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PREFACE

This volume is intended as the proceedings of Sapporo Guest House Symposium on Mathematics 22, Nonlinear Wave Equations, held on November 19 and 20 in 2006 at Sapporo Guest House.

The first Sapporo Guest House Symposium was held in 1999 by Y. Giga. We keep the size of each meeting relatively small but international. The complete list of symposia at the Sapporo Guest House is in our website:

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Hideo Kubo
Tohru Ozawa

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日 時： 2006 年 11 月 19 日 (日曜日) 13:00 ～ 11 月 20 日 (月曜日) 16:00

場 所：札幌天神山国際ゲストハウス・研修室

http://www.plaza-sapporo.or.jp/sgh/

最寄駅：地下鉄南北線・澄川駅　または　南平岸駅

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Organizers: Tohru OZAWA [Hokkaido Univ] (小澤 徹)
Tel: 011-706-2672
Hideo KUBO [Osaka Univ.] (久保 英夫)
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WKB ANALYSIS FOR THE NONLINEAR SCHRÖDINGER EQUATION AND INSTABILITY RESULTS

RÉMI CARLES

ABSTRACT. For the semi-classical limit of the cubic, defocusing nonlinear Schrödinger equation with an external potential, we explain the notion of criticality before a caustic is formed. In the sub-critical and critical cases, we justify the WKB approximation. In the super-critical case, the WKB analysis provides a new phenomenon for the (classical) cubic, defocusing nonlinear Schrödinger equation, which can be compared to the loss of regularity established for the nonlinear wave equation by G. Lebeau. We also show some instabilities at the semi-classical level.

1. INTRODUCTION

We consider the solution to the nonlinear Schrödinger equation (NLS) with a cubic, defocusing nonlinearity, and an external potential:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \varepsilon^\kappa |u^\varepsilon|^2 u^\varepsilon, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \tag{1.1}$$

$$u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}. \tag{1.2}$$

We are interested in the behavior of the solution u^ε as the positive parameter ε goes to zero. According to the cultural background, this field goes under the name of semi-classical limit, WKB analysis (which is a little bit more specific, see below), or geometrical optics. The general idea is to describe the asymptotic behavior of u^ε with a simplified model, which involves geometric quantities. In the case of (1.1), these quantities are called either classical trajectories (in view of classical mechanics), or rays (in view of geometric optics).

There are at least two motivations for such a study. We outline them here, and refer to the survey [22] for a broader discussion on this subject. The first one comes from the applied mathematics, and may find its origins in physics. In the case of (1.1), suppose that ε represents the (rescaled) Planck constant. It may be small compared to the other parameters at stake. In this case, it is sensible to consider that the asymptotic behavior of u^ε as $\varepsilon\to 0$ provides a reliable approximation of the exact solution. Hopefully, the asymptotic model is easier to describe than the initial one (1.1)–(1.2). Another motivation stems from the propagation of singularities for equation where the small parameter ε is not necessarily present initially. Most of the studies in this direction concern hyperbolic equations. However, the belief according to which Schrödinger equations share properties with hyperbolic equations in the semi-classical limit is a first hint that this field is applicable to Schrödinger equations as well (see e.g. [16, 24]). To illustrate this statement, we give a result, whose proof will be straightforward after the analysis of (1.1)–(1.2).

Theorem 1.1 ([6], Cor. 1.7). *Let $n \geqslant 3$. Consider the cubic, defocusing NLS:*

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^2 u \quad ; \quad u_{|t=0} = u_0 \,. \tag{1.3}$$

Denote $s_c = \frac{n}{2} - 1$. Let $0 < s < s_c$. We can find a family $(u_0^\varepsilon)_{0<\varepsilon\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^n)$ with

$$\|u_0^\varepsilon\|_{H^s(\mathbb{R}^n)} \to 0 \text{ as } \varepsilon \to 0 \,,$$

and $0 < t^\varepsilon \to 0$ such that the solution u^ε to (1.3) associated to u_0^ε satisfies:

$$\|u^\varepsilon(t^\varepsilon)\|_{H^k(\mathbb{R}^n)} \to +\infty \text{ as } \varepsilon \to 0 \,, \ \forall k \in \left]\frac{s}{\frac{n}{2} - s}, s\right] .$$

This result is in the same spirit as the initial breakthrough by G. Lebeau ([17, 18], see also [21]). These former results also rely on geometrical optics (in a super-critical régime; see below for this notion in the case of (1.1)). For (1.3), the above result was first established by M. Christ, J. Colliander and T. Tao [10] in the case $k = s$ (see also [3, Appendix] and [6, Appendix B]). The fact that we can go strictly below the value $k = s$ stems from an analysis of (1.1) in a case where the nonlinearity should be considered as quasilinear, and not semilinear. This result is then a consequence of the original idea of E. Grenier [14].

The rest of this text is organized as follows. In § 2, we introduce the notion of criticality for (1.1) at a formal level. In § 3, we explain how to justify this notion, and describe the asymptotic behavior of u^ε in different cases. The proof of Theorem 1.1 is given in § 4. More instability results are given in § 5.

2. WKB analysis and the notion of criticality

This section remains at a formal level only. It is a preparation to the forthcoming rigorous justifications. In a WKB analysis, one assumes for instance that the initial profile a_0^ε can be expanded as a power series in ε:

$$a_0^\varepsilon \sim a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

Note that the modulation factor ε^κ in front of the nonlinearity in (1.1) could be taken equal to one, up to replacing a_0^ε with $\varepsilon^{\kappa/2} a_0^\varepsilon$. In particular, it is clear that for large $\kappa > 0$, the nonlinearity is expected to be negligible in the limit $\varepsilon \to 0$ (at least locally in time). WKB analysis consists in seeking an approximation of the form

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x) e^{\phi(t,x)/\varepsilon},$$

where the amplitude a and the phase ϕ are independent of ε. Plugging this approximate solution into (1.1) and canceling the first powers of ε yields:

$$\mathcal{O}\left(\varepsilon^0\right): \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + V = 0 \ ; \ \phi_{|t=0} = \phi_0 \quad \text{if } \kappa > 0, \tag{2.1}$$

$$\mathcal{O}\left(\varepsilon^1\right): \quad \partial_t a + \nabla\phi \cdot \nabla a + \frac{1}{2} a \Delta\phi = \begin{cases} 0 & \text{if } \kappa > 1, \\ -i|a|^2 a & \text{if } \kappa = 1. \end{cases} ; \ a_{|t=0} = a_0 \tag{2.2}$$

This shows that the minimal value of κ for which nonlinear effects affect the solution at leading order is $\kappa = 1$. We shall consider here the following three cases: $\kappa > 1$ (sub-critical case), $\kappa = 1$ (critical case), and $\kappa = 0$ (a super-critical case). We refer to [7] for the case $0 < \kappa < 1$. The first step consists in solving (2.1). The potential V may be time dependent: $V = V(t,x)$.

Lemma 2.1. *Assume that V and ϕ_0 are smooth and sub-quadratic:*

- $V \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$, *and* $\partial_x^\alpha V \in L^\infty_{\text{loc}}(\mathbb{R}_t; L^\infty(\mathbb{R}^n_x))$ *as soon as* $|\alpha| \geqslant 2$.
- $\phi_0 \in C^\infty(\mathbb{R}^n)$, *and* $\partial_x^\alpha \phi_0 \in L^\infty(\mathbb{R}^n)$ *as soon as* $|\alpha| \geqslant 2$.

Then there exist $T > 0$ and a unique solution $\phi_{\text{eik}} \in C^\infty([0,T] \times \mathbb{R}^n)$ to (2.1). This solution is sub-quadratic: $\partial_x^\alpha \phi_{\text{eik}} \in L^\infty([0,T] \times \mathbb{R}^n)$ as soon as $|\alpha| \geqslant 2$.

The proof of this lemma relies on Hamilton–Jacobi theory. Note that the time of existence T is uniform with respect to $x \in \mathbb{R}^n$: a global inversion theorem is needed, which can be found in [23] or [11]. We refer to [7] for the proof of this lemma, and for a discussion on the optimality of the assumptions. In particular, one should not expect the above solution to remain smooth for all time. The appearance of singularities corresponds to the formation of caustics. The aim of WKB analysis is to describe the solution u^ε before a caustic is formed. See e.g. [5] for the description of a solution to (1.1)–(1.2) beyond a focal point.

To analyze (2.2), we introduce the Hamiltonian flow, on which the proof of Lemma 2.1 relies:

$$\begin{cases} \partial_t x(t,y) = \xi\,(t,y) \quad ; \quad x(0,y) = y, \\ \partial_t \xi(t,y) = -\nabla_x V\,(t, x(t,y)) \quad ; \quad \xi(0,y) = \nabla\phi_0(y). \end{cases} \tag{2.3}$$

The time T is such that the map $y \mapsto x(t,y)$ is a diffeomorphism of $\mathbb{R}^n$ for $t \in [0,T]$. The key observation is that (2.2) is a transport equation, which turns out to be an ordinary differential equation along the classical trajectories. Introduce the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t,y).$$

Denote

$$A(t,y) := a\,(t, x(t,y))\,\sqrt{J_t(y)}.$$

For $t \in [0,T]$, (2.2) is equivalent to:

$$\partial_t A = \begin{cases} 0 & \text{if } \kappa > 1, \\ -\,iJ_t(y)^{-1}\,|A|^2\,A & \text{if } \kappa = 1. \end{cases} \quad ; \quad A(0,y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly: we see that $\partial_t |A|^2 = 0$, hence, for $\kappa = 1$,

$$A(t,y) = a_0(y) \exp\left(-i \int_0^t J_s(y)^{-1}\,|a_0(y)|^2\,ds\right).$$

Inverting the map $y \mapsto x(t,y)$ yields $a(t,x)$. We see that the critical nonlinear effect is a self-modulation of the amplitude. In the context of laser physics, this phenomenon is known as *phase self-modulation* (see e.g. [25, 2, 12]).

We now turn to the super-critical case $\kappa = 0$. To illustrate the difficulty of this case, seek a more precise asymptotic expansion of u^ε:

$$u^\varepsilon(t,x) \sim \left(\mathrm{a}_0(t,x) + \varepsilon \mathrm{a}_1(t,x) + \varepsilon^2 \mathrm{a}_2(t,x) + \ldots\right) e^{i\phi(t,x)/\varepsilon}.$$

Plugging such an asymptotic expansion into (1.1) yields a shifted cascade of equations:

$$\mathcal{O}\left(\varepsilon^0\right): \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + V + |\mathrm{a}_0|^2 = 0 \; ; \; \phi_{|t=0} = \phi_0.$$

$$\mathcal{O}\left(\varepsilon^1\right): \quad \partial_t \mathrm{a}_0 + \nabla\phi \cdot \nabla \mathrm{a}_0 + \frac{1}{2}\mathrm{a}_0 \Delta\phi = -2i \mathrm{Re}\left(\mathrm{a}_0 \overline{\mathrm{a}_1}\right) \mathrm{a}_0 \; ; \; \mathrm{a}_{0|t=0} = a_0.$$

Two comments are in order. First, we see that there is a strong coupling between the phase and the main amplitude: a_0 is present in the equation for ϕ. Second, the above system is not closed: ϕ is determined in function of a_0, and a_0 is determined in function of a_1. Even if we pursued the cascade of equations, this phenomenon would remain: no matter how many terms are computed, the system is never closed (see [13]). This is a typical feature of super-critical cases in nonlinear geometrical optics (see [8, 9]).

In the case when $V \equiv 0$ and $\phi_0 \in H^s$, this problem was resolved by E. Grenier [14], by modifying the usual WKB methods (see § 3.2). Note that even though a_1 is not determined by the above system, the pair $(\rho, v) := (|\mathsf{a}_0|^2, \nabla\phi)$ solves a compressible Euler equation:

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \nabla V + \nabla \rho = 0\ ; \quad v\big|_{t=0} = \nabla\phi_0 \\ \partial_t \rho + \nabla\cdot(\rho v) = 0\ ; \quad \rho\big|_{t=0} = |a_0|^2. \end{aligned} \tag{2.4}$$

3. Rigorous WKB analysis

We outline here some results presented in details in [7].

3.1. Sub-critical and critical cases.

Let $\kappa \geqslant 1$. Under the assumptions of Lemma 2.1, we change the Cauchy problem (1.1)–(1.2): define a^ε by

$$u^\varepsilon(t,x) = a^\varepsilon(t,x) e^{i\phi_{\rm eik}(t,x)/\varepsilon}.$$

Then for $t\in[0,T]$, (1.1)–(1.2) is equivalent to:

$$\begin{aligned} \partial_t a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac12 a^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon - i\varepsilon^{\kappa-1}|a^\varepsilon|^2 a^\varepsilon, \\ a^\varepsilon_{|t=0} = a_0^\varepsilon. \end{aligned} \tag{3.1}$$

Two things must be noticed: first, the potential V and the initial phase ϕ_0 do not appear in this new problem. Second, the factors involving $\phi_{\rm eik}$ have the following features: in view of Lemma 2.1, the term $\Delta\phi_{\rm eik}$ is in $L^\infty([0,T]\times\mathbb{R}^n)$, and the operator $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ is a transport operator. We can then obtain energy estimates in Sobolev spaces, and establish:

Proposition 3.1. *Under the assumptions of Lemma 2.1, assume moreover that a_0^ε is bounded in $H^s(\mathbb{R}^n)$ uniformly for $\varepsilon\in]0,1]$, for all $s\geqslant 0$. Let $\kappa\geqslant 1$. Then for all $\varepsilon\in]0,1]$, (3.1) has a unique solution $a^\varepsilon\in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];H^s)$ for all $s>n/2$. Moreover, a^ε is bounded in $L^\infty([0,T];H^s)$ uniformly in $\varepsilon\in]0,1]$, for all $s\geqslant 0$.*

These uniform estimates allow us to neglect the term $\varepsilon\Delta a^\varepsilon$ on the right hand side of (3.1). Assume moreover the following convergence:

$$a_0^\varepsilon \to a_0 \text{ in } H^s(\mathbb{R}^n), \quad \forall s\geqslant 0. \tag{3.2}$$

Corollary 3.2. *Let $\kappa\geqslant 1$. Under the above assumptions,*

$$\|a^\varepsilon - \widetilde{a}^\varepsilon\|_{L^\infty([0,T];H^s)} \to 0 \quad \text{as } \varepsilon\to 0, \quad \forall s\geqslant 0,$$

where $\widetilde{a}^\varepsilon$ solves:

$$\partial_t\widetilde{a}^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\widetilde{a}^\varepsilon + \frac12\widetilde{a}^\varepsilon\Delta\phi_{\rm eik} = -i\varepsilon^{\kappa-1}|\widetilde{a}^\varepsilon|^2\widetilde{a}^\varepsilon \quad ; \quad \widetilde{a}^\varepsilon_{|t=0} = a_0. \tag{3.3}$$

Proceeding as in § 2, denote

$$A^\varepsilon(t,y) := \widetilde{a}^\varepsilon\left(t, x(t,y)\right)\sqrt{J_t(y)}.$$

We see that so long as $y \mapsto x(t,y)$ defines a global diffeomorphism (which is guaranteed for $t \in [0,T]$ by construction), (3.3) is equivalent to:

$$\partial_t A^\varepsilon = -i\varepsilon^{\kappa-1} J_t(y)^{-1} \left|A^\varepsilon\right|^2 A^\varepsilon \quad ; \quad A^\varepsilon(0,y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly:

$$A^\varepsilon(t,y) = a_0(y) \exp\left(-i\varepsilon^{\kappa-1}\int_0^t J_s(y)^{-1}\left|a_0(y)\right|^2 ds\right).$$

Back to the initial solution u^ε, we conclude:

Proposition 3.3. *Let $\kappa \geqslant 1$. Under the above assumptions, for all $\varepsilon \in]0,1]$, (1.1)-(1.2) has a unique solution $u^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n) \cap C([0,T];H^s)$ for all $s > n/2$. Moreover, there exist $a, G \in C^\infty([0,T]\times\mathbb{R}^n)$, independent of $\varepsilon \in]0,1]$, where $a \in C([0,T];L^2\cap L^\infty)$, and G is real-valued with $G \in C([0,T];L^\infty)$, such that:*

$$\left\| u^\varepsilon - a e^{i\varepsilon^{\kappa-1}G} e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0.$$

The profile a solves the initial value problem:

$$\partial_t a + \nabla\phi_{\rm eik}\cdot\nabla a + \frac{1}{2}a\Delta\phi_{\rm eik} = 0 \quad ; \quad a_{|t=0} = a_0, \tag{3.4}$$

and G depends nonlinearly on a:

$$a(t,x) = \frac{1}{\sqrt{J_t(y(t,x))}} a_0\left(y(t,x)\right),$$

$$G(t,x) = -\int_0^t J_s(y(t,x))^{-1}\left|a_0(y(t,x))\right|^2 ds.$$

In particular, if $\kappa > 1$, then

$$\left\| u^\varepsilon - a e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0,$$

and no nonlinear effect is present in the leading order behavior of u^ε. If $\kappa = 1$, nonlinear effects are present at leading order, measured by G.

3.2. **Super-critical case: $\kappa = 0$.** In this case, we recall the beautiful idea of E. Grenier, which makes it possible to consider the case $V \equiv 0$ and $\phi_0 \in H^s(\mathbb{R}^n)$ for sufficiently large s. The approach consists in reversing the steps of the WKB analysis: usually, one seeks an approximate solution, and then tries to show that stability arguments imply that the exact solution is well approximated by this process. To overcome the issues mentioned in § 2, the idea in [14] consists in first writing the unknown as:

$$u^\varepsilon(t,x) = a^\varepsilon(t,x)e^{i\phi^\varepsilon(t,x)/\varepsilon},$$

where the amplitude a^ε is *complex-valued* (even if a_0^ε is real-valued), and ϕ^ε is real-valued. Doing so, one introduces one degree of freedom to rewrite (1.1)–(1.2). The usual approach in physics (see e.g. [15]) consists in writing

$$\begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2}\left|\nabla \phi^\varepsilon\right|^2 + |a^\varepsilon|^2 = \varepsilon^2 \frac{\Delta a^\varepsilon}{2a^\varepsilon} \quad ; \quad \phi^\varepsilon\big|_{t=0} = \phi_0\,, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = 0 \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon. \end{cases}$$

Obviously, this approach is delicate when a^ε has zeroes; see the discussion in [13] on this subject. The choice in [14] is to write:

$$\begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2}\left|\nabla \phi^\varepsilon\right|^2 + |a^\varepsilon|^2 = 0 \quad ; \quad \phi^\varepsilon\big|_{t=0} = \phi_0\,, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{cases} \tag{3.5}$$

Inspired by the fact that the expected limit is related to the compressible Euler equation (see § 2), introduce the "velocity" $v^\varepsilon = \nabla \phi^\varepsilon$. Then (3.5) yields:

$$\begin{cases} \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + 2\,\mathrm{Re}\,(\overline{a^\varepsilon} \nabla a^\varepsilon) = 0 \quad ; \quad v^\varepsilon\big|_{t=0} = \nabla \phi_0\,, \\ \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{cases} \tag{3.6}$$

Separate real and imaginary parts of a^ε, $a^\varepsilon = a_1^\varepsilon + i a_2^\varepsilon$. Then we have

$$\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L \mathbf{u}^\varepsilon\,, \tag{3.7}$$

$$\text{with} \quad \mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v_1^\varepsilon \\ \vdots \\ v_n^\varepsilon \end{pmatrix} \quad , \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{n\times n} & \end{pmatrix},$$

$$\text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}\,{}^t\xi \\ 0 & v\cdot\xi & \frac{a_2}{2}\,{}^t\xi \\ 2a_1\,\xi & 2a_2\,\xi & v\cdot\xi I_n \end{pmatrix}.$$

The matrix $A(\mathbf{u},\xi)$ can be symmetrized by

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4}I_n \end{pmatrix}.$$

The important point to notice is that the operator L is skew-symmetric: it is invisible in the energy estimates, so the loss of derivative (it is the only operator of order two in (3.7)) is avoided. Denote $H^\infty = \cap_{s\geqslant 0} H^s(\mathbb{R}^n)$. Classical theory on symmetric hyperbolic systems yields a solution $(v^\varepsilon, a^\varepsilon)$ to (3.6). Once v^ε is known, we note that it is irrotational, so there exists ϕ^ε such that $v^\varepsilon = \nabla\phi^\varepsilon$. Up to adding a function of time only, $(\phi^\varepsilon, a^\varepsilon)$ solves (3.5).

Proposition 3.4 ([14], Th. 1.1). *Let $\kappa = 0$. Suppose that $\phi_0 \in H^\infty$, and that a_0^ε is bounded in $H^s(\mathbb{R}^n)$ uniformly for $\varepsilon \in]0,1]$, for all $s \geqslant 0$. Let $s > 2 + n/2$. There exist $T_s > 0$ independent of $\varepsilon \in]0,1]$ and $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solution to* (1.1)–(1.2) *on* $[0, T_s]$. *Moreover, a^ε and ϕ^ε are bounded in $L^\infty([0,T_s]; H^s)$, uniformly in $\varepsilon \in]0,1]$.*

Assume moreover that (3.2) holds. The solution to (3.5) formally converges to the solution of:

$$
\left\{
\begin{aligned}
&\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + |a|^2 = 0 \quad ; \quad \phi\big|_{t=0} = \phi_0 ,\\
&\partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2} a\Delta\phi = 0 \quad ; \quad a\big|_{t=0} = a_0 .
\end{aligned}
\right. \tag{3.8}
$$

Under the above assumptions, (3.8) has a unique solution $(a,\phi) \in L^\infty([0,T_*];H^m)^2$ for all $m>0$ for some $T_*>0$ independent of m (see e.g. [1, 19]).

Remark 3.5. More general nonlinearity. Suppose that we consider a more general nonlinearity:

$$
i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon .
$$

Then following the same lines as above, the symmetrizer naturally becomes

$$
S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4f'(|a^\varepsilon|^2)} I_n \end{pmatrix}.
$$

For this matrix to be positive, and to be able to estimate its time derivative, it is natural to assume $f'>0$. This corresponds to the assumption made in [14], and in [7]. For the above analysis to be valid, the nonlinearity has to be defocusing, and cubic at the origin. In particular, the WKB analysis for the quintic defocusing NLS is still an open problem. Note however that it is possible to construct solutions to the limit problem in that case (the analogue of (3.8)), thanks to the result of [20] and the geometrical analysis of § 3.1. Yet, the nonlinear change of variable of [20] is apparently incompatible with the above remark that L is skew-symmetric.

If we suppose in addition that there exists $a_0, a_1 \in H^\infty$ such that

$$
a_0^\varepsilon = a_0 + \varepsilon a_1 + o(\varepsilon) \quad \text{in } H^s,\ \forall s \geqslant 0,
$$

then we infer:

Proposition 3.6. *Let $s\in\mathbb{N}$. Then $T_s \geqslant T_*$, and there exists C_s independent of ε such that for every $0\leqslant t\leqslant T_*$,*

$$
\|a^\varepsilon(t) - a(t)\|_{H^s} \leqslant C_s\varepsilon \quad ; \quad \|\phi^\varepsilon(t) - \phi(t)\|_{H^s} \leqslant C_s\varepsilon t.
$$

Note that this suffices to describe u^ε for very small time only:

$$
\begin{aligned}
u^\varepsilon - a e^{i\phi/\varepsilon} &= a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - a e^{i\phi/\varepsilon}\\
&= \left(a^\varepsilon - a\right)e^{i\phi^\varepsilon/\varepsilon} + 2ia e^{i(\phi^\varepsilon+\phi^\varepsilon)/2\varepsilon}\sin\left(\frac{\phi^\varepsilon-\phi}{2\varepsilon}\right).
\end{aligned}
$$

The first term of the right hand side is of order $\mathcal{O}(\varepsilon)$ in $L^2\cap L^\infty$, but the second one is of order $\mathcal{O}(t)$ only: therefore, we only have

$$
u^\varepsilon(t,x) \sim a(t,x)e^{i\phi(t,x)/\varepsilon} \quad \text{for } 0\leqslant t \ll 1.
$$

To have a better error estimate, it is necessary to compute the next term in the asymptotic expansion of $(\phi^\varepsilon, a^\varepsilon)$ in powers of ε. For times of order $\mathcal{O}(1)$, the initial corrector a_1 must be taken into account:

Proposition 3.7. *Define* $(a^{(1)},\phi^{(1)})$ *by*

$$\partial_t\phi^{(1)}+\nabla\phi\cdot\nabla\phi^{(1)}+2\operatorname{Re}\left(\overline{a}a^{(1)}\right)=0,$$
$$\partial_t a^{(1)}+\nabla\phi\cdot\nabla a^{(1)}+\nabla\phi^{(1)}\cdot\nabla a+\frac{1}{2}a^{(1)}\Delta\phi+\frac{1}{2}a\Delta\phi^{(1)}=\frac{i}{2}\Delta a,$$
$$\phi^{(1)}\big|_{t=0}=0\quad;\quad a^{(1)}\big|_{t=0}=a_1.$$

Then $a^{(1)},\phi^{(1)}\in L^\infty([0,T_*];H^s)$ *for every* $s\geqslant 0$*, and*

$$\|a^\varepsilon-a-\varepsilon a^{(1)}\|_{L^\infty([0,T_*];H^s)}+\|\phi^\varepsilon-\phi-\varepsilon\phi^{(1)}\|_{L^\infty([0,T_*];H^s)}\leqslant C_s\varepsilon^2,\quad\forall s\geqslant 0\,.$$

Despite the notations, it seems unadapted to consider $\phi^{(1)}$ as being part of the phase. Indeed, we infer from Proposition 3.7 that

$$\left\|u^\varepsilon-ae^{i\phi^{(1)}}e^{i\phi/\varepsilon}\right\|_{L^\infty([0,T_*];L^2\cap L^\infty)}=\mathcal{O}(\varepsilon).$$

Relating this information to the WKB methods presented at the end of § 2, we would have:

$$\mathsf{a}_0=ae^{i\phi^{(1)}}.$$

Since $\phi^{(1)}$ depends on a_1 while a does not, we retrieve the fact that in super-critical régimes, the leading order amplitude in WKB methods depends on the initial first corrector a_1.

Remark 3.8. The term $e^{i\phi^{(1)}}$ does not appear in the Wigner measure of $ae^{i\phi^{(1)}}e^{i\phi/\varepsilon}$. Thus, from the point of view of Wigner measures, the asymptotic behavior of the exact solution is described by the Euler-type system (2.4).

Remark 3.9. If we assume that a_0 is real-valued, then so is a. If moreover a_1 is purely imaginary (for instance, if $a_1=0$), then we see that $a^{(1)}$ is purely imaginary, hence, $\phi^{(1)}\equiv 0$.

So far we have assumed $V\equiv 0$ and $\phi_0\in H^\infty$. If we try to mimic the approach of [14] for a non-trivial external potential for instance, we have to consider:

$$\partial_t\phi^\varepsilon+\frac{1}{2}\left|\nabla\phi^\varepsilon\right|^2+V+|a^\varepsilon|^2=0\quad;\quad\phi^\varepsilon\big|_{t=0}=\phi_0\,,$$
$$\partial_t a^\varepsilon+\nabla\phi^\varepsilon\cdot\nabla a^\varepsilon+\frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon=i\frac{\varepsilon}{2}\Delta a^\varepsilon\quad;\quad a^\varepsilon\big|_{t=0}=a_0^\varepsilon\,.$$

The analysis of [14] works in the same way only when $\nabla V\in L^\infty_{\rm loc}(\mathbb{R}_t;H^s(\mathbb{R}^n))$ for a sufficiently large s. To be able to consider general sub-quadratic potentials (including the harmonic oscillator), resume the assumption of Lemma 2.1, and write

$$\phi^\varepsilon=\phi_{\rm eik}+\varphi^\varepsilon.$$

Working with the unknown $(\varphi^\varepsilon,a^\varepsilon)$, we see that we are now rid of the external potential V, and of the possibly unbounded initial phase ϕ_0. The price to pay is that extra terms have appeared. The good news however is that these extra terms are *semilinear* (as in § 3.1), and can be treated by perturbative methods in energy estimates. We conclude:

Theorem 3.10. *Let* $\kappa=0$*. Under the above assumptions, there exists* $T_*>0$ *independent of* $\varepsilon\in]0,1]$ *and a unique solution* $u^\varepsilon\in C^\infty([0,T_*]\times\mathbb{R}^n)\cap C([0,T_*];H^s)$

for all $s > n/2$ to (1.1)–(1.2). *Moreover, there exist $a, \varphi \in C([0,T_*]; H^s)$ for every $s \geqslant 0$, such that:*

$$\limsup_{\varepsilon\to 0} \left\| u^\varepsilon - a e^{i(\varphi+\phi_{\rm eik})/\varepsilon} \right\|_{L^2\cap L^\infty} = \mathcal{O}(t) \quad \text{as } t \to 0.$$

Here, a and φ are nonlinear functions of $\phi_{\rm eik}$ and a_0. Finally, there exists $\varphi^{(1)} \in C([0,T_*]; H^s)$ for every $s \geqslant 0$, real-valued, such that:

$$\limsup_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant T_*} \left\| u^\varepsilon - a e^{i\varphi^{(1)}} e^{i(\varphi+\phi_{\rm eik})/\varepsilon} \right\|_{L^2\cap L^\infty} = 0.$$

The phase shift $\varphi^{(1)}$ is a nonlinear function of $\phi_{\rm eik}, a_0$ and a_1.

Remark 3.11. In [7], some assumptions on the momentum of a_0^ε are made, and not here. This is due to the fact that here, the nonlinearity that we consider is exactly cubic. When it is cubic at the origin only (see Remark 3.5), extra estimates are needed, which apparently impose some extra decay at infinity for a_0^ε, a_0 and a_1.

4. Proof of Theorem 1.1

Theorem 1.1 is a straightforward consequence of Proposition 3.7. For $a_0 \in \mathcal{S}(\mathbb{R}^n)$, let

$$u_0(x) = \lambda^{-\frac{n}{2}+s} a_0\left(\frac{x}{\lambda}\right).$$

Let $\varepsilon = \lambda^{\frac{n}{2}-1-s}$: ε and λ go simultaneously to zero, since $s < s_c$. Define

$$\psi^\varepsilon(t,x) = u^\lambda(\varepsilon t, x) = \lambda^{\frac{n}{2}-s} u\left(\lambda^{\frac{n}{2}+1-s} t, \lambda x\right).$$

It solves:

$$i\varepsilon\partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = |\psi^\varepsilon|^2\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = a_0(x). \tag{4.1}$$

The idea of the proof is that for times of order $\mathcal{O}(1)$, ψ^ε has become ε-oscillatory.

We infer from Proposition 3.7 that there exist $T > 0$ independent of $\varepsilon \in]0,1]$, and $a, \phi, \phi_1 \in C([0,T]; H^m)$ for any $m \geqslant 0$, such that:

$$\left\| \psi^\varepsilon - a e^{i\phi_1} e^{i\phi/\varepsilon} \right\|_{L^\infty([0,T];H^m)} \leqslant C_m \varepsilon^{1-m}.$$

Since the $\dot{H}^m$-norm of $a e^{i\phi_1} e^{i\phi/\varepsilon}$ is of order ε^{-m} (when ϕ is not stationary), we deduce that there exists $t \in]0,T]$ such that for any $m \geqslant 0$:

$$\|\psi^\varepsilon(t)\|_{\dot{H}^m} \approx \varepsilon^{-m}.$$

This implies:

$$\left\| u\left(\lambda^{\frac{n}{2}+1-s} t\right) \right\|_{\dot{H}^k} \approx \lambda^{s-k} \|\psi^\varepsilon(t)\|_{\dot{H}^k} \approx \lambda^{s-k}\varepsilon^{-k} = \lambda^{s-k-k\left(\frac{n}{2}-1-s\right)}.$$

The result then follows when considering the limit $\lambda \to 0$. We get exactly the statement of the theorem by replacing a_0 by $|\log\lambda|^{-1} a_0$ for instance.

Remark 4.1. The proof of ill-posed presented in [10] (see also [3, Appendix], [6, Appendix B]) consists in neglecting the Laplacian in (4.1) for very small times, and integrating explicitly an ordinary differential equation. Proving that the Laplacian is negligible stems from Gronwall lemma. Essentially, the error satisfies an inequality of the form

$$\|w^\varepsilon(t)\|_X \lesssim \varepsilon + \frac{1}{\varepsilon}\int_0^t \|w^\varepsilon(s)\|_X ds,$$

for some space X that we do not describe. The singular factor ε^{-1} is due to the ε in front of the time derivative, and to the fact that no power of ε is present in front of the nonlinearity. Therefore, Gronwall lemma yields no better than:

$$\|w^\varepsilon(t)\|_X \lesssim \varepsilon e^{Ct/\varepsilon},$$

for some $C > 0$. The error is small on an interval of the form $[0, \varepsilon|\log\varepsilon|^\theta]$ for some $\theta > 0$. This is enough to prove Theorem 1.1 for $k = s$. This analysis considers (4.1) as a semilinear equation, since the nonlinearity is viewed as a perturbation of the linear equation. To prove Theorem 1.1 for $k < s$, it seems necessary to consider (4.1) as a quasilinear equation, as was done by E. Grenier. Note also that the quasilinear approach shows that the Laplacian in (4.1) is negligible for $0 < t^\varepsilon \ll \varepsilon^{1/3}$, that is a "much larger" interval than $[0, \varepsilon|\log\varepsilon|^\theta]$ (but still very small!). See the next section.

Remark 4.2. On the other hand, Theorem 1.1 is valid only for cubic, defocusing nonlinear Schrödinger equations, while the results in [10] are valid for more general equations. This is due to the fact that the justification of super-critical nonlinear geometric optics for times $\mathcal{O}(1)$ is available only for nonlinearities which are defocusing, and cubic at the origin (see Remark 3.5). Since the proofs of ill-posedness rely on an homogeneous change of unknown function, we are left with the only possibility of an exactly cubic, defocusing nonlinearity. However, it is very likely that (an analogue of) Theorem 1.1 should be true under more general assumptions.

5. Instability for the semi-classical equation

The results we present in the paragraph are taken from [6]. We assume $V = \phi_0 = 0$ for the sake of concision. We first fix some notations.

Notation. Let $(\alpha^\varepsilon)_{0<\varepsilon\leqslant 1}$ and $(\beta^\varepsilon)_{0<\varepsilon\leqslant 1}$ be two families of positive real numbers.

- We write $\alpha^\varepsilon \ll \beta^\varepsilon$ if $\limsup\limits_{\varepsilon\to 0} \alpha^\varepsilon/\beta^\varepsilon = 0$.
- We write $\alpha^\varepsilon \lesssim \beta^\varepsilon$ if $\limsup\limits_{\varepsilon\to 0} \alpha^\varepsilon/\beta^\varepsilon < \infty$.
- We write $\alpha^\varepsilon \approx \beta^\varepsilon$ if $\alpha^\varepsilon \lesssim \beta^\varepsilon$ and $\beta^\varepsilon \lesssim \alpha^\varepsilon$.

A typical result of [6] is the following:

Theorem 5.1. *Let $n \geqslant 1$, $a_0, \widetilde{a}_0^\varepsilon \in \mathcal{S}(\mathbb{R}^n)$, where a_0 is independent of ε. Let u^ε and v^ε solve the initial value problems:*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon \ ; \ u^\varepsilon\big|_{t=0} = a_0 .$$

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = |v^\varepsilon|^2 v^\varepsilon \ ; \ v^\varepsilon\big|_{t=0} = \widetilde{a}_0^\varepsilon .$$

Assume that there exists $N \in \mathbb{N}$ and $\varepsilon^{1-\frac{1}{N}} \ll \delta^\varepsilon \ll 1$ such that:

$$\|a_0 - \widetilde{a}_0^\varepsilon\|_{H^s} \approx \delta^\varepsilon \ , \ \forall s \geqslant 0 \ ; \ \limsup_{\varepsilon\to 0} \left\| \frac{\mathrm{Re}(a_0 - \widetilde{a}_0^\varepsilon)\overline{a_0}}{\delta^\varepsilon} \right\|_{L^\infty(\mathbb{R}^n)} \neq 0. \tag{5.1}$$

Then we can find $0 < t^\varepsilon \ll 1$ such that: $\|u^\varepsilon(t^\varepsilon) - v^\varepsilon(t^\varepsilon)\|_{L^2} \gtrsim 1$. More precisely, this mechanism occurs as soon as $t^\varepsilon\delta^\varepsilon \gtrsim \varepsilon$. In particular, for all $s \geqslant 0$,

$$\frac{\|u^\varepsilon - v^\varepsilon\|_{L^\infty([0,t^\varepsilon];L^2)}}{\left\|u^\varepsilon_{|t=0} - v^\varepsilon_{|t=0}\right\|_{H^s}} \to +\infty \quad \text{as } \varepsilon \to 0 .$$

Example. Consider $a_0, b_0 \in \mathcal{S}(\mathbb{R}^n)$ independent of ε, such that $\mathrm{Re}(\overline{a_0}b_0) \not\equiv 0$, and take $\widetilde{a}_0^\varepsilon = a_0 + \delta^\varepsilon b_0$.

Example. Consider $a_0 \in \mathcal{S}(\mathbb{R}^n)$ independent of ε and $x^\varepsilon \in \mathbb{R}^n$. We can take $\widetilde{a}_0^\varepsilon(x) = a_0(x - x^\varepsilon)$, provided that $|x^\varepsilon| = \delta^\varepsilon$ and

$$\limsup_{\varepsilon\to 0} \left\| \frac{x^\varepsilon}{|x^\varepsilon|} \cdot \nabla \left(|a_0|^2\right)\right\|_{L^\infty} \neq 0.$$

This example and the analysis of [6] make it possible to refine some results of [4].

The general idea consists in using the WKB analysis in this super-critical case. Roughly speaking, we have seen that (3.8) provides a good approximation of u^ε for very small time only. Since we are interested in instabilities occurring for very small time, this is not a problem for us now. The coupling in (3.8) shows that a small perturbation of a_0 yields a small perturbation of ϕ. But when we write

$$u^\varepsilon(t,x) \sim a(t,x)e^{i\phi(t,x)/\varepsilon},$$

we see that this small perturbation is divided by ε, which goes to zero. The result may not be small...

Technically, our approach consists in resuming the result provided by Proposition 3.4. Instead of letting $\varepsilon \to 0$ in the initial data of (3.5), just neglect the skew-symmetric term (recall that we assume $\phi_0 = 0$):

$$\begin{aligned} &\partial_t \Phi^\varepsilon + \frac{1}{2}\left|\nabla\Phi^\varepsilon\right|^2 + |\mathbf{a}^\varepsilon|^2 = 0 \quad ; \quad \Phi^\varepsilon\big|_{t=0} = 0\,,\\ &\partial_t \mathbf{a}^\varepsilon + \nabla\Phi^\varepsilon\cdot\nabla\mathbf{a}^\varepsilon + \frac{1}{2}\mathbf{a}^\varepsilon\Delta\Phi^\varepsilon = 0 \quad ; \quad \mathbf{a}^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{aligned} \tag{5.2}$$

Assuming that a_0^ε is bounded in H^s for all $s \geqslant 0$, (5.2) has a unique solution $(\Phi^\varepsilon, \mathbf{a}^\varepsilon) \in L^\infty([0,T_*];H^m)^2$ for all $m > 0$ for some $T_* > 0$ independent of ε and m.

Proposition 5.2. *Let $s \in \mathbb{N}$. Then $T_s \geqslant T_*$, and there exists C_s independent of ε such that for every $0 \leqslant t \leqslant T_*$,*

$$\|a^\varepsilon(t) - \mathbf{a}^\varepsilon(t)\|_{H^s} \leqslant C_s\varepsilon t \quad ; \quad \|\phi^\varepsilon(t) - \Phi^\varepsilon(t)\|_{H^s} \leqslant C_s\varepsilon t^2.$$

The second idea consists in considering the Taylor expansion in time of $(\Phi^\varepsilon, \mathbf{a}^\varepsilon)$:

$$\Phi^\varepsilon(t,x) \sim \sum_{j\geqslant 1} t^{2j-1}\Phi_j^\varepsilon(x) \quad ; \quad \mathbf{a}^\varepsilon(t,x) \sim \sum_{j\geqslant 1} t^{2j}\mathbf{a}_j^\varepsilon(x).$$

Note that only odd powers of t are present in the expansion of Φ^ε, and even powers in that of $\mathbf{a}^\varepsilon$. This is because we have assumed $\phi_0 = 0$. Plugging these expansions into (5.2), we get formally:

$$\mathbf{a}_0^\varepsilon = a_0^\varepsilon \quad ; \quad \Phi_1^\varepsilon = -|a_0^\varepsilon|^2.$$

We can then check that a perturbation of order δ^ε of a_0^ε yields a perturbation of order δ^ε of Φ_1^ε, provided that the polarization condition (5.1) is satisfied. By induction, we see that this perturbs the other Φ_j^ε's and $\mathbf{a}_j^\varepsilon$'s by a $\mathcal{O}(\delta^\varepsilon)$. Consider the approximate solution defined by

$$u_K^\varepsilon(t,x) = a_0^\varepsilon(x)\exp\left(i\sum_{j=1}^K t^{2j-1}\Phi_j^\varepsilon(x)/\varepsilon\right).$$

Formally, we have:

$$\begin{aligned}\mathbf{a}^\varepsilon(t,x)e^{i\Phi^\varepsilon(t,x)/\varepsilon} - u^\varepsilon_K(t,x) &= \left(\mathbf{a}^\varepsilon(t,x) - a^\varepsilon_0(x)\right)e^{i\Phi^\varepsilon(t,x)/\varepsilon}\\ &\quad + a^\varepsilon_0(x)\Big(\exp\left(i\Phi^\varepsilon(t,x)/\varepsilon\right) - \exp\Big(i\sum_{j=1}^K t^{2j-1}\Phi^\varepsilon_j(x)/\varepsilon\Big)\Big)\\ &= \mathcal{O}(t^2) + \mathcal{O}\Big(\Big(\Phi^\varepsilon(t,x) - \sum_{j=1}^K t^{2j-1}\Phi^\varepsilon_j(x)\Big)/\varepsilon\Big)\\ &= \mathcal{O}(t^2) + \mathcal{O}\left(t^{2K+1}/\varepsilon\right).\end{aligned}$$

We infer that the above quantity is small for times such that $t^\varepsilon \ll 1$ and $t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}$. On the other hand, Proposition 5.2 shows that $\mathbf{a}^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ is a good approximation of u^ε for times such that $t^\varepsilon \ll 1$. Therefore, we expect

$$\left\|u^\varepsilon(t^\varepsilon,\cdot) - u^\varepsilon_K(t^\varepsilon,\cdot)\right\|_{L^2\cap L^\infty} \ll 1 \quad \text{for } t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}.$$

This can be proved by the analysis presented in § 3.2.

The case $K=1$ is of special interest. Indeed, the Laplacian plays no role in the definition of u^ε_1, and we check that it solves:

$$i\varepsilon\partial_t u^\varepsilon_1 = |u^\varepsilon_1|^2 u^\varepsilon_1 \quad ; \quad u^\varepsilon_{1|t=0} = a^\varepsilon_0.$$

This is the solution of the ordinary differential equation considered in [10] and [4]. The above analysis shows that it is a reasonable approximation of u^ε for $0 < t^\varepsilon \ll \varepsilon^{1/3}$: see Remark 4.1.

In view of Theorem 5.1, we define u^ε_K and v^ε_K in an obvious way, and we have:

$$\left\|u^\varepsilon(t^\varepsilon,\cdot) - u^\varepsilon_K(t^\varepsilon,\cdot)\right\|_{L^2\cap L^\infty} + \left\|v^\varepsilon(t^\varepsilon,\cdot) - v^\varepsilon_K(t^\varepsilon,\cdot)\right\|_{L^2\cap L^\infty} \ll 1 \quad \text{for } t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}.$$

So to prove Theorem 5.1, we just have to compare u^ε_K and v^ε_K:

$$\begin{aligned}u^\varepsilon_K(t,x) - v^\varepsilon_K(t,x) =& a_0(x)\exp\Big(i\sum_{j=1}^K t^{2j-1}\Phi_j(x)/\varepsilon\Big)\\ &- \widetilde{a}^\varepsilon_0(x)\exp\Big(i\sum_{j=1}^K t^{2j-1}\widetilde{\Phi}^\varepsilon_j(x)/\varepsilon\Big)\\ =& \left(a_0(x) - \widetilde{a}^\varepsilon_0(x)\right)\exp\Big(i\sum_{j=1}^K t^{2j-1}\Phi_j(x)/\varepsilon\Big)\\ &-\widetilde{a}^\varepsilon_0(x)\Big(\exp\Big(i\sum_{j=1}^K t^{2j-1}\Phi_j(x)/\varepsilon\Big) - \exp\Big(i\sum_{j=1}^K t^{2j-1}\widetilde{\Phi}^\varepsilon_j(x)/\varepsilon\Big)\Big).\end{aligned} \tag{5.3}$$

The first term is of order δ^ε by assumption. To estimate the second term, examine:

$$\sum_{j=1}^K t^{2j-1}\left(\Phi_j(x) - \widetilde{\Phi}^\varepsilon_j(x)\right)/\varepsilon.$$

Since we consider times such that $t^\varepsilon \ll 1$ and that we have seen that $\Phi_j - \widetilde{\Phi}_j^\varepsilon = \mathcal{O}(\delta^\varepsilon)$ for $j \geqslant 2$, the leading order term is simply:

$$t\left(\Phi_1(x) - \widetilde{\Phi}_1^\varepsilon(x)\right)/\varepsilon = \frac{t}{\varepsilon}\left(|\widetilde{a}_0^\varepsilon(x)|^2 - |a_0^\varepsilon(x)|^2\right)$$
$$= \frac{t}{\varepsilon}\left(\mathrm{Re}\left(a_0 - \widetilde{a}_0^\varepsilon\right)\overline{a_0}) + \mathcal{O}\left((\delta^\varepsilon)^2\right)\right).$$

By assumption, the modulus of (5.3) behaves like

$$\left|a_0(x)\sin\left(\frac{t\delta^\varepsilon}{\varepsilon}f(x)\right)\right|,$$

for some non-trivial function f. The conclusion of Theorem 5.1 follows easily.

References

1. S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels, InterEditions, Paris, 1991.
2. R. W. Boyd, *Nonlinear optics*, Academic Press, New York, 1992.
3. N. Burq, P. Gérard, and N. Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 255–301.
4. N. Burq and M. Zworski, *Instability for the semiclassical non-linear Schrödinger equation*, Comm. Math. Phys. **260** (2005), no. 1, 45–58.
5. R. Carles, *Geometric optics with caustic crossing for some nonlinear Schrödinger equations*, Indiana Univ. Math. J. **49** (2000), no. 2, 475–551.
6. ———, *Geometric optics and instability for semi-classical Schrödinger equations*, Arch. Ration. Mech. Anal. (2006), to appear (`doi:10.1007/s00205-006-0017-5`).
7. ———, *WKB analysis for nonlinear Schrödinger equations with potential*, Comm. Math. Phys. (2006), to appear (`doi:10.1007/s00220-006-0077-2`).
8. C. Cheverry, *Cascade of phases in turbulent flows*, Bull. Soc. Math. France **134** (2006), no. 1, 33–82.
9. C. Cheverry and O. Guès, *Counter-examples to the concentration-cancellation property*, preprint, 2005.
10. M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2006), see also `arXiv:math.AP/0311048`.
11. J. Dereziński and C. Gérard, *Scattering theory of quantum and classical N-particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.
12. P. Donnat, *Quelques contributions mathématiques en optique non linéaire*, Ph.D. thesis, École polytechnique, Palaiseau (France), 1994.
13. P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
14. E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.
15. L. Landau and E. Lifschitz, *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967, Deuxième édition, Traduit du russe par Édouard Gloukhian.
16. G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9) **71** (1992), no. 3, 267–291.
17. ———, *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège **70** (2001), no. 4-6, 267–306 (2002), Hommage à Pascal Laubin.
18. ———, *Perte de régularité pour les équations d'ondes sur-critiques*, Bull. Soc. Math. France **133** (2005), 145–157.
19. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
20. T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'équation d'Euler compressible*, Japan J. Appl. Math. **3** (1986), no. 2, 249–257.

21. G. Métivier, *Exemples d'instabilités pour des équations d'ondes non linéaires (d'après G. Lebeau)*, Astérisque (2004), no. 294, vii, 63–75.
22. J. Rauch and M. Keel, *Lectures on geometric optics*, Hyperbolic equations and frequency interactions (Park City, UT, 1995), Amer. Math. Soc., Providence, RI, 1999, pp. 383–466.
23. J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach Science Publishers, New York, 1969, Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher, Notes on Mathematics and its Applications.
24. J. Szeftel, *Propagation et réflexion des singularités pour l'équation de Schrödinger non linéaire*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 2, 573–671.
25. V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134.

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Asymptotic expansion of the solution to the nonlinear Schrödinger equation with power nonlinearity

Satoshi Masaki

1 Introduction and General result

Let us consider the asymptotic expansion of solutions to the equation

$$u = w_0 + F(u), \tag{1}$$

where u is a $\mathbb{C}$ valued function belonging to a Banach space X (such as Lebesgue space L^p) with norm $\|\cdot\|$ and F is a nonlinear mapping from X to itself. We suppose that F satisfies certain smallness conditions which lead to the contractivity of F. Smallness of $F(u)$ means that u is close to w_0. Our aim here is to construct higher order approximate solutions in terms of w_0 which approximate the nonlinearity $F(u)$.

Remark 1.1. Nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u), \qquad u(0,x) = \phi(x)$$

can be written as the following integral equation:

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta} f(u(s))ds,$$

which is of the form (1).

1.1 Asymptotic expansion of Type 1

We put the following assumptions on F, where X^m denotes the m-time direct product of X:

(A1) There exists a operator $G(u_1,\dots,u_m) : X^m \to X$ such that $F(u) = G(u,\dots,u)$, and that

$$G(u_1,\dots,u_k^1+u_k^2,\dots,u_m) = G(u_1,\dots,u_k^1,\dots,u_m) + G(u_1,\dots,u_k^2,\dots,u_m) \tag{2}$$

for any $k \in [1,m]$.

(A2) Let G be the operator associated with F as in (A1). Then, there exists a nonnegative constant ε such that

$$\|G(u_1, \dots, u_m)\| \leqslant \varepsilon \prod_{k=1}^{m} \|u_k\|. \tag{3}$$

Then, we define the higher approximate solutions inductively as follows:

$$w_n = \sum_{\substack{i_1,\dots,i_m \geqslant 0 \\ i_1+\dots+i_m=n-1}} G(w_{i_1}, \dots, w_{i_m}), \tag{4}$$

where G is the operator associated with F. We now state the main theorems.

Theorem 1.1. *Assume (A1) and (A2). Define w_n as (4). If $\|w_0\| \leqslant C_2$ for some $C_2 > 0$, then it holds for any $n \geqslant 0$ that*

$$\|w_n\| \leqslant I(n,m) C_2^{n(m-1)+1} \varepsilon^n. \tag{5}$$

Moreover, if u is a solution to the equation (1), and if u satisfies $\|u\| \leqslant C_1$ for some $C_1 > 0$, then

$$\left\| u - \sum_{k=0}^{n} w_k \right\| \leqslant I(n+1,m) C_3^{(n+1)(m-1)+1} \varepsilon^{n+1}, \tag{6}$$

where $C_3 = \max(C_1, C_2)$. The constant $I(n,m)$ appearing in (5) and (6) is defined by

$$I(0,m) = 1, \qquad I(n,m) = \sum_{i_1+\dots+i_m=n-1} \prod_{k=1}^{m} I(i_k, m).$$

Theorem 1.2. *Let $I(n,m)$ be as in Theorem 1.1. There exists a positive constant δ depending on m such that $\lim_{n\to\infty} I(n,m)\delta^n = 0$. Especially, under the setting of Theorem 1.1, if $C_3^{m-1}\varepsilon < \delta$ then we have*

$$u = \sum_{k=0}^{\infty} w_k$$

strongly in X.

1.2 Asymptotic expansion of Type 2

We next consider the case where F is less regular. We suppose that the norm $\|\cdot\|$ satisfies the following additional condition, where we denote the domain of u by D.

1. There exists a constant C such that $\big\||u|\big\| \leqslant C\|u\|$ for all $u \in X$.

2. If $|u(x)| \leqslant |v(x)|$ for all $x \in D$ then $\||u|\| \leqslant \||v|\|$.

We remark that L^p norm ($p \geqslant 1$) satisfies the above two conditions.

Let us make the following assumptions on F:

(B1) There exist an operator $G : X^m \to X$ satisfying (2) and a constant $b \in (0,1]$ such that $F(u) = G(|u|^b, \dots, |u|^b)$.

(B2) Let G be the operator associated with F as in (B1). Then, there exists a positive constant ε such that

$$\|G(u_1, \dots, u_m)\| \leqslant \varepsilon \prod_{k=1}^{m} \left\| |u_k|^{\frac{1}{b}} \right\|^b . \tag{7}$$

In this case, we define the higher approximate solution w_n ($n \geqslant 1$) from w_0 as follows:

$$w_n = \sum_{\substack{i_1+\cdots+i_m=n-1 \\ i_1,\dots,i_m \geqslant 0}} G\left(\left|\sum_{k=0}^{i_1} w_k\right|^b - \left|\sum_{k=0}^{i_1-1} w_k\right|^b, \dots, \left|\sum_{k=0}^{i_m} w_k\right|^b - \left|\sum_{k=0}^{i_m-1} w_k\right|^b \right), \tag{8}$$

where $\sum_{k=0}^{-1} w_k = 0$. Then, we have the following asymptotic expansion.

Theorem 1.3. *Assume* (*B*1) *and* (*B*$_2$). *If* $\|w_0\| \leqslant C_2$ *for some* $C_2 > 0$, *then it holds for all* $n \geqslant 0$ *that*

$$\|w_n\| \leqslant C C_2^{\gamma(n,b,m,C_2)} \varepsilon^{\theta(n,b,\varepsilon)}, \tag{9}$$

where

$$\gamma(n,b,m,c) = \begin{cases} \dfrac{b(m-1)}{1-b} - \dfrac{(mb-1)b^n}{1-b} & \text{if } 0 < b < 1, \quad c < 1, \\ 1 + bn(m-1) & \text{if } b = 1 \text{ or } c \geqslant 1, \end{cases} \tag{10}$$

$$\theta(n,b,e) = \begin{cases} \dfrac{1-b^n}{1-b} & \text{if } 0 < b < 1. \quad e < 1, \\ 1 + b(n-1) & \text{if } b = 1 \text{ or } e \geqslant 1. \end{cases} \tag{11}$$

Moreover, let u *be a solution to the equation* (1) *and satisfy* $\|u\| \leqslant C_1$ *for some* $C_1 > 0$. *Then, it holds that for any* $n \geqslant 0$ *that*

$$\left\| u - \sum_{k=0}^{n} w_k \right\| \leqslant C C_3^{\gamma(n+1,b,m,C_3)} \varepsilon^{\theta(n+1,b,\varepsilon)}, \tag{12}$$

where $C_3 = \max(C_1, C_2)$, γ *and* θ *is defined in* (10) *and* (11), *respectively, and* C *is a constant depending on* n, m, *and* b.

Remark 1.2. If $b = 1$, then the right hand sides of (9) and (12) become (5) and (6) (including their coefficients), respectively. Hence, u is exactly equal to the infinite sum of w_n converging strongly in X.

2 Examples

As applications of the above Theorems, we introduce the asymptotic expansion of the solution to the nonlinear Schrödinger equation as time t tends to $+\infty$, in some cases. Let us consider the following Cauchy problem:

$$iu_t + \Delta u = |u|^\alpha u, \quad (t,x) \in \mathbb{R}^{1+N}, \qquad u(0,x) = \phi(x), \tag{NLS}$$

where $0 < \alpha < \alpha_0(N) := 4/(N-2)$ ($\alpha_0(1) = \alpha_0(2) = \infty$) and $\phi \in \Sigma := H^1 \cap \mathcal{F}(H^1)$. $\mathcal{F}$ denotes the Fourier transform, and $\|f\|_\Sigma = \|f\|_{H^1} + \| |\cdot| f(\cdot)\|_{L^2}$. We treat the case $\alpha > \gamma(N)$, where $\gamma(N) = (2 - N + \sqrt{N^2 + 12N + 4})/2N$. In this case, the followings are known:

- The equation (NLS) has a unique global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$. Moreover, $(x + 2it\nabla)u$ belongs to $C(\mathbb{R}, L^2(\mathbb{R}^N))$.
- There exists $u^\pm \in \Sigma$ such that the solution u satisfies $\|e^{-it\Delta}u(t) - u^\pm\|_\Sigma \to 0$ as $t \to \pm\infty$.

With u^+, the solution to (NLS) satisfies the following integral equation:

$$u(t) = e^{it\Delta}u^+ + i\int_t^\infty e^{i(t-s)\Delta}|u|^\alpha u(s)ds.$$

We denote the right hand side by $w_0 + F(u)$, then it is of the form (1).

2.1 Example 1 – the case $N = 3$, $\alpha = 2$

We first consider the case where α is a positive even number. For simplicity, we let $\alpha = 2$ and $N = 3$. Since $|u|^2 u = u^2\overline{u}$, the operator

$$G(u_1, u_2, u_3) = i\int_t^\infty e^{i(t-s)\Delta}u_1 u_2 \overline{u_3} ds$$

satisfies the assumption (A1). Then, defining w_n ($n \geqslant 1$) as

$$w_n = \sum_{i_1+i_2+i_3=n-1} G(w_{i_1}, w_{i_2}, w_{i_3}),$$

we have the following proposition by Theorems 1.1 and 1.2.

Proposition 2.1. *Let $N = 3$, $\alpha = 2$, $\phi \in \Sigma$, and $2 \leqslant r \leqslant 6$. Then, for any positive integer n, it holds that*

$$\left\| u(t) - \sum_{k=0}^n w_k(t) \right\|_{L^r} = O(t^{-n\theta - 3/2 + 3/r}),$$

where $\theta = \min(5/4, 1/2 + 3/r)$. Moreover, there exists T depending only on $\|\phi\|_\Sigma$ such that

$$u(t) = e^{it\Delta}u^+ + \sum_{n=1}^\infty w_n(t)$$

holds strongly in L^r for $t \in [T, \infty)$.

2.2 Example 2 – the case $N = 4$, $\alpha = 1$

Combining asymptotic expansions of Type 1 and Type 2, we can treat general α. For instance, we show the asymptotic expansion of the solution near $t = \infty$ in the case $\alpha = 1$. We remark that $|u|u$ is not smooth. We also assume $N = 4$ because $\alpha = 1 \in (\gamma(4), \alpha_0(4))$. We first define an auxiliary multi-linear operator $G^k(u_1, \dots, u_k)$ inductively as follows:

$$G^1(u_1)(t) = i \int_t^\infty u_1(s_1)(e^{is_1\Delta}u^+)ds_1,$$
$$G^k(u_1, \dots, u_k)(t) = i \int_t^\infty u_k(s_k)\big(G_{k-1}(u_1, \dots, u_{k-1})(s_k)\big)ds_k.$$

We note that G^k are determined by u^+ only. Then, we define the higher approximate solutions $\{w_{k,l}(t)\}_{l \geqslant 0}$ as follows:

$$w_{k,0} = e^{it\Delta}u^+,$$
$$w_{k,l} = \sum_{i_1+\cdots+i_k=l-1} G^k\left(\left|\sum_{h=0}^{i_1} w_{k,h}\right| - \left|\sum_{h=0}^{i_1-1} w_{k,h}\right|, \dots, \left|\sum_{h=0}^{i_k} w_{k,h}\right| - \left|\sum_{h=0}^{i_k-1} w_{k,h}\right|\right),$$

where $\sum_{h=0}^{-1} w_{k,h} = 0$. Then, we have the following proposition:

Proposition 2.2. *Let $N = 4$, $\alpha = 1$, and $r \in [2, 4)$. Let n be any positive integer. Then, we have*

$$\left\| u(t) - e^{it\Delta}u^+ - \sum_{j=1}^{n} \sum_{kl=j} w_{k,l}(t) \right\|_{L^r} = O(t^{-n\theta-2+4/r}) \qquad (13)$$

as $t \to \infty$, where $\theta = \min(1/3, 4/r - 1) > 0$.

Remark 2.1. The nonlinearity $|u|u$ is not smooth. Nevertheless, we can construct an approximate solution with arbitrary accuracy.

Remark 2.2. Combining Theorems 1.1 and 1.3, we can treat $|u|^\alpha u$ for all $\alpha \in (\gamma(N), \alpha_0(N))$. However, if α is not an integer then the accuracy of the approximation is limited.

References

[1] N. Kita, *Sharp L^r asymptotics of small solutions to the nonlinear Schrödinger equations,* Nonlinear Anal. **52** (2003) 1365–1377.

[2] N. Kita and T. Ozawa, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations with repulsive interactions,* Comm. Contemp. Math. **7** (2005) 167–176.

[3] N. Kita and T. Wada, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations in one space dimension,* Funcialaj Ekvacioj **45** (2002) 53–69.

ON THE CUBIC NONLINEAR KLEIN-GORDON EQUATION

NAKAO HAYASHI

1. Introduction

We study the initial value problem for the cubic nonlinear Klein-Gordon equation

$$\begin{cases} v_{tt} + v - v_{xx} = \mu v^3, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ v(0) = v_0, v_t(0) = v_1, x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\mu \in \mathbf{R}$ and the initial data v_0 and v_1 are real valued functions. When $\mu < 0$, it is easy to prove the global existence in time of solutions to (1.1) in the energy space. However this approach does not explain the large time asymptotic behavior of solutions. The sharp $\mathbf{L}^\infty$ time decay estimates of solutions for the nonlinear Klein-Gordon equation in the one dimensional case were obtained by [2]. The large time asymptotic profile of small solutions to nonlinear Klein-Gordon equation was found in [1] (see also [6] for a shorter proof). In the present talk we will remove the compact support restriction on the initial data assumed in previoud papers [1], [2] and [6].

Define a new dependent variable

$$u = \frac{1}{2}\left(v + i \langle i\partial_x \rangle^{-1} v_t\right)$$

and initial data

$$u_0 = \frac{1}{2}\left(v_0 + i \langle i\partial_x \rangle^{-1} v_1\right)$$

with $\langle x \rangle = \sqrt{1 + |x|^2}$. In the case of the real-valued function v the nonlinear Klein-Gordon equation (1.1) can be rewritten as

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u), \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{1.2}$$

where $\mathcal{L} = \partial_t + i \langle i\partial_x \rangle$ and

$$\mathcal{N}(u) = 4i\mu \langle i\partial_x \rangle^{-1} (\operatorname{Re} u)^3 .$$

Then a solution of (1.1) is $v = 2 \operatorname{Re} u$.

We denote the Lebesgue space $\mathbf{L}^p = \{\phi \in \mathcal{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, with the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}} |\phi(x)|^p \, dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space is $\mathbf{H}^{m,s}_p = \{\phi \in \mathbf{L}^p; \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\}$, for $m, s \in \mathbf{R}$, $1 \le p \le \infty$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}^{m,s}_2$. The index 0 we usually omit if it does not cause a confusion.

The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \hat{\phi} = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x) \, dx,$$

This is a joint work with P.I.Naumkin.

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi) d\xi.$$

Our main result is

Theorem 1.1. *Let $u_0 \in \mathbf{H}^{4,1}$ and the norm $\|u_0\|_{\mathbf{H}^{4,1}}$ be sufficiently small. Then there exists a unique global solution u of (1.2) such that*

$$u(t) \in \mathbf{C}\left([0,\infty);\mathbf{H}^{4,1}\right)$$

and

$$\|u(t)\|_{\mathbf{H}^1_\infty} \le C(1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\| u(t) - U(t)\mathcal{F}^{-1}\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}}$$

and

$$\left\| \mathcal{F}U(-t)u(t) - \widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}},$$

where $\gamma \in \left(0,\frac{1}{4}\right)$.

From this result we see that there exists the inverse modified wave operator $\widetilde{\mathcal{W}_+}^{-1}: u_0 \in \mathbf{H}^{4,1} \to W_+ \in \mathbf{H}^{1,0}$.

Consequently we have for the Cauchy problem (1.1).

Corollary 1.2. *Let $v_0 \in \mathbf{H}^{4,1}$ and $v_1 \in \mathbf{H}^{3,1}$ be real-valued functions and the norm $\|v_0\|_{\mathbf{H}^{4,1}} + \|v_1\|_{\mathbf{H}^{3,1}}$ be sufficiently small. Then there exists a unique global solution v of the Cauchy problem (1.1) such that*

$$v \in \mathbf{C}\left([0,\infty);\mathbf{H}^{4,1}\right) \cap \mathbf{C}^1\left([0,\infty);\mathbf{H}^{3,1}\right)$$

and

$$\|v(t)\|_{\mathbf{H}^1_\infty} \le C(1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\| v(t) - 2\,\mathrm{Re}\,U(t)\mathcal{F}^{-1}\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}}$$

and

$$\left\| \mathcal{F}U(-t)v(t) - 2\,\mathrm{Re}\,\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}},$$

where $\gamma \in \left(0,\frac{1}{4}\right)$.

We use the operator

$$\mathcal{J} = \langle i\partial_x\rangle U(t)xU(-t) = \mathcal{F}^{-1}\langle\xi\rangle e^{-i\langle\xi\rangle t} i\partial_\xi e^{i\langle\xi\rangle t}\mathcal{F} = \langle i\partial_x\rangle x + it\partial_x,$$

which is an important tool for obtaining the time decay estimates of solutions and is analogous to the operator

$$x + it\partial_x = \mathcal{U}(t)x\mathcal{U}(-t)$$

in the case of nonlinear Schrödinger equations, where

$$\mathcal{U}(t) = \mathcal{F}^{-1} e^{-\frac{i}{2}|\xi|^2 t} \mathcal{F}$$

is the free Schrödinger evolution group and

$$U(t) = e^{-i\langle i\nabla\rangle t}$$

is the free Klein-Gordon evolution group, respectively. We have the commutation relation

$$[\mathcal{L}, \mathcal{J}] = \mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L} = 0,$$

since

$$[x, \langle i\partial_x\rangle^{\alpha}] = \alpha \langle i\partial_x\rangle^{\alpha-2} \partial_x.$$

However it is difficult to calculate the action of $\mathcal{J}$ on the nonlinearity $\mathcal{N}$. Therefore we use the first order differential operator

$$\mathcal{P} = t\partial_x + x\partial_t$$

which is closely related to $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$ and acts well on the nonlinearity. Moreover, it almost commutes with $\mathcal{L}$ since

$$[\mathcal{L}, \mathcal{P}] = -i \langle i\partial_x\rangle^{-1} \partial_x \mathcal{L}.$$

The operator $\mathcal{J}$ was used previously in [5] for constructing the scattering operator for the nonlinear Klein-Gordon equations with supercritical nonlinearities.

We now explain briefly our strategy. Since the nonlinear Klein-Gordon equation (1.1) is considered as the relativistic version of the nonlinear Schrödinger equation

$$\text{(1.3)} \qquad \begin{cases} iv_t + \frac{1}{2}v_{xx} = \mu |v|^2 v, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ v(0) = v_0, \ x \in \mathbf{R}, \end{cases}$$

we therefore can expect that the methods applied to the study of (1.3) could be also useful for the nonlinear Klein-Gordon equations. Thus in the construction of the inverse modified wave operator in [3] the main ideas were transformation of equation (1.3) into another one acting by $\mathcal{F}\mathcal{U}(t)$ and application of the factorization property of the free evolution group $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}\mathcal{F}\mathcal{M}(t)$, where $\mathcal{M}(t) = e^{\frac{ix^2}{2t}}$ and $\mathcal{D}\phi = \frac{1}{\sqrt{it}}\phi\left(\frac{x}{t}\right)$. So we get from (1.3)

$$\begin{aligned} i(\mathcal{F}\mathcal{U}(-t)v)_t &= \mu\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{\mathcal{M}}|v|^2 v \\ &= \mu t^{-1}\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\left|\mathcal{F}\overline{\mathcal{M}}\mathcal{U}(-t)v\right|^2 \mathcal{F}\overline{\mathcal{M}}\mathcal{U}(-t)v \\ &= \mu t^{-1}|\mathcal{F}\mathcal{U}(-t)v|^2 \mathcal{F}\mathcal{U}(-t)v + R. \end{aligned}$$

Hence the nonlinear term in (1.3) is decomposed into the resonant term

$$\mu t^{-1}|\mathcal{F}\mathcal{U}(-t)v|^2 \mathcal{F}\mathcal{U}(-t)v$$

and the remainder term R. Then the resonant term can be canceled when we change the dependent variable $\mathcal{F}\mathcal{U}(-t)v$ by

$$(\mathcal{F}\mathcal{U}(-t)v)\exp\left(\int_1^t \mu\tau^{-1}|\mathcal{F}\mathcal{U}(-\tau)v|^2 d\tau\right).$$

So the a-priori estimate of $\|\mathcal{FU}(-t)v\|_{\mathbf{L}^\infty}$ follows. In this paper we use the same ideas. We multiply both sides of (1.2) by $\mathcal{F}U(-t)$ and put $\varphi = \langle i\partial_x \rangle \mathcal{FU}(-t)v$ to get

$$
\begin{aligned}
\varphi_t &= \mathcal{F}U(-t)\langle i\partial_x\rangle \mathcal{N}(v) \\
&= \frac{3i\mu}{2t}|\varphi|^2\varphi + f(\varphi,\overline{\varphi}) + O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{4,1}}\right), \qquad (1.4)
\end{aligned}
$$

where $f(\varphi,\overline{\varphi})$ are cubic nonlinearities with some additional oscillating factors. Thus we see that the nonlinearity of the nonlinear Klein-Gordon equation can be decomposed into resonant term $\frac{3i\mu}{2t}|\varphi|^2\varphi$, nonresonant terms $f(\varphi,\overline{\varphi})$ and a remainder term $O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{4,1}}\right)$.

Remark 1.1. *Let us consider the cubic nonlinear Klein-Gordon equation*

$$
\left\{\begin{array}{c} v_{tt}+v-v_{xx}=-v_t^3, \ \ (t,x)\in\mathbf{R}\times\mathbf{R}, \\ v(0)=v_0, v_t(0)=v_1, x\in\mathbf{R}. \end{array}\right. \qquad (1.5)
$$

The usual energy method gives us the space-time estimate of solutions

$$
\int_0^t \|v_t(\tau)\|^4_{\mathbf{L}^4}\,d\tau \le C\left(\|v_0\|^2_{\mathbf{H}^1}+\|v_1\|^2_{\mathbf{L}^2}\right)
$$

which implies the dissipative property of solutions. Indeed in [7], *it was obtained the asymptotic behavior of small solutions with time decay rate* $\left(t^{-1}\log(1+t)\right)^{\frac{1}{2}}$ *under the conditions that the initial data are regular, small and have a compact support. Our method is useful to remove the condition of compactness on the data. If we put* $u=\frac{1}{2}\left(v+i\langle i\partial_x\rangle^{-1}v_t\right)$, *then for the real-valued function* v *the nonlinear Klein-Gordon equation (1.5) can be rewritten as*

$$
\left\{\begin{array}{c} \mathcal{L}u=\mathcal{N}(u), \ \ (t,x)\in\mathbf{R}\times\mathbf{R}, \\ u(0,x)=u_0(x), \ x\in\mathbf{R}, \end{array}\right. \qquad (1.6)
$$

where $\mathcal{L}=\partial_t+i\langle i\partial_x\rangle$ *and*

$$
\begin{aligned}
\mathcal{N}(u) &= \frac{1}{2}\langle i\partial_x\rangle^{-1}\left(\left(\langle i\partial_x\rangle u-\overline{\langle i\partial_x\rangle u}\right)^3\right) \\
&= -\frac{3}{2}\langle i\partial_x\rangle^{-1}|\langle i\partial_x\rangle u|^2\langle i\partial_x\rangle u + R,
\end{aligned}
$$

where R *is a remainder term. We have*

$$
\begin{aligned}
\left(\langle\xi\rangle^m \mathcal{F}U(-t)u\right)_t &= -\frac{3}{2}t^{-1}\langle\xi\rangle^{5-2m}|\langle\xi\rangle^m\mathcal{F}U(-t)u|^2\langle\xi\rangle^m\mathcal{F}U(-t)u \\
&\quad +O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{3+m,1}}\right),
\end{aligned}
$$

where $m\ge 3$. *We let* $v(t)=\langle\xi\rangle^m\mathcal{F}U(-t)u(t)$ *and*

$$
\frac{v(1)}{\left(1+2\lambda|v(1)|^2\log t\right)^{\frac{1}{2}}}=w(t)
$$

then w *satisfy*

$$
w_t=-\lambda t^{-1}|w|^2w, \ \ w(1)=v(1).
$$

Therefore the estimate

$$\left\| \langle \xi \rangle^m \mathcal{F} U(-t)\, u(t) \right\|_{\mathbf{L}^\infty} \le \frac{C\,|v(1)|}{\left(1 + 2\lambda\,|v(1)|^2 \log t\right)^{\frac{1}{2}}}$$

follows.

References

[1] J.-M. Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi-linéaire à données petites en dimension 1*, Ann. Sci. École Norm. Sup. (4) **34** (2001), pp. 1-61.

[2] V. Georgiev and B. Yordanov, *Asymptotic behaviour of the one dimensional Klein-Gordon equation with a cubic nonlinearity*, (1996).

[3] N.Hayashi and P.I.Naumkin, *Asymptotics in large time of solutions to nonlinear Schrödinger and Hartree equations*, Amer. J. Math. **120** (1998), pp. 369-389.

[4] N.Hayashi and P.I.Naumkin, *Final state problem for the cubic nonlinear Klein-Gordon equation*, preprint, (2006)

[5] N.Hayashi and P.I.Naumkin, *Scattering Operator for Nonlinear Klein-Gordon Equations*, preprint, (2006)

[6] H. Lindblad and A. Soffer, *A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation,* Lett. Math. Phys., **73** (2005), no. 3, pp. 249–258.

[7] H. Sunagawa, *Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms*, J. Math. Soc. Japan, **58**(2006), pp.379-400.

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Asymptotics of solutions to nonlinear Schrödinger equations in 3D

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1 Introduction and main results

We study the asymptotic behavior in time of small solutions to nonlinear Schrödinger equations with quadratic nonlinearities in three space dimensions:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda u^2 + \mu \overline{u}^2, & (t,x) \in \mathbf{R} \times \mathbf{R}^3, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

where $\overline{u}$ is the complex conjugate of u and λ, $\mu \in \mathbf{C}$. In [2], it was proved that global existence of small solutions and the time decay estimate

$$\|u(t)\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{3}{2}} \tag{1.2}$$

and existence of a unique final state $\phi_+ \in \mathbf{L}^2$ such that

$$\|u(t) - \mathcal{U}(t)\phi_+\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}} \quad \text{for } t > 1, \tag{1.3}$$

where the free Schrödinger evolution group $\mathcal{U}(t)$ is given by

$$\mathcal{U}(t)\psi = \frac{1}{(2\pi i t)^{\frac{n}{2}}} \int e^{\frac{i|x-y|^2}{2t}} \psi(y) dy.$$

We improve the results on asymptotic behavior (1.3) by using a special structure of nonlinearities u^2 and $\overline{u}^2$. Before stating our main results, we introduce some notations and function space which are used in this talk. Let

$$\mathcal{F}\psi \equiv \widehat{\psi} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ix\xi} \psi(x) dx$$

denote the Fourier transform of ψ and

$$\mathcal{F}^{-1}\psi = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ix\xi} \psi(\xi) d\xi$$

denote the inverse Fourier transform of ψ. The weighted Sobolev space $\mathbf{H}_p^{m,k}$ is defined by

$$\mathbf{H}_p^{m,k} = \left\{ \phi \in \mathcal{S}' : \|\phi\|_{\mathbf{H}_p^{m,k}} = \left\| (1+|x|^2)^{\frac{k}{2}} (1-\Delta)^{\frac{m}{2}} \phi \right\|_{\mathbf{L}^p} < \infty \right\},$$

with $m, k \in \mathbf{R}$ and $1 \le p \le \infty$. For simplicity, we denote $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$, $\|\cdot\|_{\mathbf{H}^{m,k}} = \|\cdot\|_{\mathbf{H}_2^{m,k}}$, $\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$. Then our first result is

Theorem 1.1. ([3]) *Let $u_0 \in \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ and $\varepsilon = \|u_0\|_{\mathbf{H}^{3,0}} + \|u_0\|_{\mathbf{H}^{1,2}}$. Then there exists an $\varepsilon > 0$ such that* (1.1) *has a unique global solution u satisfying $u \in \mathbf{C}(\mathbf{R}; \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2})$ and*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon^{\frac{1}{2}} \langle t \rangle^{-\frac{3}{2}}, \quad \|u(t)\|_{\mathbf{L}^2} \le C\varepsilon^{\frac{1}{2}}.$$

Moreover, there exists a unique final state $\widehat{\phi}_+ \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ such that

$$\begin{aligned} \Bigg\| \mathcal{F}\mathcal{U}(-t)u(t) - \widehat{\phi}_+ - \lambda 2^{-\frac{3}{2}} i^{-\frac{9}{2}} \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \widehat{\phi}_+^2\left(\frac{\xi}{2}\right) \\ - \mu 2^{-\frac{3}{2}} i^{\frac{9}{2}} \int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \overline{\widehat{\phi}}_+^2\left(-\frac{\xi}{2}\right) \Bigg\|_{\mathbf{L}^2} \le C\varepsilon t^{\theta - \frac{3}{2}} \end{aligned} \tag{1.4}$$

for $t > 1$ and

$$\|u(t) - \mathcal{U}(t)\phi_+\|_{\mathbf{L}^2} \le C\varepsilon t^{-\frac{5}{4}} \quad \text{for } t > 1, \tag{1.5}$$

where $\theta > 0$,

$$\begin{aligned} &\left\| \lambda 2^{-\frac{3}{2}} i^{-\frac{9}{2}} \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \widehat{\phi}_+^2\left(\frac{\xi}{2}\right) \right\|_{\mathbf{L}^2} + \left\| \mu 2^{-\frac{3}{2}} i^{\frac{9}{2}} \int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \overline{\widehat{\phi}}_+^2\left(-\frac{\xi}{2}\right) \right\|_{\mathbf{L}^2} \\ &\le C\varepsilon t^{-\frac{5}{4}} \quad \text{for } t > 1. \end{aligned}$$

Next we study sharp time asymptotics of solutions around the final states of nonlinear Schrödinger equations :

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda u^2 + \mu \overline{u}^2, & (t,x) \in \mathbf{R} \times \mathbf{R}^3, \\ \mathcal{U}(-\infty)u(\infty) = u_+ \end{cases} \tag{1.6}$$

which is written as the integral equation

$$u(t) = \mathcal{U}(t)u_+ + i \int_t^\infty \mathcal{U}(t-\tau)(\lambda u^2 + \mu \overline{u}^2)(\tau) d\tau, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^3, \tag{1.7}$$

for a given u_+. In particular, we consider the large time asymptotic estimates of solutions from below.

Here, we define the following function spaces

$$\mathbf{X}_{\alpha,T} = \left\{ \phi \in \mathbf{C}\left([T,\infty);\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_{\alpha,T}} < \infty \right\},$$

where

$$\|\phi\|_{\mathbf{X}_{\alpha,T}} = \sup_{t\in[T,\infty)} t^{\alpha} \|\phi(t) - u_1(t) - u_2(t) - u_3(t)\|_{\mathbf{H}^{2,0}},$$

$$u_1(t) = \mathcal{U}(t)u_+, \quad u_2(t) = i\int_t^{\infty} \mathcal{U}(t-\tau)(\lambda u_1^2 + \mu\overline{u_1}^2)d\tau,$$

$$u_3(t) = 2i\int_t^{\infty} \mathcal{U}(t-\tau)(\lambda u_1 u_2 + \mu\overline{u_1}\overline{u_2})d\tau.$$

Our next results are the following:

Theorem 1.2. ([1]) *Let $u_+ \in \mathbf{H}^{0,3} \cap \mathbf{H}^{3,0} \cap \mathbf{H}_1^{2,0}$. Then there exists a positive time T and a unique solution $u \in \mathbf{X}_{\alpha,T}$, with $\alpha \in (1,2)$, to equation (1.7). Furthermore for $\alpha \in (\frac{5}{4},2)$, if $\lambda = 0$ or $\mu = 0$, then there exist positive constants C_1, C_2 such that*

$$C_1\rho^2 t^{-\frac{3}{2}} \le \left\|\nabla^j (u(t) - u_1(t))\right\|_{\mathbf{L}^2} \le C_2\rho^2 t^{-\frac{3}{2}}$$

for $j = 1,2$, as $t \to \infty$, where $\rho = \|u_+\|_{\mathbf{H}^{0,3}} + \|u_+\|_{\mathbf{H}^{3,0}} + \|u_+\|_{\mathbf{H}_1^{2,0}}$.

Theorem 1.3. ([1]) *Let $u_+ \in \mathbf{H}^{0,3} \cap \mathbf{H}^{3,0} \cap \mathbf{H}_1^{2,0}$, and the norm $\rho = \|u_+\|_{\mathbf{H}^{0,3}} + \|u_+\|_{\mathbf{H}^{3,0}} + \|u_+\|_{\mathbf{H}_1^{2,0}}$ be sufficiently small. Then for any positive time $T > 1$ there exists a unique solution $u \in \mathbf{X}_{2,T}$ to (1.7). Furthermore if we assume that $\lambda_2 \neq 0$, then there exist positive constants C_1,C_2 such that*

$$C_1\rho^3 t^{-2} \le \left\|\nabla^j (u(t) - u_1(t) - u_2(t))\right\|_{\mathbf{L}^2} \le C_2\rho^3 t^{-2}$$

for $j = 1,2$, as $t \to \infty$.

Theorem 1.2 and Theorem1.3 are the joint work with Professors N. Hayashi and P.I. Naumkin.

Finally, we study asymptotic properties of small solutions for nonlinear Schrödinger equations:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda u^2 + \mu\overline{u}^2, \qquad (t,x) \in \mathbf{R}\times\mathbf{R}^3. \tag{1.8}$$

The purposes are twofold. One of them is to show that the inverse wave operator for (1.8) can be defined in a suitable Banach space. In order to do it, we consider the initial value problem (1.1) under the conditions that the initial function is small and in a Banach space $\mathbf{Y}$ and we find that there exists a unique $\phi_+ \in \mathbf{X}$(Banach space) such that

$$\lim_{t\to\infty} \|\mathcal{U}(-t)u(t) - \phi_+\|_{\mathbf{X}} = 0$$

which means that the operator $\mathcal{W}_+^{-1} : u_0 \mapsto \phi_+$ is well defined. We call $\mathcal{W}_+^{-1}$ the inverse wave operator. Another purpose is to consider the final states problem (1.7) for a given $u_+ \in \mathbf{X}_1$ which is small and we prove existence of a unique global solution $u(t) \in \mathbf{C}([1,\infty);\mathbf{Y}_1)$ of (1.7) for $t \geq 1$ under the condition that

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}(t)u_+\|_{\mathbf{Y}_1} = 0,$$

where Banach spaces $\mathbf{X}_1$ and $\mathbf{Y}_1$ will be defined in the theorem below precisely. Then the operator $\mathcal{W}_+ : u_+ \mapsto u(1)$ is well defined and we call $\mathcal{W}_+$ the wave operator. If we can show the above two existence results under the condition $\mathbf{X}_1 = \mathbf{X}$, we see that the operator $\mathcal{W}_+\mathcal{W}_+^{-1}$ is well defined as the mapping from the neighborhood of the origin of $\mathbf{Y}$ into $\mathbf{Y}_1$. Here we define the following function space:

$$\mathbf{Z}_T = \left\{\phi \in \mathbf{C}([T,\infty);\mathbf{L}^2) : \|\phi\|_{\mathbf{Z}_T} < \infty\right\},$$

where

$$\|\phi\|_{\mathbf{Z}_T} = \sup_{t\in[T,\infty)} \left(t^{\frac{3}{4}}\|\phi(t)\|_{\mathbf{L}^4} + t^{\frac{1}{2}}\|\phi(t)\|_{\mathbf{L}^2}\right).$$

We state our main results

Theorem 1.4. ([4]) *Let $u_0 \in \mathbf{Y} \equiv \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ and $\varepsilon = \|u_0\|_{\mathbf{Y}}$. Then there exists an $\varepsilon > 0$ such that* (1.1) *has a unique global solution $u \in \mathbf{C}(\mathbf{R};\mathbf{Y})$ which satisfies*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon^{\frac{1}{2}}\langle t\rangle^{-\frac{3}{2}}, \quad \|u(t)\|_{\mathbf{L}^2} \leq C\varepsilon^{\frac{1}{2}}.$$

Moreover, there exists a unique final state $\phi_+ \in \mathbf{X} \equiv \mathbf{H}^{1,1}$ such that

$$\left\|\mathcal{U}(-t)u(t) - \phi_+\right\|_{\mathbf{X}} \leq C\varepsilon t^{-\frac{1}{4}}$$

for $t > 1$. Namely, the inverse wave operator $\mathcal{W}_+^{-1}$ is the mapping from the neighborhood of the origin of $\mathbf{Y}$ into $\mathbf{X}$.

Theorem 1.5. ([4]) *Let $u_+ \in \mathbf{X}_1 \equiv \mathbf{H}^{1,1}$ and the norm $\rho = \|u_+\|_{\mathbf{X}_1}$ be sufficiently small. Then for any positive time $T > 1$ there exists a unique solution $u \in \mathbf{Z}_T$ to* (1.7). *Namely, the wave operator $\mathcal{W}_+$ is the mapping from the neighborhood of the origin of $\mathbf{X}_1$ into $\mathbf{Y}_1 \equiv \mathbf{L}^2$.*

By the above theorems we have

Corollary 1.1. ([4]) *The operator $\mathcal{W}_+\mathcal{W}_+^{-1}$ is well defined as the mapping from the neighborhood of the origin of $\mathbf{Y} \equiv \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ into $\mathbf{Y}_1 \equiv \mathbf{L}^2$.*

References

[1] N. Hayashi, Y. Kawahara and P.I. Naumkin, *Lower bounds of asymptotics in time of solutions to nonlinear Schrödinger equations in 3D,* Nonlinear Anal. **65** (2006), 1394-1410.

[2] N. Hayashi, T. Mizumachi and P.I. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D,* Differential Integral Equations **16** (2003), 159-179.

[3] Y. Kawahara, *Global existence and asymptotic behavior of small solutions to nonlinear Schrödinger equations in 3D,* Differential Integral Equations **18** (2005), 169-194.

[4] Y. Kawahara, *Wave and inverse wave operators for the quadratic nonlinear Schrödinger equations in 3D*, preprint.

Identities related to analytic solutions for Schrödinger equations

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We consider the analyticity of solutions to the Cauchy problem of the Schrödinger eqution:

$$\begin{aligned} i\frac{\partial}{\partial t}u + \frac{1}{2}\triangle u &= 0, \\ u(0) &= \phi. \end{aligned}$$

For $t \in \mathbb{R}^n$ let $U(t)$ be the free propagator, *i.e.*

$$u(t) = U(t)\phi = \exp(i\frac{t}{2}\triangle)\phi = \mathfrak{F}^{-1}\exp(-i\frac{t}{2}|\xi|^2)\mathfrak{F}\phi,$$

where $\mathfrak{F}$ denotes the Fourier transform. For $t \in \mathbb{R}\setminus\{0\}$, $U(t)\phi$ with $\phi \in L^1 + L^2$ has the representation

$$(U(t)\phi)(x) = (M(t)D(t)\mathfrak{F}M(t)\phi)(x),$$

where $\mathfrak{F}$ is understood to be a bounded operator from $L^1 + L^2$ to $L^\infty + L^2$,

$$\begin{aligned} (M(t)\psi)(x) &= \exp\left(\frac{i}{2t}|x|^2\right)\psi(x), \\ (D(t)\psi)(x) &= (it)^{-n/2}\psi(t^{-1}x) \end{aligned}$$

with

$$(it)^{-n/2} = \begin{cases} \left(\dfrac{1+i}{\sqrt{2}}\sqrt{t}\right)^{-n} & \text{if } t > 0, \\ \left(\dfrac{1-i}{\sqrt{2}}\sqrt{-t}\right)^{-n} & \text{if } t < 0. \end{cases}$$

Definition 1 *For any bounded convex set* $\Omega \subset \mathbb{R}^n$ *with* $0 \in \operatorname{Int}\Omega$*, its supporting function* γ_Ω *is defined by*

$$\gamma_\Omega(x) = \sup\{\, x \cdot p \,;\, p \in \Omega \,\}$$

for $x \in \mathbb{R}^n$*.*

As for the analyticity in the space variable we have the following statement.

Proposition 1 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex set satisfying* $0 \in \operatorname{Int}\Omega$*. Let* ϕ *satisfy* $e^{\gamma_\Omega}\phi \in L^1 + L^\infty$*. Then for any* $t \in \mathbb{R} \setminus \{0\}$*, the function* $\zeta \mapsto u(t,\zeta)$ *defined by*

$$u(t,\zeta) = \exp\left(\frac{i}{2t}\zeta^2\right) \cdot (it)^{-n/2} (\mathfrak{F} M(t)\phi)\left(\frac{\zeta}{t}\right)$$

is an analytic continuation of $U(t)\phi$ *on* $\mathbb{R}^n + it(\operatorname{Int}\Omega)$*, where* $\zeta^2 = \zeta \cdot \zeta = \sum_{j=1}^{n} \zeta_j^2$*.*

As for the analyticity in the time variable we have the following statement.

Proposition 2 *Let* $a > 0$ *and let* ϕ *satisfy* $e^{a|x|^2}\phi \in L^1 + L^\infty$*. Then the function* $(\tau,\zeta) \mapsto u(\tau,\zeta)$ *defined by*

$$u(\tau,\zeta) = \exp\left(\frac{i}{2\tau}\zeta^2\right) \cdot (i\tau)^{-n/2} (\mathfrak{F} M(\tau)\phi)\left(\frac{\zeta}{\tau}\right) \tag{1}$$

is analytic on $\left(\mathbb{C} \setminus \overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)}\right) \times \mathbb{C}^n$*, where*

$$\overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)} = \left\{\tau \in \mathbb{C}\,;\, \left|\tau - \frac{i}{4a}\right| \leqq \frac{1}{4a}\right\}.$$

Remark 1 u defined by (1) can be a double-valued function out of the factor $(i\tau)^{-n/2}$ provided n is odd. To be more specific, $U(t)\phi$ is connected with $-U(t)\phi$ through a mutual continuation on $\left(\mathbb{C} \setminus \overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)}\right) \times \mathbb{C}^n$ provided n is odd.

Proposition 1 and proposition 2 induce the following identities, which appear in [HS1] provided $n = 1$.

Theorem 1 *Let* $a > 0$ *and let* ϕ *satisfy* $e^{a|x|^2}\phi \in L^2$*. Then* $U(t)\phi$ *with* $t \neq 0$ *has an analytic continuation* $u(t,\cdot)$ *defined on* $\mathbb{C}^n$*, which satisfies that*

$$\begin{aligned}
&(2\pi a t^2)^{-n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\frac{\eta^2}{2at^2}\right) \left|\exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta)\right|^2 d\xi d\eta \\
= &\sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \|J^\alpha U(t)\phi\|_{L^2}^2 = \left\|e^{a|x|^2}\phi(x)\right\|_{L^2}^2,
\end{aligned} \tag{2}$$

where $J^\alpha = J^\alpha(t) = \prod_{k=1}^{n} J_k(t)^{\alpha_k}$*,* $J_k(t) = x_k + it\partial_k$ *and* $\alpha = (\alpha_1, \dots, \alpha_n)$ *is a multi-index.*

Theorem 2 *Let $a > 0$ and suppose that*

$$\left(\prod_{j=1}^{n} \frac{\sinh 2ax_j}{2ax_j}\right)^{1/2} \phi \in L^2$$

Then,

(i) *$U(t)\phi$ with $t > 0$ has an analytic continuation $u(t,\cdot)$ defined on $\mathbb{R}^n + iQ_{at}$, where*

$$Q_{at} = (-at, at)^n = \{\, \eta = (\eta_1, \dots, \eta_n)\,;\, -at < \eta_j < at,\ j = 1, \dots, n \,\}.$$

(ii) *$u(t,\cdot)$ with $t > 0$ satisfies the following identity.*

$$\begin{aligned}
&(2at)^{-n} \int_{Q_{at}} \left(\int_{\mathbb{R}^n} \left| \exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta) \right|^2 \mathrm{d}\xi \right) \mathrm{d}\eta \\
= \ & \sum_{\alpha \geq 0} \frac{(2a)^{2|\alpha|}}{\prod_{j=1}^{n}(2\alpha_j + 1)!} \|J^\alpha U(t)\phi\|_{L^2}^2 \\
= \ & \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} \frac{\sinh(2ax_j)}{2ax_j} \right) |\phi(x)|^2 \mathrm{d}x.
\end{aligned}$$

We generalize the identities above by the following statement. It is convenient to use the notation

$$\mathfrak{L}_\Omega p(x) = \int_\Omega p(y) e^{2x\cdot y} \mathrm{d}y.$$

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be a convex open set with $0 \in \mathrm{Int}\Omega$. Suppose that $p \in L^1(\Omega)$ is non-negative and that*

$$\mathrm{supp}\, p \cap \Omega = \Omega.$$

Suppose that $\mathfrak{L}_\Omega p \in L^1_{loc}$ and assume that $(\mathfrak{L}_\Omega p)\,|\phi|^2 \in L^1$. Then, $U(t)\phi$ with $t \neq 0$ has an analytic continuation $u(t,\cdot)$ defined on $\mathbb{R}^n + it\Omega$, which satisfies that

$$\begin{aligned}
&|t|^{-n} \int_{t\Omega} q(\eta/t) \left(\int_{\mathbb{R}^n} \left| \exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta) \right|^2 \mathrm{d}\xi \right) \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 \mathrm{d}x
\end{aligned} \tag{3}$$

for any $q \in L^1(\Omega)$ with $|q| \leqq p$.

Remark 2 We have some other examples for Ω, p, q, $\mathfrak{L}_\Omega p$ and $\mathfrak{L}_\Omega q$ in Theorem 3 as follows:

(i)

$$\begin{aligned}
\Omega &= Q_\pi = (-\pi, \pi)^n \\
p(x) &= \prod_{j=1}^{n} |\sin N_j x_j|, \\
q(x) &= \prod_{j=1}^{n} \sin N_j x_j, \\
\mathfrak{L}_\Omega p(x) &= \prod_{j=1}^{n} \frac{\sinh 2\pi x_j}{\tanh(\pi x_j / N_j)} \cdot \frac{2N_j}{4x^2 + N_j^2}, \\
\mathfrak{L}_\Omega q(x) &= \prod_{j=1}^{n} \frac{2(-1)^{N_j+1} N_j \sinh 2\pi x_j}{4x_j^2 + N_j^2}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\Omega &= (-\pi, \pi)^n \\
p(x) &= \prod_{j=1}^{n} |\cos N_j x_j|, \\
q(x) &= \prod_{j=1}^{n} \cos N_j x_j, \\
\mathfrak{L}_\Omega p(x) &= \prod_{j=1}^{n} \frac{2\sinh 2\pi x_j}{4x_j^2 + N_j^2} \left(2x_j + \frac{N_j}{\sinh \pi x_j / N_j} \right), \\
\mathfrak{L}_\Omega q(x) &= \prod_{j=1}^{n} \frac{4(-1)^{N_j} x_j \sinh 2\pi x_j}{4x_j^2 + N_j^2}.
\end{aligned}$$

Here $N_j \in \mathbb{N}$ for $j = 1, \ldots, n$.

References

[AHS] Hiroaki Aikawa, Nakao Hayashi and Saburou Saitoh: *Isometrical identities for the Bergman and the Szegö spaces on a sector*, J. Math. Soc. Japan **43**(1991), 195-201.

[HK1] Nakao Hayashi and Keiichi Kato: *Regularity in time of solutions to nonlinear Schrödinger equations*, Journal of Functional Analysis **128**(1995), 253-277.

[HK2] Nakao Hayashi and Keiichi Kato: *Analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations*, Commun. Math. Phys. **184**(1997), 273-300.

[HO] Nakao Hayashi and Tohru Ozawa: *On the derivative nonlinear Schrödinger equation*, Physica D **55**(1992), 14-36.

[HS1] Nakao Hayashi and Saburou Saitoh: *Analyticity and smoothing effect for the Schrödinger equation*, Ann. Inst. Henri Poincaré, Physique théorique **52**(1990), 163-173.

[HS2] Nakao Hayashi and Saburou Saitoh: *Analyticity and global existence of small solutions to some nonlinear Schrödinger equations*, Commun. Math. Phys. **129**(1990) , 27-41.

[MRZ] Yoshinori Morimoto, Luc Robbiano and Claude Zuily: *Remark on the analytic smoothing for the Schrödinger equation*, Indiana Univ. Math. J. **49**(2000), 1563-1579.

[Nak] Kuniaki Nakamitsu: *Analytic finite energy solutions of the nonlinear Schrödinger equation*, Commun. Math. Phys. **260**(2005), 117-130.

[OYY] Tohru Ozawa, Kazuyuki Yamauchi and Yasuko Yamazaki: *Analytic smoothing effect for solutions to Schrödinger equations with nonlinearity of integral type*, Osaka J. Math. **42** (2005), 737-750.

[RZ1] Luc Robbiano and Claude Zuily: *Microlocal analytic smoothing effect for the Schrödinger equation*, Duke Math. J. **100**(1999), 93-129.

[RZ2] Luc Robbiano and Claude Zuily: Analytic theory for the quadratic scattering wave front set and application to the Schrödinger equation, Astérisque **283**, Societe mathematique de France, Paris, 2002.

[S] Saburou Saitoh: Theory of reproducing kernels and its applications, Pitman Research Notes in Mathematics Series **189**, Longman Scientific and Technical, Harlow, 1988.

[T1] Hideki Takuwa: *Analytic smoothing effects for a class of dispersive equations*, Tsukuba J. Math. **28**(2004), 1-34.

[T2] Hideki Takuwa: *Microlocal analytic smoothing effects for operators of real principal type*, Osaka J. Math. **43**(2006), 13-62.

Standing waves for nonlinear Schrödinger equations with a delta function potential

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We consider the following nonlinear Schrödinger equation with a delta function potential:

$$i\partial_t u = -\partial_x^2 u + \gamma\delta(x)u + \alpha|u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where γ, $\alpha \in \mathbb{R}$, $1 < p < \infty$, and $\delta(x)$ is the delta function at $x = 0$. The formal expression $-\partial_x^2 + \gamma\delta(x)$ in (1) is formulated as a linear operator A_γ or H_γ associated with a quadratic form a_γ on $H^1(\mathbb{R})$:

$$a_\gamma(u,v) = \mathrm{Re}\left\{\int_{\mathbb{R}} \partial_x u(x)\overline{\partial_x v(x)}\,dx + \gamma u(0)\overline{v(0)}\right\}, \quad u, v \in H^1(\mathbb{R}).$$

Then, there exists a unique linear operator $A_\gamma : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ such that

$$\langle A_\gamma u, v\rangle = a_\gamma(u,v), \quad u, v \in H^1(\mathbb{R}).$$

Moreover, we define a linear operator H_γ in $L^2(\mathbb{R})$ by $H_\gamma v = -\partial_x^2 v$ for $v \in D(H_\gamma)$ with the domain

$$D(H_\gamma) = \{v \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) : \partial_x v(+0) - \partial_x v(-0) = \gamma v(0)\}.$$

Then, H_γ is a self-adjoint operator in $L^2(\mathbb{R})$, and satisfies

$$(H_\gamma u, v)_{L^2} = a_\gamma(u,v), \quad u, v \in D(H_\gamma).$$

The following spectral properties of H_γ are known: $\sigma_{\mathrm{ess}}(H_\gamma) = \sigma_{\mathrm{ac}}(H_\gamma) = [0,\infty)$, $\sigma_{\mathrm{sc}}(H_\gamma) = \emptyset$. If $\gamma \ge 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \emptyset$. If $\gamma < 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \{-\gamma^2/4\}$

with its positive normalized eigenfuction $(|\gamma|/2)^{1/2}e^{-|\gamma||x|/2}$ (see [1, Chapter I.3] for details).

In this talk, we mainly consider the case where $\gamma<0$ and $\alpha>0$ (attractive potential and repulsive nonlinearity), and study the structure and the orbital stability of standing wave solutions $e^{i\omega t}\varphi_{\omega}(x)$ for (1), where $\omega\in\mathbb{R}$ is a parameter, and $\varphi_{\omega}\in H^{1}(\mathbb{R})$ is a positive solution of the stationary problem:

$$A_{\gamma}\varphi+\omega\varphi+\alpha|\varphi|^{p-1}\varphi=0 \quad \text{in } H^{-1}(\mathbb{R}). \tag{2}$$

The local well-posedness of the Cauchy problem for (1) in the energy space $H^{1}(\mathbb{R})$ follows from an abstract result in Cazenave [2]. Moreover, there is conservation of charge and energy, i.e.,

$$\|u(t)\|_{L^{2}}=\|u_{0}\|_{L^{2}}, \quad E(u(t))=E(u_{0}), \quad \forall t\in[0,T^{*}),$$

where $u(t)$ is the solution of (1) with $u(0)=u_{0}\in H^{1}(\mathbb{R})$, $T^{*}=T^{*}(u_{0})\in(0,\infty]$ is the maximal existence time of $u(t)$, and E is defined by

$$\begin{aligned} E(v) &= \frac{1}{2}a_{\gamma}(v,v)+\frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2}\|\partial_{x}v\|_{L^{2}}^{2}+\frac{\gamma}{2}|v(0)|^{2}+\frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

(see Theorem 3.7.1 and Corollary 3.3.11 in [2]).

For the stability of standing waves for (1), the case where $\gamma<0$ and $\alpha<0$ was first studied by Goodman, Holmes and Weinstein [7] for the special case $p=3$, and then by Fukuizumi, Ohta and Ozawa [6] for general case $1<p<\infty$. The result on stability for this case is as follows. When $\omega>\gamma^{2}/4$, the stationary problem (2) has a unique positive solution φ_{ω}. If $1<p\leq 5$, the standing wave solution $e^{i\omega t}\varphi_{\omega}$ of (1) is stable for any $\omega\in(\gamma^{2}/4,\infty)$. If $p>5$, there exists $\omega^{*}=\omega^{*}(\gamma,\alpha,p)\in(\gamma^{2}/4,\infty)$ such that $e^{i\omega t}\varphi_{\omega}$ is stable for any $\omega\in(\gamma^{2}/4,\omega^{*})$, and is unstable for any $\omega\in(\omega^{*},\infty)$. Remark that for the case where $\gamma=0$ and $\alpha<0$, the standing wave solution $e^{i\omega t}\varphi_{\omega}$ is stable for any $\omega\in(0,\infty)$ if $1<p<5$, and is unstable for any $\omega\in(0,\infty)$ if $p\geq 5$. Note also that when $\gamma\in\mathbb{R}$, $\alpha=-1$ and $\omega>\gamma^{2}/4$, the positive solution φ_{ω} of (2) is given by

$$\varphi_{\omega}(x)=\left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)}\left\{\cosh\left(\frac{(p-1)\sqrt{\omega}}{2}|x|+b(\omega)\right)\right\}^{-2/(p-1)}$$

for $x \in \mathbb{R}$, where $b(\omega) = \tanh^{-1}(-\frac{\gamma}{2\sqrt{\omega}})$. For this case, the stability of $e^{i\omega t}\varphi_\omega$ is determined by the sign of the derivative of the function

$$\omega \mapsto \|\varphi_\omega\|_{L^2}^2 = C_p \omega^{\frac{5-p}{2(p-1)}} \int_{b(\omega)}^{\infty} (\cosh y)^{-4/(p-1)}\, dy,$$

where C_p is a positive constant depending only on p (see also [4]). Fukuizumi and Jeanjean [5] studies the case where $\gamma > 0$ and $\alpha < 0$.

We now state our main results.

Theorem 1 *When $\gamma < 0$, $\alpha = 1$, $1 < p < \infty$ and $0 < \omega < \gamma^2/4$, the stationary problem (2) has a unique positive solution $\varphi_\omega \in H^1(\mathbb{R})$ given by*

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)} \left\{\sinh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + c(\omega)\right)\right\}^{-2/(p-1)}$$

for $x \in \mathbb{R}$, where $c(\omega) = \tanh^{-1}(2\sqrt{\omega}/|\gamma|)$. The standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is orbitally stable for this case.

Theorem 2 *Let $\gamma < 0$, $\alpha = 1$ and $\omega = 0$. If $1 < p < 5$, the stationary problem (2) has a unique positive solution $\varphi_0 \in H^1(\mathbb{R})$ given by*

$$\varphi_0(x) = \left(\frac{2(p+1)\gamma^2}{\{4 + (p-1)|\gamma||x|\}^2}\right)^{-1/(p-1)}$$

for $x \in \mathbb{R}$. The stationary solution φ_0 of (1) is orbitally stable for this case. If $p \geq 5$, the stationary problem (2) has no nontrivial solutions in $H^1(\mathbb{R})$.

The proof of Theorem 1 is based on the fact that φ_ω is characterized by a minimizer of the minimization problem

$$\inf\{E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 : v \in H^1(\mathbb{R})\},$$

and the conservation of energy and charge (cf. Cazenave and Lions [3]). Theorem 2 is proved in a similar way.

Finally, it should be remarked that Mizumachi [8] proves the asymptotic stability of small standing waves in the energy space $H^1(\mathbb{R})$ for one dimensional nonlinear Schrödinger equation

$$i\partial_t u = -\partial_x^2 u + V(x)u + \alpha|u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

for the case $p \geq 5$ under suitable assumptions on $V(x)$.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics, 2nd ed., with an appendix by Pavel Exner*, AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 2005.

[2] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[3] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[4] F.F.G. Della Casa and A. Sacchetti, *Stationary states for non linear one-dimensional Schrödinger equations with singular potential*, Physica D **219** (2006), 60–68.

[5] R. Fukuizumi and L. Jeanjean, *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, in preparation.

[6] R. Fukuizumi, M. Ohta, and T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, preprint.

[7] R. H. Goodman, P. J. Holmes, and M. I. Weinstein, *Strong NLS soliton-defect interactions*, Physica D **192** (2004), 215–248.

[8] T. Mizumachi, *Asymptotic stability of small solitons to 1D NLS with potential*, preprint, arXiv:math.AP/060531.

Well-posedness and weak rotation limit for the Ostrovsky equation

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We consider the initial value problem for the Ostrovsky equation as follows:

$$\begin{cases} \partial_t u + c\partial_x u + \alpha u \partial_x u - \beta \partial_x^3 u = \gamma \partial_x^{-1} u, & (x,t) \in \mathbf{R} \times [0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbf{R} \end{cases} \tag{0.1}$$

where $u(x,t)$ is a real valued function, c, α, β and γ are real valued constant parameter. The Ostrovsky equation has some physical models (See, e.g. [1], [5], [8], [9], [19]). For example, it describes the gravity waves propagating down a channel under the influence of Coliolis force. The parameter γ measures the effect of the Earth's rotation and is very small in real situations. When $\gamma = 0$, (0.1) is the Korteweg-de Vries equation, which is completely integrable. However, when $\gamma \neq 0$, (0.1) is known to be not integrable ([9], [20]). Our aim is to study the effect of the rotating term, namely, to understand difference and similarities between $\gamma = 0$ and $\gamma \neq 0$. Existence and stability of solitary waves were studied in [17]. In the present paper, we consider more fundamental problems. One is the well-posendness, which means the existence of solutions and the uniqueness and the continuous dependence on initial data. The other is the convergence of the solutions when $\gamma \to 0$.

Before we proceed to our problems, for simpleness, we normalize parameters by the change of variables as follows;

$$c = 0, \quad \alpha = 1, \quad \beta = +1 \,\text{or} - 1, \quad \gamma \in \mathbf{R}.$$

We first recall the known results for the KdV equatuon ($\gamma = 0$). In [3], Bourgain proved the time local well-posedness in L^2 by introducing the Fourier restriction norm method. In [14] Kenig, Ponce and Vega proved refined bilinear estimates to extend Bourgain's result to H^s, $s > -3/4$. Earlier results can be found in [2], [10], [12], [13]. The lifetime of the solutions by the time local well-posedness results above, depends only on the size of the H^s norm of the initial data. Therefore, by combining the L^2 conservation law and time local results, we have the time global well-posedness in H^s, $s \geq 0$. Since the KdV equation on H^s for $s < 0$ has no conservation law, it seemed difficult to consider the long time behavior of solutions in H^s, $s < 0$. However, in [7], Colliander, Keel, Staffilani, Takaoka and Tao overcame this difficulty and proved the time global well-posedness with $s > -3/4$ by introducing a regularizing Fourier multiplier operator I and calculating a modified energy defined in H^s, which is called the "I-method". The value $s = -3/4$ seems to be critical. Nakanishi, Takaoka and Tsutsumi proved that the fundamental bilinear estimate used to prove the time local well-posedness fails when $s \leq -3/4$ in [18]. Christ, Colliander and Tao proved the time local well-posedness with $s \geq -3/4$ by studying the modified KdV equation and the Miura transform in [6]. They also proved the time local ill-posedness with $-1 \leq s < -3/4$ in the sense that the solution operator fails to be uniformly continuous with respect to the H^s norm (See also [4], [15], [21]).

We next recall the known results for well-posedness of the Ostrovsky equation ($\gamma \neq 0$). Assume φ is in $H^s \cap \dot{H}^{-1}$. Varlamov and Liu proved the time local well-posedness for $s > 3/2$ by the energy method in [22]. Linares and Milanés extend this result to $s > 3/4$ in [16]. Huo and Jia extend this result to $s \geq -1/8$ by the Fourier restriction norm method in [11]. In this method, a bilinear estimate plays an inportant role. We have refined the bilinear estimate (See, Proposition 0.6) to obtain the following theorem.

Theorem 0.1. *Let $\gamma \neq 0$, $\beta = +1$ or -1 and $\varphi \in H^s \cap \dot{H}^a$. If $s > \max\{-3/4, -a/2-1\}$, $-1/2 > a \geq -1$, (0.1) is time locally well-posed.*

Remark 0.1. *Since we can not apply the Miura transform for the case $\gamma \neq 0$, the time local well-posedness for $s \leq -3/4$ is still open. The ill-posedness for $s \leq -3/4$ is also still open. In the proof of the ill-posedness results and the counter examples of bilinear estimates for the KdV equation with $s < 3/4$, a low frequency part of the solution plays important role. The rotation term $\gamma\partial_x^{-1}u$ affects the low frequency part. So, the situation for $\gamma \neq 0$ is different from that for $\gamma = 0$.*

The Ostrovsky equation has the L^2 conservation law and the following a priori estimate;

$$\sup_{0 \leq t \leq T} \|u(t)\|_{\dot{H}^{-1}} \leq C\big(\|\varphi\|_{\dot{H}^{-1}} + \langle T\rangle\langle\|\varphi\|_{L^2}\rangle^2\big).$$

where $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$. Therefore we can extend the time local solutions in Theorem 0.1 to time global ones.

Theorem 0.2. *Let $\gamma \neq 0$, $\beta = +1$ or -1 and $\varphi \in H^s \cap \dot{H}^{-1}$. If $s \geq 0$, (0.1) is time globally well-posed.*

We next consider the convergence of the solutions when $\gamma \to 0$. Let u_n and v be the solutions of the following equations;

$$\begin{cases} \partial_t u_n - \beta\partial_x^3 u_n + u_n\partial_x u_n = \gamma_n \partial_x^{-1} u_n, & (x,t) \in \mathbf{R} \times [0,\infty), \\ u_n(x,0) = \varphi_n(x), & x \in \mathbf{R}, \end{cases}$$

$$\begin{cases} \partial_t v - \beta\partial_x^3 v + v\partial_x v = 0, & (x,t) \in \mathbf{R} \times [0,\infty), \\ v(x,0) = \psi(x), & x \in \mathbf{R}. \end{cases}$$

Liu and Varlamov ([17]) proved the following proposition.

Proposition 0.3. *Let $s > 3/2$, $\psi = \varphi_n \in H^s \cap \dot{H}^{-1}$ and $T > 0$. Then,*

$$\sup_{0 \le t \le T} \|v(t) - u_n(t)\|_{L^2} \to 0,$$

when $\gamma \to 0$.

The essence of their proof is to calculate the L^2 inequality for $w := v - u$ as follows;

$$\|w(T)\|_{L^2} - \|w(0)\|_{L^2} \le C\Big(\Big|\int_0^T \int w^2 \partial_x v \, dxdt\Big|\Big) + \Big|\gamma_n \int_0^T \int w \partial_x^{-1} u_n \, dxdt\Big|\Big).$$

$\int w^2 \partial_x v \, dx$ in the first term of the right-hand side is estimated by $\|w\|_{L^2}^2 \|\partial_x v\|_{L^\infty}$. The assumption $s > 3/2$ was used to control $\|\partial_x v\|_{L^\infty}$ by the Sobolev inequality. We have refined this argument, by dividing time interval $[0, T]$ into small pieces and using Proposition 0.6, to obtain the following Theorem.

Theorem 0.4. *Let $\beta = +1$ or -1, $\psi \in L^2$, $\varphi_n \in L^2 \cap \dot{H}^{-1}$ and $T > 0$. Assume that* $\max\big\{\|\psi\|_{L^2}, \|\varphi_n\|_{L^2}, \gamma_n \|\varphi_n\|_{\dot{H}^{-1}}\big\} < M$. *Then, we have*

$$\sup_{0 \le t \le T} \|v(t) - u_n(t)\|_{L^2} \le C\big(\|\psi - \varphi_n\|_{L^2} + \gamma_n \langle \|\varphi_n\|_{\dot{H}^{-1}} \rangle\big)$$

where C depends only on T and M.

Corollary 0.5. *Let $\beta = +1$ or -1, $\psi \in L^2$, $\varphi_n \in L^2 \cap \dot{H}^{-1}$ and $T > 0$. Then, we have*

$$\sup_{0 \le t \le T} \|v(t) - u_n(t)\|_{L^2} \to 0 \text{ when } \|\psi - \varphi_n\|_L^2 \to 0, \gamma_n \|\varphi_n\|_{\dot{H}^{-1}} \to 0, \gamma_n \to 0.$$

Remark 0.2. *In Theorem 0.4 and Corollary 0.5, ψ is not necessary to be in $\dot{H}^{-1}$ and $\|\varphi_n\|_{\dot{H}^{-1}}$ is not necessary to be bounded. However, the assumption $\varphi_n \in \dot{H}^{-1}$ and $\gamma_n \| \|\varphi_n\|_{\dot{H}^{-1}} \to 0$ seems to be necessary. Since we treat the nonlinear term as perturbation, we consider the following linear equations;*

$$\begin{cases} \partial_t u_n - \beta \partial_x^3 u_n = \gamma_n \partial_x^{-1} u_n, & (x,t) \in \mathbf{R} \times [0, \infty), \\ u_n(x, 0) = \varphi_n(x), & x \in \mathbf{R}, \end{cases}$$

$$\begin{cases} \partial_t v - \beta \partial_x^3 v = 0, & (x,t) \in \mathbf{R} \times [0, \infty), \\ v(x, 0) = \psi(x), & x \in \mathbf{R}. \end{cases}$$

We can explicitly solve them by the Fourier transform.

$$u_n = \mathcal{F}_\xi^{-1} \exp\big(-it(\beta\xi^3 + \gamma_n \xi^{-1})\big)\mathcal{F}_x \varphi_n, \quad v = \mathcal{F}_\xi^{-1} \exp(-it\beta\xi^3)\mathcal{F}_x \psi.$$

If $\varphi_n = \psi$, we have

$$\|u_n - v\|_{L_x^2} = \|\{\exp(-it\gamma_n \xi^{-1}) - 1\}\mathcal{F}_x \psi\|_{L_\xi^2}.$$

Therefore, we need the assumotion $\gamma_n \|\varphi_n\|_{\dot{H}^{-1}} \to 0$.

Finally, we mention the bilinear estimate, which is used to prove Theorem 0.1 and Theorem 0.4. We define the Fourier restriction norm as follows;

$$\|u(x,t)\|_{X^{s,a,b}} := \|\langle\xi\rangle^s \langle\gamma\xi^{-1}\rangle^{-a} \langle\tau+\beta\xi^3+\gamma\xi^{-1}\rangle^b \widetilde{u}\|_{L^2_{\xi,\tau}}$$

where $\widetilde{u}$ denotes the Fourier transform with respect to t and x variables.

Proposition 0.6. *Let* $0 \geq s > -a/2+b'/2-5/4$, $-b \geq a \geq -1$ *and* $\min\{b+1/4+s/3, 1\} \geq b' \geq b > 1/2$. *Then, for any* $u, v \in X^{s,a,b}$, *we have*

$$\|\partial_x(uv)\|_{X^{s,a,b'-1}} \leq C\|u\|_{X^{s,a,b}}\|v\|_{X^{s,a,b}}$$

where C *depends only on* s, a, b, b'.

References

[1] E. S. Benilov, *On the surface waves in a shallow channel with an uneven bottom*, Stud. Appl. Math. **87** (1992), no. 1, 1–14.

[2] J. L. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A **278** (1975), no. 1287, 555–601.

[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. **3** (1993), no. 3, 209–262.

[4] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data*, Selecta Math. (N.S.) **3** (1997), no. 2, 115–159.

[5] Guan-yu Chen and John P. Boyd, *Analytical and numerical studies of weakly nonlocal solitary waves of the rotation-modified Korteweg-de Vries equation*, Phys. D **155** (2001), no. 3-4, 201–222.

[6] M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. **125** (2003), no. 6, 1235–1293.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* **R** *and* **T**, J. Amer. Math. Soc. **16** (2003), no. 3, 705–749.

[8] V. M. Galkin and Yu. A. Stepanyants, *On the existence of stationary solitary waves in a rotating fluid*, Prikl. Mat. Mekh. **55** (1991), no. 6, 1051–1055.

[9] O. A. Gilman, R. Grimshaw and Yu. A. Stepanyants, *Approximate analytical and numerical solutions of the stationary Ostrovsky equation.* Stud. Appl. Math. **95** (1995), no. 1, 115–126.

[10] J. Ginibre, Y. Tsutsumi and G. Velo, *Existence and uniqueness of solutions for the generalized Korteweg de Vries equation*, Math. Z. **203** (1990), no. 1, 9–36.

[11] Z. Huo and Y. Jia, *Low-regularity solutions for the Ostrovsky equation*, Proc. Edinb. Math. Soc. (2) 49 (2006), no. 1, 87–100.

[12] T. Kato, *The Cauchy problem for the Korteweg-de Vries equation. Nonlinear partial differential equations and their applications*, College de France Seminar, Vol. I (Paris, 1978/1979), pp. 293–307, Res. Notes in Math. **53** Pitman, Boston, Mass.-London, 1981.

[13] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620.

[14] C. E. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), no. 2, 573–603.

[15] C. E. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), no. 3, 617–633.

[16] F. Linares and A. Milanés, *Local and global well-posedness for the Ostrovsky equation*, J. Differential Equations **222** (2006), no. 2, 325–340.

[17] Y. Liu and V. Varlamov *Stability of solitary waves and weak rotation limit for the Ostrovsky equation*, J. Differential Equations 203 (2004), no. 1, 159–183.

[18] K. Nakanishi, H. Takaoka and Y. Tsutsumi, *Counterexamples to bilinear estimates related with the KdV equation and the nonlinear Schrodinger equation*, IMS Conference on Differential Equations from Mechanics (Hong Kong, 1999). Methods Appl. Anal. **8** (2001), no. 4, 569–578.

[19] L. A. Ostrovsky, *Nonlinear internal waves in a rotating ocean*, Okeanologia **18** (1978), 181–191.

[20] L. A. Ostrovsky and Yu. A. Stepanyants, *Nonlinear surface and internal waves in rotating fluids*, Resarch Reports in Physics, Nonlinear Waves 3, Springer-Verlag, Berlin, Heidelberg, (1990), 106–128.

[21] N. Tzvetkov, *Remark on the local ill-posedness for KdV equation*, C. R. Acad. Sci. Paris Ser. I Math. **329** (1999), no. 12, 1043–1047.

[22] V. Varlamov and Y. Liu, *Cauchy problem for the Ostrovsky equation*, Discrete Contin. Dyn. Syst. 10 (2004), no. 3, 731–753.

ON SOME ELLIPTIC ESTIMATES

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ABSTRACT. We discuss some estimates for an elliptic system with smooth variable coefficients on a domain Ω containing the origin. The concerned estimates are invariant under a domain expansion with scale factor R such that $y = Rx$, $x, y \in \mathbf{R}^n, n \geq 3$ and a real number $R > 1$, provided the coefficients are in a homogeneous Sobolev space. We apply these invariant estimates to a three dimensional Lamé system with variable viscosity coefficients μ and μ' and obtain the local existence and uniqueness of strong solutions of heat-conducting compressible Navier-Stokes equations with density decaying at space infinity or vanishing on nonempty open subsets.

1. ELLIPTIC SYSTEM

We consider the linear system $Lu = f$ defined by

$$(Lu)^\alpha \equiv \sum_{i,j,\beta} \partial_i \left(A_{ij}^{\alpha\beta}(x)\partial_j u^\beta \right) = f^\alpha \qquad \text{in} \quad \Omega. \tag{1.1}$$

Here u is an N dimensional vector field on $\mathbb{R}^n$ for $N \geq 1, n \geq 3$ and $1 \leq i, j \leq n$, $1 \leq \alpha, \beta \leq N$. Ω is either a bounded domain in $\mathbb{R}^n$ with smooth boundary or a usual unbounded domain such as the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^{n-1} \times \mathbb{R}_+$ and an exterior domain with C^2 boundary. We assume the coefficients $A_{ij}^{\alpha\beta}$ are smooth functions of x and further assume that

- (Bound condition) $A_{i,j}^{\alpha,\beta}$ are uniformly continuous on $\overline{\Omega}$ and for some fixed constant Λ

$$\sup_{\overline{\Omega}} |A_{i,j}^{\alpha,\beta}| \leq \Lambda \quad \text{for all} \quad \alpha, \beta, i, j \tag{1.2}$$

- (Elliptic condition) for some fixed constant λ

$$\sum_{i,j,\alpha,\beta} A_{i,j}^{\alpha,\beta}(x)\xi_i^\alpha \xi_j^\beta \geq \lambda|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^{nN} \tag{1.3}$$

Here the gradient ∇ is defined by $\nabla = (\partial_1, \cdots, \partial_n)$, where $\partial_i = \frac{\partial}{\partial x_i}$.

Our main concerns are to find a condition for elliptic estimate of the operator L (defined on an bounded domain containing the origin[1]) independent of domain

[1]We assume this just for simplicity of presentation. We can proceed the domain expansion via translation which is possible because of the uniformity of A.

expansion and to show the invariance. Here by the domain expansion we mean an expansion with respect to R of scaled domain Ω_R. The scaled domain $\Omega_R, R > 1$ is defined by $\Omega_R = \{y : y = Rx, x \in \Omega\}$.

The elliptic estimate we say usually is the following: if $f \in L^q$ for some $1 < q < \infty$ and u is a solution of (1.1) in $W_0^{1,q} \cap W^{2,q}$ for a bounded Ω, then there holds

$$\|\nabla^2 u\|_{L^q} \le C(\|f\|_{L^q} + \|u\|_{L^q}), \tag{1.4}$$

where the constant C depends on $\Lambda, \lambda, n, q, \Omega, \partial\Omega$ and the modulus of continuity of A. For the details of elliptic theory, we refer the readers to the papers and books [1, 5, 7, 8, 11, 12, 13].

Throughout this article, we use the notation for Sobolev space $W^{k,q} = W^{k,q}(\Omega)$, $k \ge 0, 1 \le q \le \infty$ and $H^k = W^{k,2}$. The space $W_0^{k,q}(1 \le q < \infty)$ is the Sobolev space with trace zero on the boundary in $W^{k,q}$ sense which means that for any $u \in W_0^{k,q}$ we can find a sequence of $C^k(\overline{\Omega})$ functions which is continuously zero on the boundary converges to u in $W^{k,q}$.

The estimate (1.4) may not be invariant with respect to the scaling factor R. By the scale $y = Rx$, we have

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le \widetilde{C}(\|f\|_{L^q(\Omega_R)} + R^{-2}\|u\|_{L^q(\Omega_R)}).$$

The constant $\widetilde{C}$ may depend on R. For an estimate independent of the scale factor R, we need to scrutinize the factors on that the constant C depends and find some conditions for the invariance.

Now we introduce a candidate of condition for invariance as follows.

- (Scaling condition)

$$\begin{aligned} &\nabla A_{i,j}^{\alpha,\beta} \in L^r(\Omega) \quad \text{for some} \quad r > n \quad \text{if} \quad q \le n \\ &\nabla A_{i,j}^{\alpha,\beta} \in L^q(\Omega), \quad \text{if} \quad q > n. \end{aligned} \tag{1.5}$$

With the conditions (1.2), (1.3) and (1.5), we have

Theorem 1.1. *Let Ω be a bounded domain containing the origin and with C^2 boundary $\partial\Omega$. If $f \in L^q(\Omega_R)$ for some $1 < q < \infty$ and $u \in W_0^{1,q} \cap W^{2,q}(\Omega_R)$ is a solution of the system* (1.1) *defined on the domain $\Omega_R(R > 1)$. Then the followings hold*

(1) *If $q \le n$, then*

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\left(\|f\|_{L^q(\Omega_R)} + \|\nabla A\|_{L^r(\Omega_R)}^{\frac{r}{r-n}} \|\nabla u\|_{L^q(\Omega_R)} + \|u\|_{W^{1,q}(\Omega_R)}\right). \tag{1.6}$$

(2) *If $q < n$, then*

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\left(\|f\|_{L^q(\Omega_R)} + \|\nabla A\|_{L^r(\Omega_R)}^{\frac{r}{r-n}} \|\nabla u\|_{L^q(\Omega_R)} + \|\nabla u\|_{L^q(\Omega_R)}\right). \tag{1.7}$$

(3) *If $q > n$, then*

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\left(\|f\|_{L^q(\Omega_R)} + \|\nabla A\|_{L^q(\Omega_R)}\|\nabla u\|_{L^\infty(\Omega_R)} + \|u\|_{W^{1,q}(\Omega_R)}\right). \quad (1.8)$$

(4) *If $n = 3$ and $3 < q \le 6$, then*

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\left(\|f\|_{L^q(\Omega_R)} + \|\nabla A\|_{L^q(\Omega_R)}\|\nabla u\|_{L^\infty(\Omega_R)} + \|\nabla u\|_{L^2\cap L^q(\Omega_R)}\right). \quad (1.9)$$

Here the constant C depending only on $\Lambda, \lambda, N, n, q, r, \Omega, \partial\Omega$ and the modulus continuity of A, not on R.

Remark 1.2. *If Ω is the unit ball, then Ω_R is the ball of radius R and the constant C of each estimate in* (1.1) *independent of the radius.*

Remark 1.3. *From the invariant estimates above, we can consider a limit problem $R \to \infty$. It will be interesting to study a sequence can converge to a solution satisfying the system of equations* (1.1) *on an unbounded domain and also the estimates in Theorem 1.1.*

Proof of Theorem 1.1. For the proof, we scale functions with factor R so that $\widetilde{u}(x) = u(Rx)$, $\widetilde{A}(x) = A(Rx)$ and $\widetilde{f}(x) = R^2 f(Rx)$. Let $\widetilde{L}$ be the scaled operator such that

$$\widetilde{L}(\widetilde{u})^\alpha \equiv \sum_{i,j,\beta} \partial_i(\widetilde{A}_{i,j}^{\alpha,\beta}\widetilde{u}^\beta) = \widetilde{f}^\alpha \quad \text{on} \quad \Omega.$$

Fixing Ω, we proceed the well-known interior and boundary estimate. We first consider the interior estimate. If $\widetilde{A}$ is constant, then by the Calderon-Zygmund theory, one can show that there holds for all $v \in W_0^{2,q}(\Omega)$

$$\|\nabla^2 v\|_{L^q} \le C(n, q, \Lambda, \lambda)\|L_0 v\|_{L^q}, \quad (1.10)$$

where $(L_0 v)^\alpha = \sum_{i,j,\beta} \widetilde{A}_{i,j}^{\alpha,\beta}\partial_i\partial_j v^\beta = \sum_\beta \widetilde{A}^{\alpha,\beta}\nabla^2 v^\beta$.

Now we freeze the coefficient $\widetilde{A}$ at point $x_0 \in \Omega'$ and fix $R > 1$, where Ω' is a precompact subset of Ω (for example we can take $\Omega' = \Omega\setminus(B(0,\varepsilon_0)+\partial\Omega)$ for small ε_0, where ε_0 depends only on ρ, the modulus of continuity of A). Let L_0 be the constant coefficient operator given by $L_0 v = \widetilde{A}(x_0)\nabla^2 v$. Suppose v has support in a ball $B(x_0, \frac{\rho}{R}) \subset\subset \Omega$ (hereafter we denote $B_{\frac{\rho}{R}}$ by $B(x_0, \frac{\rho}{R})$). Then we have

$$L_0 v = (\widetilde{A}(x_0) - \widetilde{A})\nabla^2 v + \widetilde{A}\nabla^2 v,$$

and by (1.10)

$$\begin{aligned} \|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} &\le C\|L_0 v\|_{L^q(B_{\frac{\rho}{R}})} \\ &\le C(\sup_{B_{\frac{\rho}{R}}}|\widetilde{A}(x_0) - \widetilde{A}|\|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}). \end{aligned}$$

Since $\widetilde{A}$ is uniformly continuous on $\overline{\Omega'}$, there exists a positive number δ independent of R such that

$$|\widetilde{A}(x_0) - \widetilde{A}(x)| \le \frac{1}{2C} \tag{1.11}$$

if $|x - x_0| < \frac{\delta}{R}$. Actually we can choose uniform δ for any $x_0 \in \Omega'$ from the conditions (1.2) and (1.3). Hence we have

$$\|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} \le C(n,q,\lambda,\Lambda)\|\widetilde{A}\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}, \tag{1.12}$$

provided $\rho \le \delta$.

Choose a smooth cutoff function $\eta \in C_0^2(B_{\frac{\rho}{R}})$ such that $\eta = 1$ on $B_{\frac{\rho}{2R}}$, $\eta = 0$ on $B_{\frac{\rho}{R}} \setminus B_{\frac{3}{4}\frac{\rho}{R}}$ and $|\nabla\eta| \le \frac{8R}{\rho}$, $|\nabla^2\eta| \le \frac{64R^2}{\rho^2}$. Then $v = \eta\widetilde{u} \in W_0^{2,q}(\Omega)$ and we have

$$\begin{aligned}
&\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \le \|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}\\
&\le C\left(\|\eta\widetilde{A}\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}\nabla\eta\nabla\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|A(\nabla^2\eta)\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})}\right)\\
&\le C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\widetilde{A}\nabla\eta\nabla\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}(\nabla^2\eta)\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})}\right)\\
&\le C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\rho})} + \|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{1}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{1}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right),
\end{aligned}$$

provided $\rho \le \delta$.

If $q \le n$, then we use $\|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q} \le \|\nabla\widetilde{A}\|_{L^r}\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}}$. Hence, we have for $q \le n$

$$\begin{aligned}
\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \le C\Bigg(&\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\|_{L^r(\Omega)}\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})}\\
&+\frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\Bigg).
\end{aligned}$$

Since by Sobolev embedding and scaling we have

$$\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})} \le C\left(R^{\frac{n}{r}}\rho^{-\frac{n}{r}}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{u}\|^{1-\frac{n}{r}}_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\|\nabla^2\widetilde{u}\|^{\frac{n}{r}}_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right),$$

where C does not depend on ρ and R, we derive that for some $\varepsilon > 0$ and small ρ

$$\begin{aligned}
\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \le C_\varepsilon\Bigg(&\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \Big(\frac{R^{\frac{n}{r}}}{\rho^{\frac{n}{r}}}\|\nabla\widetilde{A}\|_{L^r(\Omega)} + \|\nabla\widetilde{A}\|^{\frac{r}{r-n}}_{L^r(\Omega)}\Big)\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\\
&+\frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\Bigg) + \varepsilon\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})},
\end{aligned}$$

where $C_\varepsilon = C\varepsilon^{-1}$.

If $q < n$, then since

$$\frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} \le \frac{R}{\rho}\|\widetilde{u}\|_{\frac{nq}{n-q}(B_{\frac{3}{4}\frac{\rho}{R}})} \le C\frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}$$

for some constant C independent of R and ρ, we have

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \leq C_\varepsilon \left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + (\frac{R^{\frac{n}{r}}}{\rho^{\frac{n}{r}}}\|\nabla\widetilde{A}\|_{L^r(\Omega)} + \|\nabla\widetilde{A}\|^{\frac{r}{r-n}}_{L^r(\Omega)})\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right) + \varepsilon\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}.$$

If $q > n$, then we use $\|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q} \leq \|\nabla\widetilde{A}\|_{L^q}\|\nabla\widetilde{u}\|_{L^\infty}$. Hence, we have

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \leq C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\|\nabla\widetilde{u}\|_{L^\infty(\Omega)} + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right).$$

If $n = 3$ and $3 < q \leq 6$, then since

$$\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} \leq C\left(\frac{\rho}{R}\right)^{\frac{3}{q}-\frac{1}{2}}\|\nabla\widetilde{u}\|_{L^2(B_{\frac{3}{4}\frac{\rho}{R}})}$$

for some constant C independent of R and ρ, we have

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \leq C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\|\nabla\widetilde{u}\|_{L^\infty(\Omega)} + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \left(\frac{R}{\rho}\right)^{\frac{3}{2}-\frac{3}{q}}\|\nabla\widetilde{u}\|_{L^2(B_{\frac{3}{4}\frac{\rho}{R}})}\right).$$

Fixing ρ, choose a finite covering $\{B'_k\}_{k=1}^K$ and $\{B_k\}_{k=1}^K$ of Ω' such that $B'_k = B(x_k, \frac{\rho}{2R})$ and $B_k = B(x_k, \frac{3}{4}\frac{\rho}{R})$ with $x_k \in \Omega'$, $\Omega' \subset \cup_k B'_k \subset \Omega$ and $\sum_k \chi_{B_k}(x) \leq c_n$ for some fixed positive number c_n. The number c_n can be taken by 2^n by the finite overlapping property of balls. Hence we have for $q \leq n$

$$\begin{aligned}
\|\nabla^2\widetilde{u}\|^q_{L^q(\Omega')} &\leq \sum_k \|\nabla^2\widetilde{u}\|^q_{L^q(B'_k)} \\
&\leq C_\varepsilon \sum_k \left(\|\widetilde{f}\|^q_{L^q(B_k)} + (\frac{R^{\frac{nq}{r}}}{\rho^{\frac{nq}{r}}}\|\nabla\widetilde{A}\|^q_{L^r(\Omega)} + \|\nabla\widetilde{A}\|^{\frac{rq}{r-n}}_{L^r(\Omega)})\|\nabla\widetilde{u}\|^q_{L^q(B_k)} \right. \\
&\quad \left. +(\frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|^q_{L^q(B_k)} + \frac{R^{2q}}{\rho^{2q}}\|\widetilde{u}\|^q_{L^q(B_k)}\right) + \varepsilon^q \sum_k \|\nabla^2\widetilde{u}\|^q_{L^q(B_k)} \\
&\leq C_\varepsilon c_n \left(\|\widetilde{f}\|^q_{L^q(\Omega)} + (\frac{R^{\frac{nq}{r}}}{\rho^{\frac{nq}{r}}}\|\nabla\widetilde{A}\|^q_{L^r(\Omega)} + \|\nabla\widetilde{A}\|^{\frac{rq}{r-n}}_{L^r(\Omega)})\|\nabla\widetilde{u}\|^q_{L^q(\Omega)} \right. \\
&\quad \left. +\frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|^q_{L^q} + \frac{R^{2q}}{\rho^{2q}}\|\widetilde{u}\|^q_{L^q(\Omega)}\right) + \varepsilon^q\|\nabla^2\widetilde{u}\|^q_{L^q(\Omega)}.
\end{aligned} \tag{1.13}$$

If $q < n$, then

$$\begin{aligned}
&\|\nabla^2\widetilde{u}\|^q_{L^q(\Omega')} \\
&\leq C_\varepsilon c_n \left(\|\widetilde{f}\|^q_{L^q(\Omega)} + (\frac{R^{\frac{nq}{r}}}{\rho^{\frac{nq}{r}}}\|\nabla\widetilde{A}\|^q_{L^r(\Omega)} + \|\nabla\widetilde{A}\|^{\frac{rq}{r-n}}_{L^r(\Omega)})\|\nabla\widetilde{u}\|^q_{L^q(\Omega)} \right. \\
&\quad \left. +\frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|^q_{L^q}\right) + \varepsilon^q\|\nabla^2\widetilde{u}\|^q_{L^q(\Omega)}.
\end{aligned} \tag{1.14}$$

If $q > n$, then we have

$$\|\nabla^2\widetilde{u}\|_{L^q(\Omega')}^q \le C\left(\|\widetilde{f}\|_{L^q(\Omega)}^q + \|\nabla\widetilde{A}\|_{L^q(\Omega)}^q\|\nabla\widetilde{u}\|_{L^\infty(\Omega)}^q + \frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|_{L^q(\Omega)}^q + \frac{R^{2q}}{\rho^{2q}}\|\widetilde{u}\|_{L^q(\Omega)}^q\right). \tag{1.15}$$

If $n = 3$ and $3 < q \le 6$, then

$$\|\nabla^2\widetilde{u}\|_{L^q(\Omega')}^q \le C\left(\|\widetilde{f}\|_{L^q(\Omega)}^q + \|\nabla\widetilde{A}\|_{L^q(\Omega)}^q\|\nabla\widetilde{u}\|_{L^\infty(\Omega)}^q + \frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|_{L^2\cap L^q(\Omega)}^q\right). \tag{1.16}$$

Now we let Ω_{ε_0} be the $2\varepsilon_0$ neighborhood of the boundary $\partial\Omega$, that is, $\Omega_{\varepsilon_0} = \partial\Omega + B(0, 2\varepsilon_0)$. For sufficiently small ε so that $\Omega\backslash\Omega' \subset \Omega_{\varepsilon_0}$. After domain flattening, we can proceed a boundary estimate via reflection (for this we need the żero trace condition in $W^{1,q}$). Actually we have for $q \le n$

$$\begin{aligned}
&\|\nabla^2\widetilde{u}\|_{L^q(\Omega_{\varepsilon_0})}^q \\
&\le C_{\varepsilon,\partial\Omega}\left(\|\widetilde{f}\|_{L^q(\Omega)}^q + (\frac{R^{\frac{nq}{r}}}{\rho^{\frac{nq}{r}}}\|\nabla\widetilde{A}\|_{L^r(\Omega)}^q + \|\nabla\widetilde{A}\|_{L^r(\Omega)}^{\frac{rq}{r-n}})\|\nabla\widetilde{u}\|_{L^q(\Omega)}^q \right. \\
&\qquad \left. + \frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|_{L^q}^q + \frac{R^{2q}}{\rho^{2q}}\|\widetilde{u}\|_{L^q(\Omega)}^q\right) + \varepsilon^q\|\nabla^2\widetilde{u}\|_{L^q(\Omega)}^q
\end{aligned} \tag{1.17}$$

and for $q > n$

$$\|\nabla^2\widetilde{u}\|_{L^q(\Omega_{\varepsilon_0})}^q \le C_{\partial\Omega}\left(\|\widetilde{f}\|_{L^q(\Omega)}^q + \|\nabla\widetilde{A}\|_{L^q(\Omega)}^q\|\nabla\widetilde{u}\|_{L^\infty(\Omega)}^q + \frac{R^q}{\rho^q}\|\nabla\widetilde{u}\|_{L^q(\Omega)}^q + \frac{R^{2q}}{\rho^{2q}}\|\widetilde{u}\|_{L^q(\Omega)}^q\right), \tag{1.18}$$

where $C_{\partial\Omega}$ is a constant depending on the diffeomorphism from a portion of $\partial\Omega$ to the unit ball centered at origin. The cases $q < n$ and $3 < q \le 6$ for $n = 3$ can be treated similarly to (1.14) and (1.16), respectively.

Summing every estimate case by case, and converting them back into the unscaled variables, we easily obtain the estimates (1.6)–(1.9). This completes the proof. □

Remark 1.4. *If A is constant, then in view of the proof for constant case we deduce that*

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\|f\|_{L^q(\Omega_R)}$$

for some $1 < q < \infty$ and any $R > 1$.

2. Lamé system

We apply the elliptic estimate in Theorem 1.1 to the Lamé system which is defined by

$$Lu \equiv -\mathrm{div}(2\mu du) - \nabla(\mu' \mathrm{div} u) = f \quad \text{on} \quad \Omega \subset \mathbb{R}^3, \tag{2.1}$$

where u is a n dimensional vector field, μ and μ' are smooth viscosity coefficients, du is the deformation tensor $\frac{1}{2}(\nabla u + \nabla^t u)$. If we assume that

$$0 < \lambda \leq \mu, \mu + \mu' \leq \Lambda, \tag{2.2}$$

then since the leading coefficient $A_{i,j}^{\alpha,\beta}(x)$ is

$$\mu(x)\delta_{i,j}\delta_{\alpha,\beta} + (\mu(x) + \mu'(x))\delta_{i,\alpha}\delta_{j,\beta},$$

A satisfies the conditions (1.2) and (1.3). Hence if we impose the scaling condition that

$$\begin{aligned} &\nabla\mu, \nabla\mu' \in L^r(\Omega) \quad \text{for some} \quad 3 < r \leq 6 \quad \text{if} \quad q \leq 3, \\ &\nabla\mu, \nabla\mu' \in L^q(\Omega), \quad \text{if} \quad 3 < q \leq 6, \end{aligned} \tag{2.3}$$

then from Theorem 1.1 we have the following.

Theorem 2.1. *Let Ω be a bonded domain containing the origin and with C^2 boundary. If $u \in W_0^{1,q} \cap W^{2,q}(\Omega_R)$ is a solution of (2.1) with $f \in L^q(\Omega_R)$ and μ, μ' satisfying (2.2) and (2.3), then for any $R > 1$*

(1) if $q = 2$ and $f \in L^2(\Omega_R)$, then

$$\begin{aligned} &\|\nabla^2 u\|_{L^2(\Omega_R)} \\ &\leq C \big(\|f\|_{L^2(\Omega_R)} + \|\nabla u\|_{L^2(\Omega_R)} \\ &\qquad + (\|\nabla\mu\|_{L^r(\Omega_R)} + \|\nabla\mu'\|_{L^r(\Omega_R)})^{\frac{r}{r-3}} \|\nabla u\|_{L^2(\Omega_R)} \big). \end{aligned} \tag{2.4}$$

(2) If $3 < q \leq 6$, then

$$\begin{aligned} &\|\nabla^2 u\|_{L^q(\Omega_R)} \\ &\leq C \big(\|f\|_{L^q(\Omega_R)} + \|\nabla u\|_{L^2 \cap L^q(\Omega_R)} \\ &\qquad + (\|\nabla\mu\|_{L^q(\Omega_R)} + \|\nabla\mu'\|_{L^q(\Omega_R)}) \|\nabla u\|_{L^\infty(\Omega_R)} \big). \end{aligned} \tag{2.5}$$

Here the constant C depending only on $\Lambda, \lambda, N, n, q, r, \Omega, \partial\Omega$ and the modulus continuity of μ, μ', not on R.

For constants μ and μ', see [2] and also see [3, 9] for Stokes system with variable coefficient.

The elliptic estimate of Lamé system can be applied to the following heat conducting compressible Navier-Stokes equations:

$$\begin{aligned} p &= p(\rho, \rho\theta), \\ \rho_t + \mathrm{div}(\rho u) &= 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu \mathrm{d}u) - \nabla(\mu' \mathrm{div} u) + \nabla p &= \rho f, \\ c_v\left((\rho\theta)_t + \mathrm{div}\,(\rho u\theta)\right) + (\gamma c_v \rho\theta - p)\mathrm{div}\, u - \mathrm{div}\,(\kappa\nabla\theta) & \\ = 2\mu|\mathrm{d}u|^2 + \mu'(\mathrm{div}\, u)^2 + \rho h. & \end{aligned} \tag{2.6}$$

We consider the system (2.6) supplemented with the initial and boundary value conditions:

$$\begin{aligned} (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0) \quad &\text{in} \quad \Omega, \\ (u, \theta) = (0, 0) \quad \text{on} \quad &\partial\Omega \times [0, T], \\ (\rho(t,x), u(t,x), \theta(t,x)) \to (0,0,0) \ \text{ as } \ |x| \to \infty, \ &(t,x) \in (0,T) \times \Omega. \end{aligned} \tag{2.7}$$

Here we denote by ρ, u, p and θ the unknown density, velocity, pressure and temperature fields for the fluid, respectively. The second and third equations in (2.6) are derived by the mass conservation and balance of momentum of the fluid and the forth equation is derived from the balance of energy

$$(\rho e)_t + \mathrm{div}\,(\rho e u) - \mathrm{div}(\kappa\nabla\theta) = 2\mu|\mathrm{d}u|^2 + \mu'(\mathrm{div}\, u)^2 + \rho h$$

by using the relationship $c_v = \frac{\partial e}{\partial \theta}$, where $e = e(\rho, \theta)$ is the internal energy. We assume that the viscosity coefficients $\mu = \mu(\rho, \theta)$, $\mu' = \mu'(\rho, \theta)$, specific heat at constant volume $c_v = c_v(\rho, \theta)$ and heat conductivity $\kappa = \kappa(\rho, \theta)$ are *positive* functions of ρ and θ. The factor $\gamma c_v \rho\theta - p$ follows from a result of the first law of thermodynamics law $\rho^2 \frac{\partial e}{\partial \rho} = p - \alpha K_\theta \theta$ and the relation $\alpha \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_p = \frac{\gamma c_v \rho}{K_\theta}$, where α is the thermal expansion coefficient and K_θ is the isothermal bulk modulus. The known fields f and h denote a given external force and heat source per unit mass. Finally, $(0, T) \times \Omega$ is the time-space domain for the evolution of the fluid, where T is a finite positive number and Ω is an *unbounded* domain such as the whole space $\mathbb{R}^3$ and an exterior domain with smooth boundary.

The concerning problem to this system is to show the unique solvability of strong solution for the initial density ρ_0 which is positive on Ω and decays at space infinity. Owing to the decay of the initial density, the solution ρ may decay at space infinity. Hence it is hard to expect the regularity of u and u_t which are crucial to the proof of uniqueness. To overcome this difficulty, we localize the problem and use the elliptic estimates independent of domain expansion established in Theorem 2.1. In doing this, to avoid technical difficulty, we need additional condition for the coefficients such that $c_v = c_v(\rho, \rho\theta)$, $\mu = \mu(\rho, \rho\theta)$, $\mu' = \mu'(\rho, \rho\theta)$, $\kappa = \kappa(\rho, \rho\theta)$ and assume a compatibility condition to compete with the decay of density, which can be described as follows.

$$\begin{cases} -\operatorname{div}(2\mu_0 du_0) - \nabla(\mu_0' \operatorname{div} u_0) + \nabla p_0 = \rho_0^{\frac{1}{2}} g_1 \\ -\operatorname{div}(\kappa_0 \nabla\theta_0) + p_0 \operatorname{div} u_0 - 2\mu_0 |du_0|^2 - \mu_0' (\operatorname{div} u_0)^2 = \rho_0^{\frac{1}{2}} g_2 \end{cases} \quad \text{in } \Omega \qquad (2.8)$$

some $(g_1, g_2) \in L^2$, where $\mu_0 = \mu(\rho_0, \rho_0\theta_0)$ and so on.

Now we state final result.

Theorem 2.2. *Assume that the data* $(\rho_0, u_0, \theta_0, h, f)$ *satisfies the regularity condition*

$$\rho_0 > 0, \quad \rho_0 \in L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}, \quad (u_0, \theta_0) \in D_0^1 \cap D^2,$$
$$(h, f) \in C([0,T]; L^2) \cap L^2([0,T]; L^q) \quad \text{and} \quad (h_t, f_t) \in L^2(0,T; H^{-1})$$

for some q, $3 < q \le 6$ *and the compatibility condition* (2.8). *Then there exist a small time* $T_* > 0$ *and a unique strong solution* (ρ, u, θ) *to the initial boundary value problem for* (2.6) *such that*

$$\begin{gathered} \rho \in C([0,T_*]; L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T_*]; L^2 \cap L^q), \\ (u, \theta) \in C([0,T_*]; D_0^1 \cap D^2) \cap L^2(0,T_*; D^{2,q}), \\ (u_t, \theta_t) \in L^2(0,T_*; D_0^1) \quad \text{and} \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T_*; L^2). \end{gathered} \qquad (2.9)$$

Here we used the following notations for homogeneous Sobolev space (see [6])

$$\begin{gathered} D^{k,r} = \{v \in L^1_{loc}(\Omega) : |v|_{D^{k,r}} < \infty\}, \quad D^k = D^{k,2}, \\ D_0^1 = \{v \in L^6(\Omega) : |v|_{D_0^1} < \infty \text{ and } v = 0 \text{ on } \partial\Omega\}, \\ H_0^1 = D_0^1 \cap L^2, \quad |v|_{D^{k,r}} = |\nabla^k v|_{L^r} \text{ and } |v|_{D_0^1} = |\nabla v|_{L^2}. \end{gathered}$$

The proof follows the similar line to the one in [3]. Its details will be seen in the forthcoming paper [4].

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.*, Comm. Pure Appl. Math., 17 (1964), 35-92.

[2] Y. Cho, H. J. Choe and H. Kim, *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl., 83 (2004), 243-275.

[3] Y. Cho and H. Kim, *Existence result for heat-conducting viscous incompressible fluids with vacuum*, Hokkaido Univ. Preprint Series in Math. ♯742 (2005).

[4] Y. Cho, T. Ozawa and Y. Shim, *Elliptic estimates independent of domain expansion* in preparation.

[5] A. Douglis and L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math., 8 (1955), 503-530.

[6] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, in "Springer Tracts in Natural Philosophy 38," Springer-verlag, New York, 1994.

[7] M. Giaquinta, An Introduction to the Regularity Theory for Nonlinear Elliptic Systems, Birkhauser, Basel, 1993..

[8] D. Gilbarg and N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.

[9] M. Giaquinta and G. Modica, *Nonlinear systems of the type of the stationary Navier-Stokes system*, J. Reine Angew. Math., 330 (1982), 173-214.

[10] P. -L. Lions, Mathematical Topics in Fluid Mechanics Vol. 2, Clarendon Press, Oxford, 1998.

[11] L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math., 8 (1955), 648-674.

[12] L. Nirenberg, *Estimates and existence of solutions of elliptic equations*, Comm. Pure Appl. Math., 9 (1956), 509-530..

[13] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Pisa, 13 (1959), 115-162.

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LARGE TIME BEHAVIOR OF SOLUTIONS TO THE GENERALIZED BURGERS EQUATIONS

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1 Introduction

This paper is concerned with large time behavior of the global solutions to the generalized Burgers equations:

$$u_t + (f(u))_x = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $u_0 \in L^1(\mathbb{R})$ and $f(u) = \frac{b}{2}u^2 + \frac{c}{3}u^3$ with $b \neq 0$, $c \in \mathbb{R}$. The subscripts t and x stand for the partial derivatives with respect to t and x, respectively. It is well-known that the solution of (1.1) and (1.2) tends to a nonlinear diffusion wave defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}}\chi_*\left(\frac{x}{\sqrt{1+t}}\right), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b}\frac{(e^{b\delta/2}-1)e^{-\frac{x^2}{4}}}{\sqrt{\pi} + (e^{b\delta/2}-1)\int_{x/2}^{\infty} e^{-y^2}dy}, \tag{1.4}$$

$$\delta \equiv \int_{\mathbb{R}} u_0(x)dx. \tag{1.5}$$

By the Hopf -Cole transformation in Hopf [4] and Cole [1], we see that it is a solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2}\chi^2\right)_x = \chi_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.6}$$

satisfying

$$\int_{\mathbb{R}} \chi(x,0)dx = \delta. \tag{1.7}$$

In Kawashima [8] and Nishida [11], if $u_0 \in L^1_\beta(\mathbb{R}) \cap H^1(\mathbb{R})$ for some $\beta \in (0,1)$ and $\|u_0\|_{H^1} + \|u_0\|_{L^1}$ is small, then we have

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1+\alpha}(\|u_0\|_{H^1} + \|u_0\|_{L^1_\beta}), \quad t \ge 0, \tag{1.8}$$

where $\alpha = (1-\beta)/2$. Here, for an integer $k \ge 0$, $H^k(\mathbb{R})$ denotes the space of functions $u = u(x)$ such that $\partial_x^l u$ are L^2-functions on $\mathbb{R}$ for $0 \le l \le k$, endowed with the norm $\|\cdot\|_{H^k}$, while $L^1_\beta(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u\|_{L^1_\beta} \equiv \int_{\mathbb{R}} |u|(1+|x|)^\beta dx < \infty$. However, the estimate (1.8) leads to a natural question whether it is possible to take $\alpha = 0$ in (1.8) for the extreme case $\beta = 1$ or not. An attempt to answer the question can be found in Matsumura and Nishihara [10]. To be more precise, we put $w_0(x) = \exp(-\frac{b}{2}\int_{-\infty}^x u_0(y)dy) - \exp(-\frac{b}{2}\int_{-\infty}^x \chi(y,0)dy)$ and we supposed that $w_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\|w_0\|_{H^2} + \|w_0\|_{L^1} + \|u_0\|_{L^1}$ is small. Then the estimate

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}\log(2+t)(\|w_0\|_{H^2} + \|w_0\|_{L^1} + |\delta|^{\frac{3}{2}}), \quad t \ge 0 \tag{1.9}$$

holds instead of (1.8). The aim of this paper is to show that the estimate (1.9) is actually sharp, unless $\delta = 0$ or $c = 0$. Indeed, the second asymptotic profile of large time behavior of the solutions is given by

$$\begin{aligned} V(x,t) &\equiv -\frac{cd}{12\sqrt{\pi}}V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(2+t), \quad t \ge 0, \quad x \in \mathbb{R}, \\ &= \partial_x(\eta_1(x,t)G(x,1+t))(-\frac{cd}{3})\log(2+t) \end{aligned} \tag{1.10}$$

where

$$V_*(x) \equiv (b\chi_*(x) - x)e^{-\frac{x^2}{4}}\eta_*(x), \quad \eta_*(x) \equiv \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi_*(y)dy\right),$$

and

$$d \equiv \int_{\mathbb{R}} \eta_*^{-1}(y)\chi_*^3(y)dy, \tag{1.11}$$

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \tag{1.12}$$

$$\eta_1(x,t) = \eta_*(\frac{x}{\sqrt{1+t}}). \tag{1.13}$$

Bisides, we can take the initial data from $L_1^1(\mathbb{R}) \cap H^1(\mathbb{R})$, analogously to the works of [8], [11]. And we set, for $k \geq 0$, $E_{k,\beta} \equiv \|u_0\|_{H^k} + \|u_0\|_{L^1_\beta}$. Then we have the following result.

Theorem 1.1. *Assume that* $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ *and* $E_{1,0}$ *is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution* $u(x,t)$ *satisfying* $u \in C^0([0,\infty); H^1)$ *and* $\partial_x u \in L^2(0,\infty; H^1)$. *Moreover, if* $u_0 \in L_1^1(\mathbb{R}) \cap H^1(\mathbb{R})$ *and* $E_{1,1}$ *is small, then the solution satisfies the estimate*

$$\|u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t)\|_{L^\infty} \leq CE_{1,1}(1+t)^{-1}, \quad t \geq 1. \tag{1.14}$$

Here $\chi(x,t)$ *is defined by* (1.3), *while* $V(x,t)$ *is defined by* (1.10).

REMARK 1.2. In Liu [9], the initial value problem for the Burgers equations (1.1) and (1.2) is studied, provided $c = 0$ implicity at page 42. After the proof of Theorem 2.2.1, it is mentioned, without proof, that if we assume $(1+|x|)^2|u_0(x)| \leq \tilde{\delta}$ and $\tilde{\delta}$ is small, then the estimate

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \leq C\tilde{\delta}(1+t)^{-1}, \quad t \geq 1$$

holds. However, from our result, the above estimate fails true for the case $c \neq 0$.

We remark that the estimate similar to (1.14) was obtained for other types of Burgers equation such as KdV-Burgers in Hayashi and Naumkin [3] and Kaikina and Ruiz-Paredes [5], and Benjamin-Bona-Mahony-Burgers in Hayashi, Kaikina and Naumkin [2].

2 Basic estimates

We deal with the following linearized equations which coressponds to (3.1), (3.2) below:

$$z_t = z_{xx} - (b\chi z)_x, \quad t > 0, \quad x \in \mathbb{R}, \tag{2.1}$$

$$z(x,0) = z_0(x). \tag{2.2}$$

The explicit representation formula (2.4) below plays a crucial role in our analysis. For the proof, see [6].

Lemma 2.1. *If we set*

$$U[w](x,t,\tau) = \int_{\mathbb{R}} \partial_x (G(x-y,t-\tau)\eta_1(x,t))\eta_1^{-1}(y,\tau) \int_{-\infty}^{y} w(\xi)d\xi dy, \quad 0 \le \tau < t, \quad x \in \mathbb{R}, \tag{2.3}$$

then the solutions for (2.1) *and* (2.2) *is given by*

$$z(x,t) = U[z_0](x,t,0), \quad t > 0, \quad x \in \mathbb{R}. \tag{2.4}$$

Next we introduce the decay estimates (2.5) and (2.6) below for the homogenous equation (2.1). Basic idea of the proof goes back to the estimate for the semigroup $e^{t\Delta}$ in [8]. For the proof, see [6].

Lemma 2.2. *Let $\beta \in [0,1]$, k be a positive interger and $p \in [1,\infty]$. Assume that $z_0 \in L^1_\beta(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \le Ct^{-(1-\frac{1}{p}+\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t > 0 \tag{2.5}$$

holds for any $l = 0, 1, \cdots, k$.

Lemma 2.3. *Let k be a positive interger. Assume that $z_0 \in H^k(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^2} \le C(1+t)^{-(\frac{1}{4}+\frac{l}{2})}\|z_0\|_{L^1} + Ce^{-t}\|z_0\|_{H^l}, \quad t > 0 \tag{2.6}$$

holds for any $l = 0, 1, \cdots, k$.

From Lemma 2.2 and Lemma 2.3, we get the following uniform estimate.

Corollary 2.4. *Let k be a positive integer. Assume that $z_0 \in L^1_1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot, t, 0)\|_{L^2} \le C E_{l,1}(1+t)^{-(\frac{3}{4}+\frac{l}{2})}, \quad t > 0$$

holds for any $l = 0, 1, \cdots, k$.

Next we introduce the decay estimate (2.7) below for the imhomogenous equation in the same way as Lemma 3.6 of [7]. For the proof, see [6].

Lemma 2.5. *Let k be a positive integer. Suppose $w \in C^0(0,\infty; H^k) \cap C^0(0,\infty; H^k_1)$. Then the estimate*

$$\begin{aligned}
&\left\|\partial_x^l \int_0^t U[\partial_x w(\tau)](\cdot, t, \tau)d\tau\right\|_{L^2} \\
\le\ & C\int_0^{t/2} (1+t-\tau)^{-(\frac{3}{4}+\frac{l}{2})}\|w(\cdot,\tau)\|_{L^1}d\tau \\
&+C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{l-m}{2}}\|\partial_x^m w(\cdot,\tau)\|_{L^1}d\tau \\
&+C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\|\partial_x^m w(\cdot,\tau)\|^2_{L^2}d\tau\right)^{\frac{1}{2}} \qquad (2.7)
\end{aligned}$$

holds for any $l = 0, 1, \cdots, k$. Here, $H^k_1(\mathbb{R}) \equiv \{f : L^1_{loc}(\mathbb{R}) | \ \|f\|_{H^k_1} \equiv \sum_{m=0}^{k}\|\partial_x^m f\|_{L^1} < \infty\}$.

3 Proof of Theorem 1.1

In order to prove our result, we introduce the following auxililly problem:

$$v_t = v_{xx} - (b\chi v)_x - (\frac{c}{3}\chi^3)_x, \quad t > 0, \quad x \in \mathbb{R}, \qquad (3.1)$$

$$v(x, 0) = 0. \qquad (3.2)$$

We have the following decay estimate for the solution $v(x,t)$ to the above problem. For the proof, see [6].

Lemma 3.1. *Let $l \geq 0$ be an integer. Then we have*

$$\|\partial_x^l v(\cdot,t)\|_{L^2} \leq C|\delta|^3(1+t)^{-(\frac{3}{4}+\frac{l}{2})}\log(2+t), \quad t \geq 0. \tag{3.3}$$

To prove Theorem 1.1, it is sufficent to show Proposition 3.2 and Proposition 3.3 below. For the proof, see [6].

Proposition 3.2. *Let $k \geq 1$ be an integer. Assume that $u_0 \in L^1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $E_{k,0}$ is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution $u(x,t)$ satisfying $u \in C^0([0,\infty);H^k)$ and $\partial_x u \in L^2(0,\infty;H^k)$. Moreover, if $u_0 \in L^1_1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $E_{k,1}$ is small, then the estimate*

$$\|\partial_x^l \left(u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t)\right)\|_{L^2} \leq CE_{k,1}(1+t)^{-(\frac{3}{4}+\frac{l}{2})}, \quad t \geq 0,$$

holds for $l \leq k$. In particuler,

$$\|u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t)\|_{L^\infty} \leq CE_{1,1}(1+t)^{-1}, \quad t \geq 0.$$

Here $\chi(x,t)$ is defined by (1.3)*, while $v(x,t)$ is the solution for the problem* (3.1) *and* (3.2)*.*

From Proposition 3.2 and Lemma 3.1, we can see that main term of the asymptotic expansion of $u(x,t) - \chi(x,t)$ as $t \to \infty$ is determined by the linear Cauchy problem (3.1) and (3.2).

Proposition 3.3. *Assume that $|\delta| \leq 1$. Then the estimate*

$$\|v(\cdot,t) - V(\cdot,t)\|_{L^\infty} \leq C|\delta|^3(1+t)^{-1}, \quad t \geq 1 \tag{3.4}$$

holds. Here, $v(x,t)$ is the solution for the problem (3.1) *and* (3.2)*, while $V(x,t)$ is defined by* (1.10)*.*

Although the similer estimate was shown by Lemma 3 in [5], but we need to modify the proof of it in order to avoid the logarithmic term in the right-hand side.

Finally, we mention, by heuristic considerration, how to deduce the asymptotic profile $V(x,t)$. To begin with, we consider the heat equation:

$$u_t = u_{xx} \quad t > 0, \quad x \in \mathbb{R}, \tag{3.5}$$

$$u(x,0) = u_0(x), \tag{3.6}$$

where $u_0(x) \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} u_0(x)dx \equiv \delta$. It is well-known that the solution of (3.5) and (3.6) tend to $\delta G(x,t)$. Next we consider the linearized problem (2.1) and (2.2) of (3.1) and (3.2) above:

$$\begin{aligned} &z_t = z_{xx} - (b\chi z)_x, \quad t > \tau, \quad x \in \mathbb{R}, \\ &z(x,\tau) = z_0(x). \end{aligned}$$

If we put

$$r(x,t) = \int_{-\infty}^{x} z(y,t)dy, \tag{3.7}$$

then we see from (2.1), (2.2) that $r(x,t)$ satisfies

$$\begin{aligned} &r_t = r_{xx} - b\chi r_x, \quad t > \tau, \quad x \in \mathbb{R}, \\ &r(x,\tau) = \int_{-\infty}^{x} z_0(y)dy. \end{aligned}$$

Then a direct computation yields

$$\left(\frac{r(x,t)}{\eta_1(x,t)}\right)_t = \left(\frac{r(x,t)}{\eta_1(x,t)}\right)_{xx}, \tag{3.8}$$

where η_1 is defined by (1.12). Therefore we have

$$\frac{r(x,t)}{\eta_1(x,t)} \sim M_1 G(x,t-\tau), \tag{3.9}$$

where $M_1 = \int_{\mathbb{R}} \eta_1^{-1}(x,\tau) \int_{-\infty}^{x} z(y,0)dydx$. Hence (3.7), (2.3) and (3.9) yield

$$z(x,t) = U[z_0](x,t,\tau) \sim M_1(\eta_1(x,t)G(x,t-\tau))_x. \tag{3.10}$$

Now, by the Duhamel principle, the solution $v(x,t)$ of (3.1) and (3.2) is expressed as

$$v(x,t) = \int_0^t U[\partial_x(-\frac{c}{3}\chi^3(\tau))](x,t,\tau)d\tau. \tag{3.11}$$

Note that (1.12), (1.3) and (1.11) yield $\int_{\mathbb{R}} \eta_1^{-1}(\xi,\tau)\chi^3(\xi,\tau)d\xi = d(1+\tau)^{-1}$. It follows from (3.10) and (3.11) that

$$\begin{aligned}
v(x,t) &\sim \int_0^{t/2} \partial_x(\eta_1(x,t)G(x,t-\tau)) \int_{\mathbb{R}} \eta_1^{-1}(y,\tau) \int_{-\infty}^{y} \partial_\xi(-\frac{c}{3}\chi^3(\xi,\tau))d\xi dy d\tau \\
&\sim \int_0^{t/2} \partial_x(\eta_1(x,t)G(x,t-\tau)) \int_{\mathbb{R}} \eta_1^{-1}(y,\tau)(-\frac{c}{3}\chi^3(y,\tau))dy d\tau \\
&\sim \partial_x(\eta_1(x,t)G(x,t)) \int_0^{t/2} \int_{\mathbb{R}} \eta_1^{-1}(y,\tau)(-\frac{c}{3}\chi^3(y,\tau))dy d\tau \\
&\sim \partial_x(\eta_1(x,t)G(x,t+1))(-\frac{cd}{3}) \log(2+\tau) = V(x,t).
\end{aligned}$$

Thus we find the second asymptotic profile $V(x,t)$.

References

[1] J.D. Cole: *On a quasi-linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math. **IX** (1951), 225–236

[2] N. Hayashi, E.I. Kaikina and P.I. Naumkin: *Large time asymptotics for the BBM-Burgers equation*, to appear in Ann. H. Inst. Poincaré.

[3] N. Hayashi and P.I. Naumkin: *Asymptotics for the Korteweg-de Vries-Burgers Equation*, to appear in Acta Math. Sin. Engl. Ser.

[4] E. Hopf: *The partial differential equation $u_t + uu_x = \mu u_{xx}$*, Comm. Pure Appl. Math. **3** (1950), 201–230

[5] E.I. Kaikina and H.F. Ruiz-Paredes: *Second term of asymptotics for KdVB equation with large initial data*, Osaka J. Math. **42** (2005), 407–420.

[6] M. Kato: *Large time behavior of solutions to the generalized Burgers equations*, Osaka University Research Reports in Mathematics **06-05** (2006).

[7] S. Kawashima: *The asymptotic equivalence of the Broadwell model equation and its Navier-Stokes model equations*, Japan J. Math. (N.S.) **7** (1981), 1–43.

[8] S. Kawashima: *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), 164–194.

[9] T.-P. Liu: *Hyperbolic and Viscous Conservation Laws*, in: CBMS-NSF Regional Conference Series in Applied Mathematics, **Vol. 72**, SIAM, Philadelphia, PA, 2000.

[10] A. Matsumura and K. Nishihara: *Global Solutions of Nonlinear Differential Equations –Mathematical Analysis for compressible viscous fluids–*, (in Japanese) Nippon–Hyoron-sha, Tokyo, 2004.

[11] T. Nishida: *Equations of motion of compressible viscous fluids*, in: Pattern and Waves (Ed. T. Nishida, M. Mimura, H. Fujii), Amsterdam, Tokyo: Kinokuniya/ North-Holland, 1986, 97–128