Inductive Inference of Approximations for Recursive Concepts

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Abstract

This paper provides a systematic study of inductive inference of indexable concept classes in learning scenarios where the learner is successful if its final hypothesis describes a finite variant of the target concept, i.e., learning with anomalies. Learning from positive data only and from both positive and negative data is distinguished.

The following learning models are studied: learning in the limit, finite identification, set-driven learning, conservative inference, and behaviorally correct learning.

The attention is focused on the case that the number of allowed anomalies is finite but not *a priori* bounded. However, results for the special case of learning with an *a priori* bounded number of anomalies are presented, too. Characterizations of the learning models with anomalies in terms of finite tell-tale sets are provided. The observed varieties in the degree of recursiveness of the relevant tell-tale sets are already sufficient to quantify the differences in the corresponding learning models with anomalies. Finally, a complete picture concerning the relations of all models of learning with and without anomalies mentioned above is derived.

1 Introduction

Induction constitutes an important feature of learning. The corresponding theory is called inductive inference. Inductive inference may be characterized as the study of systems that map evidence on a target concept into hypotheses about it. Investigating scenarios in which the sequence of hypotheses stabilizes to an accurate and finite description of the target concept is of particular interest. Precise definitions of the notions evidence, stabilization, and accuracy go back to Gold [11] who introduced the model of learning in the limit.

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The present paper deals with inductive inference of indexable classes of recursive concepts (indexable classes, for short). A concept class is said to be an *indexable class* if it possesses an effective enumeration with uniformly decidable membership. Angluin [2] started the systematic study of learning indexable concept classes. Her pioneering paper and succeeding publications (cf. Zeugmann and Lange [22], for an overview) attracted attention, since most natural concept classes are indexable. For example, the class of all context-sensitive, context-free, regular, and pattern languages as well as the set of all Boolean formulae expressible by a monomial, a k-CNF, a k-DNF, and a k-decision list constitute indexable classes.

As usual, we distinguish learning from positive data and learning from both positive and negative data, synonymously called learning from *text* and *informant*, respectively. A text for a concept c is an infinite sequence of elements of c such that every element from c appears eventually. Alternatively, an informant is an infinite sequence of elements exhausting the underlying learning domain that are classified with respect to their containment in the target concept.

An algorithmic learner, henceforth called *inductive inference machine* (abbr. IIM), takes as input larger and larger initial segments of a text (an informant) and outputs, from time to time, a hypothesis about the target concept. The set of all admissible hypotheses is called *hypothesis space*. When learning of indexable classes is studied, it is only natural to require the hypothesis space to be an *indexed family*, i.e., an effective enumeration with uniformly decidable membership of a (possibly) larger concept class. This assumption underlies almost all studies (cf., e.g., Angluin [2], Zeugmann and Lange [22]). However, sometimes we also consider hypotheses spaces that are not indexed families.

Gold's [11] original model requires the sequence of hypotheses to converge to a hypothesis *correctly* describing the target concept. However, from a viewpoint of potential applications, it suffices in most cases that the final hypothesis approximates the target concept sufficiently well. To capture this aspect, Blum and Blum [5] introduced a quite natural refinement of Gold's [11] model. In their setting of learning recursive functions with anomalies, it is admissible that the learner's final hypothesis may differ from the target function at finitely many data points. Case and Lynes [7] adapted this model to language learning.

Learning with anomalies has been intensively studied in the context of learning recursive functions and recursively enumerable languages (cf., e.g., Case and Smith [8], Daley [9], Fulk [10], Kinber and Zeugmann [13], Royer [18], Jain *et al.* [12] and the references therein). Preliminary results concerning the learnability of indexable classes with anomalies can be found in Tabe and Zeugmann [19]. Note that Baliga *et al.* [3] studied the learnability of indexable classes with anomalies, too. However, unlike almost all other work on learning indexable classes, Baliga *et al.* [3] allow the use of *arbitrary* hypothesis spaces including those not having a uniformly decidable membership problem. Therefore, the results from Baliga *et al.* [3] do not directly translate into the setting mainly considered in this paper, i.e., learning indexable classes with respect to indexed families as hypotheses spaces.

The present paper provides a systematic study of learning indexable concept classes with anomalies. We investigate the following variants of Gold-style concept learning: learning in the limit, finite identification, set-driven learning, conservative inference, and behaviorally correct learning. We relate the resulting models of learning with anomalies to one another as well as to the corresponding versions of learning without anomalies. The main focus of attention is put to the case that the number of allowed anomalies is finite but not *a priori* bounded. However, we also present results dealing with the case that the number of allowed anomalies is *a priori* bounded.

Finally, we mention prototypical results. In case of learning with anomalies from positive data, the learning power of set-driven learners, conservative learners, and unconstrained IIMs does coincide. In contrast, when anomaly-free learning is considered, conservative inference and set-driven learning are strictly less powerful. A further difference to learning without anomalies is obtained by showing that behaviorally correct learning with anomalies is strictly more powerful than learning in the limit with anomalies. Furthermore, if the number of allowed anomalies is finite but not *a priori* bounded, then there is no need to use arbitrary hypothesis spaces for designing superior behaviorally correct learners, thus refining the corresponding results by Baliga *et al.* [3]. However, if the number of anomalies is *a priori* bounded, it is advantageous to use arbitrary hypothesis spaces. For establishing these results, we provided characterizations of the corresponding models of learning with anomalies in terms of finite tell-tale sets (cf. Angluin [2]). The observed varieties in the degree of recursiveness of the relevant tell-tale sets are already sufficient to quantify the differences in the corresponding learning models with anomalies.

2 Preliminaries

2.1 Basic Notions

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all natural numbers and let $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. By $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we denote Cantor's pairing function. Let A and B be sets. As usual, $A \triangle B$ denotes the symmetrical difference of A and B, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$. We write $A \# B$ to indicate that $A \triangle B \neq \emptyset$. For all $a \in \mathbb{N}$, $A =^a B$ iff $card(A \triangle B) \leq a$, while $A =^* B$ iff $card(A \triangle B) < \infty$.

Any recursively enumerable set $\mathcal{X}$ is called a *learning domain*. We fix any recursive enumeration $(w_j)_{j\in\mathbb{N}}$ of $\mathcal{X}$. By $\wp(\mathcal{X})$ we denote the power set of $\mathcal{X}$. Let $\mathcal{C} \subseteq \wp(\mathcal{X})$ and let $c \in \mathcal{C}$. We refer to $\mathcal{C}$ and c as to a *concept class* and a *concept*, respectively. Sometimes we identify a concept c with its characteristic function, i.e., we write $c(x) = 1$, if $x \in c$, and $c(x) = 0$, otherwise.

We study the learnability of indexable concept classes (cf. Angluin [2]). A class of non-empty concepts $\mathcal{C}$ is said to be an *indexable concept class* iff there are an effective enumeration $(c_j)_{j\in\mathbb{N}}$ of all and only the concepts in $\mathcal{C}$ and a recursive function f such that, for all $j \in \mathbb{N}$ and all $x \in \mathcal{X}$, $f(j, x) = c_j(x)$ holds. By $\mathcal{IC}$ we denote the collection of all indexable classes.

Let $(T_j)_{j\in\mathbb{N}}$ be a family of finite sets. $(T_j)_{j\in\mathbb{N}}$ is said to be *uniformly recursively enumerable* (*recursively enumerable*, for short) iff there is an effective procedure that, on every input $j \in \mathbb{N}$, enumerates the finite set T_j. Moreover, $(T_j)_{j\in\mathbb{N}}$ is said to be *uniformly recursively generable* (*recursively generable*, for short) iff there is an effective procedure that, on every input $j \in \mathbb{N}$, generates all elements of the finite set T_j and *stops*.

2.2 Gold-Style Concept Learning

Let $\mathcal{X}$ be a learning domain, let $c \subseteq \mathcal{X}$ be a concept, and let $t = (x_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements from c such that $\{x_n \mid n \in \mathbb{N}\} = c$. Then t is said to be a *text* for c. By $\mathit{Text}(c)$ we denote the set of all texts for c. Let t be a text and let $y \in \mathbb{N}$. Then, t_y denotes the initial segment of t of length $y + 1$, and we set $\mathit{content}(t_y) = \{x_n \mid n \leq y\}$. Furthermore, let $\sigma = x_0, \ldots, x_{n-1}$ be any finite sequence. Then we use $|\sigma|$ to denote the *length* n of σ, and let $\mathit{content}(\sigma)$ denote the content of σ. Let c be a concept; then we write $\mathit{SegText}(c)$ for the set of all finite sequences of elements from c. Additionally, let t be a text and let τ be a finite sequence; then we use $\sigma \diamond t$ and $\sigma \diamond \tau$ to denote the sequence obtained by *concatenating* σ onto the front of t and τ, respectively. Furthermore, we write $\sigma \sqsubset \tau$ and $\sigma \sqsubset t$ in case that σ constitutes a proper initial segment of τ and t, respectively.

Let $(w_j)_{j\in\mathbb{N}}$ be the fixed recursive enumeration of the learning domain $\mathcal{X}$, let $c \subseteq \mathcal{X}$ be a concept, and let m be the least number such that $w_m \in c$. Then, the *canonical text* $t^c = (x_n)_{n\in\mathbb{N}}$ for c is defined as follows: $x_0 = w_m$. For all $n \in \mathbb{N}$, if $w_{n+1} \in c$ then $x_{n+1} = w_{n+1}$, otherwise $x_{n+1} = x_n$. Furthermore, for every indexable class $\mathcal{C}$ we set $\mathit{Text}(\mathcal{C}) = \bigcup_{c\in\mathcal{C}} \mathit{Text}(c)$.

Next, for every finite set $c \subseteq \mathcal{X}$, we define the *canonical arrangement* of c to be the result of the following procedure. First, compute the shortest initial segment of the canonical text t^c of c that contains all elements of c. Delete all repetitions and output the resulting sequence.

As in Gold [11], we define an *inductive inference machine* (abbr. IIM) to be an algorithmic mapping from initial segments of texts to $\mathbb{N} \cup \{?\}$. Thus, an IIM either outputs a hypothesis, i.e., a number encoding a certain computer program, or "?," a special symbol representing the case where the machine outputs "no conjecture." Note that an IIM, when learning a target class $\mathcal{C}$, is required to produce an output on every initial segment of all texts in $Text(\mathcal{C})$.

The numbers output by an IIM are interpreted with respect to a suitably chosen *hypothesis space* $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$. Since we exclusively deal with the learnability of classes $\mathcal{C} \in \mathcal{IC}$, unless otherwise stated, we assume $\mathcal{H}$ to be an indexed family, i.e., all h_j describe non-empty concepts and membership is uniformly decidable in $\mathcal{H}$. When an IIM M outputs some number j, we interpret it to mean that M hypothesizes h_j.

Let $\mathcal{C} \in \mathcal{IC}$, let $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ be a hypothesis space, and let $a \in \mathbb{N} \cup \{*\}$. If $\mathcal{C} = \{h_j \mid j \in \mathbb{N}\}$, then $\mathcal{H}$ is said to be a *class preserving* hypothesis space for $\mathcal{C}$ (cf. Lange and Zeugmann [16]). Furthermore, $\mathcal{H}$ is called *class admissible* hypothesis space for $\mathcal{C}$ with respect to a provided that, for every $c \in \mathcal{C}$, there is an index j such that $h_j =^a c$ (cf. Tabe and Zeugmann [19]). If $a = 0$, then $\mathcal{H}$ is a *class comprising* hypothesis space for $\mathcal{C}$ (cf. Lange and Zeugmann [16]).

We define convergence of IIMs as usual. Let t be a text and let M be an IIM. The sequence $(M(t_y))_{y\in\mathbb{N}}$ of M's hypotheses *converges* to a number j iff all but finitely many terms of it are equal to j.

Now, we are ready to define *learning in the limit.*

Definition 1 (Gold [11], Case and Lynes [7]). *Let $\mathcal{C} \in \mathcal{IC}$, let c be a concept, let $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ be a hypothesis space, and let $a \in \mathbb{N} \cup \{*\}$. An IIM M* ***$Lim^aTxt_{\mathcal{H}}$–learns c*** *iff, for every $t \in Text(c)$, there is a $j \in \mathbb{N}$ with $h_j =^a c$ such that the sequence $(M(t_y))_{y\in\mathbb{N}}$ converges to j.*

Furthermore, M ***$Lim^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$*** *iff M $Lim^aTxt_{\mathcal{H}}$–learns each $c' \in \mathcal{C}$.*

Finally, ***Lim^aTxt*** *denotes the collection of all classes $\mathcal{C}' \in \mathcal{IC}$ for which there are a hypothesis space $\mathcal{H}' = (h'_j)_{j\in\mathbb{N}}$ and an IIM M' that $Lim^aTxt_{\mathcal{H}'}$–learns $\mathcal{C}'$.*

Subsequently, we write $LimTxt$ instead of Lim^0Txt. We adopt this convention to all learning types defined below.

In general, it is not decidable whether or not an IIM has already converged on a text t for the target concept c. Adding this requirement to the above definition results in *finite learning* (cf. Gold [11]). The corresponding learning type is denoted by Fin^aTxt, where again $a \in \mathbb{N} \cup \{*\}$.

Definition 2 (Gold [11]). *Let $\mathcal{C}$ be an indexable class, let c be a concept, let*

$\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ *be a hypothesis space, and let* $a \in \mathbb{N}\cup\{*\}$. *An IIM M* ***Fin***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns c*** *iff, for every* $t \in \textit{Text}(c)$, *there exist* $j, m \in \mathbb{N}$ *such that* $c =^a h_j$, $M(t_r) = ?$ *for all* $r < m$, *and* $M(t_y) = j$ *for all* $y \geq m$.

Furthermore, M ***Fin***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns*** $\boldsymbol{\mathcal{C}}$ *iff M* $\textit{Fin}^a\textit{Txt}_{\mathcal{H}}$*–learns each* $c' \in \mathcal{C}$.

For every $a \in \mathbb{N} \cup \{*\}$, the resulting learning type $\textit{Fin}^a\textit{Txt}$ is defined analogously to Definition 1.

Next, we define *conservative* IIMs. Conservative IIMs maintain their actual hypothesis at least as long as they have not seen data contradicting it.

Definition 3 (Angluin [2]). *Let* $\mathcal{C} \in \mathcal{IC}$, *let c be a concept, let* $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ *be a hypothesis space, and let* $a \in \mathbb{N}\cup\{*\}$. *An IIM M* ***Consv***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns c*** *iff M* $\textit{Lim}^a\textit{Txt}_{\mathcal{H}}$*–learns c and, for all* $t \in \textit{Text}(c)$ *and for any two consecutive hypotheses* $k = M(t_y)$ *and* $j = M(t_{y+1})$, *if* $k \in \mathbb{N}$ *and* $k \neq j$, *then* $\textit{content}(t_{y+1}) \not\subseteq h_k$.

Finally, M ***Consv***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns*** $\boldsymbol{\mathcal{C}}$ *iff M* $\textit{Consv}^a\textit{Txt}_{\mathcal{H}}$*–learns each* $c' \in \mathcal{C}$.

For every $a \in \mathbb{N} \cup \{*\}$, the resulting learning type $\textit{Consv}^a\textit{Txt}$ is defined analogously to Definition 1.

Next, we define *set-driven* IIMs. Intuitively speaking, the output of a set-driven IIM depends exclusively on the *content* of its input, thereby ignoring the order as well as the frequency in which the examples occur.

Definition 4 (Wexler and Culicover [20]). *Let* $\mathcal{C} \in \mathcal{IC}$, *let c be a concept, let* $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ *be a hypothesis space, and let* $a \in \mathbb{N} \cup \{*\}$. *An IIM M* ***Sdr***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns c*** *iff M* $\textit{Lim}^a\textit{Txt}_{\mathcal{H}}$*–learns c and, for all* $t, t' \in \textit{Text}(\mathcal{C})$ *and for all* $n, m \in \mathbb{N}$, *if* $\textit{content}(t_n) = \textit{content}(t'_m)$, *then* $M(t_n) = M(t'_m)$.

Furthermore, M ***Sdr***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns*** $\boldsymbol{\mathcal{C}}$ *iff M* $\textit{Sdr}^a\textit{Txt}_{\mathcal{H}}$*–learns each* $c' \in \mathcal{C}$.

For every $a \in \mathbb{N} \cup \{*\}$, the resulting learning type $\textit{Sdr}^a\textit{Txt}$ is defined analogously to Definition 1.

Next, we relax Definition 1 by allowing the learner to converge *semantically.* That is, now it suffices that all but finitely many hypotheses do correctly approximate the target concept. The resulting learning type is referred to as *behaviorally correct learning.*

Definition 5 (Bārzdiņš [4], Case and Lynes [7]). *Let* $\mathcal{C} \in \mathcal{IC}$, *let c be a concept, let* $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ *be a hypothesis space, and let* $a \in \mathbb{N} \cup \{*\}$. *An IIM M* ***Bc***$^{\boldsymbol{a}}$***Txt***$_{\boldsymbol{\mathcal{H}}}$***–learns c*** *iff, for every* $t \in \textit{Text}(c)$ *and for all but finitely many* $y \in \mathbb{N}$, $h_{M(t_y)} =^a c$.

Furthermore, M $\boldsymbol{Bc^aTxt_{\mathcal{H}}}$*–learns* $\boldsymbol{\mathcal{C}}$ *iff M* $Bc^aTxt_{\mathcal{H}}$*–learns each* $c' \in \mathcal{C}$*.*

For every $a \in \mathbb{N} \cup \{*\}$, the resulting learning type Bc^aTxt is defined analogously to Definition 1.

Finally, we define *consistent* IIMs (cf. Gold [11]). Let $\mathcal{C} \in \mathcal{IC}$, let $\mathcal{H}$ be a hypothesis space, and let M be an IIM. Then, M is said to be consistent for $\mathcal{C}$ with respect to $\mathcal{H}$ provided that, for all $t \in \mathit{Text}(\mathcal{C})$ and all $y \in \mathbb{N}$, if $M(t_y) = k$ for some $k \in \mathbb{N}$, then $\mathit{content}(t_y) \subseteq h_k$. Intuitively speaking, the hypotheses of a consistent IIM correctly reflect the data on which they were built upon. For all $a \in \mathbb{N} \cup \{*\}$ and all learning types Lt^aTxt defined above, we let $c\text{-}Lt^aTxt$ denote the collection of all classes $\mathcal{C} \in \mathcal{IC}$ for which there are a hypothesis space $\mathcal{H}$ and a consistent IIM that $Lt^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$.

3 Learning from Positive Data

In this section, we study the power and the limitations of the various models of learning with anomalies. We relate these models to one another as well as to the different models of anomaly-free learning. We are mainly interested in the case that the number of allowed anomalies is finite but not *a priori* bounded. For giving an impression of how the overall picture changes when the number of allowed anomalies is *a priori* bounded, we also present results for this case.

Proposition 1 summarizes the known relations between the considered models of anomaly-free learning from text.

Proposition 1 (Gold [11], Lange and Zeugmann [17]).
$\mathit{FinTxt} \subset \mathit{SdrTxt} = \mathit{ConsvTxt} \subset \mathit{LimTxt} = \mathit{BcTxt} \subset \mathcal{IC}$.

In the setting of learning recursive functions, the first observation made when comparing learning in the limit with anomalies to behaviorally correct inference was the *error correcting power* of Bc-learners, i.e., $Ex^* \subseteq Bc$ (cf., e.g., Case and Smith [8]). Interestingly enough, this result did not translate into the setting of learning recursively enumerable languages from positive data. But still, a certain error correcting power is preserved in this setting, since $\mathit{LimTxt}^a \subseteq \mathit{BcTxt}^b$ provided $a \leq 2b$ (cf. Case and Lynes [7]).

When comparing learning with and without anomalies in our setting of learning indexable classes, it turns out that even finite inference may become more powerful than *Bc*-learning.

Theorem 1. $\mathit{Fin}^1\mathit{Txt} \setminus \mathit{BcTxt} \neq \emptyset$.

Proof. Let $c = \{b\}^*$ and, for all $k \in \mathbb{N}$, let $c_k = c \setminus \{b^k\}$. Let $\mathcal{C}$ be the collection of c and of all infinite concepts c_k. It is folklore that $\mathcal{C} \notin \mathit{LimTxt}$, and thus $\mathcal{C} \notin \mathit{BcTxt}$ (cf. Proposition 1). Finally, since, for all $k \in \mathbb{N}^+$, $c =^1 c_k$, an IIM that always guesses c witnesses $\mathcal{C} \in \mathit{Fin}^1\mathit{Txt}$. □

However, the opposite is also true. For instance, PAT, the well-known class of all pattern languages [1] (cf. Angluin [1]), witnesses the even stronger result:

Theorem 2. $\mathit{ConsvTxt} \setminus \mathit{Fin}^*\mathit{Txt} \neq \emptyset$.

Proof. Recall that $\mathit{PAT} \in \mathit{ConsvTxt}$ (cf. Angluin [2]). Furthermore, PAT contains a singleton language L as well as an infinite language L' with $L \subset L'$. Since every initial segment of a text for L constitutes an initial segment of a text for L' and since $L \neq^* L'$, no IIM can $\mathit{Fin}^*\mathit{Txt}$–learn L and L'. □

3.1 The Case of a Finite Number of Anomalies

As we shall see, the relations between the standard learning models change considerably, if it is no longer required that the learner almost always outputs hypotheses correctly describing the target concept. The following picture displays the established coincidences and differences by relating the models of learning with anomalies to one another and by ranking them in the hierarchy of the models of anomaly-free learning.

$$\begin{array}{ccccccccccc} \mathit{Fin}^*\mathit{Txt} & \subset & \mathit{Sdr}^*\mathit{Txt} & = & \mathit{Consv}^*\mathit{Txt} & = & \mathit{Lim}^*\mathit{Txt} & \subset & \mathit{Bc}^*\mathit{Txt} & \subset & \mathcal{IC} \\ \cup & & \cup & & \cup & & \cup & & \cup & & \\ \mathit{FinTxt} & \subset & \mathit{SdrTxt} & = & \mathit{ConsvTxt} & \subset & \mathit{LimTxt} & = & \mathit{BcTxt} & \subset & \mathcal{IC} \end{array}$$

To achieve the overall picture, we establish characterizations of all models of learning with a finite but not *a priori* bounded number of anomalies. On the one hand, we present characterizations in terms of finite tell-tale sets. On the other hand, we prove that some of the learning models coincide.

The characterizations of $\mathit{Lim}^*\mathit{Txt}$ and $\mathit{Fin}^*\mathit{Txt}$ are similar to the known characterizations of LimTxt and FinTxt (cf. Angluin [2], Lange and Zeugmann [15]).

[1] Let $\Sigma \neq \emptyset$ be a finite alphabet and let X be a countably infinite set of variables such that $\Sigma \cap X = \emptyset$. Then, every string $\pi \in (\Sigma \cup X)^+$ constitutes a pattern. The language $L(\pi)$ defined by pattern π is the set of all strings that can be obtained by replacing the variables in π by strings from Σ^+. Thereby, each occurrence of the same variable has to be replaced by the same string. Now, PAT is the set of all languages $L(\pi)$, where π is a pattern.

Proposition 2 (Tabe and Zeugmann [19]). *For all $\mathcal{C} \in \mathcal{IC}$: $\mathcal{C} \in Lim^*Txt$ iff there are an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a recursively enumerable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq c_j$,*
(2) *for all $j, k \in \mathbb{N}$, if $T_j \subseteq c_k \subseteq c_j$, then $c_k =^* c_j$.*

Theorem 3. *For all $\mathcal{C} \in \mathcal{IC}$: $\mathcal{C} \in Fin^*Txt$ iff there are an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a recursively generable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq c_j$,*
(2) *for all $j, k \in \mathbb{N}$, if $T_j \subseteq c_k$, then $c_k =^* c_j$.*

Proof. Necessity. Assume that a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and an IIM M that $Fin^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$ are given. Moreover, let $(c_j)_{j\in\mathbb{N}}$ be any indexing of $\mathcal{C}$. The family $(T_j)_{j\in\mathbb{N}}$ is defined as follows.

Let $j \in \mathbb{N}$ and let t^{c_j} be the canonical text of c_j. Since M finitely infers c_j, there exists a least $y \in \mathbb{N}$ such that $M(t^{c_j}_y) = m$ for some $m \in \mathbb{N}$. We set $T_j = content(t^{c_j}_y)$.

We have to show that $(T_j)_{j\in\mathbb{N}}$ satisfies the Properties (1) and (2). By construction, (1) is obviously fulfilled. For proving (2), let $j, k \in \mathbb{N}$ such that $T_j \subseteq c_k$. Due to our construction, there is an initial segment of c_j's canonical text t^{c_j}, say $t^{c_j}_y$, such that $content(t^{c_j}_y) = T_j$ and $M(t^{c_j}_y) = m$. Since M finitely learns c_j, we know $h_m =^* c_j$. Because of $T_j \subseteq c_k$, $t^{c_j}_y$ is also an initial segment of some text t for c_k. Taking into account that M finitely infers c_k from t and that $M(t_y) = m$, we get $h_m =^* c_k$, too.

Sufficiency. Let $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ be the hypothesis space such that $h_j = c_j$ for all $j \in \mathbb{N}$. It suffices to show that there is an IIM M that $Fin^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$. So, let $c \in \mathcal{C}$, let $t \in Text(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y do the following:
If $y = 0$ or $M(t_{y-1}) = ?$, goto (A). Otherwise, output $j = M(t_{y-1})$.
(A) For $j = 0, \dots, y$, generate T_j and test whether or not $T_j \subseteq content(t_y)$.
If there is a j passing the test, output the minimal one. Else, output "?."

One directly sees that M learns as required. □

In contrast to Proposition 1, when a finite but not *a priori* bounded number of errors in the final hypothesis is allowed, conservative IIMs become exactly as powerful as unconstrained IIMs.

Theorem 4. *$Lim^*Txt = Consv^*Txt$.*

Proof. By definition, $Consv^*Txt \subseteq Lim^*Txt$ is obvious. For the opposite direction, let $\mathcal{C} \in Lim^*Txt$, let $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ be a hypothesis space and let M be an IIM that $Lim^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$. We have to construct an IIM M' that witnesses $\mathcal{C} \in Consv^*Txt$. The conservative IIM M' uses the following hypothesis space $\mathcal{H}'$. For all $j \in \mathbb{N}$ and $x \in \mathcal{X}$, we set $h'_{j,x} = h_j \setminus \{x\}$. Let $\mathcal{H}'$ be the canonical enumeration of all those concepts $h'_{j,x}$.

Let $c \in \mathcal{C}$, let $t = (x_j)_{j\in\mathbb{N}}$ be a text for c, and let $y \in \mathbb{N}$.

IIM M': "On input t_y do the following:
If $M(t_y) = ?$, output ?. Otherwise, determine $j = M(t_y)$ and output the canonical index of h'_{j,x_0} in $\mathcal{H}'$."

By construction, M' is conservative. Since M converges on t to a hypothesis describing a finite variant of the target concept c, M' will do as well. □

The conservative IIM M' used above always outputs a hypothesis that definitely contradicts the data seen so far, and thus M' is inconsistent. Nevertheless, this slightly unconventional behavior guarantees that M' exclusively performs justified mind changes. Naturally, the question arose whether or not one can simulate the given IIM M by a learner that is both conservative and consistent. The affirmative answer is provided by Theorem 7. Before proving it, we need the following results which may be interesting in their own right.

We start with a technical lemma needed below and later.

Lemma 1. *Let $\mathcal{C} \in \mathcal{IC}$, let $a \in \mathbb{N} \cup \{*\}$, let $\mathcal{H}$ be a hypothesis space, and let M be an IIM witnessing $\mathcal{C} \in Sdr^aTxt_{\mathcal{H}}$. Then one can effectively construct a hypothesis space $\widehat{\mathcal{H}} = (\hat{h}_j)_{j\in\mathbb{N}}$ and an IIM $\widehat{M}$ such that*

(1) *$\widehat{M}$ is total,*
(2) *Let $\mathcal{X}$ be the underlying learning domain. Then $\widehat{M}$ outputs a consistent hypothesis on every finite sequence $\sigma \in SegText(\mathcal{X})$,*
(3) *$\widehat{M}$ $Sdr^aTxt_{\widehat{\mathcal{H}}}$–learns $\mathcal{C}$,*
(4) *for all $c \in \mathcal{C}$ and for all $t \in Text(c)$, if $(\widehat{M}(t_x))_{x\in\mathbb{N}}$ converges to z then $c \subseteq \hat{h}_z$.*

Proof. The hypothesis space $\mathcal{H}$ may not have a superset of some concept in $\mathcal{C}$ and may not contain a consistent hypothesis for some $\sigma \in SegText(\mathcal{X})$. This problem is solved by mixing $\mathcal{H}$ with $\widetilde{\mathcal{H}}$ and all finite sets over the learning domain $\mathcal{X}$. So, let us fix any enumeration of all finite sets over $\mathcal{X}$ such that membership in f_j is uniformly decidable for all $j \in \mathbb{N}$ and such that the cardinality of f_j is computable from j for all $j \in \mathbb{N}$. Then, we define $\widetilde{\mathcal{H}}$ to be any fixed indexing $(\tilde{h}_{\langle j,z\rangle})_{j,z\in\mathbb{N}}$ such that $\tilde{h}_{\langle j,z\rangle} = h_j \cup f_z$ and such that membership in $\tilde{h}_{\langle j,z\rangle}$ is uniformly decidable for all $j, z \in \mathbb{N}$.

The desired hypothesis space $\widehat{\mathcal{H}}$ is defined as follows. For all $j \in \mathbb{N}$, we set

$$\hat{h}_j = \begin{cases} h_{\frac{j}{3}}, & \text{if } j \equiv 0 \bmod 3 \\ \tilde{h}_{\frac{j-1}{3}}, & \text{if } j \equiv 1 \bmod 3 \\ f_{\frac{j-2}{3}}, & \text{if } j \equiv 2 \bmod 3\ . \end{cases}$$

Furthermore, we fix any enumeration $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ such that membership in c_j is uniformly decidable for all $j \in \mathbb{N}$. Now we are ready to define the desired IIM $\widehat{M}$. Let $c \subseteq \mathcal{X}$ be any concept, let $t \in \mathit{Text}(c)$, and let $x \in \mathbb{N}$.

IIM $\widehat{M}$: "On input t_x, execute Stage x.
Stage x: Determine $n_x = \mathit{card}(\mathit{content}(t_x))$; goto (1).

(1) For $i = 0, \ldots, n_x$ check whether or not $\mathit{content}(t_x) \subseteq c_i$. If such an i has been found, goto (2).
Otherwise, search the least j such that $\mathit{content}(t_x) = f_j$ and output $3j + 2$.

(2) If $M(t_x) = ?$, then goto (3). Otherwise, let $j_x = M(t_x)$. If $\mathit{content}(t_x) \subseteq h_{j_x}$ then output $3j_x$. In case $\mathit{content}(t_x) \not\subseteq h_{j_x}$ determine the least m such that $\mathit{content}(t_x) = f_m$ and output the canonical index of $\tilde{h}_{\langle j_x,m\rangle}$ in $\widehat{H}$.

(3) Find the least j such that $\mathit{content}(t_x) = f_j$ and output $3j + 2$."

It remains to show that $\widehat{M}$ satisfies Properties (1) through (4). First, let $c \subseteq \mathcal{X}$ be any concept, let $t \in \mathit{Text}(c)$, and let $x \in \mathbb{N}$. By construction, membership is uniformly decidable in $(c_j)_{j\in\mathbb{N}}$ and $(f_j)_{j\in\mathbb{N}}$. Thus, Instruction (1) is effectively executable. If $\widehat{M}$ outputs a hypothesis in Instruction (1), it is clearly consistent. If $\widehat{M}$ enters Instruction (2), then t_x is also an initial segment of some text for some concept $c \in \mathcal{C}$. Thus, $M(t_x)$ is defined. By construction, every hypothesis output in Instructions (2) or (3) is consistent, too. Since membership in $(h_j)_{j\in\mathbb{N}}$ is uniformly decidable, the tests in Instruction (2) are effectively executable. Hence, M is total and consistent. This proves Assertions (1) and (2). By assumption, M $\mathit{Sdr}^a\mathit{Txt}$–learns every $c \in \mathcal{C}$ with respect to $\mathcal{H}$. Since $\widehat{M}$ uses exclusively M and $\mathit{content}(t_x)$ for computing its hypothesis, it is set-driven, too.

Second, let $c \in \mathcal{C}$, let $t \in \mathit{Text}(c)$, and let $x \in \mathbb{N}$. It remains to show Assertions (3) and (4). Since we already know that $\widehat{M}$ is set-driven, it suffices to show that $\widehat{M}$ $\mathit{Lim}^a\mathit{Txt}$–learns $\mathcal{C}$.

If the target concept c is finite and for all $i \le \mathit{card}(c)$ we have $c \not\subseteq c_i$, then for all sufficiently large x the IIM $\widehat{M}$ determines its hypothesis on input t_x in Instruction (1), and we are already done.

If the target concept c is infinite or finite such that there is an $i \leq card(c)$ with $c \subseteq c_i$, then for all sufficiently large x the IIM $\widehat{M}$ determines its hypothesis on input t_x in Instruction (2).

By assumption, there exist $\hat{x}$ and z such that $M(t_x) = j_x = z$ for all $x \geq \hat{x}$ and $h_z =^a c$. If additionally $c \subseteq h_z$, we are done again. Otherwise, we know that $card(c \setminus h_z) \leq a$. Thus, $\widehat{M}$ performs at most a additional mind changes by combining h_z with $content(t_x)$. Finally $\widehat{M}$ converges to the canonical index $\hat{z}$ of $\tilde{h}_{\langle z,m \rangle}$ in $\widehat{\mathcal{H}}$, where m is the index of the finite set f_m such that $f_m = content(t_{x'})$, where x' is the least $x \geq \hat{x}$ with $c \setminus h_z \subseteq content(t_{x'})$. Consequently, $c =^a \hat{h}_{\hat{z}}$ and $c \subseteq \hat{h}_{\hat{z}}$. This proves Assertions (3) and (4). □

Theorem 5. *$Sdr^aTxt \subseteq c\text{-}Consv^aTxt$ for all $a \in \mathbb{N} \cup \{*\}$.*

Proof. Let $a \in \mathbb{N} \cup \{*\}$ and let $\mathcal{C} \in Sdr^aTxt$. Let $(c_j)_{j \in \mathbb{N}}$ be an indexing of $\mathcal{C}$ such that, for all $c \in \mathcal{C}$, there are infinitely many j with $c_j = c$. Moreover, let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space, and let M be a set-driven IIM that $Lim^aTxt_{\mathcal{H}}$–infers $\mathcal{C}$. Let $\mathcal{X}$ be the underlying learning domain.

By Lemma 1 we may assume that M and $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ are chosen in a way such that M always outputs a consistent hypothesis when fed any finite sequence $\sigma \in SegText(\mathcal{X})$ and that M when fed any text t for any $c \in \mathcal{C}$, converges to an index z such that $h_z =^a c$ and $c \subseteq h_z$.

Before defining the wanted conservative IIM M', we specify a suitable hypothesis space $\hat{\mathcal{H}} = (\hat{h}_{\langle i,j,k \rangle})_{i,j,k \in \mathbb{N}}$.

For the sake of readability, in the following, we consider the given set-driven IIM M to be a learning device which receives finite sets of strings as input instead of finite sequences. Let $(F_j)_{j \in \mathbb{N}}$ denote any effective repetition-free enumeration of all finite subsets of $\mathcal{X}$. We assume that, given any finite $F \subseteq \mathcal{X}$, we may effectively determine F's index $\#(F)$ in the enumeration $(F_j)_{j \in \mathbb{N}}$, i.e., $\#(F) = n$ with $F_n = F$. Let $(w_j)_{j \in \mathbb{N}}$ be the fixed enumeration of all elements in $\mathcal{X}$. Moreover, for all $c \subseteq \mathcal{X}$ and all $m \in \mathbb{N}$, we denote by $c\restriction^m$ the concept $\{w_z \mid z \leq m,\, w_z \in c\}$.

Let $i, j, k \in \mathbb{N}$. If $F_j \not\subseteq h_i \cap c_k$ or $M(F_j) \neq i$, we set $\hat{h}_{\langle i,j,k \rangle} = \{w_0\}$. Otherwise, for all $z \in \mathbb{N}$, we let $w_z \in \hat{h}_{\langle i,j,k \rangle}$ iff (i) or (ii) is fulfilled, where

(i) $w_z \in F_j$.
(ii) $w_z \notin F_j$, $w_z \in h_i \cap c_k$ and, for all $V \subseteq h_i^z \cap c_k\restriction^z$, $M(F_j \cup V) = i$.

Note that, by construction, $\hat{h}_{\langle i,j,k \rangle}$ is finite or it equals $h_i \cap c_k$. Moreover, if $\hat{h}_{\langle i,j,k \rangle} \neq \{w_0\}$, then $F_j \subseteq \hat{h}_{\langle i,j,k \rangle} \subseteq h_i \cap c_k$.

Since M is total and $(c_j)_{j\in\mathbb{N}}$ is an indexing of $\mathcal{C}$ and membership is uniformly decidable in $\mathcal{H}$, we know that $\hat{h}_{\langle i,j,k\rangle}$ is recursive. Hence, membership is uniformly decidable in $\hat{\mathcal{H}}$, too.

Next, we show that $\hat{\mathcal{H}}$ is a class comprising hypothesis space for $\mathcal{C}$. Let $c \in \mathcal{C}$ and let $k \in \mathbb{N}$ with $c_k = c$. Since M is set-driven IIM and M $\mathit{Lim}^a\mathit{Txt}$–learns c, there has to be a finite set $F \subseteq c$ such that, for all finite sets $V \subseteq c$, $M(F) = M(F \cup V) = i$, and $h_i =^a c$. Since M is consistent, $c \subseteq h_i$, and therefore $\hat{h}_{\langle i,\#(F),k\rangle} = h_i \cap c_k = c$.

Furthermore, $\hat{\mathcal{H}}$'s definition immediately implies:

Fact 1. *Let* $i, j, k \in \mathbb{N}$ *and let* V *be some finite subset of* $\mathcal{X}$*. Then, we have: If* $M(F_j) = i$*,* $F_j \cup V \subseteq c_k \cap h_i$*, and* $M(F_j \cup V) \neq i$*, then* $V \not\subseteq \hat{h}_{\langle i,j,k\rangle}$*.*

Fact 2. *Let* $i, j, k \in \mathbb{N}$ *and let* V *be some finite subset of* $\mathcal{X}$*. Then, we have: If* $M(F_j) = i$*,* $F_j \cup V \subseteq c_k \cap h_i$*, and* $M(F_j \cup V) \neq i$*, then* $\hat{h}_{\langle i,j,k\rangle}$ *is finite.*

Now, we are ready to define an IIM M' that c-$\mathit{ConsvTxt}_{\hat{\mathcal{H}}}$–learns $\mathcal{C}$. Let $c \in \mathcal{C}$, $t \in \mathit{Text}(c)$, and $y \in \mathbb{N}$.

IIM M': "On input t_y do the following:

If $y = 0$, then compute $i = M(\mathit{content}(t_y))$, $j = \#(\mathit{content}(t_y))$, and the least k with $\mathit{content}(t_y) \subseteq c_k$. Output $\langle i, j, k\rangle$. Otherwise, goto (A).

(A) Let $M'(t_{y-1}) = \langle i, j, k\rangle$. Test whether or not $\mathit{content}(t_y) \subseteq \hat{h}_{\langle i,j,k\rangle}$. If it is, output $\langle i, j, k\rangle$. Otherwise, compute $i' = M(\mathit{content}(t_y))$, $j' = \#(\mathit{content}(t_y))$ and the least $k' > k$ with $\mathit{content}(t_y) \subseteq c_{k'}$. Output $\langle i', j', k'\rangle$."

By definition, M' is consistent and it outputs a hypothesis in every step. Moreover, M' exclusively performs justified mind changes. Thus, M' is also conservative. Next, we show that M' learns c from text t. We distinguish two cases.

Case 1. c is finite.

Let y' be the least index such that $\mathit{content}(t_{y'}) = c$ and let $y \leq y'$ be the least index with $M'(t_y) = M'(t_{y'})$. Let $M'(t_y) = \langle i, j, k\rangle$. By definition, $i = M(\mathit{content}(t_y))$, $j = \#(\mathit{content}(t_y))$, and $\mathit{content}(t_y) \subseteq c_k$. Since $\mathit{content}(t_y) \subseteq \mathit{content}(t_{y'}) \subseteq \hat{h}_{\langle i,j,k\rangle} \subseteq h_i \cap c_k$, we obtain $M(\mathit{content}(t_y)) = M(\mathit{content}(t_{y'})) = i$ (cf. Fact 1). Since M is a set-driven and consistent IIM that learns c, we know that $h_i =^a c$ and $c \subseteq h_i$. Finally, since $\mathit{content}(t_{y'}) \subseteq \hat{h}_{\langle i,j,k\rangle}$ and since, by construction, $\hat{h}_{\langle i,j,k\rangle} \subseteq h_i$, we may conclude $\hat{h}_{\langle i,j,k\rangle} =^a c$, and thus we are done.

Case 2. c is infinite.

This part of the proof relies on the following claim.

Claim 1. *For all* $y, z \in \mathbb{N}$*, if* $M'(t_y) = z$ *and* $c \subseteq \hat{h}_z$*, then* $c =^a \hat{h}_z$*.*

Without loss of generality, let y be the least index with $M'(t_y) = z$. By definition, $z = \langle i, j, k\rangle$, where $i = M(\mathit{content}(t_y))$, $j = \#(\mathit{content}(t_y))$, and $\mathit{content}(t_y) \subseteq c_k$. Suppose that $c \subseteq \hat{h}_z$ and $c \neq^a \hat{h}_z$. Note that, by construction, $c \subseteq \hat{h}_{\langle i,j,k\rangle} \subseteq h_i \cap c_k$, and therefore, $h_i \neq^a c$. Now, since M learns c, there are $r, i' \in \mathbb{N}$ such that $M(\mathit{content}(t_{y+r})) = i'$ and $h_{i'} =^a c$. Clearly, $i' \neq i$. Finally, because of $\mathit{content}(t_{y+r}) \subseteq c \subseteq c_k \cap h_i$, $\hat{h}_{\langle i,j,k\rangle}$ must be finite (cf. Fact 2). Since c is infinite, $\hat{h}_{\langle i,j,k\rangle}$ cannot constitute a proper superset of c, a contradiction, and the claim follows.

It remains to show that there are indices y, z such that $M'(t_y) = z$, $\hat{h}_z =^a c$, and $c \subseteq \hat{h}_z$. Since M' is conservative, this will suffice.

Again, since M is set-driven and $\mathit{Lim}^a\mathit{Txt}$–learns c, there has to be a finite set $F \subseteq c$ such that, for all finite sets $V \subseteq c$, $M(F) = M(F \cup V) = i$, and $h_i =^a c$. Since M is consistent, $c \subseteq h_i$ holds.

Next, let y be the least index such that $F \subseteq \mathit{content}(t_y)$ and let $\langle i_y, j_y, k_y\rangle = M'(t_y)$. Obviously, if $c \subseteq \hat{h}_{\langle i_y,j_y,k_y\rangle}$, then, by Claim 1, we are immediately done. Otherwise, by Claim 1, we may assume that $c \setminus \hat{h}_{\langle i_y,j_y,k_y\rangle} \neq \emptyset$. Hence, there is a least $y' > y$ such that $\mathit{content}(t_{y'}) \not\subseteq \hat{h}_{\langle i_y,j_y,k_y\rangle}$ and thus M' performs a mind change, i.e., it computes $\langle i_{y'}, j_{y'}, k_{y'}\rangle = M'(t_{y'})$.

Now, by the choice of y, we know that $i_{y'} = i$. Moreover, $j_{y'} = \#(\mathit{content}(t_{y'}))$ and, even more important, $k_{y'}$ is the least index such that $k_{y'} > k_y$ and $\mathit{content}(t_{y'}) \subseteq c_{k_{y'}}$. Now, recall that, by the choice of the indexing $(c_j)_{j\in\mathbb{N}}$, there is a least index $\hat{k} > k_y$ such that $c_{\hat{k}} = c$. Hence, we may conclude that $k_y < k_{y'} \leq \hat{k}$. As above, there are two cases to distinguish. First, if $c \subseteq \hat{h}_{\langle i_{y'},j_{y'},k_{y'}\rangle}$, then, again by Claim 1, we are directly done. Second, if $c \not\subseteq \hat{h}_{\langle i_{y'},j_{y'},k_{y'}\rangle}$, there is a least $y'' \in \mathbb{N}$ such that $\mathit{content}(t_{y''}) \not\subseteq \hat{h}_{\langle i_{y'},j_{y'},k_{y'}\rangle}$. Again, by definition, M' changes its mind to $\langle i, \#(\mathit{content}(t_{y''})), k_{y''}\rangle$, where $k_{y''}$ is the least index with $k_{y'} < k_{y''} \leq \hat{k}$ and $\mathit{content}(t_{y''}) \subseteq c_{k_{y''}}$.

Finally, by simply iterating this argumentation and by taking into consideration that, for all finite sets V with $\mathit{content}(t_y) \subseteq V \subseteq c$, $\hat{h}_{\langle i,\#(V),\hat{k}\rangle} = c$, one directly sees that M' eventually outputs a hypothesis z with $c \subseteq \hat{h}_z$ and $\hat{h}_z =^a c$, and thus, we are done. □

Theorem 6. $\mathit{Lim}^*\mathit{Txt} \subseteq \mathit{Sdr}^*\mathit{Txt}$.

Proof. Let $\mathcal{C} \in Lim^*Txt$. By Proposition 2, there is an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a recursively enumerable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that, for all $j,k \in \mathbb{N}$, (1) and (2) are fulfilled, where

(1) $T_j \subseteq c_j$.
(2) If $T_j \subseteq c_k \subseteq c_j$, then $c_k =^* c_j$.

For all $j, y \in \mathbb{N}$, we let $T_j^{(y)}$ denote the finite subset of T_j that is enumerated within y steps. Note that, for all $j, y, y' \in \mathbb{N}$, it is decidable whether or not $T_j^{(y)} = T_j^{(y')}$. For technical reasons, it is convenient to assume that, for all $j \in \mathbb{N}$, $T_j \neq \emptyset$ and $T_j^{(0)} = \emptyset$. Clearly, this assumption is justified, since $\mathcal{C}$ exclusively contains non-empty concepts.

Before we define a set-driven IIM M' that learns $\mathcal{C}$, we define a consistent IIM M that $Lim^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$, where $\mathcal{H} = (c_j)_{j\in\mathbb{N}}$. The required set-driven IIM M' will use M as a subroutine.

Let $c \in \mathcal{C}$, let $t \in Text(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y proceed as follows: For all $j \leq y$, test whether or not $T_j^{(y)} \subseteq content(t_y) \subseteq c_j$. If there is a j passing this test, then output the minimal one. Otherwise, determine the minimal j with $content(t_y) \subseteq c_j$ and output j."

The verification that M behaves as required is straightforward.

We continue in defining a hypothesis space $\mathcal{H}' = (h'_{\langle j,k\rangle})_{j,k\in\mathbb{N}}$ and a set-driven IIM M' that $Lim^*Txt_{\mathcal{H}'}$–learns $\mathcal{C}$. Let $(w_j)_{j\in\mathbb{N}}$ be an effective enumeration of all elements in $\mathcal{X}$. Let $j, k \in \mathbb{N}$. For all $z \in \mathbb{N}$, we let $w_z \in h_{\langle j,k\rangle}$ iff one of the Conditions (i) and (ii) is fulfilled, where

(i) $z \leq k$ and $w_z \in c_j$.
(ii) $z > k$, $w_z \in c_j$, and $T_j^{(z)} = T_j^{(k)}$.

Now, one easily verifies that membership is uniformly decidable in $\mathcal{H}'$. Since $(c_j)_{j\in\mathbb{N}}$ is an indexing of $\mathcal{C}$ and since all the sets T_j are finite, we may immediately conclude:

Fact 1. *For all $j \in \mathbb{N}$ and all $k \in \mathbb{N}$, if $T_j^{(k)} = T_j$, then $h'_{\langle j,k\rangle} = c_j$.*

Fact 2. *For all $j, k \in \mathbb{N}$, if $T_j^{(k)} \neq T_j$, then $h'_{\langle j,k\rangle}$ is finite.*

Now, we are ready to define M'. So, let $c \in \mathcal{C}$, let $t \in Text(c)$, and let $y \in \mathbb{N}$.

IIM M': "On input t_y do the following:
Compute the canonical arrangement $\sigma = x_0, \ldots, x_r$ of $content(t_y)$. Deter-

mine $j = M(\sigma)$. Test whether or not $T_j^{(r)} \subseteq content(\sigma)$. In case it is, goto (A). Else, goto (B).
(A) Determine $m = \min\{n \mid T_j^{(n)} = T_j^{(r)}\}$ and output $\langle j, m\rangle$.
(B) Output $\langle j, 0\rangle$."

By definition, M' is set-driven. We have to show that M' learns as required. For that purpose, we distinguish two cases.

Case 1. *c is infinite.*

Recall that M, when fed the canonical text for c, where all repetitions are deleted, converges to a hypothesis j with $c_j =^* c$. Due to M's definition we have $T_j \subseteq c$. Thus, M' converges to $\langle j, m\rangle$, where $m = \min\{n \mid T_j^{(n)} = T_j\}$. Hence, by Fact 1, $h'_{\langle j,m\rangle} = c_j$, and we are done.

Case 2. *c is finite.*

Let σ be the canonical arrangement of the elements of c. By definition, M' converges on t to $\langle j, m\rangle = M'(\sigma)$. We claim that $h'_{\langle j,m\rangle} =^* c$. First, assume that the hypothesis $\langle j, m\rangle$ was build in accordance with (B). Hence, $m = 0$. Since, by assumption, $T_j^{(0)} \neq T_j$, we obtain, via Fact 2, that $h'_{\langle j,m\rangle}$ is finite. Hence, $h'_{\langle j,m\rangle} =^* c$. Second, suppose the hypothesis $\langle j, m\rangle$ was build due to (A). Now, if $T_j^{(m)} \neq T_j$, the same arguments yield $h'_{\langle j,m\rangle} =^* c$. Finally, consider the case that $T_j^{(m)} = T_j$. Since $M(\sigma) = j$, we get $T_j \subseteq content(\sigma) = c$. Since M is consistent, we know that $c = content(\sigma) \subseteq c_j$. Hence, by Property (2) of the recursively enumerable family $(T_j)_{j\in\mathbb{N}}$, we may conclude that $c_j =^* c$. □

Theorem 7. *c-Consv*Txt = Consv*Txt.*

Proof. The theorem is a direct consequence of Theorem 5 and 6 above. □

As a closer look at the demonstration of Theorem 6 shows, every unconstrained IIM M can be replaced by an IIM M' that is simultaneously set-driven and consistent and that is at least as powerful as M. Moreover, Theorems 5 and 6 and the definitions of the relevant learning types allow the following corollary.

Corollary 8. *Sdr*Txt = Lim*Txt.*

Note that Corollary 8 contrasts the fact that set-drivenness is a severe restriction in case that anomalies are inadmissible (cf. Lange and Zeugmann [17]).

However, there are differences between conservative inference and set-driven learning, on the one hand, and learning in the limit, on the other hand, which we point out next. While learning in the limit is invariant to the choice of the hypothesis space (cf. Tabe and Zeugmann [19]), conservative inference and set-driven learning are not. In order to design most powerful learners that are conservative and set-driven, respectively, it is sometimes inevitable to select a hypothesis space that contains concepts which are not subject to learning.

Theorem 9.

(1) *There is an indexable class $\mathcal{C} \in Consv^*Txt$ such that, for all class preserving hypothesis spaces $\mathcal{H}$ for $\mathcal{C}$, $\mathcal{C}$ is not $Consv^*Txt_{\mathcal{H}}$–learnable.*
(2) *There is an indexable class $\mathcal{C} \in Sdr^*Txt$ such that, for all class preserving hypothesis spaces $\mathcal{H}$ for $\mathcal{C}$, $\mathcal{C}$ is not $Sdr^*Txt_{\mathcal{H}}$–learnable.*

Proof. We present a class $\mathcal{C} \in \mathcal{IC}$ that simultaneously witnesses (1) and (2). For this purpose, let $(M_j)_{j\in\mathbb{N}}$ be an effective enumeration of all IIMs. Without loss of generality we may assume that each M_j is total, i.e., M_j, when fed any finite sequence of elements from $\mathcal{X}$, outputs a number. Moreover, let $\mathcal{X} = \{b, d\}^*$.

The underlying idea is as follows: Given any $j \in \mathbb{N}$, we define a particular indexable class $\mathcal{C}_j$ such that M_j either does not witness $\mathcal{C}_j \in Lim^*Txt$ or M_j is not conservative (set-driven) provided it uses any class preserving hypothesis space for $\mathcal{C} = \bigcup_{j\in\mathbb{N}} \mathcal{C}_j$. For showing that M_j violates the constraints a conservative (set-driven) IIM has to fulfill some *a priori* knowledge about the semantics of M_j's hypotheses is required. In order to provide this knowledge we choose the following approach.

Let $(\varphi_j)_{j\in\mathbb{N}}$ be any acceptable programming system of all partial recursive predicates and let $(\Phi_j)_{j\in\mathbb{N}}$ be any fixed associated complexity measure (cf. Blum [6]). Let $(w_j)_{j\in\mathbb{N}}$ be the fixed recursive enumeration of the elements of $\mathcal{X}$. For every $j \in \mathbb{N}$, let $c(\varphi_j) = \{w_m \mid m \in \mathbb{N},\ \varphi_j(m)\downarrow,\ \varphi_j(m) = 1\}$. Then, we use $\mathcal{H} = (c(\varphi_j))_{j\in\mathbb{N}}$ as a universal hypothesis space, i.e., if any of the enumerated IIMs outputs a hypothesis, say k, then we interpret it to mean that the IIM is guessing the concept $c(\varphi_k)$. Note that $\mathcal{H}$ is *not* an indexed family. The following lemma guarantees that this approach is successful.

Lemma 2. *Let $\mathcal{C}'$ be any indexable class over the learning domain $\mathcal{X}$, let $\mathcal{H}' = (h'_j)_{j\in\mathbb{N}}$ be a hypothesis space, and let M' be any total IIM that $Lim^*Txt_{\mathcal{H}'}$–learns $\mathcal{C}'$. Then, there exists an IIM M which $Lim^*Txt_{\mathcal{H}}$–learns $\mathcal{C}'$.*

Proof of the lemma. For all $j, m \in \mathbb{N}$ we define $p_j(m) = 1$ iff $w_m \in h'_j$. Since membership is uniformly decidable in $\mathcal{H}'$, $(p_j)_{j\in\mathbb{N}}$ is an effective enumeration of recursive predicates. By the choice of $(\varphi_j)_{j\in\mathbb{N}}$, there is a recursive compiler f such that, for all $j \in \mathbb{N}$, $p_j = \varphi_{f(j)}$. Given this compiler f, one can easily define an IIM M which $Lim^*Txt_{\mathcal{H}}$–learns $\mathcal{C}'$. Let $c \in \mathcal{C}'$, $t \in text(c)$, and $y \in \mathbb{N}$.

IIM M: "On input t_y proceed as follows:

Determine $j = M'(t_y)$ and output $f(j)$."

Obviously, M learns $\mathcal{C}'$ as required. Note that our transformation guarantees any additional constraint met by M' is satisfied by M, too. In particular, if M' is conservative (set-driven), then M is also conservative (set-driven). Moreover, if $\mathcal{H}'$ is a class preserving hypothesis space for $\mathcal{C}$, then M outputs exclusively indices for concepts belonging to $\mathcal{C}$. Thus, Lemma 2 is proved. □

So, let $j \in \mathbb{N}$. As a rule, $\mathcal{C}_j$ exclusively contains at most two different concepts c and c', where c is an infinite concept and c' is a finite one. In order to answer the question how to define c and c', the following procedure is used.

Subsequently, we use the following notation. For all $m, j \in \mathbb{N}$, let $c(\varphi_j)^+\!\restriction^m = \{w_n \mid n \leq m,\ \Phi_j(n) \leq m, \varphi_j(n) = 1\}$. Note that, by the properties of a complexity measure, the set $c(\varphi_j)^+\!\restriction^m$ is recursive in m and j.

Fix $j \in \mathbb{N}$.

Stage 0.
Set $c = \{b^j d^z \mid z \in \mathbb{N}\}$, $\sigma = \sigma' = b^j d^0$, and $w = b^j d$. Goto Stage 1.
Stage $k+1$.
Set $\sigma' = \sigma' \diamond b^j d^{k+1}$. If $M_j(\sigma) \neq M_j(\sigma')$, goto (A). Otherwise, goto (B).
(A) Set $w = b^j d^{k+2}$, $\sigma = \sigma'$, and goto Stage $k+2$.
(B) Let $z = M(\sigma)$. Test whether or not $w \in c(\varphi_z)^+\!\restriction^{k+1}$. If it is, set $c' = \mathit{content}(\sigma)$ and finish the definition of $\mathcal{C}_j$. Otherwise, goto Stage $k+2$.

We set $\mathcal{C} = \bigcup_{j\in\mathbb{N}} \mathcal{C}_j$ and claim that $\mathcal{C}$ witnesses Assertions (1) and (2) above. Clearly, $\mathcal{C}$ is indexable, since the $\mathcal{C}_j$'s are indexable uniformly in j.

By Theorem 4 and Corollary 8, it suffices to show that $\mathcal{C} \in \mathit{Lim}^*\mathit{Txt}$. However, we even show that $\mathcal{C}$ is LimTxt–identifiable. The desired IIM M works as follows. On input t_y, M determines the unique $j \in \mathbb{N}$ such that $t_y \in \mathit{Text}(\mathcal{C}_j)$. Now, M uses y steps of computation to simulate the procedure defined above in order to decide whether or not $\mathcal{C}_j$ contains a finite concept c'. If y steps of computation do not suffice for making this decision, M guesses the infinite concept $c \in \mathcal{C}_j$. If M has verified that there is a finite concept $c' \in \mathcal{C}_j$, it tests whether or not $\mathit{content}(t_y) = c'$. In case it is, M guesses c'; otherwise, M guesses c. Obviously, M learns as required, and thus we are done.

Next we complete the proof of (1). Suppose that there are a class preserving hypothesis space $\mathcal{H}'$ for $\mathcal{C}$ and a conservative IIM M' that $\mathit{Lim}^*\mathit{Txt}_{\mathcal{H}'}$–learns $\mathcal{C}$. Without loss of generality, we assume that M' is total. By Lemma 2, there is a conservative IIM M that $\mathit{Lim}^*\mathit{Txt}_{\mathcal{H}}$–learns $\mathcal{C}$ and that outputs exclusively indices for concepts belonging to $\mathcal{C}$. Now, let $j \in \mathbb{N}$ be fixed such that $M_j = M$. We claim that M cannot learn the concepts in $\mathcal{C}_j$ as required.

Let $c = \{b^j d^z \mid z \in \mathbb{N}\}$ and let t^c be the canonical text for c. We distinguish two cases.

Case 1. *The construction of $\mathcal{C}_j$ does not terminate.*

Clearly, in case that M, when fed t^c, changes its mind infinitely often, it cannot learn c. Hence, there is a least y such that, for all $y' \geq y$, $M(t^c_y) = M(t^c_{y'})$. Let $z = M(t^c_y)$. Since the construction of $\mathcal{C}_j$ does not terminate, we know that $b^j d^{y+1} \notin c(\varphi_z)$. Since, in addition, $c(\varphi_z) \in \mathcal{C}$, we may conclude that $c(\varphi_z) \neq^* c$, and therefore M fails to learn c from its canonical text.

Case 2. *The construction of $\mathcal{C}_j$ terminates.*

Hence, $\mathcal{C}_j$ contains a finite concept c'. Let $c' = \{b^j d^0, \ldots, b^j d^y\}$. By construction, we know that, for $z = M(t^c_y)$, it has been verified that $b^j d^{y+1} \in c(\varphi_z)$. Moreover, since $c(\varphi_z) \in \mathcal{C}$, we obtain $c(\varphi_z) = c$. By definition, $c' \subseteq c$. Since M is conservative, it converges to z when fed the text $t' = t^c_y \diamond b^j d^0, b^j d^0, \ldots$ for c'. But $c \neq^* c'$, and thus M cannot learn c', a contradiction.

Finally, the same argumentation applies *mutatis mutandis* to complete the verification of (2). Only the following minor modification is necessary. In Case 2, one has to stress the argument that $c' = content(t^c_y)$ to show that M converges to z when fed the text $t' = t^c_y \diamond b^j b^0, b^j d^0, \ldots$ for c' provided that M is set-driven. We omit further details. □

For anomaly-free learning, the analogue of Theorem 9 holds as well (cf. Lange and Zeugmann [16]).

Next we study behaviorally correct identification. As we shall see, finite tell-tale sets form a conceptual basis that is also well-suited to characterize the collection of all Bc^*Txt–identifiable indexable classes. Now the existence of the corresponding tell-tale sets is already sufficient.

Theorem 10. *For all $\mathcal{C} \in \mathcal{IC}$: $\mathcal{C} \in Bc^*Txt$ iff there is an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq c_j$,*
(2) *for all $j, k \in \mathbb{N}$, if $T_j \subseteq c_k \subseteq c_j$, then $c_k =^* c_j$.*

Proof. Necessity. Let $\mathcal{H}$ be a hypothesis space and let M be an IIM that $Bc^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$. Moreover, let $(c_j)_{j\in\mathbb{N}}$ be an indexing of $\mathcal{C}$. Let $j \in \mathbb{N}$. Since M $Bc^*Txt_{\mathcal{H}}$–learns c_j, there is some finite sequence $\sigma \in SegText(c_j)$ such that, for all finite sequences $\tau \in SegText(c_j)$ and all $k \in \mathbb{N}$, if $k = M(\sigma \diamond \tau)$, then $h_k =^* c_j$ (cf. Jain *et al.* [12]). We claim that $T_j = content(\sigma)$ will do. Suppose that there is a $k \in \mathbb{N}$ such that $T_j \subseteq c_k$, $c_k \subset c_j$, and $c_k \neq^* c_j$. Due to the choice of σ and since $c_k \subset c_j$, one directly sees that M fails to learn c_k on each of its texts having the initial segment σ, a contradiction.

Sufficiency. We define an appropriate hypothesis space $\mathcal{H} = (h_{\langle j,k \rangle})_{j,k \in \mathbb{N}}$. Let $(F_j)_{j \in \mathbb{N}}$ be an effective enumeration of all finite subsets of $\mathcal{X}$ and let $(w_j)_{j \in \mathbb{N}}$ be the fixed recursive enumeration of all elements in $\mathcal{X}$. For the sake of readability, we use the following notions and notations.

First, for all $c \subseteq \mathcal{X}$ and all $z \in \mathbb{N}$, we let $c\!\upharpoonright^z = \{w_r \mid r \leq z,\, w_r \in c\}$. Second, for all $j,k,z \in \mathbb{N}$, we let $S_{(j,k,z)}$ be the set of all indices $r \leq k$ that meet (i) $F_j \subseteq c_r$ and (ii), for all $r' < r$ with $c_{r'} \supseteq F_j$, $c_r\!\upharpoonright^z \subseteq c_{r'}\!\upharpoonright^z$.

Now we are ready to define the required hypothesis space $\mathcal{H}$. Let $j,k \in \mathbb{N}$. We define the characteristic function of $h_{\langle j,k \rangle}$ as follows. If $S_{(j,k,z)} = \emptyset$, we set $h_{\langle j,k \rangle}(w_z) = -$. Otherwise, i.e., $S_{(j,k,z)} \neq \emptyset$, we let $n = \max\{r \mid r \in S_{(j,k,z)}\}$ and set $h_{\langle j,k \rangle}(w_z) = c_n(w_z)$.

Since membership is uniformly decidable in $(c_j)_{j \in \mathbb{N}}$, we know that $\mathcal{H}$ is an admissible hypothesis space.

The desired IIM M is defined as follows. Let $c \in \mathcal{C}$, $t \in \mathit{Text}(c)$, and $y \in \mathbb{N}$.

IIM M': "On input t_y proceed as follows:
  Determine $j \in \mathbb{N}$ with $F_j = \mathit{content}(t_y)$ and output $\langle j, y \rangle$."

We claim that M $\mathit{Bc}^*\mathit{Txt}_{\mathcal{H}}$–learns c.

Let $m = \min\{r \mid c_r = c\}$. Since $t \in \mathit{Text}(c)$, there is a least $y' \geq m$ such that, for all $k' < m$, $\mathit{content}(t_{y'}) \subseteq c_{k'}$ implies $c_{k'} \supseteq c_m$. By assumption, there is some finite tell-tale set T_m for $c = c_m$. Again, since $t \in \mathit{Text}(c)$, there is a least $y'' \geq y'$ such that $T_m \subseteq \mathit{content}(t_{y''})$. Fix any $y \geq y''$ and consider $\langle j, y \rangle = M(t_y)$. We claim that $h_{\langle j,y \rangle} =^* c$. This can be seen as follows.

Let $z' \in \mathbb{N}$. By the choice of y', $m \in S_{(j,y,z')}$. Moreover, $S_{(j,y,z')}$ is finite and $S_{(j,y,z')} \supseteq S_{(j,y,z'+1)}$. Hence, there is some $n \geq m$ such that, for almost all z, $n = \max\{r \mid r \in S_{(j,y,z)}\}$. By definition of $\mathcal{H}$, we know that $h_{\langle j,y \rangle} =^* c_n$. Since, for all $z \in \mathbb{N}$, $m, n \in S_{(j,y,z)}$ and since $n \geq m$, we conclude $c_m \supseteq c_n$. By $\mathcal{H}$'s definition, we have $\mathit{content}(t_y) \subseteq c_n$, and thus, by the choice of y'', $T_m \subseteq c_n$. Hence, Condition (2) guarantees that $c_n =^* c$, and therefore $h_{\langle j,y \rangle} =^* c$. □

Note that Baliga *et al.* [3] have been shown recently that the same characterizing conditions as in Theorem 10 completely describe the collection of all indexable classes that are $\mathit{Bc}^*\mathit{Txt}$–learnable with respect to *arbitrary* hypothesis spaces [2]. Hence, our result refines theirs by showing that, in order to

[2] That means, hypothesis spaces that do not necessarily admit a decidable membership problem.

*Bc*Txt*–identify an indexable class, it is always possible to select a hypothesis space with uniformly decidable membership. However, as we see next, it is inevitable to select the actual hypothesis space appropriately.

Theorem 11. *There is an indexable class $\mathcal{C} \in Bc^*Txt$ such that, for all class preserving hypothesis spaces $\mathcal{H}$ for $\mathcal{C}$, $\mathcal{C}$ is not $Bc^*Txt_{\mathcal{H}}$–learnable.*

Proof. The required class $\mathcal{C} \in \mathcal{IC}$ is defined as follows. Let $(M_j)_{j\in\mathbb{N}}$ be an effective enumeration of all IIMs. Without loss of generality we assume that each M_j is total, i.e., M_j, when fed any finite sequence $\sigma \in SegText(\mathcal{X})$, outputs a number. Moreover, let $\mathcal{X} = \{b, d\}^*$.

The proof idea is as follows. For any $j \in \mathbb{N}$, we define a class $\mathcal{C}_j \in \mathcal{IC}$ such that M_j fails to Bc^*Txt–identify $\mathcal{C}_j$ for every class preserving hypothesis space for $\mathcal{C} = \bigcup_{j\in\mathbb{N}} \mathcal{C}_j$. Some *a priory* knowledge about M_j's hypotheses is necessary. For getting it, we use the same approach as in the proof of Theorem 9.

Fix an acceptable programming system $(\varphi_j)_{j\in\mathbb{N}}$ and an associated complexity measure $(\Phi_j)_{j\in\mathbb{N}}$. Let $(w_j)_{j\in\mathbb{N}}$ be the fixed recursive enumeration of all elements in $\mathcal{X}$, and let $c(\varphi_j) = \{w_m \mid m \in \mathbb{N},\ \varphi_j(m)\downarrow,\ \varphi_j(m) = 1\}$ for all $j \in \mathbb{N}$. Then, we use $\mathcal{H} = (c(\varphi_j))_{j\in\mathbb{N}}$ as a universal hypothesis space, i.e., if any of the enumerated IIMs outputs a hypothesis, say k, then we interpret it to mean that the IIM is guessing the concept $c(\varphi_k)$.

For all $j \in \mathbb{N}$, the definition of $\mathcal{C}_j$ is performed in stages. Furthermore, for all $m, j \in \mathbb{N}$, we set $c(\varphi_j)^+\lceil^m = \{w_n \mid n \le m,\ \Phi_j(n) \le m,\ \varphi_j(n) = 1\}$ and $c(\varphi_j)^-\lceil^m = \{w_n \mid n \le m,\ \Phi_j(n) \le m,\ \varphi_j(n) = 0\}$. Again, by the properties of a complexity measure, the sets $c(\varphi_j)^+\lceil^m$ and $c(\varphi_j)^-\lceil^m$ are recursive in m and j.

Fix $j \in \mathbb{N}$.

Stage 0.
  Define c_0 by setting $c_0 = \{b^j d^z \mid z \in \mathbb{N}\}$. Furthermore, set $\sigma = b^j d^0$, set $\max = 0$ and goto Stage 1.
Stage $k+1$.
  Set $m = 0$ and execute Instruction (A).
  (A) For all $y \le m$, execute the test (α).
    (α) Set $\sigma_y = \sigma \diamond b^j d^{\max+0}, \dots, b^j d^{\max+y}$ and determine $r_y = M_j(\sigma_y)$. Test whether or not $b^j d^{\max+1} \in c(\varphi_{r_y})^+\lceil^m$.
    In case there is some y passing this test, fix the least one, say y^*, set $\max = \max + y^*$, and execute Instruction (B).
    Otherwise, set $m = m + 1$ and execute Instruction (A) again.
  (B) Start the definition of c_{k+1} and set $c_{k+1} = \{b^j d^z \mid z \le \max\}$. Set $n = 0$ and execute Instruction (C).

(C) For all $\ell \leq n$, execute the following test (β).

(β) Set $\sigma'_\ell = \sigma_{y^*} \diamond \underbrace{b^j d^0, \ldots, b^j d^0}_{\ell-times}$ and determine $r'_\ell = M_j(\sigma'_\ell)$. Test whether there is a $z \leq \max + n$ such that $b^j d^z \in c(\varphi_{r'_\ell})^- \lceil^{max+n}$.

If no ℓ passes this test, set $n = n + 1$, and execute Instruction (C) again. Otherwise, fix the least ℓ that passes this test. Complete the definition of c_{k+1} by setting $c_{k+1} = c_{k+1} \cup \{b^{\langle k,n \rangle}\}$. Set $\sigma = \sigma'_\ell$, and goto Stage $k + 2$.

We let $\mathcal{C} = \bigcup_{j \in \mathbb{N}} \mathcal{C}_j$ and claim that $\mathcal{C}$ possesses the above property. Clearly, $\mathcal{C}$ is an indexable class.

By applying Theorem 10, one easily verifies that $\mathcal{C} \in Bc^*Txt$. To see this, let $j \in \mathbb{N}$. For all finite concepts $c_i \in \mathcal{C}_j$, we let $T_{c_i} = c_i$. We distinguish the following cases. First assume that $\mathcal{C}_j$ contains only finitely many concepts, say $c_0, \ldots, c_k$. For the infinite concept c_0, we let $T_{c_0} = \{b^j d^m\}$, where $m = \max\{z \mid b^j d^z \in c_k\} + 1$. Second, consider the case that $\mathcal{C}_j$ contains infinitely many concepts. Then $T_{c_0} = \{b^j d^0\}$ obviously suffices.

Next we show that, for all class preserving hypothesis spaces $\mathcal{H}'$, $\mathcal{C}$ is not $Bc^*Txt_{\mathcal{H}'}$-learnable. Suppose that there are a class preserving hypothesis space $\mathcal{H}' = (h'_j)_{j \in \mathbb{N}}$ and an IIM M' that $Bc^*Txt_{\mathcal{H}'}$–learns $\mathcal{C}$. Without loss of generality we may assume that M' is total. Applying similar arguments as in the proof of Lemma 2, it can be shown that there is an IIM M that $Bc^*Txt_{\mathcal{H}}$–learns $\mathcal{C}$ and that outputs exclusively indices for concepts belonging $\mathcal{C}$. Now, let $j \in \mathbb{N}$ be fixed such that $M_j = M$. We claim that M cannot learn all concepts in $\mathcal{C}_j$.

Case 1. *$\mathcal{C}_j$ contains infinitely many concepts.*

Note that, in the definition of $\mathcal{C}_j$, σ tends to become a text for $c_0 \in \mathcal{C}_j$. Moreover, in every Stage k with $k \geq 1$, it has been verified that there exists some y_k such that, for $r_k = M(\sigma_{y_k})$, $c(\varphi_{r_k}) \neq c_0$ (cf. Instruction (C)). Moreover, we know that $c(\varphi_{r_k}) \in \mathcal{C}$. The latter yields $c(\varphi_{r_k}) \neq^* c_0$. By construction, M, when fed σ, guesses infinitely often a concept that is not a finite variant of c_0, and thus it fails to learn c_0 on σ, a contradiction.

Case 2. *$\mathcal{C}_j$ contains finitely many concepts.*

First, consider the case that $\mathcal{C}_j$ contains only the infinite concept c_0. Hence, while executing Instruction (A) in Stage 1, a text t for c_0 is formed on which M almost always guesses a concept that is not a finite variant of c_0. To see this, note that, for all but finitely many r which M outputs when fed t, it must be the case that $b^j d^1 \notin c(\varphi_r)$. Since $c(\varphi_r) \in \mathcal{C}$, we may conclude that $c(\varphi_r) \neq^* c_0$, and thus M cannot learn c_0 on t.

Second, let $c_1, \ldots, c_k$ be the finite concepts belonging to $\mathcal{C}_j$. Now assume that Stage k does not terminate. Then, while executing Instruction (C) in Stage k, a text t for c_k is formed on which M almost always guesses a concept that is not a finite variant of c_k. To see this, note that, for all but finitely many r' which M outputs when fed t, $c_0 \subseteq c(\varphi_{r'})$ must be the case. Moreover, $c(\varphi_{r'}) \in \mathcal{C}$, and therefore $c(\varphi_{r'}) \neq^* c_k$. Hence, M fails to learn c_k on t, a contradiction.

Finally, consider the case that Stage $k+1$ does not terminate. Hence, while executing Instruction (A) in Stage $k+1$, a text t for c_0 is formed on which M almost always guesses a concept that is not a finite variant of c_0. To see this, note that, for all but finitely many r which M outputs when fed t, it must hold that $b^j d^{m+1} \notin c(\varphi_r)$, where $m = \max\{z \mid b^j d^z \in c_k\}$. Since $\mathcal{H}'$ is a class preserving hypothesis space, we have $c(\varphi_r) \in \mathcal{C}$. This again yields $c(\varphi_r) \neq^* c_0$, contradicting the assumption that M learns c_0 from every text for it. □

In contrast, since $BcTxt = LimTxt$, it can easily be shown that $BcTxt$ is invariant to the choice of the hypothesis space (cf. Lange and Zeugmann [16], for the relevant details). To be complete, note that there are indexable classes which are not Bc^*Txt-identifiable (cf. Jain *et al.* [12], Exercise 6-9(c)).

Proposition 3. $Bc^*Txt \subset \mathcal{IC}$.

3.2 The Case of an a priori Bounded Number of Anomalies

Next we turn our attention to the case that the number of allowed anomalies is *a priori* bounded. For learning in the limit, the situation remains unchanged.

Proposition 4 (Tabe and Zeugmann [19]). *For all $\mathcal{C} \in \mathcal{IC}$ and all $a \in \mathbb{N}$: $\mathcal{C} \in Lim^aTxt$ iff there are an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a recursively enumerable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq c_j$,*
(2) *for all $j, k \in \mathbb{N}$, if $T_j \subseteq c_k \subseteq c_j$, then $c_k =^a c_j$.*

Surprisingly, the situation changes already, if finite inference is considered. In order to design powerful finite learners it is inevitable to use hypothesis spaces that contain concepts that are not subject to learning.

Theorem 12. *For all $a \in \mathbb{N}^+$ there is an indexable class $\mathcal{C}' \in Fin^aTxt$ such that, for all class preserving hypothesis spaces $\mathcal{H}$ for $\mathcal{C}'$, $\mathcal{C}'$ is not $Fin^aTxt_{\mathcal{H}}$–learnable.*

Proof. We consider the case of $a = 1$, only. The adaptation to the general case is obvious. For all $k \in \mathbb{N}$, we set $c_k = \{b\}^* \setminus \{b^k\}$. Let $\mathcal{C}$ be the collection of all concepts c_k. One the one hand, one immediately sees that $\mathcal{C} \in Fin^1Txt$. On the other hand, since, for all distinctive concepts $c, c' \in \mathcal{C}$, $c \neq^1 c'$, it is not hard to verify that there is no IIM that Fin^1Txt-learns $\mathcal{C}$ and that outputs exclusively indices for concepts in $\mathcal{C}$. We omit the details. □

As a kind of side-effect, one obtains the following characterization for finite inference with an *a priori* bounded number of anomalies.

Theorem 13. *For all $\mathcal{C} \in \mathcal{IC}$ and all $a \in \mathbb{N}$: $\mathcal{C} \in Fin^aTxt$ iff there are a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and a recursively generable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that:*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq h_j$,*
(2) *for all $c \in \mathcal{C}$, there is a $j \in \mathbb{N}$ such that $T_j \subseteq c$,*
(3) *for all $j \in \mathbb{N}$ and all $c \in \mathcal{C}$, if $T_j \subseteq c$, then $c =^a h_j$.*

Proof. Necessity. Assume that a hypothesis space $\mathcal{H}' = (h'_j)_{j\in\mathbb{N}}$ and an IIM M that $Fin^aInf_{\mathcal{H}'}$–learns $\mathcal{C}$ are given. Moreover, let $(c_j)_{j\in\mathbb{N}}$ be any indexing of $\mathcal{C}$.

We define the hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and the family $(T_j)_{j\in\mathbb{N}}$ as follows: Let $j \in \mathbb{N}$ and let t^{c_j} be the canonical text of c_j. Since M finitely infers c_j, there exists a least $y \in \mathbb{N}$ such that $M(t^{c_j}_y) = m$ for some $m \in \mathbb{N}$. We set $h_j = h'_m \cup content(t^{c_j}_y)$ and $T_j = content(t^{c_j}_y)$.

We have to show that $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and $(T_j)_{j\in\mathbb{N}}$ fulfill the announced properties. By construction, (1) and (2) are trivially fulfilled. Next we show (3). Suppose $j \in \mathbb{N}$ and $c \in \mathcal{C}$ such that $T_j \subseteq c$. By construction, there is an initial segment of c_j's canonical text t^{c_j}, say $t^{c_j}_y$, such that $T_j = content(t^{c_j}_y)$ and $M(t^{c_j}_y) = m$. Since M finitely learns c_j, we have $h'_m =^a c_j$. We conclude that $t^{c_j}_y$ is also an initial segment of some text t' for c, since $T_j \subseteq c$. Taking into account that M finitely infers c when fed t' and that $M(t'_y) = m$, we obtain $h'_m =^a c$. Since $h_j = h'_m \cup content(t^{c_j}_y)$ and $content(t^{c_j}_y) \subseteq c$, this gives us $h_j =^a c$.

Sufficiency. Let $a \in \mathbb{N}$. It suffices to prove that there is an IIM M that $Fin^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$. Let $c \in \mathcal{C}$, let $t \in Text(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y do the following:
 If $y = 0$ or $M(t_{y-1}) = ?$, goto (A). Otherwise, output $j = M(t_{y-1})$.
 (A) For $j = 0, \dots, y$, generate T_j and test whether or not $T_j \subseteq content(t_y)$.
  If there is a j fulfilling the test, output the minimal one. Else, output ?."

One directly sees that M learns as required. □

As we shall see next, when behaviorally correct learning, conservative inference, and set-driven learning are considered, the overall picture changes, if there is an *a priori* fixed bound on the number of allowed anomalies.

On the one hand, Case and Lynes' [7] result that, for all $a \in \mathbb{N}$, $Lim^{2a}Txt \subseteq Bc^aTxt$ easily translates into our setting of learning indexable classes. Surprisingly, the opposite is also true, i.e., every IIM that Bc^aTxt–learns a target indexable class can be simulated by a learner that $Lim^{2a}Txt$–learns the same class, as expressed by the following theorem.

Theorem 14. *For all $a \in \mathbb{N}$: $Bc^aTxt = Lim^{2a}Txt$.*

Proof. Let $a \in \mathbb{N}$. As mentioned above, $Lim^{2a}Txt \subseteq Bc^aTxt$ can be shown by using the ideas from Case and Lynes [7] (see also Jain *et al.* [12]).

Next we verify that $Bc^aTxt \subseteq Lim^{2a}Txt$. Let $\mathcal{C} \in Bc^aTxt$, let $\mathcal{H}$ be a hypothesis space, and let M be an IIM that $Bc^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$. Since membership is uniformly decidable in $\mathcal{H}$, the set $\{(j,k) \mid h_j \neq^{2a} h_k\}$ is recursively enumerable. If $\{(j,k) \mid h_j \neq^{2a} h_k\} = \emptyset$ then the wanted IIM M' witnessing $C \in Lim^{2a}Txt_{\mathcal{H}}$ simply always outputs 0 and the theorem follows.

Now assume $\{(j,k) \mid h_j \neq^{2a} h_k\} \neq \emptyset$. Then there is a total recursive function $f : \mathbb{N} \to \mathbb{N}^2$ such that $\{f(n) \mid n \in \mathbb{N}\} = \{(j,k) \mid h_j \neq^{2a} h_k\}$.

The required IIM M' also uses the hypothesis space $\mathcal{H}$. Let $c \in \mathcal{C}$, $t \in Text(c)$, and $y \in \mathbb{N}$.

IIM M': "On input t_y proceed as follows:

If $y = 0$, set $z = 0$, determine $j_0 = M(t_0)$, and output j_0. Otherwise, goto (A).

(A) Determine $j = M'(t_{y-1})$. For all $s = z, \dots, y$, determine $j_s = M(t_s)$, and test whether or not $(j, j_s) \in \{f(n) \mid n \leq y\}$. In case there is no j_s passing this test, then output j. Otherwise, set $z = y$ and output j_y."

Since M $Bc^aTxt_{\mathcal{H}}$–learns c, there is has to be a least y such that, for all $y', y'' \geq y$, $h_{M(t_{y'})} =^a c$ and $h_{M(t_{y'})} =^{2a} h_{M(t_{y''})}$. Consequently, M', when fed t, converges to a hypothesis j that meets $h_j =^{2a} c$. □

Applying Proposition 4, we may conclude:

Corollary 15. *For all $\mathcal{C} \in \mathcal{IC}$ and all $a \in \mathbb{N}$: $\mathcal{C} \in Bc^aTxt$ iff there are an indexing $(c_j)_{j \in \mathbb{N}}$ of $\mathcal{C}$ and a recursively enumerable family $(T_j)_{j \in \mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq c_j$,*
(2) *for all $j, k \in \mathbb{N}$, if $T_j \subseteq c_k \subseteq c_j$, then $c_k =^{2a} c_j$.*

The latter corollary nicely contrasts the results in Baliga *et al.* [3]. When arbitrary hypothesis spaces are admissible (see above), there is no need to add any recursive component, i.e., the existence of the corresponding tell-tale sets is again sufficient.

Moreover, Theorem 14 can be used to show that Lim^*Txt is an upper bound for behaviorally correct inference with an *a priori* bounded number of anomalies.

Theorem 16. $\bigcup_{a\in\mathbb{N}} Lim^aTxt = \bigcup_{a\in\mathbb{N}} Bc^aTxt \subset Lim^*Txt$.

Proof. $\bigcup_{a\in\mathbb{N}} Lim^aTxt = \bigcup_{a\in\mathbb{N}} Bc^aTxt$ follows directly via Theorem 14. Moreover, by definition, $\bigcup_{a\in\mathbb{N}} Lim^aTxt \subseteq Lim^*Txt$. Hence, it remains to provide an indexable class $\mathcal{C} \in Lim^*Txt$ such that, for all $a \in \mathbb{N}$, $\mathcal{C} \notin Lim^aTxt$.

We let $\mathcal{C}_{cof}$ be the collection of all co-finite concepts c with $c \subseteq \{b\}^*$. On the one hand, one easily sees that $\mathcal{C}_{cof}$ is even Fin^*Txt–identifiable. On the other hand, suppose that there is some $a \in \mathbb{N}$ such that $\mathcal{C}_{cof} \in Lim^aTxt$. By Theorem 4, for $c = \{b\}^*$, there must be a finite set $T_c \subseteq c$ such that, for all $c' \in \mathcal{C}_{cof}$, $T_c \subseteq c' \subseteq c$ implies $c' =^a c$. Clearly, such a finite set cannot exist. □

Next we deal with conservative inference and set-driven learning.

Theorem 17. *For all* $a \in \mathbb{N}$*:* $Lim^aTxt \subset Consv^{a+1}Txt \subset Lim^{a+1}Txt$.

Proof. Let $a \in \mathbb{N}$. The same idea as in the demonstration of Theorem 4 applies to show that $Lim^aTxt \subseteq Consv^{a+1}Txt$. Next, $Consv^{a+1}Txt \setminus Lim^aTxt \neq \emptyset$ can be shown by appropriately adapting the idea used to show Theorem 1. Furthermore, $Consv^{a+1}Txt \subseteq Lim^{a+1}Txt$ follows directly from the definitions.

It remains to show $Lim^{a+1}Txt \setminus Consv^{a+1}Txt \neq \emptyset$. For this purpose, let $(M_j)_{j\in\mathbb{N}}$ be an effective enumeration of all IIMs. Using the same idea as in the proof of Lemma 1 we may assume that each M_j is total, i.e., M_j, when fed any finite sequence of elements from $\mathcal{X}$, outputs a hypothesis. Moreover, let $\mathcal{X} = \{b, d\}^*$.

The underlying proof idea is as follows: Given any $j \in \mathbb{N}$, we define an indexable class $\mathcal{C}_j$ such that M_j either does not witness $\mathcal{C} \in Lim^{a+1}Txt$ or M_j is not conservative. For showing that M_j violates the constraints a conservative IIM has to fulfill some *a priori* knowledge about the semantics of M_j's hypotheses is required. We provide this knowledge by using the same universal hypothesis space $\mathcal{H} = (c(\varphi_j))_{j\in\mathbb{N}}$ as in the demonstration of Theorems 9 and 11.

Let $(\sigma_k)_{k\in\mathbb{N}}$ be any effective enumeration of all finite sequences of elements from $\{b^jd^z \mid z \in \mathbb{N}\}$. Moreover, for all $m, j \in \mathbb{N}$, the concepts $c(\varphi_j)^+\lceil^m$ and $c(\varphi_j)^-\lceil^m$ are defined analogously as in the proof of Theorem 11. Again, by the properties of a complexity measure, both sets are recursive in m and j.

So, let $j \in \mathbb{N}$. As a rule, the required indexable class $\mathcal{C}_j$ contains all infinite concepts $c \subseteq \{b^jd^z \mid z \in \mathbb{N}\}$ that meet $card(\{b^jd^z \mid z \in \mathbb{N}\} \setminus c) \leq a + 1$. In addition, $\mathcal{C}_j$ may contain a finite concept $c' \subseteq \{b^jd^z \mid z \in \mathbb{N}\}$. In order to answer the question of whether or not there is at most one finite concept c' belonging to $\mathcal{C}_j$ and of how to define c', the following procedure is used.

Initially, we set $k = 0$ and $P_{-1} = \emptyset$.

Stage k.

Determine $z = M_j(\sigma_k)$ and goto (A).

(A) Determine the least $m \in \mathbb{N}$ such that (i) or (ii) is fulfilled, where

(i) $\mathit{content}(\sigma_k) \subseteq c(\varphi_z)^+\lceil^m$.

(ii) $\mathit{content}(\sigma_k) \cap c(\varphi_z)^-\lceil^m \neq \emptyset$.

If (i) happens, set $P_k = P_{k-1} \cup \{k\}$. If (ii) happens, set $P_k = P_{k-1}$. Execute Instruction (B).

(B) For all $r \in P_k$, execute (β).

(β) Determine $z_r = M(\sigma_r)$ and $S_r = \{b^j d^n \mid b^j d^n \in c(\varphi_{z_r})^+\lceil^k\}$. Test whether or not $\mathit{card}(S \setminus \mathit{content}(\sigma_r)) \geq a + 2$.

In case an r has been found, fix the least one, say r', set $c' = \mathit{content}(\sigma_{r'})$, and finish the definition of c'. Otherwise, goto Stage $k + 1$.

After a bit of reflection, one sees that $\mathcal{C}_j$ constitutes an indexable class. We let $\mathcal{C} = \bigcup_{j \in \mathbb{N}} \mathcal{C}_j$ and claim that $\mathcal{C} \in \mathit{Lim}^{a+1}\mathit{Txt} \setminus \mathit{Consv}^{a+1}\mathit{Txt}$.

Claim 1. $\mathcal{C} \notin \mathit{Consv}^{a+1}\mathit{Txt}$.

Suppose there are a hypothesis space $\mathcal{H}' = (h'_j)_{j \in \mathbb{N}}$ and a total IIM M' that $\mathit{Consv}^{a+1}\mathit{Txt}_{\mathcal{H}'}$–learns $\mathcal{C}$. By an argumentation similar to the one proving Lemma 2, there is a total IIM M that $\mathit{Consv}^{a+1}\mathit{Txt}_{\mathcal{H}}$–learns $\mathcal{C}$. Let $j \in \mathbb{N}$ with $M_j = M$. We show that M fails to $\mathit{Consv}^{a+1}\mathit{Txt}_{\mathcal{H}}$–identify $\mathcal{C}_j$.

First, we show that $\mathcal{C}_j$ contains a finite concept. Suppose the converse. Since M exclusively outputs indices of recursive concepts, we know that every stage terminates. Let $c = \{b^j d^z \mid z \in \mathbb{N}\}$. Since M learns $\mathcal{C}_j$ there has to be a finite sequence $\sigma_k \in \mathit{SegText}(c)$ such that, for $z = M(\sigma_k)$, $c(\varphi_z) =^a c$ as well as, for all finite sequences $\tau \in \mathit{SegText}(c)$, $j = M(\sigma_k \diamond \tau)$. Obviously, $c(\varphi_z) =^{a+1} c$ implies $\mathit{card}(c(\varphi_z) \setminus \mathit{content}(\sigma_k)) \geq a + 2$. Moreover, it can be shown that $\mathit{content}(\sigma_k) \subseteq c(\varphi_z)$, and therefore $k \in P_k$. To see this, suppose $\mathit{content}(\sigma_k) \setminus c(\varphi_z) \neq \emptyset$. Let $x \in \mathit{content}(\sigma_k) \setminus c(\varphi_z)$. Moreover, let $S \subseteq c$ such that $\mathit{card}(S) = a+1$, $S \subseteq c(\varphi_z)$, and $S \cap \mathit{content}(\sigma_k) = \emptyset$. Since $c(\varphi_z) =^{a+1} c$, such set S must exist. Now, let $c' = c \setminus S$ and consider M when fed any text t for c' that begins with σ_k. Since $c' \subseteq c$ and by the properties of σ_k, M converges on t to z. Because of $S \subseteq c(\varphi_z) \setminus c'$ and $x \in c' \setminus c(\varphi_z)$, we obtain $c(\varphi_z) \neq^{a+1} c'$, and therefore M fails to learn $c' \in \mathcal{C}$, a contradiction. Consider $S = \{b^j d^n \mid b^j d^n \in c(\varphi_z)^+_{k'}\}$ for $k' \geq k$. By construction, there must be a $k' \geq k$ such that $\mathit{card}(S \setminus \mathit{content}(\sigma_k)) \geq a + 2$. Consequently, the finite concept c' is defined at the latest in Stage k'.

Second, we show that M fails to learn c'. Let c' be defined in Stage k''. Let r

be the least index in $P_{k''}$ such that, for $z_r = M(\sigma_r)$, it has been verified that $card(c(\varphi_{z_r}) \setminus content(\sigma_r)) \geq a + 2$. By construction, $c' = content(\sigma_r)$. Since M is a conservative IIM and since $content(\sigma_r) \subseteq c(\varphi_{z_r})$, M must converge to z_r when fed any text t for c' that has the initial segment σ_r. Hence, M cannot learn c', a contradiction. Claim 1 follows.

Claim 2. $\mathcal{C} \in Lim^{a+1}Txt$.

The desired IIM M, on input t_y, computes the unique $j \in \mathbb{N}$ such that $content(t_y) \subseteq \{b^j d^z \mid z \in \mathbb{N}\}$. Now M uses y steps of computation to simulate the procedure defined above for deciding whether $\mathcal{C}_j$ contains a finite concept. If y steps of computation do not suffice to make this decision, M guesses $\{b^j d^z \mid z \in \mathbb{N}\}$. If it has been verified that there is a finite concept $c' \in \mathcal{C}_j$, M tests whether or not $content(t_y) = c'$. If it is, M guesses c'; else, M guesses $\{b^j d^z \mid z \in \mathbb{N}\}$. Obviously, M learns as required, and thus we are done. □

In contrast to Theorem 7, it is no longer possible to replace a conservative learner by an equally powerful IIM that is both conservative and consistent. To our knowledge, this is the first result that proves that consistency severely restricts the general learning power when learning of indexable classes is considered. On the other hand, if one is considering exclusively polynomial-time computable learners then consistency has been known to be a severe restriction. That is, the indexable class *PAT* is *not consistently* learnable in polynomial-time from informant [3], provided $\mathcal{P} \neq \mathcal{NP}$ (cf. [21]). But *PAT* is *non-consistently* in polynomial-time, even from text (cf. [14]).

Theorem 18. $LimTxt \setminus \bigcup_{a \in \mathbb{N}} c\text{-}Consv^aTxt \neq \emptyset$.

Proof. The required class $\mathcal{C}_{consv}$ is defined as follows. Fix an acceptable programming system $(\varphi_j)_{j \in \mathbb{N}}$ and an associated complexity measure $(\Phi_j)_{j \in \mathbb{N}}$. For all $k \in \mathbb{N}$, let $c_k = \{b^k d^z \mid z \in \mathbb{N}\}$. Moreover, for all $k \in \mathbb{N}$ with $\varphi_k(k) \downarrow$ and all $j \leq \Phi_k(k)$, let $c_{k,j} = \{b^k d^z \mid z \leq j\}$. Finally, let $\mathcal{C}_{consv}$ be the collection of all those concepts c_k and $c_{k,j}$.

It is well-known that $\mathcal{C}_{consv} \in LimTxt$ (cf. Lange and Zeugmann [16]). Let $a \in \mathbb{N}$. Since the halting problem is undecidable, $\mathcal{C}_{consv} \notin c\text{-}Consv^aTxt$ follows by contraposition of the following claim.

Claim. If there is a consistent IIM that witnesses $\mathcal{C}_{consv} \in Consv^aTxt$, then one can effectively construct an algorithm deciding, for all $k \in \mathbb{N}$, whether or not $\varphi_k(k) \downarrow$.

Suppose that there are a hypothesis space $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ and a consistent IIM M that $Consv^aTxt_{\mathcal{H}}$–learns $\mathcal{C}_{consv}$. We define an algorithm $\mathcal{A}$ that solves the

[3] cf. Section 4 for a formal definition of informant

halting problem.

Algorithm $\mathcal{A}$: On input k execute (A) and (B):

(A) For $z = 0, 1, \ldots$, execute (α1) until (α2) happens.

(α1) Set $t_z = b^k d^0, b^k d^1, \ldots, b^k d^z$ and $S_z = \{b^k d^r \mid z+1 \leq r \leq 2z\}$. Determine $j_z = M(t_z)$.

(α2) $card(S_z \cap h_{j_z}) \geq a+1$ is verified.

(B) Test whether or not $\Phi_k(k) \leq z$. In case it is, output "$\varphi_k(k) \downarrow$." Otherwise, output "$\varphi_k(k) \uparrow$."

We verify $\mathcal{A}$'s correctness as follows. Let $k \in \mathbb{N}$. Since M learns c_k, there has to be some $y \in \mathbb{N}$ such that, for $j_y = M(t_y)$, we must have $h_{j_y} =^a c_k$. Hence, (α2) must happen, and thus algorithm $\mathcal{A}$ terminates on input k.

Suppose that $\varphi_k(k) \downarrow$, but $\mathcal{A}$ outputs "$\varphi_k(k) \uparrow$." Let z be fixed such that, for $j_z = M(t_z)$, $card(S_z \cap h_{j_z}) \geq a+1$. By construction, $\Phi_k(k) > z$, and thus $c_{k,z} = \{b^k d^r \mid r \leq z\} \in \mathcal{C}$. The consistent IIM M, when successively fed the text $t = b^k d^0, b^k d^1, \ldots, b^k d^z \diamond b^k d^0, b^k d^0, \ldots$ for $c_{k,z}$, has output a number j_z such that $c_{k,z} \subseteq h_{j_z}$ and $h_{j_z} \neq^a c_{k,z}$. Since M is conservative, it converges to j_z on t, and thus fails to $Consv^aTxt_{\mathcal{H}}$–learn $c_{k,z}$. □

We directly conclude:

Corollary 19. *For all $a \in \mathbb{N}^+$: c-$Consv^aTxt \subset Consv^aTxt$.*

Note that *c-ConsvTxt* = *ConsvTxt* (cf., e.g., Lange and Zeugmann [22]).

In contrast, one immediately sees that set-driven learning fits in the usual pattern that consistency does not limit the learning capabilities when learning of indexable classes is concerned.

Proposition 5. *For all $a \in \mathbb{N}$: c-$Sdr^aTxt = Sdr^aTxt$.*

Comparing Corollary 19 and Proposition 5, one may readily expect that the learning power of conservative learners and set-driven IIMs does not coincide, if the final hypothesis is allowed to have an *a priori* bounded number of anomalies. This is indeed the case as our next theorem shows.

Theorem 20. *$Consv^1Txt \setminus \bigcup_{a\in\mathbb{N}} Sdr^aTxt \neq \emptyset$.*

Proof. We claim that the indexable class $\mathcal{C}_{consv}$ (cf. the proof of Theorem 18) witnesses the stated separation. First, by Theorem 17, $\mathcal{C}_{consv} \in LimTxt$ implies $\mathcal{C}_{consv} \in Consv^1Txt$. Second, let $a \in \mathbb{N}$. Now $\mathcal{C}_{consv} \notin Sdr^aTxt$ can easily be shown by reducing the halting problem to the learning problem on hand. It is not hard to see that the algorithm $\mathcal{A}$ defined in the demonstration of

Theorem 18 can be used to establish the announced reduction, too. □

Combining these insights with the fact that $Sdr^aTxt \subseteq Consv^aTxt$ for all $a \in \mathbb{N}^+$ (cf. Theorem 5), one arrives at the following result.

Corollary 21. *For all $a \in \mathbb{N}^+$: $Sdr^aTxt \subset Consv^aTxt$.*

As we shall see, set-driven learners are exactly as powerful as learning machines that are both conservative and consistent. To prove this equivalence, we use the following characterization of c-$Consv^aTxt$.

Theorem 22. *For all $\mathcal{C} \in \mathcal{IC}$ and all $a \in \mathbb{N}$: $\mathcal{C} \in c$-$Consv^aTxt$ iff there are a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and a recursively generable family $(T_j)_{j\in\mathbb{N}}$ of finite sets such that*

(1) *for all $j \in \mathbb{N}$, $T_j \subseteq h_j$,*
(2) *for all $c \in \mathcal{C}$, there is a $j \in \mathbb{N}$ such that $T_j \subseteq c \subseteq h_j$,*
(3) *for all $j \in \mathbb{N}$ and all $c \in \mathcal{C}$, if $T_j \subseteq c \subseteq h_j$, then $c =^a h_j$.*

Proof. Necessity. Let $\mathcal{C} \in c$-$Consv^aTxt$. Hence, there are a hypothesis space $\hat{\mathcal{H}} = (\hat{h}_j)_{j\in\mathbb{N}}$ and a consistent IIM M that $Consv^aTxt_{\hat{\mathcal{H}}}$–learns $\mathcal{C}$. Using the ideas from the proof of Lemma 1, we can assume M to be total. We define a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and a family of finite sets $(T_j)_{j\in\mathbb{N}}$ as follows.

Let $(c_j)_{j\in\mathbb{N}}$ be any indexing of $\mathcal{C}$. For all $c \in \mathcal{C}$, let t^c be the canonical text of c. Now, let $r, x \in \mathbb{N}$. Let $k = M(t^{c_r}_x)$. We set $h_{\langle r,x\rangle} = \hat{h}_k$ and $T_{\langle r,x\rangle} = content(t^{c_r}_x)$.

Obviously, $(h_{\langle r,x\rangle})_{r,x\in\mathbb{N}}$ is an indexable family of recursive concepts. Furthermore, $(T_{\langle r,x\rangle})_{r,x\in\mathbb{N}}$ is recursively generable and all sets $T_{\langle r,x\rangle}$ are finite. It remains to show that $\mathcal{H} = (h_{\langle r,x\rangle})_{r,x\in\mathbb{N}}$ and $(T_{\langle r,x\rangle})_{r,x\in\mathbb{N}}$ fulfill the announced properties. By construction, (1) is satisfied, since M is consistent.

For proving (2), let $c \in \mathcal{C}$. We have to show that there is at least one index j such that $T_j \subseteq c \subseteq h_j$. Let $r \in \mathbb{N}$ be fixed such that $c_r = c$. Since M has to infer c from t^c, there have to be $k, x \in \mathbb{N}$ such that $\hat{h}_k =^a c$ and, for all $y \geq x$, $M(t^c_y) = k$. Since M is consistent, we know that $c \subseteq \hat{h}_k$. By definition, $h_{\langle r,x\rangle} = \hat{h}_k$. Hence, $T_{\langle r,x\rangle} = content(t^c_x) \subseteq c \subseteq h_{\langle r,x\rangle}$, and we are done.

Finally, we prove (3). Suppose that there are $r, x \in \mathbb{N}$ and some $c \in \mathcal{C}$ such that $T_{\langle r,x\rangle} \subseteq c \subseteq h_{\langle r,x\rangle}$ and $h_{\langle r,x\rangle} \neq^a c$. Let $k = M(t^{c_r}_x)$. Since $c \subset h_{\langle r,x\rangle} = \hat{h}_k$ and since M is conservative, M converges to k when fed any text t for c having the initial segment $t^{c_r}_x$. However, because of $h_{\langle r,x\rangle} \neq^a c$, M cannot $Consv^aTxt_{\hat{\mathcal{H}}}$–identify c from t, a contradiction.

Sufficiency. We define a consistent IIM M that $Consv^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$. So, let

$c \in \mathcal{C}$, let $t \in \mathit{Text}(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y proceed as follows:
If $y = 0$ or $M(t_{y-1}) = ?$, then goto (B). Otherwise, goto (A).
(A) Let $j = M(t_{y-1})$. Test whether or not $\mathit{content}(t_y) \subseteq h_j$. In case it is, output j. Otherwise, goto (B).
(B) For $j = 0, \ldots, y$, generate T_j and test whether or not $T_j \subseteq \mathit{content}(t_y)$ and $\mathit{content}(t_y) \subseteq h_j$. In case there exists a j fulfilling the test, output the minimal one. Otherwise, output ?."

Since all sets T_j are uniformly recursively generable and finite, we see that M is an IIM. By definition, M is consistent. Moreover, M changes its mind only in case it detects an inconsistency in (A). Hence, M learns conservatively provided it converges on t to a correct hypothesis.

Claim 1. *M converges on t.*

Let $k = \min\{z \mid T_z \subseteq c,\ c \subseteq h_z,\ h_z =^a c\}$. Consider $T_0, \ldots, T_k$. Since $t \in \mathit{Text}(c)$, there must be a $y \geq k$ such that $T_k \subseteq \mathit{content}(t_y) \subseteq h_k$. That means, at least after having fed t_y to M, the IIM M outputs a number. Furthermore, since, for all $y' \geq y$, $T_k \subseteq \mathit{content}(t'_y) \subseteq h_k$, the IIM M never changes its mind to some $j > k$ when processing any initial segment $t_{y'}$. Finally, M changes its mind iff it receives some string that is misclassified by its current guess. Since M is consistent, any hypothesis once rejected is never repeated in some subsequent step. Since at least k can never be rejected, M has to converge.

Claim 2. *If M converges, say to j, then $h_j =^a c$.*

Suppose that M converges on t to j and $h_j \neq^a c$.

Case 1. $c \setminus h_j \neq \emptyset$.

There is at least one element $x \in c \setminus h_j$ that has to appear eventually, i.e., $x \in \mathit{content}(t_y)$ for some y. Thus, $\mathit{content}(t_y) \not\subseteq h_j$, a contradiction.

Case 2. $h_j \setminus c \neq \emptyset$.

We may restrict ourselves to the case $c \subset h_j$, since otherwise we are again in Case 1. Since, for all sufficiently large y, $T_j \subseteq \mathit{content}(t_y) \subseteq c$, we obtain $T_j \subseteq c$. By Property (3), we may conclude $c =^a h_j$, a contradiction. □

Theorem 23. *For all $a \in \mathbb{N}$: $c\text{-}\mathit{Consv}^a\mathit{Txt} = \mathit{Sdr}^a\mathit{Txt}$.*

Proof. Let $a \in \mathbb{N}$. $\mathit{Sdr}^a\mathit{Txt} \subseteq c\text{-}\mathit{Consv}^a\mathit{Txt}$ has already been verified in the demonstration of Theorem 5. It remains to show that $c\text{-}\mathit{Consv}^a\mathit{Txt} \subseteq \mathit{Sdr}^a\mathit{Txt}$.

Let $\mathcal{C} \in c\text{-}Consv^aTxt$. By Theorem 22, there are a recursively generable family $(T_j)_{j\in\mathbb{N}}$ of finite sets and a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ such that

(1) for all $j \in \mathbb{N}$, $T_j \subseteq h_j$,
(2) for all $c \in \mathcal{C}$, there is a $j \in \mathbb{N}$ such that $T_j \subseteq c \subseteq h_j$,
(3) for all $j \in N$ and all $c \in \mathcal{C}$, if $T_j \subseteq c \subseteq h_j$, then $c =^a h_j$.

Having a closer look at the demonstration of Theorem 22 one immediately sees that there is an indexing $(c_j)_{j\in\mathbb{N}}$ of $\mathcal{C}$ and a total recursive function f such that, for all $j \in \mathbb{N}$, $T_{f(j)} \subseteq c_j$, $c_{f(j)} \subseteq h_j$ and $c_{f(j)} =^a h_j$. We first define a new recursively generable family $(T'_j)_{j\in\mathbb{N}}$ of finite sets. Afterwards we use the family $(T'_j)_{j\in\mathbb{N}}$ to create a rearrangement-independent [4] IIM M that $Consv^aTxt_{\mathcal{H}}$–learns $\mathcal{C}$. In a concluding step, we construct a set-driven learner M' which witnesses $\mathcal{C} \in Lim^aTxt$.

We define the new family of finite tell-tale sets as follows. For all $j \in \mathbb{N}$, we set $T'_j = \bigcup_{n\leq j} T_n \cap c_{f(j)}$. Obviously, $(T'_j)_{j\in\mathbb{N}}$ is also a recursively generable family of finite sets that fulfills Conditions (1) and (3). Moreover, by the properties of f, Condition (2) is satisfied as well.

The desired IIM M is defined as follows. Let $c \in \mathcal{C}$, let $t \in Text(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y do the following:
For all $k \leq y$, generate T'_k and test whether or not $T'_k \subseteq content(t_y) \subseteq h_k$. In case there is a k fulfilling the test, output the minimal one. Otherwise, output ?."

M is rearrangement-independent by definition. Moreover, M is consistent.

Claim 1. *M is conservative.*

Let $k, y \in \mathbb{N}$ such that $M(t_y) = k$ and $M(t_{y+1}) \neq M(t_y)$. Now, if $M(t_{y+1}) = ?$, we directly obtain $content(t_{y+1}) \not\subseteq h_k$. Next, let $M(t_{y+1}) = j$ for some $j \in \mathbb{N}$. It remains to show that $content(t_{y+1}) \not\subseteq h_k$.

Let $k < j$. By M's definition, $content(t_{y+1}) \not\subseteq h_k$. Now let $j < k$ and suppose that $content(t_{y+1}) \subseteq h_k$. By definition, M has verified that $T'_j \subseteq content(t_{y+1}) \subseteq h_j$. Since $j < k$, since $T'_j \subseteq content(t_{y+1})$, and since, by assumption, $content(t_{y+1}) \subseteq h_k$, we obtain $T'_j \subseteq T'_k$. By definition of M, $M(t_y) = k$ implies $T'_k \subseteq content(t_y)$, and thus $T'_j \subseteq content(t_y)$. Because of $j < k$, we may conclude that $M(t_y) = j$, contradicting $M(t_y) = k$. Therefore Claim 1 is proved.

[4] An IIM M is said to be rearrangement independent for $\mathcal{C}$ provided that, for all $t, t' \in Text(\mathcal{C})$ and all $y \in \mathbb{N}$, if $content(t_y) = content(t'_y)$, then $M(t_y) = M(t'_y)$.

Claim 2. *M learns c from t.*

Let $m = \min\{z \mid T'_z \subseteq c,\, c \subseteq h_z,\, h_z =^a c\}$. By Condition (3), we obtain that $c \setminus h_j \neq \emptyset$ for all $j < m$ provided $T'_j \subseteq c$. Consequently, every possible candidate hypothesis $j < m$ must be abandoned at some time. Thus, M converges to m. This proves Claim 2.

To sum up, M is a rearrangement-independent machine that $\mathit{ConsvTxt}^a_{\mathcal{H}}$–learns $\mathcal{C}$. We continue by defining a hypothesis space $\mathcal{H}'$ and the required set-driven IIM M' such that M' $\mathit{LimTxt}^a_{\mathcal{H}'}$–learns $\mathcal{C}$.

Let $\mathcal{H}' = (h'_j)_{j\in\mathbb{N}}$ be the canonical enumeration of all concepts in $\mathcal{C}$ and of all finite concepts over the learning domain $\mathcal{X}$. Before defining M', we need the notion of the *repetition free version* of a given text t, denoted by $\mathit{rfv}(t)$.

Let $t = (x_j)_{j\in\mathbb{N}}$ be any text. Initially, we set $\mathit{rfv}(t_0) = x_0$ and proceed inductively. For all $y \in \mathbb{N}$, we set $\mathit{rfv}(t_{y+1}) = \mathit{rfv}(t_y)$, if $x_{y+1} \in \mathit{content}(\mathit{rfv}(t_y))$. Otherwise, we set $\mathit{rfv}(t_{y+1}) = \mathit{rfv}(t_y) \diamond x_{y+1}$. Obviously, given any initial segment t_y of a text t, one can effectively compute $\mathit{rfv}(t_y)$.

Now, we are ready to define M'. So, let $c \in \mathcal{C}$, $t \in \mathit{Text}(c)$, and $y \in \mathbb{N}$.

IIM M': "On input t_y do the following:
Compute $\mathit{rfv}(t_y)$. If $M(\mathit{rfv}(t_y)) = ?$, then output the canonical index of $\mathit{content}(t_y)$ in $\mathcal{H}'$. Otherwise, fix $j = M(\mathit{rfv}(t_y))$ and output the canonical index of h_j in $\mathcal{H}'$."

We show that M' learns as required.

Claim 3. *M' is set-driven.*

Let $t, t' \in \mathit{Text}(\mathcal{C})$ and let $x, y \in \mathbb{N}$ such that $\mathit{content}(t_x) = \mathit{content}(t'_y)$. By definition, $|\mathit{rfv}(t_x)| = |\mathit{rfv}(t'_y)|$. Therefore, $M(\mathit{rfv}(t_x)) = M(\mathit{rfv}(t'_y))$, since M is rearrangement-independent, and thus $M(t_x) = M(t'_y)$.

Claim 4. *M' learns c from t.*

We distinguish two cases.

Case 1. *c is finite.*

Then there exists an $x \in \mathbb{N}$ such that $\mathit{content}(t_y) = c$. On the one hand, if $M(\mathit{rfv}(t_y)) = ?$, then, by definition, M' converges to the canonical index of the finite concept $\mathit{content}(t_y)$ in $\mathcal{H}'$. On the other hand, if $M(\mathit{rfv}(t_y)) = j$, then $c \subseteq h_j$, since M is consistent. Since M is conservative, it converges to j when fed any text for c that has the initial segment t_y. Hence, $h_j =^a c$, and thus M' behaves as required.

Case 2. *c is infinite.*

Since c is infinite, $rfv(t)$ constitutes a text for c. Recall that M learns c from $rfv(t)$. Consequently, there have to be $y, k \in \mathbb{N}$ such that, for all $r \in \mathbb{N}$, $M(rfv(t)_{y+r}) = k$ and $h_k =^a c$. Hence, past point y, M' always outputs the canonical index of h_k in $\mathcal{H}'$, and thus M' infers c. This proves Claim 4.. □

When learning with an *a priori* bounded number of allowed anomalies is considered, it can be shown that there is an infinite hierarchy of more and more powerful set-driven, conservative, limit, and behaviorally correct learners, respectively, parameterized in the number of allowed anomalies. The following theorem provides the missing piece to establish the existence of these infinite hierarchies.

Theorem 24. *For all* $a \in \mathbb{N}$*:* $Fin^{2a+1}Txt \setminus Bc^aTxt \neq \emptyset$.

Proof. Let $a \in \mathbb{N}$. We let $\mathcal{C}_a$ be the collection of all infinite concepts $c \subseteq \{b\}^*$ that meet $card(\{b\}^* \setminus c) \leq 2a + 1$. One the one hand, one easily sees $\mathcal{C} \in Fin^{2a+1}Txt$. On the other hand, suppose that $\mathcal{C}_a \in Bc^aTxt$. Now, by Corollary 15, for $c = \{b\}^*$, there has to be a finite set $T_c \subseteq c$ such that, for all $c' \in \mathcal{C}_a$, $T_c \subseteq c' \subseteq c$ implies $c' =^{2a} c$. Obviously, such a finite set cannot exist, and thus we are done. □

We conclude this section by providing, for all $a \in \mathbb{N}$, a characterization of the collection of all $Consv^aTxt$–identifiable classes.

Theorem 25. *For all* $\mathcal{C} \in \mathcal{IC}$ *and all* $a \in \mathbb{N}$*:* $\mathcal{C} \in Consv^aTxt$ *iff there are a hypothesis space* $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$*, a computable relation* $\prec$ *over* $\mathbb{N}$*, and a recursively generable family* $(T_j)_{j\in\mathbb{N}}$ *of finite sets such that*

(1) *for all* $c \in \mathcal{C}$*, there is a* j *such that* $T_j \subseteq c$*, and* $h_j =^a c$*,*
(2) *for all* $c \in \mathcal{C}$*, all* $k \in \mathbb{N}$*, and all finite sets* $A \subseteq c$*, if* $T_k \subseteq c$ *and* $h_k \neq^a c$*, then there is a* j *such that* $k \prec j$*,* $A \subseteq T_j$*, and* $h_j =^a c$*,*
(3) *for all* $c \in \mathcal{C}$*, there is no infinite sequence* $(k_r)_{r\in\mathbb{N}}$ *such that, for all* $r \in \mathbb{N}$*,* $k_r \prec k_{r+1}$ *and* $\bigcup_{r\in\mathbb{N}} T_{k_r} = c$*,*
(4) *for all* $c \in \mathcal{C}$ *and all* $k, j \in \mathbb{N}$*, if* $k \prec j$ *and* $T_j \subseteq c$*, then* $T_j \setminus h_k \neq \emptyset$*.*

Proof. Necessity. Let $\mathcal{C} \in Consv^aTxt$. Therefore, there are a hypothesis space $\hat{\mathcal{H}} = (\hat{h}_j)_{j\in\mathbb{N}}$ and an IIM M that $Consv^aTxt_{\hat{\mathcal{H}}}$–learns $\mathcal{C}$. Without of loss of generality, we may assume that M is total. First, we construct a hypothesis space $\tilde{\mathcal{H}} = (\tilde{h}_j)_{j\in\mathbb{N}}$ and a recursively generable family $(\tilde{T}_j)_{j\in\mathbb{N}}$ of finite sets. Then we describe a procedure enumerating a certain subset of $\tilde{\mathcal{H}}$ that forms the required hypothesis space $\mathcal{H}$. Finally, we define the required computable relation $\prec$.

Let $(\sigma_j)_{j\in\mathbb{N}}$ be an effective enumeration of all finite, non-null sequences of elements from the underlying learning domain $\mathcal{X}$ such that, for all $m, n \in \mathbb{N}$, $\sigma_m \sqsubset \sigma_n$ implies $m < n$. Furthermore, for all $n, y \in \mathbb{N}$, we set $\tilde{h}_{\langle n,y\rangle} = \hat{h}_n$. The family $(\tilde{T}_{\langle n,y\rangle})_{n,y\in\mathbb{N}}$ is defined as follows. For all $n, y \in \mathbb{N}$, we set

$$\tilde{T}_{\langle n,y\rangle} = \begin{cases} \mathit{content}(\sigma_y), & \text{if } M(\sigma_y) = n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly, $(\tilde{T}_{\langle n,y\rangle})_{n,y\in\mathbb{N}}$ is a uniformly recursively generable family of finite sets.

Claim. For all $c \in \mathcal{C}$, there are $n, y \in \mathbb{N}$ such that $\tilde{h}_{\langle n,y\rangle} =^a c$ and $\tilde{T}_{\langle n,y\rangle} \neq \emptyset$.

Let t^c be the canonical text of c. Since M learns c, there are $n, z \in \mathbb{N}$ such that $M(t^c_z) = n$ and $\hat{h}_n =^a c$. Let $y \in \mathbb{N}$ with $\sigma_y = t^c_z$. By construction, $\tilde{T}_{\langle n,y\rangle} \neq \emptyset$ as well as $\tilde{h}_{\langle n,y\rangle} =^a c$, and thus the claim follows.

We proceed with the definition of the desired hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and the relation $\prec$. For this purpose, we define a recursive function f as follows. Set $f(0) = k$, where k is the least index with $\tilde{T}_k \neq \emptyset$. Note that, by the the claim above, such an index k must exist. For all $j \geq 1$, set

$$f(j) = \begin{cases} j, & \text{if } \tilde{T}_j \neq \emptyset, \\ f(j-1), & \text{otherwise.} \end{cases}$$

For all $j \in \mathbb{N}$, we define $h_j = \tilde{h}_{f(j)}$ and $T_j = \tilde{T}_{f(j)}$. Let $k, j \in \mathbb{N}$ and let $m, n, y, z \in \mathbb{N}$ be the uniquely determined numbers such that $f(k) = \langle m, y\rangle$ and $f(j) = \langle n, z\rangle$. Then, we let $k \prec j$ iff $m \neq n$ and $\sigma_y \sqsubset \sigma_z$.

Clearly, $(T_j)_{j\in\mathbb{N}}$ is a uniformly recursively generable family of finite sets and the relation $\prec$ is computable. It remains to show that (1) to (4) are fulfilled. Obviously, (1) is a direct consequence of the claim above.

We next verify (2). Let $c \in \mathcal{C}$, let $A \subseteq c$ be any finite set, and let $k \in \mathbb{N}$ be any index such that $T_k \subseteq c$ and $h_k \neq^a c$. We have to show that there is an index j such that $k \prec j$, $A \subseteq T_j$, and $h_j =^a c$. Due to our construction, we have $T_k = \tilde{T}_{f(k)}$ and $h_k = \tilde{h}_{f(k)}$. Let $m, y \in \mathbb{N}$ be the uniquely determined numbers with $f(k) = \langle m, y\rangle$. We know that $M(\sigma_y) = m$ and $c \neq^a \hat{h}_m$. Moreover, $T_k = \mathit{content}(\sigma_y) \subseteq c$. Hence, σ_y is an initial segment of a text for c. Let t^c be the canonical text of c. Since $A \subseteq c$, there exists a $b \in \mathbb{N}$ such that $A \subseteq \mathit{content}(t^c_b)$. Thus, there has to be an $r \in \mathbb{N}$ such that, for $n = M(\sigma_y \diamond t^c_{b+r})$, the condition $\hat{h}_n =^a c$ is satisfied, since M has to learn c from every text for it. Furthermore, since $\sigma_y \diamond t^c_{b+r}$ is a finite sequence, there exists an index z with

$\sigma_z = \sigma_y \diamond t^c_{b+r}$. By construction, we get $\tilde{T}_{\langle n,z\rangle} = \mathit{content}(\sigma_z) \neq \emptyset$, $A \subseteq \tilde{T}_{\langle n,z\rangle}$, and $\tilde{h}_{\langle n,z\rangle} =^a c$. Thus, there is a number j such that $f(j) = \langle n, z\rangle$. Since $\sigma_y \sqsubset \sigma_z$ and $m \neq n$, we obtain $k \prec j$, and therefore (2) is proved.

We proceed with the demonstration of (3). Looking at the definition of the relation $\prec$, one sees that $k \prec j$ implies $T_k \subseteq T_j$. Suppose there is an infinite sequence $(k_r)_{r\in\mathbb{N}}$ such that $k_r \prec k_{r+1}$ and $\bigcup_{r\in\mathbb{N}} T_{k_r} = c$. Since $T_{k_r} \subseteq T_{k_{r+1}}$, in the limit we get a text t for $c \in \mathcal{C}$ on which M changes its mind infinitely often, a contradiction. Hence, (3) is proved.

Finally we show (4). Let $c \in \mathcal{C}$, and let $k, j \in \mathbb{N}$ such that $k \prec j$ and $T_j \subseteq c$. Furthermore, let $m, n, y, z \in \mathbb{N}$ be the uniquely determined numbers such that $f(k) = \langle m, y\rangle$ and $f(j) = \langle n, z\rangle$. By definition of the relation $\prec$, we get $\sigma_y \sqsubset \sigma_z$ as well as $m \neq n$. Moreover, by the definition of the tell-tale family, $M(\sigma_y) = m$ and $M(\sigma_z) = n$. Since $T_j = \mathit{content}(\sigma_z)$ and $T_j \subseteq c$, we see that σ_z is an initial segment of some text t for $c \in \mathcal{C}$ on which M successively outputs m and n. Since M is conservative, we obtain $T_j \setminus \hat{h}_m \neq \emptyset$. Finally, by construction, we have $h_k = \tilde{h}_{\langle m,y\rangle} = \hat{h}_m$, and thus $T_j \setminus h_k \neq \emptyset$. Hence, (4) follows.

Sufficiency. It suffices to define an IIM M that $\mathit{Consv}^a\mathit{Txt}_{\mathcal{H}}$–learns $\mathcal{C}$. Let $c \in \mathcal{C}$, let $t \in \mathit{Text}(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y do the following:

If $y = 0$ or $y > 0$ and $M(t_{y-1}) = ?$, then goto (A). Otherwise, goto (B).

(A) Search for the least $k \leq y$ such that $T_k \subseteq \mathit{content}(t_y)$. In case it is found, set $y_k = y$ and output k. Otherwise, output ?.

(B) Let $k = M(t_{y-1})$. Search for the least $j \leq y$ such that $k \prec j$ and $\mathit{content}(t_{y_k}) \subseteq T_j \subseteq \mathit{content}(t_y)$. In case such j is found, set $y_j = y$ and output j. Otherwise, output k."

Since all sets T_j are uniformly recursively generable and finite, and since the relation $\prec$ is computable, we directly obtain that M is an IIM. Moreover, Condition (1) guarantees that M outputs at least once a hypothesis. We proceed in showing that M $\mathit{Consv}^a\mathit{Txt}_{\mathcal{H}}$–learns c from t.

Claim 1. *If M converges, say to k, then $h_k =^a c$.*

Note that $T_k \subseteq c$, since otherwise k cannot be any of M's guesses. Suppose $h_k \neq^a c$. By (2), there is an index j such that $k \prec j$, $\mathit{content}(t_{y_k}) \subseteq T_j$, and $h_j =^a c$. Hence, there is a $y \in \mathbb{N}$ with $\mathit{content}(t_{y_k}) \subseteq T_j \subseteq \mathit{content}(t_y)$. Thus, $M(t_y) \neq k$, contradicting the assumption that M converges to k.

Claim 2. *M is conservative.*

This is an immediate consequence of (4) and the definition of M.

Claim 3. *M converges on t.*

Observe that M outputs at least once a hypothesis, say k. As long as M does not find a j such that $k \prec j$ and $content(t_{y_k}) \subseteq T_j \subseteq content(t_x)$, this hypothesis is repeated. Hence, as long as M finds only finitely many j's in (B), it converges. Consequently, if M does not converge, it finds an infinite sequence $(k_r)_{r\in\mathbb{N}}$ such that $k_r \prec k_{r+1}$ for all $r \in \mathbb{N}$. But every mind change implies an update of the value of the variable y_{k_r}. Thus, for all $z \in \mathbb{N}$, there exist $y_{k_r}, y \in \mathbb{N}$ with $content(t_z) \subseteq content(t_{y_{k_r}}) \subseteq T_{k_r} \subseteq content(t_y)$. Therefore, we immediately obtain $\bigcup_{r\in\mathbb{N}} T_{k_r} = c$, a contradiction to (3). This proves the claim, and hence the verification of the sufficiency part is completed. □

Compared to the learning devices introduced in the other characterization theorems, the IIM defined in the proof of Theorem 25 uses a different technique to detect that its actual hypothesis may be incorrect. Clearly, no IIM can prove that its actual guess is really correct, unless it finitely learns. Hence, the machine has to collect evidence allowing it to decide whether or not it should prefer a new hypothesis instead of maintaining its actual one. The machine defined in the proof of Theorem 25 achieves this goal by using *a priori* knowledge concerning both the hypothesis space and the family of tell-tale sets. This *a priori* knowledge is provided by the computable relation $\prec$.

4 Learning from Positive and Negative Data

In this section, we briefly summarize the results that can be obtained when learning with anomalies from both positive and negative examples is studied.

Let $\mathcal{X}$ be the learning domain, let $c \subseteq \mathcal{X}$ be a concept, and let $i = ((x_n, b_n))_{n\in\mathbb{N}}$ be any infinite sequence of elements from $\mathcal{X} \times \{+,-\}$ such that $content(i) = \{x_n \mid n \in \mathbb{N}\} = \mathcal{X}$, $content^+(i) = \{x_n \mid n \in \mathbb{N}, b_n = +\} = c$, and $content^-(i) = \{x_n \mid n \in \mathbb{N}, b_n = -\} = \mathcal{X} \setminus c = \bar{c}$. Then, we refer to i as an *informant*. By $Info(c)$ we denote the set of all informants for c. Moreover, let $i = ((x_n, b_n))_{n\in\mathbb{N}}$ be an informant and let $y \in \mathbb{N}$. Then, i_y denotes the initial segment of i of length $y + 1$. By $content(i_y)$, $content^+(i_y)$, and $content^-(i_y)$ we denote the sets $\{x_j \mid j \leq y\}$, $\{x_j \mid j \leq y,\, b_j = +\}$, and $\{x_j \mid j \leq y,\, b_j = -\}$, respectively. Let $(w_j)_{j\in\mathbb{N}}$ be the fixed enumeration of $\mathcal{X}$. Then, for every concept $c \subseteq \mathcal{X}$, we define the *canonical informant* to be the sequence $(w_j, c(w_j))_{j\in\mathbb{N}}$.

For all $a \in \mathbb{N} \cup \{*\}$, the learning models Fin^aInf, Sdr^aInf, $Consv^aInf$, Lim^aInf and Bc^aInf are defined analogously as their text counterparts by replacing text by informant.

Since, for all $\mathcal{C} \in \mathcal{IC}$, $\mathcal{C} \in$ *ConsvInf* as well as $\mathcal{C} \in$ *SdrInf* (cf. Gold [11]), we may easily conclude:

Corollary 26.
For all $a \in \mathbb{N} \cup \{*\}$: $\mathit{ConsvInf} = \mathit{Consv}^a\mathit{Inf} = \mathit{Sdr}^a\mathit{Inf} = \mathit{Lim}^a\mathit{Inf} = \mathit{Bc}^a\mathit{Inf}$.

Next we study finite learning with anomalies. As in the case of learning from positive data, there is a difference between finite learning with an *a priori* bounded number of allowed anomalies and finite learning with a bounded number of allowed anomalies. While the latter is invariant to the choice of the hypothesis space, the former is not.

Theorem 27. *For all* $\mathcal{C} \in \mathcal{IC}$: $\mathcal{C} \in \mathit{Fin}^*\mathit{Inf}$ *iff there are an indexing* $(c_j)_{j\in\mathbb{N}}$ *of* $\mathcal{C}$ *and a recursively generable family* $(S_j)_{j\in\mathbb{N}}$ *of finite sets such that*

(1) *for all* $j, k \in \mathbb{N}$, *if* $S_j \cap c_k = S_j \cap c_j$, *then* $c_k =^* c_j$.

Proof. Necessity. Assume that a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and an IIM M that $\mathit{Fin}^*\mathit{Inf}_{\mathcal{H}}$–learns $\mathcal{C}$ are given. Moreover, let $(c_j)_{j\in\mathbb{N}}$ be any indexing of $\mathcal{C}$. The family $(S_j)_{j\in\mathbb{N}}$ is defined as follows.

Let $j \in \mathbb{N}$ and let i^{c_j} be the canonical informant of c_j. Since M finitely infers c_j, there exists a least $y \in \mathbb{N}$ such that $M(i^{c_j}_y) = m$ for some $m \in \mathbb{N}$. We set $S_j = \mathit{content}(i^{c_j}_y)$.

We have to show that $(S_j)_{j\in\mathbb{N}}$ fulfills Property (1). Suppose $j, k \in \mathbb{N}$ such that $S_j \cap c_k = S_j \cap c_j$. By construction, there is an initial segment of c_j's canonical informant i^{c_j}, say $i^{c_j}_y$, such that $\mathit{content}(i^{c_j}_y) = S_j$ and $M(i^{c_j}_y) = m$. Now, M finitely learns c_j, thus $h_m =^* c_j$. Since $S_j \cap c_k = S_j \cap c_j$, $i^{c_j}_y$ is also an initial segment of some informant i for c_k. But M finitely infers c_k when fed i and $M(i_y) = m$. Hence, we obtain $h_m =^* c_k$.

Sufficiency. We set $\mathcal{H} = (c_j)_{j\in\mathbb{N}}$ and prove that there is an IIM M that $\mathit{Fin}^*\mathit{Inf}_{\mathcal{H}}$–learns $\mathcal{C}$. So, let $c \in \mathcal{C}$, let $i \in \mathit{Info}(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input i_y do the following:
If $y = 0$ or $M(i_{y-1}) = ?$, goto (A). Otherwise, output $j = M(i_{y-1})$.
(A) For $j = 0, \dots, y$, generate S_j and test whether or not $S_j \cap c_j \subseteq \mathit{content}^+(i_y)$ and $S_j \cap \bar{c}_j \subseteq \mathit{content}^-(i_y)$. In case there is a j fulfilling the test, output the minimal one. Otherwise, output ?."

One directly sees that M learns as required. □

The next result provides some evidence that it is a bit more complicated to characterize $\mathit{Fin}^a\mathit{Inf}$ for any $a \in \mathbb{N}^+$.

Theorem 28. *Let $a \in \mathbb{N}^+$. There is an indexable class $\mathcal{C} \in Fin^aInf$ such that, for all class preserving hypothesis spaces $\mathcal{H}$ for $\mathcal{C}$, $\mathcal{C}$ is not $Fin^aInf_{\mathcal{H}}$–learnable.*

Proof. We discuss the case of $a = 1$ only. The adaptation to the cases of $a \in \mathbb{N}$, $a > 1$, should be obvious.

Fix an acceptable programming system $(\varphi_j)_{j\in\mathbb{N}}$ and an associated complexity measure $(\Phi_j)_{j\in\mathbb{N}}$. For all $k \in \mathbb{N}$, we let $c_k = \{b^k d^j \mid j \in \mathbb{N}\}$. The required indexable class $\mathcal{C}$ is defined as follows. For all $k \in \mathbb{N}$ with $\varphi_k(k)\uparrow$, $\mathcal{C}$ contains the concept c_k, while, for all $k \in \mathbb{N}$ with $\varphi_k(k)\downarrow$, $\mathcal{C}$ contains the concepts $c'_k = c_k \setminus \{b^k d^{\Phi_k(k)+1}\}$ and $c''_k = c_k \setminus \{b^k d^{\Phi_k(k)+2}\}$.

It is not hard to see that $\mathcal{C} \in Fin^1Inf$. Suppose, there are a class preserving hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and an IIM M that $Fin^1Inf_{\mathcal{H}}$–infers $\mathcal{C}$. Then, the following algorithm $\mathcal{A}$, based on $\mathcal{H}$ and M, solves the halting problem.

Algorithm $\mathcal{A}$: On input k proceed as follows:

For $z = 0, 1, \ldots$, execute (α) until $(\beta1)$ or $(\beta2)$ happens.

(α) Test whether or not $\Phi_k(k) \le z$. In case it is not, fix the initial segment $i^{c_k}_z$ of c_k's canonical informant i^{c_k} and determine $M(i^{c_k}_z)$.

$(\beta1)$ $\Phi_k(k) \le z$ has been verified. Then, output "$\varphi_k(k)\downarrow$" and stop.

$(\beta2)$ $M(i^{c_k}_z) \neq\ ?$ has been verified. Then, output "$\varphi_k(k)\uparrow$" and stop.

The verification of $\mathcal{A}$'s correctness is straightforward. □

Analogously to Theorem 13, finite learning with an *a priori* bounded number of allowed anomalies can be characterized as follows.

Theorem 29. *For all $\mathcal{C} \in \mathcal{IC}$ and all $a \in \mathbb{N}$: $\mathcal{C} \in Fin^aInf$ iff there are a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ and a recursively generable family $(S_j)_{j\in\mathbb{N}}$ of finite sets such that:*

(1) *for all $c \in \mathcal{C}$, there is a $j \in \mathbb{N}$ such that $S_j \cap c = S_j \cap h_j$,*
(2) *for all $j \in \mathbb{N}$ and all $c \in \mathcal{C}$, if $S_j \cap c = S_j \cap h_j$, then $c =^a h_j$.*

Proof. The theorem can easily be proved by combining the ideas from the demonstration of Theorems 13 and 27. We omit the details. □

Next we show that the known inclusions $FinTxt \subset FinInf \subset ConsvTxt$ (cf. Lange and Zeugmann [15]) generalize as follows.

Theorem 30. *$Fin^aTxt \subset Fin^aInf \subset Consv^aTxt$ for all $a \in \mathbb{N} \cup \{*\}$.*

Proof. By definition, $Fin^aTxt \subseteq Fin^aInf$ for all $a \in \mathbb{N} \cup \{*\}$. Let $\mathcal{C}$ be the collection of all singleton concepts $\{b^k\}$, $k \in \mathbb{N}^+$, and of $\{b\}^+$. One easily verifies that $\mathcal{C} \in FinInf \setminus Fin^*Txt$.

Next let $c = \{b\}^*$ and let $c_k = \{b^0, \ldots, b^k, d^k\}$ for all $k \in \mathbb{N}$. Furthermore, let $\mathcal{C}_{sep}$ be the collection of all finite concept c_k, $k \in \mathbb{N}$, and of c. It is not hard to see that $\mathcal{C}_{sep} \in \mathit{ConsvTxt}$. Moreover, one directly sees that there cannot be a finite set S for the concept c satisfying Property (1) of Theorem 27. Hence, we have $\mathcal{C}_{sep} \notin \mathit{Fin}^*\mathit{Inf}$.

We verify $\mathit{Fin}^a\mathit{Inf} \subseteq \mathit{Consv}^a\mathit{Txt}$ for all $a \in \mathbb{N}$. Let $\mathcal{C} \in \mathit{Fin}^a\mathit{Inf}$. By Theorem 29, there is a hypothesis space $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ and a recursively generable family $(S_j)_{j \in \mathbb{N}}$ of finite sets such that:

(1) for all $c \in \mathcal{C}$, there is a $j \in \mathbb{N}$ such that $S_j \cap c = S_j \cap h_j$,
(2) for all $j \in \mathbb{N}$ and all $c \in \mathcal{C}$, if $S_j \cap c = S_j \cap h_j$, then $c =^a h_j$.

The required conservative IIM M also uses the hypothesis space $\mathcal{H}$ and is defined as follows.

Let $c \in \mathcal{C}$, let $t \in \mathit{Text}(c)$, and let $y \in \mathbb{N}$.

IIM M: "On input t_y do the following:
If $y = 0$ or $M(t_{y-1}) = ?$, goto (A). Otherwise, set $j = M(t_{y-1})$ and test whether or not $S_j \cap \mathit{content}(t_y) \subseteq h_j$. In case it is, output j. Otherwise, goto (A).
(A) For $j = 0, \ldots, y$, generate S_j and test whether or not $S_j \cap h_j \subseteq \mathit{content}(t_y)$ and $S_j \cap \mathit{content}(t_y) \subseteq h_j$. In case there exists a j fulfilling the test, output the minimal one. Otherwise, output ?."

By definition, M performs exclusively justified mind changes, and thus it is conservative. It suffices to show that M learns as required.

Let $k = \min\{j \mid S_j \cap h_j = S_j \cap c\}$. Since M never outputs a hypothesis that has been rejected once, it is not hard to see that M must converge, say to k'. By construction, we know that $S_{k'} \cap h_{k'} \subseteq c$. Moreover, for almost all $y \in \mathbb{N}$, $S_j \cap \mathit{content}(t_y) \subseteq h_j$ is fulfilled. Combining this with $S_{k'} \cap c \subseteq h_{k'}$, we may conclude that $S_{k'} \cap h_{k'} = S_{k'} \cap c$. Hence, by Condition (2), we obtain $c =^a h_{k'}$.

Finally, $\mathit{Fin}^*\mathit{Inf} \subseteq \mathit{Consv}^*\mathit{Txt}$ can be shown by applying similar arguments as above. We omit the details. □

Note that it is not hard to verify that the results obtained so far prove the existence of an infinite hierarchy of more and more powerful finite learners parameterized in the number of allowed anomalies.

5 Conclusions

The present paper provided a systematic study of inductive inference of approximations for recursive concepts. These approximations have been allowed to describe a finite variant of the target concept as well as a variant that has at most an *a priori* bounded number of anomalies. We studied finite inference, set-driven identification, conservative inference, learning in the limit and behaviorally correct learning. Thus, our work completes to a large extent the study of learning indexable classes with respect to their principal inferability.

Looking at results previously obtained in the field of inductive inference with anomalies, some of our results could have been expected. For example, the infinite hierarchies for finite learning, conservative inference, set-driven identification, learning in the limit and behaviorally correct inference in the number of allowed anomalies are not surprising. But there are a several results, at least we did not conjecture.

First, the equality $Sdr^*Txt = Consv^*Text = Lim^*Txt$ nicely contrasts the severe restriction caused by the requirement to learn conservatively in the anomaly-free case. Second, as far as we are aware of, within the setting of learning indexable classes till now consistency did not constitute a restriction to the learning power. However, as Theorem 18 shows, conservative inference with an *a priori* bounded number of allowed anomalies *cannot* always be achieved.

Finally, our characterization theorems complete the picture that has been obtained since Angluin's [2] pioneering paper. All learning models considered can be characterized by using finite tell-tale sets. Abstracting from technical details, if these sets are recursive, conservative learning is possible. For recursively enumerable tell-tale sets learning in the limit can be achieved. Furthermore, the pure existence of such tell-tale sets is sufficient to design behaviorally correct learners (see also Baliga *et al.* [3]). Concerning the latter result, our main contribution here is the proof that Bc^*Txt–identification for indexable classes can *always* be achieved by using a hypothesis space with uniformly decidable membership.

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