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CONVERGENCE OF PHASE–FIELD APPROXIMATIONS TO THE GIBBS–THOMSON LAW

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Abstract. We prove the convergence of phase-field approximations of the Gibbs–Thomson law. This establishes a relation between the first variation of the Van-der-Waals–Cahn–Hilliard energy and the first variation of the area functional. We allow for folding of diffuse interfaces in the limit and the occurrence of higher-multiplicities of the limit energy measures. We show that the multiplicity does not affect the Gibbs–Thomson law and that the mean curvature vanishes where diffuse interfaces have collided.

We apply our results to prove the convergence of stationary points of the Cahn–Hilliard equation to constant mean curvature surfaces and the convergence of stationary points of an energy functional that was proposed by Ohta–Kawasaki as a model for micro-phase separation in block-copolymers.

1. Introduction

Phase separation is a common phenomenon in many areas of the sciences. Alloys studied in material sciences, melting and solidification processes, or block-copolymers investigated in physical chemistry, they all show the coexistence of two or more phases, separated by thin *transition layers*. The main approaches to describe phase transitions are on the one hand *sharp interface models* and on the other hand *diffuse interface models*, also referred to as 'phase field' or 'Ginzburg-Landau' models. The relation between both kinds of models remains an outstanding question. Rigorous passages to the sharp interface limit are often difficult and generalized formulations for the limit problems are necessary to obtain the convergence of diffuse approximations. However, care has to be taken that solutions satisfy the equations in a reasonably strong sense.

The goal of the present paper is to prove the convergence of diffuse approximations of the so-called *Gibbs–Thomson law*, which states that the mean curvature of the phase boundary is given as the trace of a function in the bulk. Our result relates the first variation of the Van-der-Waal-Cahn–Hilliard energy, which is the common root of most phase field models, to the first variation of the area functional. To the best of our knowledge, we give the first satisfactory solution in the case that diffuse interfaces collapse or cancel each other in the limit.

Before stating the main result we describe the setting and background of the problem.

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1.1. **Phase fields, sharp interfaces, and the Gibbs–Thomson law.** The diffuse interface approach is based on a free energy that acts on *smooth phase fields* and that was proposed by Van-der-Waals [33] and later Cahn–Hilliard [7]. In a normalized form this energy is given by

$$E_\varepsilon(u) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)dx, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter and W is a nonnegative 'double-well potential' with value zero if and only if $u = \pm 1$. Domains where $u \approx 1$ or $u \approx -1$ represent two coexisting phases, separated by *diffuse interfaces*. Formal arguments show that E_ε favors transition layers with a thickness of order ε. Hence, as ε tends to zero the diffuse interfaces become sharp.

One naturally associated quantity to the Cahn–Hilliard energy is its L^2-functional derivative, which often corresponds to the *chemical potential*,

$$f_\varepsilon = -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u). \tag{1.2}$$

In many applications f_ε is given by means of other quantities and a certain control on f_ε is available. The corresponding functional derivative of the surface area functional, evaluated at a smooth compact hypersurface Σ, is given by the mean curvature of Σ and (1.2) formally corresponds to the following equation, in solidification processes known as *Gibbs–Thomson law* (and we will adopt this term throughout the paper),

$$H = \sigma f, \tag{1.3}$$

with a surface tension coefficient $\sigma > 0$. The Gibbs–Thomson law relates the local geometry of the phase boundary to a function $f : \Omega \to \mathbb{R}$ in the bulk, for example the temperature or the chemical potential.

1.2. **Main results.** Let us first state our main result in a concise form (we will prove a slightly stronger statement, given in Theorem 3.2).

Theorem 1.1. *Suppose $p > n$ and let sequences of functions $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{3,p}(\Omega)$ and functions $\{f_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega)$ be given such that* (1.2) *holds and such that*

$$E_\varepsilon(u_\varepsilon) \le \Lambda, \tag{1.4}$$

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega), \tag{1.5}$$

$$f_\varepsilon \to f \quad \text{weakly in } W^{1,p}(\Omega) \tag{1.6}$$

as $\varepsilon \to 0$. Then u is of bounded variations and takes only values in $\{-1, 1\}$. Moreover there exists a unique generalized mean curvature H of the 'phase boundary' $\Sigma := \Omega \cap \partial^*\{u = 1\}$ and

$$\sigma H = f \quad \text{holds } \mathcal{H}^{n-1}\text{-almost everywhere on } \Sigma. \tag{1.7}$$

Here $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds$ is the surface tension coefficient and the sign of H is chosen positive for spherical $\{u = 1\}$.

This Theorem uses generalized formulations for the phase boundary and the mean curvature. The notion of mean curvature is based on a measure-theoretic approach and was introduced by the first author in [31]. We refer to the appendix A for the exact definition.

Remark 1.2. For a sequence $(u_\varepsilon)_{\varepsilon>0}$ that satisfies the uniform energy bound (1.4) and a sequence $(f_\varepsilon)_{\varepsilon>0}$ that is uniformly bounded in $W^{1,p}(\Omega)$ there exists a subsequence $\varepsilon \to 0$ such that (1.5), (1.6) hold. Besides these uniform bounds no other conditions, such as energy minimality, are required to apply Theorem 1.1. For this reason the result is relevant to a large class of stationary and time-dependent problems. In section 7 we use our results to characterize the limit of stationary points of the Cahn-Hilliard functional (1.1) and to prove the convergence of stationary points in a model for block-copolymers.

The assumption (1.6) on the chemical potentials f_ε is still restrictive. We conjecture that the (weak) convergence of f_ε in $W^{1,p}(\Omega)$ with $p > n/2$ would suffice to conclude (1.7). However, our techniques yet require the continuity of f, which is ensured only if $p > n$. For the Cahn–Hilliard equation for example the natural regularity of the chemical potential is $W^{1,2}(\Omega)$ in space. Hence, our result does not apply in this case.

1.3. **Related results and main techniques.** Since the fundamental work of Modica and Mortola [22, 21] on the convergence of E_ε to the area functional the relation between their first variations has drawn attention. Modica [21] and Sternberg [40] proved that minimizers of E_ε under a volume constraint converge to area-minimizing hypersurfaces with an integral constraint. Luckhaus–Modica [19] then showed that the Lagrange-multipliers associated with the volume constraint converge to the constant mean curvature of the limiting hypersurface. Ilmanen [17] considered the corresponding L^2-gradient flows and proved the convergence of the Allen–Cahn equation to the mean-curvature flow, in the varifold formulation of Brakke [5]. Convergence of various other phase field problems to the corresponding sharp interface models have been shown either formally or rigorously [8, 1, 27, 6, 18, 9, 38], sometimes in quite involved weak formulations.

The second author considered, partly in joint-work with Hutchinson, the convergence of diffuse interface approximations of the Gibbs–Thomson law, under different assumptions on the chemical potential [16, 41, 42]. However, the Gibbs–Thomson relation is only verified in an (in some respect unsatisfactory) multiplicity-dependent formulation, see (1.9) and the discussion below.

Schätzle [35] considered a sequence of hypersurfaces with mean curvature given by a Sobolev function in the ambient space and obtained that the Gibbs–Thomson law holds in the limit in a rather clean varifold formulation. In [35] the chemical potentials f_ε in (1.6) need only to converge in a Sobolev space $f \in W^{1,p}(\Omega)$ with $p > n/2$, *c.f.* Remark 1.2.

Geometric Measure Theory provides suitable generalized formulations in spaces that allow for the compactness of approximations. Luckhaus–Modica [19] and Luckhaus–Sturzenhecker [20] introduced a weak formulation of the Gibbs–Thomson law (1.3) for characteristic functions of bounded variation. This formulation is rather natural and has the advantage of being based directly on the phase function. However, justifying the Gibbs–Thomson law in the limit of approximations requires the additional assumption that no cancellation of (diffuse) interfaces occurs. Unfortunately, this property does in general not hold [34].

To master such cancellations Ilmanen [17] used a *varifold-approach*. He considered the limit of the *diffuse surface-area measures* (*energy measures*)

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\,\mathcal{L}^n. \tag{1.8}$$

The idea behind is that this limit makes information visible that is lost in the limit of the phase fields: Where cancellation of the approximate phase boundaries occurs the limit μ of the measures μ_ε carries a higher multiplicity. The support of μ eventually extends the limit phase boundary by *hidden boundaries*. Showing that the limit measure is in fact given as a integer-rectifiable varifold with a weak mean curvature vector, the Gibbs–Thomson law can be verified in a varifold formulation.

This strategy was used for various problems by Chen [9], Soner [39], Hutchinson–Tonegawa [16], Tonegawa [41, 42], and others. However, in none of these papers the problem of higher multiplicity was completely solved. Typically the convergence of the diffuse phase fields and the diffuse surface-area measures μ_ε is shown and the rectifiability of the limit μ as well as the existence of a weak mean curvature H_μ is obtained. Still, the Gibbs–Thomson law holds only in a multiplicity-dependent formulation

$$H_\mu \,=\, (N\sigma)^{-1} f \quad \mathcal{H}^{n-1}\text{-almost everywhere on } \partial^*\{u=1\}, \tag{1.9}$$

where N is the density function of the measure μ. This formulation is for two reason unsatisfactory: First the Gibbs–Thomson law should be satisfied by the *phase boundary* rather than by the (in view of the applications) 'obscure' measure μ. Secondly, the density function N should not affect the Gibbs–Thomson law. To prove the full results (1.9) has to be complemented by

$$f \,=\, H_\mu \,=\, 0 \quad \mathcal{H}^{n-1}\text{-almost everywhere on } \partial^*\{u=1\}\cap\{N>1\}. \tag{1.10}$$

In recent years progress has been made on this issue. Schätzle [35] proves the Gibbs–Thomson law in the limit of an approximation by hypersurfaces: There the weak mean curvature H_μ of the limit measure μ satisfies (1.3) and (1.10). It was then shown by the first author [31] that H_μ is in fact a property of the phase boundary $\partial^*\{u=1\}$, see Appendix A. This is crucial in order to apply the (stationary) convergence result [35] to evolution problems [31, 32].

The higher-multiplicity problem is even more challenging in the context of the sharp interface limit of diffuse approximations, due to the singular nature of this limit process. The three main ingredients of our proof are first an earlier result of the second author [42] on the convergence of certain phase field equations with chemical potential. This ensures rectifiability, existence of a weak mean curvature with appropriate regularity and the multiplicity-dependent Gibbs–Thomson relation (1.9). The second ingredient is the fine local analysis of Schätzle [36] on rectifiable measures with sufficiently regular weak mean curvature. The third important argument is a comparison principle for the phase fields u_ε and diffuse approximations of suitably constructed comparison graphs.

1.4. **Organization of the paper.** In the next section we will precisely formulate our assumptions and introduce some notations. Section 3 states our main result. A localization step in Section 4 prepares a contradiction argument that we will use in Section 5 to prove our main Theorem 3.2. We first assume that a certain comparison principle, which is given in Proposition 6.11, holds. Section 6 is then devoted to the proof of this Proposition. Finally we give in Section 7 two applications of Theorem 1.1 and we recall in the appendix the definition of a generalized mean curvature for phase boundaries that we will use.

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2. Notations and assumptions

We state first all assumptions and definitions, including those already appeared in the introduction.

Assumption 2.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz-boundary and the standard double-well potential W given by*

$$W(r) := \frac{1}{4}(1-r^2)^2.$$

We define an energy functional E_ε on $W^{1,2}(\Omega)$,

$$E_\varepsilon(u) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\,dx. \tag{2.1}$$

Suppose $p > n$ and let sequences $(u_\varepsilon)_{\varepsilon>0} \subset W^{3,p}(\Omega)$ and $(f_\varepsilon)_{\varepsilon>0} \subset W^{1,p}(\Omega)$ be given such that

$$E_\varepsilon(u_\varepsilon) \le \Lambda \quad \text{for all } \varepsilon > 0, \tag{2.2}$$

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon) = f_\varepsilon \quad \text{in } \Omega. \tag{2.3}$$

Assume further that

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega), \qquad u \in BV(\Omega,\{-1,1\}), \tag{2.4}$$

$$f_\varepsilon \to f \quad \text{weakly in } W^{1,p}(\Omega). \tag{2.5}$$

We may generalize W to be any C^3-function with two non-degenerate minima and one local maximum, so that the results in [42] apply.

We next associate *diffuse surface-area measures* and appropriate varifolds to the functions u_ε.

Definition 2.2. *For u_ε we define Radon-measures μ_ε on Ω,*

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n, \tag{2.6}$$

and $(n-1)$-varifolds V_ε on $G^{n-1}(\Omega)$,

$$V_\varepsilon(\zeta) = \int_{G^{n-1}(\Omega)} \zeta(x,S)\,dV_\varepsilon(x,S) := \int_\Omega \zeta(x,\nu_\varepsilon^\perp(x))\,d\mu_\varepsilon(x) \tag{2.7}$$

for all $\zeta \in C_c^0(G^{n-1}(\Omega))$, where

$$\nu_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \text{ if } \nabla u_\varepsilon \neq 0, \qquad \nu_\varepsilon = (1,0,\dots,0)^T \text{ otherwise.}$$

Remark 2.3. By the Sobolev embedding Theorem, (2.5), and $p > n$ it follows that

$$f_\varepsilon, f \in C^{0,\alpha}(\overline{\Omega}), \tag{2.8}$$

$$f_\varepsilon \to f \quad \text{in } C^{0,\beta}(\overline{\Omega}) \tag{2.9}$$

for $\alpha := 1 - n/p$, all $0 \le \beta < \alpha$, and a subsequence $\varepsilon \to 0$.

Moreover, by (2.2) there exists a subsequences $\varepsilon \to 0$ and a Radon-measure μ on Ω such that

$$\mu_\varepsilon \to \mu \quad \text{as Radon-measures on } \Omega. \tag{2.10}$$

Here and in the following we often do not relabel subsequences. In particular we assume from now on that (2.9), (2.10) hold for the whole sequence $\varepsilon \to 0$.

Finally we define the mean-curvature operator H for graphs: for $p \in \mathbb{R}^{n-1}$, $X \in \mathcal{S}(n-1)$ set

$$\mathsf{H}(p, X) := (1+|p|^2)^{-\frac{3}{2}} \Big(\operatorname{tr} X + |p|^2 \big(Id - \frac{p}{|p|} \otimes \frac{p}{|p|} \big) : X \Big).$$

3. Statement of results

The first conclusion we draw is a direct consequence of previous results of the second author.

Theorem 3.1 ([42])**.** *Let Assumption 2.1 hold, let μ satisfy* (2.10)*, and set*

$$\sigma := \int_{-1}^{1} \sqrt{\frac{1}{2} W(s)}\, ds. \tag{3.1}$$

Then $(2\sigma)^{-1}\mu$ is $(n-1)$-integer-rectifiable,

$$\theta^{(n-1)}(\mu, \cdot) = N(\cdot) 2\sigma, \quad N \text{ integer-valued.}$$

Moreover μ has weak mean curvature $H_\mu \in L^\infty(\mu)$, and

$$H_\mu = \frac{f}{N(\cdot)\sigma} \nu \tag{3.2}$$

holds μ-almost everywhere, where $\nu = \frac{\nabla u}{|\nabla u|}$ on $\partial^* \{u = 1\}$ and $\nu = 0$ elsewhere.

Proof. See [42]. □

Our main results are summarized in the following theorem.

Theorem 3.2. *Let Assumption 2.1 hold, let μ satisfy* (2.10)*, and let N denote the multiplicity function of μ as in Theorem 3.1. Then*

(1) μ-almost everywhere in $\{N(\cdot) \ge 3 \text{ odd}\}$

$$H_\mu = 0, \quad f = 0. \tag{3.3}$$

(2) μ-almost everywhere in $\{N(\cdot) \ge 2 \text{ even}\}$

$$H_\mu = 0, \qquad f \le 0 \text{ in } \{u = 1\} \cap \operatorname{supp}(\mu), \quad f \ge 0 \text{ in } \{u = -1\} \cap \operatorname{supp}(\mu). \tag{3.4}$$

(3) the function $H : \partial^* \{u = 1\} \to \mathbb{R}$ defined as

$$H := H_\mu \cdot \frac{\nabla u}{|\nabla u|} \tag{3.5}$$

is the generalized mean curvature of $\partial^* \{u = 1\}$ in the sense of Definition A.2.

(4) Finally

$$\sigma H = f \tag{3.6}$$

holds $\mathcal{H}^{n-1}$-almost everywhere on $\partial^* \{u = 1\} \cap \Omega$.

4. Localization

In this section we show that we can restrict ourselves to a 'generic' local situation, where the support of μ is well described in terms of graphs. We then apply a result of Schätzle [36] that gives a fine description of the varifold μ.

Lemma 4.1. *It is sufficient to prove* (3.3), (3.4) *for μ-almost all* generic points, *that are those points $x_0 \in \Omega$ satisfying*

$$T_{x_0}\mu \text{ exists}, \tag{4.1}$$

$$N_0 := N(x_0) \in \mathbb{N}\,, \tag{4.2}$$

$$\theta\big(\mu, \{N(\cdot) = N_0\}, x_0\big) = 1. \tag{4.3}$$

In addition we may assume without loss of generality that

$$x_0 \in \operatorname{supp}(\mu) \cap \partial^*\{u = 1\} \quad \text{if } N(x_0) \text{ is odd}, \tag{4.4}$$

$$\lim_{\varrho\to 0}\Big(\sup\big\{\varrho^{-1}\operatorname{dist}(x, T_{x_0}\mu) : x \in \operatorname{supp}(\mu) \cap B^n_\varrho(x_0)\big\}\Big) = 0. \tag{4.5}$$

Proof. We show that (4.1)-(4.5) hold μ-almost everywhere in $\{N(\cdot) \geq 2\}$.

Since μ is integer-rectifiable (4.1), (4.2) are satisfied μ-almost everywhere. Since the set $\{\theta^{n-1}(\mu, \cdot) = N_0\}$ is μ-measurable (4.3) holds for μ-almost all points. By [42, Theorem 1] $\mathcal{H}^{n-1}$-almost all $x \in \operatorname{supp}(\mu)$ with odd density $N(x)$ belong to $\partial^*\{u = 1\}$. Finally (4.5) follows from [37, Lemma 17.11]. □

We fix x_0 such that (4.1)-(4.5) hold. After applying a suitable translation and rotation we may assume that $x_0 = 0$ and

$$T_0\mu = \mathbb{R}^{n-1} \times \{0\}. \tag{4.6}$$

To apply a contradiction argument we assume (3.3), (3.4) to be false.

Assumption 4.2. *Suppose that*

$$N_0 \geq 2 \text{ and } \quad f(0) \neq 0. \tag{4.7}$$

By (2.8), (4.5) there exists $\varrho_0 > 0$ such that

$$|f - f(0)| \leq \frac{1}{27}|f(0)| \quad \text{on } \overline{B_{\varrho_0}(0)} \times [-5\varrho_0, 5\varrho_0], \tag{4.8}$$

$$\operatorname{supp}(\mu) \cap \big(B_{\varrho_0}(0) \times (-5\varrho_0, 5\varrho_0)\big) \subset B_{\varrho_0}(0) \times (-\varrho_0, \varrho_0). \tag{4.9}$$

We distinguish four cases depending on whether $f(0) < 0$ or $f(0) > 0$ and whether $u = 1$ or $u = -1$ in the region 'above' $\operatorname{supp}(\mu)$. In the following we consider the case that

$$f(0) > 0, \tag{4.10}$$

$$u(y,t) = 1 \quad \text{for } y \in B_{\varrho_0}(0), t > \varrho_0. \tag{4.11}$$

That implies that

$$u(y,t) = \begin{cases} -1 & \text{if } N_0 \text{ is odd}, \\ 1 & \text{if } N_0 \text{ is even} \end{cases} \qquad \text{for all } y \in B_{\varrho_0}(0), t < -\varrho_0. \tag{4.12}$$

The other cases can be treated analogously or follow from a symmetry argument. By (2.9) we obtain that

$$|f_\varepsilon - f| \leq \frac{1}{27} f(0) \quad \text{on } \overline{B_{\varrho_0}(0)} \times [-5\varrho_0, 5\varrho_0] \tag{4.13}$$

for all $\varepsilon > 0$ sufficiently small.

In the next step we apply a result of Schätzle [36] on the local structure of the measure μ. First we need some definitions.

Definition 4.3. *We define the upper and lower height-functions* $\varphi_+, \varphi_- : B^{n-1}_{\varrho_0}(0) \to [-\infty, \infty]$,

$$\varphi_+(y) := \sup\big\{t \in (-5\varrho_0, 5\varrho_0) : (y,t) \in \mathrm{supp}(\mu) \cap \big(B^{n-1}_{\varrho_0}(0) \times \mathbb{R}\big)\big\}, \tag{4.14}$$

$$\varphi_-(y) := \inf\big\{t \in (-5\varrho_0, 5\varrho_0) : (y,t) \in \mathrm{supp}(\mu) \cap \big(B^{n-1}_{\varrho_0}(0) \times \mathbb{R}\big)\big\}, \tag{4.15}$$

with the convention that the supremum over an empty set is $-\infty$ *and the infimum over an empty set is* $+\infty$.

Moreover we set

$$F_+(y) := \begin{cases} \frac{f(y,\varphi_+(y))}{N(y,\varphi_+(y))\sigma} & \text{if } N(y,\varphi_+(y)) \text{ is odd,} \\ 0 & \text{if } N(y,\varphi_+(y)) \text{ is even,} \end{cases} \tag{4.16}$$

and

$$F_-(y) := \begin{cases} \frac{f(y,\varphi_-(y))}{N(y,\varphi_-(y))\sigma} & \text{if } N(y,\varphi_-(y)) \text{ is odd,} \\ 0 & \text{if } N(y,\varphi_-(y)) \text{ is even.} \end{cases} \tag{4.17}$$

Proposition 4.4. *The upper height-functions* $\varphi_\pm$ *are twice approximately differentiable* $\mathcal{L}^{n-1}$*-almost everywhere in* $\{\varphi_\pm \in \mathbb{R}\}$ *with*

$$-\mathsf{H}(\nabla\varphi_+, D^2\varphi_+) = -F_+, \tag{4.18}$$

$$-\mathsf{H}(\nabla\varphi_-, D^2\varphi_-) = -(-1)^{1+N_0}F_-, \tag{4.19}$$

Moreover, for all $\frac{n-1}{2} < s < \infty$ *the upper height-function* φ_+ *is an* $W^{2,s}$*-viscosity subsolution of*

$$-\mathsf{H}(\nabla\varphi_+, D^2\varphi_+) \le -F_+ \tag{4.20}$$

and the lower height-function φ_- *is a* $W^{2,s}$*-viscosity supersolution of*

$$-\mathsf{H}(\nabla\varphi_-, D^2\varphi_-) \ge -(-1)^{1+N_0}F_- \tag{4.21}$$

Proof. Since $H_\mu \in L^\infty(\mu)$ we obtain from [36, Theorem 6.1] that

$$H_\mu(\cdot,\varphi_+(\cdot)) = \nabla\cdot\Big(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\Big)\frac{1}{\sqrt{1+|\nabla\varphi_+|^2}}\begin{pmatrix}-\nabla\varphi_+\\ 1\end{pmatrix} \tag{4.22}$$

$\mathcal{L}^{n-1}$-almost everywhere in $\{\varphi_+ \in \mathbb{R}\}$ and that φ_+ is for all $\frac{n-1}{2} < s < \infty$ a $W^{2,s}$-viscosity subsolution of

$$-\nabla\cdot\Big(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\Big) \le -H_\mu(\cdot,\varphi_+(\cdot))\frac{1}{\sqrt{1+|\nabla\varphi_+|^2}}\begin{pmatrix}-\nabla\varphi_+\\ 1\end{pmatrix}. \tag{4.23}$$

From (3.2), (4.22) we deduce that

$$\frac{f(\cdot,\varphi_+(\cdot))}{N(\cdot,\varphi_+(\cdot))\sigma}\nu(\cdot,\varphi_+(\cdot)) = \mathsf{H}(\nabla\varphi_+, D^2\varphi_+)\frac{1}{\sqrt{1+|\nabla\varphi_+|^2}}\begin{pmatrix}-\nabla\varphi_+\\ 1\end{pmatrix}. \tag{4.24}$$

Next we observe that (4.11) implies

$$\nu(y,\varphi_+(y)) = \frac{\nabla u}{|\nabla u|}(y,\varphi_+(y)) = \big(1+|\nabla\varphi_+|^2\big)^{-\frac{1}{2}}(-\nabla\varphi_+(y),1)^T \tag{4.25}$$

for $\mathcal{L}^{n-1}$ almost all $y \in \{\varphi_+ \in \mathbb{R}\}$ such that $(y, \varphi_+(y)) \in \partial^*\{u = 1\}$, and $\nu = 0$ otherwise. Since up to a μ-nullset $\partial^*\{u = 1\} = \{N \geq 1 \text{ odd}\}$ we obtain from (4.16) and (4.24), (4.25) that (4.18) holds. (4.20) follows by the same arguments. To obtain (4.19), (4.21) we observe that

$$\nu(y, \varphi_-(y)) \,=\, \frac{\nabla u}{|\nabla u|}(y, \varphi_-(y)) \,=\, (-1)^{1+N_0}\big(1 + |\nabla\varphi_-|^2\big)^{-\frac{1}{2}}(-\nabla\varphi_-(y), 1)^T \tag{4.26}$$

and we proceed as above. □

We choose below a 'good point' for which we derive a contradiction to Assumption 4.2. Before, we need another definition.

Definition 4.5. *We say that a function ψ has a second-order Taylor expansion at a point $y_1 \in \mathbb{R}^{n-1}$ if there exist $p \in \mathbb{R}^{n-1}, X \in \mathcal{S}(n-1)$ such that*

$$\psi(y) - \psi(y_1) - p \cdot (y - y_1) - \frac{1}{2}(y - y_1) \cdot X(y - y_1) \,=\, o(|y - y_1|^2). \tag{4.27}$$

We then set $\nabla\psi(y_1) := p$, $D^2\psi(y_1) := X$.

Lemma 4.6. *There exists a point $y_1 \in B_{\varrho_0}(0)$ such that*

$$\varphi_+(y_1) \,=\, \varphi_-(y_1), \tag{4.28}$$

$$x_1 \,:=\, (y_1, \varphi_\pm(y_1)) \text{ is a generic point}, \tag{4.29}$$

$$\theta^{n-1}(\mu, x_1) \,=\, N_0, \tag{4.30}$$

$$\varphi_\pm \text{ have a second-order Taylor expansion at } y_1, \tag{4.31}$$

$$-\,\mathsf{H}(\nabla\varphi_\pm(y_1), D^2\varphi_\pm(y_1)) \,=\, -F_\pm(y_1). \tag{4.32}$$

Proof. Since the weak mean curvature H_μ belongs to $L^\infty(\mu)$ by (3.2) and since $0 \in \mathbb{R}^n$ is a generic point we can apply [35, Lemma 3.4] (see also Step 3 in the proof of [32, Lemma 3.2]) and obtain that the set

$$\Sigma_0 \,:=\, \Big\{x = (y, \varphi_\pm(y)) : y \in B^{n-1}_{\varrho_0}(0) \cap \{\varphi_+ = \varphi_-\},\\ x \in \operatorname{supp}(\mu) \text{ is generic, } N(x) \,=\, N_0\Big\}$$

has full density with respect to μ in $0 \in \mathbb{R}^n$. This property was essentially deduced from a tilted version of Brakke's Lipschitz Approximation Theorem. The curvature bound ensures a strong control on the approximations, see [35, 31] for the details. From the Coarea Formula we then deduce that (4.28)-(4.30) holds in a set with full $\mathcal{L}^{n-1}$-density in $0 \in \mathbb{R}^{n-1}$. Finally, (4.31), (4.32) are satisfied $\mathcal{L}^{n-1}$ almost everywhere in $\{\varphi_+ = \varphi_-\}$ by [36, Proposition 4.1] and (4.18), (4.19). □

5. Proof of Theorem 3.2

We fix $y_1 \in B_{\varrho_0}(0)$ such that (4.28)-(4.32) hold and consider the second order Taylor approximation of $\varphi_\pm$ at y_1,

$$P_1(y) \,:=\, \varphi_\pm(y_1) + \nabla\varphi_\pm(y_1) \cdot (y - y_1) + \frac{1}{2}(y - y_1) \cdot D^2\varphi_\pm(y_1)(y - y_1). \tag{5.1}$$

From (4.31) we then deduce that

$$|P_1(y) - \varphi_\pm(y)| \,=\, o(|y - y_1|^2). \tag{5.2}$$

Lemma 5.1. *For all $\omega > 0$ there is $\varrho_1 > 0$ such that*

$$B_{\varrho_1}(y_1) \subset B_{\varrho_0}(0), \tag{5.3}$$

$$\big|P_1(y) - \varphi_+(y)\big| < \omega |y - y_1|^2 \quad \text{on } B_{\varrho_1}(y_1) \tag{5.4}$$

and such that for any $0 < \varrho < \varrho_1$ there exists a unique solution $\psi \in C^\infty(\overline{B_\varrho(y_1)})$ of

$$-\mathsf{H}(\nabla \psi, D^2 \psi) = -\frac{2f(y_1)}{3\sigma} \qquad \text{in } B_\varrho(y_1), \tag{5.5}$$

$$\psi(y) = P_1(y) + \omega \varrho^2 \quad \text{on } \partial B_\varrho(y_1). \tag{5.6}$$

Proof. Since $y_1 \in B_{\varrho_0}(0)$ and by (5.2) for any $\varrho_1 > 0$ sufficiently small the properties (5.3), (5.4) hold. Since P_1 is smooth and since the right-hand side of equation (5.5) is constant we deduce from [13, Theorem 16.9] that for all

$$\varrho \le \varrho_1, \quad \varrho_1 = \varrho_1(n, \sigma, f(y_1))$$

a unique solution $\psi \in C^{2,\gamma}(\overline{B_\varrho(y_1)})$, $0 < \gamma < 1$, of (5.5), (5.6) exists. The higher regularity of ψ follows from standard elliptic theory and the smoothness of the data in (5.5), (5.6). □

The next Proposition is the heart of the contradiction argument. It relies on the fact that the approximations u_ε behave as if the curvature of the limit interface is given by f/σ rather than by $f/(N\sigma)$.

Proposition 5.2. *Let ψ be as in Lemma 5.1. Then*

$$\psi \ge \varphi_+ \quad \text{in } B_\varrho(y_1). \tag{5.7}$$

The proof of this Proposition uses a comparison between u_ε and approximations v_ε of $2\mathcal{X}_E - 1$, where E is the region above the graph of ψ. We postpone this proof to section 6 and continue the proof of Theorem 3.2.

Lemma 5.3. *For all $0 < \omega < 1$ there exists $\tilde\varrho_1 > 0$ such that for all $0 < \varrho < \tilde\varrho_1$ the function*

$$\eta(y) := P_1(y) - 2\omega(\varrho^2 - |y - y_1|^2) \tag{5.8}$$

satisfies for all $y \in B_\varrho(y_1)$

$$-\mathsf{H}(\nabla \eta, D^2 \eta) \ge -F_\pm(y_1) - 5(n-1)\omega. \tag{5.9}$$

Proof. We compute that for $y \in B_\varrho(y_1)$

$$\begin{aligned} |\nabla \eta(y) - \nabla \varphi_\pm(y_1)| &= |D^2 \varphi_\pm(y_1)(y - y_1) + 4\omega(y - y_1)| \\ &\le \big(|D^2 \varphi_\pm(y_1)| + 4\big)\varrho, \\ |D^2 \eta(y)| &\le |D^2 \varphi_\pm(y_1)| + 4. \end{aligned}$$

Hence, we can choose $\tilde\varrho_1 = \tilde\varrho_1(\nabla \varphi_\pm(y_1), D^2 \varphi_\pm(y_1), \omega)$ such that

$$|\mathsf{H}(\nabla \eta, D^2 \eta) - \mathsf{H}(\varphi_\pm(y_1), D^2 \eta)| \le (n-1)\omega \quad \text{in } B_\varrho(y_1)$$

for all $0 < \varrho < \tilde\varrho_1$. This implies that

$$\begin{aligned} -\mathsf{H}(\nabla \eta, D^2 \eta) &\ge -\mathsf{H}(\nabla \varphi_\pm(y_1), D^2 \varphi_\pm(y_1)) - 4\omega \mathsf{H}(\nabla \varphi_\pm(y_1), Id) - (n-1)\omega \\ &\ge -\mathsf{H}(\nabla \varphi_\pm(y_1), D^2 \varphi_\pm(y_1)) - 5(n-1)\omega, \end{aligned}$$

where we have used that $\mathsf{H}(p, Id) \le n - 1$ for all $p \in \mathbb{R}^{n-1}$. □

Proof of Theorem 3.2. Choose $0 < \omega < \frac{f(y_1)}{30(n-1)\sigma}$ and $0 < \varrho < \min(\varrho_1, \tilde{\varrho}_1)$. Let ψ, η be the functions constructed in Lemma 5.1 and Lemma 5.3. We then obtain from (5.5), (5.9), the definition of $F_\pm$, (4.7), and $f(y_1) > 0$ that

$$\begin{aligned} -\mathsf{H}(\nabla\psi, D^2\psi) + \mathsf{H}(\nabla\eta, D^2\eta) &\leq -\frac{2f(y_1)}{3\sigma} + \frac{f(y_1)}{N(y_1, \varphi_\pm(y_1))\sigma} + 5(n-1)\omega \\ &\leq -\frac{2f(y_1)}{3\sigma} + \frac{f(y_1)}{2\sigma} + 5(n-1)\omega \\ &< 0. \end{aligned} \tag{5.10}$$

Since $|\nabla\psi|, |\nabla\eta|$ are uniformly bounded the maximum principle [13, Theorem 10.1] implies that $\psi - \eta$ has no interior maximum. In particular,

$$\begin{aligned} \psi(y_1) - \varphi_\pm(y_1) + 2\omega\varrho^2 &= \psi(y_1) - \eta(y_1) \\ &\leq \sup_{\partial B_\varrho(y_1)} (\psi - \eta) = \omega\varrho^2 \end{aligned}$$

and we deduce that

$$\psi(y_1) - \varphi_\pm(y_1) < 0,$$

which is a contradiction to (5.7).

This shows by Assumption 4.2 and (4.10), (4.11) that

$$f \leq 0 \quad \text{if } N_0 \geq 2 \text{ and } u = 1 \text{ in } \{t > \varphi_+(y)\}. \tag{5.11}$$

By a symmetry argument $u \mapsto -u$ it follows that

$$f \geq 0 \quad \text{if } N_0 \geq 2 \text{ and } u = -1 \text{ in } \{t > \varphi_+(y)\}. \tag{5.12}$$

As we explain in Remark 5.4 below we obtain also

$$f \geq 0 \quad \text{if } N_0 \geq 3 \text{ is odd and } u = 1 \text{ in } \{t > \varphi_+(y)\} \tag{5.13}$$

and, again by symmetry, that

$$f \leq 0 \quad \text{if } N_0 \geq 3 \text{ is odd and } u = -1 \text{ in } \{t > \varphi_+(y)\}. \tag{5.14}$$

Putting together (5.11)-(5.14) and using (3.2) we deduce the conclusion (1), (2) of Theorem 3.2. The conclusion (3) follows from Proposition A.1, and the statement (4) is deduced from (1), (3) and (3.2). □

Remark 5.4. In the case that $f(0) < 0$ and $u = 1$ 'above' φ_+ one considers for suitably small $\varrho > 0$ the solution $\tilde{\psi}$ of

$$\begin{aligned} -\mathsf{H}(\nabla\tilde{\psi}, D^2\tilde{\psi}) &= -\frac{2f(y_1)}{3\sigma} \qquad \text{in } B_\varrho(y_1), \\ \tilde{\psi}(y) &= P_1(y) - \omega\varrho^2 \quad \text{on } \partial B_\varrho(y_1) \end{aligned}$$

and the function $\tilde{\eta}$,

$$\tilde{\eta}(y) := P_1(y) + 2\omega(\varrho^2 - |y - y_1|^2)$$

To derive a contradiction the corresponding statement to Proposition 5.2 is needed, that is

$$\tilde{\psi} \leq \varphi_- \quad \text{in } B_\varrho(y_1).$$

In the case that $N_0 \geq 1$ this property can be proved in the same way as we will prove Proposition 5.2: One constructs smooth approximations v_ε of the function $2\mathcal{X}_{\{t>\tilde{\psi}(y)\}} - 1$ and uses a comparison principle to obtain $u_\varepsilon \leq v_\varepsilon$. However, these

arguments do not apply if $N_0 \geq 2$ is even, since in that case $u_\varepsilon \approx 1$ is larger than $2\mathcal{X}_{\{t>\tilde{\psi}(y)\}} - 1$ in the region 'below' $\tilde{\psi}$.

6. Proof of Proposition 5.2

Assume that (5.7) does not hold, that is

$$\sup_{B_\varrho(y_1)} (\varphi_+ - \psi) > 0. \tag{6.1}$$

By (5.4) and (5.6)

$$\varphi_+ < \psi \quad \text{on } \partial B_\varrho(y_1). \tag{6.2}$$

Since ψ is continuous and φ_+ is upper-semicontinuous there exist $0 < \varrho_3 < \varrho_2 < \varrho$ such that

$$\sup_{B_{\varrho_3}(y_1)} (\varphi_+ - \psi) > 0, \tag{6.3}$$

$$\varphi_+ < \psi \quad \text{on } B_\varrho(y_1) \setminus B_{\varrho_3}(y_1). \tag{6.4}$$

As explained before we will use that u_ε behaves as if the curvature of the sharp interface limit is given by f/σ, instead of $f/(N\sigma)$. In a first step we construct functions v_ε such that

$$-\varepsilon\Delta v_\varepsilon + \frac{1}{\varepsilon} W'(v_\varepsilon) \leq \frac{7}{9} f(y_1), \tag{6.5}$$

$$v_\varepsilon \to 2\mathcal{X}_{\{(y,t):t>\psi(y)\}} - 1. \tag{6.6}$$

In the second step we will apply a comparison principle to $u_\varepsilon, v_\varepsilon$ to obtain a contradiction in the limit $\varepsilon \to 0$.

6.1. **Construction of v_ε.** The two ingredients to construct v_ε are a *modified distance* function from graph(ψ) and the optimal profile and first order-correction of the one-dimensional minimisation problem associated to the Cahn–Hilliard functional.

Definition 6.1. *We define*

$$M := \operatorname{graph}\big(\psi \lfloor B_\varrho(y_1)\big)$$

and denote by $\mathsf{d} := \operatorname{dist}(M, \cdot)$ *the signed distance function from M, taken positive in the region 'above' M. Moreover we let $\Pi_M : \mathbb{R}^n \to M$ be the orthogonal projection onto M and $(\kappa_i)_{i=1,\dots,n-1}$ the principal curvatures of M. Finally we define for $x \in \mathbb{R}^n$, $(y,t) = \Pi_M(x)$*

$$\tilde{\kappa}_i(x) := \kappa_i(y,t),$$

which is well-defined in a neighborhood of M.

Remark 6.2. Since ψ is smooth we deduce that M is a smooth hypersurface and that there exists $\delta > 0$, $\delta = \delta(\|\psi\|_{C^2(B_\varrho(y_1))}, \varrho_2)$ such that the distance function d is unique and smooth in a neighborhood

$$G = \big\{x \in B_{\varrho_2}(y_1) \times (-5\varrho_0, 5\varrho_0) : |\mathsf{d}(x)| < \delta\big\}. \tag{6.7}$$

Moreover

$$\Delta \mathsf{d} = \sum_{i=1}^{n-1} \tilde{\kappa}_i + O(\mathsf{d}) = \frac{2f(y_1)}{3\sigma} + O(\mathsf{d}). \tag{6.8}$$

holds in G [13, Lemma 14.17].

We turn to the optimal profile for the one-dimensional minimisation in the Cahn–Hilliard energy.

Remark 6.3. Let $\phi_0 : \mathbb{R} \to [-1, 1]$ be the *optimal profile*, that is the solution of

$$-\phi_0'' + W'(\phi_0) = 0, \tag{6.9}$$

$$\phi_0(-\infty) = -1, \quad \phi_0(+\infty) = 1, \tag{6.10}$$

and let $\phi_1 : \mathbb{R} \to \mathbb{R}$ be the first order correction (see [26]),

$$-\phi_1'' + W''(\phi_0)\phi_1 = \phi_0' + \sigma, \tag{6.11}$$

$$\phi_1(\pm\infty) = \frac{\sigma}{W''(\pm 1)}. \tag{6.12}$$

Since the distance function d is smooth only in a neighborhood of graph(ψ) we have to modify the distance function.

Definition 6.4. *For $\varepsilon > 0$ we choose $\delta(\varepsilon) > 0$ such that*

$$\delta(\varepsilon) \to 0, \ \frac{\delta(\varepsilon)}{\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0 \tag{6.13}$$

and such that the conditions

$$\frac{1}{\delta(\varepsilon)}\phi_0'\big(\varepsilon^{-1}\delta(\varepsilon)\big), \ \frac{1}{\varepsilon}\phi_0''\big(\varepsilon^{-1}\delta(\varepsilon)\big) \to 0 \quad \text{as } \varepsilon \to 0 \tag{6.14}$$

are satisfied. Moreover we choose smooth functions $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$, with

$$\beta_\varepsilon(r) = \begin{cases} r & \text{for} \quad |r| \le \frac{\delta(\varepsilon)}{3} \\ -\delta(\varepsilon) & \text{for} \quad r \le -2\delta(\varepsilon) \\ \delta(\varepsilon) & \text{for} \quad r \ge 2\delta(\varepsilon) \end{cases} \tag{6.15}$$

such that

$$0 \le \beta_\varepsilon' \le 1, \tag{6.16}$$

$$0 \ge \beta_\varepsilon'' \ge -\frac{3}{\delta(\varepsilon)}. \tag{6.17}$$

We then define the modified distance functions d_ε*,*

$$d_\varepsilon(x) := \beta_\varepsilon(\mathsf{d}(x)). \tag{6.18}$$

Remark 6.5. We observe that by (6.13)

$$\{|\mathsf{d}| \le 2\delta(\varepsilon)\} \subset G \tag{6.19}$$

for all $\varepsilon \le \varepsilon_0(\|\psi\|_{C^2(B_\varrho(y_1))})$ and deduce that d_ε is smooth for sufficiently small $\varepsilon > 0$. We compute that

$$\nabla d_\varepsilon = \beta_\varepsilon'(\mathsf{d})\nabla \mathsf{d}, \tag{6.20}$$

$$\begin{aligned} \Delta d_\varepsilon &= \beta_\varepsilon''(\mathsf{d}) + \beta_\varepsilon'(\mathsf{d})\Delta \mathsf{d} \\ &= \beta_\varepsilon''(\mathsf{d}) + \beta_\varepsilon'(\mathsf{d})\Big(\frac{2f(y_1)}{3\sigma} + O(\mathsf{d})\Big), \end{aligned} \tag{6.21}$$

where we have used (6.8). For $\varepsilon < \varepsilon_0(\psi)$ we obtain that

$$\beta_\varepsilon' \circ \mathsf{d} = 0 \quad \text{in the set } \big\{|\mathsf{d}| > 2\delta(\varepsilon)\}\big\}$$

and we deduce from (4.8), (6.16), (6.17) that

$$|\Delta d_\varepsilon| \le C\big(1+\frac{1}{\delta(\varepsilon)}\big). \tag{6.22}$$

We are now ready to define v_ε.

Definition 6.6. *Let $\varepsilon_0 = \varepsilon_0(\|\psi\|_{C^2(B_\varrho(y_1))})$ be chosen such that (6.19) holds. We then define $v_\varepsilon : B_{\varrho_2}(y_1)\times\mathbb{R}\to\mathbb{R}$,*

$$v_\varepsilon(x) := \phi_0(\varepsilon^{-1}d_\varepsilon(x)) + \varepsilon\phi_1(\varepsilon^{-1}d_\varepsilon(x))\frac{2}{3\sigma}f(y_1). \tag{6.23}$$

6.2. **Comparison of $u_\varepsilon, v_\varepsilon$.**

6.2.1. *Subsolution property.* We are going to show that v_ε is a suitable subsolution of a (diffuse) constant curvature equation.

We first compute that, using (6.11),

$$\begin{aligned}-\varepsilon\Delta v_\varepsilon =& -\frac{1}{\varepsilon}\phi_0''(\varepsilon^{-1}d_\varepsilon)|\nabla d_\varepsilon|^2 - \phi_0'(\varepsilon^{-1}d_\varepsilon)\Delta d_\varepsilon\\ &+\frac{2f(y_1)}{3\sigma}\Big(-W''(\phi_0(\varepsilon^{-1}d_\varepsilon))\phi_1(\varepsilon^{-1}d_\varepsilon)+\phi_0'(\varepsilon^{-1}d_\varepsilon)+\sigma\Big)|\nabla d_\varepsilon|^2\\ &-\varepsilon\phi_1'(\varepsilon^{-1}d_\varepsilon)\frac{2f(y_1)}{3\sigma}\Delta d_\varepsilon\end{aligned} \tag{6.24}$$

and

$$\frac{1}{\varepsilon}W'(v_\varepsilon) = \frac{1}{\varepsilon}W'(\phi_0(\varepsilon^{-1}d_\varepsilon)) + W''(\phi_0(\varepsilon^{-1}d_\varepsilon))\frac{2f(y_1)}{3\sigma}\phi_1(\varepsilon^{-1}d_\varepsilon)+O(\varepsilon). \tag{6.25}$$

Using (6.22) we deduce from (6.24), (6.25) that

$$\begin{aligned}-\varepsilon\Delta v_\varepsilon+\frac{1}{\varepsilon}W'(v_\varepsilon) =& -\frac{1}{\varepsilon}\phi_0''\big(|\nabla d_\varepsilon|^2-1\big)-\phi_0'\big(\Delta d_\varepsilon-\frac{2f(y_1)}{3\sigma}|\nabla d_\varepsilon|^2\big)\\ &-\frac{2f(y_1)}{3\sigma}\phi_1W''(\phi_0)(|\nabla d_\varepsilon|^2-1)\\ &+\frac{2}{3}f(y_1)|\nabla d_\varepsilon|^2+o(\varepsilon\delta(\varepsilon)^{-1}).\end{aligned} \tag{6.26}$$

Proposition 6.7. *For all $0<\varepsilon<\varepsilon_0$, $\varepsilon_0=\varepsilon_0(\psi,W)$,*

$$-\varepsilon\Delta v_\varepsilon+\frac{1}{\varepsilon}W'(v_\varepsilon)\le\frac{7}{9}f(y_1). \tag{6.27}$$

Proof. We check (6.27) in the different regions.

Step1. In the region $\{|\mathsf{d}|\le\frac{1}{3}\delta(\varepsilon)\}$ holds $|\nabla d_\varepsilon|=1$ and we obtain from (6.21), (6.26) that

$$\begin{aligned}-\varepsilon\Delta v_\varepsilon+\frac{1}{\varepsilon}W'(v_\varepsilon) =& \frac{2}{3}f(y_1)+O(\mathsf{d})+o(\varepsilon\delta(\varepsilon)^{-1})\\ \le& \frac{2}{3}f(y_1)+O(\delta(\varepsilon))+o(\varepsilon\delta(\varepsilon)^{-1}).\end{aligned}$$

Therefore (6.27) holds for $\varepsilon>0$ sufficiently small.

STEP2. In $\{\mathsf{d} \geq 2\delta(\varepsilon)\}$ we obtain

$$\begin{aligned}
&-\varepsilon\Delta v_\varepsilon + \frac{1}{\varepsilon}W'(v_\varepsilon)\\
&= \frac{1}{\varepsilon}W'\Big(\phi_0(\varepsilon^{-1}\delta(\varepsilon)) + \varepsilon\phi_1(\varepsilon^{-1}\delta(\varepsilon))\frac{2}{3\sigma}f(y_1)\Big)\\
&= \frac{1}{\varepsilon}W'\big(\phi_0(\varepsilon^{-1}\delta)\big) + W''\big(\phi_0(\varepsilon^{-1}\delta)\big)\phi_1(\varepsilon^{-1}\delta)\frac{2}{3\sigma}f(y_1) + O(\varepsilon)\phi_1(\varepsilon^{-1}\delta). \qquad (6.28)
\end{aligned}$$

From (6.9), (6.14) and (6.12), (6.13) we deduce that (6.27) holds in $\{\mathsf{d} \geq 2\delta(\varepsilon)\}$ for sufficiently small $\varepsilon > 0$. By similar calculations we obtain (6.27) also in the region $\{\mathsf{d} \leq -2\delta(\varepsilon)\}$.

STEP3. Let us now consider the set $\{\frac{1}{3}\delta(\varepsilon) \leq \mathsf{d} \leq 2\delta(\varepsilon)\}$ and estimate the different terms in (6.26). We first obtain from (6.14) that in this region

$$-\frac{1}{\varepsilon}\phi_0''(\varepsilon^{-1}d_\varepsilon)\big(|\nabla d_\varepsilon|^2 - 1\big) \to 0 \quad \text{as } \varepsilon \to 0. \qquad (6.29)$$

Next we compute that, using (6.20), (6.21) and (6.17),

$$\begin{aligned}
&\Big| -\phi_0'(\varepsilon^{-1}d_\varepsilon)\big(\Delta d_\varepsilon - \frac{2f(y_1)}{3\sigma}|\nabla d_\varepsilon|^2\big)\Big|\\
&= \phi_0'(\varepsilon^{-1}d_\varepsilon)\Big|\beta_\varepsilon''(\mathsf{d}) + \beta_\varepsilon'(\mathsf{d})\Big(\frac{2f(y_1)}{3\sigma} + O(\mathsf{d})\Big) - \frac{2f(y_1)}{3\sigma}\beta_\varepsilon'(\mathsf{d})^2\Big|\\
&\leq \phi_0'((3\varepsilon)^{-1}\delta(\varepsilon))\Big(\frac{3}{\delta(\varepsilon)} + \frac{4f(y_1)}{3\sigma} + O(\delta(\varepsilon))\Big).
\end{aligned}$$

Hence, by (6.14)

$$-\phi_0'(\varepsilon^{-1}d_\varepsilon)\big(\Delta d_\varepsilon - \frac{2f(y_1)}{3\sigma}|\nabla d_\varepsilon|^2\big) \to 0 \text{ as } \varepsilon \to 0. \qquad (6.30)$$

Finally we observe that in $\{\frac{1}{3}\delta(\varepsilon) \leq \mathsf{d} \leq 2\delta(\varepsilon)\}$

$$a_\varepsilon := \frac{\phi_1(\varepsilon^{-1}d_\varepsilon)W''(\phi_0(\varepsilon^{-1}d_\varepsilon))}{\sigma} = 1 + o(1) \qquad (6.31)$$

by (6.12), (6.13) and we deduce for the last line in (6.26) that

$$\frac{2}{3}f(y_1)\Big((-a_\varepsilon + 1)|\nabla d_\varepsilon|^2 + a_\varepsilon\Big) = \frac{2}{3}f(y_1) + o(1). \qquad (6.32)$$

We obtain from (6.26) and (6.29), (6.30), (6.32) that (6.27) holds in $\{\frac{1}{3}\delta(\varepsilon) \leq \mathsf{d} \leq 2\delta(\varepsilon)\}$ for all $\varepsilon > 0$ sufficiently small. By similar considerations we prove (6.27) also in $\{-\delta(\varepsilon) \leq \mathsf{d} \leq -\frac{1}{3}\delta(\varepsilon)\}$. □

6.2.2. *Comparison in the bulk regions.*

Lemma 6.8. *As $\varepsilon \to 0$*

$$u_\varepsilon \to 1 \quad \textit{uniformly on each compact subset of } \{t > \varphi_+(y)\}, \qquad (6.33)$$

$$u_\varepsilon \to (-1)^{N_0} \quad \textit{uniformly on each compact subset of } \{t < \varphi_-(y)\}. \qquad (6.34)$$

and

$$v_\varepsilon \to 1 \quad \textit{uniformly on each compact subset of } \{t > \psi(y)\}, \qquad (6.35)$$

$$v_\varepsilon \to -1 \quad \textit{uniformly on each compact subset of } \{t < \psi(y)\}. \qquad (6.36)$$

Proof. By [42, Proposition 4.2] either $u_\varepsilon \to 1$ or $u_\varepsilon \to -1$ uniformly on each compact subset of $\Omega \setminus \operatorname{supp}\mu$. By (4.11), (4.12) we conclude (6.33), (6.34). The construction of v_ε yields (6.35), (6.36). □

For $\varepsilon > 0$ such that (4.8), (4.13) holds we deduce that

$$f_\varepsilon \ > \ \frac{8}{9} f(y_1) \quad \text{ in } B_{\varrho_0}(0) \times (-5\varrho_0, 5\varrho_0). \tag{6.37}$$

In fact, in this region we compute that

$$\frac{f(y_1)}{f_\varepsilon} \ \le \ \frac{|f(y_1) - f(0)| + f(0)}{f(0) - |f - f(0)| - |f_\varepsilon - f|} \ \le \ \frac{28}{25} \ < \ \frac{9}{8}.$$

Lemma 6.9. *Choose $c_0 > 0$ such that*

$$W''(r) \ \ge \ \frac{W''(1)}{2} > 0 \quad \text{ for all } |r| \ge 1 - c_0 \tag{6.38}$$

and set $\Omega' := B_{\varrho_0}(0) \times (-5\varrho_0, 5\varrho_0)$.

Let $\lambda_{\varepsilon,+}, \lambda_{\varepsilon,-}$ be the positive and negative solution, respectively, of

$$\frac{1}{\varepsilon} W'(\lambda_{\varepsilon,\pm}) \ = \ \frac{8}{9} f(y_1). \tag{6.39}$$

Then there exists for any bounded domain $U \subset\subset \Omega'$ and any $k \in \mathbb{N}$ a constant $C_k = C_k(n, U, \Omega')$ such that for all $\varepsilon < \varepsilon_0(k, c_0)$

$$u_\varepsilon \ \ge \ \lambda_{\varepsilon,-} - C_k \varepsilon^k \quad \text{ in } U. \tag{6.40}$$

Assume (6.38), (6.39) *and in addition that there exists $\Omega'' \subset \Omega'$ such that*

$$u_\varepsilon \ \ge \ 1 - c_0 \quad \text{ in } \Omega''. \tag{6.41}$$

Then there exists for any bounded domain $U \subset\subset \Omega''$ and any $k \in \mathbb{N}$ a constant $C_k = C_k(n, U, \Omega'')$ such that

$$u_\varepsilon \ \ge \ \lambda_{\varepsilon,+} - C_k \varepsilon^k \quad \text{ in } U. \tag{6.42}$$

Proof. We first show the second conclusion. We deduce from (6.37), (6.39) that

$$-\varepsilon \Delta(u_\varepsilon - \lambda_{\varepsilon,+}) + \frac{1}{\varepsilon}\Big(W'(u_\varepsilon) - W'(\lambda_{\varepsilon,+})\Big) \ \ge \ 0. \tag{6.43}$$

Consider first bounded domains Ω_1, Ω_2 such that

$$U \ \subset \ \Omega_1 \ \subset\subset \ \Omega_2 \ \subset \Omega''.$$

Chose a cut-off function $\phi \in C_c^\infty(\Omega'')$ such that $0 \le \phi \le 1$ and

$$\phi \ = \ 1 \ \text{ in } \Omega_1, \qquad \phi \ = \ 0 \ \text{ in } \Omega'' \setminus \Omega_2.$$

Next we define

$$(u_\varepsilon - \lambda_{\varepsilon,+})_- \ := \ \min(0, u_\varepsilon - \lambda_{\varepsilon,+}) \ \le \ 0,$$

we multiply (6.43) by $(u_\varepsilon - \lambda_{\varepsilon,+})_-\phi^2$, and integrate over Ω_2. We then deduce that

$$\begin{aligned}\int_{\Omega_2} \varepsilon|\nabla(u_\varepsilon - \lambda_{\varepsilon,+})_-|^2\phi^2 \;\le\; &-\int_{\Omega_2} \varepsilon(u_\varepsilon - \lambda_{\varepsilon,+})_-\nabla u_\varepsilon\cdot 2\phi\nabla\phi\\ &-\int_{\Omega_2}\frac{1}{\varepsilon}\Big(W'(u_\varepsilon) - W'(\lambda_{\varepsilon,+})\Big)(u_\varepsilon - \lambda_{\varepsilon,+})_-\phi^2\\ \le\; &\int_{\Omega_2}\varepsilon|\nabla(u_\varepsilon - \lambda_{\varepsilon,+})_-|^2\phi^2 + \int_{\Omega_2}\varepsilon|\nabla\phi|^2(u_\varepsilon - \lambda_{\varepsilon,+})_-^2\\ &-\frac{1}{2\varepsilon}W''(1)\int_{\Omega_2}(u_\varepsilon - \lambda_{\varepsilon,+})_-^2\phi^2, \qquad (6.44)\end{aligned}$$

where in the last line we have used (6.38), (6.41). We therefore obtain that

$$\frac{1}{2}W''(1)\int_{\Omega_1}(u_\varepsilon - \lambda_{\varepsilon,+})_-^2 \;\le\; \varepsilon^2\|\nabla\phi\|^2_{L^\infty(\Omega_2)}\int_{\Omega_2}(u_\varepsilon - \lambda_{\varepsilon,+})_-^2. \qquad (6.45)$$

Choosing now bounded domains Ω_j, $j = 1, ..., k+1$, such that

$$U \;=\; \Omega_1 \;\subset\subset\; \Omega_2 \;\subset\subset\; ... \;\subset\subset\; \Omega_{k+1} \;=\; \Omega''$$

and iterating the procedure above we deduce that

$$\int_U (u_\varepsilon - \lambda_{\varepsilon,+})_-^2 \;\le\; C(k,U,\Omega'')\varepsilon^{2k}\int_{\Omega''}(u_\varepsilon - \lambda_{\varepsilon,+})_-^2 \;\le\; C(k,U,\Omega'')\varepsilon^{2k}. \qquad (6.46)$$

Assume now that for a $x_1 \in U$

$$u_\varepsilon(x_1) \;<\; \lambda_{\varepsilon,+} - c_1\varepsilon^k, \qquad (6.47)$$

choose $r = r(U,\Omega'')$ such that $B_{r\varepsilon}(x_1) \subset \Omega''$, and consider the scaled functions $\tilde u, \tilde f : B_r^n(0) \to \mathbb{R}$,

$$\tilde u(x) \;:=\; u_\varepsilon(x_1 + \varepsilon x), \qquad \tilde f(x) \;:=\; f_\varepsilon(x_1 + \varepsilon x).$$

Then $\tilde u, \tilde f$ satisfy the equation

$$-\Delta\tilde u \;=\; -W'(\tilde u) + \varepsilon\tilde f.$$

Since the right-hand side is uniformly bounded we deduce that $\tilde u \in W^{2,q}(B_r(0))$ for all $1 \le q < \infty$ and by the Sobolev inequality that

$$\begin{aligned}|\nabla\tilde u| \;&\le\; c_2(n,r) \quad \text{on } B_r(0),\\ |\nabla u_\varepsilon| \;&\le\; \frac{c_2(n,r)}{\varepsilon} \quad \text{on } B_{\varepsilon r}(x_1).\end{aligned}$$

Thus (6.47) gives us for all $x \in B_{r\varepsilon^{k+1}}(x_1)$

$$u_\varepsilon(x) \;=\; u_\varepsilon(x_1) + u_\varepsilon(x) - u_\varepsilon(x_1) \;<\; \lambda_{\varepsilon,+} - (c_1 - c_2(n,r))\varepsilon^k$$

and we compute that for $c_1 > c_2(n,r)$

$$\int_U (u_\varepsilon - \lambda_{\varepsilon,+})_-^2 \;\ge\; \int_{B_{r\varepsilon^{k+1}}(x_1)}(c_1 - c_2(n,r))^2\varepsilon^k \;\ge\; (c_1 - c_2(n,r))^2\omega_n\varepsilon^{k+n(k+1)}r^n.$$

On the other hand, by (6.46) with $2k$ replaced by $k + n(k+1)$ we obtain that

$$\int_U (u_\varepsilon - \lambda_{\varepsilon,+})_-^2 \;\le\; \tilde C(k,U,\Omega'')\varepsilon^{k+n(k+1)},$$

which gives a contradiction for all $c_1 = c_1(n,k,U,\Omega'')$ sufficiently large.

To prove (6.40) we first observe that

$$-\varepsilon\Delta(-1+c_0)+\frac{1}{\varepsilon}W'(-1+c_0)\ \geq\ \frac{8}{9}f(y_1)$$

for all $\varepsilon>0$ sufficiently small. Since the minimum of two supersolutions is a supersolution we deduce that

$$\tilde{u}_\varepsilon\ :=\ \min(u_\varepsilon,-1+c_0)$$

satisfies $\tilde{u}_\varepsilon\in W^{1,2}(\Omega')$ and

$$\begin{aligned}-\varepsilon\Delta\tilde{u}_\varepsilon+\frac{1}{\varepsilon}W'(\tilde{u}_\varepsilon)\ &\geq\ \frac{8}{9}f(y_1),\\ \tilde{u}_\varepsilon\ &\leq\ -1+c_0\end{aligned}$$

Then we can prove by the same arguments as for (1) that

$$\tilde{u}_\varepsilon\ \geq\ \lambda_{\varepsilon,-}-C_k\varepsilon^k\quad\text{ in } U.$$

Since $\lambda_{\varepsilon,-}-C_k\varepsilon^k<-1+c_0$ for $\varepsilon<\varepsilon_0(k)$ this proves (6.40). □

We will employ a comparison principle on the cylinder $B_{\varrho_2}(y_1)\times(-4\varrho_0,4\varrho_0)$. We first control the difference $u_\varepsilon-v_\varepsilon$ on the top and the bottom, starting with the following lemma.

Lemma 6.10. *Let $\lambda_{\varepsilon,+},\lambda_{\varepsilon,-}$ be the positive and negative solution of* (6.39) *and let $\beta_{\varepsilon,\pm}$ denote the values of v_ε 'away' from M,*

$$\beta_{\varepsilon,+}\ :=\ \phi_0(\varepsilon^{-1}\delta(\varepsilon))+\varepsilon\phi_1(\varepsilon^{-1}\delta(\varepsilon))\frac{2}{3\sigma}f(y_1),\tag{6.48}$$

$$\beta_{\varepsilon,-}\ :=\ \phi_0(-\varepsilon^{-1}\delta(\varepsilon))+\varepsilon\phi_1(-\varepsilon^{-1}\delta(\varepsilon))\frac{2}{3\sigma}f(y_1).\tag{6.49}$$

Then there exists $\gamma>0$, $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$

$$\lambda_{\varepsilon,+}-\beta_{\varepsilon,+}\ \geq\ \gamma\varepsilon,\tag{6.50}$$

$$\lambda_{\varepsilon,-}-\beta_{\varepsilon,-}\ \geq\ \gamma\varepsilon.\tag{6.51}$$

Proof. By a Taylor approximation

$$\begin{aligned}W'(\beta_{\varepsilon,\pm})\ &=\ W'\big(\phi_0(-\varepsilon^{-1}\delta(\varepsilon))\big)+W''\big(\phi_0(-\varepsilon^{-1}\delta(\varepsilon))\big)\cdot\varepsilon\phi_1(-\varepsilon^{-1}\delta(\varepsilon))\frac{2}{3\sigma}f(y_1)\\ &\quad+O(\varepsilon^2)\\ &=\ \frac{2}{3}f(y_1)\varepsilon+o(\varepsilon),\end{aligned}$$

where we have used that

$$W'\big(\phi_0(-\varepsilon^{-1}\delta(\varepsilon))\big)\ =\ o(\varepsilon)$$

by (6.9), (6.14) and that

$$W''\big(\phi_0(-\varepsilon^{-1}\delta(\varepsilon))\big)\phi_1(-\varepsilon^{-1}\delta(\varepsilon))\frac{1}{\sigma}\ \to\ 0\quad\text{ as }\varepsilon\to 0$$

by (6.12), (6.13).

We therefore deduce that

$$(\lambda_{\varepsilon,\pm} - \beta_{\varepsilon,\pm}) \int_0^1 W''\big(s\lambda_{\varepsilon,\pm} + (1-s)\beta_{\varepsilon,\pm}\big)\, ds \;=\; W'(\lambda_{\varepsilon,\pm}) - W'(\beta_{\varepsilon,\pm})$$
$$= \frac{1}{6} f(y_1)\varepsilon + o(\varepsilon). \tag{6.52}$$

Since $\lambda_{\varepsilon,-}, \beta_{\varepsilon,-}$ converge to -1 as $\varepsilon \to 0$ and since $W''(-1) > 0$ we deduce from (6.52) that (6.51) holds for $\gamma > 0$ and $\varepsilon > 0$ sufficiently small. Analogously we obtain (6.50). □

Proposition 6.11. *For all $\varepsilon > 0$ sufficiently small we obtain that*

$$u_\varepsilon \;\geq\; v_\varepsilon \quad \text{ in } \overline{B_{\varrho_2}(y_1)} \times [-4\varrho_0, 4\varrho_0]. \tag{6.53}$$

Proof. Let us define the sets

$$U \;:=\; B_{\varrho_2}(y_1) \times (-4\varrho_0, 4\varrho_0), \qquad \Omega' \;:=\; B_{\varrho}(y_1) \times (-5\varrho_0, 5\varrho_0).$$

Consider for $s > 0$ the *shifted functions* $v_\varepsilon^{(s)}$,

$$v_\varepsilon^{(s)}(y,t) \;:=\; v_\varepsilon(y, t-s) \quad \text{ for } (y,t) \in \Omega',$$

and the function

$$\Phi(s) \;:=\; \min_{\overline{U}} (u_\varepsilon - v_\varepsilon^{(s)}).$$

Assume now $\varepsilon < \varepsilon_1$, where we choose $\varepsilon_1 > 0$ below, and that (6.53) is not satisfied, hence

$$\Phi(0) \;<\; 0. \tag{6.54}$$

The definition of v_ε in (6.15), (6.23) implies that we can choose $s_0 > 0$, $s_0 = s_0(\varrho_0)$ such that for all $\varepsilon > 0$ sufficiently small

$$v_\varepsilon^{(s_0)} \;=\; \beta_{\varepsilon,-} \quad \text{ in } \Omega'. \tag{6.55}$$

Applying then Lemma 6.9 with U, Ω' as above and $k = 2$ we deduce that

$$u_\varepsilon \;\geq\; \lambda_{\varepsilon,-} - O(\varepsilon^2) \quad \text{ in } U. \tag{6.56}$$

Therefore (6.51) and (6.55), (6.56) imply that

$$u_\varepsilon - v_\varepsilon^{(s_0)} \;\geq\; \lambda_{\varepsilon,-} - O(\varepsilon^2) - \beta_{\varepsilon,-} \;\geq\; \gamma\varepsilon - O(\varepsilon^2)$$

and we deduce that

$$\Phi(s_0) \;>\; \frac{\gamma}{2}\varepsilon \tag{6.57}$$

for all $\varepsilon < \varepsilon_1$ and $\varepsilon_1 > 0$ chosen suitably small.

Since Φ is continuous (6.54), (6.57) imply the existence of $s_* > 0$ and $x_* \in \overline{U}$ such that

$$0 \;=\; (u_\varepsilon - v_\varepsilon^{(s_*)})(x_*) \;=\; \min_{\overline{U}} (u_\varepsilon - v_\varepsilon^{(s_*)}). \tag{6.58}$$

We first prove that $x_* \in U$. With this aim we consider the different parts of ∂U.

(1) For $\varepsilon_1 > 0$ chosen suitably small we have

$$v_\varepsilon^{(s_*)} \,=\, \beta_{\varepsilon,-} \quad \text{ in } B_{\varrho_2}(y_1) \times (-5\varrho_0, -4\varrho_0)$$

and therefore, by Lemma 6.9 and Lemma 6.10,

$$u_\varepsilon \,\geq\, v_\varepsilon^{(s_*)} + \frac{\gamma}{2}\varepsilon \quad \text{ in } B_{\varrho_2}(y_1) \times (-5\varrho_0, -4\varrho_0)$$

for $\varepsilon < \varepsilon_1$ and $\varepsilon_1 > 0$ suitably small, see the argument above. This shows that

$$x_* \,\notin\, \overline{B_{\varrho_2}(y_1)} \times \{-4\varrho_0\}. \tag{6.59}$$

(2) For $\varepsilon_1 > 0$ sufficiently small we obtain from Lemma 6.8 and Lemma 6.9, applied to

$$U_+ \,=\, B_{\varrho_2}(y_1) \times (2\varrho_0, 4\varrho_0), \quad \Omega_+'' \,=\, B_\varrho(y_1) \times (\varrho_0, 5\varrho_0), k = 2$$

that

$$u_\varepsilon \,\geq\, \lambda_{\varepsilon,+} - O(\varepsilon^2) \quad \text{ in } U_+.$$

By (6.50) this implies that

$$u_\varepsilon \,\geq\, \beta_{\varepsilon,+} + \gamma_\varepsilon - O(\varepsilon^2) \,\geq\, \beta_{\varepsilon,+} + \frac{\gamma}{2}\varepsilon \,\geq\, v_\varepsilon + \frac{\gamma}{2}$$

holds in U_+, hence

$$x_* \,\notin\, \overline{B_{\varrho_2}(y_1)} \times \{4\varrho_0\}. \tag{6.60}$$

(3) By (6.4) there exists bounded domains U_1, Ω_1, U_2 such that

$$U_1 \,\subset\subset\, \Omega_1 \,\subset\subset\, B_\varrho(y_1) \setminus \overline{B_{\varrho_3}(y_1)} \times (-5\varrho_0, 5\varrho_0) \tag{6.61}$$

$$\Omega_1 \,\subset\subset\, \{t > \varphi_+(y)\}, \tag{6.62}$$

$$U_2 \,\subset\subset\, \{t < \psi(y)\}, \tag{6.63}$$

$$\partial B_{\varrho_2}(y_1) \times [-4\varrho_0, 4\varrho_0] \,\subset\, U_1 \cup U_2. \tag{6.64}$$

By similar arguments as above we first prove that $u_\varepsilon > v_\varepsilon^{(s_*)}$ in U_1. First we obtain from Lemma 6.8 that

$$u_\varepsilon \,\geq\, 1 - c_0 \quad \text{ in } \Omega_1$$

and applying Lemma 6.9 and Lemma 6.10 with $k = 2$ we deduce that in U_1

$$u_\varepsilon \,\geq\, \lambda_{\varepsilon,+} - O(\varepsilon^2) \,>\, v_\varepsilon^{(s_*)}. \tag{6.65}$$

Since $s_* > 0$ we obtain that for $\varepsilon < \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(U_2, \varrho_2)$ is chosen sufficiently small,

$$v_\varepsilon^{(s_*)} \,\leq\, \beta_{\varepsilon,-} + o(\varepsilon) \,\leq\, \lambda_{\varepsilon,-} - \gamma\varepsilon + o(\varepsilon) \,<\, u_\varepsilon \tag{6.66}$$

in U_2. By (6.64), (6.65), (6.66) we deduce that

$$u_\varepsilon \,>\, v_\varepsilon^{(s_*)} \quad \text{ on } \partial B_{\varrho_2}(y_1) \times (-4\varrho_0, 4\varrho_0). \tag{6.67}$$

By (6.58)-(6.60), (6.67) we get that $u_\varepsilon - v_\varepsilon^{(s_*)}$ has an interior minimum with value zero at x_*. Using (2.3), (6.37), and (6.27) we therefore deduce that

$$\begin{aligned} 0 \;&\geq\; -\varepsilon\Delta(u_\varepsilon - v_\varepsilon^{(s_*)})(x_*) \\ &\geq\; f_\varepsilon(x_*) - \frac{7}{9}f(y_1) - \frac{1}{\varepsilon}W'(u_\varepsilon(x_*)) + \frac{1}{\varepsilon}W'(v_\varepsilon^{(s_*)}(x_*)) \\ &\geq\; \frac{1}{9}f(y_1). \end{aligned}$$

This finally gives a contradiction and proves Proposition 5.2. □

7. Applications

7.1. **Stationary solutions for the Cahn-Hilliard equation.** Consider the Cahn-Hilliard equation [7]

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta f_\varepsilon, \qquad f_\varepsilon \;=\; -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon) \tag{7.1}$$

with Neumann boundary conditions for u_ε and f_ε.

The stationary solutions of (7.1) are those with constant f_ε,

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon) \;=\; \lambda_\varepsilon, \quad \lambda_\varepsilon \in \mathbb{R}. \tag{7.2}$$

This is also the Euler–Lagrange equation of the volume constrained minimization problem for the Cahn–Hilliard energy (2.1),

$$\inf\big\{E_\varepsilon(u) \,:\, u \in H^{1,2}(\Omega), \int_\Omega u \;=\; m\big\}. \tag{7.3}$$

To better understand stationary solutions of the Cahn–Hilliard equation the asymptotic of (7.2) is analyzed in a couple of papers. The behavior of (locally) energy minimizing solutions of (7.2) is well understood [21, 40, 19, 16]. In this case sequence $(u_\varepsilon)_{\varepsilon>0}$ with uniformly bounded energy converge to a constant-mean curvature hypersurface with single-multiplicity. This hypersurface is smooth except for a closed set of dimension at most $n-8$.

Solutions of the Cahn–Hilliard equation are observed to undergo pattern similar to unstable equilibria [14] and the behavior of general stationary points is another question of interest. However, this situation is more difficult due to the possibility of higher-multiplicity surfaces in the limit. Hutchinson–Tonegawa [16] showed that the limit is given by an integer-rectifiable varifold and that the weak mean curvature exists. However this mean curvature is only *locally* constant, the constant depending on the multiplicity. A higher multiplicity is also an obstacle to use Allard's regularity theory [3] and to obtain the smoothness of the limit.

As a corollary of Theorem 3.2 we can improve the previous results.

Theorem 7.1. *Consider a sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^{1,2}(\Omega)$ with a fixed volume constraint $\int_\Omega u_\varepsilon = m$, and a sequence $(\lambda_\varepsilon)_{\varepsilon>0} \subset \mathbb{R}$ such that (7.2) is satisfied. Suppose further that*

$$E_\varepsilon(u_\varepsilon) \;\leq\; \Lambda \quad \text{for all } \varepsilon > 0. \tag{7.4}$$

Then there exists a subsequence $\varepsilon \to 0$, a function $u \in BV(\Omega, \{-1,1\})$, and $\lambda := \lim_{\varepsilon\to 0}\lambda_\varepsilon$, such that $u_\varepsilon \to u$ in $L^1(\Omega)$. The phase boundary $\partial^*\{u=1\} \cap \Omega$ has constant mean curvature $\sigma H = \lambda$.

In the case that $\lambda \neq 0$ the phase boundary is up to a $\mathcal{H}^{n-1}$-nullset a smooth hypersurface. The energy measures μ_ε as defined in (2.6) converge to a measure μ that is up to the factor 2σ integer-rectifiable, has constant mean curvature $\sigma H = \lambda$ and multiplicity one $\mathcal{H}^{n-1}$-almost everywhere on $\partial^*\{u = 1\}$. Moreover 'hidden boundaries' can only occur in one phase and have zero mean curvature:

$$\mu(K) \,=\, 0 \quad \begin{cases} \text{for all compact sets } K \subset \{u=1\}^o & \text{if } \lambda > 0, \\ \text{for all compact sets } K \subset \{u=-1\}^o & \text{if } \lambda < 0, \end{cases} \tag{7.5}$$

$$H \,=\, 0 \quad \mathcal{H}^{n-1}\text{-almost everywhere on } \operatorname{supp}(\mu) \setminus \partial^*\{u=1\}. \tag{7.6}$$

Proof. It follows from [9, Lemma 3.4] that $|\lambda_\varepsilon| \leq c(m,\Lambda)$ and we may choose a subsequence such that $\lambda = \lim_{\varepsilon\to 0} \lambda_\varepsilon$ exists.

We therefore can apply Theorem 3.2 and obtain that there exists a subsequence $\varepsilon \to 0$ and limits u, μ of $u_\varepsilon, \mu_\varepsilon$. Moreover, $u \in BV(\Omega, \{-1,1\})$ and $(2\sigma)^{-1}\mu$ is an integer-rectifiable varifold with weak mean curvature

$$\sigma H \,=\, \lambda \quad \mathcal{H}^{n-1} - \text{almost everywhere on } \partial^*\{u=1\}.$$

Next it follows from Theorem 3.2 that $\lambda = 0$ on the parts of $\partial^*\{u=1\} \cap \Omega$ with odd multiplicity larger than 1, which shows that in the case $\lambda \neq 0$ the phase boundary is given as a constant curvature varifold with unit multiplicity. By Allard's regularity theory [3] we conclude the smoothness of the phase boundary. Finally (7.5), (7.6) follow from (3.4). □

In general dimension we can not insure good regularity of the hidden boundaries, due to the lack of regularity theory for general stationary integral varifold. Only for $n = 2$ we can conclude that $\operatorname{spt}\mu \cap K$ is given by straight line segments with possible junction points for all compact sets $K \subset \Omega \setminus \partial^*\{u=1\}$ [2].

7.2. **Critical points of the Ohta-Kawasaki functional.** The micro-phase separation of block copolymers exhibits the formation of complex patterns. Ohta–Kawasaki [25] and later Bahiana–Oono [4] used a phase-field like approach and proposed a free energy that is after a suitable rescaling given by

$$F_\varepsilon(u) \,=\, \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u) + \frac{1}{2}|\nabla v|^2 \Big) dx, \tag{7.7}$$

where $v(\cdot) = v[u](\cdot)$ is the solution of

$$-\Delta v \,=\, u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \qquad \nabla v \cdot \nu_\Omega \,=\, 0 \quad \text{on } \partial\Omega. \tag{7.8}$$

The functional F_ε extends the Cahn–Hilliard energy by a non-local term that describes long-range interactions between chains of macromolecules. For a derivation of this model by a density-functional approach see [11]. The set of (local) minimizers of F_ε is extremely rich and (7.7) has drawn quite some attention [10, 24, 29, 30]. The Gamma-limit of F_ε as $\varepsilon \to 0$ and the convergence of the corresponding H^{-1} gradient-flow that was proposed by Nishiura and Ohnishi [23] are also well-studied [28, 12, 15].

Critical points of F_ε under a volume-constraint satisfy the Euler–Lagrange equation

$$-\varepsilon\Delta u + \frac{1}{\varepsilon} W'(u) + v[u] \,=\, \lambda, \tag{7.9}$$

where $\lambda \in \mathbb{R}$ is a Lagrange-multiplier.

As a corollary of our results we obtain the convergence of stationary points of F_ε.

Theorem 7.2. *Assume that we have a sequence $(u_\varepsilon)_{\varepsilon>0}$ such that*

$$F_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{L^\infty(\Omega)} \leq \Lambda \quad \text{for all } \varepsilon > 0 \tag{7.10}$$

and such that (7.9) holds for Lagrange multipliers $\lambda_\varepsilon \in \mathbb{R}$ and the solutions v_ε of

$$-\Delta v_\varepsilon = u_\varepsilon - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \quad \text{in } \Omega, \qquad \nabla v_\varepsilon \cdot \nu_\Omega = 0 \quad \text{on } \partial\Omega. \tag{7.11}$$

Then there exists a subsequence $\varepsilon \to 0$, a number $\lambda \in \mathbb{R}$, and a function $u \in BV(\Omega, \{-1,1\})$ such that $\lambda = \lim_{\varepsilon\to 0}\lambda_\varepsilon$ and $u_\varepsilon \to u$ in $L^1(\Omega)$. Moreover $v_\varepsilon \to v$ in $C^{1,\alpha}(\bar\Omega)$ for all $0<\alpha<1$ and v solves (7.8).

The energy measures μ_ε as defined in (2.6) converge to a measure μ that is up to the factor 2σ integer-rectifiable and has a weak mean curvature that satisfies

$$\sigma H = \begin{cases} -v+\lambda & \mathcal{H}^{n-1}-\text{almost everywhere on } \partial^*\{u=1\}, \\ 0 & \mathcal{H}^{n-1}-\text{almost everywhere on } \operatorname{supp}(\mu)\setminus\partial^*\{u=1\}. \end{cases} \tag{7.12}$$

Finally $\partial^*\{u=1\}$ has multiplicity one $\mathcal{H}^{n-1}$-almost everywhere in the set $\{v\neq\lambda\}$ and this part of the phase boundary is a $C^{3,\alpha}$-surface for all $\alpha<1$, except for a set of $\mathcal{H}^{n-1}$-measure zero.

Proof. By standard elliptic theory we obtain from (7.10), (7.11) that v_ε is uniformly bounded in $W^{2,p}(\Omega)$ for all $1\leq p<\infty$. Therefore Theorem 3.2 applies and we can repeat the arguments of the proof of Theorem 7.1. We omit the details here. □

If u_ε has in addition a local energy minimizing property for F_ε we can draw stronger conclusions: Then μ has multiplicity one μ-almost everywhere and is $C^{3,\alpha}$-smooth, see the arguments in [16].

Appendix A. A generalization of mean curvature to general phase boundaries

Proposition A.1 (see [31, Proposition 3.1])**.** *Let $\Omega\subset\mathbb{R}^n$ be open, $E\subset\Omega$, and $\mathcal{X}_E\in \mathrm{BV}(\Omega)$. Assume that there are two integral $(n-1)$-varifolds μ_1,μ_2 on Ω such that for $i=1,2$ the following hold:*

$$\partial^* E \subset \operatorname{supp}(\mu_i), \tag{A.1}$$

$$\mu_i \text{ has locally bounded first variation with mean curvature vector } \vec{H}_{\mu_i}, \tag{A.2}$$

$$\vec{H}_{\mu_i} \in \mathrm{L}^s_{\mathrm{loc}}(\mu_i),\ s>n-1,\ s\geq 2. \tag{A.3}$$

Then

$$\vec{H}_{\mu_1}|_{\partial^*E} = \vec{H}_{\mu_2}|_{\partial^*E}$$

is satisfied $\mathcal{H}^{n-1}$-almost everywhere on ∂^*E.

This proposition justifies the following definition.

Definition A.2. *Let $E\subset\Omega$ and $\mathcal{X}_E\in\mathrm{BV}(\Omega)$, and assume that there exists an integral $(n-1)$-varifold μ on Ω satisfying (A.1)–(A.3). Then we call*

$$\vec{H} := \vec{H}_\mu|_{\partial^*E}$$

the generalized mean curvature vector of $\partial^* E$ and define a scalar mean curvature by

$$H := \vec{H} \cdot \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|} \text{ on } \partial^* E.$$

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