On the discriminants of transformation equations of modular forms

by

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Introduction.

The purpose of this note is to add some supplementary results to our previous paper [3]. We have proved there that the transformation equations of certain modular forms can be expressed by special values of the zeta functions of those forms. At the symposium, we talked about this result. Here we give some results obtained after that.

Let f be a modular form on $\Gamma_0(p)$ of weight k. We assume that p is an odd prime throughout the paper. Then the transformation equation of f is defined by

$$\Phi(X;f) = \prod_{\alpha \in \Gamma_0(p)\backslash SL_2(Z)} (X - f|_k\alpha) = 0,$$

where $(f|_k\gamma)(z) = \det(\gamma)^{k/2} f((az+b)/(cz+d))(cz+d)^{-k}$ for

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(R) = \{\gamma \in GL_2(R) \mid \det(\gamma) > 0\}$.

Obviously all coefficients σ_μ of $\Phi(X;f)$ are modular forms on $SL_2(Z)$. We call that the transformation equation $\Phi(X;f) = 0$ is Z-rational if all coefficients σ_μ have Z-rational Fourier expansions as modular forms (see §1, for the Z-rationality of Fourier expansions). Then one of our results is

Theorem 1. *If the transformation equation* $\Phi(X;f) = 0$ *of* f *is* Z*-rational and if* p *is an odd prime, then the discriminant* D *of* $\Phi(X;f)$ *is expressed as*

$$D = \begin{cases} (-1)^{(p-1)/2} p^p \Delta^{p+1} h^2, & \text{if } f \text{ is a cusp form,} \\ (-1)^{(p-1)/2} p^p \Delta^{p-1} h^2, & \text{otherwise,} \end{cases}$$

where h *is a modular form on* $SL_2(Z)$ *with a* Z*-rational Fourier expansion and* Δ *is Ramanujan's function* $\exp(2\pi iz)\prod_{n=1}^{\infty}(1-\exp(2\pi inz))^{24}$.

Especially, when f is the special cusp form discussed in [3], we even know the divisibility of the above form h by $\Delta^{(p+1)/2}$ (Proposition 9). We will prove this theorem in §2.

In Proposition 4, we will also show under the assumption in Theorem 1 the following congruence relation:

$$\Phi(X;f) \equiv (X - \sigma_1)(X^p - \sigma_p) \mod p.$$

Here both σ_1 and σ_p are certain modular forms on $SL_2(Z)$. Though this result follows from the Eicher-Shimura congruence relation [7, Theorem 7.9], we will give an elementary proof without using their result.

In [3], we considered the transformation equation $\Phi(X;f) = 0$ for $f = gE^*_{\lambda,p}$, where g is a cusp form on $\Gamma_0(p)$ and $E^*_{\lambda,p}$ is a certain Eisenstein series. In §3, we will show that for a certain choice of g, the transformation equation $\Phi(X;gE^*_{\lambda,p}) = 0$ of $gE^*_{\lambda,p}$ is Z-rational. In §4, we will give numerical examples of the transformation equations for the above $gE^*_{\lambda,p}$ and the specialized equations at several elliptic curves (see [3, §3], for the definition of the specialized equation at an elliptic curve).

§1. Congruence relation of transformation equations.

Let p be an odd prime and $\Gamma_0(p)$ be a subgroup of $SL_2(Z)$ defined by

$$\Gamma_0(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid \quad c \equiv 0 \mod p \}.$$

We denote by $M_k(\Gamma_0(p))$ the vector space of modular forms on $\Gamma_0(p)$ of weight k. Hence an element f of $M_k(\Gamma_0(p))$ is a function on the upper half complex plane H with the following three properties:

(i) f is holomorphic on H;

(ii) $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(p)$;

(iii) $f|_k\gamma$ has the Fourier expansion of the form: $\Sigma_{n=0}^{\infty} a_\gamma(n) e(nz/p)$ at $i\infty$ for all $\gamma \in SL_2(Z)$ ($e(z) = \exp(2\pi iz)$).

Moreover, if $a_\gamma(0) = 0$ for all $\gamma \in SL_2(Z)$, then f is called a cusp form. The subspace of $M_k(\Gamma_0(p))$ consisting of all cusp forms is denoted by $S_k(\Gamma_0(p))$.

Let Λ be a subring of C and f be a function on H with a Fourier expansion of the form: $\Sigma_{n=0}^{\infty} a(n) e(nz/N)$ for some positive integer N. Then we say that f is Λ-rational if a(n) belongs to Λ for any n. Let $g(z) = \Sigma_{n=0}^{\infty} b(n) e(nz/N)$ be another Λ-rational function and m be an ideal of Λ. Then we write $f \equiv g \mod m$ if $a(n) \equiv b(n) \mod m$ for all n. Further, for any field-automorphism σ of C, we define an action of σ on f by

$$f^\sigma(z) = \Sigma_{n=0}^{\infty} a(n)^\sigma e(nz/N).$$

Though the following lemma is a corollary of [8, Theorem 5], we give an elementary proof.

Lemma 2. Let f be an element of $M_k(\Gamma_0(p))$. Then for any field-automorphism σ of C, we have that

$$(f|_k\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix})^\sigma = f^\sigma|_k\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Proof. It is sufficient to show this for a basis of $M_k(\Gamma_0(p))$. First let us show the relation for Eisenstein series. We know from [4, Satz 11] that the subspace of $M_k(\Gamma_0(p))$ spanned by all Eisenstein series has a basis consisting of the following functions:

$k > 2$: $G_k(z) = -B_k/2k + \Sigma_{n=1}^{\infty}\{\Sigma_{0<d|n} d^{k-1}\}e(nz)$

and $G_k(pz)$

$k = 2$: $E_p(z) = G_2(z) - pG_2(pz)$.

Here B_k is the k-th Bernoulli number and

$$G_2(z) = i/\{4\pi(z-\bar{z})\} - 1/24 + \Sigma_{n=1}^{\infty}\{\Sigma_{0<d|n} d\}e(nz).$$

Put $\tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Since $G_k(z)$ is an Eisenstein series on $SL_2(Z)$ of weight k, we have that $G_k|_k\tau = G_k$, and therefore, $G_k(pz)|_k\tau = p^{-k}G_k(z/p)$. Thus both $G_k|_k\tau$ and $G_k(pz)|_k\tau$ are Q-rational. Then the desired result follows for $G_k(z)$ and $G_k(pz)$. A similar argument shows that the lemma holds also for E_p. Next let us show the relation for a basis of cusp forms. The theory of primitive forms shows that a basis of $S_k(\Gamma_0(p))$ is given by $f_i(z)$, $f_i(pz)$ and $g_j(z)$. Here $f_i(z)$ and $g_j(z)$ are primitive forms of conductor 1 and p, respectively. Our assertion holds for such cusp forms f_i. In fact, it is known that f_i^σ are again primitive forms of conductor 1. Let us write $f(z) = f_i(pz)$. Then we have that

$$(f|_k\tau)(z) = p^{-k}f_i(z/p)$$

and

$$(f|_k\tau)^\sigma = p^{-k} f_i{}^\sigma(z/p).$$

Since $f^\sigma(z) = f_i{}^\sigma(pz)$, we see that

$$(f^\sigma|_k\tau)(z) = p^{-k} f_i{}^\sigma(z/p).$$

It follows that $(f|_k\tau)^\sigma = f^\sigma|_k\tau$.

Now let us show the relation for primitive forms g_j of conductor p. We know from [1, Lemma 3] that

$$g_j|_k\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma g_j \quad \text{with } \gamma = \pm 1.$$

Since γ is expressed as

$$\gamma = -a(p)p^{1-k/2}$$

with the p-th Fourier coefficient $a(p)$ of g_j and since $g_j{}^\sigma$ is again a primitive form of conductor p, we have that

$$g_j{}^\sigma|_k\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma g_j{}^\sigma.$$

Thus we observe that

$$g_j|_k\tau = \gamma p^{-k/2} g_j(z/p)$$

and

$$g_j{}^\sigma|_k\tau = \gamma p^{-k/2} g_j{}^\sigma(z/p).$$

This concludes the proof of Lemma 2.

Lemma 3. Let f be an element of $M_k(\Gamma_0(p))$. Then the transformation equation $\Phi(X;f) = 0$ of f is Z-rational if and only if both f and $f|_k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are Z-rational.

Proof. Let us write the Fourier expansions of f and $f|_k\tau$ as

$$(1.1_a) \quad f(z) = \Sigma_{n=0}^{\infty} a(n)e(nz),$$

and

$$(1.1_b) \quad (f|_k\tau)(z) = \Sigma_{n=0}^{\infty} b(n)e(nz/p),$$

respectively. Further let us write the transformation equation

of f as

$$(1.2)\qquad \Phi(X;f) = X^{p+1} + \Sigma_{\mu=1}^{p+1} (-1)^{\mu}\sigma_{\mu}X^{p+1-\mu}.$$

First let us show that if both f and $f|_k\tau$ are Z-rational, then all coefficients σ_μ are Z-rational. Let us define a set R by

$$(1.3)\qquad R = \{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \tau_u = \begin{pmatrix}0 & -1\\ 1 & u\end{pmatrix} \mid u = 0,1,2,\cdots,p-1\}.$$

Then, since p is a prime, the set R gives a complete set of representatives for $\Gamma_0(p)\backslash SL_2(Z)$ (e.g., [5, Lemma 2.2]). Since $\tau_u = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$, it follows from (1.1_b) that for any integer u,

$$(1.4)\qquad (f|_k\tau_u)(z) = \Sigma_{n=0}^{\infty}\zeta^{nu}b(n)e(nz/p),$$

where $\zeta = e(1/p)$. Since all Fourier coefficients b(n) belong to Z, the modular forms $f|_k\tau_u$ are $Z[\zeta]$-rational. Thus coefficients σ_μ are $Z[\zeta]$-rational. In fact, coefficients σ_μ are symmetric functions in $\{ f|_k\alpha\}_{\alpha\in R}$. On the other hand, (1.4) shows that for any field-automorphism ρ of C, $\{(f|_k\alpha)^{\rho}\}_{\alpha\in R} = \{f|_k\alpha\}_{\alpha\in R}$. Thus we have that $\sigma_\mu{}^{\rho} = \sigma_\mu$. This shows that the modular forms σ_μ are Z-rational. Conversely assume that all the coefficients σ_μ are Z-rational. Then we have that for any field-automorphism ρ of C,

$$(1.5)\qquad \Pi_{\alpha\in R}(X - f|_k\alpha) = \Pi_{\alpha\in R}(X - (f|_k\alpha)^{\rho}).$$

Especially, we have that $f^{\rho} = f$ or $f^{\rho} = f|_k\tau_{u_0}$ for some u_0. Assume that $f^{\rho} = f|_k\tau_{u_0}$. Since $f^{\rho}|_k\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} = f^{\rho}$ and $\tau_{u_0} = \tau\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{u_0}$, we see

(1.6) $f^{\rho} = f|_k\tau.$

Therefore we have

(1.7) $f|_k\tau_u = f|_k\tau = f^{\rho}$ for any u.

Moreover Lemma 2 and (1.6) show that

(1.8) $(f|_k\tau)^{\rho} = f^{\rho}|_k\tau = f.$

Thus we have that

(1.9) $(f|_k\tau_u)^{\rho} = (f|_k\tau)^{\rho} = f.$

Rewriting the equation (1.5) through (1.7) and (1.9), we obtain that

$$(X - f)(X - f^{\rho})^p = (X - f^{\rho})(X - f)^p.$$

Hence we have that for any field-automorphism ρ,

(1.10) $f^{\rho} = f.$

Namely, f is Q-rational. On the other hand, the vector space $M_k(\Gamma(p))$ has a Z-rational basis (e.g., [8, (9)]), where $\Gamma(p)$ is the subgroup of $SL_2(Z)$ defined by

$$\Gamma(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid a \equiv d \equiv 1, \quad b \equiv c \equiv 0 \mod p\}.$$

Thus we can write f as

$$f(z) = c\Sigma_{n=0}^{\infty} d(n)e(nz)$$

with a rational number c and rational integers d(n). Let us consider the equation $\Phi(X;f) = 0$ over the ring Q[[q]] of formal power series in $q = e(z)$. Then f is integral over Z[[q]] and is contained in its quotient field. In fact, all coefficients of $\Phi(X;f)$ are Z-rational from the assumption, and therefore, they belong to Z[[q]] for $q = e(z)$. Since Z is principal, Z[[q]]

is a normal ring (e.g., [2, VII.3 Exercise 9c]), and this shows that f is Z-rational. Lemma 2 and (1.10) show that for any field-automorphism ρ,

$$(f|_k\tau)^\rho = f|_k\tau.$$

Therefore a similar argument as above in $Z[[q^{1/p}]]$ shows that $f|_k\tau$ is again Z-rational. Note that $f|_k\tau$ belongs to $M_k(\Gamma(p))$. This concludes the proof of Lemma 3.

Proposition 4. *Let f be an element of $M_k(\Gamma_0(p))$ and let us write the transformation equation of f as*

$$\Phi(X;f) = X^{p+1} + \Sigma_{\mu=1}^{p+1}(-1)^\mu\sigma_\mu X^{p+1-\mu}.$$

If $\Phi(X;f) = 0$ is Z-rational, then we have a congruence:

$$\Phi(X;f) \equiv (X-\sigma_1)(X^p-\sigma_p) \mod p.$$

Proof. Let us write $\zeta = e(1/p)$ and p be the unique prime ideal of $Z[\zeta]$ which divides p. Then we know that $\zeta \equiv 1 \mod p$. Since $f|_k\tau$ is Z-rational by Lemma 3, the Fourier expansion (1.4) of $f|_k\tau_u$ shows that

$$f|_k\tau_u \equiv f|_k\tau \mod p$$

for any τ_u. Thus we have that

$$\Phi(X;f) \equiv (X-f)(X-f|_k\tau)^p \mod p$$

$$\equiv (X-f)(X^p-(f|_k\tau)^p) \mod p.$$

Especially, we have that

$$\sigma_1 \equiv f \mod p$$

and

$$\sigma_p \equiv (f|_k\tau)^p \mod p.$$

Thus we have that

$$\Phi(X;f) \equiv (X - \sigma_1)(X^p - \sigma_p) \mod p.$$

Here all coefficients of $\Phi(X;f)$ are Z-rational, and especially both σ_1 and σ_p are Z-rational. Therefore we have that

$$\Phi(X;f) \equiv (X - \sigma_1)(X^p - \sigma_p) \mod p.$$

§2. Proof of Theorem 1.

By the definition of the discriminant, we have

$$D = \Pi(f|_k\alpha - f|_k\beta)^2,$$

where the product is taken over all non-ordered pairs (α,β) with $\alpha \neq \beta$ in the representative set R as in (1.3). Then we see that

$$D = \Pi_{u=0}^{p-1}(f - f|_k\tau_u)^2 \cdot \Pi_{0\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v)^2.$$

Obviously D is a modular form on $SL_2(Z)$ of weight $kp(p+1)$. Let us put

$$\delta = \Pi_{u=0}^{p-1}(f - f|_k\tau_u) \cdot \Pi_{0\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v).$$

Then we see $D = \delta^2$. First let us show

Lemma 5. δ *is a modular form on* $SL_2(Z)$.

Proof. It is sufficient to prove that $\delta|_m\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \delta$ and $\delta|_m\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \delta$ for $m = kp(p+1)/2$. Put $\sigma = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. The right multiplication of σ on the coset space $\Gamma_0(p)\backslash SL_2(Z)$ induces a permutation on the representative set R, and $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is transformed to $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, τ_{p-1} to τ_0, and τ_u to τ_{u+1} for $u = 0,1,\cdots,p-2$. Thus we observe that the first factor $\Pi_{u=0}^{p-1}(f - f|_k\tau_u)$ of δ is invariant under σ, and that

$$\{\Pi_{0\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v)\}|_{kp(p-1)/2}\sigma$$

$$= (-1)^{p-1}\Pi_{0\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v).$$

Since p is odd, this shows that $\delta|_m\sigma = \delta$. Also, τ induces a permutation on R, and $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is transformed to τ_0, τ_0 to $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$,

and τ_u to $\tau_{\nu(u)}$ for $u = 1,2,\cdots,p-1$. Here $\nu(u)$ is a integer in $\{1, 2, \cdots, p-1\}$ satisfying $\nu(u)u \equiv -1 \mod p$. Rewriting δ as $\Pi_{u=0}^{p-1}(f - f|_k\tau_u)\cdot\Pi_{v=1}^{p-1}(f|_k\tau_0 - f|_k\tau_v)\cdot\Pi_{1\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v)$, we observe that

$$\delta|_m\tau = -\Pi_{u=0}^{p-1}(f-f|_k\tau_u)\cdot\Pi_{v=1}^{p-1}(f|_k\tau_0-f|_k\tau_v)$$

$$\cdot\Pi_{1\leqq u<v\leqq p-1}(f|_k\tau_{\nu(u)} - f|_k\tau_{\nu(v)}).$$

Thus, in order to prove that $\delta|_m\tau = \delta$, we have to show that the last factor $\Pi_{1\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v)$ of δ is alternating under the permutation ν; namely, it is sufficient to prove that the permutation ψ on $(Z/pZ)^\times$ defined by $\psi(a) = -1/a$ for $a \in (Z/pZ)^\times$ is an odd permutation. Since $\psi^2 = id.$, we see that

$$\psi = \Pi(a,b),$$

where the product is taken over all the transpositions (a,b) between a and b with $ab = -1$ and $a \neq b$. The number ℓ of the elements in $(Z/pZ)^\times$ with $a^2 = -1$ is 2 or 0 according as $p \equiv 1 \mod 4$ or not. Thus $(p-1-\ell)/2$ is odd. Since the number of the transpositions in ψ is $(p-1-\ell)/2$, ψ is an odd permutation. This concludes the proof of Lemma 5.

We claim that

(2.1) $\delta = cg$, <u>where</u> c <u>is a constant with</u> $c^2 = (-1)^{(p-1)/2}p$ <u>and</u> g <u>is a</u> Z-<u>rational modular form on</u> $SL_2(Z)$.

In fact, both f and $f|_k\tau$ are Z-rational by Lemma 3. Thus, using the Fourier expansion (1.4) of $f|_k\tau_u$, we can find polynomials $\beta_n(x,y)$ in $Z[x,y]$ so that

(2.2) $f|_k\tau_u - f|_k\tau_v = (\zeta^u - \zeta^v)\Sigma_{n=1}^{\infty}\beta_n(\zeta^u,\zeta^v)e(nz/p)$

for all τ_u and τ_v. Put $c = \Pi_{0\leq u<v\leq p-1}(\zeta^u - \zeta^v)$. Then we see that

$$c^2 = (-1)^{p(p-1)/2}\Pi_{0\leq u<v\leq p-1}(\zeta^u - \zeta^v)(\zeta^v - \zeta^u)$$

$$= (-1)^{(p-1)/2}\Pi_{\substack{u,v=0\\ u\neq v}}^{p-1}(\zeta^u - \zeta^v)$$

$$= (-1)^{(p-1)/2}\Pi_{u=0}^{p-1}\{\zeta^u\Pi_{v=1}^{p-1}(1 - \zeta^v)\}$$

$$= (-1)^{(p-1)/2}p^p.$$

Further put $g = \Pi_{u=0}^{p-1}(f - f|_k\tau_u)\cdot\Pi_{0\leq u<v\leq p-1}\{\Sigma_{n=1}^{\infty}\beta_n(\zeta^u,\zeta^v)e(nz/p)\}$. Then $\Pi_{u=0}^{p-1}(f - f|_k\tau_u)$ is Z-rational as shown below. For any field-automorphism ρ of C, we defined the action of ρ on a Fourier series $\varphi = \Sigma_{n=0}^{\infty}c(n)e(nz/p)$ by $\varphi^{\rho} = \Sigma_{n=0}^{\infty}c(n)^{\rho}e(nz/p)$. Then the Fourier expansions (1.1_a) and (1.4) of f and $f|_k\tau_u$ show that any automorphism ρ induces a bijection of the set $\{f - f|_k\tau_u \mid u = 0, 1,2,\cdots,p-1\}$. Thus $\Pi_{u=0}^{p-1}(f - f|_k\tau_u)$ is Q-rational. Moreover, all Fourier coefficients of $f - f|_k\tau_u$ belong to $Z[\zeta]$. Thus $\Pi_{u=0}^{p-1}(f - f|_k\tau_u)$ is Z-rational. A similar argument combined with the symmetricity in x and y of the polynomial $\beta_n(x,y)$ shows that $\Pi_{0\leq u<v\leq p-1}\{\Sigma_{n=1}^{\infty}\beta_n(\zeta^u,\zeta^v)e(nz/p)\}$is also Z-rational. Therefore g is Z-rational. Note that $\delta = cg$. Then Lemma 5 shows that g is a Z-rational modular form on $SL_2(Z)$. Thus (2.1) is established.

Now we prove the divisibility of g by the power of Δ as indicated in the theorem. Let us put $h = g/\Delta^{(p-1)/2}$. Since the Fourier expansion of g starts from $e(\frac{p-1}{2}z)$, h is still a modular

form. In fact, the Fourier expansion (2.2) of $f|_{k\tau_u} - f|_{k\tau_v}$ starts from $e(z/p)$, and therefore, by counting the number of such factors in the product for g, we know that the Fourier expansion of g starts from $e(\frac{p-1}{2}z)$. Further the cusp form Δ is nowhere vanishing on H and its Fourier expansion starts from $e(z)$. Thus h is holomorphic on H and even at $i\infty$. Since the first coefficient of the Forier expansion of Δ is equal to 1, h is again Z-rational. Next we assume that f is a cusp form. Then, since the Fourier expansion of the first factor $\Pi_{u=0}^{p-1}(f - f|_{k\tau_u})$ of g starts from $e(z)$, that of g starts from $e(\frac{p+1}{2}z)$ or higher term. A similar argument as above is still valid. This completes the proof of Theorem 1.

We remark the following direct consequence of Theorem 1, which may be well known:

Corollary 6. Let us consider the specialized equation of f at an elliptic curve E defined over Q. For the definition of the specialized equation and the details, see [3, §3]. Under the same notation as in [3, §3], if the specialized equation $\Phi(X;f,E) = 0$ is irreducible, then the prime p always ramifies in the splitting field of the equation.

Remark. The above proof shows that without assuming the Z-rationality of transformation equation, we may express D as

$$D = \Delta^{p-1}h^2$$

with a modular form h on $SL_2(Z)$.

§3. The transformation equation of $gE^*_{\lambda,p}$.

In this section, let us consider in more detail the transformation equation of $gE^*_{\lambda,p}$ as in [3]. Here g is a cusp form in $S_\ell(\Gamma_0(p))$ and $E^*_{\lambda,p}$ is the Eisenstein series defined by

$$E^*_{\lambda,p}(z) = \Sigma_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)}(cz+d)^{-\lambda}, \qquad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}),$$

where λ is an even integer > 2 and $\Gamma_\infty = \{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n\in Z\}$. The Eisenstein series $E^*_{\lambda,p}$ is a modular form on $\Gamma_0(p)$ of weight λ, and is expressed as follows (see [9, p.794]):

$$(3.1) \qquad E^*_{\lambda,p}(z) = \frac{2\lambda}{(p^\lambda-1)B_\lambda}\{G_\lambda(z) - p^\lambda G_\lambda(pz)\},$$

where B_λ is the λ-th Bernoulli number and

$$(3.2) \qquad G_\lambda(z) = -B_\lambda/2\lambda + \Sigma_{n=1}^{\infty}\{\Sigma_{0<d|n} d^{\lambda-1}\}e(nz).$$

Let us take a cusp form $\varphi(z) = \Sigma_{n=1}^{\infty}a(n)e(nz)$ in $S_\ell(\Gamma_0(p))$ with the following three properties:

(3.3_a) $\varphi|_\ell\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma\varphi$ for $\gamma = \pm 1$;

(3.3_b) $a(1) = 1$;

(3.3_c) φ is Z-rational.

For example, any Q-rational primitive form in $S_\ell(\Gamma_0(p))$ satisfies these conditions. Moreover, we can construct another example of such cusp forms. We will give this example at the end of this section.

Let us put

$$(3.4) \qquad g = \frac{p^{\ell/2}}{d}N_\lambda\varphi,$$

where we write

$$(3.5)\qquad (p^{\lambda}-1)B_{\lambda}/2\lambda = N_{\lambda}/D_{\lambda}$$

with mutually prime integers N_{λ} and D_{λ}, and d is the greatest common divisor of $p^{\ell/2}$ and D_{λ}. Then we have

Proposition 7. The transformation equation of $gE^{*}_{\lambda,p}$ is Z-rational.

Proof. It is sufficient to show that $gE^{*}_{\lambda,p}$ and $gE^{*}_{\lambda,p}|_{\ell+\lambda}\tau$ are Z-rational (see Lemma 3). Here $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see from (3.1), (3.2) and (3.5) that

$$(3.6)\qquad E^{*}_{\lambda,p}(z) = \frac{D_{\lambda}}{N_{\lambda}}\{\frac{N_{\lambda}}{D_{\lambda}} + \Sigma_{n=1}^{\infty} b(n)e(nz)\}$$

with $b(n) \in Z$. Thus $N_{\lambda}E^{*}_{\lambda,p}$ is Z-rational, and therefore, $gE^{*}_{\lambda,p}$ is Z-rational. Since $G_{\lambda}(z)$ is a modular form in $M_{\lambda}(SL_2(Z))$, we have that

$$(3.7)\qquad G_{\lambda}(pz)|_{\lambda}\tau = p^{-\lambda}G_{\lambda}(z/p),$$

and hence, (3.1) shows that

$$(3.8)\qquad (N_{\lambda}E^{*}_{\lambda,p}|_{\lambda}\tau)(z) = D_{\lambda}\{G_{\lambda}(z) - G_{\lambda}(z/p)\}.$$

On the other hand, it follows from (3.3_a) that

$$(3.9)\qquad (\varphi|_{\ell}\tau)(z) = \gamma p^{-\ell/2}\varphi(z/p).$$

Thus $gE^{*}_{\lambda,p}|_{\ell+\lambda}\tau$ is again Z-rational. In fact, both the modular forms $\varphi(z/p)$ and $G_{\lambda}(z) - G_{\lambda}(z/p)$ are Z-rational; therefore, the modular form:

$$(3.10)\qquad (gE^{*}_{\lambda,p}|_{k}\tau)(z) = \gamma\frac{D_{\lambda}}{d}\varphi(z/p)\{G_{\lambda}(z) - G_{\lambda}(z/p)\}$$

is Z-rational, where $k = \ell+\lambda$. This is what we wanted to show.

Proposition 8. Let us write the transformation equation of $gE^*_{\lambda,p}$ as

$$\Phi(X;gE^*_{\lambda,p}) = X^{p+1} + \Sigma_{\mu=1}^{p+1}(-1)^{\mu}\sigma_{\mu}X^{p+1-\mu}.$$

Then the modular form σ_p has a Fourier expansion of the form:

$$(3.11)\qquad \sigma_p(z) = -\gamma(\frac{D_\lambda}{d})^p e(2z) + \Sigma_{n=3}^{\infty}c(n)e(nz)$$

with rational integers $c(n)$.

Proof. Since σ_p is the p-th elementary symmetric function in $\{gE^*_{\lambda,p}|_k\alpha\}_{\alpha\in R'}$ we have by (1.3) that

$$\sigma_p = gE^*_{\lambda,p}\{\Sigma_{u=0}^{p-1}\Pi_{\substack{v=0\\v\neq u}}^{p-1}(gE^*_{\lambda,p}|_k\tau_v)\} + \Pi_{u=0}^{p-1}(gE^*_{\lambda,p}|_k\tau_u).$$

Here $\tau_u = \begin{pmatrix}0 & -1\\ 1 & u\end{pmatrix}$. Since $\tau_u = \tau\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$, it follows from (3.10) that

$$(3.12)\qquad (gE^*_{\lambda,p}|_k\tau_u)(z) = \gamma\frac{D_\lambda}{d}\wp((z+u)/p)\{G_\lambda(z) - G_\lambda((z+u)/p)\}.$$

Thus we can find polynomials $\beta_n(x)$ in $Z[x]$ so that (3.12) is rewritten as

$$(3.13)\qquad (gE^*_{\lambda,p}|_k\tau_u)(z) = -\gamma\frac{D_\lambda}{d}\Sigma_{n=2}^{\infty}\beta_n(\zeta^u)e(nz/p)$$

for all τ_u. Here $\zeta = e(1/p)$. Especially we have the second polynomial $\beta_2(x) = x^2$. Therefore we have that

$$(3.14)\qquad \Pi_{u=0}^{p-1}(gE^*_{\lambda,p}|_k\tau_u)(z) = -\gamma(\frac{D_\lambda}{d})^p\Sigma_{n=2p}^{\infty}w(n)e(nz/p)$$

with rational integers $w(n)$. Note $w(2p) = 1$. On the other hand, since the Fourier expansion of $gE^*_{\lambda,p}$ starts from $e(z)$, (3.13) shows that

$$(3.15)\qquad gE^*_{\lambda,p}\{\Sigma_{u=0}^{p-1}\Pi_{\substack{v=0\\v\neq u}}^{p-1}(gE^*_{\lambda,p}|_k\tau_v)\}(z) = \Sigma_{n=3p-2}^{\infty}w'(n)e(nz/p)$$

with rational integers $w'(n)$. Thus, considering $w(2p) = 1$ and $3p-2 > 2p$, we see that the Fourier expansion of σ_p starts from $e(2z)$ with the coefficient $-\gamma(\frac{D_\lambda}{d})^p$. Moreover, it follows from Proposition 7 that σ_p is Z-rational modular form. This concludes the proof of Proposition 8.

Remark. Our modification of φ as in (3.4) is best possible. Analyzing carefully the above proof of Proposition 7, one see that if both $c\varphi E^*_{\lambda,p}$ and $c\varphi E^*_{\lambda,p}|_k\tau$ are Z-rational for a constant c, then c is a rational integer and a multiple of $p^{\ell/2}N_\lambda/d$.

Proposition 9. The discriminant D of $\Phi(X; gE^*_{\lambda,p})$ is expressed as

$$D = (-1)^{(p-1)/2} p p_\Delta^{2(p+1)} h^2,$$

where h is a Z-rational modular form on $SL_2(Z)$.

Proof. The Fourier expansion (3.13) of the modular form $gE^*_{\lambda,p}|_k\tau_u$ shows that both the Fourier expansions of the modular forms $g - gE^*_{\lambda,p}|_k\tau_u$ and $gE^*_{\lambda,p}|_k\tau_u - gE^*_{\lambda,p}|_k\tau_v$ start from $e(2z/p)$ for any τ_u and τ_v. Thus that of the modular form $\Pi_{u=0}^{p-1}(gE^*_{\lambda,p} - gE^*_{\lambda,p}|_k\tau_u)\cdot\Pi_{0\leqq u<v\leqq p-1}(gE^*_{\lambda,p}|_k\tau_u - gE^*_{\lambda,p}|_k\tau_v)$ starts from $e((p+1)z)$. Then a similar argument as in the proof of Theorem 1 shows our assertion.

Now let us give examples of the cusp forms φ satisfying the conditions $(3.3_{a,b,c})$ when there exists a primitive form of conductor p. Let us take a primitive form f in $S_\ell(\Gamma_0(p))$ of conductor p and write the Fourier expansion of f as

$$f(z) = \Sigma_{n=1}^{\infty} b(n)e(nz).$$

(For the primitiveness of cusp forms, see, for example, [9, p.789].) We denote by M the module generated over Z by all $b(n)$ in C and by K the field generated by M. For any isomorphism σ of K into C, we define the conjugate f^σ of f by

$$f^\sigma(z) = \Sigma_{n=1}^{\infty} b(n)^\sigma e(nz).$$

As is well known, f^σ is again a primitive form in $S_\ell(\Gamma_0(p))$. We define a cusp form $Tr(\alpha f)$ in $S_\ell(\Gamma_0(p))$ for any α in K by

$$Tr(\alpha f) = \Sigma \alpha^\sigma f^\sigma,$$

where σ runs over all isomorphisms of K into C. Since f is primitive, it follows from [1, Lemma 3] that

$$(3.16_a) \qquad f|_\ell \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma f, \quad \gamma = \pm 1.$$

Moreover we have that for any conjugate f^σ,

$$(3.16_b) \qquad f^\sigma|_\ell \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma f^\sigma$$

for the above γ, because γ is expressed as

$$(3.17) \qquad \gamma = -b(p)p^{1-\ell/2}.$$

Thus we see that for any α in K,

$$(3.18) \qquad Tr(\alpha f)|_\ell \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \gamma Tr(\alpha f).$$

We see easily that

(3.19) $Tr(\alpha f)$ is Z-rational if and only if α belongs to $D = \{\beta \in K \mid Tr_{K/Q}(\beta x) \in Z \text{ for all } x \in M\}$.

Proposition 10. Let us put

$U = \{\alpha \in D \mid Tr(\alpha f)$ has a Fourier expansion of the form: $e(z) + \Sigma_{n=2}^{\infty} c(n)e(nz)$ with rational integers $c(n)\}$,

and

$$V = \{\alpha \in D \mid Tr_{K/Q}(\alpha) = 0\}.$$

Then we have

(1) U is not empty;

(2) V is isomorphic to Z^{d-1} for $d = [K:Q]$;

(3) $U = \alpha_0 + V$ for any element α_0 of U.

Proof. Let α be an element of D. Since $b(1) = 1$ and $Tr(\alpha f) = \Sigma_{n=1}^{\infty} Tr_{K/Q}(\alpha b(n))e(nz)$, we see that

(3.20) α belongs to U if and only if $Tr_{K/Q}\alpha = 1$.

Since M generates K and since M is a Z-free module, M is isomorphic to Z^d. Let $\{\omega_i\}_{i=1}^d$ be a Z-basis of M and $\{\eta_i\}_{i=1}^d$ be the dual basis of $\{\omega_i\}$ with respect to $Tr_{K/Q}$; hence we have

(3.21) $Tr_{K/Q}(\omega_i\eta_j) = \delta_{ij}$.

Then we know that

(3.22) $D = \Sigma_{i=1}^d Z\eta_i$ (direct sum).

Since 1 (=b(1)) belongs to M, we may write as $1 = \Sigma_{i=1}^d m_i\omega_i$ for some rational integers m_i. Then (3.21) shows that $m_i = Tr_{K/Q}\eta_i$ for any i. Namely, we have that

(3.23) $1 = \Sigma_{i=1}^d (Tr_{K/Q}\eta_i)\omega_i$.

Let c be the greatest common divisor of $\{Tr_{K/Q}\eta_i\}_{i=1}^d$. Since f is primitive , all Fourier coefficients b(n) of f, and therefore, all ω_i are algebraic integers. This combined with (3.23) shows that c is equal to 1. Thus considering (3.22), we know that

(3.24) $Tr_{K/Q}D = Z$.

Especially there exists an element α_0 of D such that $\mathrm{Tr}(\alpha_0 f)$ belongs to U. Since $V \otimes_Z Q$ is isomorphic to Q^{d-1}, we see the assertion (2). The third assertion is clear from (3.20).

Now let us put $\varphi = \mathrm{Tr}(\alpha f)$ for any element α of U. Then (3.18), (3.19) and (3.20) show that φ satisfies the conditions $(3.3_{a,b,c})$.

§4. Numerical examples.

In this section, we are going to give several numeriacal examples of the transformation equations $\Phi(x;gE^*_{\lambda,p}) = 0$ and the specialized equations $\Phi(X;gE^*_{\lambda,p},E) = 0$ at various elliptic curves E defined over Q. See [3, §3] for the definition of the specialized equation at an elliptic curve. For simplicity, we consider only the case dim $S_\ell(\Gamma_0(p)) = 1$. Thus we may take as φ in (3.4) with $(3.3_{a,b,c})$ the unique primitive form in $S_\ell(\Gamma_0(p))$. Let us modify φ as in (3.4) and write the modified modular form as g.

Let us explain how to read the table given below by taking the following case I as an example. This case is the restatement of the example given in our previous paper [3, §5]. We will add several new examples here. We use the same notation in §3 and write simply G, H, and D for $12g_2$, $216g_3$, and Δ, respectively. Here g_2 (resp., g_3) is the Eisenstein series in $M_4(SL_2(Z))$ (resp., $M_6(SL_2(Z))$) whose constant term of the Fourier expansion is equal to 1/12 (resp., 1/216). Thus they are Q-rational.

Case I. $p = 5$, $\ell = 4$, $\lambda = 4$, $k = 8$.

$g = -5\cdot 13\varphi$.

X^6	1
X^5	0
X^4	$-25GD$
X^3	$-1440D^2$
X^2	$155G^2D^2$
X	$GH^2D^2 + 18096GD^3$
1	$65H^2D^3 + 538240D^4$

The above table can be read that the transformation equation $\Phi(X;gE^*_{4,5})$ is given by the polynomial

$$(4.1)\quad X^6 -25GDX^4 -1440D^2X^3 +155G^2D^2X^2 +(GH^2D^2 +18096GD^3)X +(65H^2D^3 +538240D^4).$$

Thus, for example, the monomial $-25GD$ given at the right-hand side of X^4 is the isobaric polynomial of the coefficient of X^4.

$\mathrm{Tr}(X)$ 0

$\mathrm{Tr}(X^2)$ $50GD = 2^{-30}\cdot 3\cdot 14!\cdot \dfrac{2^{20}}{3^6\cdot 7^2\cdot 11\cdot 13} f_{16}$

$\mathrm{Tr}(X^3)$ $4320D^2 = 2^{-46}\cdot 3\cdot 22!\Sigma_\sigma\left(\dfrac{2^{29}}{3^8\cdot 5^3\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19\sqrt{144169}} f_{24}\right)^\sigma$

$\mathrm{Tr}(X^4)$ $630G^2D^2 = 2^{-62}\cdot 3\cdot 30!$

$$\cdot\Sigma_\sigma\left(\frac{2^{34}}{3^{14}\cdot 5^6\cdot 7^3\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\sqrt{18295489}} f_{32}\right)^\sigma$$

$\mathrm{Tr}(X^5)$ $-5GH^2 +89520GD^3$

$$= 2^{-78}\cdot 3\cdot 38!\Sigma_\sigma\left(\frac{-2^{43}(\alpha -3537792)}{3^{18}\cdot 5^7\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)} f_{40}\right)^\sigma$$

$$N(\alpha-3537792) = 2^{18}\cdot 3^7\cdot 7^2\cdot 11\cdot 23\cdot 31\cdot 73\cdot 2161$$

$\mathrm{Tr}(X^6)$ $7610H^2D^3 +16815360D^4$

$$= 2^{-94}\cdot 3\cdot 46!\Sigma_\sigma\left(\frac{2^{56}(5117\alpha +17457217536)}{3^{21}\cdot 5^9\cdot 7^5\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\psi'(\alpha)} f_{48}\right)^\sigma$$

$$N(5117\alpha+17457217536)$$

$$= -2^{37}\cdot 3^8\cdot 11\cdot 383^2\cdot 312951285187012426 5857$$

Here $\mathrm{Tr}(X^{\mu})$ indicates the μ-th power sum $\mathrm{Tr}(gE^{*}_{4,5})^{\mu}$ of all the roots of the transformation equation given in (4.1), and the corresponding isobaric polynomial is given at the right-hand side of $\mathrm{Tr}(X^{\mu})$. As we have seen in [3], the power sum $\mathrm{Tr}(gE^{*}_{4,5})^{\mu}$ can be expressed as

$$\mathrm{Tr}(gE^{*}_{4,5})^{\mu} = 2^{-2(8\mu-1)}\cdot 3\cdot(8\mu-2)!\,\Sigma_{f}\, P(8\mu)\frac{D(8\mu-1,f,g^{\mu}E^{*}_{4,5}{}^{\mu-1})}{\pi^{8\mu}<f,f>}f.$$

(See [3, Theorem] for the notation.) After the isobaric polynomial in the table, we have given this expression of power sum. (This expression is not given in [3, §5].) Thus, for example, in the expression corresponding to $\mathrm{Tr}(X^{5})$, the value $-2^{43}(\alpha-3537792)/\{3^{18}\cdot 5^{7}\cdot 7^{5}\cdot 11^{3}\cdot 13^{2}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)\}$ gives the special value $\dfrac{D(39,f_{40},g^{5}E^{*}_{4,5}{}^{4})}{\pi^{40}<f_{40},f_{40}>}$. Here f_{40} indicates a primitive form in $S_{40}(SL_{2}(Z))$; α is a generator of the field $K(f_{40})$ generated over Q by all Fourier coefficients of f_{40}; ψ indicates the characteristic polynomial of α and $\psi'(x) = d\psi/dx$. Further, in the above expression of $\mathrm{Tr}(X^{5})$, the summation is over all isomorphisms σ of $K(f_{40})$ into C. Note that in the limit of the calculation we have done, all the primitive forms in $S_{m}(SL_{2}(Z))$ are conjugate under the automorphisms of C. We denote by $N(\gamma)$ the norm of an algebraic number γ, for example, $N(\alpha-3537792)$ indicates the norm of the number $\alpha-3537792$. If the factors in the listed numbers are less than 10^{10}, then they are primes; otherwise, we do not know whether they are prime or not.

Let us now list the characteristic polynomials $\psi(x)$ and their discriminants $D(\psi)$ of a generator α of the fields $K(f_{m})$:

m	$\psi(x)$ and $D(\psi)$
40	$\psi(x) = x^3-548856x^2-810051757056x+213542160549543936$ $D(\psi) = 2^{26}\cdot3^{12}\cdot5^2\cdot7^2\cdot13^2\cdot73\cdot59077\cdot92419245301$
48	$\psi(x) = x^4-5785560x^3-467142374034432x^2+1426830562183253852160x$ $+3297913828840214320807673856$ $D(\psi) = 2^{70}\cdot3^{22}\cdot5^6\cdot7^6\cdot31\cdot383^2\cdot102107536163441411992455248734239414994 39$
50	$\psi(x) = x^3+24225168x^2-566746931810304x-13634883228742736412672$ $D(\psi) = 2^{32}\cdot3^{12}\cdot5^4\cdot7^4\cdot1228462869413174261940 1$
60	$\psi(x) = x^5+449691864x^4-220945018405443379 2x^3$ $-736010060393513697870348288x^2$ $+810634763334812972416233648439689216x$ $+263222216157060824115203098902237248565018624$ $D(\psi) = 2^{148}\cdot3^{38}\cdot5^8\cdot7^8\cdot17^4\cdot23\cdot1019$ $\cdot651916320472103878902727074480503094855670432357132070088$ $298897328058850220694574730171748779559 7$*

* This number 65191···5597 is a number of 97-figures.

Case II. $p = 5,\ \ell = 4,\ \lambda = 6,\ k = 10.$

$g = 5^2\cdot31\varphi$

X^6	1
X^5	0
X^4	$-145G^2D$
X^3	$587520HD^2$
X^2	$3635GH^2D^2 - 377403840GD^3$
X	$G^2H^3D^2 + 6290064G^2HD^3$
1	$-775H^4D^3 -7058849600H^2D^4$

$\mathrm{Tr}(X)$ $\quad 0$

$\mathrm{Tr}(X^2)$ $\quad 290G^2D = 2^{-38}\cdot 3\cdot 18!\cdot\dfrac{2^{23}\cdot 29}{3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17}\,f_{20}$

$\mathrm{Tr}(X^3)$ $\quad -1762560HD^2 = 2^{-58}\cdot 3\cdot 28!\,\Sigma_\sigma\left(\dfrac{-2^{35}}{3^{11}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 19\cdot 23\sqrt{51349}}\,f_{30}\right)^\sigma$

$\mathrm{Tr}(X^4)$ $\quad 27510GH^2D^2 + 1582277760GD^3$

$$= 2^{-78}\cdot 3\cdot 38!\,\Sigma_\sigma\left(\frac{2^{44}(131\alpha+1196402688)}{3^{17}\cdot 5^7\cdot 7^4\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)}\,f_{40}\right)^\sigma$$

$$N(131\alpha+1196402688) = -2^{27}\cdot 3^7\cdot 7^2\cdot 3833\cdot 32619042931$$

$\mathrm{Tr}(X^5)$ $\quad -5G^2H^3D^2 - 457402320G^2HD^3$

$$= 2^{-98}\cdot 3\cdot 48!$$

$$\cdot\Sigma_\sigma\left(\frac{-2^{52}(\alpha-8757800448)}{3^{23}\cdot 5^9\cdot 7^6\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\psi'(\alpha)}\,f_{50}\right)^\sigma$$

$$N(\alpha-8757800448) = 2^{21}\cdot 3^7\cdot 5^2\cdot 19\cdot 73\cdot 4235321855794559$$

$\mathrm{Tr}(X^6)$ $\quad 2939450H^4D^3 + 1421841072000H^2D^4 + 585580127846400D^5$

$$= 2^{-118}\cdot 3\cdot 58!$$

$$\cdot\Sigma_\sigma\left(\frac{2^{68}(168536131\alpha^2+47995636461477888\alpha+99935035640221872905256 96)}{3^{27}\cdot 5^{11}\cdot 7^8\cdot 11^5\cdot 13^4\cdot 17^3\cdot 19^3\cdot 23^2\cdot 29^2\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\psi'(\alpha)}\right.$$

$$\left.\cdot f_{60}\right)^\sigma$$

$$N(168536131\alpha^2+47995636461477888\alpha+99935035640221872905256 96)$$

$$= -2^{105}\cdot 3^{23}\cdot 5^3\cdot 11^2\cdot 13\cdot 17^4$$

$$\cdot 1165842531588761730920594563390304066538263826553725077932$$

$$2558076295700706 2391$$

Case III. $P = 5,\ \ell = 4,\ \lambda = 8,\ k = 12.$

$g = -5\cdot 13\cdot 313\varphi$

X^6 1

X^5 $60480D$

X^4 $-625H^2D + 1301832000D^2$

X^3 $117113760H^2D^2 + 11768083937280D^3$

X^2 $69755H^4D^2 + 1728323786880H^2D^3 + 39309437117214720D^4$

X $H^6D^2 + 17889611952H^4D^3 - 1824903062774784 0H^2D^4$
$+ 1541762666850543 2064D^5$

1 $20345H^6D^3 + 335091233981440H^4D^4 + 66604523265119232 00H^2D^5$
$+ 3217592173497380241408 0D^6$

$Tr(X)$ $-60480D$

$Tr(X^2)$ $1250H^2D + 1054166400D^2$

$Tr(X^3)$ $-464741280H^2D^2 - 20325436323840D^3$

$Tr(X^4)$ $502230H^4D^2 + 27308861671680H^2D^3 + 411430804078878720D^4$

$Tr(X^5)$ $-5H^6D^2 - 552459647760H^4D^3 - 100172160150266880 0H^2D^4$
$-852820366590697439232 0D^5$

$Tr(X^6)$ $226940810H^6D^3 + 101092055900113920H^4D^4$
$+ 3012447762017718128640 0H^2D^5 + 17866412661784867206856704 0D^6$

Case IV. $P = 5$, $\ell = 4$, $\lambda = 10$, $k = 14$.

$$g = 5^2 \cdot 71 \cdot 521\varphi$$

$$X^6 \quad 1$$

$$X^5 \quad 0$$

$$X^4 \quad -2545GH^2D - 604109741760GD^2$$

$$X^3 \quad 25344112320H^3D^2 - 211931520573911040HD^3$$

$$X^2 \quad 1207235G^2H^4D^2 - 27110066987928960G^2H^2D^3 - 183934239995711768371200G^2D^4$$

$$X \quad GH^7D^2 + 22434273283920GH^5D^3 + 5557901335458375149568GH^3D^4 - 3067140238776492879943434240GHD^5$$

$$1 \quad -924775H^8D^3 - 217790930732661680000H^6D^4 - 13773337937065083738654720000H^4D^5 - 13353978977429466150344392704000H^2D^6$$

$$\mathrm{Tr}(X) \quad 0$$

$$\mathrm{Tr}(X^2) \quad 5090GH^2D + 1208219483520GD^2$$

$$\mathrm{Tr}(X^3) \quad -76032336960H^3D^2 + 635794561721733120HD^3$$

$$\mathrm{Tr}(X^4) \quad 8125110G^2H^4D^2 + 114590105122832640G^2H^2D^3 + 80347085617695248314368000G^2D^4$$

$$\mathrm{Tr}(X^5) \quad -5GH^7D^2 - 434675195691600GH^5D^3 - 10164580382184703017984000GH^3D^4 + 641683050942935873278036869120GHD^5$$

$$\mathrm{Tr}(X^6) \quad 14539127450H^8D^3 + 249071858318180032320000H^6D^4 + 72718216860527846662995763200H^4D^5 + 64254070184347969126043594348298240000H^2D^6 + 87714639016992770475227268903610613760000D^7$$

Case V. $p = 5,\ \ell = 4,\ \lambda = 12,\ k = 16.$

$$g = -5\cdot 31\cdot 601\cdot 691\varphi$$

$$X^6 \quad 1$$

$$X^5 \quad 81829440GD$$

$$X^4 \quad -10225G^2H^2D + 1216590866568000G^2D^2$$

$$X^3 \quad 2910441252960H^4D^2 + 6127470158334076661760H^2D^3 - 32282327635049729294991360D^4$$

$$X^2 \quad 20065355GH^6D^2 + 204260280738724336320GH^4D^3 + 1025675627148717142617040896 0GH^2D^4 + 92754860107460880754044689448960GD^5$$

$$X \quad G^2H^8D^2 + 30194230743474480G^2H^6D^3 - 161607524021417153848134758 4G^2H^4D^4 + 42117422128340913862098973248061 44G^2H^2D^5 - 13223615489524979651841803535733751808G^2D^6$$

$$1 \quad 64370105H^{10}D^3 + 18604647958742074084998 40H^8D^4 + 14358613619629720425298218824908 80H^6D^5 + 27232467565920521205584946614609903 6160H^4D^6 + 10889017808078270477869270369367230513 1520H^2D^7 + 87186102896322601284933986395964551565672 4480D^8$$

$Tr(X)$ $-81829440GD$.

$Tr(X^2)$ $20450G^2H^2D + 4262875517577600G^2D^2$

$Tr(X^3)$ $-11241441830880H^4D^2 - 267658181885197261025280H^2D^3$
$- 333901542596495424649297920D^4$

$Tr(X^4)$ $128839830GH^6D^2 + 359708164048445224320GH^4D^3$
$+ 171765241739983259380327621017 60GH^2D^4$
$+ 153486008873427298587164791249305 60GD^5$

$Tr(X^5)$ $-5G^2H^8D^2 - 33433437443669640 0G^2H^6D^3$
$- 623256201795001793666231808 0G^2H^4D^4$
$- 110521725422678931260034397042851151872 0G^2H^2D^5$
$- 70447410787570457585274157497757672144896 0G^2D^6$

$Tr(X^6)$ $907155508010H^{10}D^3 + 506425506648200761436428 80H^8D^4$
$+ 841008155152186300489845505206681 60H^6D^5$
$+ 711391932787509952337600204957843023859495731 20H^4D^6$
$+ 152077104881047937683933892892491027290971061616640H^2D^7$
$+ 558823976148858965042637471021663302378110420883865 60D^8$

. . . . . .

Case VI. $p = 5$, $\ell = 6$, $\lambda = 4$, $k = 10$.

$g = -5^2 \cdot 13\varphi$.

X^6 1

X^5 0

X^4 $-55G^2D$

X^3 $-41040HD^2$

X^2 $395GH^2D^2 - 7266240GD^3$

X $-G^2H^3D^2 + 121104G^2HD^3$

1 $-325H^4D^3 - 2691200H^2D^4$

$\mathrm{Tr}(X)$ $\quad 0$

$\mathrm{Tr}(X^2)$ $\quad 110G^2D = 2^{-38}\cdot 3\cdot 18!\cdot\dfrac{2^{23}}{3^9\cdot 5^2\cdot 7^2\cdot 13\cdot 17}\,f_{20}$

$\mathrm{Tr}(X^3)$ $\quad 123120HD^2 = 2^{-58}\cdot 3\cdot 28!\,\Sigma_\sigma\left(\dfrac{2^{31}}{3^{11}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 23\sqrt{51349}}\,f_{30}\right)^\sigma$

$\mathrm{Tr}(X^4)$ $\quad 4470GH^2D^2 + 39519360GD^3$

$$= 2^{-78}\cdot 3\cdot 38!\,\Sigma_\sigma\left(\frac{2^{44}(149\alpha+142350336)}{3^{17}\cdot 5^7\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)}\,f_{40}\right)^\sigma$$

$$N(149\alpha+142350336) = -2^{25}\cdot 3^7\cdot 5^2\cdot 7^2\cdot 19\cdot 8389\cdot 89003$$

$\mathrm{Tr}(X^5)$ $\quad 5G^2H^3D^2 + 10680480G^2HD^3$

$$= 2^{-98}\cdot 3\cdot 48!$$

$$\cdot\Sigma_\sigma\left(\frac{2^{52}(\alpha-180741120)}{3^{23}\cdot 5^9\cdot 7^6\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\psi'(\alpha)}\,f_{50}\right)^\sigma$$

$$N(\alpha-180741120) = 2^{22}\cdot 3^7\cdot 31\cdot 1223\cdot 18919300277$$

$\mathrm{Tr}(X^6)$ $\quad 204350H^4D^3 + 8391590400H^2D^4 + 5137086873600D^5$

$$= 2^{-118}\cdot 3\cdot 58!$$

$$\cdot\Sigma_\sigma\left(\frac{2^{68}(6637423\alpha^2+2121380494196736\alpha-67543443341033481437184)}{3^{27}\cdot 5^{11}\cdot 7^9\cdot 11^5\cdot 13^4\cdot 17^3\cdot 19^3\cdot 23^2\cdot 29^2\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\psi'(\alpha)}\,f_{60}\right)^\sigma$$

$$N(6637423\alpha^2+2121380494196736\alpha-67543443341033481437184)$$

$$= 2^{107}\cdot 3^{23}\cdot 5^3\cdot 7^5\cdot 17^4\cdot 181\cdot 233$$

$$\cdot 450456472421115659925683391938601142706911756985491297538017303$$

Case VII. $p = 5$, $\ell = 6$, $\lambda = 6$, $k = 12$.

$g = 5^3 \cdot 31\varphi$

X^6 1

X^5 $-10800D$

X^4 $-175H^2D + 39994560D^2$

X^3 $-954000H^2D^2 - 58506624000D^3$

X^2 $4595H^4D^2 + 5976610560H^2D^3 + 29346922598400D^4$

X $-H^6D^2 - 26913792H^4D^3 + 1347784593408H^2D^4$

1 $3875H^6D^3 + 35294248000H^4D^4$

$Tr(X)$ $10800D$

$Tr(X^2)$ $350H^2D + 36650880D^2$

$Tr(X^3)$ $8532000H^2D^2 + 139408128000D^3$

$Tr(X^4)$ $42870H^4D^2 + 70958165760H^2D^3 + 5542558119936000D^4$

$Tr(X^5)$ $5H^6D^2 + 2374938960H^4D^3 + 433666967592960H^2D^4$
$+ 223776852111360000D^5$

$Tr(X^6)$ $5935550H^6D^3 + 42310659964800H^4D^4 + 2330955736245043200H^2D^5$
$+ 9081391088816750592000D^6$

Case VIII. $p = 11,\ \ell = 2,\ \lambda = 4,\ k = 6.$

$g = -11 \cdot 61\varphi$

X^{12} 1

X^{11} 0

X^{10} $11088D$

X^{9} $-9075HD$

X^{8} $-5962H^2D + 24952224D^2$

X^{7} $-77H^3D - 67215456HD^2$

X^{6} $37678773H^2D^2 + 25829299584D^3$

X^{5} $-17237913H^3D^2 - 119108926464HD^3$

X^{4} $2011493H^4D^2 + 104087609758H^2D^3 - 68766745458048D^4$

X^{3} $-55913H^5D^2 - 44737025102H^3D^3 + 9874684797536HD^4$

X^{2} $440H^6D^2 + 6582378638H^4D^3 - 52178539740844H^2D^4$
$+ 21138255578398464D^5$

X $-H^7D^2 - 308633685H^5D^3 + 6499878090033H^3D^4$
$- 209148873196874 88HD^5$

1 $- 671H^6D^3 - 207290985242H^4D^4 + 1480882485474007H^2D^5$
$- 37778664373067911 04D^6$

$Tr(X)$ 0

$Tr(X^2)$ $-22176D$

$Tr(X^3)$ $27225HD$

$Tr(X^4)$ $23848H^2D + 146078592D^2$

$Tr(X^5)$ $385H^3D - 167040720HD^2$

$Tr(X^6)$ $-375645699H^2D^2 - 1221354706176D^3$

$Tr(X^7)$ $493425009H^3D^2 + 1841678960352HD^3$

$Tr(X^8)$ $131680032H^4D^2 + 4757929285648H^2D^3 + 11020319744130048D^4$

$Tr(X^9)$ $4634883H^5D^2 - 9276821678520H^3D^3 - 21750837501488832HD^4$

$Tr(X^{10})$ $25245H^6D^2 + 340701804850H^4D^3 - 45226189141798280H^2D^4$
$- 972272507295947996 16D^5$

$Tr(X^{11})$ $11H^7D^2 + 4400455461072H^5D^3 + 12710957710878949 5H^3D^4$
$+ 23914064600479375 2384HD^5$

$Tr(X^{12})$ $825050231500H^6D^3 - 19665598331764275H^4D^4$
$+ 39324813923947628670 0H^2D^5 + 84518045159262798708 5568D^6$

In what follows, we are going to give the specialized equations $\Phi(X;gE^*_{\lambda,p},E)= 0$ at several elliptic curves E defined over Q. Again let us explain how to read the table given below for specialized equations. We first list the curves where we specialize the transformation equations in Case I-V:

Case A : $y^2 = 4x^3 - 2^2\cdot 3^{-1}x + 3^{-3}\cdot 19$ (11A);

Case B : $y^2 = 4x^3 - 2^2 x + 1$ (37A);

Case C : $y^2 = 4x^3 - 2^3\cdot 3^{-1}\cdot 5x + 3^{-3}\cdot 251$ (37B);

Case D : $y^2 = 4x^3 + 2^3\cdot 3x - 2^3$;

Case E : $y^2 = 4x^3 + 1$ (27A).

The curve in Case A is isogeneous to the modular curve $X_0(11)_{/Q}$ ($\cong H/\Gamma_0(11)$). This curve is referred in [10] as 11A. The example of Case A is the restatement of [3, §5]. The curves in Case B and Case C correspond the distinct non-isogeneous factors of the jacobian variety of $X_0(37)_{/Q}$. The curve in Case D is found in Serre [6, 5.9.2], which has potential everywhere good reduction. The curve in Case E has complex multiplication under $Q(\sqrt{-3})$. In the following table, we list the specialized equations of the transformation equations already listed above at these elliptic curves. In Case A, as is well known, all the specialized equations of level 5 are reducible; so, we here list only one of them which corresponds to that in Case I. All the factors of the equations listed below are irreducible over Q.

Case A. $G = 2^4$ $H = -2^3 \cdot 19$ $D = -11$

(I) $X^6+4400X^4-174240X^3+4801280X^2-340643072X+5881529280$

$=(X-22)(X^5+22X^4+4884X^3-66792X^2+3331856X-267342240)$

Case B. $G = 2^4 \cdot 3$ $H = -2^3 \cdot 3^3$ $D = 37$

(I) $X^6-44400X^4-1971360X^3+488897280X^2+47063460096X+1162360730560$

Discriminant

$= 2^{36} \cdot 3^{12} \cdot 5^5 \cdot 11^6 \cdot 37^{12} \cdot 42044237^2$

Constant term $= 2^6 \cdot 5 \cdot 37^3 \cdot 71711$

(II) $X^6-12360960X^4-173732014080X^3-906454164234240X^2-158592818333712384X$

-617317300619300044800

Discriminant

$= 2^{90} \cdot 3^{30} \cdot 5^5 \cdot 37^{12} \cdot 431^2 \cdot 175158867454805351481672$

Constant term $= -2^{19} \cdot 3^6 \cdot 5^2 \cdot 37^3 \cdot 1275457$

(III) $X^6+2237760X^5+1781129088000X^4+603569053249044480X^3$

$+77756911326531739975680X^2-52461909281646516043461427 2X$

$+10547030308145620659892484308992 0$

Discriminant

$= 2^{156} \cdot 3^{60} \cdot 5^5 \cdot 37^{12} \cdot 5387^2 \cdot 11719^2 \cdot 191320980941323573505961215 3^2$

Constant term $= 2^{33} \cdot 3^{12} \cdot 5 \cdot 37^3 \cdot 71711 \cdot 1272109$

(IV) $X^6-3969747023192064 0X^4+231840328266709697101824 0X^3$

$-795716495364315743888001623654 40X^2$

$+215475800430255113967103637484510117888X$

$-180650394914609884769327204746079333724979200$

Discriminant

$= 2^{156} \cdot 3^{90} \cdot 5^5 \cdot 11^2 \cdot 37^{12}$

$\times \binom{16926206015272284315501000059831565580855098524819080 28}{37328703}^2$

Constant term $= -2^{30} \cdot 3^{18} \cdot 5^2 \cdot 37^3 \cdot 2777 \cdot 6469 \cdot 19089662430217$

(V) $X^6+145329085440X^5+38373416724797816832000X^4$

$-460214266235592346411892893286400X^3$

$+351785812651605713387572533809327772794880X^2$

$-467907236561224841504789599481108148857636609064960X$

$+5075737063108711398438669685930545872239299744266893393920$

Discriminant

$= 2^{210}\cdot 3^{102}\cdot 5^5\cdot 37^{12}\cdot 73^2\cdot 39521^2$

$\times\left(\begin{matrix}16388230918081261220040555436226618726893298260185323 38\\ 16031863967223997469326369 21\end{matrix}\right)^2$

Constant term $= 2^{43}\cdot 3^{18}\cdot 5\cdot 37^3\cdot 661\cdot 889713298204304238228020812 9$

Case C. $G = 2^5\cdot 5$ $H = -2^3\cdot 251$ $D = 37$

(I) $X^6-148000X^4-1971360X^3+5432192000X^2+1029841968640X+14284097373120$

Discriminant

$= 2^{36}\cdot 5^5\cdot 37^{12}\cdot 97^2\cdot 251^4\cdot 158512865466953^2$

Constant term $= 2^6\cdot 3\cdot 5\cdot 37^3\cdot 293749$

(II) $X^6-137344000X^4-1615064279040X^3+151709417062400X^2$

$-1666186390720675840 0X-5398007785715322716160 0$

Discriminant

$= 2^{96}\cdot 5^{11}\cdot 37^{12}\cdot 251^2\cdot 84649^2\cdot 80524545706391590943442857^2$

Constant term $= -2^{18}\cdot 3\cdot 5^2\cdot 37^3\cdot 103\cdot 251^2\cdot 8353$

(III) $X^6+2237760X^5+1688966528000X^4+1242544486072320000X^3$

$+428210896271006105600000X^2-122101944360234810500710400000X$

$+121548575717669223512628264960000 00$

Discriminant

$= 2^{186}\cdot 5^{35}\cdot 37^{12}\cdot 149^2\cdot 251^4\cdot 613^2\cdot 7681^2\cdot 85999^2\cdot 2828711435333527499047^2$

Constant term $= 2^{30}\cdot 3\cdot 5^6\cdot 11\cdot 37^3\cdot 433421914559869$

(IV) $X^6-13238494652416000X^4+212749014779446225305600000X^3$

$-10242321004657701912901058560000000X^2$

$-6666338187777448119334115989258240000000X$

$-17176597045167377150442215146863628124160000000$

Discriminant

$= 2^{156}\cdot 5^{41}\cdot 37^{12}\cdot 223^2\cdot 251^2$

$\times\left(\begin{matrix}21041388636101115247169710167790485273645676234370155644 \\ 7 \\ 5299986747399133\end{matrix}\right)^2$

Constant term $= -2^{30}\cdot 3\cdot 5^8\cdot 37^3\cdot 251^2\cdot 4277631339834905360883 1$

(V) $X^6+484430284800X^5+426370910950650675200000X^4$

$+11910133365775079136433274880000000X^3$

$+13457249056881351216480342923175526400000000X^2$

$+28017443414101677967629098569361263396126720000000000X$

$+17931826045430048857989553905829178499337154671411200000000000$

Discriminant

$= 2^{216}\cdot 5^{55}\cdot 37^{12}\cdot 251^4\cdot 3001^2$

$\times\left(\begin{matrix}59569294896991541925368974827809306186927780234914489024 \\ 56440039394814730210943947368314410 3\end{matrix}\right)^2$

Constant term $= 2^{42}\cdot 3\cdot 5^{11}\cdot 37^3\cdot 809$

$\times 679234447403551875231178057851449$

Case D. $G = -2^5\cdot 3^2$ $H = 2^6\cdot 3^3$ $D = -2^6\cdot 3^5$

(I) $X^6-111974400X^4-348285173760X^3+3109490031329280X^2+19395514284707414016X$

$+30756189783160164188160$

Discriminant

$= 2^{156}\cdot 3^{102}\cdot 5^5\cdot 523^2\cdot 1993^2$

Constant term $= 2^{30}\cdot 3^{20}\cdot 5\cdot 31\cdot 53$

(II) $X^6+187042037760X^4+24554940634497024 0X^3-40959998252641947746304 0X^2$

$-339101151063969167461004 0832X$

$-12329834303145689520938944561152 00$

Discriminant

$$= 2^{228}\cdot 3^{132}\cdot 5^{5}\cdot 13^{2}\cdot 1766601958386631369872$$

Constant term $= -2^{45}\cdot 3^{26}\cdot 5^{2}\cdot 551461$

(III) $X^6-940584960X^5+31489623510220800 0X^4-441808308744218926173388 80X^3$

$+22801325884247010780508536491212 80X^2$

$-17214744129018591819208640165526254911488X$

$+43732973945707100175825194219369249263344156672 0$

Discriminant

$$= 2^{306}\cdot 3^{162}\cdot 5^{5}\cdot 2857^{2}\cdot 128103862043615822809259620839427^{2}$$

Constant term $= 2^{60}\cdot 3^{32}\cdot 5\cdot 31\cdot 53\cdot 509\cdot 48955757$

(IV) $X^6+42080459238584604426240X^4+13775520156142163718733347107635 20X^3$

$-89221522557725558133720282371231512044503040X^2$

$-13935080050601902328495731343754388260702333021057024X$

$-55300532147549500076664990648195586731052493633265100180684800$

Discriminant

$$= 2^{368}\cdot 3^{192}\cdot 5^{5}\cdot 47^{2}\cdot 109^{2}$$

$$\times 44275612064289958736301204301726556694592007012186072876292$$

Constant term $= -2^{72}\cdot 3^{38}\cdot 5^{2}\cdot 107\cdot 3240694984266366049$

(V) $X^6+366512097853440X^5+24406304373574174539723571200X^4$

$-19572908894197542838868422888455095608934 40X^3$

$+23787048189330037674267146417212680864094364958279598080X^2$

$-16467673098362564917405712521433419400135894794798454628675502473216X$

$+29463512571207486730328650782376415071679202494411628848219380064330$
22156472320

Discriminant

$$= 2^{438}\cdot 3^{222}\cdot 5^{5}$$

$$\times\left(\frac{17055481350680990459643415602664324740824343889635566755 10}{86293800093349973130262537 07}\right)^{2}$$

Constant term $= 2^{87}\cdot 3^{44}\cdot 5\cdot 19\cdot 9787\cdot 207953625880836441264743 41$

Case E. $G = 0$ $H = -2^3 \cdot 3^3$ $D = -3^3$

(I) $X^6-1049760X^3+226351350720$

Discriminant = $2^{36} \cdot 3^{66} \cdot 5^5 \cdot 11^6 \cdot 17^6$

Constant term = $2^6 \cdot 3^{12} \cdot 5 \cdot 11^3$

(II) $X^6-92513249280X^3-174990344338597478400$

Discriminant = $2^{96} \cdot 3^{96} \cdot 5^{13} \cdot 7^6 \cdot 41^6 \cdot 61^6$

Constant term = $-2^{18} \cdot 3^{18} \cdot 5^2 \cdot 41^3$

(III) $X^6-1632960X^5+949822848000X^4-227647896698880000X^3$

$+193035855972630528000000X^2-6744756780841215983616000000X$

$+839433158209838194989465600000000$

$= (X^2-544320X+20323353600)^3$

Discriminant of the irreducible factor = $2^{18} \cdot 3^8 \cdot 5^3$

Constant term of the irreducible factor = $2^{10} \cdot 3^8 \cdot 5^2 \cdot 11^2$

(IV) $X^6-901218987881625600000000X^3$

$-19837121300256932128368336568320000000000000$

Discriminant = $2^{156} \cdot 3^{156} \cdot 5^{65} \cdot 17^6 \cdot 23^6 \cdot 59^6 \cdot 71^6 \cdot 70157^6$

Constant term = $-2^{30} \cdot 3^{30} \cdot 5^{13} \cdot 59^3 \cdot 71^3$

(V) $X^6-22783187826815470647902208000000X^3$

$+42066388595442440479420138302171596988528721920000000000000$

Discriminant

$= 2^{216} \cdot 3^{186} \cdot 5^{55} \cdot 7^6 \cdot 13^6 \cdot 173^6 \cdot 521^6 \cdot 4519^6$

$\times 2070362216376807869728367^2$

Constant term = $2^{42} \cdot 3^{36} \cdot 5^{11} \cdot 521^3 \cdot 4519^3$

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