On the Rheological Behavior of Snow under High Stresses*

By

Bruno SALM†
Federal Institute for Snow and Avalanche Research, Weissfluhjoch/Davos, Switzerland

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Abstract

An attempt is made to establish a failure criterion for snow. The classical failure criteria take only the elastic part of the stress work into consideration. It is supposed that this does not suffice for snow because of its pecular structure of the material. This structure is assumed to consist of "lines" of snow grains on which the stresses are concentrated. One part of them, the so-called "elastic lines" behave like a HOOKE body whereas so-called "viscous lines" exhibit non-NEWTONIAN behavior. A breakdown of this "viscous lines" happens as soon as a critical power is reached, which is in contradiction to most other materials where a body with pure viscous deformation has theoretically no upper limit of stresses. The load capacity of the "elastic lines" is limited by the critical, conserved stress energy. The postulate of the critical power confines the stresses in the "viscous lines" by a critical amount of the rate of the dissipated part of the work.

The total strength of snow F_s under an uniaxial state of stress is therefore given by

$$F_s = kE_{1(v)}^{\frac{1}{2}} + \frac{\kappa^2}{\lambda v + Ce^{aF_s}}$$

where the first term on the right represents the part of the "elastic lines" and the second term the one of the "viscous lines". k, κ^2, λ, C and a are constants, $E_{1(v)}$ is the Young's modulus and v is the rate of deformation.

Tests, with constant rates of deformation, are showing a good agreement with the theory.

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Contents

I. Introduction

The aim of the present study is an investigation of the rheological behavior of snow under high stresses *i.e.* in the vicinity of states of stress where strength of the snow type in question is reached. It seems obvious that first of all one has to check how far the methods already developed in rheology can be applied to snow. Rheologically speaking, deposited snow is a *sol* (REINER, 1958) with the ice grains as dispersed phase and two dispersion mediums: One of them is the air and water vapor with which the pores are filled. The other consists of the thin quasi-liquid layers of atomic disorder, forming the bonds between adjacent grains or crystals (the ice grains are either composed of single crystals or of close groupings of several crystals). As the smallest volume elements under consideration are much larger than the dispersed elements *i.e.* the ice grains, the material in question is *quasi-homogeneous*, so that it can be treated with the methods for continuous media. Hitherto no methods are known for the derivation of mechanical properties from the actual snow structure. Therefore snow represents more or less a "black box", at least as far as our problem is concerned. The usual way is to substitute for the real structure a *mechanical model* which is supposed to behave analogously to the real structure. Snow under low stresses and limited total deformation is well represented by a BURGER's body, first introduced for snow by DE QUERVAIN (1946). However, as stress increases our material exhibits a distinct *non-Newtonian* behavior.

So far available methods of rheology are applicable. But snow shows some properties which seem *fundamentally different from the materials usually considered in rheology.*

One of them is the behavior under the isotropic part of the stress tensor. For "usual" materials to a first approximation the work accomplished by the hydrostatic pressure is entirely conserved and vanishes in a closed cycle. Whereas in the case of snow this work is almost completely dissipated as long as the load duration is not too short. This characteristic property is closely connected with the fact, that the density of seasonal snow —we shall restrict ourselves to this type— is relatively low compared to that of ice, so that there are large pore spaces into which grains can move with much freedom.

Another property, different from other materials is *strength.* This problem will form the centre of the present considerations. Deformations or rates of deformation of any body are limited by strength. The following definition of this quantity will be used: Strength of snow is reached as soon as under a certain state of stress for the first time the original structure is destroyed in a certain area or volume, which must comprise a minimum number of grains. Destruction means a quick disconnection of the bonds between grains so that the fundamental requirement for a *stable* material —forces must increase with increasing deformation or rate of deformation— is no more fulfilled (ZIEGLER, 1963).

The classical statement concerning strength is, that to deform a body plastically or to break it, the external forces must perform work. HENCKY and HUBER postulated that a material is made to yield plastically or to break respectively, when the *distortional stress work* per unit volume reaches a certain limit (REINER, 1958). REINER and WEISSENBERG made this postulate more precise in saying that the specific work in question is *exclusively the elastic part of the work* (REINER, 1958). In other words the dissipated stress work is converted into heat and therefore not available to produce fracture. This leads to the statement, that a Newtonian or non-Newtonian liquid has no distortion strength *i.e.* there exists no upper limit for deformation or rate of deformation for such a body. Finally it is to mention that DRUCKER and PRAGER introduced in their criterion also the influence of the isotropic part of the stress tensor (DRUCKER and PRAGER, 1952).

In the next section the classical failure criteria will be treated mathematically. However it will be shown later that none of them is in accordance with experimental results so that one has to make an attempt to find a fundamentally new criterion (Section III).

II. The classical failure criteria applied to snow

We assume with REINER and WEISSENBERG that *only the elastic part* of the work accomplished by the stresses has to be taken into account. The dissipated part of the *work* is of no importance because otherwise snow would break under any small stress as soon as a certain limit of this quantity would be reached. As in all viscous liquids this part of the work is continuously converted into heat so that stationary, stable states are conceivable.

As it already has been stated before, the behavior of snow under hydrostatic stresses is much different from that of other materials. In the "normal" case strength under isotropic pressure is theoretically unlimited, while for snow with its large pore spaces, which are all connected with each other, there exists such a limit. It is therefore certainly not sufficient to consider only the distortional stress work, we have of course to introduce the *total* elastic part of it.

An important remaining problem is which *type of elasticity*, *i.e.* the instantaneous elastic strain, fits best the real characteristic of snow. Hitherto all authors assumed —without saying— the simplest type of hypoelastic materials, the HOOKE-body (REINER, 1958). Although we have some hints that in fact it is more complicated, we shall assume the same, first of all for the reason of simplicity.

When we express the deformation U_{ij} in terms of the stress T_{ij}, *Hooke's law* is given by

$$U_{ij}=\frac{1}{2\mu}\left(T_{ij}-\frac{\lambda}{3\lambda+2\mu}T_{kk}\delta_{ij}\right) \qquad \text{(II. 1)}$$

where λ and μ are the *Lamé's constants* and δ_{ij} means the Kronecker delta. The work $\mathcal{E}$ done in a volume element becomes

$$\mathcal{E}=\frac{1}{4\mu}\left(T_{ij}T_{ij}-\frac{\lambda}{3\lambda+2\mu}T_{ii}T_{jj}\right) \qquad \text{(II. 2)}$$

where $\mathcal{E}$ is also called the *stress energy* (PRAGER, 1961).

Taking into consideration, that

$$T_{ij}T_{ij}=2\mathscr{T}'_{(2)}+\frac{1}{3}\mathscr{T}^2_{(1)} \qquad \text{(II. 3)}$$

and

$$\frac{\mu}{\lambda}=\frac{1}{2}(m-2), \qquad \text{(II. 4)}$$

where $\mathscr{T}'_{(2)}$ denotes the second invariant of the deviator of the stress tensor, $\mathscr{T}_{(1)}$ the first invariant of the stress tensor itself and m the *Poisson's ratio*, we obtain

$$\bar{\mathcal{E}} = \frac{1}{4G}\left(2\mathcal{T}'_{(2)} + \frac{m-2}{3(m+1)}\mathcal{T}^2_{(1)}\right). \tag{II. 5}$$

In (II. 5) Lamé's constant μ is replaced by the *shear modulus* G. If only the distortional stress work is taken into consideration, thus $\mathcal{T}_{(1)}=0$, we get the well known *von MISES yield condition*

$$\mathcal{T}'_{(2)} = 2G\bar{\mathcal{E}} \tag{II. 6}$$

von MISES derived his condition before HENCKY, but not from energy considerations. This is the reason that he wrote on the right hand of (II. 6) K^2, the square of the yield stress for simple shear, instead of the constants of the material G and $\bar{\mathcal{E}}$.

For the uniaxial state of stress in which only T_{11} differs from zero, (II. 5) becomes

$$T^2_{11} = 4G\bar{\mathcal{E}}\left(1+\frac{1}{m}\right). \tag{II. 7}$$

When the shear modulus G is replaced by *Young's modulus E*, (II. 7) yields

$$T^2_{11} = 2E\bar{\mathcal{E}} \tag{II. 8}$$

As our tests predominantly were carried out under uniaxial states of stress, we shall readily be able to examine the latter relation.

The DRUCKER-PRAGER criterion is similar to (II. 5), but it is not derived from energy considerations. It has the advantage, that it reduces to the *COULOMB-MOHR criterion* for the case of plane deformation (DRUCKER and PRAGER, 1952). The DRUCKER-PRAGER condition is usually written as

$$\mathcal{T}'^{1/2}_{(2)} + \alpha\mathcal{T}_{(1)} = k \tag{II. 9}$$

where α is a positive constant of the material and k is the yield stress for simple shearing stress. For the uniaxial state of stress in which only T_{11} differs from zero, (II. 9) becomes

$$T_{11} = \frac{k}{\left(\frac{1}{\sqrt{3}} + \alpha\right)}, \tag{II. 10}$$

which is in principle the same as (II. 8). We shall however prefer (II. 8) because Young's modulus is known from the tests, which is not the case for k and α.

III. The postulate of the critical power

As we already noticed in the first section, the classical failure criteria do not fit the experimental results. Many researchers already have found before, that strength of snow depends strongly on the *rate of deformation* (FURUKAWA, 1958), (KINOSITA, 1967), (MELLOR and SMITH, 1966), whereas in the classical criteria *e.g.* in (II. 8) or (II. 10) this quantity does not appear. It is however to mention that the present tests are showing, that *e.g.* the right hand of (II. 8) is not a constant of the material as it seems. The reason lies in the fact, that Young's modulus depends on the rate of deformation but not in a manner that (II. 8) is fulfilled. We shall return to this subject again with the discussion of the experimental results. We note this fact at this place only to give reasons for the following attempt to establish a new more general criterion.

The assumptions and theoretical considerations mentioned below are based on preliminary tests and it will be shown that they fit the main tests too.

Let us consider a certain place in the ice network of snow where deformation takes place (*e.g.* in a bond between two grains). The first fundamental theorem of thermodynamics, the energy theorem, stipulates that

$$\rho \dot{e} = T_{kl} V_{kl} - q_{k,k} \qquad \text{(III. 1)}$$

where e is the specific internal energy (the point denotes the material rate of change), ρ is the density, V_{kl} is the rate of deformation and q is the vector of the heat flux (subscripts after the comma denote derivatives).

We assume that the term $T_{kl} V_{kl}$—the *power of the stresses on the rate of deformation*—remains constant for a certain time Δt. When $\Delta t \to 0$ only *inviscid* elastic deformations in the sense of a hypoelastic material are taking place. The power becomes infinitely large but the amount of the accomplished work is finite and equals the hypoelastic deformation energy. In this case no heat flux occurs. If on the other hand Δt exceeds a certain amount a steady state is reached and the term on the left side of (III. 1) vanishes. Therefore all the produced work—except the conserved hypoelastic deformation energy—is continuously transfered to colder parts in the form of heat. (Obviously a notation "*quasi*-steady state" would be better because strictly speaking pure steady states are impossible). Between the described extremes additional elastic deformations connected with energy dissipation are occuring. (For snow under low stresses represented by the KELVIN-part of BURGER's body). This work is therefore not conserved and heat flux is to be expected.

Considering only *viscid* deformations we can state that at places in the ice network where deformations are concentrated, there is a *competition between the material rate of change of internal energy* (left side of (III. 1) *and the heat flux* (second term of the right of (III. 1)) *to colder parts.* With increasing heat production in the weak places of the ice network temperature is growing in such places and the snow loses gradually the resistance to the stresses. Strength is reached as soon as the resistance is no more large enough to fulfill the condition of a stable material (in most cases the resistance vanishes entirely). *We suppose that this critical point is in connection with a critical power of the stresses on the rate of deformation.* This is of course only a question of a hypothesis and our postulate cannot be proved, at least for the present time. All considerations based on (III. 1) are merely qualitative ones.

In order to formulate the postulate of the critical power we need a *mechanical model* which represents in a realistic but manageable way snow under high stresses. This model cannot in a simple manner be composed of the 3 elementary rheological bodies—the NEWTON's, ST. VENANT and HOOKE body—because of the distinct non-linear behavior. To avoid a too big number of elementary bodies on which our material should be composed of, we introduce more complicated "elementary" bodies from which we require a certain relation between deformations and stresses. These relations will be empirical ones.

Because of the difficulties to be expected for the three-dimensional states we shall restrict ourselves to *uniaxial* states of stress.

We start with a consideration of the BURGER's body (Fig. 1) under constant rate of deformation. From that we shall try to find a suitable model for high stresses. Resolving the differential equation for a constant velocity of deformation v, following REINER (1958), we obtain the results mentioned below. The velocity of the HOOKE element (denoted by H) is

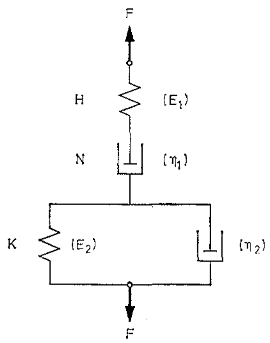


Fig. 1. BURGER's body

$$\dot{x}_H = \frac{\dot{F}}{E_1} = \frac{v}{E_1}\left[\frac{E_1 - \frac{\eta_1}{T_2}}{1-\frac{T_1}{T_2}} e^{-\frac{t}{T_1}} + \frac{\frac{\eta_1}{T_2} - \frac{E_1 T_1}{T_2}}{1-\frac{T_1}{T_2}} e^{-\frac{t}{T_2}}\right] \qquad \text{(III. 2)}$$

For the velocity of the NEWTON element (denoted by N) we get

$$\dot{x}_N = \frac{F}{\eta_1} = \frac{v}{\eta_1}\left[\frac{\frac{\eta_1 T_1}{T_2} - T_1 E_1}{1-\frac{T_1}{T_2}} e^{-\frac{t}{T_1}} + \frac{T_1 E_1 - \eta_1}{1-\frac{T_1}{T_2}} e^{-\frac{t}{T_2}} + \eta_1\right] \qquad \text{(III. 3)}$$

Finally we get for the velocity of the KELVIN body (denoted by K)

$$\dot{x}_K = v - (\dot{x}_H + \dot{x}_N) \tag{III. 4}$$

F is the time dependent force, E_1 the "gross" Young's modulus of H, η_1 the coefficient of viscosity of N, and E_2 and η_2 the corresponding constants of K. t means the time, and T_1 and T_2 are relaxation times, both depending on the Young's moduli E_1, E_2 and also on the coefficients of viscosity η_1 and η_2.

A qualitative representation of the velocity distribution in a BURGER's body is given in Fig. 2.

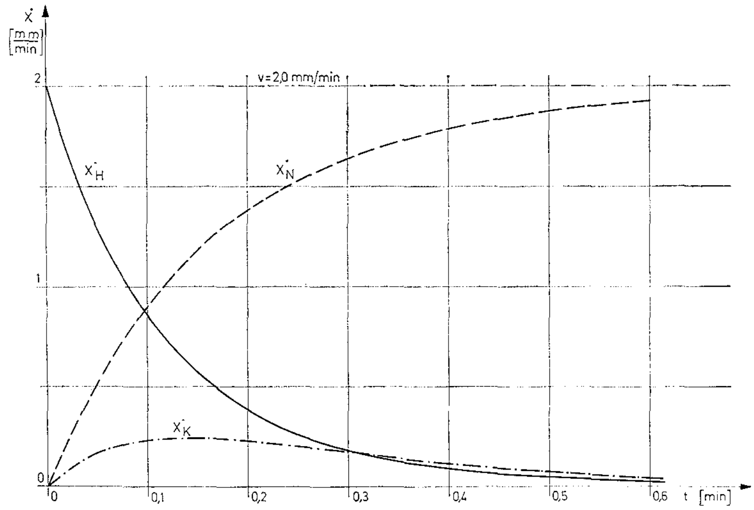


Fig. 2. Distribution of the velocities in the BURGER's body

In the proposed *model for high stresses* set forth in the following, two elements of it are conserving and two are dissipating energy, the same conditions as in the BURGER's body.

The *inviscid* part is given by a HOOKE body, with

$$\dot{x}_H = \frac{\dot{F}}{E_{1(v)}} \tag{III. 5}$$

where $E_{1(v)}$ is the "gross" Young's modulus which may also depend on the given rate of deformation v.

The *first viscid* element is a non-NEWTONIAN body (denoted by NN) characterized by the relation

$$\dot{x}_{NN} = Ce^{aF} \tag{III. 6}$$

where C and a are positive constants of the material. Supposition (III. 6) is nothing but an approximation to the well known hyperbolic sine relation for the case of high stresses.

For the *second viscid* part we have to substitute for the KELVIN body a type of non-KELVIN body (denoted by NK). As even in the case of linear elements the relations become intricate as (III. 4) the only way is to search for a simple but realistic assumption. We set

$$\dot{x}_{NK} = \lambda \dot{x}_{H} \qquad \text{(III. 7)}$$

where λ is a positive constant*. As $\dot{x}_{NK}$ is proportional to $\dot{x}_{H}$, the boundary condition $\dot{x}_{NK}=0$ for $t\to\infty$ is fulfilled. Furthermore (III. 7) expresses a decreasing coefficient of viscosity η_2 for increasing speed of deformation. In the case of $\dot{F}\to\infty$ our non-KELVIN body becomes inviscid, therefore the second boundary condition $\dot{x}_{NK}=0$ for $t\to 0$ is not fulfilled. Hence we have to replace (III. 7) by the boundary value in the case of extremely large force rates.

The differential equation for our non-BURGER's liquid is

$$v = \dot{x}_{H} + \dot{x}_{NK} + \dot{x}_{NN} = (1+\lambda)\frac{\dot{F}}{E_{1(v)}} + Ce^{aF} \qquad \text{(III. 8)}$$

Integrating this equation we find

$$\frac{E_{1(v)}}{1+\lambda} vt = F + \frac{1}{a} \ln \frac{C_1}{v - Ce^{aF}}$$

The initial condition $F=F_0$ for $t=0$ furnishes

$$\frac{1}{a} \ln C_1 = -F_0 + \frac{1}{a} \ln (v - Ce^{aF_0}) \qquad \text{(III. 9)}$$

thus the solution becomes

$$\frac{E_{1(v)}}{1+\lambda} vt = (F-F_0) + \frac{1}{a} \ln\left(\frac{v - Ce^{aF_0}}{v - Ce^{aF}}\right) \qquad \text{(III. 10)}$$

In the vicinity of $\dot{F}\to\infty$ the solution of (III. 8) reduces to

$$x_{H} = \frac{F}{E_{1(v\to\infty)}} \qquad \text{(III. 11)}$$

According to (III. 8) or (III. 10) we obtain for the force rate

$$\dot{F} = \frac{E_{1(v)}}{1+\lambda} (v - Ce^{aF}) \qquad \text{(III. 12)}$$

which becomes for $t=0$, the moment where the velocity is increased from

*) There is no correlation with Lamé's constant in (II. 1)

v_0 to $v_0+\Delta v$

$$\dot{F}_0=\frac{E_{1(v)}\Delta v}{1+\lambda} \tag{III. 13}$$

With vanishing initial force and velocity (III. 12) yields

$$\dot{F}_0=\frac{E_{1(v)}(v-C)}{1+\lambda} \tag{III. 14}$$

instead of

$$\dot{F}_0=E_{1(v)}v \tag{III. 15}$$

which is due to the fact that (III. 6) holds only for high stresses and that λ should vanish for this point of time.

We obtain a characteristic time T_{rel} for our liquid when we compute the time until the stationary force F_∞—given by the condition $v=\dot{x}_{NN}$—is reached under a constant force rate (III. 15)

$$T_{rel}=\frac{F_\infty}{E_{1(v)}v}=\frac{\frac{1}{a}\ln\frac{v}{C}}{E_{1(v)}v} \tag{III. 16}$$

which corresponds to the relaxation time of a MAXMELL body with

$$\frac{F_\infty}{v}=\eta_1=\text{constant}$$

Therefore we can call (III. 16) the relaxation time of our body.

When at the time $t=0$ the constant deformation velocity v drops to zero, (III. 8) furnishes the following relation between decreasing force and time, as long as the force is not too low

$$t=\frac{1+\lambda}{CaE_{1(v)}}(e^{-aF}-e^{-aF_0}) \tag{III. 17}$$

Finally we obtain from (III. 10) an expression for the time t_s until strength F_s is reached. When we assume that this happens during the time that $\dot{x}_{NN}$ remains small we have

$$t_s=\frac{1+\lambda}{vE_{1(v)}}(F_s+F_0) \tag{III. 18}$$

A qualitative representation of the velocity distribution for our non-BURGER's body, according to (III. 10) with vanishing F_0, is given in Fig. 3.

We are now in a position to formulate the postulate of the critical power.

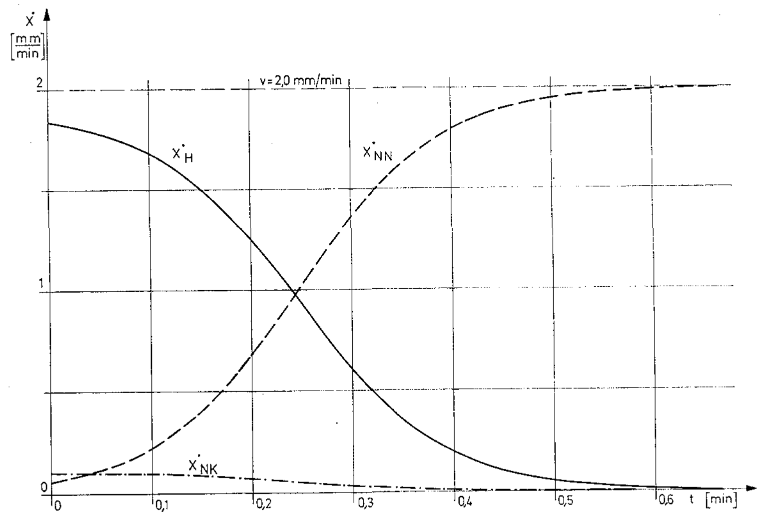


Fig. 3. Distribution of the velocities in the non-BURGER's body

With this end in view we write the sum of the power accomplished in the elements of our non-BURGER's body in the form

$$\mathscr{P} = \frac{\mathrm{d}\bar{\mathscr{E}}}{\mathrm{d}t} = \frac{\mathrm{d}\bar{\mathscr{E}}^{\mathrm{i}}}{\mathrm{d}t} + \frac{\mathrm{d}\bar{\mathscr{E}}^{\mathrm{r}}}{\mathrm{d}t} = F^{\mathrm{i}}\dot{x}^{\mathrm{i}} + F^{\mathrm{r}}\dot{x}^{\mathrm{r}} \qquad \text{(III. 19)}$$

where the index i denotes the power connected with irreversible changes of state and r the one with reversible changes.

The *reversible* part of (III. 19) can be written as

$$\frac{\mathrm{d}\bar{\mathscr{E}}^{\mathrm{r}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int F^{\mathrm{r}}\,\mathrm{d}x^{\mathrm{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F^{\mathrm{r}2}}{2E_1} + \lambda\frac{F^{\mathrm{r}2}}{2E_1}\right) \qquad \text{(III. 20)}$$

where the second term in the bracket on the right vanishes in the vicinity of $F \to \infty$ (see comment on (III. 7)). From (III. 20) and the postulate of the critical work we obtain the limiting force $F^{\mathrm{r}}_{\mathrm{s}}$

$$F^{\mathrm{r}}_{2} = \sqrt{\frac{2E_1\bar{\mathscr{E}}^{\mathrm{r}}_{\mathrm{c}}}{1+\lambda}} \qquad \text{(III. 21)}$$

which corresponds with (II. 8) for vanishing λ. $\bar{\mathscr{E}}^{\mathrm{r}}_{\mathrm{c}}$ denotes the critical work.

The *irreversible* part of (III. 19) and the postulate of the critical power yield the limiting force $F^{\mathrm{i}}_{\mathrm{s}}$

$$F^{\mathrm{i}}_{\mathrm{s}} = \frac{\mathscr{P}^{\mathrm{i}}_{\mathrm{c}}}{\dot{x}^{\mathrm{i}}} = \frac{(1+\lambda)\mathscr{P}^{\mathrm{i}}_{\mathrm{c}}}{\lambda v + Ce^{aF^{\mathrm{i}}_{\mathrm{s}}}}\,. \qquad \text{(III. 22)}$$

according to (III. 6), (III. 7) and the relation

$$v = \dot{x}_H + \dot{x}_{NK} + \dot{x}_{NN} .$$

(III. 22) does not hold in the vicinity of $t \to 0$ because of the fact that during the start of loading the velocities of the irreversible parts do not differ from zero and therewith the power vanishes too in reality.

It is obvious that for an explanation of the fracture mechanism a model as given in Fig. 1 cannot suffice because it is too simple. Such a model behaves analogously to the real structure only when forces and rates of deformation and their relationship to time are under consideration. Our idea about fracture in snow is the following.

When a force is applied to snow by a given rate of deformation v, the sequence of the phenomena occuring in the ice lattice may be as follows.

In a *first stage* immediately after loading the stresses are concentrated on certain "primary elastic lines" of ice grains which are in such a position that they cannot move without a deformation of the ice crystals themselves. The response to this type of deformation is purely elastic, therefore the stress energy is entirely conserved until a critical value is reached and the crystals start to deform plastically under constant stresses (behavior of a perfectly plastic material). During the deformation new "secondary elastic lines" are continuously activated and are also conserving stress energy up to the critical value. As long as in no crystal the critical energy is reached this first stage is represented by the HOOKE element in our non-BURGER's body. The speciality of this HOOKE element is however that the amount of deformation of the spring is *confined* by the critical work.

In a *second stage* the elastic lines may deform now partly viscid (due to the changed form of the lines) or new "visco-elastic lines" of the ice lattice will be created. This stage (including the irreversible part of the first stage) is represented by the non-KELVIN body in our non-BURGER's body.

Finally purely "viscous lines" will be activated in a last *third stage*, which correponds to the non-NEWTON element.

When the rate of deformation v is very high, failure occurs in the first stage. At first the "primary elastic lines" will break and all other lines after that in a progressive fracture. With decreasing rate of deformation the secondary elastic lines receive an increasing part of the load before failure. Thus we expect rising Young's modula and herewith growing expression (III. 21) for decreasing velocity.

The smaller v the more the visco-elastic and the viscous lines are activated and contribute to an increase in the load capacity according to (III. 22).

The maximum strength will be reached when all possible lines are activated and the actual work and power exceed slightly their critical values.

According to (III. 21) and (III. 22) we write therefore for the total strength of snow F_s

$$F_s = kE_{1(v)}^{1/2} + \frac{\kappa^2}{\lambda v + Ce^{aF_s}} \qquad \text{(III. 23)}$$

where k and κ are positive constants.

The present failure criterion states that *strength of snow consists in the sum of two parts, one resulting from the postulate of critical work and the other from the postulate of critical rate of dissipation work.*

It would be desirable to generalize (III. 23) for the three-dimensional case.

IV. Experiments

IV. 1. *Apparatus*

The apparatus used for the tests was a "TOM 10 000" produced by the SHINKOH Company, Japan (Figs. 4 and 5). The cross-head of this testing machine can be moved with the following 21 constant velocities

500 / 300 / 200 / 100 mm/minute

with the multiples

10^0 / 10^{-1} / 10^{-2} / 10^{-3} / 10^{-4}

and with 0.005 mm/minute.

The forces were measured by load cells equipped with strain gauges. For the compression tests a load cell with a load capacity of 1000 kgf (where kgf means kilogramme force) was used. This cell was mounted at the bottom of the apparatus, whereas for the tension tests a load cell with a capacity of 500 kgf was fixed at the top of the machine.

On a recording instrument the forces were recorded as a function of deformation or time respectively. The chart speed could be controlled in both ways: One to set it proportional to the speed of the cross-head by different multiples and the other to set it at different constant speeds independent on the cross-head speed.

The recorder has different possibilities to amplify the force measurements. For the maximum amplification the full scale range is 50^{-1} of the load capacity. The accuracy until 10^{-1} of the load cell capacity is $\pm 0.5\%$ and beyond this value ± 1.0 till 1.5% of the respective full scale load.

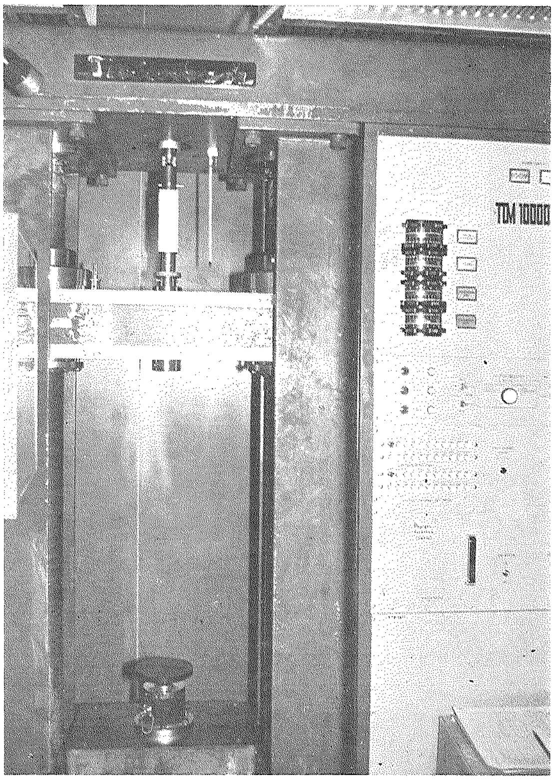


Fig. 4. Apparatus "TOM 10 000". Cross-head in the upper position for tension tests. At the bottom the load cell for compression tests. On the right hand in the middle the buttons for the selection of the constant velocities

A calibration of the load cells and the recorder was carried out by known weights.

Special tests were aimed at the measurement of the necessary time until the nominal cross-head speed is reached, starting from standstill. These tests were inevitable because no datas were available from the manufacturer. The results—which were of some importance for the test procedures— are given in Table 1.

IV. 2. *Test snow*

The snow for the tests was harvested during the winter 1969/70 at Nakayama-pass (800 m height above sea level) in the south of Sapporo and then stored in cardboard boxes at about −15°C.

Snow layers of sufficient thickness were thought out to get samples

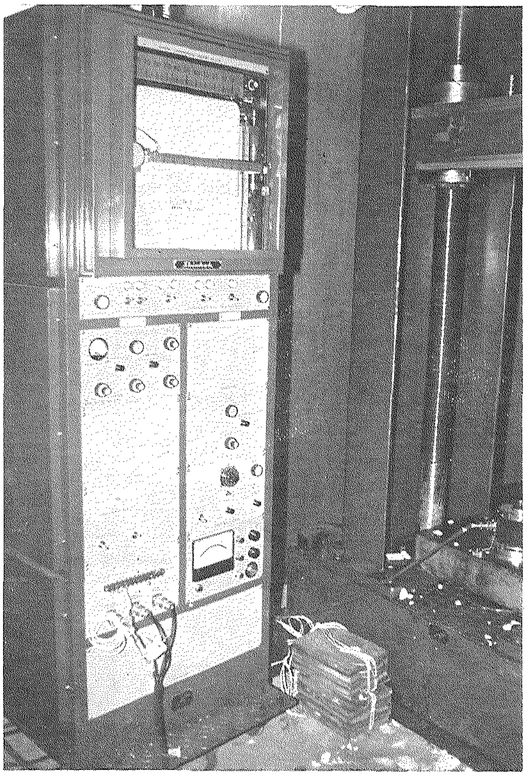


Fig. 5. Recording instrument of "TOM 10 000"

Table 1.

Cross-head speed	Necessary time and path until nominal speed is reached	
mm/minute	time in seconds	path in millimeter
500	0.20	0.8
300	0.17	0.5
200	0.15	0.3
100	0.09	0.1
50	0.03	0.01
30	0.02	0.01
20	0.01	0.01
10	0.005	0.005

which are so much as possible homogenous in structure, grain shape and density. The homogeneity of the samples seems to be one of the most important requirement for any tests in snow mechanics. If this is not fulfilled one gets no more conclusive tests because of the too large scattering in retests. The harvested snow seemed to be fairly homogenous but unfortunately there were remarkable daily fluctuations in temperature during storage. Therefore at the moment of the tests some samples were more metamorphosed than others, depending on the location within the pile of the cardboard boxes (stronger metamorphosis along the boundary surface to the air).

Three types of snow were tested, for preliminary tests snow type 1—which was about one year old—and for the main tests type 2 and 4, harvested on February 2nd and April 8th, 1970 respectively. Type 3, collected on March 6th 1970, we could no more test because of lack of time.

IV. 3. *Test samples*

(1) *Preparation of the snow samples*

The standard dimension of an undisturbed snow sample was 150 mm in height and 50 mm in width (square cross-sectional area). Generally they were cut out in such a way, that the long axis was parallel to the snow layering. In order to avoid crack formations and not to disturb the natural snow too much it seems imperative to shun too big forces. Taking into consideration this requirement it appeared that the best method for cutting out is to saw the snow with a metal saw. This is the reason for the choice of a square cross-sectional area. Attempts to cut out the samples by a circular snow sampler were unsuccessful.

In order to save time for the sample preparation and to get the nominal dimensions with relative high precision, special wooden boxes were manufactured in the workshop (Fig. 6). Thereby the most important points are that the planes perpendicular to the long axis of the sample are really planes and not curved surfaces and that they are in parallel to each other. Through this planes the forces are transmitted from the cross head and load cell respectively to the snow. If therefore there is no parallelism, the distribution of the stresses becomes nonuniformly which leads to too small strength values. It will however be shown later, that the parallelism of the planes is a necessary but not sufficient condition. The reason lies in the fact, that the snow structure is weakened near the surface as an effect of cutting. Therefore the snow in question seems to

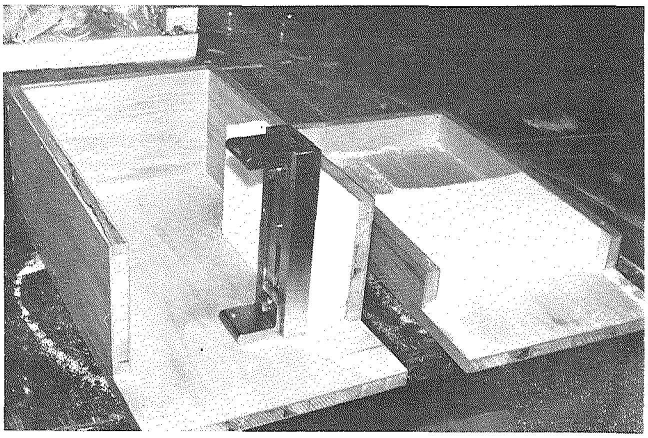


Fig. 6. Wooden boxes for the preparation of the snow samples

have a lower strength value than the actual one, according to this weak layers on both sides of the sample. By prior weighting these layers can be reinforced so that they reach the strength of the rest of the snow sample.

A period of several hours was allowed for the samples to reach equilibrium with the room temperature before the test.

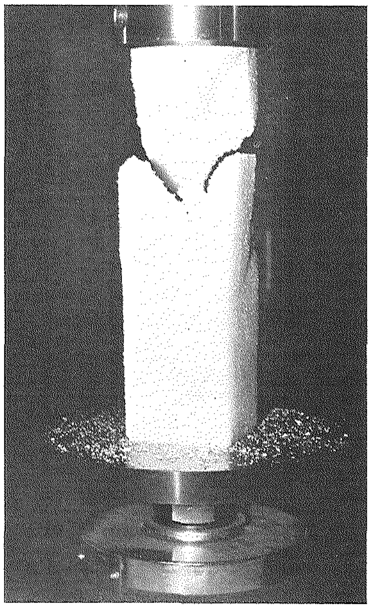


Fig. 7. Unconfined compression test No. 70.4272.2 with a typical fracture surface of snow type 4

(2) *Unconfined compression tests*

We weighed every sample and measured the width in both directions by a caliper rule before and after the test. In order to prevent a penetration of the jaw into the snow, two glass plates of known thickness were placed underneath.

(3) *Confined compression tests*

For the confined compression tests 4 metal plates were used to prevent a lateral movement of the snow. In order to adapt the inside width of these plates to the actual width of the sample, two plates in pairs were connected by 4 screws (Fig. 8). A block of metal was arranged to transmit the forces from the crosshead to the snow. The plates were coated with Teflon so as to reduce friction between snow and plates as good as possible.

Every sample was weighed and the inside width was measured in both directions by a caliper rule prior to the tests.

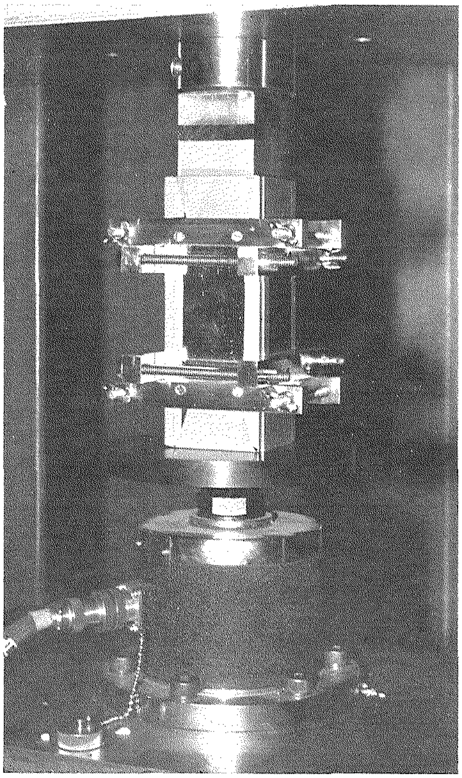


Fig. 8. Device for confined compression test

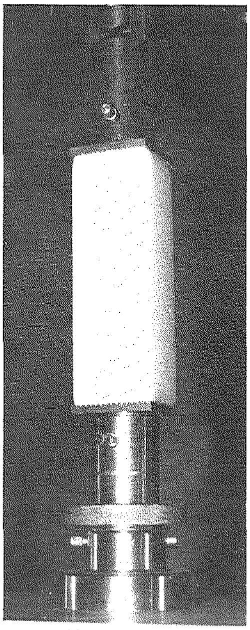


Fig. 9. Tension test

(4) *Tension tests*

The main problem for tension tests is to freeze the snow together with the connecting pieces, transmitting the forces to the load cell and to the cross-head respectively. This connection has to be at least as strong as the rest of the snow sample but on the other hand there should not be too much ice formation because of the different mechanical behavior of ice and snow. In order to increase the contact surface between snow and metal the connectors were roughened. With latticed grooves (1 mm depth) the surface was about 50% enlarged.

An important matter is furthermore not to transmit moments to the snow. Otherwise the distribution of the stresses would be no more uniform. Therefore pin-jointed connectors were constructed.

The following procedure for the freezing was chosen: The connectors were heated until a certain temperature—determined by preliminary tests—was reached. The member sensing the temperature (thermistor) was introduced through a hole into the connector, so that the temperature was measured about in the centre of it.

IV. 4. *Test procedures*

In order to formulate (III. 23) in a quantitative way, one has to measure the constants characterizing the non-BURGER's body a, C, λ and $E_{1(v)}$. If furthermore k and κ^2 are known, strength can be given as a function of the rate of deformation.

In our tests v is known and a, C, $E_{1(v)}$ and F_s could be measured. Therefrom k, λ, and κ^2 can be computed (it is impossible to measure λ directly). The main purpose of the present tests was to investigate how far k, λ, and κ^2 are really constants. It can be expected that these quantities depend at least on *snow type* (density, structure, grain shape and dimensions) and on *temperature*. They may also depend on the *state of stress*. If this would be true—the tests are demonstrating in a clear way that it is true—then they are not constants of the material but functions of the invariants of the stress tensor. Certainly not acceptable would however be a dependance of the mentioned quantities on the rate of deformation. In this case our theory, set forth in the previous section, would be invalidated.

A serious difficulty in making tests with snow is the continuous change in the properties of snow during the investigation. In short tests as in ours (duration varying from a few minutes up to about one hour), this change is probably predominantly due to a change in volume. Although this change was relatively small (less than approximately 15%, in most cases about 5%) its influence had to be taken into account. Another conceivable influence is that of "stress history" *i.e.* the way how stresses are applied during the test.

The present tests consist in two series. In a *first group* F and F_s were measured as a function of speed of deformation v and of deformation Δl of the sample. The measured values were then reduced to the undeformed state *i.e.* $\Delta l=0$. For this reduction we have to establish a relation. (III. 10) postulates that $\dot{F}$ vanishes after a sufficient long time. The tests are however showing that in fact $\dot{F}$ becomes very small after a certain time but generally always differs from zero. This effect, attributed to the change in density, seems to be governed by the differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}l}=\alpha F \qquad \text{(IV. 1)}$$

l is the actual length of the snow sample and α is a constant depending on the state of stress, the type of snow and on temperature. (IV .1) written in differences yields

$$\frac{\Delta F}{\Delta l}=\frac{F-F_0}{\Delta l}=\alpha F_0 \qquad \text{(IV. 2)}$$

where ΔF is the difference between the force F, occuring at a deformation Δl of the sample and F_0 at the beginning of the (theoretical) steady state.

(IV. 1) is in accordance with results of KOJIMA (1958) and YOSIDA (1963) who obtained from observations on the natural settlement of seasonal snows the following relation between viscosity η and density ρ

$$\eta=\eta_0 e^{c\rho} \qquad \text{(IV. 3)}$$

η_0 and c are constants. The differential equation of (IV. 3) is

$$\frac{\mathrm{d}\eta}{\mathrm{d}\rho}=c\eta \qquad \text{(IV. 4)}$$

or written in differences

$$\frac{\Delta\eta}{\Delta\rho}=c\eta_0 \qquad \text{(IV. 5)}$$

When we define viscosity as the quotient of the force during the quasi-steady state and the rate of deformation and furthermore we take into consideration that $\Delta\rho$ is to a first approximation proportional to Δl—the factor of proportionality depends on the state of stress—we obtain again (IV. 2).

For the dependence of $F_{s\Delta l}$ on Δl the same relation as (IV. 2) was chosen *i.e.*

$$F_{s\Delta l}=F_{so}+\alpha F_{so}\Delta l=F_{so}+A\Delta l \qquad \text{(IV. 6)}$$

where $F_{s\Delta l}$ denotes strength at a deformation Δl of the sample and F_{so} is the reduced value for $\Delta l=0$. A means the product of α and F_{so} (α has not the same magnitude as in (IV. 2)).

In the first group of tests the influence of stress history was investigated too in reaching a certain deformation on different ways *i.e.* with different velocities.

In a *second group* of tests the dependence of strength on rate of deformation was measured. In section IV. 3. (1) it has already been mentioned, that because of the cutting, the surfaces of the snow samples are considerably weakened. A prior weighting was therefore imperative. In Fig. 10 a sample with and without prior weighting are compared. Both samples had been sprayed with diluted ink in order to demonstrate differences in density. Snow type, temperature and velocity were in both cases equal (snow type 2, temperature $-9.2°C$, $v=3$ mm min^{-1}) except prior weighting with 0.5 mm min^{-1} of one sample. In the latter case no failure occured whereas the other sample broke continously in a surface layer. The recorded force dropped in a certain frequency to zero so that a "saw tooth" shaped

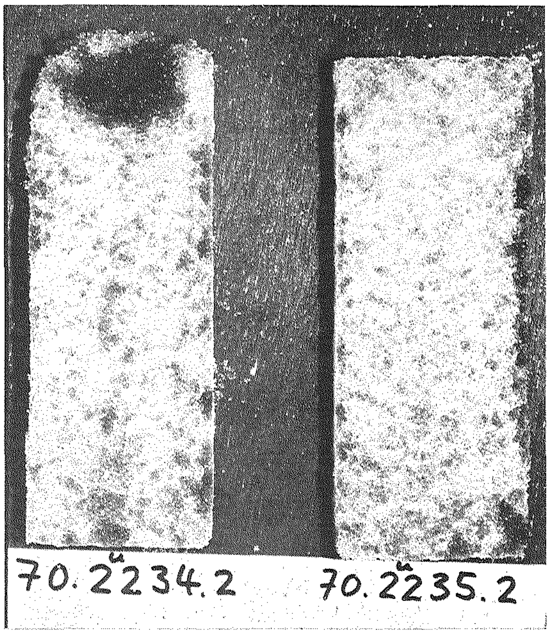


Fig. 10. Influence of prior weighting. Sample on the left without and on the right with prior weighting. Snow type 2

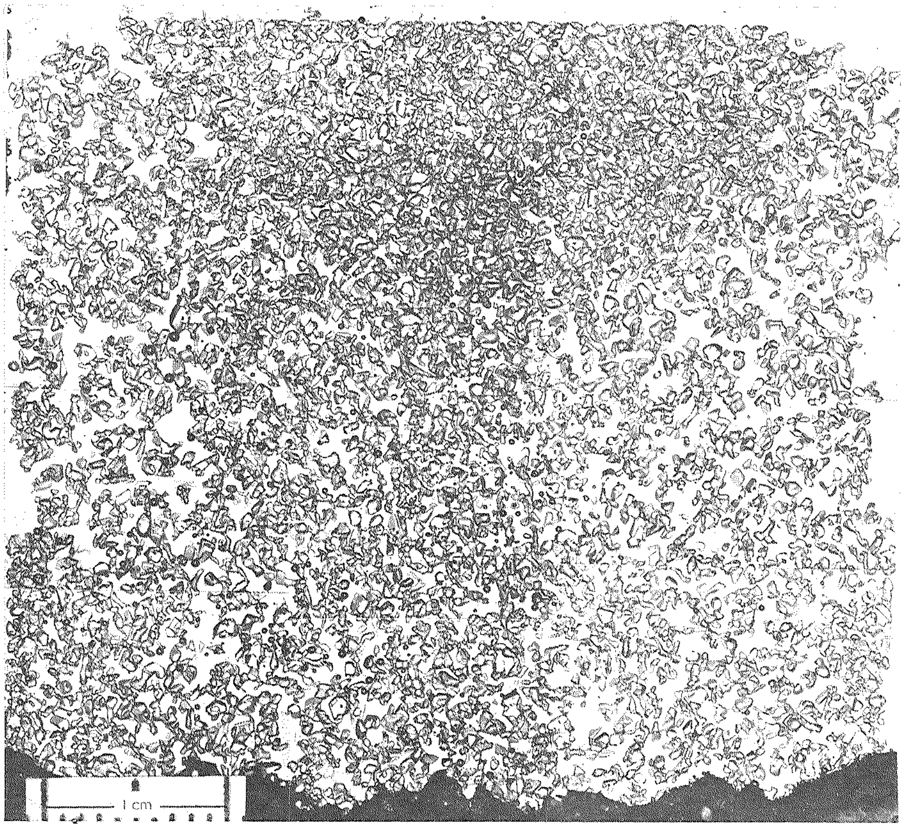


Fig. 11. Thin section of a sample without prior weighting. A triangular compressed zone can be easily recognized in the upper part. Snow type 2

pressure curve occured. The continuous failures, obviously due to stress concentrations, effected a zone of compression easily discernible in Fig. 10 and in the thin section of Fig. 11.

Before the strength test the cross head was moved backwards in accordance with Table 1, that it reaches nominal speed before touching the snow sample.

IV. 5. *Test results*

The number of a tested snow sample is xx.yzzz.a. xx means the year, y the snow type (from type 2, two nearly identical layers were investigated 2° and 2ᵘ). zzz is the number of the test and a denotes the state of stress: 1 is tension, 2 unconfined and 3 confined compression.

First of all some remarks to special tests devoted to the problem of the relation between forces in the quasisteady state and the deformation. Fig. 12 and Fig. 13 are showing records of tests where snow samples were deformed to a large amount under tensile and compressive forces smaller than strength. The tensile test Fig. 12 shows a decrease of the force with growing deformation. The increase of the

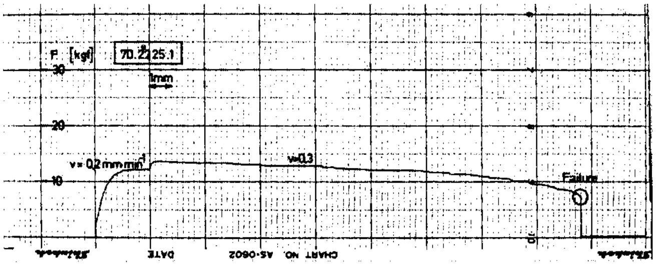


Fig. 12. Tensile test with large deformation

volume in this test was about 8%. Failure occured probably because the majority of the "viscous lines" of the ice lattice became straight lines during deformation and therefore no possibilities existed for any further viscous extension. Fig. 13 shows two unconfined compressive

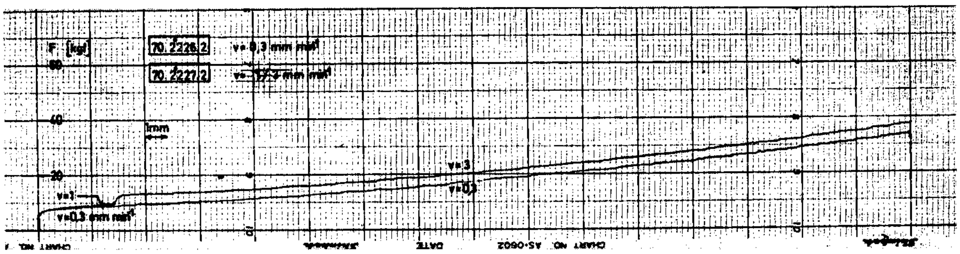
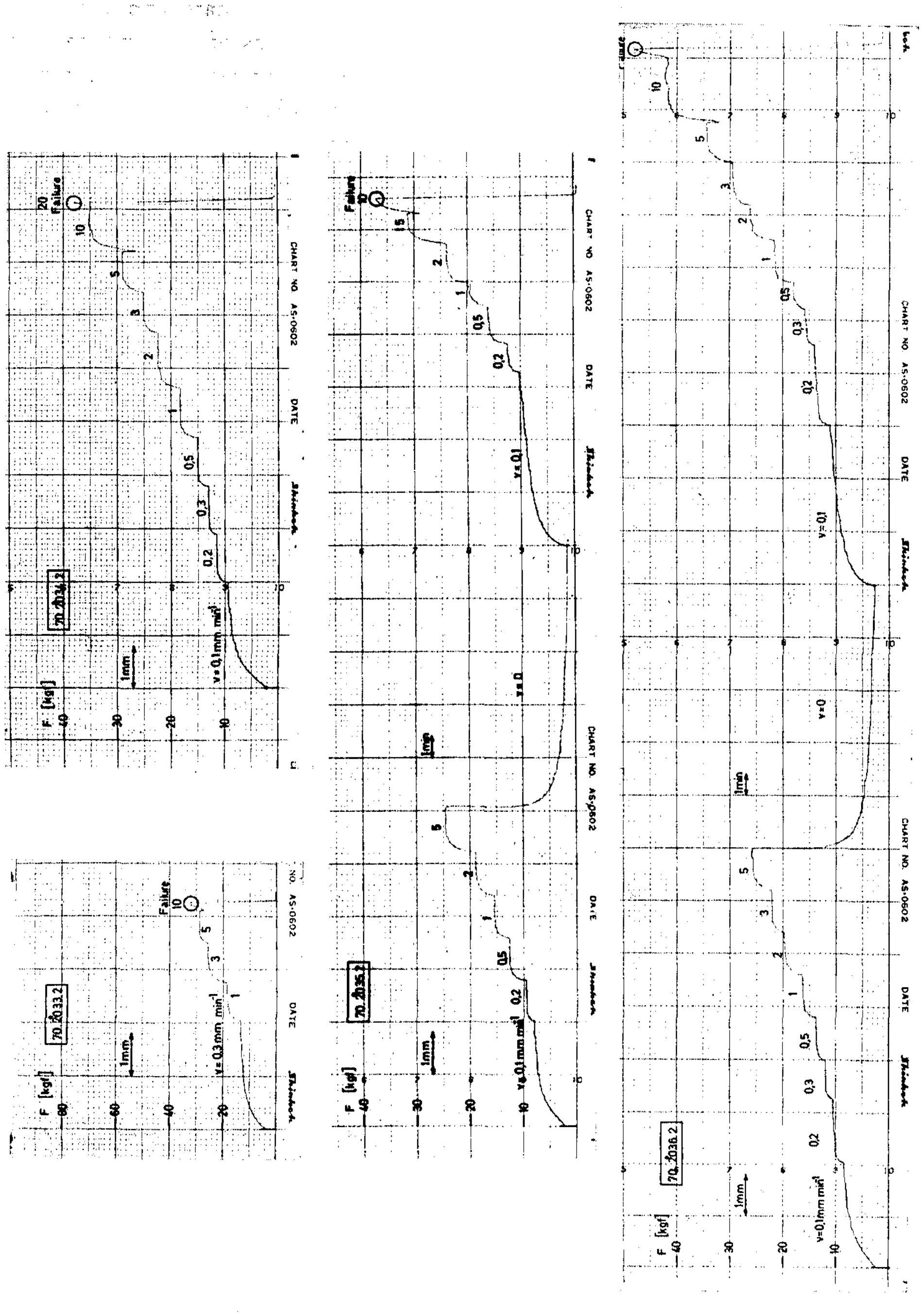


Fig. 13. Two unconfined compression tests with large deformation

tests with different velocities but with the same starting point on the chart. The shape of both curves is in agreement with relation (IV. 2): Factor α remained fairly constant during the tests and was independent on velocity. In either case the decrease in volume was about 15%. Of course no failure occured in these compression tests.

Fig. 14 shows an example of the *first group* tests. The measured mean values of α are:


F [kgf]
1mm
v=0.1mm min-1
70.2034.2
Failure
70.2033.2
v=0.3mm min-1
70.2035.2
1min
v=0.1
v=0
70.2036.2
CHART NO. AS-0602
DATE

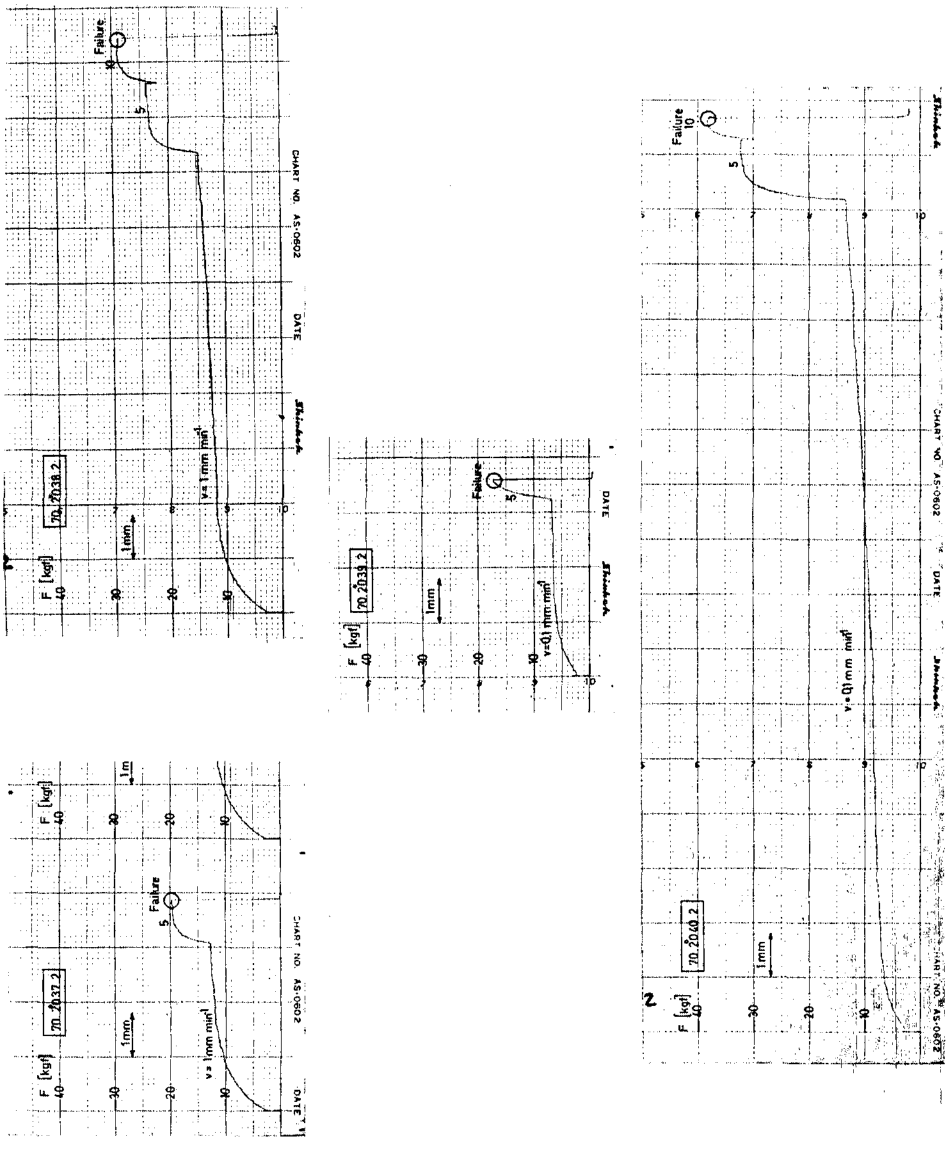


Fig. 14. Typical first group tests

Table 2.

Experiment	α [mm^{-1}]
70.2°035.2	+ 0.0320
70.2°036.2	+ 0.0288
70.2°038.2	+ 0.0312
70.2°040.2	+ 0.0431

Fig. 15, 16, 17 and 18 represent a semilogarithmic plot of the reduced forces and the velocities. The straight lines according to equation (III. 6) were determined by a least square fit. Here we detect an extreme interesting fact: *The constants a and C of relation* (III. 6) *are changing their values rather abruptly when a certain rate of deformation is exceeded.* This phenomenon—it is not due to the change of density because this effect has been already eliminated—was more or less observed in all tests. It seems that a change of deformation mechanism takes place as soon as some critical rates of defor-

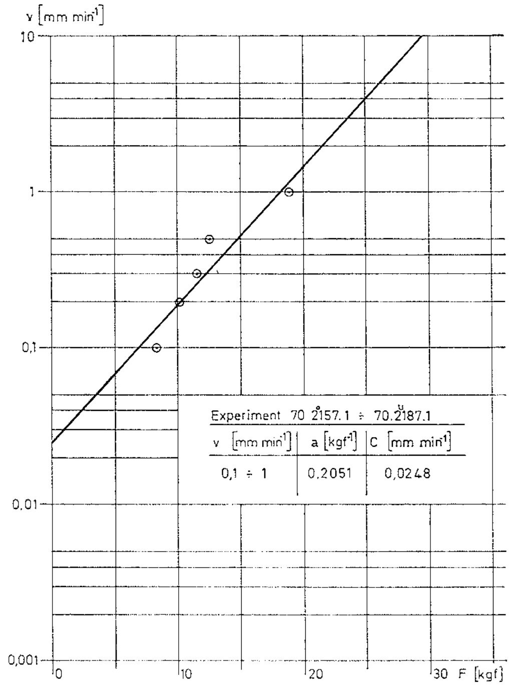


Fig. 15. Constants of the non-NEWTONIAN body for tension. Snow type 2° and 2ᵘ. Temperature −8.7°C

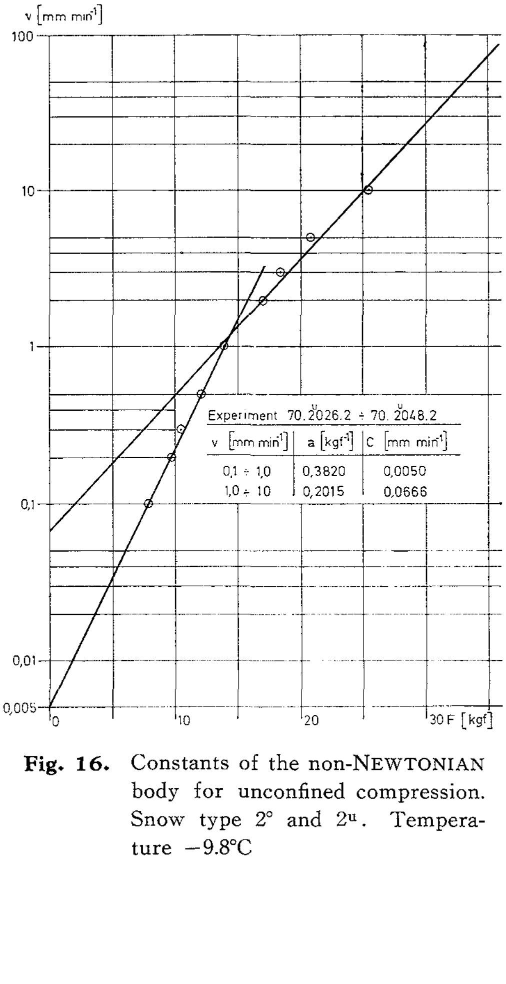


Fig. 16. Constants of the non-NEWTONIAN body for unconfined compression. Snow type 2° and 2ᵘ. Temperature −9.8°C

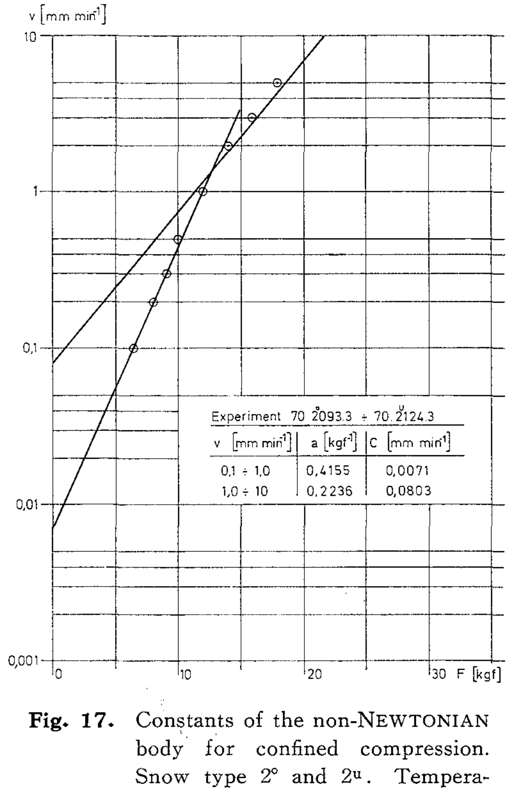


Fig. 17. Constants of the non-NEWTONIAN body for confined compression. Snow type 2° and 2ᵘ. Temperature −9.2°C

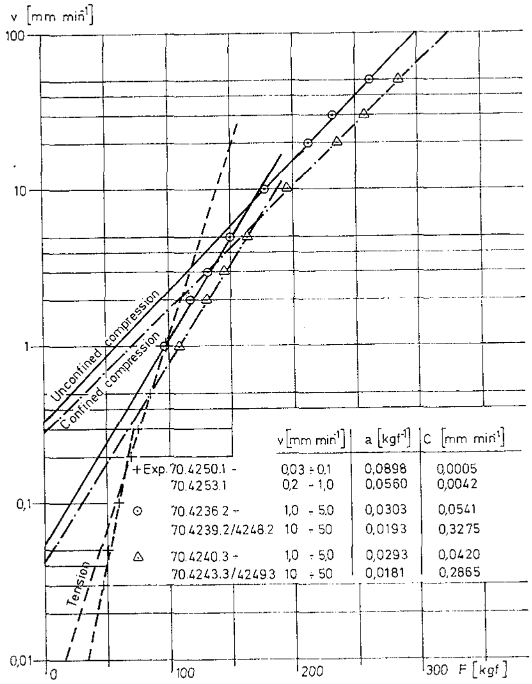


Fig. 18. Constants of the non-NEWTONIAN body for tension, unconfined and confined compression. Snow type 4. Temperature −10.3°C

mation are reached. Therefore one state of stress seems to be associated with two or even more values of a and C respectively. For our failure criterion we have to make use of the values a and C belonging to the highest velocities.

Fig. 19, 20, 21 and 22 show the relation of strength and deformation. They also were calculated by a least square fit. The quantities A and F_{so} refer to equation (IV. 6). ρ is the density, R the mean hardness of the snow samples after the tests measured with KINOSITA's hardness meter. T means temperature.

A comparison between the theoretical and measured relation between forces, rate of deformation and deformation is given in Fig. 23. The theoretical relaxation curve according to equation (III. 17) is given too. It is obvious that the theoretical curves do not agree well with the measured ones in the case of low stresses because of the assumptions of the present theory.

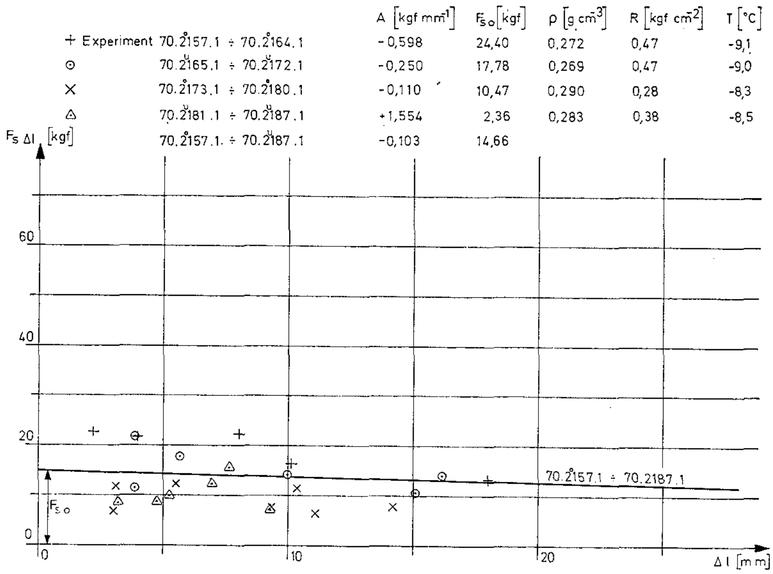


Fig. 19. Tensile strength as a function of deformation. Snow type 2° and 2ᵘ

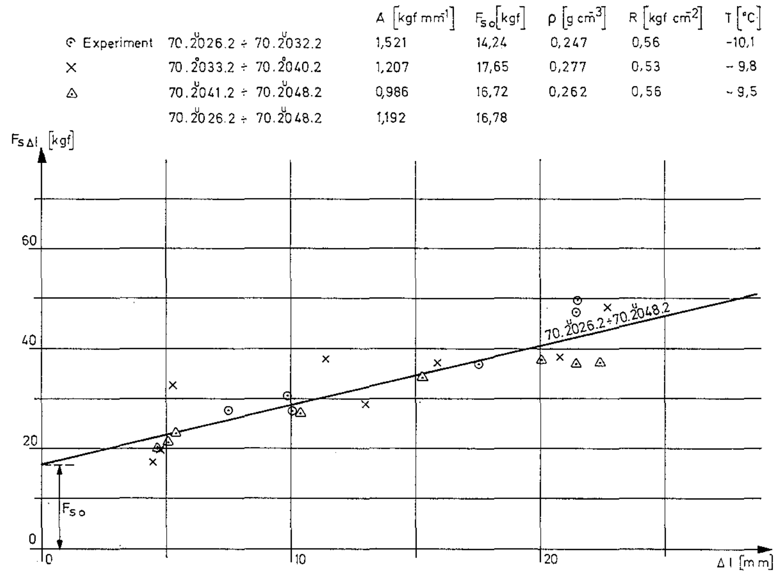


Fig. 20. Unconfined compressive strength as a function of deformation. Snow type 2° and 2ᵘ

		A [kgf mm⁻¹]	F_{so} [kgf]	ρ [g cm⁻³]	R [kgf cm⁻²]	T [°C]
+ Experiment	70.2093.3 ÷ 70.2100.3	1,604	9,64	0,284	0,37	-8,9
⊙	70.2101.3 ÷ 70.2108.3	1,545	11,39	0,272	0,42	-8,9
×	70.2109.3 ÷ 70.2116.3	1,936	21,49	0,275	0,73	-9,7
△	70.2117.3 ÷ 70.2124.3	1,748	19,10	0,271	0,48	-9,4
	70.2093.3 ÷ 70.2124.3	1,750	15,12			

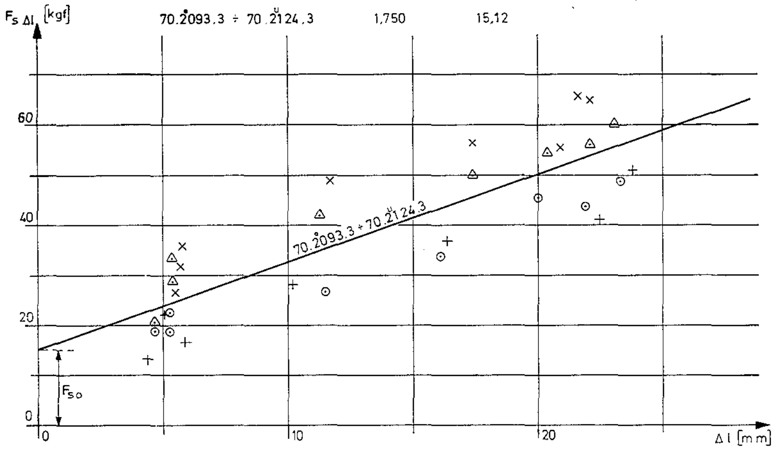


Fig. 21. Confined compressive strength as a function of deformation. Snow type 2° and 2^u

		A [kgf mm⁻¹]	F_{so} [kgf]	ρ [g cm⁻³]	R [kgf cm⁻²]	T [°C]
+ Experiment	70.4250.1 ÷ 70.4253.1	-0,330	93,32	0,423	24	-10,3
⊙	70.4236.2 ÷ 70.4239.2/4248.2	3,141	196,73	0,424	31	-10,3
△	70.4240.3 ÷ 70.4243.3/4249.3	7,302	333,09	0,431	45	-10,3

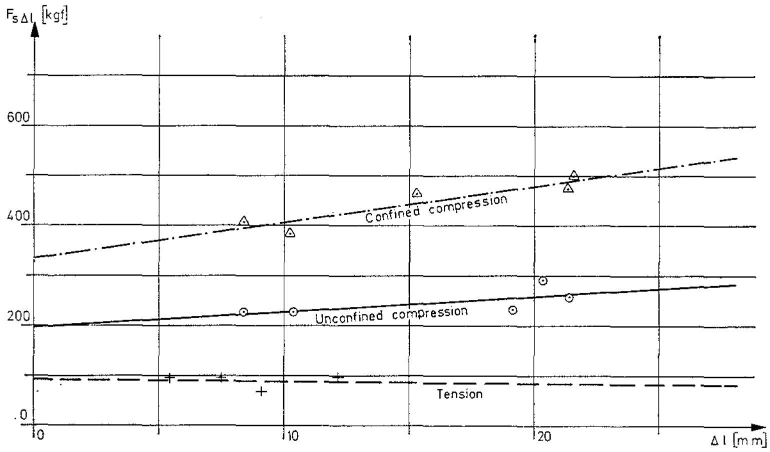


Fig. 22. Tensile, unconfined and confined compressive strength as a function of deformation. Snow type 4

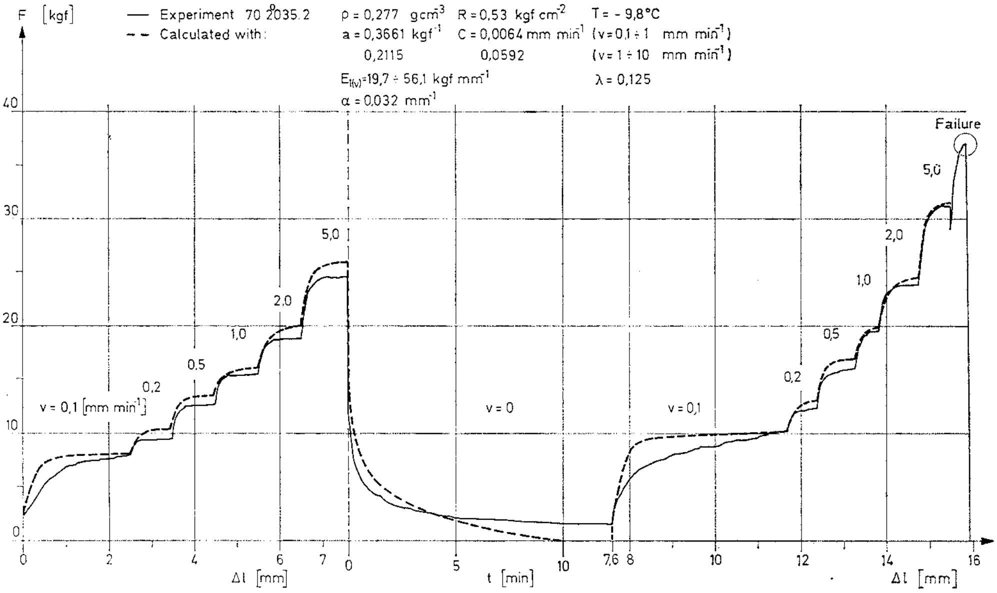


Fig. 23. Comparison of the actual behavior of snow with the rheological model

Concerning the *stress history* it is to notice that *no influence was observed* neither on α nor on $F_{s\,\Delta l}$ when considering this quantities as a function of deformation. The same is true for the constants a and C. This independence on stress history was discovered in all our tests. It seems therefore that the change of the different properties of snow during short time tests, say with a duration of less than a few hours, depends only on *one* parameter, the *density*.

Two examples of the *second group* tests are given in Figs. 24 and

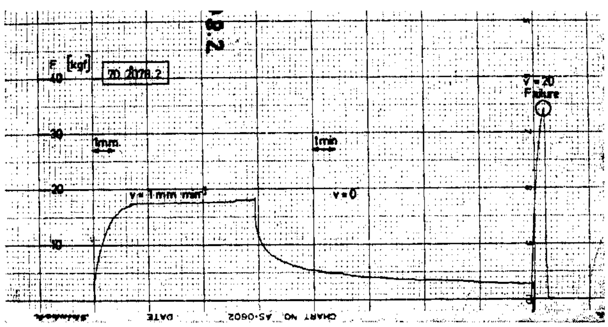


Fig. 24. Strength test of snow in the second group tests. Failure with a rate of deformation of 20 mm min^{-1}

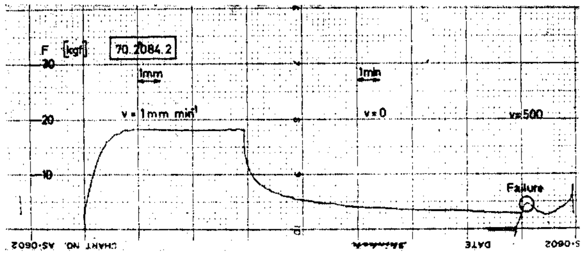


Fig. 25. Second group test. Failure with a rate of deformation of 500 mm min^{-1}

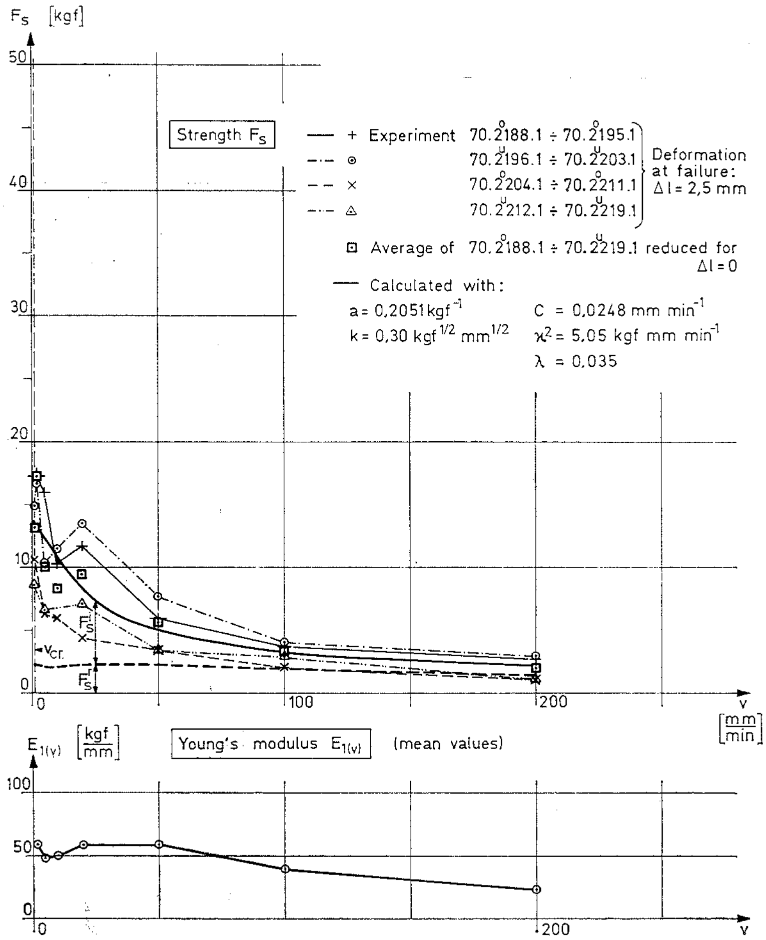


Fig. 26. Tensile strength. Snow type 2° and 2^{u}. Temperature $-9.5°C$

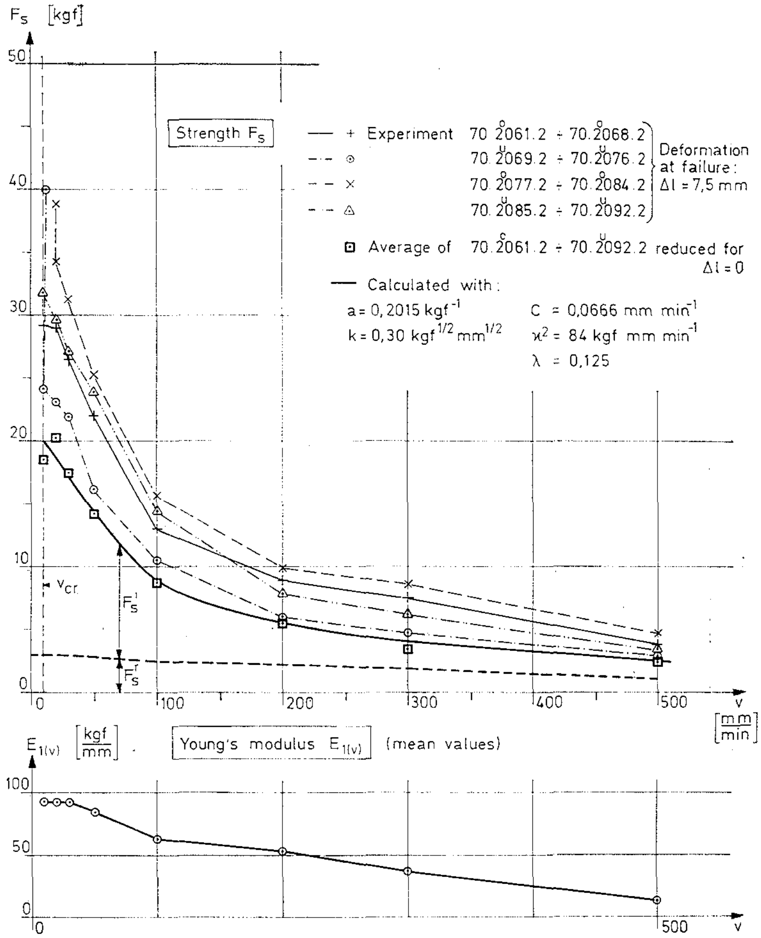


Fig. 27. Unconfined compressive strength. Snow type 2° and 2^{u}. Temperature -8.9°C

25. The records are showing the prior weighting with 1 mm min^{-1} and afterwards the relaxation curve. The increase of the force until strength is reached was always used for the measurement of $E_{1(v)}$. Interesting seems to look more closely at the needed time for fracture *i.e.* the time from beginning until end of breakdown. This failure time is considerably longer for 500 mm min^{-1} than for 20 mm min^{-1} which can be interpreted as an effect of the progressive fracture under high rates of deformation discussed in section III.

The final results of the tests are given in Figs. 26, 27, 28 and 29.

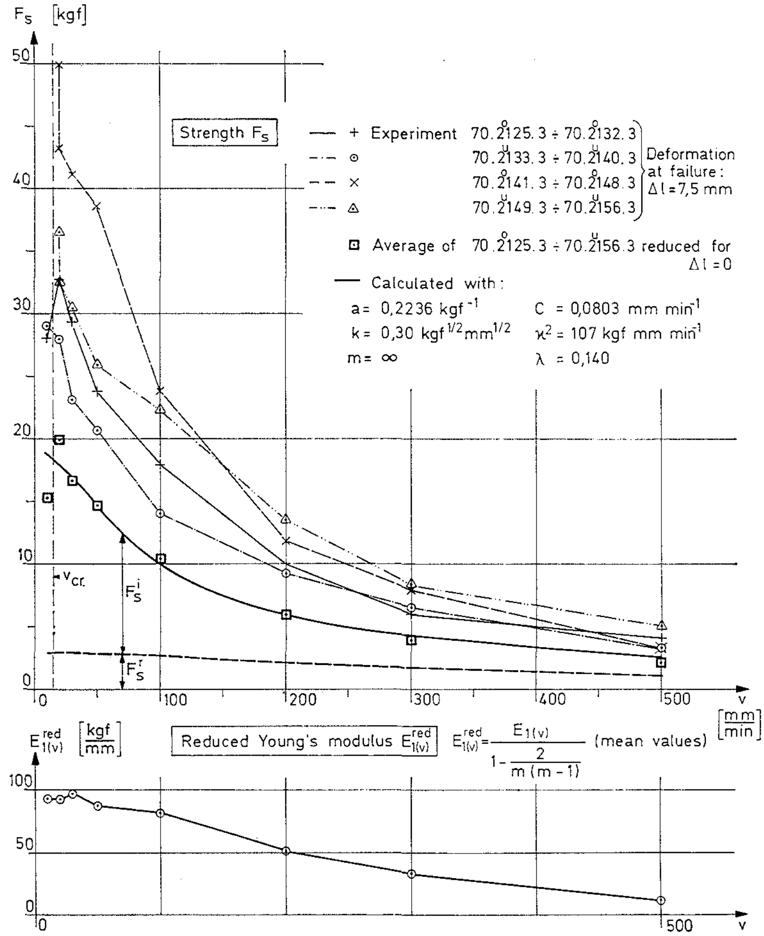


Fig. 28. Confined compressive strength. Snow type 2° and 2^{u}. Temperature $-9.6°C$

Values for k, κ^2 and λ were seeked in way that the theoretical curves given by equation (III. 23) are fitting the measured values as well as possible. In doing so k could be determined only in a more or less arbitrary way. For a better determination of k additional tests under extreme high rates of deformation would be necessary so that strength would consist merely in the component F_s^r. Remarkable are the differences of κ^2 and λ in different states of stress.

Some peculiarities are appearing in the *confined compression tests*, because here the state of stress is a three-dimensional one and we

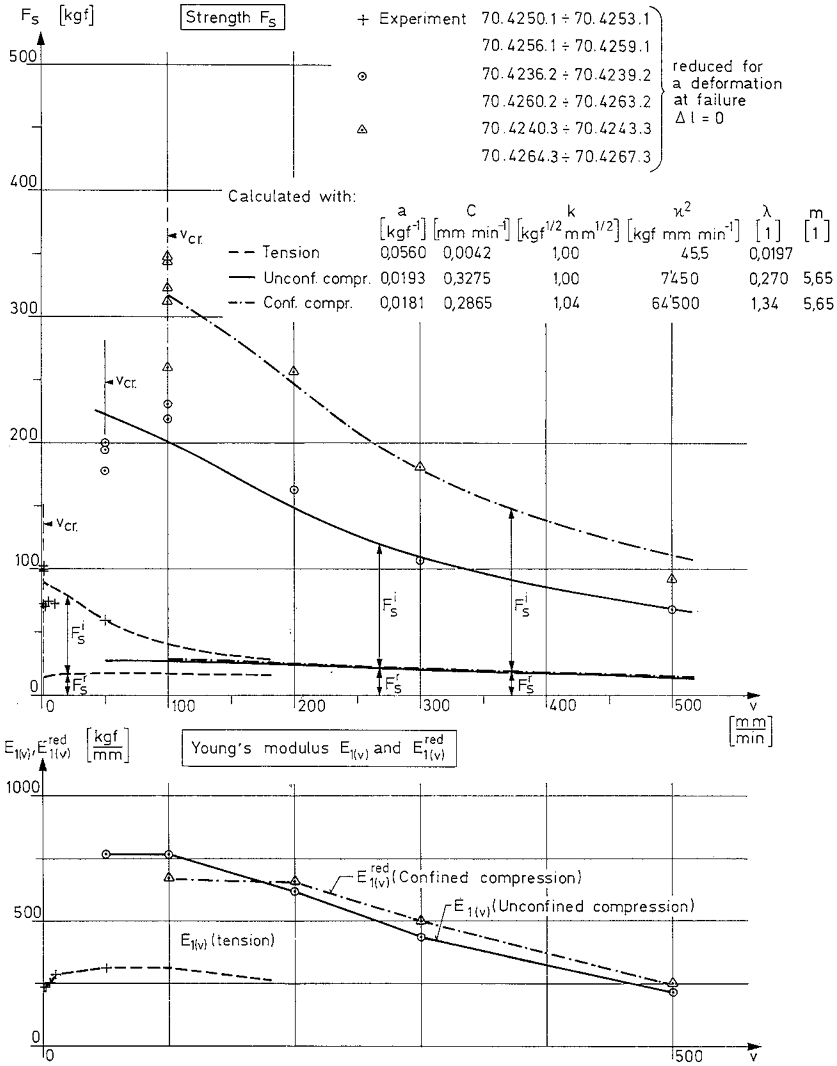


Fig. 29. Tensile, unconfined and confined compressive strength. Snow type 4. Temperature −10·2°C

have therefore to adapt our one-dimensional model to this state. For the *reversible* part of strength we find instead of the first term on the right of (III. 23)

$$F_s^r = +k\sqrt{\varkappa E_{1(v)}\frac{m-1}{(m-2)\left(1+\frac{1}{m}\right)}} \qquad \text{(IV. 7)}$$

where m is the (elastic) Poisson's ratio, which can be found by comparing the reduced Young's modulus of the confined compression test

$$E_{1(\nu)}^{\text{red}} = \frac{E_{1(\nu)}}{1 - \dfrac{2}{m(m-1)}} \tag{IV. 8}$$

with the real Young's modulus $E_{1(\nu)}$. The values of m found in such a way are given in the figures. Concerning the *irreversible* part of strength it is to mention that (III. 6) holds too for confined compression, therefore the corresponding values of a and C were introduced in the second term of the right of (III. 23).

The present results are demonstrating in a clear way that the von MISES or DRUCKER-PRAGER and with the latter also the COULOMB-MOHR criterion are not in a position to explain strength of snow. It

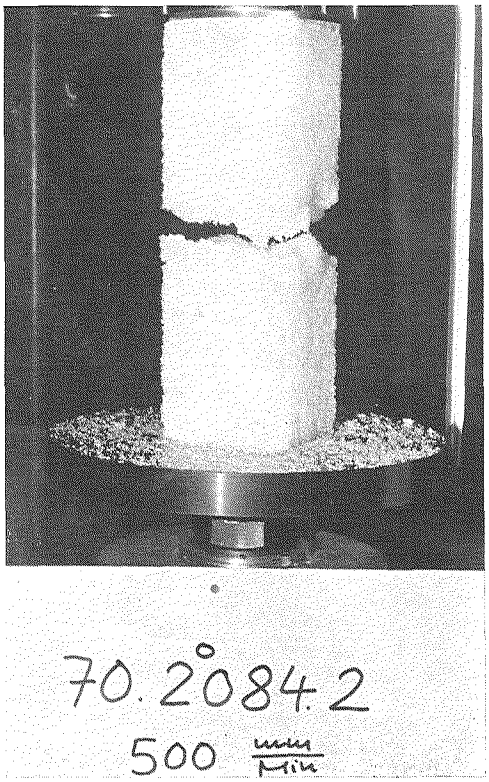


Fig. 30. Unconfined compression test No. 70.2°084.2. Rate of deformation 500 mm min⁻¹. Typical fracture surface of snow type 2. (compare Fig. 25)

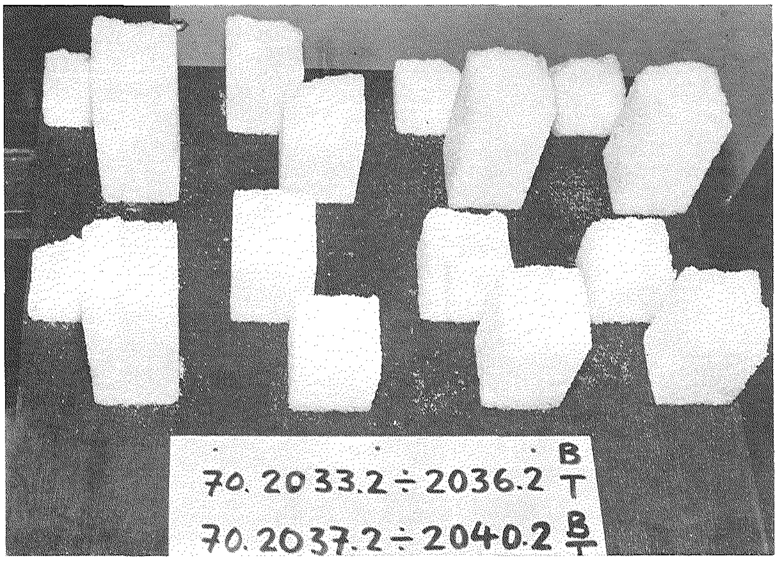


Fig. 31. Fracture surfaces of the first group test No. 70.2°033.2–70.0°040.2. Temperature −9.8°C

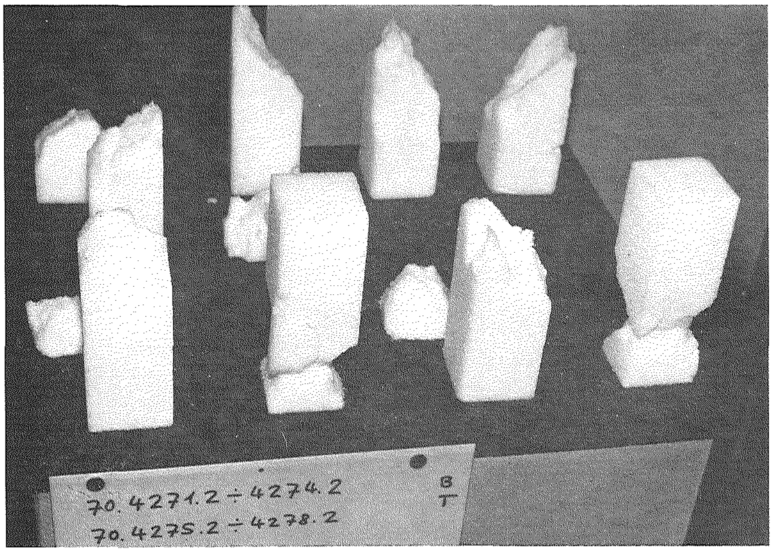


Fig. 32. Fracture surfaces of the first (front row) and second group test No. 70.4271.2–70.4278.2. Temperature −5.4°C

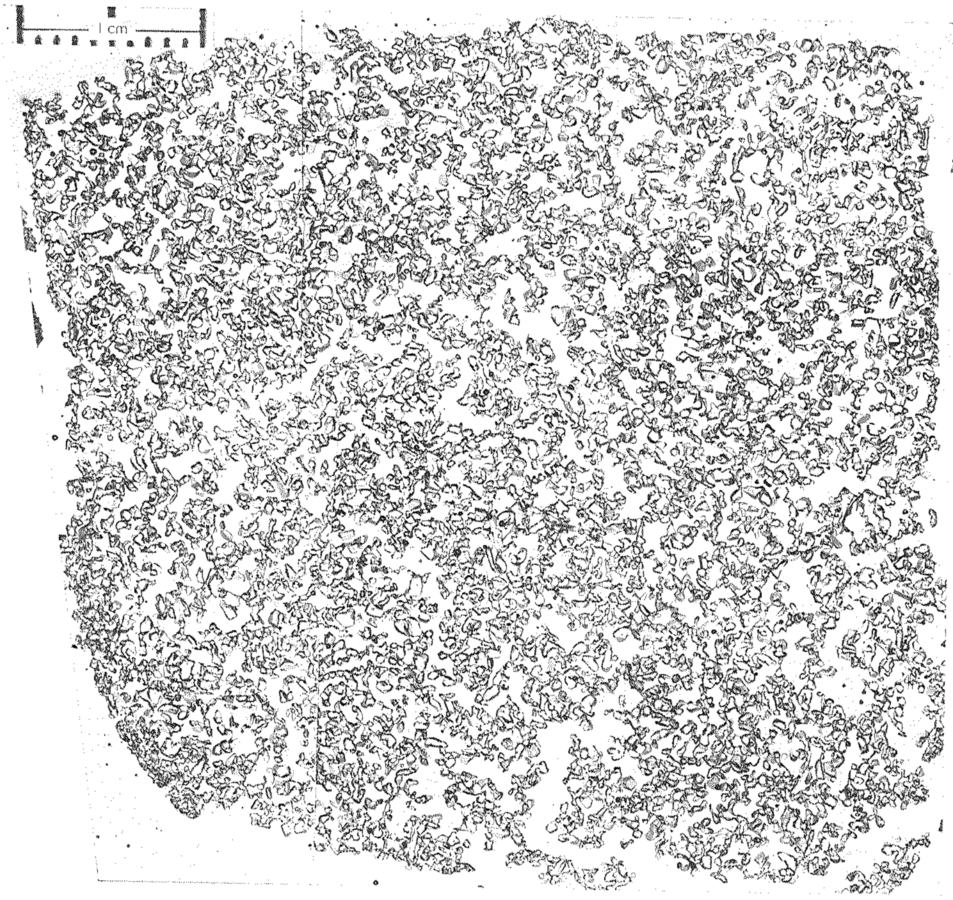


Fig. 33. Thin section of snow type 2°

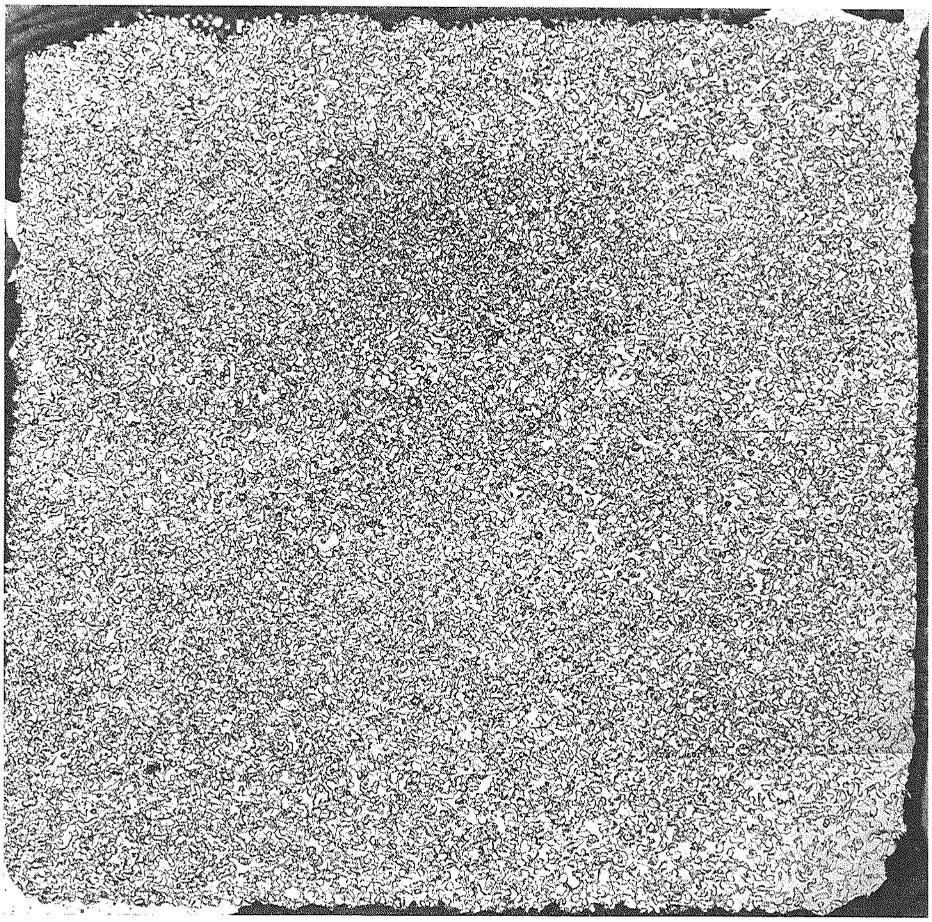


Fig. 34. Thin section of snow type 4

seems however that a combination of the postulates of the critical power and critical work, set forth in section III, fit the properties of snow.

The infiuence of *temperature* was investigated in a few unconfined compression tests with snow type 4. It turned out that α of equation (IV. 2) is a function of temperature as it is shown below.

Table 3.

Experiment	Temperature {°C}	α {mm^{-1}}
70.4238/4239.2	− 10.3	+ 0.00153
70.4248.2		
70.4273/4274.2	− 5.4	+ 0.00666
70.4281/4282.2	− 2.7	+ 0.01143

The same is true for α refering to equation (IV. 6), whereas C does not

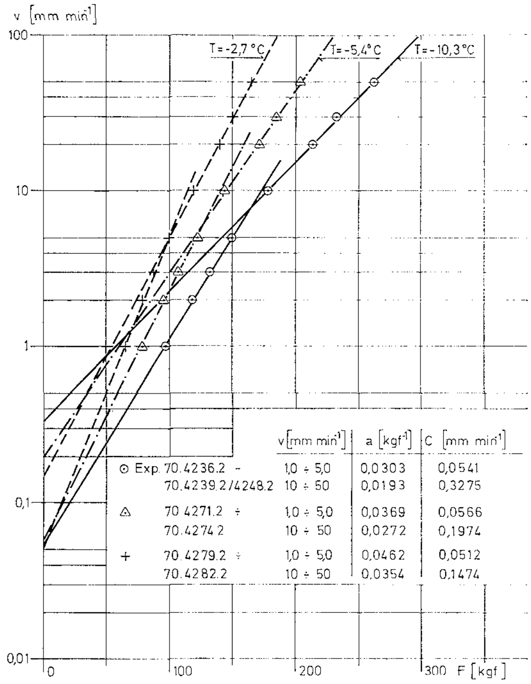


Fig. 35. Constants of the non-NEWTONIAN body for unconfined compression at different temperatures. Snow type 4

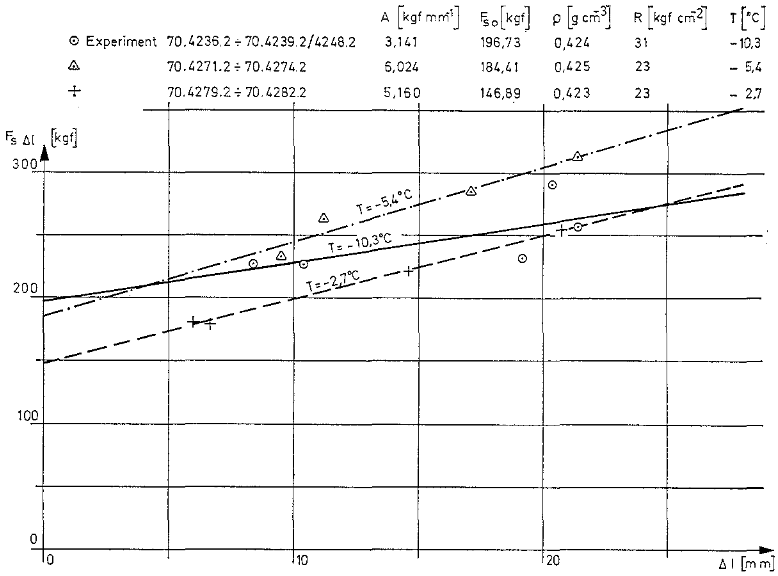


Fig. 36. Unconfined compressive strength as a function of deformation and temperature. Snow type 4

seem to depend on temperature (Figs. 35 and 36). The temperature variation of strength appears as a function of λ only (Fig. 37).

Finally a batch of samples was tested with reference to *mechanical isotropy*. The dimension of this samples was 50 mm in height and width because one standard sample was cut into 3 pieces and each of

Table 4.

Experiment	$v\left[\frac{mm}{min}\right]$	F^{I}	F^{II}	F^{III} [kgf]	
70.2°901.2-2°903.2	0.3	15.4	12.0	15.5	
	2	26.1	20.6	26.7	
	3	29.4	23.6	29.7	
		32.8	27.0	33.5	(strength)
70.2^{u}907.2-2^{u}909.2	0.3	17.7	12.2	15.6	
	2	30.4	22.2	27.2	
	3	34.2	25.6	30.6	
		37.1	29.5	35.4	(strength)
70.4910.2-4912.2	1	128	125	123	
		236	232	235	(strength)

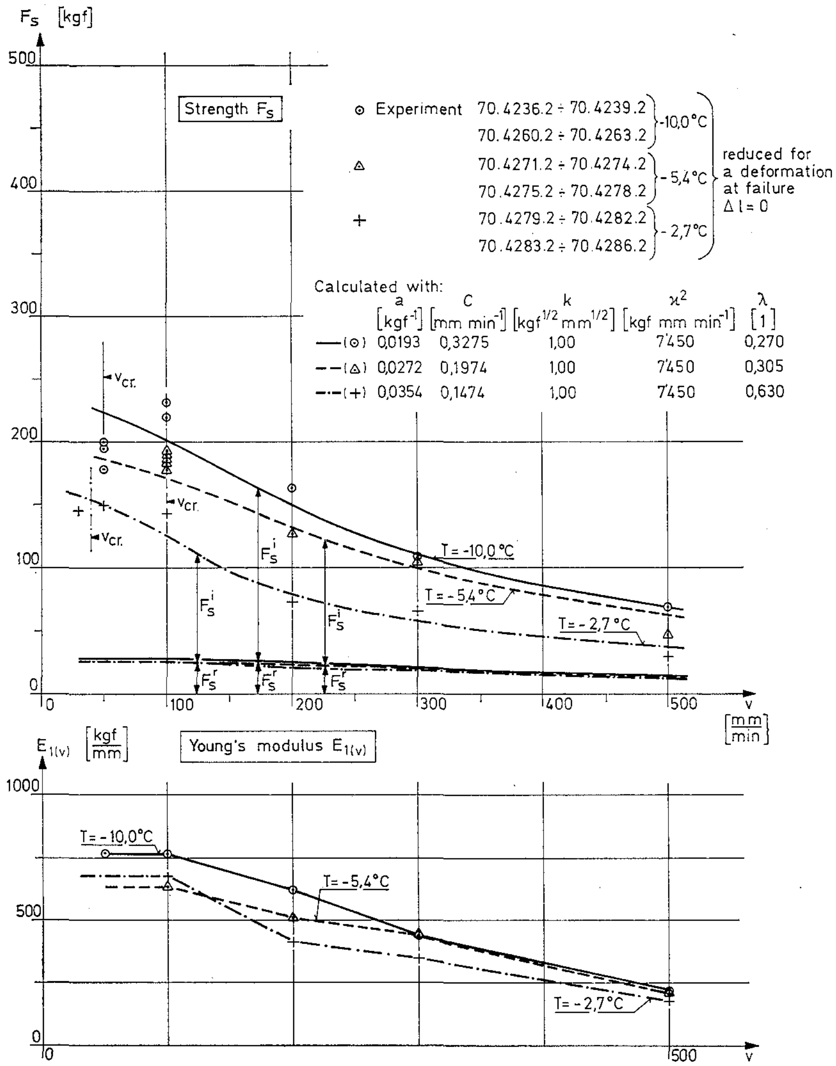


Fig. 37. Unconfined compressive strength as a fnnction of temperature. Snow type 4

them was loaded differently refering to snow layering. Results are given in table 4. Index I means direction of force parallel to the snow layer and indices II and III mean directions perpendicular to direction I. These results are proving that our test snow is not far away from being isotropic.

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References

DE QUERVAIN, M. 1946 Kristallplastische Vorgange im Schneeaggregat II. *Interner Bericht SLF* Nr. **24**, Weissfluhjoch, Davos, 1–41.

DRUCKER, D. C. and PRAGER, W. 1952 Soil mechanics and plastic analysis or limit design. *Quart. Appl. Mech.*, **10**, No. 2. 157–165.

FURUKAWA, G. 1958 A study on the shearing resistance for snow deposite. *Seppyo (Japanese Soc. of Snow and Ice)*, **20**, No. 4, 1–11.

KINOSITA, S. 1967 Compression of snow at constant speed. *In* Physics of Snow and Ice, Part **2**, (H. Ôura, *ed.*), Inst. Low Temp. Sci., Sapporo, 911–927.

KOJIMA, K. 1958 Sekisetsusō no nensei asshuku. IV (Viscous compression of natural snow layers. IV). *Low Temp. Sci.*, **A 17**, 53–64.

MELLOR, M. and SMITH, J. H. 1966 Strength studies on snow. *In* Scientific Aspects of Snow and Ice Avalanches, Davos, IASH Publ., **69**, 100–113.

PRAGER, W. 1961 Einführung in die Kontinuumsmechanik. Birkhäuser Verlag, Basel und Stuttgart, 139–168.

REINER, M. 1958 Rheology. *In* Encyclopedia of Physics (S. Flugge, *ed*). VI. Elasticity and Plasticity, Springer-Verlag, Berlin-Göttingen-Heidelberg, 434–550.

YOSIDA, Z. 1963 Physical properties of snow. *In* Ice and Snow (W. D. Kingery, *ed.*), M. I. T. Press, Cambridge, Mass., 485–527.

ZIEGLER, H. 1963 Some extremuum principles in irreversible thermodynamics with application to continuum mechanics. *In* Progress in Solid Mechanics (I. N. Sneddon and R. Hill, *ed.*), IV, North-Holland Publishing Company, Amsterdam, 91 pp.