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Tight Money Policies and Inflation Revisited*

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Abstract

This paper reconsiders the link between tight money policies and inflation in the spirit of Sargent and Wallace's (1981) influential paper "Some Unpleasant Monetarist Arithmetic". A standard neoclassical model with production, capital, bonds, and return-dominated currency is used to study the long-run effects on inflation of a tightening of monetary policy engineered via a open market sale of bonds. The potential for tight money policies to be inflationary (unpleasant arithmetic) is shown to exist even when the real interest rate is below the growth rate of the economy. Equilibria exhibiting unpleasant arithmetic can be stable. In contrast, when monetary policy is conducted via an inflation target rule, the only stable equilibrium is the one that exhibits pleasant arithmetic. The two monetary policy rules therefore produce sharply different predictions about the likely observability of unpleasant arithmetic in real-world economies.

JEL Codes: E52, E63, E31.

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1 Introduction

Conventional wisdom suggests that a tightening of monetary policy ought to be accompanied by a reduction in inflation. There is ample evidence, at least from highly inflationary economies like Germany and Argentina in the post-war period, to support such a notion. A counterexample to this well-accepted line of thinking is provided by Sargent and Wallace (1981) in their classic paper "Some Unpleasant Monetarist Arithmetic". In that paper, they show that in a standard model where monetary policy cannot affect the fixed real interest rate, it is quite possible for a tightening of monetary policy to precipitate an increase in inflation ("unpleasant monetarist arithmetic"). A tightening of money in their model reduces revenue, causing debt to accumulate, which at some future date must be serviced by a higher inflation tax. This "spectacular" result is, however, predicated on the satisfaction of one crucial yet controversial condition: the after-tax real rate of interest has to be higher than the real rate of growth of the economy.[1]

Subsequent to the publication of the Sargent and Wallace paper, a lively debate has ensued in which a number of researchers have questioned the relevance of this condition for real world economies, and have gone on to wonder if unpleasant arithmetic is nothing but a theoretical curiosum. Darby (1984) and more recently, Espinosa and Russell (1998), have pointed out, for example, that for the post-war U.S, the after-tax real rate of return on government debt is close to zero and that the real growth rate of GDP is over 3 percent. The implication is clear: since we live in a low real-interest-rate world, the only arithmetic we are likely to observe is of the "pleasant" variety. Miller and Sargent (1984), using insights from a Diamond (1965)-style model with an endogenous real interest rate, responded by showing that unpleasant arithmetic may obtain even when the real interest rate is *less* than the growth rate of the economy.[2] Wallace (1984) uses a pure-exchange

[1] The Sargent-Wallace result also presumes that it is the fiscal authority that has the dominant upper hand in determining the course of deficits.

[2] Throughout the paper, we use the term "real interest rate" to signify the real return on one-period government bonds, and not the real marginal product of capital.

model with an endogenous real rate to show that if the policy instrument is the bonds to money ratio (and not the money growth rate as in Sargent and Wallace, 1981), then across steady states, unpleasant arithmetic is possible when the real rate exceeds the growth rate of the economy. Espinosa and Russell (1998) use the Wallace (1984) setup to show that both pleasant and unpleasant arithmetic are possibilities when the real rate is below the growth rate of the economy. Bhattacharya, Guzman, and Smith (1998), using the bonds-money ratio as the policy variable, and a model similar to Wallace (1984), showed that unpleasant arithmetic was possible in a low real-rate economy where the fixed marginal product of capital is higher than the growth rate of the economy.

This paper is part of this aforementioned line of research. It seeks to understand the long-run impact of monetary policy on inflation in low real-interest-rate economies. The model used is a hybrid of Diamond (1965) and Wallace (1984). The former component allows us to replace the assumption of a linear production technology in Sargent and Wallace (1981) with the assumption of a neoclassical production technology. As a result, the real interest rate is no longer fixed. The latter component permits us to study equilibria where money is held even when dominated by competing stores of value. In our setup the single final good may be used to purchase government bonds or it may be stored to yield capital the following period. The latter activity is assumed to intermediated, and it is also subject to a standard currency reserve requirement. Capital dominates money in rate of return. We focus on equilibria where the reserve requirement binds. In these equilibria, bonds dominate money in rate of return, and the marginal product of capital exceeds the real return on bonds. It remains possible for the real rate of return on government bonds to stay below the long run real rate of growth (unity in our case) of the economy.

Following Wallace (1984), we assume that the monetary authority chooses the bonds-money ratio in the initial period and keeps the ratio fixed for all time. Changes in this bond-money ratio may be thought of as permanent open market operations. As in Wallace (1984), a tightening of monetary policy is modeled as a permanent cut in this ratio. In such a world, and for fairly general

specifications of preferences and technology, we provide a complete characterization of the necessary and sufficient conditions needed to obtain unpleasant monetarist arithmetic across steady states. In particular, we show that if household saving is mildly sensitive to its return, then multiple equilibria (with the same bonds-money ratio) are possible. At some of these equilibria, the volume of seigniorage revenue the government collects is increasing in the inflation rate. These are the ones that are the most empirically relevant. At these equilibria, tight money necessarily reduces the capital stock and may even be inflationary. In particular, if the initial real interest rate exceeds the growth rate of the economy, then the unpleasant arithmetic of Sargent and Wallace necessarily obtains. Recall, however, that our principal focus is on equilibria with real interest rates below the economy's growth rate. We show that even in this case, unpleasant arithmetic remains a strong possibility.

Sharper results are obtained when saving is return-invariant. In this case, there is a unique stationary monetary competitive equilibrium (if any). At this equilibrium, an increase in the inflation rate necessarily raises the volume of seigniorage revenues. This equilibrium may have an associated real interest rate that is high or low. If the initial real interest rate is greater than the growth rate of the economy, unpleasant arithmetic holds. Even at the equilibrium with a real interest rate less than the growth rate, it is still possible for unpleasant (and pleasant) arithmetic to obtain.[3] Specializing to a model economy with Cobb-Douglas technology and logarithmic utility, we go on to show that pleasant (unpleasant) arithmetic is a strong possibility for low (high) values of monetary-policy tightness.[4] Two additional points deserve mention here. Unlike Bhattacharya, Guzman and Smith (1998), our results do not require the marginal product of capital to exceed the economy's growth rate. Similarly, for tight money to be disinflationary, unlike Wallace (1984),

[3]This result has been obtained in slightly different environments by Miller and Sargent (1984), Bhattacharya, Guzman, and Smith (1998), and Espinosa and Russell (1998, 2001).

[4]In a model without production and capital, Espinosa and Russell (1998) show that under low interest rates, a small tightening of money is disinflationary if initially money is not too tight.

we do not require that the marginal product of capital be less than the growth rate.[5]

The paper closest in spirit and form to our current endeavour is Espinosa and Russell (2001).[6] They consider a model that is almost identical to ours except for the following features. First, and foremost, their specification of the monetary policy rule is different from ours: they assume that monetary policy is conducted by fixing the money growth rate and allowing the bonds-money ratio to adjust accordingly. Second, our specification of preferences and technology is more general; they focus on return-invariant saving and Cobb-Douglas technology throughout. Third, they use reserve requirements assumptions under which the return rates on capital and bonds are equal and then introduce "capital intermediation costs" to create a wedge between them. In contrast, we make assumption on reserve requirements that produce this wedge without the need for such costs. The last two differences are relatively trivial, but not the first.

Received wisdom from related overlapping generation models suggests that, under a fixed money growth rule only the equilibria exhibiting *pleasant* arithmetic are dynamically stable (sinks). Espinosa and Russell (2001) have confirmed this and have gone on to raise concerns about the actual observability of unpleasant arithmetic. That is, they have tried to rule out the possibility of observing unpleasant arithmetic on stability grounds.[7] Our results indicate that under *their* assumptions on preferences, but with a fixed bonds-money ratio rule, there is a unique low-real rate steady state; this steady state is empirically relevant, dynamically stable (a sink), and exhibits unpleasant arithmetic. Under more general preference (and production) assumptions, and under the fixed bonds-money ratio rule, we find that all empirically relevant equilibria are saddles (dynamically approachable from specific initial conditions), including those exhibiting unpleasant arithmetic. The upshot is that if the fixed bonds-money ratio rule formulation is realistic, then there is no

[5] In fact, pleasant arithmetic is not a possibility in Bhattacharya, Guzman, and Smith (1998) under their assumption that the real return to storage exceeds the growth rate of the economy.

[6] We thank the referees for drawing our attention to this concurrently but independently written paper.

[7] Espinosa and Russell (2001) do not present a formal stability analysis in the more general, return-dependent saving case. Numerical experiments, however, indicate that their stability results are robust to this generalization.

presumption in favor of pleasant arithmetic steady states on stability grounds.

To further facilitate comparison with Espinosa and Russell (2001), we rework our model assuming return-invariant saving but with a strict inflation target policy rule.[8] By means of numerical examples, we show that multiple monetary stationary equilibria are now possible, and these may all have real interest rates less than the economy's growth rate. In the case where there are exactly two equilibria, which is the case Espinosa and Russell (2001) focus on, we prove that the equilibrium with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Moreover, the equilibrium exhibiting pleasant arithmetic is the *only* dynamically stable equilibrium. Thus, our results confirm the Espinosa and Russell (2001) finding that under a strict inflation target rule, the only arithmetic one is likely to observe is pleasant.

In sum, the two different monetary policy rules studied here produce sharply different predictions about the likely observability of unpleasant arithmetic. With a fixed bonds-money ratio rule and return-invariant saving, it is quite possible for a low real-rate equilibrium to exhibit unpleasant arithmetic and be dynamically stable. With a fixed inflation rate rule and return-invariant saving, it is *not* possible for a low real-rate equilibrium to exhibit unpleasant arithmetic and be dynamically stable. While our result somewhat strengthens the case for the observability of unpleasant arithmetic, the Espinosa and Russell result definitely weakens it. In a broader sense, our results indicate that the question of observability of unpleasant arithmetic goes well beyond a simple comparison between the real interest rate and the economy's growth rate.

The rest of the paper is organized as follows. Section 2 describes the model under a fixed bonds-money ratio rule. Section 3 defines a monetary equilibrium and explores its properties. Section 4 sets out the main results concerning the long-run effects of open market operations. Section 5 sketches the main result from the model under a strict inflation rate target rule and establishes

[8] Espinosa and Russell (2001) study a fixed-money-growth rule, not a fixed-inflation rule. These rules are entirely equivalent in steady states, but not so along the transition. The stability properties of steady states under the two rules are very similar.

how the predictions for the likely observability of unpleasant arithmetic differs sharply across the two policy rules. Appendix A contains the details of the model under a strict inflation rate target rule. The proofs of some important results are relegated to the remaining appendices.

2 The Model

2.1 Environment

We consider an economy consisting of an infinite sequence of two-period lived overlapping generations, an initial old generation, and an infinitely-lived government. Let $t = 1, 2, ...$ index time. At each date t, a new generation comprised of N identical members appears. Each agent is endowed with one unit of labor when young and is retired when old. In addition, the initial old agents are endowed with $M_0 > 0$ units of fiat currency and $k_1 > 0$ units of capital.

There is a single final good produced using a standard neoclassical production function $F(K_t, L_t)$ where K_t denotes the capital input and L_t denotes the labor input at t. Let $k_t \equiv K_t/L_t$ denote the capital-labor ratio (capital per young agent). Then, output per young agent at time t may be expressed as $f(k_t)$ where $f(k_t) \equiv F(K_t/L_t, 1)$ is the intensive production function. We assume that $f(0) = 0$, $f' > 0 > f''$, and that the usual Inada conditions hold. The final good can either be consumed in the period it is produced, or it can be stored to yield capital the following period. For reasons of analytical tractability, capital is assumed to depreciate 100% between periods.

Let c_{1t} (c_{2t}) denote the consumption of the final good by a representative young (old) agent born at t. All such agents have preferences representable by the utility function $U(c_{1t}, c_{2t})$ where U is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments.

Finally, the government has a constant net-of-interest per young agent deficit of $g \geq 0$ in each period. The government does not levy any direct taxes and hence finances this deficit every period entirely by issuing bonds and money. Let M_t denote the aggregate stock of money outstanding at the end of period t, and B_t denote the outstanding per capita supply of bonds (in nominal terms)

where $B_0 = 0$.

2.2 Markets

Young agents supply their labor endowment inelastically in competitive labor markets, earning a wage income of ω_t at time t, where

$$\omega_t \equiv \omega(k_t) = f(k_t) - k_t f'(k_t) \qquad \forall t > 1. \tag{1}$$

In addition, capital is traded in competitive capital markets, and earns a gross real return of q_{t+1} between t and $t+1$, where

$$q_{t+1} = f'(k_{t+1}). \tag{2}$$

It is assumed that private agents do not have direct access to productive capital – presumably because of a minimum size restriction, as in Freeman (1987). They may hold either bank deposits, D_t, or government bonds, B_t. The bonds are not intermediated. Let p_t denote the time t price level, R^d_{t+1} denote the gross real interest rate on bank deposits, and ρ_{t+1} be the gross real interest rate on government bonds. Denote $d_t \equiv D_t/Np_t$ and $b_t \equiv B_t/Np_t$.

Each young agent[9] born at date $t \geq 1$ chooses how much bonds and deposits to hold to maximize $U(c_{1t}\,, c_{2t})$ subject to

$$c_{1t} + d_t + b_t \leq \omega_t \tag{3}$$

and

$$c_{2t} \leq R^d_{t+1} d_t + \rho_{t+1} b_t. \tag{4}$$

For both bonds and deposits to be held, $R^d_{t+1} = \rho_{t+1}$ must hold. Let $S_t \equiv d_t + b_t$ denote savings of a young agent. Then the agent's problem is simply to choose S_t to maximize $U\left[\omega(k_t) - S_t, \rho_{t+1} S_t\right]$.

[9] The initial old agents face the following budget constraint: $c_{20} \leq q_1 k_1 + M_0/Np_1$. Thus, at date 1, the initial old agents receive capital income $q_1 k_1$ and nominal money M_0 and they consume it all.

Let

$$S(\omega(k_t), \rho_{t+1}) \equiv \arg\max\ U\left[\omega(k_t) - S_t, \rho_{t+1} S_t\right] \tag{5}$$

The function S summarizes an agent's optimal saving behavior.

At several points below, we will be assuming (for reasons of analytical simplicity) that saving is not too sensitive to a change in its return. Formally stated, we assume that the interest elasticity of saving is less than unity, so that[10]

$$S_\rho \frac{\rho}{S} < 1. \tag{A.1}$$

We now turn to the determination of the interest on deposits (and bonds). As stated earlier, private agents cannot save in the form of capital directly. Banks arise to perform a simple intermediation function. They collect the individual deposits from all young agents and invest it in capital on their behalf. In return, private agents are promised a competitive return of R^d_{t+1} per unit deposited. We assume that banks are subject to a currency reserve requirement. For every unit of funds invested in the capital production technology, they are required by law to hold a fixed fraction λ of that amount in the form of currency reserves. That is,

$$m_t \geq \lambda k_{t+1}, \tag{6}$$

where $\lambda \in (0, 1)$. Let k_{t+1} denote capital, and m_t denote real reserves. Then, the banks' balance sheet requires $d_t \equiv k_{t+1} + m_t$. If

$$q_{t+1} > \frac{p_t}{p_{t+1}} \tag{A.2}$$

holds, then capital dominates money in rate of return, and the reserve requirement (6) binds. In passing, note that (A.2) also implies that $f'(k_{t+1}) > \rho_{t+1}$. Henceforth, we exclusively focus on

[10] We have chosen to make this assumption solely to obtain sharp analytical results. There is enough evidence to suggest that this assumption is empirically quite plausible. The consequences of abandoning this assumption are spelt out in Kudoh (2000).

equilibria that satisfy (A.2).[11]

Let $\phi \equiv \frac{1}{1+\lambda}$. Then ϕ represents the fraction of deposits held in the form of capital, while $(1-\phi)$ can be interpreted as the fraction of deposits required to be held as reserves. Then $(1-\phi)$ may be interpreted as a conventional reserve requirement. The real interest rate on bank deposits is then given by a weighted sum of returns to capital and money, *i.e.*,

$$\rho_{t+1} = \phi q_{t+1} + (1-\phi)\left(\frac{p_t}{p_{t+1}}\right) \tag{7}$$

must hold. It follows that if (A.2) holds, then $\rho_{t+1} > p_t/p_{t+1}$ holds too (*i.e.*, in the equilibria we consider, bonds always dominate money in rate of return). It is the presence of a binding reserve requirement that creates a wedge between the return to capital and that on bonds.[12]

2.3 The Government

The government finances a constant net-of-interest per young person deficit of $g \geq 0$ every period by issuing one-period default-free bonds and by printing money. The government's flow budget constraint is

$$p_t g = M_t - M_{t-1} + B_t - I_t B_{t-1}, \tag{8}$$

where I_t is the gross nominal interest rate on bonds. Equation (8) states that the government finances its expenditures and interest obligations on outstanding government debt, from seignorage revenue earned by money creation and from the sale of new bonds. Using $m_t \equiv M_t/Np_t$, $b_t \equiv$

[11] It is important to note that fiat money is valued in this setup solely because of the presence of a reserve requirement. In other words, if money is dominated in return and *no* legal restrictions (e.g., reserve requirements) are present, the demand for money would be zero. Since the current purpose is to study alternative modes of deficit finance, we restrict our attention only on equilibria where a positive demand for money (reserves) exists.

[12] In Espinosa and Russell (2001), capital intermediation costs creates a similar wedge between the return to capital and the return to bonds. This particular difference between our models does not seem to create any major qualitative differences in the results.

B_t/Np_t, and $\rho_t \equiv I_t p_{t-1}/p_t$, we can rewrite (8) as[13]

$$g = m_t - \left(\frac{p_{t-1}}{p_t}\right) m_{t-1} + b_t - \rho_t b_{t-1} \quad \forall t \geq 2. \tag{9}$$

Following Sargent and Wallace (1981), we assume that the fiscal authority in time 1 chooses the primary deficit g for all t and that it is the monetary authority that is forced to design its policies so that (9) is satisfied each period.[14] Following Wallace (1984),[15] Bhattacharya, Guzman, and Smith (1998), among others, we assume that the treasury fixes at date 1 the ratio of government debt to money, μ, where

$$\mu \equiv \frac{B_t}{M_t} > 0 \quad \forall t \geq 1. \tag{10}$$

The central bank conducts its monetary policy by changing the nominal money stock at all future dates to ensure the ratio is always kept fixed at μ.[16] Henceforth, we will refer to this type of monetary policy as a fixed bond-money ratio rule. It may be convenient to think of variations in μ as *permanent* open market operations. We will sometimes interpret an increase in μ as a tight money policy. Finally, at date 1, the central bank also sets the reserve requirement $(1 - \phi)$.[17]

3 Equilibrium

3.1 Characterization

We begin by formally defining a monetary competitive equilibrium for our economy.

[13] Since $B_0 = 0$, the government budget constraint at date 1 is

$$g = \frac{M_1 - M_0}{p_1} + \frac{B_1}{p_1}$$

which may be rewritten as $p_1 g + M_0 = M_1 + B_1$. In equilibrium, this pins down the initial price level.

[14] For a recent treatment of the "game of chicken" between the treasury and the central bank, see Barnett (2001).

[15] Strictly speaking, Wallace (1984), as well as Espinosa and Russell (1998), study a rule that fixes the ratio of the nominal *face* value of bonds to the nominal money stock.

[16] In Section 5, we will compare our results under this fixed μ rule with those obtained under a rule where the central bank sets the inflation rate and allows the bond-money ratio to reach the treasury's target of μ at some date in the future.

[17] Bhattacharya and Haslag (2000) discuss issues of budget arithmetic in a model where the central bank conducts monetary policy by directly choosing the reserve requirement.

Definition 1 *A binding monetary equilibrium is a set of sequences for allocations* $\{S_t\}$, $\{b_t\}$, $\{k_t\}$, $\{m_t\}$, *prices* $\{q_t\}$, $\{\omega_t\}$, $\{p_t\}$, $\{\rho_t\}$, *and the initial conditions* $M_0 > 0$, $k_1 > 0$, $B_0 = 0$ *such that (a) factor markets clear, i.e., (1)-(2) hold, (b) asset market clears:* $m_t + k_{t+1} + b_t = S\left(\omega_t, \rho_{t+1}\right)$, *c)* $p_t \in \Re_{++}$ *for all* t, *(d) the government's budget constraints at date 1,* $p_1 g + M_0 = M_1 + B_1$, *and at dates* $t > 1$, *i.e., (9) hold, and (e)* $f'(k_{t+1}) > p_t/p_{t+1} > 0$.

Noting that $b_t + d_t = \left[\frac{1+\mu(1-\phi)}{\phi}\right] k_{t+1}$, we can rewrite the asset market clearing condition as

$$[1 + \mu(1-\phi)]k_{t+1} = \phi S\left(\omega\left(k_t\right), \rho_{t+1}\right). \tag{11}$$

The equilibrium gross return on real balances (the inverse of the gross rate of inflation) for given ϕ and g can now be determined as

$$\frac{p_{t-1}}{p_t} = \frac{(1+\mu)k_{t+1} - \mu\phi k_t f'(k_t) - \frac{\phi}{1-\phi}g}{[1+\mu(1-\phi)]k_t}. \tag{12}$$

Substituting (12) in (7), it is possible to write (7) as

$$\rho_t = \frac{\phi k_t f'(k_t) + (1-\phi)(1+\mu)k_{t+1} - \phi g}{[1+\mu(1-\phi)]k_t}. \tag{13}$$

(13) describes the equilibrium rate of return on government bonds as a function of the capital-labor ratio. Therefore, (11) and (13), along with the restrictions embedded in Definition 1 jointly constitute the equilibrium conditions for this economy. Together they determine the time path of the capital-labor ratio, $\{k_t\}_{t=1}^{\infty}$, given k_1 and the government's prior pre-committed choices of μ, ϕ, M_0, and g.

Note that in order to describe the entire price sequence $\{p_t\}_{t=1}^{\infty}$, it is necessary to pin down the price level at date 1, which is an endogenous variable. The initial price level is given by

$$p_1 = \frac{M_0}{m_1 + b_1 - g}, \tag{14}$$

where $m_1 + b_1 = (1+\mu)\, k_2$. Here, g is determined by the fiscal authority and M_0, the initial quantity of money, is determined by the monetary authority.

3.2 Steady State Equilibria

We first explore the issue of existence of stationary equilibria. In a steady state, $k_t = k$ holds. Define $A \equiv 1 + \mu(1-\phi)$. Then we may rewrite (11) and (13) as

$$Ak = \phi S(\omega(k), \rho) \tag{15}$$

and

$$\rho = \rho(k) \equiv 1 + \frac{\phi}{A}\left[f'(k) - 1 - \frac{g}{k}\right], \tag{16}$$

respectively. For future reference, note that

$$\rho'(k) = \frac{\phi}{A}\left[f''(k) + \frac{g}{k^2}\right], \tag{17}$$

and that $\rho'(k) < 0$ holds for sufficiently small values of g.

Define R_m to be the stationary gross real return on money and $\pi \equiv 1/R_m$ to be the gross inflation rate. Then, it follows from (12) that

$$R_m = \frac{(1+\mu) - \mu\phi f'(k)}{A} - \frac{\frac{\phi}{1-\phi}g}{Ak} \tag{18}$$

Combining (16) and (15) yields

$$k = \frac{\phi}{A} S(\omega(k), \rho(k)) \equiv \Omega(k). \tag{19}$$

The solution to (19) describes the steady state capital-labor ratio, k^*, where $k^* = \Omega(k^*)$. It is easy to verify that in a binding monetary equilibrium, k^* must additionally satisfy

$$1 - \frac{\phi g}{(A-\phi)k^*} < f'(k^*) < \frac{(A-\phi)k^* - \phi g}{\mu\phi(1-\phi)k^*}. \tag{20}$$

For future reference, denote the slope of the function $\Omega(k)$ by $\Theta(k)$. As we shall see below, the size of $\Theta(k)$ is singularly important in determining the properties of various equilibria.

It is easy to check that[18]

$$\Theta(k) = \frac{\phi}{A}\left[S_\omega \omega'(k) + S_\rho \rho'(k)\right]. \tag{21}$$

Figure 1 illustrates several configurations of the function $\Omega(k)$. For $g = 0$, $\Omega(k)$ is concave and strictly increasing in k. In this case, as is well-known, there is a unique steady state if $k/\omega(k)$ is increasing in k.[19] Similarly, if saving is return-invariant, *i.e.*, $S_\rho = 0$, then there is necessarily an unique equilibrium (just as in Espinosa and Russell, 2001). Consider the more general case where $S_\rho > 0$ and $g > 0$. As g increases, $\Omega(k)$ shifts down in a manner illustrated in Figure 1; multiple steady state equilibria are easily possible here. In contrast with the standard Diamond model with fiat money however, not all candidate stationary equilibria in our model may satisfy (20).

Lemma 1 *If saving is interest-invariant, then the steady state equilibrium k^* is necessarily unique, and is characterized by $\Theta(k^*) < 1$. If saving is increasing in its return and $g > 0$, then a unique equilibrium (characterized by $\Theta(k^*) \lesseqgtr 1$) is possible; similarly, multiple equilibria (characterized by $\Theta(k^*) \lesseqgtr 1$) with the same bonds-money ratio are also possible.*

The proof of Lemma 1 readily follows from arguments that are standard and discussed in Azariadis (1993). Also see Figure 1.

3.2.1 Examples

We now explore the issue of existence by means of some illustrative numerical examples.

1. Suppose the production function and utility function is specified as follows:

$$f(k) = ak^\alpha \tag{22}$$

[18] $\Theta(k)$ may be also be written as a combination of various elasticities

$$\Theta(k) = \left(S_\omega \frac{\omega}{S}\right)\left(\omega_k \frac{k}{\omega}\right) + \left(S_\rho \frac{\rho}{S}\right)\left(\rho_k \frac{k}{\rho}\right).$$

[19] As is easily verified, $k/\omega(k)$ is increasing in k if $f(k) = k^\alpha$, $\alpha \in (0,1)$. However, for $f(k)$ specified as in (24) below, $k/\omega(k)$ is decreasing (increasing) in k for small (large) values of k. See von Thadden (1999) for details.

and

$$U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}. \tag{23}$$

Suppose $\alpha = 0.33$, $\phi = 0.9$, $a = 8$, $\beta = 0.96$, $\gamma = 0.97$, $\mu = 12$, and $g = 1.66$. Then, the unique steady state capital stock is given by $k^* = 1.10$. That is, even with return-dependent saving, an unique steady state is possible.

2. Suppose we retain (23) but respecify the production function as follows:

$$f(k) = \left(a_1 k^{\delta} + a_2\right)^{\frac{1}{\delta}} \tag{24}$$

The parameters of the economy are as follows: $a_1 = 0.11$, $a_2 = 0.25$, $\mu = 0.57$, $g = 0$, $\phi = 0.65$, $\gamma = 0.97$, $\beta = 0.89$, and $\delta = -0.71$. Under this choice of parameterization, it can be checked that $k^* = 0.12$ and $k^* = 0.31$ are the two steady states.

3.3 Seigniorage

The government's ability to make use of currency and (possibly) bond seigniorage to raise revenue obviously faces certain endogenous restrictions. In particular, there may arise a tension between the inflation tax base and the inflation tax rate when it comes to raising a fixed amount of seigniorage. We turn to these issues below. First, we compute the total seigniorage raised in steady states in the economy. From (9), it follows that the total seigniorage collected at a steady state k^* is given by

$$\widetilde{H}(R_m, k; \mu) \equiv \lambda k^* \left[1 + \mu - AR_m - \mu\phi f'(k^*)\right]. \tag{25}$$

Substitute (7) into (15) to obtain

$$Ak^* = \phi S\left[\omega(k^*), \phi f'(k^*) + (1-\phi) R_m\right], \tag{26}$$

which describes the relationship between k^* and R_m, for given μ. From (26), it is easy to show that

$$\frac{dR_m}{dk^*} = \frac{A\left[1 - \Theta(k^*)\right] + \phi\left[\rho'(k^*) - \phi f''(k^*)\right] S_\rho}{\phi(1-\phi) S_\rho}. \tag{27}$$

From (27), it is then apparent that if $\Theta(k^*) < 1$ holds, then $dR_m/dk^* < 0$ holds at any k^*, given μ. Thus, for given μ, (26) implicitly defines $k^* = h(R_m; \mu)$, where $h'(R_m; \mu) > 0$ if $\Theta(k^*) < 1$. Then, we substitute $k^* = h(R_m; \mu)$ into (25) to obtain the expression for total seigniorage:

$$H(R_m; \mu) \equiv \widetilde{H}(R_m, h(R_m; \mu); \mu) = \lambda h(R_m; \mu) \left[1 + \mu - AR_m - \mu\phi f'(h(R_m; \mu))\right]. \tag{28}$$

The right hand side of (28) is the conventional total seigniorage Laffer curve (drawn with the stationary return to money on the horizontal axis).[20] For future reference, a steady state k^* is henceforth described to be on the "good side" of the total seigniorage Laffer curve if an increase in the inflation rate necessarily raises the volume of seigniorage revenue at that steady state.

Proposition 1 *On the (H, R_m) space, a steady state k^* is on the "good side" of the total seigniorage Laffer curve, i.e., $(\partial H(R_m; \mu)/\partial R_m < 0)$ holds if $\Theta(k^*) < 1$.*

Proposition 1 states that $\Theta(k^*) < 1$ is sufficient for an increase in the inflation rate to raise the volume of seigniorage revenue. Such steady states are also the ones that are regarded widely as being the most empirically realistic.[21] Henceforth, we focus our attention only on those equilibria that are on the "good side" of the seigniorage Laffer curve. That is, we will maintain that

Assumption 1 $\Theta(k^*) < 1$ *holds.*

3.4 Bond Seigniorage

In our analysis, the real interest rate is endogenous and may be greater or less than the growth rate of the economy. Equilibrium bond holdings are also endogenous. As such, the revenue from the sale of bonds may go up or down when the interest rate changes. The revenue from the sale of

[20] This is by no means the only way to draw a Laffer curve. Espinosa and Russell (2001) draw Laffer curves with the real interest rate on the horizontal axis while others present them with the inflation rate on the horizontal axis. Our formulation follows Sargent (1987).

[21] If saving is return-invariant, there is a unique steady state, and it is always on the good side of the total seigniorage Laffer curve.

bonds in steady states is given by

$$s_b \equiv b(\rho)(1-\rho) = \mu\lambda k(1-\rho(k)). \tag{29}$$

For future reference, we collect some information about the volume of bond seigniorage in this economy in the lemma below.

Lemma 2 *a) At steady state k^*, the volume of bond seigniorage decreases with an increase in the interest rate on bonds if and only if*

$$\frac{1-\rho(k^*)}{k^*\rho'(k^*)} \leqslant 1. \tag{30}$$

b) Define

$$\Gamma(k^*) \equiv \mu\rho'(k^*) - \frac{\phi}{1-\phi}\frac{g}{(k^*)^2}.$$

For small g, $\rho'(k^*) < 0$ holds (see (17)), and hence $\Gamma(k^*) < 0$ holds.

3.5 Dynamic Properties of Equilibria

As is clear from Lemma 1 above, multiple stationary equilibria are possible in our model. More specifically, if saving is return-dependent, then it is possible that multiple stationary equilibria (some characterized by $\Theta(k^*) < 1$ and others characterized by $\Theta(k^*) > 1$) exist. As such, it becomes crucial to know which steady state a particular economy is going to approach, given an arbitrary initial capital-labor ratio, k_1. In the appendix, we prove the following proposition describing the local stability properties of the stationary equilibria.[22]

Proposition 2 *All equilibria characterized by $\Theta(k^*) < 1$ are saddles.* [23]

[22] An extensive discussion of stability properties of steady states in low-order non-linear dynamical systems is presented in Azariadis (1993; Chapter 6).

[23] It is not possible to prove in general that with return-dependent saving, equilibria with $\Theta(k^*) > 1$ are sources. Two comments are in order here. First, it seems that for valid equilibria with $\Theta(k^*) > 1$ to exist, it is necessary that $k/\omega(k)$ be decreasing in k somewhere. A production function that satisfies this condition is (24). Second, using this production function and (23) where $\beta \in (0,1)$ and $\gamma \in (0,1)$, Bhattacharya (1996) proves that $\phi > \left(\frac{1+\mu}{2+\mu}\right)$ and $\gamma \in (0.5, 1]$ are sufficient to ensure that equilibria with $\Theta(k^*) > 1$ are sources.

Corollary 1 *If saving is return-invariant, k^* is unique and stable.*

Under return-dependent saving, steady states that have $\Theta(k^*) < 1$ are saddles. They are unstable in the sense that they cannot be approached from an *arbitrary* set of initial conditions. However, they are approachable from correctly-chosen initial conditions that place the economy on the stable manifold or the saddle path. In this sense, steady states in the return-dependent saving case that are characterized by $\Theta(k^*) < 1$ are considered to be "potentially" dynamically approachable even though they are unstable. We think of "dynamic approachability" as a "weaker" form of stability.

If saving is interest-invariant, the case considered by Espinosa and Russell (2001), then there is a unique equilibrium (if any), and it is a sink (locally stable and approachable from an arbitrary set of initial condtions).[24] The bottom line is that generalizing preferences to include return-dependent saving "weakens" the stability of the good-side-of-the-Laffer-curve steady states (from sinks to saddles).

For future reference, let us collect all our results in one place here. Thus far, we have shown that if monetary policy is conducted using a fixed bonds-money rule, and saving is return-invariant, then there is a unique (if any) dynamically stable long-run equilibrium. At that equilibrium, increasing the money growth rate always raises revenue (across steady states) for the government. Suppose instead that saving is return-dependent. Then, all equilibria where $\Theta(k^*) < 1$ holds are saddles and hence are approachable from some set of initial conditions. Below we show that such steady states may also exhibit unpleasant arithmetic. The implication is that unpleasant arithmetic steady states are approachable from an arbitrary set of initial conditions if saving is return-invariant and are approachable from *some* set of initial conditions if saving is return-dependent. Hence, there can be no presumption against unpleasant arithmetic purely on stability grounds.

[24] To foreshadow, in Section 5 below, we show that were the central bank to follow a fixed inflation rate rule instead, then even with return-invariant saving and reasonable looking parameters, there are two steady states, one stable and the other unstable (though both are potentially dynamically approachable).

4 Open Market Operations

4.1 Unpleasant and Pleasant Arithmetic

Following Sargent and Wallace (1981), Wallace (1984), and Bhattacharya, Guzman, and Smith (1998), we now investigate how changes in the bond-money ratio, μ, affect the steady state equilibrium levels of the rate of inflation. As stated earlier, an increase in μ corresponds to a contractionary (and permanent) open market operation. In other words, tight money policies involve increases in μ.

First, we investigate the effects of a change in μ on the steady state capital stock. Notice that (19) gives a relationship between k^* and μ that is independent of R_m. Then, using (19), it is easily established that

$$\frac{dk^*}{d\mu} = \frac{(1-\phi)\left(\frac{S_\rho}{S}(\rho-1)-1\right)k^*}{A\left[1-\Theta\left(k^*\right)\right]}. \tag{31}$$

Lemma 3 *Under Assumption (A.1), $dk/d\mu < 0$ at a steady state characterized by $\Theta(k^*) < 1$.*

The proof of Lemma 3 follows straightforwardly from (31). This result is crucial to much of what follows. What it says is that if monetary policy is conducted using a fixed bond-money rule and saving is relatively insensitive to its return (Assumption A.1), then at any steady state on the good side of the total seigniorage Laffer curve a marginal tightening of monetary policy reduces the long-run capital stock. Two effects are at work here. When the treasury increases the supply of bonds, agents buy up this debt only at a higher promised return on bonds. With return-invariant saving, a shift towards more bonds in private portfolios causes bank deposits get crowded out. At a given return on money, the bank is forced to lower its capital holdings (because of the reserve requirement) in order to raise the return on deposits so as to match up with the new higher return on bonds.

To study the effects of a marginal tightening on the steady state rate of inflation, transform

(18) into an expression for the steady state inflation rate:

$$\pi = \frac{[1+\mu(1-\phi)]k}{(1+\mu)k - \mu\phi k f'(k) - \frac{\phi}{1-\phi}g}. \tag{32}$$

At a steady state k^* characterized by $\Theta(k^*) < 1$, (31) implicitly defines $k^* = x(\mu)$, where $x'(\mu) < 0$. Substitute $k^* = x(\mu)$ into (32) to obtain

$$\pi = \frac{1+\mu(1-\phi)}{(1+\mu) - \mu\phi f'(x(\mu)) - \frac{\phi}{1-\phi}\frac{g}{x(\mu)}}. \tag{33}$$

Equation (33) gives us an expression for the gross inflation rate as a function of μ. Then, the effect of an increase in μ on the inflation rate is captured by:

$$\frac{d\pi}{d\mu} = \frac{(\rho-1)A + \left[\mu\phi f''(k) - \frac{\phi}{1-\phi}\frac{g}{k^2}\right]Ax'(\mu)}{\left[(1+\mu) - \mu\phi f'(k) - \frac{\phi}{1-\phi}\frac{g}{k}\right]^2}. \tag{34}$$

Using (31), and Lemma 2, we can simplify (34) to:

$$\frac{1}{\pi^2}\frac{d\pi}{d\mu} = (\rho-1) + \Gamma(k^*)\frac{dk^*}{d\mu} \tag{35}$$

For sufficiently small values of g, we know from Lemma 2 that $\Gamma(k^*) < 0$ holds. It follows that the sign of $\frac{d\pi}{d\mu}$ crucially depends on whether the initial real interest rate $\rho \gtrless 1$ and on the sign of $\frac{dk^*}{d\mu}$ (which, in turn, is naturally connected to the stability properties of k^*).

Proposition 3 *Consider a steady state k^* characterized by $\Theta(k^*) < 1$. At that steady state, (a) if $\rho > 1$, then $d\pi/d\mu > 0$ for all μ implying that unpleasant arithmetic necessarily obtains (b) if $\rho < 1$, then $d\pi/d\mu$ may be either positive or negative implying that either pleasant or unpleasant arithmetic is possible.*

Corollary 2 *Consider a steady state k^* characterized by $\Theta(k^*) < 1$. At that steady state, suppose $\rho < 1$. Then, a tight money policy engineered by a permanent open market operation raises the steady state inflation rate if and only if*

$$1 - \rho < \Gamma(k^*)\frac{dk^*}{d\mu}.$$

The proof of Proposition 3 follows straightforwardly from (35). Part (a) of Proposition 3 says the following. If the return on government bonds exceeds the growth rate of the economy (here unity), and saving is relatively insensitive to its return, then any equilibria on the good side of the total seigniorage Laffer curve will necessarily exhibit unpleasant monetarist arithmetic. The intuition is clear. Suppose the treasury conducts an open market sale of new bonds. If the initial promised return on these bonds exceeds the growth rate of the economy, then, interest payments on outstanding debt will eventually outstrip the revenue from the fresh sale of these new bonds. The central bank eventually will have to oblige by printing money to cover the shortfall. It will succeed in its efforts only on the good side of the total seigniorage Laffer curve. Part (a) of Proposition 3 therefore extends Sargent and Wallace's (1981) classic result on unpleasant monetarist arithmetic to fairly general neoclassical economies of the Diamond (1965) variety.[25]

4.2 Real interest rate and the growth rate

As mentioned in the introduction, the condition $\rho > 1$ is not satisfied for the post-war U.S. Miller and Sargent (1984) and later Bhattacharya, Guzman, and Smith (1998) consider the possibility of resurrecting unpleasant arithmetic by weakening this unrealistic high real rate condition. Part (b) of Proposition 3 is in that vein. It states that even if the return on bonds is less than the growth rate of the economy, it is still possible that a tight money policy may raise the long-run inflation rate.

The intuition for Part (b) of Proposition 3 is as follows. When $\rho < 1$ obtains, the government raises a positive revenue from the sale of bonds. The question is whether following an open market sale of debt, this bond seigniorage is large enough to finance the government's primary deficit without eventually necessitating an increase in currency seigniorage. Suppose, as in the experiment of Proposition 3 that the government sells new debt. On the one hand, this directly increases the

[25] Bhattacharya (1996; Chapter 4) shows a similar result for an identical model with CRRA utility and CES preferences.

bond seigniorage $b(1-\rho)$; this is the "tax base effect" of Espinosa and Russell (1998). On the other hand, for reasons discussed earlier, this reduces the capital stock, and in turn increases the interest rate, which reduces the volume of bond seigniorage (the "tax rate effect"). On net, the latter effect can sometimes outweigh the former. Additionally, the crowding out of capital reduces reserves, and hence reduces the inflation tax base. As such, on the good side of the total seigniorage Laffer curve, the central bank still has to make up the shortfall by raising the inflation rate.

Corollary 2 attempts to somewhat sharpen the requirements for obtaining unpleasant arithmetic in the case when $\rho < 1$ obtains. Notice that for small g, we know from Lemma 2 that $\Gamma(k^*) < 0$ holds. We also know from Lemma 2 that $\Gamma(k^*) < 0$ implies that the volume of bond seigniorage decreases with an increase in the bond interest rate at such a steady state. Additionally, since we are focusing our attention only on dynamically approachable steady states, it follows from Proposition 3 that $\frac{dk^*}{d\mu} < 0$ holds. Then, it is clear that the necessary and sufficient condition for unpleasant arithmetic is the one spelt out in Corollary 2.

It is useful to collect all the qualifiers needed to obtain unpleasant arithmetic in an economy where the real interest rate is below its growth rate. First, it is necessary that monetary policy has real effects: specifically, a marginal tightening of the monetary policy must reduce the capital stock. Without this, unpleasant arithmetic is obtained only if the real rate of return on bonds exceeds the growth rate of an economy. The next qualifier is that a higher interest rate on bonds reduces bond seigniorage. In this connection, there are two effects to consider. One is that a tight money policy (higher reliance on bonds) increases bond seigniorage. The other is that, the tight money policy reduces the capital stock, raises the return on bonds, and thereby reduces bond seigniorage. The final qualifier needed is that the latter effect exceeds the former, which is the condition in Corollary 2. In plain English, this last condition requires the economy to be on the "bad" side of a bond-seigniorage Laffer curve (see (29)) drawn on the (s_b, μ) space.[26] Parenthetically, it can be

[26] At such an equilibrium, a tightening of monetary policy (sale of bonds) causes the steady state revenue from bonds to fall. This case has been extensively studied in Espinosa and Russell (2001).

noted that a necessary condition required to obtain pleasant arithmetic when $\rho < 1$ is that the economy to be on the "good" side of a bond-seigniorage Laffer curve drawn on the (s_b, μ) space. This is exactly the type of condition that Miller and Sargent (1984) had in mind.

It seems possible that the effect on the inflation rate of a marginal tightening of monetary policy may therefore depend on the existing extent of monetary policy tightness (as in Espinosa and Russell, 1998). The next lemma formalizes all this.

Lemma 4 *Define $\hat{\mu}$ to be the solution to $(1+\hat{\mu})k(\hat{\mu}) - \hat{\mu}\phi k(\hat{\mu}) f'(k(\hat{\mu})) = \frac{\phi g}{1-\phi}$. Then, at a steady state characterized by $\Theta(k^*) < 1$ and $\rho < 1$, for sufficiently small g,*

$$\lim_{\mu \to 0} \frac{d\pi}{d\mu} < 0$$

and

$$\lim_{\mu \to \hat{\mu}} \frac{d\pi}{d\mu} > 0$$

holds, where $\frac{d\pi}{d\mu}$ is derived from (35). In other words, pleasant arithmetic obtains only for sufficiently low values of the bonds-money ratio.

4.3 Comparison

How does our result compare to the one presented in Bhattacharya, Guzman, and Smith (1998). The crucial difference between our two models is that, by virtue of their assumption of a linear-storage saving technology, the real interest rate in their model is fixed. The driving force of their result is then the "tax base effect"of Espinosa and Russell (1998). A higher bond-money ratio through open market operations crowds out deposits from the portfolio of private agents. This reduction in deposits causes a reduction in the volume of reserves held by banks. Consequently, the inflation tax base falls forcing the central bank to raise the inflation rate. In contrast, in our Diamond (1965) economy, an additional "interest rate effect" emerges, and this effect, which is

necessarily absent in the Bhattacharya, Guzman, and Smith (1998) framework, may reduce the volume of bond seigniorage (as well as currency seigniorage).

Dating back to Wallace (1984), researchers have studied the connection between the pleasantness of the arithmetic and the initial marginal product of capital at the equilibrium being studied. In this context, notice that in our model [as is clear from (16)], when $g > 0$ holds, $f'(k) > 1$ does not necessarily imply that $\rho > 1$ holds. Thus Part (b) of Proposition 3 shows that it is possible for tight money to have inflationary consequences even when the return on bonds is lower and the marginal product of capital is higher than the growth rate of the economy.[27] This has two important implications. One is that in contrast to Bhattacharya, Guzman, and Smith (1998), $f'(k) > 1$ is *not* required for tight money to be inflationary. In words, the marginal product of capital being greater than the growth rate is not necessary to obtaining unpleasant monetarist arithmetic. The other implication is that $f'(k) < 1$ is not required for tight money to be disinflationary.[28] That is, for tight money to reduce the long-run inflation rate, it is not necessary that the marginal product of capital be less than the growth rate of the economy.

How does our result compare with those in Espinosa and Russell (2001)? Recall that Espinosa and Russell consider an alternative policy rule. In their setup, the central bank changes the money growth rate so as to satisfy the Treasury's target level of the bond-money ratio. What Espinosa and Russell find is that with return-invariant saving, in cases where there are two equilibria, both characterized by real rates that are less than unity, it is the lower (higher) real-rate equilibrium that is dynamically stable (unstable) and exhibits pleasant (unpleasant) arithmetic. Since real interest rates in most real-world economies are quite low, their result has the implication that the only equilibrium one is likely to observe in actuality will exhibit pleasant arithmetic. In other words,

[27] For example, consider the specification of preferences and technology as described in Section 4.4 below. Let the parameters of the economy be given by $f(k) = 4k^{0.35}$, $\phi = 0.97$, $g = 0.03$, $s = 0.55$. Then, for $\mu = 0.2$, there is a unique steady state defined by $k^* = 1.63$, $\rho^* = 0.98$ and $f'(k^*) = 1.01$. At this steady state, an increase in μ increases the inflation rate.

[28] In Bhattacharya, Guzman, and Smith (1998), as in Wallace (1984) and others, tight money is necessarily disinflationary if the marginal product of capital is below unity.

they rule out the possibility of unpleasant arithmetic on stability grounds. In sharp contrast, our Proposition 3, along with Proposition 2, suggests that with return-invariant saving, an equilibrium with a low real interest rate which exhibits *unpleasant* arithmetic is stable, and hence potentially observable.

4.4 An example with return-invariant saving

In this subsection, we illustrate some of our main results thus far by focussing on an example with return-invariant saving. This will also facilitate the comparison with results obtained by Espinosa and Russell (1999) and others in the literature. We use the following specification: $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$, which (using (5)) implies that $S = \beta\omega(k)/1+\beta = s\omega(k)$. Furthermore, assume $f(k) = ak^{\alpha}$, $\alpha \in (0,1)$. From (19), it follows then that

$$k^* = \left[\frac{s\phi a(1-\alpha)}{1+\mu(1-\phi)}\right]^{\frac{1}{1-\alpha}}$$

and therefore that k^* is unique. Also $\rho(k^*) \lesseqgtr 1$ is possible. The law of motion for the capital stock is given by

$$k_{t+1} = \frac{s\phi a(1-\alpha)k_t^{\alpha}}{1+\mu(1-\phi)}$$

from where it is clear that k^* is stable since $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k=k^*} = \alpha \in (0,1)$. It is easy to check that $\frac{\partial k^*}{\partial \mu} < 0$ holds. The steady state inflation rate (using (33)) is

$$\pi(\mu) = \frac{[1+\mu(1-\phi)]}{(1+\mu) - \mu\phi a\alpha(k^*)^{\alpha-1} - \frac{\phi}{1-\phi}\frac{g}{k^*}}$$

Example 1 *Let the parameters of the economy be given by* $f(k) = 4k^{0.35}$, $\phi = 0.97$, $g = 0.03$, $s = 0.7$. *Then, for* $\mu = 1.1$, *there is a unique steady state,* $k^* = 2.28$.[29] *As illustrated in Figure 2, a tight money policy engineered by an increase in the bonds-money ratio causes the inflation rate to fall in the range* $\mu \in (0,4)$ *and causes it to rise thereafter.*

[29] Similarly, for $\mu = 7.12$, there is a unique steady state, $k^* = 1.78$.

4.5 Transitional Dynamics

In this subsection, we study the behavior of inflation and the real interest rate on the transition path from one steady state to another by means of numerical experiments. The basic idea of these experiments is as follows. Take a set of parameters (including the bond-money ratio) and find a unique monetary stationary equilibrium. Then set the stationary capital-labor ratio as the initial condition, and change the bonds-money ratio at date 1. Then keep track of the time paths for key endogenous variables.

The first example is intended to replicate Sargent and Wallace's (1981) unpleasant monetarist arithmetic. The parameters for this example are: $\phi = 0.9$, $\alpha = 0.2$, $a = 2$, $\beta = 0.97$, $g = 0.01$, $M_0 = 1$. Initially set $\mu = 9$. Under these specifications, there is a unique monetary steady state equilibrium given by $k^* = 0.29$, $\rho^* = 1.02$, and $\pi^* = 1.88$. Now raise μ from 9 to 10, given $k_1 = k^* = 0.29$ and $M_0 = 1$ as the initial conditions. The results are reported in Figures 3a. As is evident, k decreases over time, and ρ increases over time without a jump. Since we assume 100% depreciation, these transition paths are rather rapid. Since $\rho > 1$, bond seigniorage is negative for all t. Note that the inflation rate jumps up at the time the new policy is implemented, and then it declines over time until it reaches the new steady state. At the new steady state, $k^* = 0.27$, $\rho^* = 1.04$, and $\pi^* = 3.87$. The initial price level can be computed as $p_1 = 3.08$.

The next example is intended to explore the possibility of obtaining pleasant arithmetic in low real-interest rate economies. The parameters chosen are: $\phi = 0.9$, $\alpha = 0.2$, $a = 5$, $\beta = 1$, $g = 0.2$. The initial value of μ is 2 and $M_0 = 1$. Figures 3b shows the case where μ increases from 2 to 3 while Figure 3c presents the case where μ increases from 5 to 6.[30] As suggested by Lemma 4, "pleasant" arithmetic obtains for these low (2 to 3) values of μ. Note that the gross inflation rate decreases monotonically along the transition path, and bond seigniorage is positive at all dates. According to Figure 3c, however, as μ increases from 5 to 6, the inflation rate starts to rise. This

[30] For $\mu = 2$, $k^* = 1.66$, $\rho^* = 0.66$, and $\pi^* = 1.68$. For $\mu = 3$, $k^* = 1.50$, $\rho^* = 0.72$, and $\pi^* = 1.53$. For $\mu = 5$, $k^* = 1.26$, $\rho^* = 0.80$, and $\pi^* = 1.84$. For $\mu = 6$, $k^* = 1.16$, $\rho^* = 0.84$, and $\pi^* = 2.48$.

is an illustration of unpleasant arithmetic in low real interest rate economies. The behavior of the inflation rate is similar to the one for high interest rates; inflation jumps up and then declines over time.

5 Strict inflation targeting

Thus far, we have considered a setup where the government conducts monetary policy by fixing the bonds-money ratio at the start of time and then choosing a nominal money stock then and all future dates that is consistent with its choice of the bonds-money ratio. In short, the principal instrument of monetary policy was the bonds-money ratio. Under this fixed bonds-money ratio rule, we derived the following result. If saving is return-invariant, then there is a unique monetary stationary equilibrium (if any) and it is dynamically stable. Unpleasant arithmetic is possible with low or high real rates; in particular, real world economies may observe unpleasant arithmetic. This section examines the sharp contrast between this result and those obtained in another version of the model that is virtually identical to this one except that the monetary policy rule is different. In particular, we consider an alternative rule wherein the government fixes the inflation rate at date 1 and allows the bonds-money ratio to adjust gradually to its new level. This is very similar to the constant money growth rate policy rule described in Espinosa and Russell (2001).

In Appendix A, we briefly sketch the model under the new monetary policy rule. Here, we report our main findings. First, we show that a strict inflation target rule does not uniquely define a equilibrium value for the bonds-money ratio. That is, a fixed inflation rate or money-growth rule can easily produce multiple equilibria even with return invariant saving. Example 2 illustrates. This possibility does not exist with a fixed bonds-money ratio rule as we have seen in Section 4.4. Second, we ask: what happens to real interest rates when the government follows a tight money policy? The conventional wisdom is that of pleasant arithmetic which, under a fixed inflation rate rule, may be characterized as tight money policies causing the real interest rate to rise. Then,

unpleasant arithmetic would be defined as a situation where tight money policies precipitate a fall in the real interest rate. We find (in various numerical examples) that in situations where there are two stationary equilibria both with low (less than unity) real interest rates, the one with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Third, the low real rate equilibrium is a sink (dynamically stable) while the higher real rate equilibrium is not (it is a saddle, and hence potentially dynamically approachable). The equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. Hence, as Espinosa and Russell (2001) find, the only arithmetic one is likely to observe in low real-rate economies is of the pleasant type.

6 Conclusion

A recap of our results and comparison across the two monetary policy rules is in order here. First, consider the fixed bonds-money ratio rule version of our model. In that setting, if saving is return-invariant, then there is a unique monetary stationary equilibrium (if any) and it is dynamically stable. If the initial real interest rate is greater than unity, unpleasant arithmetic is observed. If, as is empirically relevant, the real interest rate is less than unity at that steady state, it is still possible for unpleasant arithmetic to be observed. That is, it is not possible to rule out unpleasant arithmetic in real world economies on stability grounds. If saving is return-dependent, then multiple equilibria are possible. Of these, those that are on the good side of the seigniorage Laffer curve, are saddles and are hence potentially dynamically approachable from some set of initial conditions. Additionally, these equilibria may exhibit unpleasant arithmetic.

In the strict inflation-targeting rule version of the model, we find the following. Even if saving is return-invariant, multiple equilibria (if any) are possible and these may all have real interest rates less than unity. The equilibrium with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Moreover, the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. As such, it is possible (as Espinosa and Russell, 2001 have done) to rule out unpleasant

arithmetic in real world economies on stability grounds.

The bottom line is this. Using the fixed bonds-money ratio rule, we can produce an equilibrium that is unique, stable, and exhibits unpleasant arithmetic. Using a money-growth rule, the only stable equilibrium is one that exhibits pleasant arithmetic. The two monetary policy rules therefore produce sharply different predictions about the likely observability of unpleasant arithmetic. In a broader sense, our results indicate that the question of observability of unpleasant arithmetic goes beyond a simple comparison between the real interest rate and the economy's growth rate.

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Appendix

A The model with strict inflation targeting

For sake of brevity, we sketch the main elements of the model and the important results below, and relegate most of the details to Bhattacharya and Kudoh (2001). As stated in the body of the paper, the model presented below is identical to the one presented thus far except for the specification of monetary policy. Specifically, assume that the monetary authority sets the inflation rate at date 1,

$$\frac{p_{t+1}}{p_t} = \Pi \geq 1 \quad \forall t \geq 1 \tag{36}$$

and implements it by changing the money growth rate according to

$$\frac{M_{t+1}}{M_t} = \frac{M_{t+1}/p_{t+1}}{M_t/p_t}\frac{p_{t+1}}{p_t} = \Pi\frac{m_{t+1}}{m_t} = \Pi\frac{k_{t+2}}{k_{t+1}} \tag{37}$$

where the last equality uses (6) at equality. Note that in a steady state equilibrium (and not otherwise), this inflation setting rule is identical to the constant money growth rule adopted in Espinosa and Russell (2001).

For ease of comparison, we too assume that saving is return-invariant. That is, $S\left(\omega\left(k_t\right), \rho_{t+1}\right) = s\omega\left(k_t\right)$ where $s \equiv \frac{\beta}{1+\beta} \in (0,1)$. Then, a binding monetary competitive equilibrium of this economy [using (7), (11) and (9)] is characterized by:

$$m_t + k_{t+1} + b_t = s\omega\left(k_t\right), \tag{38}$$

$$\rho_{t+1} = \phi f'\left(k_{t+1}\right) + \frac{1-\phi}{\Pi}, \tag{39}$$

$$g = m_t - \frac{1}{\Pi}m_{t-1} + b_t - \rho_t b_{t-1}, \tag{40}$$

and (37). Substitute (37) into (38) to obtain

$$b_t = s\omega\left(k_t\right) - \frac{1}{\phi}k_{t+1}. \tag{41}$$

Substitute (38) and (39) into (40) to obtain

$$g = \frac{1-\phi}{\phi}k_{t+1} - \frac{1-\phi}{\phi\Pi}k_t + b_t - \left[\phi f'(k_t) + \frac{1-\phi}{\Pi}\right] b_{t-1}. \tag{42}$$

Analogous to (11) and (13), equations (41) and (42) jointly constitute the equilibrium conditions for this economy.

In a steady state, (41) implies

$$b = s\omega(k) - \frac{1}{\phi}k \equiv B(k) \tag{43}$$

and (42) may be rearranged to yield

$$b = \frac{g - \left(1 - \frac{1}{\Pi}\right)\frac{1-\phi}{\phi}k}{1 - \phi f'(k) - \frac{1-\phi}{\Pi}} = \frac{g - \left(1 - \frac{1}{\Pi}\right)\frac{1-\phi}{\phi}k}{1-\rho} \equiv \Psi(k). \tag{44}$$

A valid monetary competitive equilibrium is a solution to (43) and (44) that satisfy $b > 0$, $k > 0$, and $f'(k) > \frac{1}{\Pi}$. We are principally interested in solutions where $\rho < 1$ obtains. Several configurations of the intersection of the $b = B(k)$ and the $b = \Psi(k)$ loci are possible; we are particularly interested in the configuration illustrated in Figure 4. Small values of the reserve requirement, intermediate values of the government deficit and the savings rate are conducive to producing this particular configuration.

Example 2 *Let the parameters of the economy be given by* $\Pi = 1.3$, $f(k) = 4k^{0.35}$, $\phi = 0.97$, $g = 0.03$, *and* $S = s\omega(k)$ *where* $s = 0.7$. *Then, there are exactly two valid steady states; the high capital steady state has* $k_h = 2.28$ ($\rho_l = 0.81$) *and the low one has* $k_l = 1.78$ ($\rho_h = 0.95$). *The situation is illustrated in Figure 5.*

It is important to note that the parameters of the economy outlined in Example 2 produce two values of the long-run bonds-money ratio ($\mu = 7.11$ and $\mu = 1.1$). The latter coincides with the unique equilibrium presented in Example 1. In other words, a fixed money-growth rule does not uniquely define a equilibrium value for the bonds-money ratio. In general, it is the case that a

fixed inflation rate or money-growth rule can easily produce multiple equilibria even with return invariant saving. This possibility does not exist with a fixed bonds-money ratio rule as we have seen in Section 4.4.

Having characterized the stationary equilibria, we now ask the question: what happens to real interest rates when the government follows a tight money policy? In the present setup, a tight money policy would imply a cut in the inflation rate at all dates. Again, the reason for this focus is the ease of comparison with the results in Espinosa and Russell (2001). In their terminology, the conventional wisdom is that of pleasant arithmetic which, under a fixed inflation rate rule, may be characterized as tight money policies causing the real interest rate to rise. Then, unpleasant arithmetic would naturally be defined as a situation where tight money policies precipitate a fall in the real interest rate. What Espinosa and Russell find is that in situations where there are two stationary equilibria both with low (less than unity) real interest rates, the one with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Not surprisingly, given the ample similarity between this current setup and their paper, we find the same result which we simply state here for completeness.

Lemma 5 *Let the setting be the same as in Example 2. A marginal tightening of the monetary policy achieved via a cut in the inflation rate causes the real interest rate to rise (fall) at* ρ_l *(*ρ_h*). In other words, a marginal tightening produces pleasant arithmetic at the low real-rate equilibrium and produces unpleasant arithmetic at the high real-rate equilibrium.*

As an illustration, consider the situation described in Example 2. If, ceteris paribus, Π is cut to 1.2, it is easily verified that ρ_h falls to 0.94 and ρ_l rises to 0.83.

Finally, we undertake a quick study of the local stability properties of the two steady state equilibria. Recall, that the laws of motion are given by (41) and (42). Then, linearizing around a

steady state (k^*, b^*) yields the Jacobian matrix J

$$J(k^*, b^*) = \begin{pmatrix} -\left[\frac{1-\phi}{\phi\Pi} + \phi f''(k^*) b^*\right] + s\omega'(k^*) & -\rho^* \\ \frac{1}{\phi}\left[\frac{1-\phi}{\phi\Pi} + \phi f''(k^*) b^*\right] + \left(\frac{-1+\phi}{\phi}\right) s\omega'(k^*) & \frac{\rho^*}{\phi} \end{pmatrix}$$

A very likely scenario is the combination of a saddle and a sink as is illustrated in the example below.

Example 3 *Let the parameters of the economy be the same as in Example 2 above. There are exactly two steady states, the high capital steady state has* $k_h = 2.28$ $(\rho_l = 0.81)$ *and the low one has* $k_l = 1.78$ $(\rho_h = 0.95)$. *Then,*

$$J(\rho_h) = \begin{pmatrix} .619\,65 & -.956\,6 \\ -.200\,93 & .986\,19 \end{pmatrix}$$

holds, with associated eigenroots 0.32774 *and* 1.2781 *(indicating that* ρ_h *is a saddle). Similarly,*

$$J(\rho_l) = \begin{pmatrix} .374\,45 & -.816\,72 \\ -1.375\,3 \times 10^{-2} & .841\,98 \end{pmatrix}$$

with associated eigenroots 0.35154, *and* 0.86488 *(indicating that* ρ_l *is a sink). Hence the configuration is exactly as described in the phase diagram illustrated in Figure 6.*

In other words, as is the case in Espinosa and Russell (2001), the low real rate equilibrium is dynamically stable while the higher real rate equilibrium is not (it is potentially dynamically approachable though). In conjunction with Lemma 5, it follows that the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. It is this last observation that Espinosa and Russell (2001) use to make the argument that real world economies with low real interest rates are likely to exhibit only pleasant arithmetic.[31]

[31] If the central bank is following a fixed- inflation rate rule, then it does not know which time path for k (and other real variables) to expect, because there is no unique path toward a steady state if it is a sink. But when it tries to tighten money, it is sure that pleasant arithmetic will obtain. On the other hand, if the central bank follows a fixed bonds-money ratio rule, it knows the unique equilibrium path to the steady state since it is a saddle, but then it no longer knows if pleasant or unpleasant arithmetic will obtain. This would be another way of characterizing the differences between our result and those obtained in Espinosa and Russell (2001).

In sum, if saving is return-invariant, then there is likely to be multiple monetary stationary equilibria (if any) and these may all have real interest rates less than unity. The equilibrium with the lower (higher) interest rate exhibits pleasant (unpleasant) arithmetic. Moreover, the equilibrium exhibiting pleasant arithmetic is the only dynamically stable equilibrium. As such, it is possible to rule out unpleasant arithmetic in real world economies on stability grounds. The two monetary policy rules therefore produce sharply different predictions about the likely observability of unpleasant arithmetic.

B Proof of Proposition 1

Differentiate (28) with respect to R_m to obtain

$$\frac{\partial H}{\partial R_m} = \left[\frac{g}{k} - \lambda\mu\phi k f''(k)\right] h'(R_m;\mu) - \lambda k A, \tag{45}$$

where we have used (25) to write $\lambda\left[1+\mu - AR_m - \mu\phi f'(k)\right] = g/k$. Substituting (27) into (45), we have

$$\frac{\partial H}{\partial R_m} = \frac{\left[\frac{g}{k} - \lambda\mu\phi k f''(k)\right]\phi(1-\phi)S_\rho}{A(1-\Theta(k)) + \phi\left[\rho'(k) - \phi f''(k)\right]S_\rho} - \lambda k A.$$

Thus, $\partial H/\partial R_m < 0$ if

$$\lambda k A^2 (1-\Theta(k)) > \left[\frac{g}{k} - \lambda\mu\phi k f''(k)\right]\phi(1-\phi)S_\rho - \left[\rho'(k) - \phi f''(k)\right]\phi\lambda k A S_\rho = 0, \tag{46}$$

where we have substituted (17) into the right-hand-side of (46). This proves that at a steady state characterized by $\Theta(k) < 1$, $\partial H/\partial R_m < 0$ at that steady state. ■

C Proof of Lemma 2

a) It is easily established (using (29)) that

$$\frac{\partial s_b}{\partial \rho} = \mu\lambda\left\{\frac{1-\rho(k^*)}{\rho'(k^*)} - k^*\right\}$$

Then the rest is immediate.

b) Substitute (17) into the expression for $\Gamma(k^*)$ to get

$$\begin{aligned}\Gamma(k^*) &= \mu\frac{\phi}{A}\left[f''(k^*)+\frac{g}{(k^*)^2}\right]-\frac{\phi}{1-\phi}\frac{g}{(k^*)^2}\\ &= \mu\frac{\phi}{A}f''(k^*)-\frac{1}{(1-\phi)A}\frac{\phi g}{(k^*)^2}<0.\end{aligned}$$

D Proof of Proposition 2

To study the local dynamical properties of this system, we linearize the system of nonlinear difference equations (11) and (13) around a steady state. First, to reduce notation we define $\psi(k,\rho)\equiv A\rho-\phi f'(k)-\phi k f''(k)$. Notice that since $\rho=\phi f'(k)+(1-\phi)R_m$ in a binding steady state,

$$\psi(k,\rho)\equiv\mu(1-\phi)\phi f'(k)+(1-\phi)AR_m-\phi k f''(k)>0.$$

Local dynamics of the system is described by the linear difference equations, $Xz_{t+1}=Yz_t$,where

$$X\equiv\begin{pmatrix}A & -\phi S_\rho\\ (1-\phi)(1+\mu) & 0\end{pmatrix},\ Y\equiv\begin{pmatrix}\phi S_\omega\omega'(k) & 0\\ \psi(k,\rho) & Ak\end{pmatrix},\ \text{and } z_t\equiv\begin{pmatrix}k_t-k\\ \rho_t-\rho\end{pmatrix}.$$

The inverse of matrix X exists if $S_\rho>0$. Then, we obtain $z_{t+1}=\left(X^{-1}Y\right)z_t=Jz_t$. It is straightforward to compute the Jacobian matrix:

$$J=\frac{1}{(1-\phi)(1+\mu)\phi S_\rho}\begin{pmatrix}\phi S_\rho\psi(k,\rho) & \phi AkS_\rho\\ -(1-\phi)(1+\mu)\phi S_\omega\omega'(k)+A\psi(k,\rho) & A^2k\end{pmatrix}.$$

Let $D(k)$ denote the determinant of J as a function of the steady state capital stock, and $T(k)$ denote the trace of J. Then it is easily checked that

$$D(k)=\frac{\phi AkS_\omega\omega'(k)}{(1-\phi)(1+\mu)\phi S_\rho}>0,\ \text{and } T(k)=\frac{\phi S_\rho\psi(k,\rho)+A^2k}{(1-\phi)(1+\mu)\phi S_\rho}>0.$$

It is therefore easily established that $1+T(k)+D(k)>0$. Use $\psi(k,\rho)=(1-\phi)(1+\mu)-Ak\rho'(k)$ to obtain

$$1-T(k)+D(k)=-\frac{A^2k\left[1-\Theta(k)\right]}{(1-\phi)(1+\mu)\phi S_\rho},$$

which is negative if and only if $\Theta(k)<1$. This proves that all k^* characterized by $\Theta(k^*)<1$ are saddles. ■

E Proof of Lemma 4

From (35), it follows that

$$\frac{1}{A}\frac{d\pi}{d\mu}=\frac{(\rho-1)+\left[\mu\phi f''(k)-\frac{\phi}{1-\phi}\frac{g}{k^2}\right]\frac{dk^*}{d\mu}}{\left[(1+\mu)-\mu\phi f'(k)-\frac{\phi}{1-\phi}\frac{g}{k}\right]^2} \tag{47}$$

Using Proposition 3, it follows that $\frac{dk^*}{d\mu}<0$ holds. As $\mu\to 0$ and for small enough g, the denominator of (47) remains finite, and the numerator of (47) is then clearly negative when $\rho<1$ holds. As $\mu\to\hat{\mu}$, the denominator goes to zero and hence the rest follows.

Figure 1. Steady state equilibria

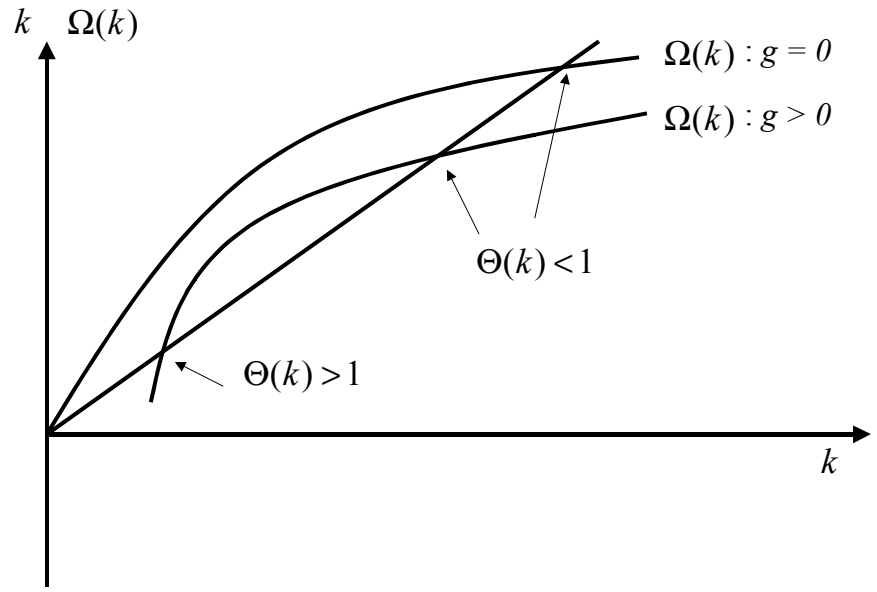


Figure 2. Steady state inflation as a function of the bonds-money ratio (Example 1)

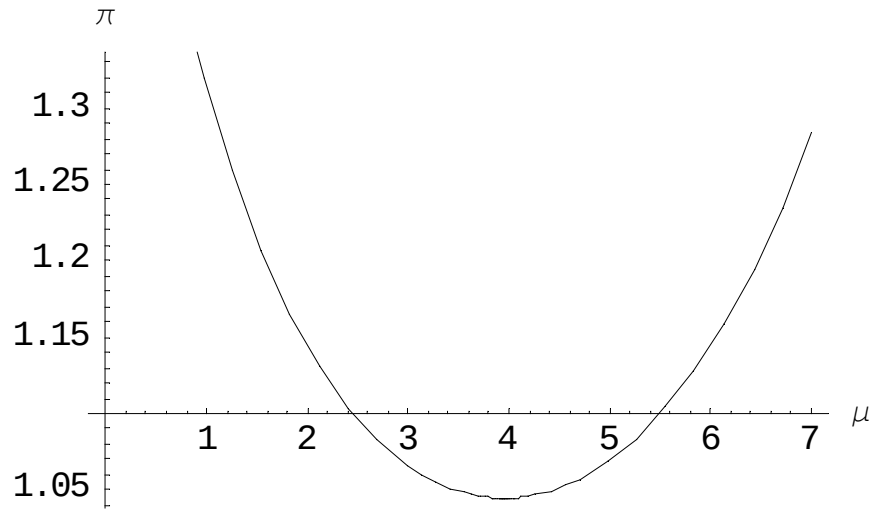


Figure 3a. Tight money raises inflation under $\rho > 1$.

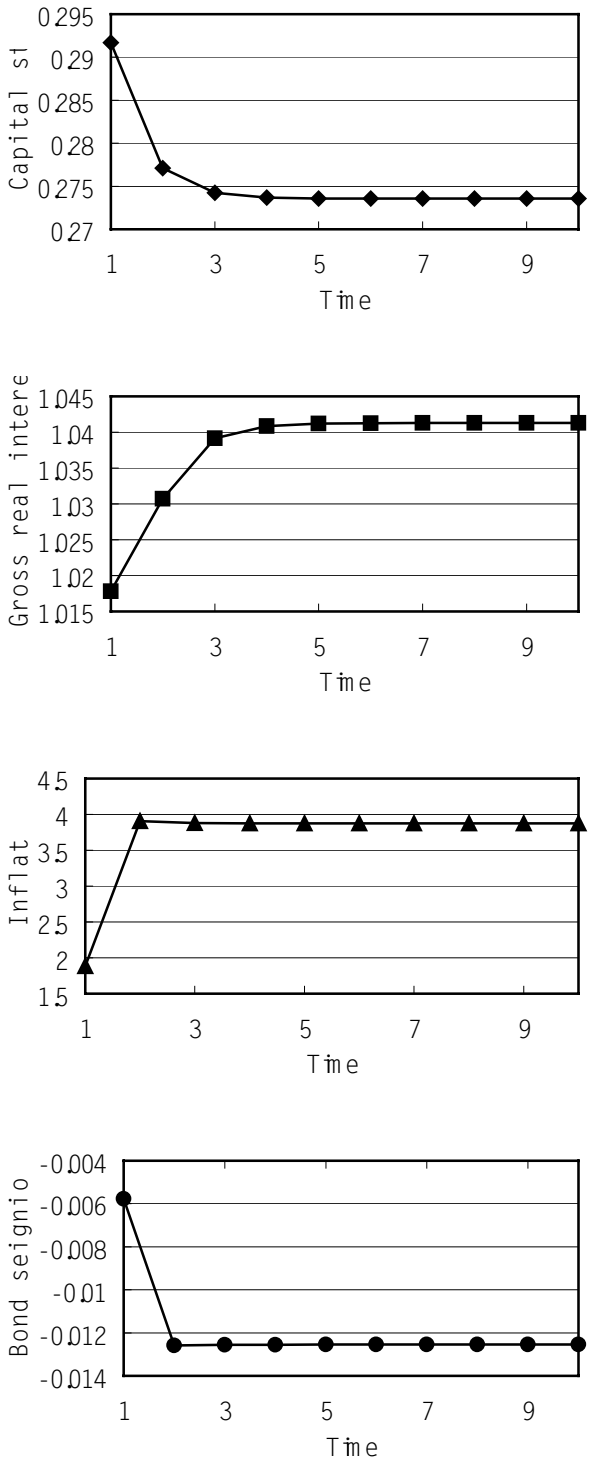
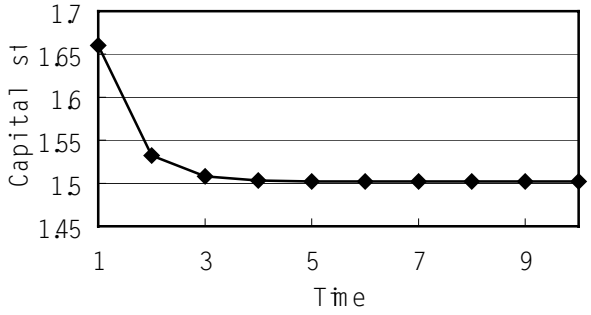
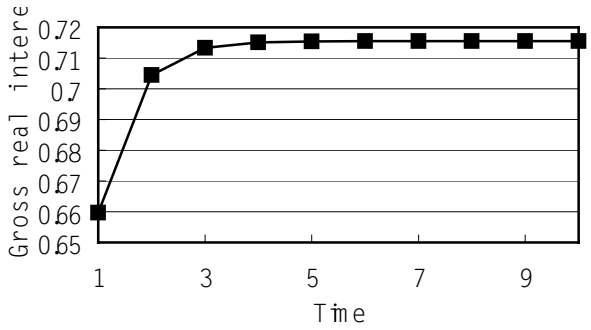
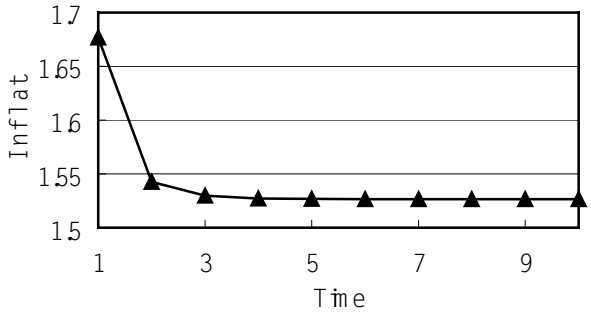


Figure 3b. Tight money reduces inflation under $\rho < 1$.







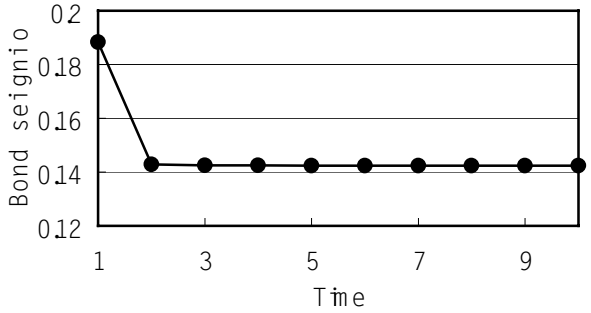
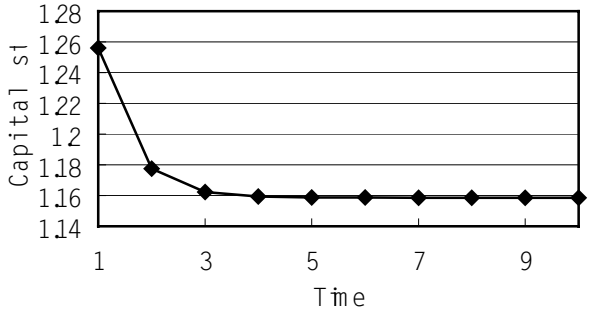
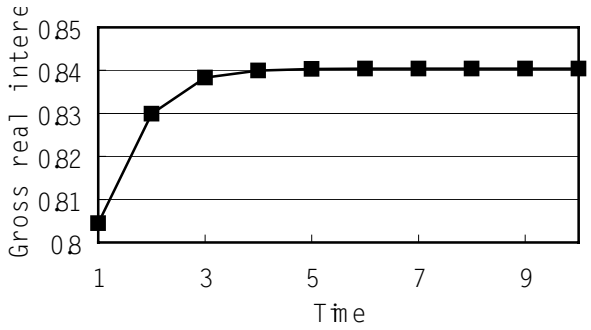
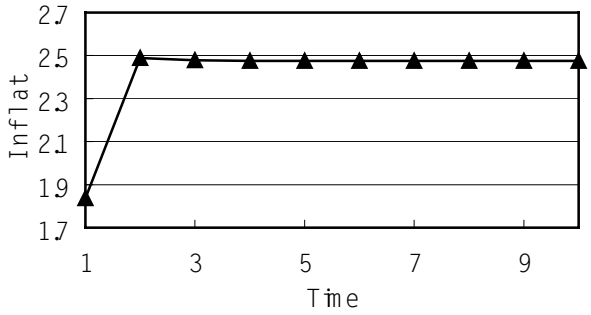


Figure 3c. Tight money raises inflation under $\rho < 1$.







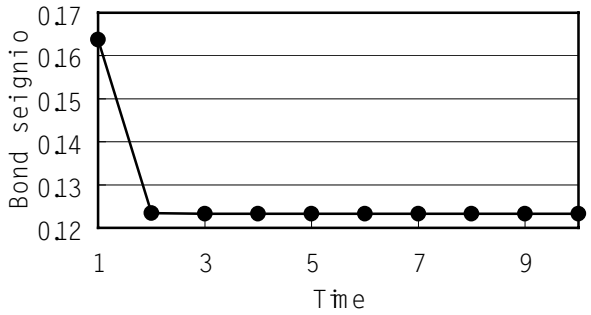


Figure 4. Steady state equilibria under strict inflation targeting

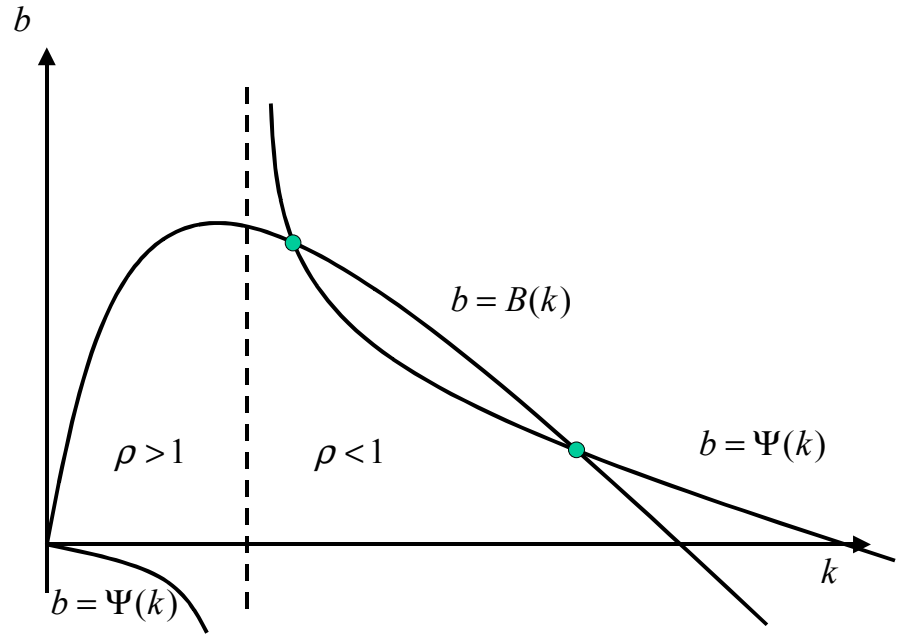


Figure 5. Steady state equilibria under strict inflation targeting (Example 2)

