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Title	A NOTE ON THE HAWKINS-SIMON CONDITIONS
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Citation	HOKUDAI ECONOMIC PAPERS, 3, 46-48
Issue Date	1972
Doc URL	https://hdl.handle.net/2115/30645
Type	departmental bulletin paper
File Information	3_P46-48.pdf



A NOTE ON THE HAWKINS-SIMON CONDITIONS^{*)}

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The object of the present note is to provide an alternative proof of the theorem that the Hawkins-Simon conditions are necessary and sufficient for the static Leontief system to have positive solutions. Originally, the theorem is proved by Hawkins and Simon [2], and subsequently by Morishima [5], and Nikaido [6], [7] in a more general setting. Most of the books referring to the theorem, however, confine their remarks to the two-sector case for the convenience of a diagrammatic exposition.¹⁾ The following proof will be performed in the wake of the original version by Hawkins and Simon.

Let us consider the system of nonhomogeneous linear equations:

$$(1) \quad \sum_{j=1}^m a_{ij} x_j = k_i \quad (i = 1, \dots, m)$$

with $a_{ij} \leq 0$ for all $i \neq j$. Then, the theorem to be proved is as follows:

THEOREM: A necessary and sufficient condition that the x_i satisfying (1) be all positive for any positive k_i is that all principal minors of the square matrix $A = (a_{ij})$ are positive, i.e.,

$$(2) \quad \begin{vmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} \end{vmatrix} > 0 \quad (s = 1, \dots, m)$$

Proof. We first prove that the Hawkins-Simon conditions (2) is necessary for the system (1) to have solutions $x_i > 0$ for any $k_i > 0$, and then that (2) is sufficient.

(*Necessity*) If the system (1) has positive solutions, the first equation of the system (1) can be rewritten as

$$(3) \quad a_{11} = \frac{1}{x_1} (k_1 - \sum_{j=2}^m a_{1j} x_j) > 0.$$

This implies that (2) holds when $s=1$.

By using the first equation of the system (1), x_1 can be eliminated from the remaining equations. Thus, we obtain the subsystem of equations:

$$(4) \quad \sum_{j=2}^m \left(a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right) x_j = k_i - \frac{a_{i1}}{a_{11}} k_1 \quad (i = 2, \dots, m)$$

^{*)} The authors are indebted to Professor T. Shirai for suggestions which led them to write this note.

¹⁾ See, for example, Dorfman, Samuelson, and Sollow ([1], Chap. 9), Kuenne ([3], Chap. 6), and Morishima ([4], Chap. 2).

For simplicity, the system (4) is denoted by

$$(5) \quad \sum_{j=2}^m a_{ij}^{(1)} x_j = k_i^{(1)} \quad (i = 2, \dots, m)$$

In view of the conditions that $a_{ij} \leq 0$ for all $i \neq j$, and $a_{11} > 0$, we conclude that

$$(6) \quad a_{ij}^{(1)} \leq 0 \quad \text{for all } i = j, \quad \text{and} \quad k_i^{(1)} > 0.$$

If the system (1) has positive solutions, the first equation of the subsystem (5) can be rewritten as

$$(7) \quad a_{22}^{(1)} = \frac{1}{x_2} \left(k_2^{(1)} - \sum_{j=3}^m a_{2j}^{(1)} x_j \right) > 0.$$

It is easily seen that this implies that the condition (2) holds for $s=2$, because

$$(8) \quad a_{22}^{(1)} = \frac{1}{a_{11}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Similarly, by using the first equation of the subsystem (5), x_2 (the first variable of the subsystem) can be eliminated from the remaining $(m-2)$ equations. Thus, we obtain the second subsystem of $(m-2)$ equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of the system (1) or (5). In general, by using the first equation of the s -th subsystem of $(m-s)$ equations, x_{s+1} (the first variable of the s -th subsystem) can be eliminated out of the remaining $(m-s-1)$ equations. Thus, we shall obtain the $(s+1)$ -th subsystem of $(m-s-1)$ equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of earlier subsystems.

If the system (1) has positive solutions, the first equation of the $(s-1)$ -th subsystem can be rewritten as

$$(9) \quad a_{ss}^{(s-1)} = \frac{1}{x_s} \left(k_s^{(s-1)} - \sum_{j=s+1}^m a_{sj}^{(s-1)} x_j \right) > 0, \quad (s=2, \dots, m)$$

The above procedure of elimination is equivalent to such operations that a multiple of the elements of one row of the coefficient matrix (a_{ij}) is added to the corresponding elements of other rows. According to the well-known property of determinants, this does not alter the values of the principal minors of (a_{ij}) . Thus, we have

$$(10) \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ 0 & a_{22}^{(1)} & \cdots & a_{2s}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{ss}^{(s-1)} \end{vmatrix} \\ = a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{s-1, s-1}^{(s-2)} a_{ss}^{(s-1)} \quad (s=2, \dots, m)$$

This completes the proof of necessity.

(Sufficiency) In view of (10), we have

$$(11) \quad a_{11} > 0, \quad a_{22}^{(1)} > 0, \dots, a_{m-1, m-1}^{(m-2)} > 0, \quad \text{and} \quad a_{mm}^{(m-1)} > 0,$$

if all principal minors of the matrix (a_{ij}) are positive.

To begin with, let us consider the last (*i.e.*, the $(m-1)$ -th) subsystem:

$$(12) \quad a_{mm}^{(m-1)} x_m = k_m^{(m-1)},$$

where $k_m^{(m-1)} > 0$.

By virtue of (11), we can divide both sides of the equation (12) by $a_{mm}^{(m-1)}$ and obtain

$$(13) \quad x_m > 0.$$

In order to work out the proof by mathematical induction, suppose that we have already obtained solutions:

$$(14) \quad x_m > 0, \quad x_{m-1} > 0, \dots, x_{s+1} > 0.$$

Now, let us consider the first equation of the $(s-1)$ -th subsystem:

$$(15) \quad a_{ss}^{(s-1)} x_s = k_s^{(s-1)} - \sum_{j=s+1}^m a_{sj}^{(s-1)} x_j,$$

where the nonhomogeneous term $k_s^{(s-1)}$ is positive and the off-diagonal coefficients $a_{sj}^{(s-1)}$ are nonpositive. By (14), the right-hand-side of the equation is positive. By virtue of (11), we can divide both sides of the equation by $a_{ss}^{(s-1)}$ and obtain

$$(16) \quad x_s > 0.$$

Thus the proof is complete, Q.E.D.

References

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