On the Generalization of the Theory of the Brownian Motion for the Case of Distributed Statistical Phenomena.[1)]

by

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It is attempted to extend the theory of the Brownian motion based upon the ordinary Langevin equation to the case of distributed statistical phenomena, that is, the case in which the Langevin equation becomes partial differential equation containing the space derivatives. Its general form in the one-dimensional case is

$$\frac{\partial^n u}{\partial t^n}+\cdots\quad +a_1\frac{\partial u}{\partial t}+a_0u=b_m\frac{\partial^m u}{\partial x^m}+\cdots\quad +b_1\frac{\partial u}{\partial x}+F(t,x).$$

At first the symbolic (operational) method to solve such a equation is developed as an extention of the one applicable to the ordinary linear differential equations with constant coefficients, the one which I have proposed in my previous paper.[2)] Then making use of it the generalized Fokker-Planck equation is set up, from which the theory of surmise and prediction of stochastically perturbed distributed system is developed in a similar way to Prof Imahori's theory of prediction, which he has successfully used in the long-period weather-forecasting. It is concluded that we have only to measure the (stochastical) temporal variation of the spacial Fourier spectrum of the phenomenon in question, and, applying the ordinary method to each component of the spectrum, obtain the characteristic constants as the function of frequency, from which the coefficients of the generalized Langevin equation are determined. This amounts to saying that, in the case of the approximate prediction in Imahori's sense, we have only to superpose the results obtained for each component by his method.

Introduction

The existing theory of the Brownian motion consists in deducing the so-called Fokker-Planck (F-P) equation from the basic Langevin equation which is supposed

1) Contribution NO. 141 from the Institute of Low Temperature Science.

2) Contrib. from the Inst. Low Temp. Sci., Hokkaido Univ. No. 3 (1952). This paper will be referred as I in the following.

to describe the microscopic mechanism of the Brownian motion. The simplest case is that in which we have the ordinary linear equation of the first order such as

$$\frac{du}{dt} + \beta u = F(t), \tag{1}$$

where $F(t)$ is the so-called purely random function which is Gaussian and has the white spectrum with intensity $2D$. In such a case the corresponding F-P equation for the transition probability function $P(u_0/u,\ t)$ takes the form

$$\frac{\partial P}{\partial t} = \beta \frac{\partial}{\partial u}(uP) + D\frac{\partial^2 P}{\partial u^2}, \tag{2}$$

whose solution is

$$P(u_0/u, t) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-(x-\bar{x})^2/2\sigma^2\right], \tag{3}$$

where $\bar{x}=x_0 \exp(-\beta t)$ and $\sigma^2=(D/\beta)[1-\exp(-2\beta t)]$, showing that the phenomenon is Markoffian with the spectrum $4D/(\beta^2+\omega^2)$ and the correlation function $\rho(t)=\exp(-\beta t)$.

Now it can be shown that any process which is described either by the ordinary linear differential equation of higher order

$$\frac{d^n u}{dt^n} + a_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_1\frac{du}{dt} + a_0 u = F(t), \tag{4}$$

or by the system of such equations of the first order

$$\frac{du_i}{dt} + \sum_j a_{ij} u = F_i(t), \tag{5}$$

may be treated in terms of the basic simple Langevin equation of the form (1). This amounts to saying that the equation or the system of equations such as (4) or (5) are always equivalent to the system of the equations of type (1).

Any process which is described by the ordinary linear differential equation may thus be regarded as the superposition of many simple Markoffian processes described by the basic Langevin equation of the type (1). Even the non-linear process can, at least approximately, be regarded as such a superposition. This method of treating has been rigorously formulated and applied successfully to the problem of long-period weather-forecasting by Prof. Imahori.[3]

3) Imahori K. and Kobayashi T. 1951. J. Met. Soc. Jap., **29**, 365-378.

The problem of treating the statistical phenomena of "lumped-constant" type is thus completely solved, at least in principle. There remains, however, vast field of more complicated statistical phenomena, whose overall features are described not by the ordinary differential equations but by the partial differential equations. The phenomena of turbulence, for instance, which are described by non-linear partial differential equations, can hardly be treated in practical cases by the existing theories based on the non-linear Navier-Stokes' equation. It is to be expected, however, that they can approximately be treated as the distributed statistical phenomena which are described by the linear parital differential equations with constant coefficients. And furthermore, the fact that the linear partial differential equations can — at least formally — be transformed into the form (1)[4] suggests the possibility to generalize the afore-mentioned point of view for the case of phenomena of "distributed type". Such a generalization is the main topic of this paper (§2 et. seq.). In §1 the principle of the method in the lumped-constant case is given only for the sake of comparison.

§ 1. The Lumped-Constant Case.

The simplest example is furnished by the equation of harmonic oscillator

$$\frac{d^2 y}{d t^2} + \beta \frac{d y}{d t} + \omega_0^2 y = 0 . \tag{6}$$

If we put

$$y = u(t) e^{-\frac{\beta}{2} t} , \tag{7}$$

this equation reduces to simpler form:

$$\left. \begin{aligned} \frac{d^2 u}{d t^2} &= \gamma^2 u , \\ \gamma^2 &= \frac{\beta^2}{4} - \omega_0^2 , \end{aligned} \right\} \tag{8}$$

the solution of which, corresponding to the initial condition

$$u(0) = u_0 , \qquad u'(0) = u_0' , \tag{9}$$

is given by

$$\left. \begin{aligned} u &= \frac{u_0}{2} (e^{\gamma t} + e^{-\gamma t}) + \frac{u_0'}{2\gamma} (e^{\gamma t} - e^{-\gamma t}) , \\ u' &= \frac{\gamma u_0}{2} (e^{\gamma t} - e^{-\gamma t}) + \frac{u_0'}{2} (e^{\gamma t} + e^{-\gamma t}) . \end{aligned} \right\} \tag{10}$$

4) Hori J.: loc. cit.

This can be written in a more compact form:

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} \cosh \gamma t & \dfrac{\sinh \gamma t}{\gamma} \\ \gamma \sinh \gamma t & \cosh \gamma t \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}. \tag{11}$$

The transformation matrix is, in another form,

$$e^{t \begin{pmatrix} 0 & 1 \\ \gamma^2 & 0 \end{pmatrix}} = e^{tN}. \tag{12}$$

Thus we can write the equation (8) in the form:

$$\frac{d}{dt} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \gamma^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}. \tag{13}$$

This is nothing other than the compact representation of the initial equation (8) written as

$$\left.\begin{aligned} \frac{du}{dt} &= u', \\ \frac{du'}{dt} &= \gamma^2 u. \end{aligned}\right\} \tag{14}$$

If we introduce the vector quantities

$$F(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \quad F(0) = \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}, \tag{15}$$

we can write the equation (8) in still more compact form

$$\frac{dF(t)}{dt} = \boldsymbol{N} F(t). \tag{16}$$

Now we transform the "infinitesimal generator matrix" $\boldsymbol{N}$ into the diagonal form, the eigenvalues of which are given by the roots of

$$\begin{vmatrix} -\lambda & 1 \\ \gamma^2 & -\lambda \end{vmatrix} = 0, \tag{17}$$

or

$$\lambda_1 = \gamma\,, \qquad \lambda_2 = -\gamma\,. \tag{18}$$

Then the equation (15) becomes

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{19}$$

or

$$\left.\begin{aligned} \frac{dz_1}{dt} &= \gamma z_1, \\ \frac{dz_2}{dt} &= -\gamma z_2\,, \end{aligned}\right\} \tag{19'}$$

where

$$\left.\begin{aligned} z_1 &= \gamma u + u'\,, \\ z_2 &= -\gamma u + u'\,. \end{aligned}\right\} \tag{20}$$

Thus our original equation of the second order (8) has been transformed into the system of the equation of the first order in standard form (19').

It proceeds in much the same way when the perturbing random force is introduced. Then we have, instead of (15),

$$\frac{d}{dt}\begin{pmatrix} u \\ u' \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \gamma^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}, \tag{21}$$

corresponding to the introduction of $F(t)$ in the right-hand side of (8) or of the second equation of (18). Again transforming to the diagonal form we get

$$\left.\begin{aligned} \frac{dz_1}{dt} - \gamma z_1 &= F(t)\,, \\ \frac{dz_2}{dt} + \gamma z_2 &= F(t)\,. \end{aligned}\right\} \tag{22}$$

Hence we obtain two simple Markoffian processes independent of each other.

We can obtain the resolution more directly in terms of the coordinate y. Using (7), (9), and (10), we get

$$\begin{pmatrix} y' \\ y \end{pmatrix} =$$

$$\begin{pmatrix} e^{-\frac{\beta}{2}t}\cosh\gamma t + \frac{\beta e^{-\frac{\beta}{2}t}}{2}\frac{\sinh\gamma t}{\gamma} & \frac{e^{-\frac{\beta}{2}t}}{\gamma}\sinh\gamma t \\ e^{-\frac{\beta}{2}t}\gamma\sinh\gamma t + e^{-\frac{\beta}{2}t}\frac{\beta}{2}\cosh\gamma t - \frac{\beta}{2}e^{-\frac{\beta t}{2}}\cosh\gamma t - \frac{\beta^2}{4\gamma}e^{-\frac{\beta t}{2}}\sinh\gamma t & e^{-\frac{\beta}{2}t}\cosh\gamma t - \frac{\beta e^{-\frac{\beta}{2}t}}{2\gamma}\sinh\gamma t \end{pmatrix}$$

$$\times \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}. \tag{23}$$

This transfer matrix has the infinitesimal generator

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\beta \end{pmatrix}, \tag{24}$$

the eigenvalues of which are found to be

$$\lambda = -\frac{\beta}{2} \pm \gamma, \tag{25}$$

and when we put

$$\left.\begin{aligned} z_1 &= \left(\frac{\beta}{2} + \gamma\right)y + y', \\ z_2 &= \left(\frac{\beta}{2} - \gamma\right)y + y', \end{aligned}\right\} \tag{26}$$

the equation (6), completed by the random force term, is resolved into

$$\left.\begin{aligned} \frac{dz_1}{dt} - \left(-\frac{\beta}{2} + \gamma\right)z_1 &= F(t), \\ \frac{dz_2}{dt} - \left(-\frac{\beta}{2} - \gamma\right)z_2 &= F(t). \end{aligned}\right\} \tag{27}$$

The transformation (26) just corresponds to the one which has been found by Wang-Uhlenbeck.[5)]

5) Wang M. C. and Uhlenbeck G. E. 1945 Rev. Mod. Phys. **17**, 323, eq. (52).

From (24) we see that we can express the equation (6) as

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\beta \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}, \tag{28}$$

which is, as before, equivalent to the transcribed initial equation (6):

$$\left.\begin{aligned} &\frac{dy}{dt} = y', \\ &\frac{dy'}{dt} + \beta y' + \omega_0^2 y = 0. \end{aligned}\right\} \tag{29}$$

The above examples are merely the special cases of the following general consequences. We can rewrite the general linear differential equation with constant coefficients of the n th order

$$\frac{d^n u}{dt^n} + a_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + \cdots \quad + a_1 \frac{du}{dt} + a_0 u = 0, \tag{30}$$

into the form

$$\left.\begin{aligned} &\frac{du}{dt} = u', \\ &\frac{du'}{dt} = u'', \\ &\quad \vdots \\ &\frac{du^{(n-2)}}{dt} = u^{(n-1)}, \\ &\frac{du^{(n-1)}}{dt} + a_{n-1} u^{(n-1)} + a_{n-2} u^{(n-2)} + \cdots \quad + a_1 u' + a_0 u = 0, \end{aligned}\right\} \tag{31}$$

or

$$\frac{d}{dt}\begin{pmatrix} u \\ n' \\ u'' \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\cdots & \\ 0 & 0 & 1 & 0\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \\ 0 & 0 & 0 & \cdots 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}\begin{pmatrix} u \\ u' \\ u'' \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{pmatrix}. \tag{32}$$

This matrix is the infinitesimal generator of the transfer matrix between the initial and the present value of the vector $\boldsymbol{u}=(u, u', u'', \ldots, u^{(n-1)})$, that is, when we write (32) as

$$\frac{d\boldsymbol{u}}{dt}=\boldsymbol{A}\boldsymbol{u}, \tag{33}$$

we have, as the solution of the initial value problem,

$$\boldsymbol{u}=e^{tA}\boldsymbol{u}_0. \tag{34}$$

The eigenvalues of $\boldsymbol{A}$ are the roots of the equation

$$\boldsymbol{A}-\lambda\boldsymbol{I}=0, \tag{35}$$

or

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots\quad +a_1\lambda+a_0=0. \tag{36}$$

This is nothing but the so-called characteristic equation of the differential equation (30). Denoting these eigenvalues by $\lambda_i(i=1, \ldots, n)$, we obtain the system of equations

$$\frac{dz_i}{dt}-\lambda_i z_i=0, \qquad (i=1, \cdots, n) \tag{37}$$

which is equivalent to the original equation (30), where z_i's are obtained by proper linear transformations from the original coordinates $u, u', \ldots, u^{(n-1)}$, the transformation matrix being calculated to be

$$\boldsymbol{C}=\begin{pmatrix} \lambda_1^{n-1}+a_{n-1}\lambda_1^{n-2}+\cdots+a_1 & \lambda_1^{n-2}+a_{n-1}\lambda_1^{n-3}+\cdots+a_2, \cdots & \lambda_1+a_{n-1} & 1 \\ \lambda_2^{n-1}+a_{n-1}\lambda_2^{n-2}+\cdots+a_1 & \lambda_2^{n-2}+a_{n-1}\lambda_2^{n-3}+\cdots+a_2 \cdots & \lambda_2+a_{n-1} & 1 \\ \cdots\cdots\cdots & & & \\ \cdots\cdots\cdots & & & \\ \lambda_n^{n-1}+a_{n-1}\lambda_n^{n-2}+\cdots+a_1 & \lambda_n^{n-2}+a_{n-1}\lambda_n^{n-3}+\cdots+a_2 \cdots & \lambda_n+a_{n-1} & 1 \end{pmatrix} \tag{38}$$

If we start from equation (4) instead of (30), we shall naturally obtain

$$\frac{dz_i}{dt}-\lambda_i z_i=F(t), \qquad (i=1, \cdots, n), \tag{39}$$

with the solution

$$\bar{z}_i=e^{\lambda_i t}z_{i0}, \tag{39'}$$

the values of λ_i's and the coefficients of the linear transformation remaining the same.

§ 2. The Partial Differential Equations.

The completely analogous argument as the above can be applied also to the two-dimensional linear partial diffierential equations with constant coefficients of arbitrary order, if we use the spacial Fourier-spectrum. The most general equation of such a type is written as

$$\frac{\partial^n u}{\partial t^n} + a_{n-1}\frac{\partial^{n-1} u}{\partial t^{n-1}} + \cdots + a_1\frac{\partial u}{\partial t} + a_0 u = b_m \frac{\partial^m u}{\partial x^m} + b_{m-1}\frac{\partial^{m-1} u}{\partial x^{m-1}} + \cdots + b_1 \frac{\partial u}{\partial x}. \tag{40}$$

In order that the following treatment is applicable, however, when $n=m=$even, the coefficient b_m must be positive so that the equation is of the hyperbolic or parabolic type, since only in that case it describes the phenomenon that develops temporally.

If we take the Fourier transform with respect to the space coordinate x of both sides of (40), it becomes

$$\left.\begin{aligned} &\frac{d^n U}{dt^n} + a_{n-1}\frac{d^{n-1} U}{dt^{n-1}} + \cdots + a_1 \frac{dU}{dt} + \left[a_0 - \sum_{\nu=1}^{m} b_\nu (i\omega)^\nu\right] U = 0, \\ &U(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} u(x)\, dx. \end{aligned}\right\} \tag{41}$$

As this is just the same as the equation (30), except that a_0 is replaced by $a_0 - \sum_{\nu=1}^{m} b_\nu (i\omega)^\nu$, we can solve the equation (40) with the initial conditions

$$\left.\begin{aligned} u(x,t)|_{t=0} &= u_0, \\ u'(x,t)|_{t=0} &= u_0', \\ &\cdots\cdots\cdots\cdots \\ u^{(n-1)}(x,t)|_{t=0} &= u_0^{(n-1)}, \end{aligned}\right\} \tag{42}$$

by the method that follows. First we set up the infinitesimal generator

$$A = \begin{pmatrix} 0 & 1 & 0 & 0\cdots & \\ 0 & 0 & 1 & 0\cdots & \\ \cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots & \\ \cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots & \\ & & & \cdots\cdots 0 & 1 \\ -a_0 + \sum_{v=1}^{m} b_v (i\,\omega)^v & -a_1 & -a_2 & \cdots\cdots & -a_{n-1} \end{pmatrix} \tag{43}$$

and then calculate all the elements $m_{11}(\omega)$, $m_{12}(\omega)$, ..., $m_{1n}(\omega)$ of the first row of the exponential matrix e^{tA}. Then the solution of (41) will be given by

$$U(\omega) = m_{11}\, U_0(\omega) + m_{12}\, U_0'(\omega) + \cdots \quad + m_{1n} U_0^{(n-1)}(\omega), \tag{44}$$

where $U_0(\omega)$, $U_0'(\omega)$, ... are the Fourier transforms of the initial function given by (42). The desired solution of the equation (40) is then given by the reciprocal Fourier transform of the right-hand side of (44).

This method is the generalization of the one I have stated in my previous paper I, which referred to the case when only one (but of arbitrary order) time derivative* was present.

As a simple example let us solve the initial value problem of the so-called telegraphists' equation in its general form:

$$a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} + c\,u = \frac{\partial^2 u}{\partial x^2}, \quad u(x,t)|_{t=0} = u_0\,,\ u'(x,t)|_{t=0} = u_0'\,. \tag{45}$$

Taking the Fourier transform with respect to x of the both sides it becomes

$$\frac{\partial^2 U}{\partial t^2} + \frac{2b}{a}\,\frac{\partial U}{\partial t} + \frac{c+\omega^2}{a}\,U = 0\,. \tag{46}$$

This is the same equation as (1) in the preceding section. The infinitesimal generator is

$$A = \begin{pmatrix} 0 & 1 \\ -\dfrac{c+\omega^2}{a} & -\dfrac{2b}{a} \end{pmatrix}, \tag{47}$$

corresponding to (24), and the elements of the first row of the matrix e^{tA} are found

* In the paper I it was taken as the space derivative.

to be

$$\left.\begin{aligned} m_{11}(\omega) &= e^{-\frac{b}{a}t}\cosh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}+\frac{b}{a}e^{-\frac{b}{a}t}\sqrt{a}\,\frac{\sinh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}}{\sqrt{\alpha^2-\omega^2}}, \\ m_{12}(\omega) &= \sqrt{a}\,e^{-\frac{b}{a}t}\,\frac{\sinh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}}{\sqrt{\alpha^2-\omega^2}}, \\ \alpha^2 &= \frac{b^2-ac}{a}, \end{aligned}\right\} \quad (48)$$

as can easily be seen from (23). Hence

$$U=\left[e^{-\frac{b}{a}t}\cosh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}+\frac{b}{\sqrt{a}}e^{-\frac{b}{a}t}\,\frac{\sinh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}}{\sqrt{\alpha^2-\omega^2}}\right]U_0$$
$$+\sqrt{a}\,e^{-\frac{b}{a}t}\,\frac{\sinh\frac{t}{\sqrt{a}}\sqrt{\alpha^2-\omega^2}}{\sqrt{\alpha^2-\omega^2}}\,U_0'. \quad (49)$$

Transforming the both sides of (46) back to the coordinate x, we have

$$u(t,x)=\frac{1}{2}e^{-\frac{b}{a}t}\Bigg[u_0\left(x+\frac{t}{\sqrt{a}}\right)+u_0\left(x-\frac{t}{\sqrt{a}}\right)$$
$$+\sqrt{a}\int_{x-t/\sqrt{a}}^{x+t/\sqrt{a}}u_0(\alpha)\frac{\partial}{\partial t}I_0\sqrt{\frac{(b^2-ac)}{a}\left\{\frac{t^2}{a}-(x-\alpha)^2\right\}}\,d\alpha$$
$$+\sqrt{a}\int_{x-t/\sqrt{a}}^{x+t/\sqrt{a}}\left\{u_0'(\alpha)+\frac{b}{a}u_0(\alpha)\right\}I_0\sqrt{\frac{(b^2-ac)}{a}\left\{\frac{t^2}{a}-(x-\alpha)^2\right\}}\,d\alpha\Bigg], \quad (50)$$

where I_0 is the Bessel function of the first kind for imaginary argument. This agrees with the result obtained by the ordinary method. It is seen that our method leads very directly to the result provided that the Fourier transforms of m's are known.

We may also, as before, diagonalize the matrix (47) so as to obtain the eigenvalues

$$\left.\begin{aligned} \lambda_1 &= -\frac{b}{a}+\frac{\sqrt{b^2-(c+\omega^2)^2}}{a}, \\ \lambda_2 &= -\frac{b}{a}-\frac{\sqrt{b^2-(c+\omega^2)^2}}{a}, \end{aligned}\right\} \quad (51)$$

and the transformation

$$\left.\begin{aligned} Z_1 &= \frac{b+\sqrt{b^2-(c+\omega^2)^2}}{a}\,U+U'\,, \\ Z_2 &= \frac{b-\sqrt{b^2-(c+\omega^2)^2}}{a}\,U+U'\,. \end{aligned}\right\} \tag{52}$$

Then the equation (46) becomes

$$\left.\begin{aligned} \frac{dZ_1}{dt}-\lambda_1 Z_1 &= 0\,, \\ \frac{dZ_2}{dt}-\lambda_2 Z_2 &= 0\,, \end{aligned}\right\} \tag{53}$$

with the solutions

$$\left.\begin{aligned} Z_1 &= e^{\lambda_1 t} Z_{10}\,, \\ Z_2 &= e^{\lambda_2 t} Z_{20}\,. \end{aligned}\right\} \tag{54}$$

If we introduce the random force $F(t, x)$ into the right hand side of equation (45), we have, as before,

$$\left.\begin{aligned} \frac{dZ_1}{dt}-\lambda_1 Z_1 &= F(\omega, t)\,, \\ \frac{dZ_2}{dt}-\lambda_2 Z_2 &= F(\omega, t)\,, \end{aligned}\right\} \tag{55}$$

where $F(\omega,t)$ is the Fourier transform of the function $F(x,t)$, and

$$\overline{F(\omega,t)F(\omega,t')} = \sigma(\omega)\,\delta(t-t')\,. \tag{56}$$

We have explicitly indicated that the random force is now in general the function of both time and space. The spacial distribution of the external force is assumed to change from time to time purely at random so that the equations (56) may hold good. The intensity σ will in general be the function of ω. The solution of (55) is

$$\left.\begin{aligned} \overline{Z}_1 &= e^{\lambda_1 t} Z_{10}\,, \\ \overline{Z}_2 &= e^{\lambda_2 t} Z_{20}\,. \end{aligned}\right\} \tag{57}$$

Transforming back to the original coordinate system, we obtain

$$\left.\begin{aligned}\overline{U} &= \frac{a}{2\sqrt{b^2-(c+\omega^2)^2}}(\overline{Z}_1-\overline{Z}_2)=\frac{a}{2\sqrt{b^2-(c+\omega^2)^2}}(e^{\lambda_1 t}Z_{10}-e^{\lambda_2 t}Z_{20}),\\ \overline{U}' &= \frac{b+\sqrt{b^2-(c+\omega^2)^2}}{2\sqrt{b^2-(c+\omega^2)^2}}\overline{Z}_1+\frac{-b+\sqrt{b^2-(c+\omega^2)^2}}{2\sqrt{b^2-(c+\omega^2)^2}}\overline{Z}_2\\ &= \frac{b+\sqrt{b^2-(c+\omega^2)^2}}{2\sqrt{b^2-(c+\omega^2)^2}}e^{\lambda_1 t}Z_{10}+\frac{-b+\sqrt{b^2-(c+\omega^2)^2}}{2\sqrt{b^2-(c+\omega^2)^2}}e^{\lambda_2 t}Z_{20}.\end{aligned}\right\} \tag{58}$$

The general case can also be treated completely in the same way as the case of ordinary differential equation except that the coefficient a_0 is to be replaced by $a_0-\sum_{\nu=1}^{m} b_\nu (i\omega)^\nu$ if we only use the Fourier transform $U(t, \omega)$ of the original function $u(t, x)$. That is, certain linear combinations of $U(t, \omega)$ and its time derivatives form the mutually independent Markoffian stochastic processes each having characteristic correlation parameter λ_i, and the original U's are the superposition of n mutually independent Markoffian stochastic processes. It is to be noted that all the λ_i's and consequently all the elements of the transformation matrix $\boldsymbol{C}$ are the functions of the frequency ω. (It is clear that if we try to treat the problem in terms of the original function u of time and space coordinates, the matter will become very much complicated, involving the complicated operators which are not the mere multipliers but the trnascendental function of them, and almost intractable except in the simplest case).

The above results are in accordance with the theory of the semi-group transformations which I have briefly accounted in I. At least in the case of the telegraphists' equation it was demonstrated that the solution of equation of the type (45) can be written in the form of semi-group transformation:

$$\boldsymbol{u}(x,t) = e^{tB} f(x) = \boldsymbol{T}(t) f(x), \tag{59}$$

where

$$\left.\begin{aligned}\boldsymbol{u}(x,t) &= \begin{pmatrix} u(x,t) \\ u'(x,t)\end{pmatrix},\\ f(x) &= \begin{pmatrix} u(x,0) \\ u'(x,0)\end{pmatrix},\end{aligned}\right\} \tag{60}$$

and $\boldsymbol{B}$ is the infinitesimal generator of the semi-group operator $\boldsymbol{T}(t)$:

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ -\dfrac{c-\dfrac{\partial^2}{\partial x^2}}{a} & -\dfrac{2b}{a}\end{pmatrix}, \tag{61}$$

The Fourier transform $\boldsymbol{U}$ of $\boldsymbol{u}$ can be written as

$$\boldsymbol{U}(\omega, t) = e^{tA} f(\omega), \tag{62}$$

where

$$f(\omega) = \begin{pmatrix} U(\omega, 0) \\ U'(\omega, 0) \end{pmatrix}, \tag{63}$$

in accordance with the theorem of factor-function transformation.

Also in the general case it may be surmised that the solution of the equation of the type (4) can be written in the form of the semi-group transformation

$$\boldsymbol{u}(x, t) = e^{tB} f(x) = \boldsymbol{T}(t) f(x), \tag{64}$$

where $\boldsymbol{B}$ is the infinitesimal generator of the operator $\boldsymbol{T}(t)$ given by the Fourier transform of the matrix operator (43), and

$$\boldsymbol{u}(x, t) = \begin{pmatrix} u(x, t) \\ u'(x, t) \\ \vdots \\ u^{(n-1)}(x, t) \end{pmatrix}, \quad f(x) = \begin{pmatrix} u(x, 0) \\ u'(x, 0) \\ \vdots \\ u^{(n-1)}(x, 0) \end{pmatrix}, \tag{65}$$

and the theorem of factor-function transformation naturally applies. Of course the rigorous mathematical foundation is required in order that the logical consistency may be established, but it may well be inferred that our surmise is true from the assumed hyperbolic nature of the equation (40) and the general premise of Huygens' principle.

§ 3. The Theory of Surmise and Prediction.

Once the results such as the equations (55) and (57) are obtained, the theory of the surmise of the properties of a statistically perturbed distributed system (i.e. the surmise of the partial differential equation which is to describe such a system, from the observed stochastic time variation of the spectrum with respect to the space coordinate) can be established, provided that the assumption made above concerning the character of the random force is tolerable. It goes exactly in the same way as described in §3 of Prof. Imahori's paper (loc. cit.), apart from minor differences.

First we can set up the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\sum_i \lambda_i \frac{\partial}{\partial z_i}[z_i P] + \frac{1}{2}\sigma(\omega)\sum_{ij}\frac{\partial^2 P}{\partial z_i \partial z_j}, \tag{66}$$

whose solution, i.e. the probability distribution function $P(Z, t)$ is an n-dimensional Gaussian distribution with the average values

$$\overline{Z}_i = Z_{i0}\, e^{\lambda_i t}, \qquad i = 1, \cdots, n, \tag{67}$$

and the variances

$$\overline{(Z_i-\overline{Z}_i)(Z_j-\overline{Z}_j)} = -\frac{\sigma(\omega)}{\lambda_i+\lambda_j}\left[1 - e^{(\lambda_i+\lambda_j)t}\right],$$

$$i, j = 1, 2, \cdots, n. \tag{68}$$

Transforming back to the original coordinates and replacing $U, U', U'', \cdots, U^{(n-1)}$ by $U_1, U_2, U_3, \cdots, U_n$,we obtain the expressions:

$$\left.\begin{aligned}
&P = \frac{1}{(2\pi)^{n/2}\Delta^{1/2}} \exp\left[-\frac{1}{2\Delta}\sum_{ij}\tilde{\Delta}_{ij}(U_i-\overline{U}_i)(U_j-\overline{U}_j)\right],\\
&\qquad \Delta = \mathrm{Det}(D_{ij}),\\
&\qquad \tilde{\Delta}_{ij}: \text{cofactor of } ij \text{ element in } \Delta,\\
&\overline{U}_i = \sum_{jk} c_{ji}\, c_{jk}\, U_{k0}\, e^{\lambda_j t},\\
&\overline{(U_i-\overline{U}_i)(U_j-\overline{U}_j)} = D_{ij} = -\sum_{kl} c_{ki}\, c_{lj}\frac{\sigma(\omega)}{\lambda_k+\lambda_e}\left[1-e^{(\lambda_k+\lambda_l)t}\right],\\
&\qquad c_{ij}: ij \text{ element of the matrix } C \text{ given by (38).}
\end{aligned}\right\} \tag{69}$$

$$U_i(t) = \sum_j c_{ji}\int_0^{\infty} F(\omega, t-t')\, e^{\lambda_j t'}\, dt' = \sum_j c_{ji}\int_{-\infty}^{t} F(\omega, t')\, e^{\lambda_j(t-t')}\, dt'. \tag{70}$$

$$\left.\begin{aligned}
Q_{ij} &= [U_i(t+\tau)\, U_j(t)]_t\\
&= \sum_{kl}\frac{c_{ki}\, c_{lj}\, \sigma(\omega)}{\lambda_k+\lambda_l}\, e^{-\lambda_l\tau}, \quad \text{for} \quad \tau < 0,\\
&= \sum_{kl}\frac{c_{ki}\, c_{lj}\, \sigma(\omega)}{\lambda_k+\lambda_l}\, e^{+\lambda_k\tau}, \quad \text{for} \quad \tau > 0.
\end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned}
&\frac{dQ_{ij}}{d\tau} - \sum_k a_{ik}\, Q_{kj} = 0, \quad \text{for} \quad \tau > 0,\\
&\qquad i, j = 1, 2, \cdots, n.
\end{aligned}\right\} \tag{72}$$

The last equation (72) serves to determine the elements a_{ij} of the infinitesimal generator matrix $\boldsymbol{A}$ given by (43). If this is done successfully, the surmise problem is completely solved, i. e. the coefficients a_i of the time derivatives and those b_i of the space derivatives as well of the required partial differential equation are uniquely determined.

Since the equations (72) for $i \neq n$ are merely the identities becanse of the fact that the elements of the matrix $\boldsymbol{A}$ down to the $(n-1)$ th row are all zero except $a_{i,\,i+1}$ $(i=1, \ldots\ldots n-1)$, the number of unknown quantities and of the corresponding equations reduces to n instead of n^2 in the more general case considered by Prof. Imahori (loc. cit.). New complication arises, however, in our case. The frequency dependence is involved only in one element a_{n1}, to which the result of the calculation must conform ; that is to say, the selection of the number n will play a very important role, upon which the efficiency of the calculation largely depends. The order n may be estimated approximately by the following procedure. First we measure the stochastic time variation of the spectrum $U(\omega)$, and calculate its correlation function as a frequency ω. Then we try to represent it as a superposition of simple Markoffian correlation functions $e^{\lambda_i t}$ with appropriate coefficients and eigenvalues λ_i. If we succeed in doing this strictly or with tolerable approximation, the number of these resolved components will give the order n. This point would however require the practical test.

If we can determine or surmise the n, or more specifically the detailed temporal structure of the required partial differential equation strictly or with sufficient approximation from some physical intuition, the problem becomes more reduced. In the former case we should measure the correlation functions up to the predetermined order and solve the equation (72), which will lead to a satisfactory result in so far as the presupposed number of the order is correct. In the latter case, since the coefficient a's are known, we have only to determine the coefficient b's from the equation (72).

The simplest case is that in which only the first order time derivative appears. That is, we suppose that the system in question is described by the equation of the type

$$\frac{\partial u}{\partial t} = b_m \frac{\partial^m u}{\partial x^m} + b_{m-1} \frac{\partial^{m-1} u}{\partial x^{m-1}} + \cdots + b_1 \frac{\partial u}{\partial x} + b_0 u + F(x,t). \tag{73}$$

Then the matrix operator A reduces simply to a multiplier

$$\boldsymbol{A} = \sum_{\nu=0}^{m} b_\nu (i\,\omega)^\nu \equiv P(i\,\omega), \tag{74}$$

and this is moreover the eigenvalue itself. Accordingly the mean value of the

Fourier transform $U(\omega, t)$ of the solution $u(x, t)$ of (73), with the initial condition $u(x, 0)$ with its Fourier mate $U_0(\omega)$, is given by

$$\overline{U}(\omega, t) = e^{P(i\omega)t} U_0(\omega). \tag{75}$$

Thus we have only to measure the self correlation function of U, from which the polynomial $P(i\omega)$ can directly be obtained.

In the afore-mentioned general case there remains a difficulty of practical nature, which lies in measuring the stochastic time variation of the spacial spectrum of the extended field. This difficulty might be avoided by making use of the spacial variation of the temporal spectrum instead of the temporal variation of the spacial spectrum, the former being far more tractable experimentally than the latter. The theory of this method may presumably obtained by merely exchanging the spacial and temporal coordinates with each other in the above arguments, namely, by replacing the assumption of the temporal homogeneity which is expressed by the constancy of the strength of the random force, by that of the spacial homogeneity. There appear, however, in this case new problems concerning the nature of the random force, which require further investigations. And, even when this latter method could be successfully developed, its inherent dimensional limitation (i.e. the limitation to the one-dimensional case) will seriously restrict its domain of applicability, while the above method applies equally well to any systems with arbitrary boundary conditions and dimension, as will be shown in the next section. Thus it is still highly desirable to develop the practical method by which we can mesure effectively the spacial spectrum of the extended field and follow it temporally.

It is clear that our method can be utilized also for the purpose of prediction. The theory of prediction on this basis can be developed wholly along the line of Prof. Imahori's approximation method. It is quite evident that we have only to apply his method to each component of the spacial Fourier spectrum of $u(x, t)$, and thereafter superpose them to obtain the future value of $u(x, t)$. It is interesting to note that this corresponds to the so-called multiple prediction with continuous (or non-denumerably infinite) multiplicity. It is expected that the ordinary multiple prediction problem with discrete (finite or denumerably infinite) multiplicity can be treated, in some favorable cases, as the special case of the former.

§ 4. Spacially Higher Dimensional Cases.

More general cases in which the dimension of the space in which the phenomenon in question displays itself is more than one can, at least formally, be treated in completely the same way. The general equation for such cases is

$$\frac{\partial^n u}{\partial t^n}+a_{n-1}\frac{\partial^{n-1}u}{\partial t^{n-1}}+\cdots+a_1\frac{\partial u}{\partial t}+a_0 u$$
$$=\sum_{i_1+\cdots+i_m=k} b_{i_1}\cdots_{i_m}\frac{\partial^k u}{\partial x_1^{i_1}\cdots\partial x_m^{i_m}}+\cdots\cdots \tag{76}$$

the number of spacial dimension being supposed to be m, and the order to be k*. The dots indicates the terms whose order is smaller than k, down to 1. If we transform the variable u on each side of this equation to its m-dimensional Fourier mate $U(\omega_1, \ldots, \omega_m)$, we obtain

$$\frac{\partial^n U}{\partial t^n}+\cdots$$
$$+a_1\frac{\partial U}{\partial t}+\left(a_0-\sum_{i_1+\cdots+i_m=k} b_{i_1},\cdots_{i_m}(i\omega_1)^{i_1}\cdots(i\omega_m)^{i_m}-\cdots\right)=0. \tag{77}$$

Thus we can solve the equation (76) ,with the initial conditions

$$\left.\begin{aligned} &u(x_1,\cdots,x_m,t)|_{t=0}=u_0,\\ &u'(x_1,\cdots,x_m,t)|_{t=0}=u_0',\\ &\cdots\cdots\cdots\cdots\cdots\\ &u^{(n-1)}(x_1,\cdots,x_m,t)|_{t=0}=u_0^{(n-1)}, \end{aligned}\right\} \tag{78}$$

by setting up the infinitesimal generator, whic is obtained from (43) by replacing the $(n, 1)$-element by $-a_0+\sum_{i_1+\cdots+i_m=k} b_{i_1}\cdots_{i_m}(i\omega_1)^{i_1}\cdots(i\omega_m)^{i_m}$, and calculating the first row of the matrix e^{tA}. Formulation analogous to (44) and application of the reciprocal m-dimensional Fourier transform will then yield the desired solution, if the values of the coefficients of the initial equation fulfille the proper conditions.

The theory of surmise and prediction in such a case can be established in a completely similar way as in §3, the only difference being that now the U_i's $(i=1, \ldots, n)$ are the functions of m frequeneies $\omega_1, \ldots\ldots, \omega_m$, and the element a_{n1} of the infinitesimal generator matrix $\boldsymbol{A}$ should become the polynomial of kth degree in ω_ν's. The Coefficients a_i's and $b_{i_1},\cdots_{i_m}$'s in the required equation can

* In order that the argument which follows may be correct in strict mathematical sense, some restricting conditions for the coefficients b_ν's — supposedly such that the equation should be of totally hyperbolic type — might have to be imposed ; but such a consideration would involve the thorough-going mathematical argument, which is here momentarily omitted. We restrict ourselves to the purely formal argument, whose mathematical justification or criticism is left to later investigations.

thus at least in principle be determined completely by the same procedure as in §3 ; only the calculation becomes far more intricate, the problem of determination of the order n becomes more subtle, and the experimental measurement of the spacial spectrum becomes more difficult.

This treatment can only be applied to the case in which the representation by m-dimensional Fourier-transform or Fourier series is possible, that is the case in which the space in question is the whole Euclidean m-dimensional space or the rectangular part of it. But once we perform this procedure in such a simple case, and succeed in surmising the partial differential equation which would describe the system under investigation, then we can calculate the behaviour of this system under arbitrary boundary conditions by solving the equation now obtained under new conditions (by the method, for example, of eignefunctions, if the obtained spacial differential operator is self-adjoint).

I hope that the method proposed above will effectively be applied to the investigations of the phenomenon of turbulence and the other stochastically perturbed distributed systems. More profound investigations, both from theoretical and practical point of view, would however be necessary — mathematically rigorous foundation, reduction to the practically more tractable approximate formulation, etc. — before the method becomes to be able to claim its utility. In later papers I will report further investigations, with the practical applications of the method.

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