = 1, \;\; N_N = 0\,.$$

Case D: If $D(h_-^*, h_+^*) = 0$ and $h_-^* < -b$, then

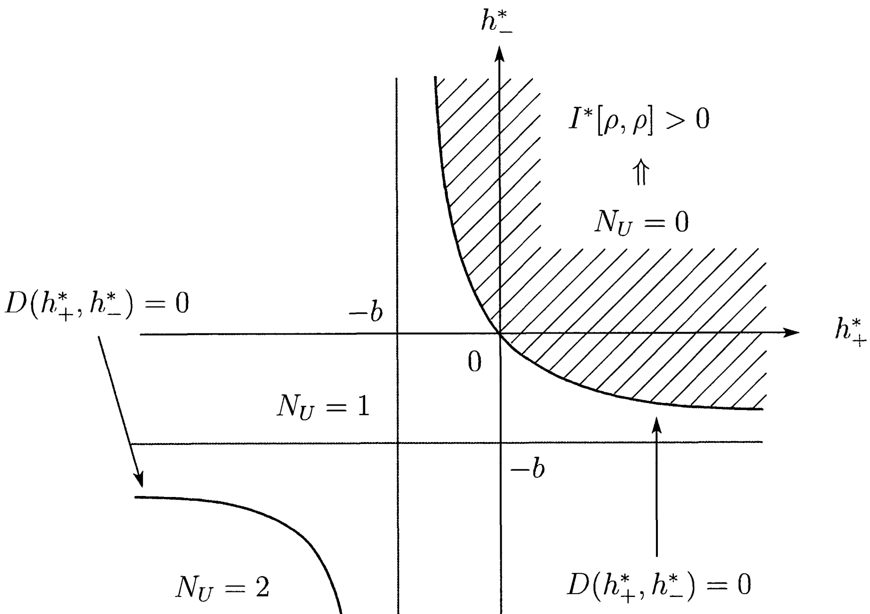
$$N_U = 1, \;\; N_N = 1.$$

Case E: If $D(h_-^*, h_+^*) > 0$ and $h_-^* < -b$, then

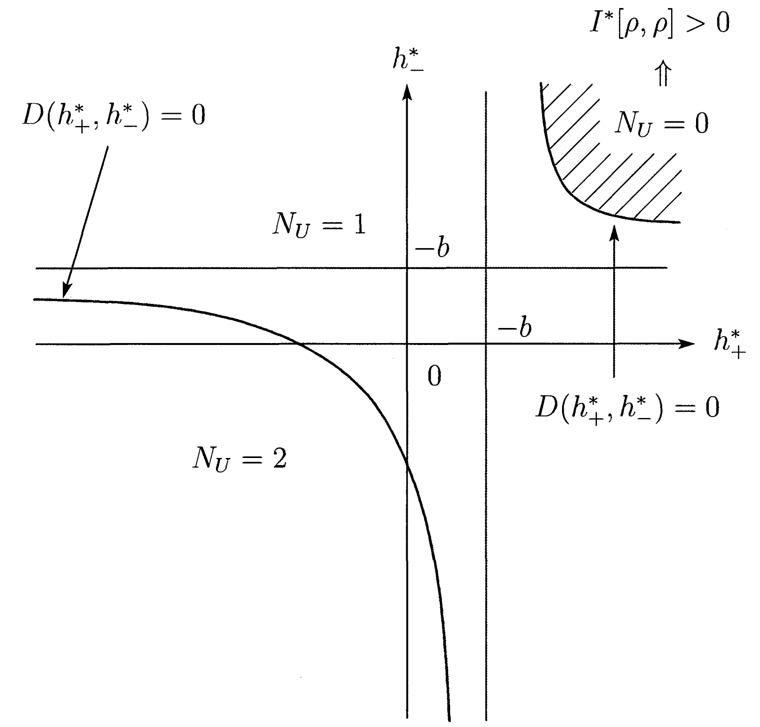
$$N_U = 2, \;\; N_N = 0.$$



Case: $L^* < \pi/|\kappa^*|$



Case: $L^* = \pi/|\kappa^*|$



Case: $L^* > \omega_0/|\kappa^*|$ where $\omega_0\ (> \pi)$ filfills $b = 0$.

4 Global existence and nonlinear stability

Let $L[\Gamma_t]$ be the length of Γ_t. Then we obtain the following lemma.

Lemma 4.1 *A smooth solution of (2.5) fulfills*

(i) $\displaystyle \frac{d}{dt}L[\Gamma_t] + \int_{r_-(t)}^{r_+(t)} \kappa_s^2\, ds = 0,$

(ii) $$\frac{d}{dt}\int_{r_-(t)}^{r_+(t)} \kappa_s^2\, ds = -2\left[\int_{r_-(t)}^{r_+(t)} \{V_s^2 - \kappa^2 V^2\}\, ds + h_+(V^2|_{s=r_+(t)}) + h_-(V^2|_{s=r_-(t)})\right] + \int_{r_-(t)}^{r_+(t)} \kappa_s^2 \kappa V\, ds.$$

Here $h_\pm$ are the curvature of $\partial\Omega$ at $\Gamma_t \cap \partial\Omega$.

Furthermore, we have the following result.

Lemma 4.2 (T. Vogel [8]) *Assume that a stationary solution Γ^* is given such that the bilinear form $I^*[\rho,\rho]$ is positive. Let $\rho \in C^1[\ell_-, \ell_+]$ be a function describing a curve close to Γ^* as in Section 2. In addition, assume that the curve parametrized by $\Psi(\sigma, \rho(\sigma))$ in Section 2 includes the same area as Γ^*. Then there exists a $\gamma^* > 0$ such that*

$$L[\Gamma_t] \geq L^* + \bar{c}\|\rho\|_{H^1}^2$$

provided that $\|\rho\|_{C^1} < \gamma^*$, where $L[\Gamma_t]$ is the length of Γ_t and L^* is the length of Γ^*.

Lemma 4.1 and Lemma 4.2 implies a priori estimates when the initial curve $\Gamma(0)$ is close to a stationary curve Γ^*, which satisfies that the bilinear form $I^*[\rho,\rho]$ is positive. Then we obtain the following result.

Theorem 4.3 (Global existence) *Let $\rho_0 \in C^3[\ell_-,\ell_+]$ fulfills the compatibility conditions and let*

$$\Lambda(t) := \|\kappa_s(t)\|_{L^2}^2 + L[\Gamma_t] - L^*.$$

Assume that a stationary solution Γ^* is given such that the bilinear form $I^*[\rho,\rho]$ is positive. Then, there exist $\gamma_0 > 0$ and $\delta_0 > 0$ such that if $\|\rho_0\|_{C^1} < \gamma_0$ and $\Lambda(0) < \delta_0$, the problem (2.5) admits a unique global solution.

The following theorem shows nonlinear stability of a stationary solution Γ^* which satisfies that the bilinear form $I^*[\rho,\rho]$ is positive.

Theorem 4.4 *Let the assumptions of Theorem 4.3 hold and let Γ^* be a stationary solutions such that the bilinear form $I^*[\rho,\rho]$ is positive. Then we obtain*

$$\|\rho(t)\|_{H^3} \to 0 \quad as \quad t \to \infty.$$

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The condition on the stability of stationary lines in a curvature flow in the whole plane

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1. Introduction

In this note we study a curvature flow in the whole plane. For a given constant $k \in \mathbb{R}$, we are concerned with a Cauchy problem

$$\frac{u_t}{\sqrt{1+u_x^2}} = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + k, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2}$$

We study stability of the traveling wave $u(x,t) = kt$ for spatially non-decaying initial values. Especially we focus on stability of the stationary solution $u(x,t) \equiv 0$ in the case of $k = 0$.

Our motivation to study this problem comes from the theory of interfacial phenomena, which is a part of mathematical science and is concerned with formation and development of a "shape" in biological, chemical, and physical fields.

Let $D(t)$ be a moving domain in $\mathbb{R}^n$ with a smooth boundary $\Gamma(t) = \partial D(t)$. Let $\boldsymbol{\nu}$ be the unit normal vector on $\Gamma(t)$ pointing from $D(t)$ to $D(t)^{\mathrm{C}}$. We consider an interface $\Gamma(t)$ governed by *the mean curvature flow* with constant driving force $k \in \mathbb{R}$. Namely, we have

$$V = -H + k, \tag{3}$$

where V is the velocity of $\Gamma(t)$ along $\boldsymbol{\nu}$, and $H = \operatorname{div} \boldsymbol{\nu}$. This model appears in several fields. One of them is the dynamics of interfaces in an excitable media, for example, Belousov-Zhabotinsky reaction [3, 17]. It also appears in the dynamics of interfaces in the Allen-Cahn equations. See [4] for instance. Moreover it appears in the reaction-diffusion systems of a competition type. See [8].

In this note, we consider a curve $\Gamma(t)$ in $\mathbb{R}^2$ governed by the curvature flow (3). We deal with the case where $\Gamma(t)$ is represented by a graph on the x-axis; in other words, an initial curve $\Gamma(0)$ is given by a function $y = u_0(x), x \in \mathbb{R}$, and a moving curve $\Gamma(t), t > 0$ is expressed by a function $y = u(x,t), x \in \mathbb{R}, t > 0$. Under these assumptions, the curvature flow (3) is rewritten as the Cauchy problem (1) - (2).

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1.1. **Related works.** A pioneering work for (1) - (2) with $k = 0$ is given by Ecker and Huisken [7]. They showed the existence of extracting self-similar solutions and the large time behavior of the solution, where an unbounded initial value $u_0(x)$ satisfies linear growth and further assumptions. Ishimura [10] also studied the same problem in detail.

For the case of $k \neq 0$, the classification of the traveling wave of (1) - (2) is obtained in [3, 15]. Ninomiya and Taniguchi [16] studied stability of a traveling wave of (1) - (2) that is called *the V-shaped front.* They showed that the V-shaped front is asymptotically stable for spatially decaying initial perturbations, and that it is not asymptotically stable for some initial perturbations similar to $\varphi^*(x)$ in Proposition 1.2 stated below.

1.2. **The heat equation.** As is mentioned above, we are interested in stability of the traveling wave $u(x,t) = kt$ of (1) - (2). In this connection, we mention some results for stability of $h(x,t) \equiv 0$ of the Cauchy problem of the heat equation

$$h_t = h_{xx}, \qquad x \in \mathbb{R},\ t > 0, \tag{4}$$

$$h(x,0) = \varphi(x), \quad x \in \mathbb{R}. \tag{5}$$

It is well known that the solution $h(x,t)$ of (4) - (5) is expressed by the heat kernel, and that $h(x,t)$ converges to $h(x,t) \equiv 0$ uniformly if an initial value decays to zero at infinity. To be more precise, stability of the stationary solution $h(x,t) \equiv 0$ of (4) - (5) is as follows.

Proposition 1.1 ([12, 18, 9, 11]). *The solution $h(x,t)$ to the Cauchy problem of the heat equation* (4) - (5) *satisfies* $\lim_{t\to\infty} h(0,t) = 0$*, if and only if* $\varphi(x)$ *satisfies*

$$\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \varphi(y)\,dy = 0. \tag{6}$$

Moreover $h(x,t)$ *satisfies* $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |h(x,t)| = 0$*, if and only if* $\varphi(x)$ *satisfies*

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{2R} \left| \int_{x-R}^{x+R} \varphi(y)\,dy \right| = 0. \tag{7}$$

Note that this is the sufficient and necessary condition for the asymptotic stability of $h(x,t) \equiv 0$. Collet and Eckmann [6] showed an example where an initial value does not satisfy the criterion (6) and the solution $h(x,t)$ oscillates forever. The initial value $\varphi^*(x)$ defined below does not decay but oscillates slower and slower as $|x| \to \infty$.

Proposition 1.2 ([6]). *Let* $L_n = n!$ *and define an even function* $\varphi^*(x) \in C^\infty(\mathbb{R})$ *that satisfies* $|\varphi^*(x)| \le 1$ *for* $x \in \mathbb{R}$ *and*

$$\varphi^*(x) = (-1)^n, \quad x \in [\,L_n + 2^n,\ L_{n+1} - 2^{n+1}\,]$$

for $n \ge 5$*. Then the solution* $h(x,t)$ *of* (4) - (5) *with* $h(x,0) = \varphi^*(x)$ *satisfies*

$$\liminf_{t\to\infty} h(0,t) = -1, \quad \limsup_{t\to\infty} h(0,t) = 1.$$

1.3. **Notation.** In what follows, $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, and $W^{1,\infty}(\mathbb{R})$ denote the Lebesgue or Sobolev spaces. For $\gamma \in (0,1)$, $C^\gamma(\mathbb{R})$ denotes the Hölder space, that is, the space of functions that are bounded and uniformly Hölder continuous with exponent γ on $\mathbb{R}$. $C^{2+\gamma}(\mathbb{R})$ means the space of functions with $u, u', u'' \in C^\gamma(\mathbb{R})$. For a domain $R_T = \mathbb{R} \times [0,T]$, $C^{\gamma,\gamma/2}(R_T)$ denotes the space of functions that are bounded and uniformly Hölder continuous with exponent γ and $\gamma/2$ with respect to x and t, respectively on R_T. $C^{2+\gamma,1+\gamma/2}(R_T)$ means the space of functions with $u, u_x, u_{xx}, u_t \in C^{\gamma,\gamma/2}(R_T)$.

2. Stability of the traveling line $u(x,t) = kt$

Now we show some results for the case of $k \in \mathbb{R}$. First we consider stability with spatially decaying initial values. The following result is similar to that for the Cauchy problem of the heat equation.

Theorem 2.1 (Nara and Taniguchi [13]). *Let $\gamma \in (0,1)$ be an arbitrary number. Suppose that an initial value $u_0 \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} u_0(x) = 0$. Then the solution $u(x,t)$ of (1) - (2) exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt| = 0.$$

Especially, if u_0 belongs to $C^{2+\gamma}(\mathbb{R}) \cap L^1(\mathbb{R})$, the solution $u(x,t)$ satisfies the estimate

$$\sup_{x\in\mathbb{R}} |u(x,t) - kt| \leq C(1+t)^{-\frac{1}{2}}, \quad t > 0,$$

where C is a constant depending only on k and u_0.

Next we show a result with spatially non-decaying initial values. For this purpose we recall the definition of *an almost periodic function.*

Definition 2.2. *A continuous function $f(x) : \mathbb{R} \to \mathbb{R}$ is called an almost periodic function (in the sense of Bohr) if, for every $\varepsilon > 0$, there exists $\ell(\varepsilon) > 0$ such that, for every $p \in \mathbb{R}$, an interval $[p, p+\ell(\varepsilon)]$ contains at least one number q with*

$$|f(x-q) - f(x)| < \varepsilon \quad \text{for all} \quad x \in \mathbb{R}. \tag{8}$$

For any almost periodic function f, there exists a mean $\mathcal{M}\{f\}$ defined by

$$\mathcal{M}\{f\} = \lim_{R\to\infty} \frac{1}{R} \int_s^{s+R} f(x)\,dx,$$

where the convergence is uniform with respect to $s \in \mathbb{R}$, and the limit is independent of s.

By this definition, every periodic function is an almost periodic function. Moreover if f and g are both almost periodic functions, $f(x) + g(x)$ is an almost periodic function, where $\mathcal{M}\{f+g\} = \mathcal{M}\{f\} + \mathcal{M}\{g\}$ holds true. Note that a non-periodic function $f(x) = \sin x + \sin\sqrt{2}x$ is an almost periodic function with $\mathcal{M}\{f\} = 0$. For further details, see [1, 2, 5] for instance.

The following result says that one of the sufficient conditions for asymptotic stability of $u(x,t) = kt$ is that the initial value converges to an almost periodic function asymptotically as $|x| \to \infty$.

Theorem 2.3 (Nara and Taniguchi [13]). *Let $\gamma \in (0,1)$ be an arbitrary number. Assume that $f \in C^{2+\gamma}(\mathbb{R})$ is an almost periodic function, and that $g \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} g(x) = 0$. Then the solution $u(x,t)$ of (1) - (2) with $u_0 = f + g$ exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - (kt + \mu)| = 0$$

for a constant μ that depends only on k and f, and is independent of g. Here μ satisfies

$$\inf_{x\in\mathbb{R}} f(x) \leq \mu \leq \sup_{x\in\mathbb{R}} f(x).$$

In addition, for each f, the constant μ is a nondecreasing function of $k \in \mathbb{R}$. Especially $\mu = \mathcal{M}\{f\}$ holds true when $k = 0$.

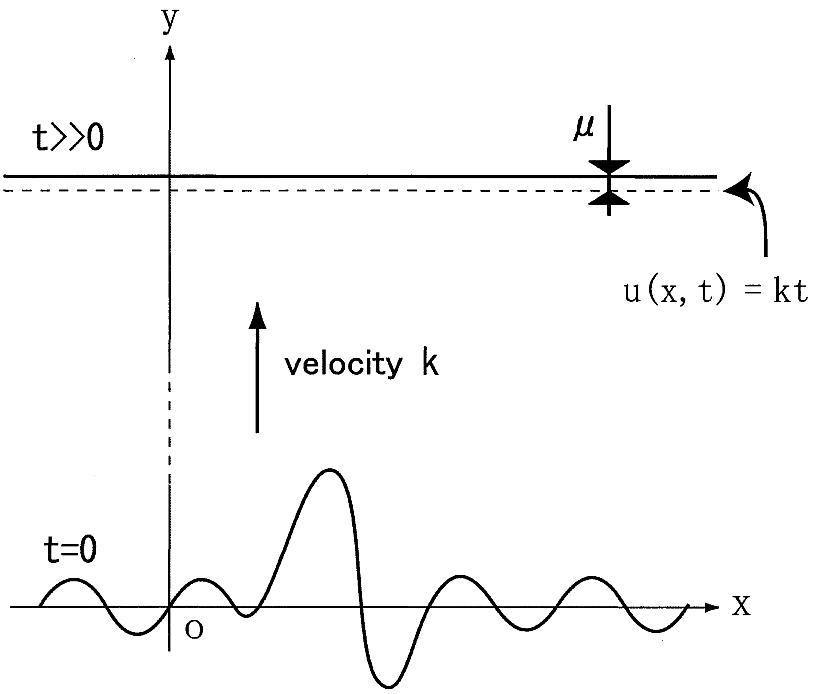

FIGURE 1. Stability of $u(x,t) = kt$ with a spatially non-decaying initial value.

Generically $\mu \neq \mathcal{M}\{f\}$ holds true if $k \neq 0$. Indeed, for $k > 0$ and $u_0 = \sin x$, we have $\mu > 0$ by using the periodic boundary condition at $x = 0,\ 2\pi$. Thus μ differs from $\mathcal{M}\{f\} = 0$ in this case.

3. Stability of the stationary line $u(x,t) \equiv 0$

Now we show a detailed result for the problem (1) - (2) with $k = 0$. Namely, we consider stability of the stationary line $u(x,t) \equiv 0$ of a Cauchy problem

$$\frac{u_t}{\sqrt{1+u_x^2}} = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}, \quad x \in \mathbb{R},\ t > 0, \tag{9}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{10}$$

Our main result is as follow. This theorem says that the solution of (9) - (10) converges uniformly to the solution of the Cauchy problem of the heat equation with the same initial value, if the initial value belongs to $C^{2+\gamma}(\mathbb{R})$.

Theorem 3.1 (Nara and Taniguchi [14]). *Let $\gamma \in (0,1)$ be an arbitrary number. Assume that $u_0 \in C^{2+\gamma}(\mathbb{R})$. Then the solution $u(x,t)$ of (9) - (10) exists up to $t = \infty$. Moreover it satisfies*

$$\sup_{x\in\mathbb{R}} \left| u(x,t) - \int_{\mathbb{R}} \Gamma(x-y,t)\, u_0(y)\, dy \right| \le C_1 t^{-\frac{1}{4}} \sqrt{\log t}, \quad t > 2$$

for a constant $C_1 > 0$ depending only on u_0. Here $\Gamma(\xi,\tau)$ is the heat kernel given by $\Gamma(\xi,\tau) = 1/\sqrt{4\pi\tau}\exp(-\xi^2/(4\tau))$.

Thus the large time behavior of the solution of (9) - (10) is derived directly from that of the Cauchy problem of the heat equation with the difference of order $O(t^{-1/4}\sqrt{\log t})$. Here we give an outline of the proof. The following proposition reduces the problem of asymptotic behavior to that of decay estimates for the derivatives.

Proposition 3.2. *Assume that the solution $u(x,t)$ of (9) - (10) with $u_0 \in C^{2+\gamma}(\mathbb{R})$ satisfies*

$$\sup_{x\in\mathbb{R}} \left|u_{xx}(x,t)(u_x(x,t))^2\right| \le C_2 t^{-\beta}, \quad t \ge 1$$

for some constants $C_2 > 0$ and $\beta > 1$. Then $u(x,t)$ satisfies

$$\sup_{x\in\mathbb{R}} |u(x,t) - h(x,t)| \le \left(\frac{\|u_0'\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}} + \frac{C_2}{\beta - 1}\right) t^{-\frac{1}{2}(1-\frac{1}{\beta})} + C_3 t^{-\frac{3}{2}+\frac{1}{\beta}}, \quad t \ge 1,$$

for a constant $C_3 > 0$ depending only on u_0 and C_2. Here $h(x,t)$ is defined by

$$h(x,t) = \int_{\mathbb{R}} \Gamma(x-y,t)\, u_0(y)\, dy.$$

Proposition 3.2 is proved by deriving a decay estimate for the second term of the right hand side of

$$u(x,t) = \int_{\mathbb{R}} \Gamma(x-y,t)\, u_0(y)\, dy - \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s)\, \frac{u_{yy}u_y^2}{1+u_y^2}\, dy\, ds.$$

Note that the solution $u(x,t)$ of (9) - (10) is expressed as above, since $u_t = u_{xx}/(1+u_x^2) = u_{xx} - u_{xx}u_x^2/(1+u_x^2)$. By this proposition, the proof of Theorem 3.1 is given by the decay estimates for $|u_x|$ and $|u_{xx}|$ stated below.

Proposition 3.3. *The solution $u(x,t)$ of (9) - (10) with $u_0 \in C^{2+\gamma}(\mathbb{R})$ satisfies*

$$\sup_{x\in\mathbb{R}} |u_x(x,t)| \le C_4(1+t)^{-\frac{1}{2}}, \quad t \ge 0$$

for a constant C_4 depending only on u_0. Moreover there exists a constant $C_5 > 0$ depending only on u_0 such that, for any constant δ with $0 < \delta < 1/4$, the solution $u(x,t)$ of (9) - (10) satisfies

$$\sup_{x\in\mathbb{R}} |u_{xx}(x,t)| \le \frac{C_5}{\sqrt{\delta}}(1+t)^{-1+\delta}, \quad t > 0.$$

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Elliptic overdetermined boundary problems

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When studying the well-posedness of elliptic boundary value problems on a compact smooth manifold with boundary it is convenient to relax the requirement of existence and uniqueness, and allow boundary problem operators to be Fredholm. The ellipticity of a boundary problem operator consists of both ellipticity of the given differential operator on the manifold and ellipticity of the boundary conditions. The latter is called the Shapiro-Lopatinskij condition. It is known [1] that in case of square systems (as many equations as unknowns) ellipticity of a boundary problem operators is equivalent to the Fredholm property of them in appropriate Sobolev spaces.

Douglis and Nirenberg [4] generalized the notion of ellipticity of operators. They introduced some weights for equations and unknowns so that their symbol contains information also about derivatives which are not of maximal order. It turns out the theory of Shapiro–Lopatinskij conditions can be generalised also to this case.

Now to analyse overdetermined PDEs in general it is important to check if the system is involutive, and if not then transform it to the involutive form. The technical definition of the involutive form is quite complicated (see [8], [11] and [10] and for the actual definition). The geometrical definition is based on representing the PDE system as subbundle of some appropriate jet space. However, essentially the involutivity means that one has to find all integrability conditions (or differential consequences) of the given system up to some order. Under some appropriate hypothesis one can show that this can be done in a constructive way; this is sometimes called the Cartan–Kuranishi completion algorithm.

The transformation to the involutive form usually requires the use of symbolic computation. In practice to complete a system to the involutive form one may use DETools package [2] in computer algebra system MuPAD [5].

Now when we complete our system to involutive form we see that we no longer need the weights of Douglis and Nirenberg: in [6] we showed that any system that is elliptic with respect to the generalised definition becomes elliptic in the standard sense when completed to the involutive form. So the apparent generality of ellipticity is just the result of restricting the attention to square systems. Moreover, we gave examples of operators that are not elliptic (even in the generalised sense) but whose involutive form are elliptic. Hence when determining ellipticity of the operator one should consider its involutive form and check the classical ellipticity.

It turns out that for technical reasons it is convenient to further transform the involutive system to a normalised system. Roughly speaking, an operator is normalised if it is a first order involutive operator and there are no (explicit or implicit) algebraic (i.e., non-differential) relations between dependent variables. A boundary value problem operator is normalised if the system is normalised and the boundary conditions contain only differentiation in directions tangent to the boundary. Evidently this transformation is constructive so there is no loss of generality in assuming that our system is normalised.

Next we have to construct the compatibility complex. Let us first consider the operator A itself. For simplicity we will only discuss the case when the operators (and later boundary operators) have constant coefficients. In this case we can compute the compatibility complex by simply computing the free resolution of the module generated by the rows of A. Incidentally this shows that the length of the compatibility complex is at most the number of independent variables. However, to study boundary value problems we need to compute the compatibility operators involving the boundary operators. This is not as straightforward as the simple free resolution, and in fact here we need the notion of normalised operator to perform this task. Anyway the problem can be formulated again with modules, and choosing suitable module orderings we can compute the necessary information by Gröbner basis techniques, cf. [7].

Now note that in the overdetermined case the cokernel is typically infinite dimensional, so the standard Fredholm property fails. However, when we construct the compatibility complex to the boundary value problem operator we may inquire if the resulting complex has finite dimensional cohomologies. If this is the case we say that the compatibility complex is Fredholm and the boundary problem is well-posed. The natural question arises under which condition the compatibility complex is Fredholm. In the talk we give the answer to this problem.

Moreover to study elliptic boundary problem operators and the corresponding compatibility operators we need the parametrices of these operators. These parametrices are *Boutet de Monvel operators*, cf. [3] and [9]. Hence we may as well consider a more general problem, i.e. to find a condition under which a complex of Boutet de Monvel operators is Fredholm in appropriate Sobolev spaces.

We go further and we observe that from the point of view of functional analysis, instead of complexes it is much more natural to consider sequences operators such that the composition of two consecutive operators is small in some reasonable sense, e.g. a compact operator. Such sequences are called *quasicomplexes.* Indeed, perturbation of a single Fredholm operator by a compact operator leads to a Fredholm operator. However, most perturbations of complexes lead out of the class of complexes, but it turns out that Fredholm quasicomplexes are stable under compact perturbations.

We prove that ellipticity of such quasicomplexes (i.e. the exactness of both interior and boundary symbol sequences) implies Fredholm property. Then as a very important consequence, we get that ellipticity of compatibility complex for an elliptic overdetermined operator is sufficient condition for compatibility complex to be Fredholm. We construct a parametrix for elliptic quasicomplexes on manifold with boundary which is analogous to the parametrix used in Hodge theory for elliptic complexes. We show how to introduce Euler characteristic for elliptic quasicomplexes.

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Spectral properties of the operators generated by PDE systems of stratified ideal and compressible fluids

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We establish the localization and the structure of the spectrum of normal vibrations described by systems of partial differential equations modelling small displacements of stratified fluid in the homogeneous gravity field. We also compare the spectral properties of gravitational and rotational operators. The similarity of the essential spectrum for stratified and rotational flows corresponds to the analogy in the propagation of gravitational and Coriolis waves in viscous fluids, whose consideration includes the study of qualitative properties of the solutions, such as existence, uniqueness, smoothness, asymptotics, etc. We also obtain a solution of the Cauchy problem for a system of an exponentially stratified fluid in the gravity field in the form of singular integrals, taken in the Cauchy principal value sense, when singularities are removed by a ball, that is, isotropically. If the initial data have a specified smoothness, the solution is written in the form of integrals with weak singularities of the kernels. Both these forms of solutions enable exact L_p estimates $(p>1)$ to be obtained. We also compare the spectrum of the operators which describe stratified ideal and compressible fluids acting in bounded and unbounded domains.

Keywords: - Partial differential equations, essential spectrum, Sobolev spaces, stratified fluid, internal waves.

1. Introduction

Let us consider a PDE system which describes small displacements of an exponentially stratified ideal fluid in the gravity field

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g\rho + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2 \rho_*}{g} u_3 = 0 \\ div\, \vec{u} = 0 \end{cases} \quad , \qquad (1)$$

the system which describes the same motions for a compressible fluid

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g\rho + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2 \rho_*}{g} u_3 = 0 \\ \dfrac{\partial p}{\partial t} + div\, \vec{u} = 0 \end{cases} , \qquad (2)$$

and the system which describes the movement of a rotational fluid

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \nabla p = 0 \\ div\, \vec{u} = 0 \end{cases} . \qquad (2^*)$$

Here $x \in \Omega \subset R^3$, $t \ge 0$, $\vec{u}(x,t) = (u_1, u_2, u_3)$ is the velocity field, $p(x,t)$ is the scalar field of the dynamic pressure , $\rho(x,t)$ is the dynamic density, $\vec{\omega} = (0,0,\omega)$, $\vec{e}_3 = (0,0,1)$, and ρ_* , μ, g , N , ω are positive constants. The equations (1) are deduced under the assumption that the function of stationary distribution of density is performed by $\rho_* e^{-Nx_3}$. The system (2) describes the rotation over the vertical axis, $\vec{\omega} \times \vec{u}$ is the vector product in R^3 .

The systems (1) and (2) were studied from different angles , some of the results may be found in [2]- [5].

In [2] we prove that the essential spectrum of normal vibrations for the operators generated by (2*), is the interval of the real axis $[-\omega, \omega]$, and we also construct an explicit example of non-uniqueness for the spectral parameter belonging to the essential spectrum.

Let us observe the analogy between the singular solutions for the systems (1) and (2*):

$$E(x,t) = \frac{1}{4\pi |x_3|} \int_0^{\frac{Nt|x_3|}{|x|}} J_0(\alpha) d\alpha \ , \qquad E(x,t) = \frac{1}{4\pi |\bar{x}|} \int_0^{\frac{\omega t|\bar{x}|}{|x|}} J_0(\alpha) d\alpha \ , \ |\bar{x}| = \sqrt{x_1^2 + x_2^2} \ .$$

It seems appropriate to express the conjecture that the operators generated by the system (1) should possess spectral properties, analogous to the system (2*) , namely, the essential spectrum of such operators should be the interval [$-N,N$] . In this paper we prove that this conjecture is true.

For system (1) we will construct the explicit form of the solution of the Cauchy problem in form of singular integrals which will allow us to obtain L_p -estimates by means of the Calderón-Zygmund Theorem.

2. Spectral Problem Formulation

Let us consider the system

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g \rho + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2 \rho_*}{g} u_3 = 0 \\ div\, \vec{u} = 0 \end{cases} \quad . \qquad (4)$$

Differentiating the second equation of (4) with respect to t , we obtain

$$\begin{cases} \dfrac{\partial^2 \vec{u}}{\partial t^2} + \vec{e}_3 N^2 u_3 + \nabla P = 0 \\ div\, \vec{u} = 0 \end{cases} \quad , \qquad (5)$$

where $P = \dfrac{1}{\rho_*}\dfrac{\partial p}{\partial t}$. For the system (4), let us consider the boundary value problem

$$\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0 \quad , \qquad (6)$$

where $\vec{n}$ is the vector of the external normal for the bounded domain $\Omega \subset R^3$.

Let $G(\Omega)$ be the space of potential fields in $L_2(\Omega)$:

$$G_2(\Omega) = \{\vec{u} \in L_2(\Omega) : \vec{u} = \nabla \varphi; \varphi \in W_2^1(\Omega)\}.$$

Furthermore, let $\overset{0}{J}(\Omega)$ be the space of solenoidal fields :

$$\overset{0}{J}(\Omega) = \{\vec{u} \in C^1(\Omega) : div\, \vec{u} = 0, \vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0\}.$$

Finally, let us introduce the space $J_2(\Omega)$ as a closure of $\overset{0}{J}(\Omega)$ in the norm of $L_2(\Omega)$. It can be shown ([1]), that $L_2(\Omega)$ permits the following orthogonal decomposition:

$$L_2(\Omega) = J_2(\Omega) \oplus G_2(\Omega) .$$

Let P be the operator of the orthogonal projection of $L_2(\Omega)$ onto $J_2(\Omega)$. Now, let us define the operator B :

$$B\vec{u} = P\{u_3 \vec{e}_3\}$$

with the domain $D(B) = J_2(\Omega)$.

Thus, the system (5) transforms into

$$\begin{cases} \dfrac{\partial^2 \vec{u}}{\partial t^2} + N^2 B\vec{u} = 0 \\ \vec{u} \in J_2(\Omega) \end{cases} \quad . \qquad (7)$$

For the system (7) we consider the problem of normal vibrations

$$\vec{u}(x,t) = \vec{v}(x) e^{i\lambda t} \quad . \qquad (8)$$

Therefore, we can finally write the system (7) as

$$\begin{cases} \lambda^2 \vec{v} - N^2 B\vec{v} = 0 \\ \vec{v} \in J_2(\Omega) \end{cases} \quad . \qquad (9)$$

Our aim is to investigate the spectrum of the operator B . From the physical point of view, the separation of variables (8) serves as a tool to establish the possibility to represent every non-stationery process described by (4) as a linear superposition of the normal vibrations. The knowledge of the spectrum of the normal vibrations, its structure and localization, may be very useful for studying the stability of the flows. Finally, the spectrum of operator B is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator B.

3. Spectral Problem Solution for Ideal Fluid

Lemma 1. B is a positive self-adjoint operator in $J_2(\Omega)$.

Lemma 2. The kernel of B is the subspace $H_J(\Omega)$ which consists of all elements of $J_2(\Omega)$ with trivial third component.

Corollary. $\lambda = 0$ is an eigenvalue of infinite multiplicity for B. Its corresponding eigenvectors compose all the subspace $H_J(\Omega)$.

Let us consider the same separation of variables for the function $P(x,t)$:

$$P(x,t) = q(x)e^{i\lambda t}, \quad q \in W_2^1(\Omega).$$

If $q(x)$ is a solution of the system

$$\begin{cases} -\lambda^2 v_1 + \dfrac{\partial q}{\partial x_1} = 0 \\ -\lambda^2 v_2 + \dfrac{\partial q}{\partial x_2} = 0 \\ \left(-\lambda^2 + N^2\right)v_3 + \dfrac{\partial q}{\partial x_3} = 0 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases}, \qquad (10)$$

then $q(x)$ satisfies the equation $\Delta q = -div\left(N^2 v_3 \vec{e}_3\right)$, which implies

$$div\left(N^2 v_3 \vec{e}_3 + \nabla q\right) = 0.$$

Thus, the projection operator B obtains its explicit form as $N^2 B\vec{v} = N^2 v_3 \vec{e}_3 + \nabla q$.

We shall establish now the structure of the spectrum of the operator B.

Theorem 3. The essential spectrum of the operator $N^2 B$ is the interval of the real axis $[-N, N]$. Moreover, the points $0, \pm N$ are eigenvalues of infinite multiplicity.

To prove the theorem, we construct an explicit Weyl sequence.

We recall the following criterion of an essential spectrum, which is attributed to Weyl: A necessary and sufficient condition that a real finite value μ be a point of the essential spectrum of a self-adjoint operator B is that there exist a sequence of elements $x_n \in D(B)$ such that

$$\|x_n\| = 1, x_n \to 0 \text{ weakly and}$$
$$\|(B - \mu I)x_n\| \to 0$$

Remark 4. Let us consider the system (9) in the whole space $(\Omega = R^3)$. Since the orthogonal decomposition $L_2(\Omega) = J_2(\Omega) \oplus G_2(\Omega)$ holds true for $\Omega = R^3$, we have that the result of Theorem 3 is valid for $\Omega = R^3$.
We observe that, in this case, the system (9) is equivalent to the scalar equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \left(\frac{\mu}{N^2 - \mu}\right)\frac{\partial^2 u}{\partial x_3^2} = 0 .$$

4. Spectral Problem for Compressible fluid

Now, we consider the case of compressible stratified fluid

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g \rho + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2 \rho_*}{g} u_3 = 0 \\ \dfrac{\partial p}{\partial t} + div(\rho_* \vec{u}) = 0 \end{cases}$$

Without loss of generality, we may put $\rho_* = 1$, $g = 1$ (which can be achieved introducing new unknown functions and renaming them as follows):

$$\vec{u} := \rho_* \vec{u} , \quad \rho := g\rho .$$

Thus, the previous system is transformed into

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 \rho + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \dfrac{\partial p}{\partial t} + div(\vec{u}) = 0 \end{cases} . \qquad (11)$$

Let us consider the following problem of normal vibrations:

$$\begin{aligned} &\vec{u}(x,t) = \vec{v}(x) e^{-\lambda t} \\ &\rho(x,t) = N v_4(x) e^{-\lambda t} \quad , \quad \lambda \in C , \\ &p(x,t) = v_5(x) e^{-\lambda t} \end{aligned}$$

associated to the boundary conditions $\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0$.
We denote $\widetilde{v} = (v_1, v_2, v_3, v_4, v_5)$ and write the system (11) in the matrix form

$$L\widetilde{u} = 0 , \qquad (12)$$

where $L = M - \lambda I$,

$$M=\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}.$$

We shall investigate the spectrum of the operator M.
The domain of the operator M can be naturally defined as follows:

$$D(M)=\left\{\begin{matrix}\vec{u}\in L_2(\Omega)\ |\exists f\in L_2(\Omega): \\ (\vec{u},\nabla\varphi)=(f,\varphi)\forall\varphi\in W_2^1(\Omega)\end{matrix}\right\}\times W_2^1(\Omega)\times W_2^1(\Omega).$$

Lemma 5 . The operator M is skew-selfadjoint.

Since the spectrum of skew-selfadjoint operators belongs to the imaginary axis, then, for the problem of normal vibrations we can introduce the spectral parameter $\mu : \lambda = i\mu$, and prove the following result

Theorem 6. The essential spectrum of the operator M is the interval of the real axis $[-N,N]$. Moreover, the points $\mu = 0,\pm N$ are eigenvalues of infinite multiplicity.

Let us consider the system (12) in $\Omega = R^3$. For the normal vibrations problem we have the system

$$(M^* - i\mu I)\vec{u} = 0,$$

where the matrix M^* is the same matrix M, and the domain of M^* is defined as follows: $D(M^*)=\left\{\begin{matrix}\vec{u}\in L_2(R^3)\ |\exists f\in L_2(R^3): \\ (\vec{u},\nabla\varphi)=(f,\varphi)\forall\varphi\in W_2^2(R^3)\end{matrix}\right\}\times W_2^2(R^3)\times W_2^2(R^3)$

Theorem 7. The essential spectrum of the operator M^* is all the real axis. Moreover, the points μ such that $\mu\notin[-N,N]$, belong to the continuous spectrum of the operator M^*.

5. Construction of solutions for the Cauchy problem

We consider a system of equations of the form

$$\begin{cases} \rho_* \dfrac{\partial v_1}{\partial t} + \dfrac{\partial p}{\partial x_1} = 0 \\ \rho_* \dfrac{\partial v_2}{\partial t} + \dfrac{\partial p}{\partial x_2} = 0 \\ \rho_* \dfrac{\partial v_3}{\partial t} + g\rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2\rho_*}{g} v_3 = 0 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases} \qquad (13)$$

in the domain $\{x \in R^3, t > 0\}$, where $\vec{v}(x,t)$ is a velocity field with components $v_1(x,t), v_2(x,t), v_3(x,t)$, $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamic density and $\rho_* \ g, N$ are positive constants .

Let us consider first the Cauchy problem for (13) :

$$\begin{aligned} \vec{v}\big|_{t=0} &= \vec{v}^0(x) \\ \rho\big|_{t=0} &= 0 \end{aligned} \quad . \qquad (14)$$

Differentiating the fourth equation of (12) with respect to t , we obtain

$$\begin{cases} \dfrac{\partial^2 \vec{v}}{\partial t^2} + \nabla P + \vec{e}_3 N^2 v_3 = 0 \\ div(\vec{v}) = 0 \end{cases} , \qquad (15)$$

where $\vec{e}_3 = (0,0,1)$, $P = \dfrac{1}{\rho_*}\dfrac{\partial p}{\partial t}$.

If we denote by P the operator of the orthogonal projection of $L_2(R^3)$ onto $J_2(R^3)$, then we can define the following operator A :

$$A\vec{v} = P\{v_3 \vec{e}_3\} \qquad (16)$$

with the domain $D(A) = J_2(R^3)$.

Since $\|A\vec{v}\| \le \|\vec{v}\|$, the norm of the operator A is not greater than unity. Thus, the equation

$$\frac{d^2\vec{v}}{dt^2} + A\vec{v} = 0 \ , \ \vec{v}\big|_{t=0} = \vec{v}^0 \ , \ \frac{d\vec{v}}{dt}\bigg|_{t=0} = \vec{v}_1^{\,0} \qquad (17)$$

has the solution $\qquad \vec{v} = \vec{v}^0 \cos ANt + \dfrac{\vec{v}_1^{\,0}}{N} \sin ANt =$

$$= \vec{v}^0 - \frac{N^2 t^2}{2!} A^2 \vec{v}^0 + \frac{N^4 t^4}{4!} A^4 \vec{v}^0 - .. \\ .. + \frac{1}{N}\left[\frac{Nt}{1!} A\vec{v}_1^{\,0} - \frac{N^3 t^3}{3!} A^3 \vec{v}_1^{\,0} + \frac{N^5 t^5}{5!} A^5 \vec{v}_1^{\,0} - .. \right] . \qquad (18)$$

The series (18) represent the solution of the Cauchy problem. It is easy to see that the problem is well-posed.

Now we shall construct the explicit form of the solution of the Cauchy problem for (13).

For system (13) we consider the initial conditions (14) and the additional conditions of the absence of the rotational component in (x_1, x_2)

$$\frac{\partial v_1^0}{\partial x_2} - \frac{\partial v_2^0}{\partial x_1} = 0 \ , \qquad (19)$$

together with the natural condition

$$div(\vec{v}^0) = 0 \ . \qquad (20)$$

So that we shall be dealing with convergent integrals, we assume that the initial data have, for example, continuous second derivatives and decrease sufficiently rapidly at infinity together with their derivatives up to the second order. Using the Fourier transform with respect to x, the Laplace transform with respect to t and the conditions (19), (20), we obtain the solution of our problem in the form

$$\hat{\vec{v}}(\xi,\lambda)=\frac{\lambda|\xi|^2}{\lambda^2|\xi|^2+|\xi'|^2}\hat{\vec{v}}^0(\xi)\ ,\ \hat{\rho}(\xi,\lambda)=\frac{|\xi|^2\hat{v}_3^0(\xi)}{g\left(\lambda^2|\xi|^2+|\xi'|^2\right)}\ ,\ \hat{p}(\xi,\lambda)=\frac{i\xi_3\hat{v}_3^0(\xi)}{\lambda^2|\xi|^2+|\xi'|^2}\ ,$$

where $\xi=(\xi_1,\xi_2,\xi_3)$, $|\xi|^2=\sum_{k=1}^{3}\xi_k^2$, $|\xi'|^2=\sum_{k=1}^{2}\xi_k^2$. After an inverse Laplace transform we obtain the solution in the form

$$\hat{\vec{v}}(\xi,t)=\hat{\vec{v}}^0(\xi)\cos\frac{|\xi'|}{|\xi|}t,\ \hat{\rho}(\xi,t)=\hat{v}_3^0(\xi)\frac{|\xi|}{g|\xi'|}\sin\frac{|\xi'|}{|\xi|}t,\ \hat{P}(\xi,t)=\hat{v}_3^0(\xi)\frac{i\xi_3}{|\xi'||\xi|}\sin\frac{|\xi'|}{|\xi|}t\ . \quad (21)$$

We now find the inverse Fourier transform of the required solution. We first obtain the solution in the form of integrals with weak singularities of the kernels. For this we seek a vector $\vec{v}(x,t)$ expressed in terms of the Laplace operator and a function $p(x,t)$ expressed in terms of the first derivatives of the initial data. As we see from (21), it is sufficient to calculate only two kernels

$$K_1(x-y,t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}e^{i(\xi,x-y)}\frac{1}{|\xi|^2}\cos\frac{|\xi'|}{|\xi|}td\xi\ , \quad (22)$$

$$K_2(x-y,t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}e^{i(\xi,x-y)}\frac{1}{|\xi'||\xi|}\sin\frac{|\xi'|}{|\xi|}td\xi\ . \quad (23)$$

We note that K_2 is the primitive of K_1 with respect to t , and it is therefore sufficient to calculate only one of these integrals.

An integral of type (23) is calculated in [11] by means of Sonine′s formulas for Bessel functions and is given by

$$K_2(x-y,t)=\frac{1}{4\pi}\frac{1}{r}\int_0^t J_0(t-\tau)J_0\left(\frac{\rho\tau}{r}\right)d\tau, \quad (24)$$

where $\rho^2=(x_3-y_3)^2$, $r^2=\sum_{k=1}^{3}(x_k-y_k)^2$, and J_0 is the Bessel function of order zero. Therefore

$$K_1(x-y,t)=\frac{1}{4\pi}\frac{1}{r}J_0\left(\frac{\rho t}{r}\right)-\frac{1}{4\pi}\frac{1}{r}\int_0^t J_1(t-\tau)J_0\left(\frac{\rho\tau}{r}\right)d\tau\ . \quad (25)$$

If we now use (24) and (25), the solution of the Cauchy problem for the system (5) can be written in the form

$$\vec{v}(x,t)=\iiint_{R^3}\left\{-\Delta\vec{v}^0(y)K_1(x-y,t)\right\}dy\ , \quad (26)$$

$$P(x,t)=\iiint_{R^3}\left\{\frac{\partial v_3^0}{\partial y_3}K_2(x-y,t)\right\}dy\ . \quad (27)$$

6. L*p*- estimates for a solution of the Cauchy problem

To obtain L_p-estimates for a solution of the Cauchy problem we shall show that the kernels which are used in writing out the solution and its derivatives, satisfy the conditions of the Calderón-Zygmund Theorem. We write (26) in the form of convolution:

$$\vec{v}(x,t)=\vec{v}^{\,0}(x)\Phi(t)+\left(\vec{v}^{\,0}*\Gamma\right)_{R^3} \ , \qquad (28)$$

where

$$\left(\vec{v}^{\,0}*\Gamma\right)_{R^3}=\iiint\limits_{R^3}\vec{v}^{\,0}(y)\Gamma(x-y,t)dy \ , \qquad (29)$$

$$\Gamma(x,t)=G(x,t)-\int_0^t J_1(t-\tau)G(x,\tau)d\tau \ , \qquad (30)$$

The infinite triple integrals are taken in the sense of principal value, and from (26), after the corresponding differentiation, we obtain

$$G(x,t)=\frac{1}{4\pi}\left[\frac{t^2\left(x_1^2+x_2^2\right)}{r^5}J_0\left(\frac{\rho t}{r}\right)+\frac{t}{r^3}\left(\frac{\rho}{r}+\frac{r}{\rho}\right)J_0'\left(\frac{\rho t}{r}\right)\right] . \qquad (31)$$

It is easy to see that Γ is infinitely differentiable function of t (a singularity in the space $x=\left(x_1,x_2,x_3\right)$ does not increase on differentiation with respect to t).

We examine the properties of the kernel Γ for any finite t : $0\le t\le T<\infty$.

1) Γ is a homogeneous function of x of degree -3 . The proof is obvious on nothing that Bessel functions of the argument $\rho t/r$ are homogeneous functions of degree zero.

2) Γ may be put in the form $\Gamma(x,t)=\dfrac{\tilde{\Omega}(x,t)}{r^3}$, where

$$\tilde{\Omega}(x,t)=\Omega(x,t)-\int_0^t J_1(t-\tau)\Omega(x,\tau)d\tau \ ,$$

$$\Omega(x,t)=\frac{1}{4\pi}\left[\frac{t^2\left(x_1^2+x_2^2\right)}{r^2}J_0\left(\frac{\rho t}{r}\right)+\frac{t\left(r^2+\rho^2\right)}{r\rho}J_0'\left(\frac{\rho t}{r}\right)\right] .$$

3) The integrals of $\tilde{\Omega}(x,t)$ over the unit sphere are zero.

Thus, the conditions of the Calderón-Zygmund Theorem [12] are satisfied.

If we denote by $W_{p,t,x}^{k,l}\left(E_4^T\right)$ the Sobolev space of functions having k derivatives with respect to t and l derivatives with respect to x which are pth power summable, then we have proved the following theorem.

Theorem 8.

If the initial data satisfy $\vec{v}^{\,0}(x)\in W_p^l\left(R^3\right)$ *and if* $\vec{v}(x,t)$, $P(x,t)$ *is a solution of the problem (19),(23),(24) for which the norms given below are finite, then the following estimates will hold :*

$$\|\vec{v}\|_{W^{k,l}_{p,t,x}(E_4^T)} \le C_1(p,T)\|\vec{v}^{\,0}\|_{W_p^l(R^3)}$$

$$\|\nabla P\|_{W^{k,l}_{p,t,x}(E_4^T)} \le C_2(p,T)\|\vec{v}^{\,0}\|_{W_p^l(R^3)}$$

where the constants $C_i(p,T)$ depend only on p and, in general, on T (where $0 \le t \le T < \infty$), k and l.

7. Conclusions

The importance of construction of a Weyl sequence for such problems is that, it is "almost" a solution of a system of partial differential equations. And, for λ belonging to the essential spectrum of B , the Weyl sequence represent explicit examples of non-uniqueness of the solutions, due to the arbitrarity of the function ψ .

As we have seen, the solutions of the considered problems are closely related to the function

$$V = \frac{1}{r}J_0\left(\frac{\rho}{r}t\right) = \frac{1}{r}J_0(t\cos\theta).$$

Let us discuss the conduct of the function V as a function of t . We consider a sphere of a constant radius. On the sphere, for every t , the function V depends only on the polar angle θ . The argument of the Bessel function on the sphere changes from 0 to t . With t growing, we will have more and more waves generated by maxima and minima of the Bessel function, all of them situated between the pole and the equator of the sphere. The waves will appear on the pole and then will move towards the equator, accumulating but not disappearing. Thus large waves will generate more and more short ones.

The remarkable analogy of gravitational and rotational waves discussed above could serve as an example of how mathematical description of physical forces of different origin may help us to understand the unity of the Nature's manifestations.

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A class of nonlocal nonlinear boundary value problems with definite integrals

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We have been interested in complete global bifurcation structures of several nonlocal nonlinear boundary problems arising in various fileds. We show four typical examples.

1. Oseen's spiral flow

The first problem is related with the Oseen's spiral flow [12]. Find a function $U(x)$ such that

$$(\mathrm{O}) \quad \begin{cases} U_{xx} + A\,U - U^2 + C = 0, \quad x \in (-\pi, \pi), \\ C := \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} U(x)^2 \, dx, \\ U(-\pi) = U(\pi), \quad U_x(-\pi) = U_x(\pi), \\ \displaystyle\int_{-\pi}^{\pi} U(x) \, \mathrm{d}x = 0 \end{cases}$$

for arbitrarily fixed A.

It is easily seen that U $\equiv$ 0 is the trivial solution of the above problem for any fixed A. Okamoto [11] started to investigate the global bifurcation structure of this problem. Moreover, Ikeda-Mimura-Okamoto [4] obtained the asymptotic shape of solutions as $A \to -\infty$.

Let us recall the standard notation of complete elliptic integrals:

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} \, \mathrm{d}\varphi, \qquad k \in [0, 1),$$

$$E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi} \, \mathrm{d}\varphi, \qquad k \in [0, 1).$$

Jacobi's elliptic functions $\mathrm{sn}(x, k)$ and $\mathrm{cn}(x, k)$ with the modulus k are defined as follows:

$$\mathrm{sn}^{-1}(z, k) := \int_0^z \frac{1}{\sqrt{(1 - \xi^2)(1 - k^2 \xi^2)}} d\xi, \qquad z \in [0, 1],\ k \in [0, 1),$$

and

$$\mathrm{cn}^2(z,k) = 1 - \mathrm{sn}^2(z,k).$$

We note that

$$E(0) = K(0) = \frac{\pi}{2},\ \ E(1) = 1,\ \ K(k) \sim \frac{1}{2}\log\left(\frac{16}{1-k^2}\right) \quad \text{as } k \to 1.$$

Ikeda-Kondo-Okamoto-Yotsutani [3] have parameterized all solutions (A, U) of (O) in terms of the elliptic functions, and clarified the global bifurcation structure by the following Theorems 1 and 2.

Theorem 1 *All the solution* (A, U) *of* (O) *are parameterized by*

$$\{(n^2 A(k), n^2 U(nx - x_0; A(k)) : 0 < k < 1, -\pi < x_0 \leq \pi, n = 1, 2, 3 \cdots\}$$

where

$$\begin{aligned} A(k) &:= \frac{4K(k)}{\pi^2}\left(3E(k) + (k^2 - 2)K(k)\right), \\ U(x; A(k)) &:= -\frac{6k^2 K(k)^2}{\pi^2}\,\mathrm{cn}^2\left(\frac{K(k)}{\pi}x, k\right) \\ &\qquad + \frac{6K(k)}{\pi^2}\{E(k) - (1-k^2)K(k)\}. \end{aligned}$$

Theorem 2 *The function* $A(k)$ *is strictly monotone decreasing in* $k \in (0,1)$*. It also satisfies* $\lim_{k\to 0} A(k) = 1$ *and* $\lim_{k\to 1} A(k) = -\infty$.

2. Ginzburg-Landau equation

The second problem is related with structure of stationary solutions in S^1 of the Ginzburg-Landau equation. Find a function $u(x)$ such that

$$(\mathrm{GL}) \quad \begin{cases} u_{xx} - \frac{C^2}{u^3} + \lambda(1 - u^2)u = 0, x \in [-\pi, \pi], \\ C := 2m\pi\left(\int_{-\pi}^{\pi} \frac{1}{u^2} dx\right)^{-1}, \\ U(-\pi) = U(\pi), \quad U_x(-\pi) = U_x(\pi), \\ u > 0 \quad \text{in } [-\pi, \pi]. \end{cases}$$

where m is a given integer and λ is a bifurcation parameter.

The structure of solutions is similar to that of Oseen's spiral flow, though the analysis is more difficult. Kosugi-Morita-Yotsutani [5] have clarified the global bifurcation structure of this problem. (See, also Kosugi-Morita-Yotsutani [6].)

We briefly explain about the original equation. Consider the following Ginzburg-Landau equation:

$$\begin{cases} \psi_{\mathrm{xx}} + \lambda(1 - |\psi|^2)\psi = 0, \quad \mathrm{x} \in (-\pi, \pi) \\ \psi(-\pi) = \psi(\pi), \quad \psi_{\mathrm{x}}(-\pi) = \psi_{\mathrm{x}}(\pi). \end{cases}$$

We here assume that $|\psi| > 0$ and ψ is written as the form

$$\psi = u(x)\exp(i\theta(x)),$$

where u and θ are both real-valued smooth functions. Clearly the equation is equivalent the following system:

$$\begin{cases} u_{xx} - {\theta_x}^2 u + \lambda(1 - u^2)u = 0, \quad x \in (-\pi, \pi), \\ (u^2\theta_x)_x = 0, x \in (-\pi, \pi), \\ U(-\pi) = U(\pi), \quad U_x(-\pi) = U_x(\pi), \\ \theta(\pi) - \theta(-\pi)u = 2m\pi, \quad \theta_x(-\pi) = \theta_x(\pi), \end{cases}$$

where m is an integer. Thus, $\theta_x = C/u^2$ for a constant C and hence we obtain (P).

3. Minimum energy curve

The third problem is related to find the minimum energy curve for given the length L and area M, which K.Watanabe [14] started to investigate.

For given $L > 0$ and $M > 0$ with $L^2 - 4\pi M > 0$, find a function $\kappa(s)$ such that

$$\text{(E)} \quad \begin{cases} \left\{\kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa\right\}_s = 0, s \in [0, L], \\ \mu := \frac{1}{L^2 - 4\pi M}\left\{M\int_0^L \kappa(s)^3 ds - \frac{L}{2}\int_0^L \kappa(s)^2 ds\right\}, \\ \kappa(0) = \kappa(L), \quad \kappa_s(0) = \kappa_s(L), \\ \int_0^L \kappa(s) ds = 2\pi. \end{cases}$$

Murai-Matsumoto-Yotsutani [10] have clarified the global bifurcation strucure of this problem, though terribly complicated calculations are needed.

4. Cross-diffusion

The final problem is a limiting equation for the Shigesada-Kawasaki-Teramoto model with cross-diffusion [13]. This problem is the hardest.

Find $(v(x), \tau)$ such that $\tau > 0$, and

$$
\text{(C)} \quad \begin{cases} \displaystyle\int_0^1 \frac{\tau}{v(x)} \left(a_1 - b_1 \frac{\tau}{v(x)} - c_1 v(x) \right) dx = 0, \\ d_2 v_{xx} + v \cdot \left(a_2 - b_2 \frac{\tau}{v} - c_2 v \right) = 0 \quad \text{in } (0,1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v > 0 \quad \text{on } [0,1]. \end{cases}
$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_2$ are given positive constants.

We briefly explain the original equation. In 1979, Kawasaki–Shigesada–Teramoto proposed a cross-diffusion system

$$
\begin{cases} u_t = \{(d_1 + \rho_{12} v) u\}_{xx} + u(a_1 - b_1 u - c_1 v) \quad (0 < x < 1, t > 0), \\ v_t = \{(d_2 + \rho_{21} u) v\}_{xx} + v(a_2 - b_2 u - c_2 v) \quad (0 < x < 1, t > 0), \\ u_x(0,t) = v_x(0,t) = 0, \ u_x(1,t) = v_x(1,t) = 0 \quad (0 < x < 1, t > 0), \\ u(x,0) = u_0(x) \geq 0, \ v(x,0) = v_0(x) \geq 0 \quad (0 < x < 1), \end{cases}
$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are positive constants, ρ_{12} and ρ_{21} are nonnegative constants, and $u_0(x)$ and $v_0(x)$ are nonnegative initial data.

This is a mathematical model to explain the segregation phenomena. Mathematical study of cross-diffusion equations was begun by M. Mimura in 1980 (see, e.g., [9]). There are various results concerning the existence of solutions to time-dependent problem (see, e.g., [1], [2] and references therein), and stationary problems. Sharp existence and non-existence results of stationary solutions are not known.

The limiting equation (S) was discovered by Lou-Ni [7] as a limiting equation when cross-diffusion effect $\rho_{12} \to \infty$. Actually, we see from the numeriacl computations that solutions of (S) approximate stable stationary solutions of the original time-dependent problem. Thus, it is important to know the structure of solutions of (S).

Let us put

$$
A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}.
$$

We mainly concentrate on the case $B < C$ (strong competition case).

It seems that the following conjectures hold.

Conjecture 1: Suppose that $B < C$. For any d_2 with

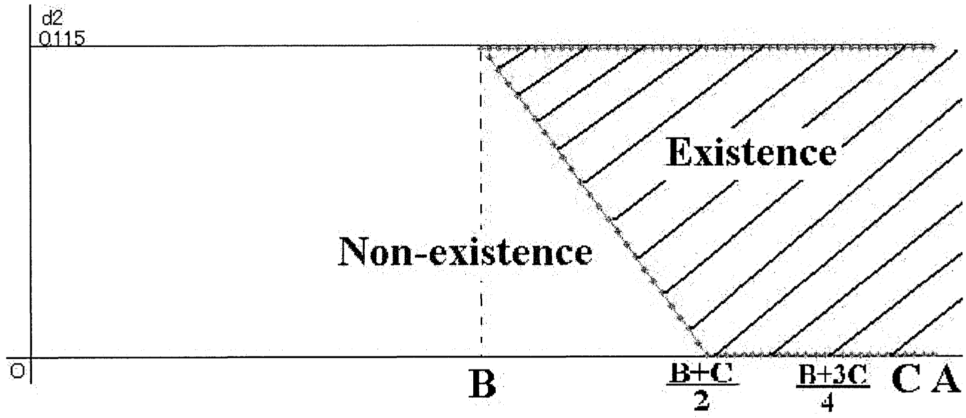
$$\max\{0, \frac{B+C-2A}{C-B}\}\ \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

there exists the unique solution $(v(x), \tau)$ of (S).

Conjecture 2: Existence, non-existence and uniqueness: Suppose that $B < C \leq 7B/3$. (S)has a solution $(v(x), \tau)$ if and only if d_2 satisfies

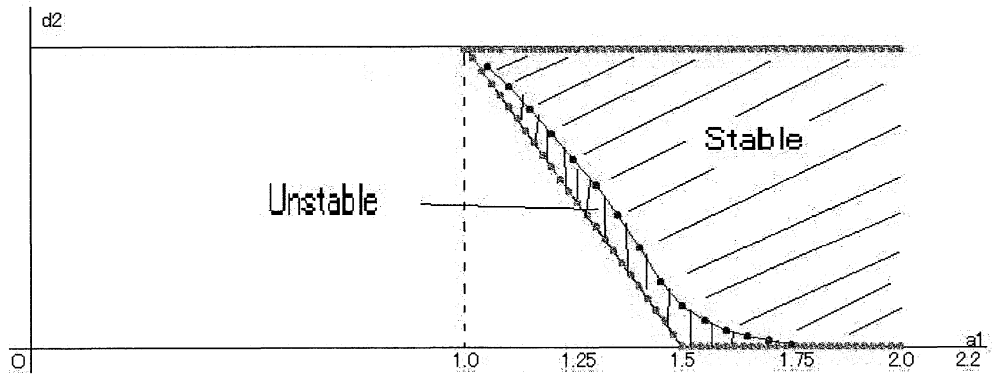
$$\max\{0, \frac{B+C-2A}{C-B}\}\ \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, the solution is unique.



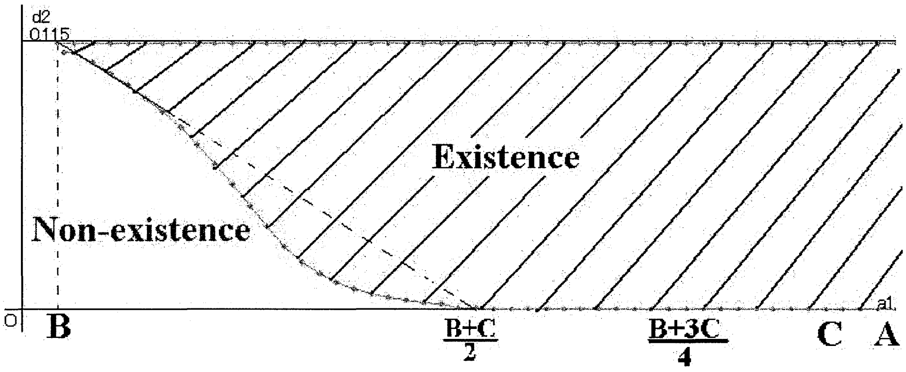
$B < C \leq 7B/3, \quad a_2 = b_2 = c_2 = 1$

Stability:



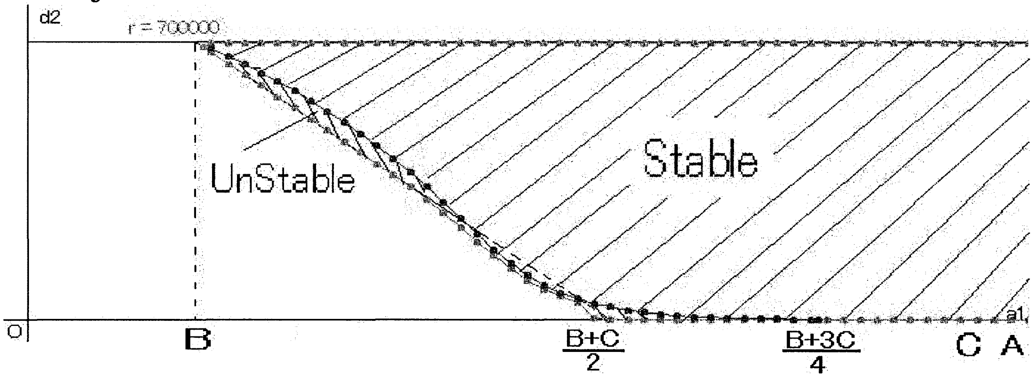
$B = 1, C = 2, \quad a_2 = b_2 = c_2 = 1$

Conjecture 3: Suppose that $C > 7B/3$. There exists the only one connected non-empty open set D such that (S) has exactly two solutions $(v(x), \tau)$ if and only if $d_2 \in D$.



$C > 7B/3, \quad a_2 = b_2 = c_2 = 1$

Stability:



$C > 7B/3, \quad a_2 = b_2 = c_2 = 1$

Lou-Ni-Yotsutani [8] have almost clarified the existence and the shape of solutions as follows.

Theorem 3 (Existence) *Suppose that $B < C$. If*

$$\max\{0, \frac{B+C-2A}{C-B}\}\ \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

then there exists a solution $(v(x), \tau)$ of (S).

Theorem 4 (Nonexistence) *Suppose that $B < C$.*
(i) If $d_2 \geq a_2/\pi^2$, then there exists no solution of (S).
(ii) If $A < (B+C)/2$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exiists no solution of (S) *for $d_2 \in (0, d_2^*]$.*
(iii) If $A < B$, there exists no solution of (S).

The following theorems give the shape of solutions.

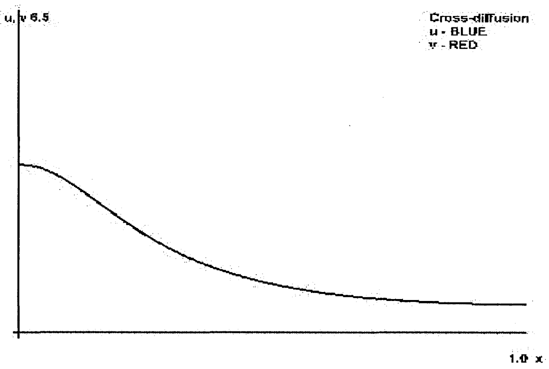
Theorem 5 (Shape of solutions as $d_2 \to a_2/\pi^2$) *Let $(v(x, d_2), \tau(d_2))$ be solutions of* (S). *If $A \geq B$, then*

$$v(x; d_2) \to 0,$$

$$\frac{v(x; d_2) - v(0; d_2)}{v(1; d_2) - v(0; d_2)} \to \sin^2(\frac{\pi}{2}x),$$

$$\frac{\tau(d_2)}{v(x; d_2)} \to \frac{a_2}{b_2} \cdot \frac{A/B + \sqrt{(A/B)^2 - A/B}}{1 + 2\{A/B - 1 + \sqrt{(A/B)^2 - A/B}\} \sin^2(\frac{\pi}{2}x)}$$

uniformly on $[0, 1]$ as $d_2 \to a_2/\pi^2$.



Theorem 6 (Shape of solutions as $d_2 \to 0$ *for* $A < (B+3C)/4$) *Let* $(v(x,d_2),\tau(d_2))$ *be solutions of* (S). *If* $A < (B+3C)/4$ *and* $B \neq C$, *then*

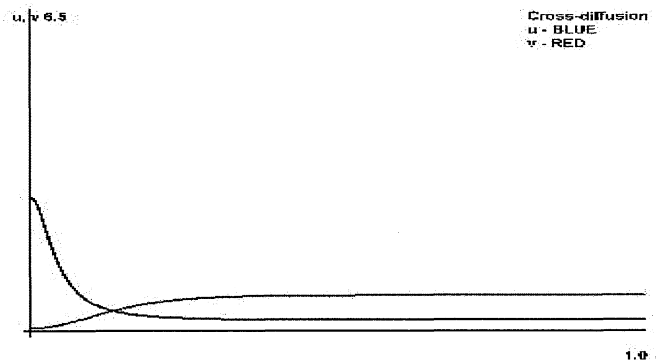
$$v(0;d_2) \to 2 \cdot \frac{a_2}{c_2} \cdot \frac{\frac{B+3C}{4} - A}{C-B},$$

$$v(x;d_2) \to \frac{a_2}{c_2} \cdot \frac{A-B}{C-B} \text{for} \quad x > 0,$$

$$\frac{\tau(d_2)}{v(0;d_2)} \to \frac{a_2}{2c_2} \cdot \frac{C-A}{C-B} \cdot \frac{A-B}{\frac{B+3C}{4} - A},$$

$$\frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{b_2} \cdot \frac{C-A}{C-B} \text{for} \quad x > 0,$$

as $d_2 \to 0$.

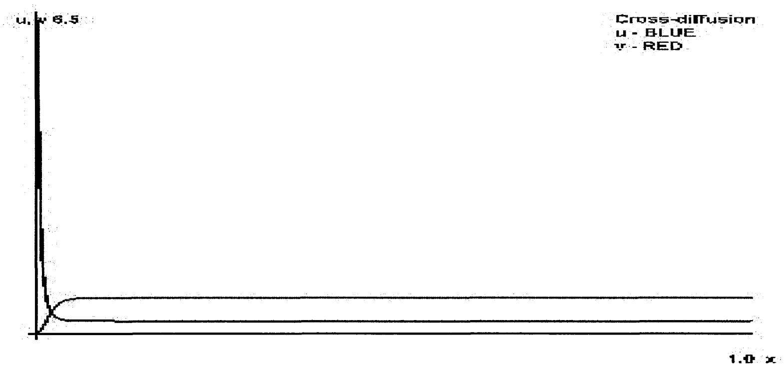


Theorem 7 (Shape of solutions as $d_2 \to 0$ *for* $A \geq (B+3C)/4$) *Let* $(v(x,d_2),\tau(d_2))$ *be solutions of* (S). *If* $B < C$ *and* $A \geq (B+3C)/4$, *then*

$$v(0;d_2) \to 0, \qquad v(x;d_2) \to \frac{3a_2}{4c_2} \quad \text{for } x > 0,$$

$$\frac{\tau(d_2)}{v(0;d_2)} \to \infty, \quad \frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{4c_2} \quad \text{for } x > 0,$$

as $d_2 \to 0$.



Now, we will discuss about the uniqueness and non-uniquess. The following result are a part of joint projects with W.-M. Ni.

Theorem 8 *Suppose that* $B < C$. *If* d_2 *is sufficiently small, the solution* $(v(x), \tau)$ *is unique for any given* A.

Theorem 9 *Suppose that* $C > 7B/3$. *There exists an open set* D *such that* (S) *has at least two solutions* $(v(x), \tau)$ *for* $d_2 \in D$.

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Variational aspects of the exisntence of $\mathbf{L}^\infty$ bounds for global solutions of some semilinear parabolic equations

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1 Introduction

In this note, we discuss the existence of an L^∞-global bounds of time global solutions of the following semilinear parabolic equation (P).

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $\lambda \in \mathbb{R}$. We denote the critical exponent of the Sobolev embedding $H_0^1 \hookrightarrow L^r$ by $2^*(= 2N/(N-2))$. We assume that $q \in (2, 2^*)$ and, for simplicity, $u_0 \in C(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$.

Then our problem reads as follows:

$$
(\mathrm{P}) \quad \begin{cases} \partial u/\partial t = \Delta u + \lambda u + u|u|^{q-2} & \text{in} \quad \Omega \times (0, T_m), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T_m), \\ u = u_0 & \text{in} \quad \Omega \times \{0\}. \end{cases}
$$

Here T_m denotes the maximal existence time of the classical solution of (P). We assume that, in the main part of this paper, $T_m = \infty$.

Problem (P) appears in various kinds of fields of physics and engineering. For example, in the context of the chemical reaction theory, (P) is called a diffusion-reaction equation. In this context, the first term describes the diffusion while the second term the reaction. Since the reaction in (P) is of

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positive feedback type, the asymptotic behavior of solutions of (P) is not so clear.

We here briefly review some known results. For the simplicity, in the rest of this section, we assume that $\lambda = 0$. A global-in-time solution u of (P) is said to have an L^∞-global bounds if there exists $C > 0$ such that $\sup_{t\geq 0} \|u(t)\|_\infty < C$. The existence of an L^∞-global bounds for the subcritical case (in the sense of the Sobolev embedding) is now well-known:

Proposition 1.1 (Subcritical case) [2]
Suppose that $q \in (2, 2^*)$. *Then any global-in-time solution* u *of* (P) *has an* L^∞*-global bounds.*

But the existence of L^∞-global bounds (for all global-in-time solutions) is no longer true for the critical case:

Proposition 1.2 (Critical case) [3]
Suppose that $q = 2^*$. *Let* Ω *be a ball. Then there exists a radially symmetric function* $u_0 \in L^\infty$ *which gives a global-in-time solution* u *of* (P) *with*

$$\|u(t)\|_\infty \to \infty \text{ as } t \to \infty.$$

Hence it is quite natural to seek the condition which assures the existence of an L^∞-global bounds in the critical case.

The main purpose of this note is to shed some new light on and to give an answer for this problem from the variational analytical point of view. From this viewpoint, we can provide a necessary and sufficient condition of the existence of an L^∞-global bounds for both of the subcritical and the critical problems in a unifies way.

2 Main Result

Hereafter we always assume that u denotes a global-in-time solution of (P).

Multiplying (P) by $\partial u(t)/\partial t$ and integrating it over Ω, we have

$$\left\| \frac{\partial u(t)}{\partial t} \right\|_2^2 = -\frac{d}{dt} J_\lambda(u(t)), \tag{1}$$

where J_λ denotes the energy (or Lyapunov) functional associated to (P) defined by

$$J_\lambda(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{q}\|u\|_q^q.$$

Hence $J_\lambda(u(t))$ is nonincreasing in t. Moreover, it is well known that

$$\text{if } T_m = \infty, \text{ then } J_\lambda(u(t)) \geq 0 \text{ for } t \in [0, \infty), \tag{2}$$

see e.g. [11].

By (1) and by (2), for any (global) solution u of (P), there exists $d \geq 0$ such that

$$\lim_{t\to\infty} J_\lambda(u(t)) = d. \tag{3}$$

In order to state our main result, we have to recall some notion from variational analysis.

Definition 2.1 ((PS)-condition)

Let $S \subset H_0^1$.

(a) (u_n) *is said to be a Palais-Smale sequence of* J_λ *at level* d *in* S ($(\mathrm{PS})_d$-*sequence in* S) *if*

$$(u_n) \subset S, \quad J_\lambda(u_n) \to d, \quad (dJ_\lambda)_{u_n} \to 0 \text{ in } (H_0^1)^*$$

where $(dJ_\lambda)_{u_n}$ *denotes the Fréchet derivative of* J_λ *at* u_n *in* H_0^1.

(b) J_λ *is said to satisfy the Palais-Smale condition at level* d *in* S ($(\mathrm{PS})_d$-*condition in* S) *if any* $(\mathrm{PS})_d$-*sequence in* S *contains a strongly convergent subsequence in* H_0^1.

Let u be a (global) solution of (P). We introduce the $(\mathrm{PS})_d$-condition along the orbit u.

Definition 2.2 ((PS)-condition along the orbit)

J_λ *is said to satisfy the Palais-Smale condition along* u ((PS)-*condition along* u) *when* J_λ *satisfies the* $(\mathrm{PS})_d$-*condition in* $S = \{u(t); t \in (0, \infty)\}$ *where* d *is given by* (3).

Remark 2.1

It is easy to see that J_λ *satisfies the* (PS)-*condition along* u *if there exists* U *such that* J_λ *satisfies the* $(\mathrm{PS})_d$-*condition in* U *and* $\{u(t);\ t \in (0, \infty)\} \subset U$.

Our main theorem gives a sufficient and necessary condition (on u_0 or u and J_λ) for the existence of an L^∞-global bounds of u in terms of the (PS)-condition. Observe that our main theorem does not require the subcriticality of q.

Theorem 2.1 (Main Theorem)
Let $q \in (2, 2^*]$ and $d = \lim_{t\to\infty} J_\lambda(u(t))$. Then the following assertion (a) *and* (b) *are equivalent.*
(a) *J_λ satisfies the $(\mathrm{PS})_d$-condition along u.*
(b) *u has an L^∞-global bounds.*

Now we shall see some corollaries which follow easily from the main theorem.

For $q < 2^*$, it is well known that J_λ satisfies the $(\mathrm{PS})_d$-condition for any $d \in \mathbb{R}$ and for any $\lambda \in \mathbb{R}$, see e.g. [12, Chapter II, Proposition 2.2]. Hence by Remark 2.1 and by Theorem 2.1, we again obtain Proposition 1.1.

Corollary 2.1 (Subcritical case, Proposition 1.1)
Let $q \in (2, 2^*)$ and let $\lambda \in \mathbb{R}$. Then u has an L^∞-global bounds.

Let $q = 2^*$ and $d < S^{N/2}/N$. Then it is well known that J_λ satisfies the $(\mathrm{PS})_d$-condition, see [8]. Hence Remark 2.1 and Theorem 2.1 yield the following.

Corollary 2.2 (Critical case, Brezis-Nirenberg type)
Let $q = 2^*$, $\lambda \in \mathbb{R}$ and $d < S^{N/2}/N$. Then u has an L^∞-global bounds.

Let $\Omega_a := \{x \in \mathbb{R}^m; 1 < |x|_{\mathbb{R}^m} < 2\}$ be an annulus, $k \in \mathbb{N}$ and $\Omega := \Omega_a \times \cdots \times \Omega_a$ (k times). Also let $G := SO(m) \oplus \cdots \oplus SO(m)$ (k fold). Here we recall that J_λ satisfies the $(\mathrm{PS})_d$-condition in the G-invariant subspace of H_0^1. It is also obvious that if u_0 is G-invariant, then the corresponding solution of (P) is also G-invariant. Hence by Remark 2.1 and Theorem 2.1, we have:

Corollary 2.3 (Critical, G-invariant case)
Let $q = 2^*$ and $\lambda \in \mathbb{R}$. Let Ω and G be as above and u_0 be a G-invariant function. Then u has an L^∞-global bounds.

As for the solution which blows up in infintie time (see e.g. Proposition 1.2), Theorem 2.1 yields:

Corollary 2.4 (Infinite time blow up solution)

Let $q \in (2, 2^*]$. Assume that u blows up in infinite time in the L^∞-sense. Then J_λ does not satisfy the $(\mathrm{PS})_d$*-condition along u.*

3 Proof of Theorem 2.1

Now let us give the sketch of the proof of Theorem 2.1. In the following, $q_0 := N(q-2)/2$ (which is the critical exponent of (P) as a parabolic problem, see [6]).

The proof of (a)⇒(b) consists of two steps. The first step, Proposition 3.1, involves the compactness propertiy of the orbit in L^{q_0}. In the latter step, we establish the relation between the existence of an L^∞-global bounds and the compactness of the orbit in L^{q_0} (Proposition 3.2). The proof of Theorem 2.1 is in the last of this section. In this section, u always denotes a global-in-time solution of (P).

Proposition 3.1

Let $q \in (2, 2^*]$. Assume that J_λ satisfies $(\mathrm{PS})_d$*-condition along u. Then for any $t_n \to \infty$, there exists a subsequence of (t_n) (still denoted by the same symbol) and $u \in L^{q_0}$ such that $u(t_n) \to u$ strongly in L^{q_0}.*

Proof.

Take any $t_n \to \infty$ and let $u_n(s) := u(t_n - 1/2 + s)$ for $s \in [0, 1]$. Then it is easy to see that there exists a subsequence of (t_n) (still denoted by the same symbol) and $L \subset [0, 1]$ with measure zero such that, for all $s \in [0, 1] \setminus L$,

$$(u_n(s)) \text{ is a } (\mathrm{PS})_d\text{-sequence,} \tag{1}$$

$$\|u_n(s)\|_q^q \to d/(1/2 - 1/q) \text{ as } n \to \infty. \tag{2}$$

By the assumption of the Proposition and (1), $(u_n(s))$ is relatively compact in H_0^1 for $s \in [0, 1] \setminus L$. Hence by the continuity of $H_0^1 \hookrightarrow L^q$, $K(s) := \overline{\{u_n(s)\}}^{L^q}$ is a compact set in L^q. Then by Theorem 1 of [6], for any $\varepsilon > 0$ and for any $s \in [0, 1] \setminus L$, there exists $\delta(\varepsilon, s) := \delta(\varepsilon, K(s)) > 0$ such that

$$\|u_n(s + \sigma)\|_q^q \leq \|u_n(s)\|_q^q + \varepsilon/2, \quad \forall n, \ \forall \sigma \in [0, \delta(\varepsilon, s)]. \tag{3}$$

Consequently, we find that

$$\|u_n(s)\|_q^q \leq d/(1/2 - 1/q) + \varepsilon, \quad \forall s \in [1/4, 3/4], \ \forall n > N \tag{4}$$

for some N. Hence, the decreasing property of $J_\lambda(u(t))$ in t together with (4) yields

$$\|\nabla u_n(s)\|_2 \leq C, \quad \forall s \in [1/4, 3/4],\ \forall n > N \tag{5}$$

for some $C > 0$.

Case 1. Assume that $q < 2^*$. Then $q_0(:= N(q-2)/2) < 2^*$. Hence by the compactness of $H_0^1 \hookrightarrow L^{q_0}$ and by (5) with $s = 1/2$, we have the conclusion (recall that $u_n(1/2) = u(t_n)$).

Case 2. Assume that $q = 2^*$. Note that, in this case, $q_0 = q = 2^*$. Hereafter we denote both of q and q_0 by 2^*. By (5), by the continuity of $H_0^1 \hookrightarrow L^{2^*}$ and by the compactness of $H_0^1 \hookrightarrow L^2$, we can find $u(1/2)$ such that

$$u_n(1/2) \rightharpoonup u(1/2) \text{ weakly in } L^{2^*}, \tag{6}$$

$$u_n(1/2) \to u(1/2) \text{ strongly in } L^2 \tag{7}$$

as $n \to \infty$, taking subsequence if necessary (recall that $u_n(1/2) = u(t_n)$). Especially by (6) and (4),

$$\|u(1/2)\|_{2^*}^{2^*} \leq \|u_n(1/2)\|_{2^*}^{2^*} + o(1) \leq d/(1/2 - 1/q) + o(1) \tag{8}$$

as $n \to \infty$.

Take any $\sigma \in [1/4, 3/4] \setminus L$. Then by (1) and by the assumption of the Proposition, $(u_n(\sigma))$ has a strongly convergent subsequence in H_0^1. Hence, there exists $u(\sigma) \in H_0^1$ such that

$$u_n(\sigma) \to u(\sigma) \text{ strongly in } L^{2^*} \text{ and in } L^2 \tag{9}$$

taking further subsequence if necessary. Especially by (2) and by (9), we have

$$d/(1/2 - 1/q) = \|u_n(\sigma)\|_{2^*}^{2^*} + o(1) = \|u(\sigma)\|_{2^*}^{2^*} \tag{10}$$

as $n \to \infty$.

Moreover, by (7) and (9),

$$\begin{aligned} \|u(\sigma) - u(1/2)\|_2 &\leq \|u(\sigma) - u_n(\sigma)\|_2 + \left\|\frac{\partial u_n}{\partial s}\right\|_{L^2(0,1;L^2)} \\ &\quad + \|u(1/2) - u_n(1/2)\|_2 \\ &= o(1), \end{aligned} \tag{11}$$

thus we have

$$u(1/2) = u(\sigma). \tag{12}$$

Hence by (10), (12) and (8),

$$\begin{aligned} d/(1/2 - 1/q) &= \|u(\sigma)\|_{2^*}^{2^*} = \|u(1/2)\|_{2^*}^{2^*} \le \|u_n(1/2)\|_{2^*}^{2^*} + o(1) \\ &\le d/(1/2 - 1/q) \end{aligned}$$

as $n \to \infty$. Therefore combining this relation with (6), we have $u(t_n) = u_n(1/2) \to u(1/2)$ strongly in $L^{2^*} = L^{q_0}$, thus the conclusion. ∎

Proposition 3.2

Assume that for any $t_n \to \infty$, there exists a subsequence of (t_n) (still denoted by the same symbol) and u such that $u(t_n) \to u$ in L^{q_0}. Then u has an L^∞-global bounds.

Proof.

Assume that the conclusion is false. Then there exist $(x_n) \subset \Omega$ and $t_n \to \infty$ such that

$$\|u(t_n)\|_\infty \to \infty, \ \sup_{t\in(0,t_n]} \|u_n(t)\|_\infty = \|u(t_n)\|_\infty, \ \|u(t_n)\|_\infty/2 \le |u(x_n, t_n)|. \tag{13}$$

Let y, s, v_n be

$$y = \lambda_n(x - x_n), \ s = \lambda_n^2(t - t_n), \ \lambda_n^{2/(q-2)} v_n(y, s) = u(x, t)$$

for λ_n with $\lambda_n^{2/(q-2)} = \|u(t_n)\|_\infty$. Note that by virtue of the choice of λ_n and (13), we have $\lambda_n \to \infty$ and

$$\sup_{s\in[-1,0]} \|v_n(s)\|_\infty \le \|v_n(0_s)\|_\infty = 1, \tag{14}$$

$$|v_n(0_y, 0_s)| \ge 1/2. \tag{15}$$

By the boundedness of Ω and the homogeneous Dirichlet condition, we can assume that $x_n \to x \in \operatorname{int}\Omega$ taking subsequence if necessary, see e.g. [5] or [9]. By (14), $\|v_n\|_{L^\infty(-1,\delta:L^\infty)} < 2$ holds for some $\delta > 0$ which is independent of n. Then, by the standard parabolic estimate, we see that

$$v_n \to v \text{ in } C_{\text{loc}}(\mathbb{R}^N \times (-1, \delta)) \tag{16}$$

holds for $v \in C_{\mathrm{loc}}(\mathbb{R}^N \times (-1,\delta))$.

Also by the straightforward calculation using (3),

$$\begin{aligned} \left\| \frac{\partial v_n}{\partial s} \right\|^2_{L^2(-1,\delta;L^2)} &= \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t_n-1/\lambda_n^2, t_n+\delta/\lambda_n^2;L^2)} \\ &= J_\lambda(u(t_n - 1/\lambda_n^2)) - J_\lambda(u(t_n + \delta/\lambda_n^2)) \\ &\to d - d = 0 \end{aligned}$$

follows. Hence the same argument as in (11) implies that v is independent of s. Moreover by (15) and by (16), $|v(0_y)| \geq 1/2$. Therefore there exists $R > 0$ sufficientlly small such that

$$\|v\|_{q_0, B(0;R)} =: \eta > 0. \tag{17}$$

Since $x \in \operatorname{int}\Omega$, $B(x;\varepsilon) \subset \Omega$ holds for small ε. Observe that for large n, $B(x_n; R/\lambda_n) \subset B(x,\varepsilon)$. Then by (17) and (16),

$$\begin{aligned} 0 &< \eta = \|v\|_{q_0,B(0;R)} = \|v_n(0_s)\|_{q_0,B(0;R)} + o(1) \\ &= \|u(t_n)\|_{q_0,B(x_n;R/\lambda_n)} + o(1) \leq \|u(t_n)\|_{q_0,B(x;\varepsilon)} + o(1) \end{aligned} \tag{18}$$

for small $\varepsilon > 0$.

On the other hand, the assumption of the Proposition yields

$$\|u(t_n)\|_{q_0,B(x;\varepsilon)} \overset{n\to\infty}{\to} \|u\|_{q_0,B(x;\varepsilon)} \overset{\varepsilon\to 0}{\to} 0$$

along an appropriate subsequence, which is absurd in view of (18). ∎

Proof of Theorem 2.1 The assertion (a) ⇒ (b) immediately follows from Proposition 3.1 and 3.2.

The assertion (b) ⇒ (a) follows from a typical argument for the verification of (PS)-condition in the variational analysis. ∎

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