Analysis of Varying Sounds, A New Photographic Method with Applications to Typical Examples

By

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(*With Plate V–XXII*)

Abstract

After a brief critical review of the existing methods for the analysis of varying sounds, in this paper a general theory is given of the underlying fundamental problems, with particular attention to the uncertainty relation $\Delta\nu\Delta t \approx 1$ and its physical meaning. A new photographic method is described which is specially adapted to varying sound analysis. It consists of a photographic plate (analysing plate) with sinusoidally light transmitting property along one direction, and a photographic film moving uniformly along the same direction at the back of the plate. The wave-length on the analysing plate varies continuously in the direction perpendicular to the sinusoidal variation. If the light intensity falling upon the plate is modulated by the sound wave which is to be analysed, one obtains on the developed film stroboscopically resolved components of the original complex wave. The resolving power $\Delta\nu$ and the time constant Δt may be changed by preparing a suitable analysing plate, so that one may imitate the ear in analysing sounds. Numerous photographs are given of typical examples such as limited wave trains, amplitude modulation, interrupted waves, phase change, beats, change of frequency and trillers, and also two examples applied to spoken words.

§ 1. Introduction

The application of Fourier's theorem to various phenomena in physics and other branches of science has been one of the most influential methods of fundamental importance. There are numer-

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ous means for practical application of the theorem to actual problems, each having its characteristic feature and special adaptability to different purposes. One point of defect common to most of these Fourier analysers or frequency analysers is the ineffectiveness found in various degrees in the case of non-periodic or quasi-periodic phenomena such as transient vibration in electrical circuits or speech sounds in daily conversation.

Hickman[1] in his acoustic spectrometer used a series of tuned reeds with small mirrors, which are driven electromagnetically by an electric current having the wave form to be analysed. By observing or photographing the images of an illuminated slit reflected by the mirrors, the resonance frequencies and their amplitudes are determined. When the wave form varies with time, the corresponding variation in the spectrum may be observed. But in this case, the number of reeds and relative frequency intervals on one side and time constant of each resonator on the other, must be properly adjusted to obtain reasonable results. The difficulty of changing this "analysing characteristics" for different purposes limits the field of application of this method.

The same resonance principle is also employed in the tone-frequency spectrometer of Freysted and others.[2] Instead of the mechanical resonators of Hickman a relatively small number (22 or 27) of resonating electric circuits is used. The output of each filter is successively applied with necessary amplification to one pair of deflecting plates in a Braun tube, while the other pair serves to separate each component. The time required for one complete analysis over the acoustic frequency range is about 0.1 second. Thus a small film camera making about ten exposures per sec. is used in the case of varying sound. The separations of frequencies and the precise description of time variation are however, far from the ideal condition $\varDelta\nu\varDelta t\approx 1$.[3] While this instrument has much convenience in practical use, it is not adapted to an exact measurement.

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1) C. N. Hickman: J. A. S. A., 6, 108. 1934.

2) E. Freystedt: ZS. f. Techn. Phys., 16, 533. 1935.

3) The same remark may be applied to the search-tone method of Grützmacher and others—in case this is applied to varying sound by some necessary modifications—, and also to the ultrasonic diffraction method of Meyer and Thienhaus.

An improvement of fundamental importance is found in the octave-filter oscillograms of Trendelenburg[4] and Vierling.[5] The out-put of 6 or 8 octave-filters is fed into a multi-element oscillograph, so that each octave range is simultaneously recorded. This process has theoretically no weak point, and also is very convenient for practical use. But any change of analysing characteristics is almost impossible, while a higher resolving power in frequency is frequently more desirable.

Meyer and Thienhaus[6] used in analogy to the optical diffraction grating an ultrasonic sound wave of 45 k.c. modulated by the sound to be analysed. The side-band embracing frequencies of 45–50 k.c. is diffracted by a concave diffraction grating of about 1 cm grating constant and 3 m long, a small resonated condenser microphone being used as a detector. The resolving power of frequency is 125 ~, and the time required for one complete analysis about 0.1 sec. As remarked before, a definite drawback is found in the impossibility of getting the entire spectrum simultaneously. In this connection the recent work of R. Pohlman[7] on the photography of ultrasonic sound image seems to throw light upon further development of this method.

An interesting idea was reported by Brown[8] of using sound picture film of variable density type as a diffraction grating of monochromatic light. If the light transmitting characteristic of a grating is sinusoidal, instead of black and white as in the usual optical gratings, the diffracted light contains only the first order spectrum besides the undeflected beam. A sound film is so prepared that the transmitted light varies in its amplitude (not intensity) as the wave to be analysed, and then this is used as a diffraction grating. The diffraction pattern gives the complex structure of the wave resolved into its components. The analysing characteristic of this method is determined by the length of the film used as diffraction grating, so that it may be changed at will for different

4) F. Trendelenburg and E. Franz: J.A.S.A., 7, 142. 1935. ZS. f Techn. Phys., 16, 513. 1935. Wiss. Veröff. Siemens., 15, 78. 1936.

5) O. Vierling: ZS. f. Techn. Phys., 16, 528. 1935.

6) E. Meyer and E. Thienhaus: ZS. f. Techn. Phys., 15, 630. 1934. E. Meyer: J.A.S.A., 7, 88, 1935.

7) R. Pohlman: ZS. f. Phys., 113, 697. 1939.

8) D. Brown: Nature, 140, 1099. 1937. Proc. Roy. Soc., 51, 244. 1939.

purposes. The same principle was previously employed by Germansky[9], using diffraction of light by a reflecting surface, but the preparation of the reflecting plate involved an elaborate procedure. Owing to small diffraction angles and the consequent low dispersion of the sound spectrum obtained, there remain in this method of light diffraction some technical difficulties to overcome.

The present author[10] has recently used a very simple photographic method of frequency analysis which gives time variations in amplitudes, frequencies and phases of the components of complex sound simultaneously as it is being produced. The analysing characteristic may be designed to meet any special purpose, so that an analysis of sound analogous to the ear may be obtained. This paper deals mainly with a general theory of varying sound illustrated by photographs which were obtained by the application of the new method to various model examples. Practical applications to actual problems will be dealt with in later papers.

§ 2. General Theory of Varying Sound Analysis

The principle of various frequency analysers including Fourier- and periodogramme-analysis by graphical method, lies in carrying out an approximate or somewhat modified calculation of the integral of the form

$$A(\nu) = \int f(t')e^{-2\pi i\nu t'}dt' , \tag{2,1}$$

where $f(t')$ is the function to be analysed, and $A(\nu)$ a complex quantity giving amplitude and phase of the harmonic component as a function of frequency ν. The integration limits are taken either for one period of $f(t')$ when this is periodic, or otherwise for a sufficiently large interval of time. In the former case the integral is usually divided by the period to give correct amplitude, but this additional factor can be omitted, for one may simply extend Fourier's integral theorem to periodic case by using Dirac's δ-function. Accordingly, the latter interpretation is always assumed in this paper, and integrals without any specified integration limits are to be understood as extending over the entire range of time from

9) B. Germansky: Ann. d. Phys., 7, 453. 1930.
10) K. Imahori: Nature, 144, 708. 1939.

$-\infty$ to $+\infty$. The function $f(t')$ must also be properly defined in the entire domain. The spectrum thus obtained is in general continuous with more or less sharply defined maxima and shows line structure in special ideal cases. While in optical researches spectroscopists always interpret their spectra in this manner, we find in acoustical phenomena special interest in the growth and decay of sound waves, so that a time-analysis of sound spectrum becomes indipensable.

For this purpse an integral of the form

$$A(\nu, t) = \int f(t')e^{-2\pi i\nu(t'-t)}dt' \qquad (2,2)$$
$$= A(\nu)e^{2\pi i\nu t}$$

is considered. This may be regarded as representing the spectrum at $t'=t$, while (2,1) the spectrum at $t'=0$. The difference is an additive phase constant $2\pi\nu t$. It must be remarked, however, that this difference is of purely mathematical nature, because the same difference is effected by a change of time origin. A physical meaning arises when one passes into the case of varying spectra.

In order to distinguish spectra at different times one must limit the integral of (2,2) to a certain interval in the neighbourhood of $t'=t$. Or more generally, by assuming a "weight function" $\Delta(\nu, t'-t)$ which, considered as a function of $t'-t$ and depending on a parameter ν, has finite values —for example unity— in the neighbourhood of $t'=t$ and is negligibly small outside this domain, a "weighted integral" of the form

$$A(\nu, t) = \int f(t')\Delta(\nu, t'-t)e^{-2\pi i\nu(t'-t)}dt' \qquad (2,3)$$

is used to represent the spectrum in the neighbourhood of t. In the limiting case where the weight function approaches Dirac's δ-function such that

$$\delta(t'-t) = 0 \quad \text{for} \quad t' \neq t \qquad (2,4)$$

and

$$\int \delta(t'-t)dt' = 1\,, \qquad (2,4a)$$

the integral (2,3) is reduced to

$$A(\nu,\, t) = \int f(t')\delta(t'-t)e^{-2\pi i\nu(t'-t)}dt' = f(t)\,. \tag{2,5}$$

Perfect description of time variation is thus obtained, while the other extreme case where the spectrum is completely resolved is represented by (2,2), and the square of the amplitude of each component is given by

$$A(\nu,\, t)A^*(\nu,\, t) = A(\nu)A^*(\nu)\,,$$

where * denotes a complex conjugate quantity. The intermediate cases are characterised by the functional form of the weight function $\varDelta(\nu,\, t'-t)$. Simple examples which are frequently used in this paper are

$$\varDelta(\nu,\, t'-t) = 1 \quad \text{for} \quad |\,t'-t\,| \leq \frac{\tau}{2} \tag{2,6a}$$

$$= 0 \quad \text{otherwise}\,;$$

and
$$\varDelta(\nu,\, t'-t) = e^{-k^2(t'-t)^2}\,. \tag{2,6b}$$

The "time constants" τ and $2/k$ in these expressions represent the time intervals in which the weight function has appreciable values and may be considered to depend upon the frequency ν.

To find a physical interpretation of $A(\nu,\, t)$ given by (2,3) a case is examined in which $f(t')$ is an impulsive motion at an instant $t'=t_1$. Such an impulsive motion may be expressed by a Dirac's function, such that[11]

$$f(t') = \delta(t'-t_1)\,.$$

Then (2,3) reduces to

$$A(\nu,\, t) = \int \delta(t'-t_1)\varDelta(\nu,\, t'-t)e^{-2\pi i\nu(t'-t)}dt'$$

$$= \varDelta(\nu,\, t_1-t)\,e^{-2\pi i(t_1-t)}\,. \tag{2,7}$$

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11) Generally a function f(t) may be regarded as a superposition of δ-function such that

$$f(t) = \int f(t')\delta(t'-t)dt'\,.$$

This is a harmonic motion the amplitude of which is given by $\Delta(\nu, t_1 - t)$ and so has finite value in the neighbourhood of $t = t_1$, the time interval of which is equal to the time constant of Δ-function. Two consecutive impulses occurring at instants t_1 and t_2 will therefore be separated in $A(\nu, t)$, if the time difference $|t_2 - t_1|$ is larger than the time constant Δt. Motions occurring within Δt give rise to expressions corresponding to (2,7) which "interfere" and so finite amplitudes are found in some selected values of frequency ν. Thus the function defined by (2,3) represents 1) considered as a function of ν, the spectrum in the neighbourhood of time t, and 2) considered as a function of t, the time variation of each component of the spectrum. The uncertainty in time of the order of magnitude Δt which is the characteristic time constant of Δ-function employed in the analysis must always be kept in mind.

The spectrum thus obtained depends of course upon particular selection of the weight function $\Delta(\nu, t' - t)$. To see this dependence and to compare it with the ideal spectrum given by (2,2), another case is considered in which $f(t')$ is a harmonic function of frequency ν_1 such as

$$f(t') = e^{2\pi i\nu_1 t'} .$$

Then (2,2) and (2,3) are reduced to

$$A(\nu, t) = \int e^{2\pi i\nu_1 t'} e^{-2\pi i\nu(t'-t)} dt'$$

$$= \delta(\nu_1 - \nu) e^{2\pi i\nu_1 t}, \tag{2,8}$$

and

$$A(\nu, t) = \int e^{2\pi i\nu_1 t'} \Delta(\nu, t' - t) e^{-2\pi i\nu(t'-t)} dt'$$

$$= D(\nu, \nu - \nu_1) e^{2\pi i\nu_1 t}, \tag{2,9}$$

where

$$D(\nu, \nu') = \int \Delta(\nu, t') e^{-2\pi i\nu' t'} dt' . \tag{2,10}$$

Thus a line spectrum (2,8) becomes a more or less broad continuous band (2,9), the amplitude and phase distribution being given by the function $D(\nu, \nu - \nu_1)$. In special cases where the Δ-function is given by (2,6a) or (2,6b), (2,10) is easily calculated such that

$$D(\nu, \nu') = \frac{1}{\pi\nu'}\sin \pi\nu'\tau\,, \tag{2,11a}$$

and
$$D(\nu, \nu') = \frac{\sqrt{\pi}}{k}e^{-\frac{\pi^2}{k^2}\nu'^2} \tag{2,11b}$$

respectively. In general the time constants τ and $2/k$ depend upon ν. The curves are shown in Fig. 1 for fixed values of τ and k. The amplitude is at its maximum at $\nu' = 0$, and the "band width" $\Delta\nu$ is of the order of $2/\tau$ and $2k/\pi$ respectively. Thus the well known relation

$$\Delta\nu\,\Delta t \approx 1 \tag{2,12}$$

is obtained which characterises the results of sound analysis. Two harmonic motions of types $e^{2\pi i\nu_1 t}$ and $e^{2\pi i\nu_2 t}$ are separated in the spectrum only when the frequency difference $|\nu_1 - \nu_2|$ is larger than $\Delta\nu$ which satisfies (2,12). When this is smaller than $\Delta\nu$ an interference occurs to result in a beat, and is observed as a harmonic motion with varying amplitude. Thus the lack in the resolving power of frequency is complemented by the description of time variation.

For a more general case in which $\Delta(\nu, t')$ is an arbitrary (real and positive) function having finite value near $t' = 0$ and being sensibly equal to zero outside the interval $|t'| \leq \Delta t/2$, the same relation will be found to hold by the following elementary reasoning. In the integral (2,10) the function $e^{-2\pi i\nu' t'}$ is a periodic function of t' with variable period $1/\nu'$, so that for sufficiently large values of ν' compared with $1/\Delta t$ this integral will vanish approximately if the variation in time of the Δ-function is simply monotonic. For $\nu' = 0$ the integral has a maximum, for

$$\left|\int \Delta(\nu, t')e^{-2\pi i\nu' t'}dt'\right| \leq \int \left|\Delta(\nu, t')e^{-2\pi i\nu' t'}\right| dt'$$
$$\leq \int \Delta(\nu, t')dt'\,.$$

The width $\Delta\nu$ of the band $\mathrm{D}(\nu, \nu')$ will then be estimated by taking the integral for one complete period of $e^{-2\pi i\nu' t'}$ which is just involved

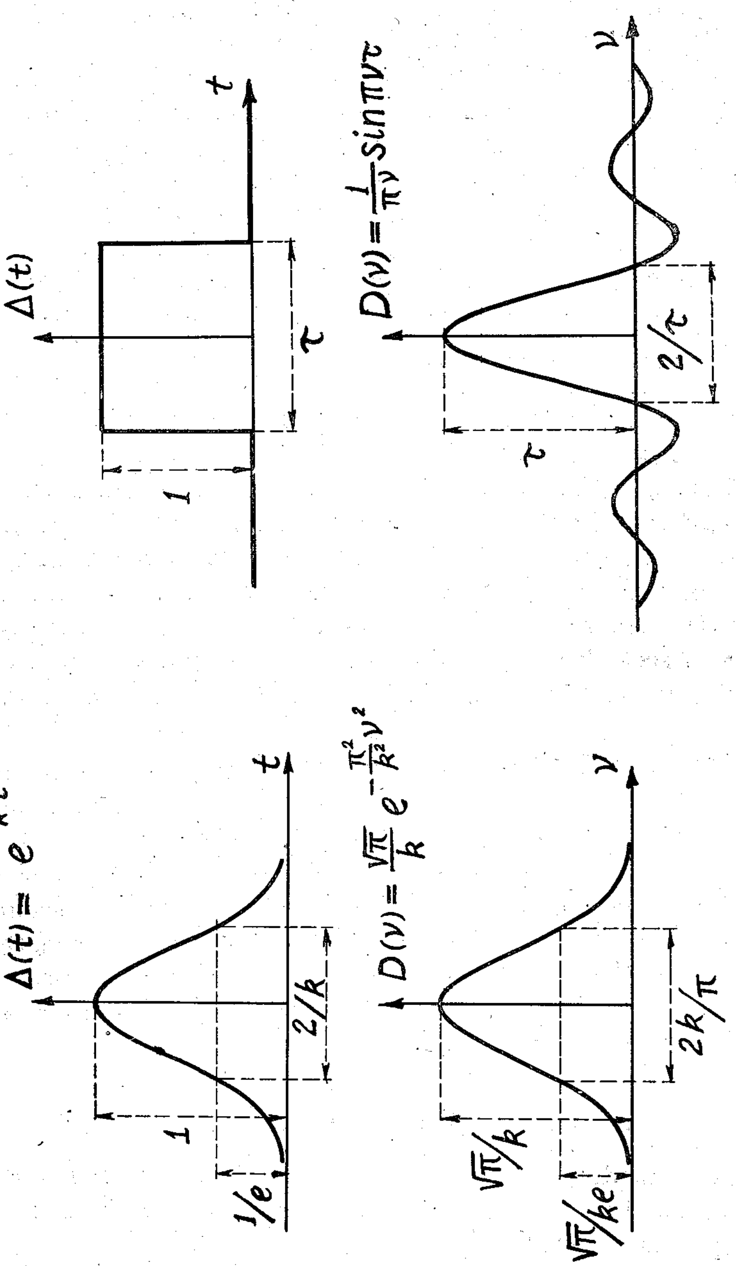

Fig. 1. Weight Functions and Their Spectra.

in the interval Δt, so that the integral becomes quite small. This yields the value $\nu' = \Delta\nu = 1/\Delta t$ as before.

The phase of $D(\nu, \nu')$ at $\nu' = 0$ is zero for a real function $\Delta(\nu, t')$ and so agrees with the ideal spectrum with infinite frequency discrimination. But as is shown by case (2.11a), it is different for other values of ν', and if the Δ-function is complex, even for $\nu' = 0$ a different phase is obtained. All these deviations which arise from the selection of a particular form of the Δ-function must be kept in mind in order not to deduce erroneous conclusions in interpreting an experimental result of sound analysis.

The relation (2,12) is usually called the uncertainty relation. The physical meaning of the word "uncertainty" in the case of sound analysis is to be understood as follows. The results of a sound analysis with a given value Δt of time constant have physical significance, in the sense that they are independent of the functional form of the Δ-function employed, when and only when a variation in time large compared to Δt, or a difference in spectrum with frequency difference larger than $\Delta\nu \approx 1/\Delta t$, is asked. It is meaningless to inquire about a time variation within a smaller interval than Δt, or to seek for a fine structure exceeding the lower limit $\Delta\nu$. The situation is different, however, when the analysing characteristics of a sound analyser are the object of investigation. This is the case, for example, when the function of the ear as a sound analyser is investigated. Every detail of the analysis obtained has its significance in describing how a sound is heard.

In concluding this brief theoretical treatment of varying sound analysis, it should be remarked that in the above description the time constant was supposed to depend only on frequency and not on time. A generalization in which this is variable with time may be possible. But as it seems superfluous for the present purpose of sound analysis, it was left untouched altogether.

§ 3. Photographic Method of Sound Analysis

An elementary description of the method has been given in a previous note[12]. A more detailed account is given here in connection with the general theory developed in § 2.

12) K. Imahori: l. c.

The method consists of three parts;

1) Modulation of light intensity by the sound to be analysed.

2) An "analysing plate" through which the modulated light passes, the plate acting somewhat like a stroboscopic plate with continuously varying spacings. See Plate V.

3) A moving photographic film on which the optical image of the analysing plate is formed by a photographic lense. A photograph is obtained which shows stroboscopically analysed components of the original light intensity variation.

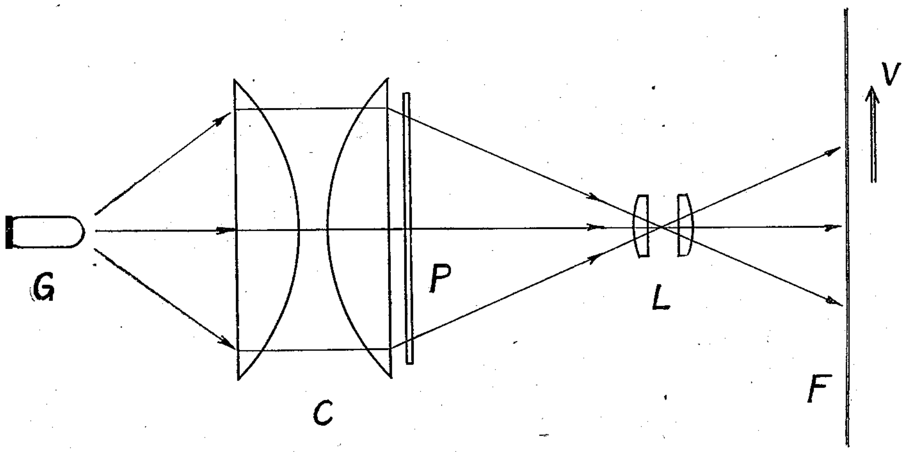

Fig. 2. Photographic Sound Analyser.

The arrangement of each part is shown diagrammatically in Fig. 2. G is a glow lamp for light source with modulated intensity, such as used in sound film recording. A large condenser lense C is used to concentrate the light on a photographic lense L which focuses an image of the analysing plate P on a moving photographic film F. An illustration of the analysing process is given in Fig. 3, in which it is shown how a series of short period exposures with equal intervals results in an overlapping of each exposure so that a stroboscopic frequency analysis is effected. Along the portion of the film, where the spacing of lines in the optical image of the analysing plate is such that this is just equal to the displacement of the film during each consecutive exposure, a periodic intensity variation with the same spacing is obtained on development. In order to eliminate a simultaneous appearance of other periodicities which are observed in the usual stroboscopic observations, the analysing plate is so prepared that the transmitted light intensity varies sinusoidally in the direction of the moving film. The reason will become clear in the following general treatment.

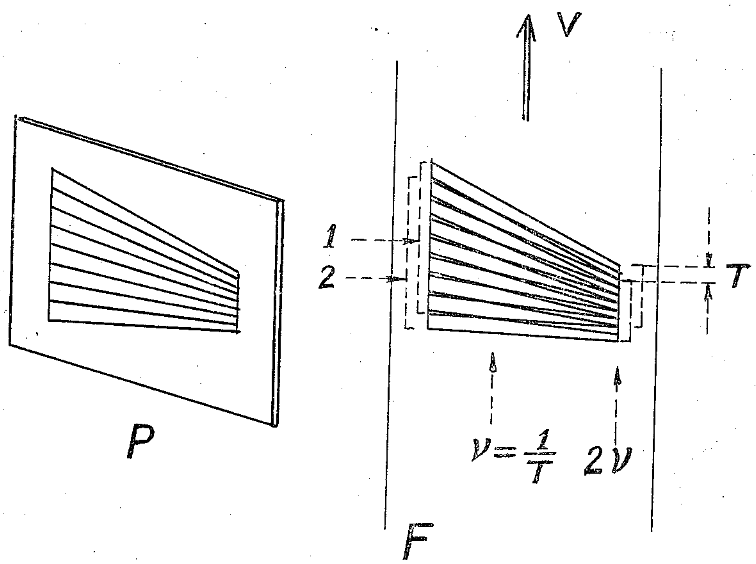

Fig. 3. Illustration of the Analysing Process.

Lete $f(t')$ denote the pressure variation of a sound which is received by a microphone with subsequent electrical amplification and finally transformed into a light intensity variation expressed by such a form as

$$P+Qf(t')\geq 0 \tag{3,1}$$

The constant term P is necessary, for the light intensity cannot take negative values, so that (3,1) must always be positive. The light beam illuminates uniformly the entire area of the analysing plate and the transmitted light undergoes a change in intensity by a factor

$$A+B\cos\frac{2\pi}{d}x\geq 0 \tag{3,2}$$

according to different positions on the plate. The coordinate x in this expression is taken in the plane of optical image of the analysing plate and is measured along the direction of motion of the moving photographic film. The wavelength d may in general be a monotonically increasing or decreasing function of y which is measured in the same plane and perpendicular to x axis, but only the simplest case of a linear function is used at present. Fixing one's attention for a moment on some special value of d, the amount of light per unit area received by a point on the moving film will be shown to be equal to

$$\int_{x=x_1}^{x=x_2}\left\{P+Qf(t')\right\}\left\{A+B\cos\frac{2\pi}{d}x\right\}dt', \tag{3,3}$$

where
$$x=\xi+Vt' \tag{3,3a}$$

represents the coordinate of the moving point on the film at time t', so that ξ is the coordinate at time $t'=0$, and V the velocity of the moving film. x_1 and x_2 are the end points between which the sinusoidal light intensity variation is found. By using the weight function described in § 2, (3,3) may be generalised to a form expressed by the integral

$$\int\left\{P+Qf(t')\right\}\Delta(y,x)\left\{A+B\cos\frac{2\pi}{d}x\right\}dt', \tag{3,4}$$

where the integral is to be taken from $t'=-\infty$ to $+\infty$. In practice this is realised by preparing a suitable mask or a second photographic plate, which covers the analysing plate and limits the portion of the latter so that the transmitted light is reduced in its intensity by a factor $\Delta(y, x)$. Some examples of these "weighted analysing plates" are shown in Plate V. They are:

1) $$\Delta(y,x)=1 \quad \text{for} \quad |x|\leq\frac{N}{2}d,$$
$$=0 \quad \text{for} \quad |x|>\frac{N}{2}d,$$

where N is the total number of periods and is independent of y, so that $\Delta\nu/\nu$ is constant for all ν.

2) $$\Delta(y,x)=1 \quad \text{for} \quad |x|\leq\frac{1}{2}V\tau,$$
$$=0 \quad \text{for} \quad |x|>\frac{1}{2}V\tau,$$

In this case V_τ is independent of y, so that $\Delta\nu \approx \frac{1}{\Delta t}$ is constant for all ν.

3) $$\Delta(y, x) = e^{-k^2x^2},$$

where k is taken inversely proportional to ν, so that

$$\Delta\nu/\nu = \text{const.}$$

4) The same as (3), but k is taken independent of ν. $\Delta t = \text{const.}$

In order to bring (3, 4) to a complex representation commonly employed in expressing periodic phenomena a variable transformation defined by

$$\xi/d = -\nu t, \qquad V/d = \nu \tag{3,5}$$

is applied and an expression

$$B(\nu, t) = \int \{P + Qf(t')\} \Delta(\nu, t'-t) \{A + Be^{-2\pi i\nu(t'-t)}\} dt' \tag{3,6}$$

is obtained which finally reduces to a sum of the following four terms:

$$A(\nu, t) = QB \int f(t') \Delta(\nu, t'-t) e^{-2\pi i\nu(t'-t)} dt', \tag{3,7a}$$

$$B(\nu) = PA \int \Delta(\nu, t'-t) dt', \tag{3,7b}$$

$$C(\nu, t) = QA \int f(t') \Delta(\nu, t'-t) dt', \tag{3,7c}$$

$$D(\nu) = PB \int \Delta(\nu, t'-t) e^{-2\pi i\nu(t'-t)} dt'. \tag{3,7d}$$

The first expression (3, 7a) is of the form treated in the preceding section and so represents the time variation of the sound spectrum, which is the required result. The constant term (3, 7b) represents a mean exposure determined by the form of Δ-function and results in a uniform blackness independent of time in the developed photograph. In addition to these the contributions of $C(\nu, t)$ and $D(\nu)$ given by (3,7c) and (3,7d) must be taken into consideration. $D(\nu)$ represents the spectrum of the weight function $\Delta(\nu, t')$ which has also been made use of in § 2. If the functional form of

$\varDelta(\nu, t')$ as a function of t' is chosen such that this has no appreciable components in the spectral region to be investigated, the effect of (3, 7d) will be completely eliminated. There occur, however, some cases where this condition is not satisfied, so that care must be taken in these cases not to make any erroneous interpretation of the photograph. In this connection it may be remarked that of the two forms of $\varDelta$-function (2, 6a) and (2, 6b) the latter is more recommendable because of its simple spectrum as shown by (2, 11b).

The physical meaning of (3, 7c) may easily be understood if one reminds himself of the sound recording process by the variable density method used in talkie film. In the integral (3, 7c) $\varDelta(\nu, t'-t)$ defines the light intensity distribution across the slit width, and $f(t')$ the time variation in the incident light intensity, so that $C(\nu, t)$ corresponds to the density variation in the recorded sound film. For a sufficiently small value of the time constant $\varDelta t$ determined by the $\varDelta$-function, $C(\nu, t)$ shows, as well as $A(\nu, t)$, the exact form of $f(t')$. For moderate or large values of $\varDelta t$, only that part of variation involved in $f(t')$ is recorded which is slow compared to variations within a time interval smaller than $\varDelta t$. $C(\nu, t)$ is thus distinguished from $A(\nu, t)$, the variation in time of the latter being in general more rapid than the former. This property of $C(\nu, t)$ may sometimes be of use in a case where the spectral region analysed by a given analysing plate and a given speed of the moving film[13] does not cover the entire frequency range of interest. Information in the frequency range lower than the analysed limit may be obtained from $C(\nu, t)$.

A simpler interpretation of (3, 6) is possible if this is divided into two parts:

$$B(\nu, t) = A\int\left\{P+Qf(t')\right\}\varDelta(\nu, t'-t)dt' \\ +B\int\left\{P+Qf(t')\right\}\varDelta(\nu, t'-t)e^{-2\pi i\nu(t'-t)}dt'. \quad (3,8)$$

The first term in the right hand side of this equation represents a recording of a phenomenon expressed by $P+Qf(t')$, while the second term analyses it. They are however not independent processes

13) The spectral region analysed by a given plate may be changed at will by changing the film speed.

and coincide with each other when the time constant becomes sufficiently small.

On developing the exposed film one obtains a negative in which is observed a blackness distribution corresponding to the amount of light received. Now the blackness or the density of an exposed photographic material is usually defined as the common logarithm of the opacity, and its dependence on the amount of light received is represented by the characteristic curve of the material. In Fig. 4 let the curve *ABC* represent the characteristic curve of the

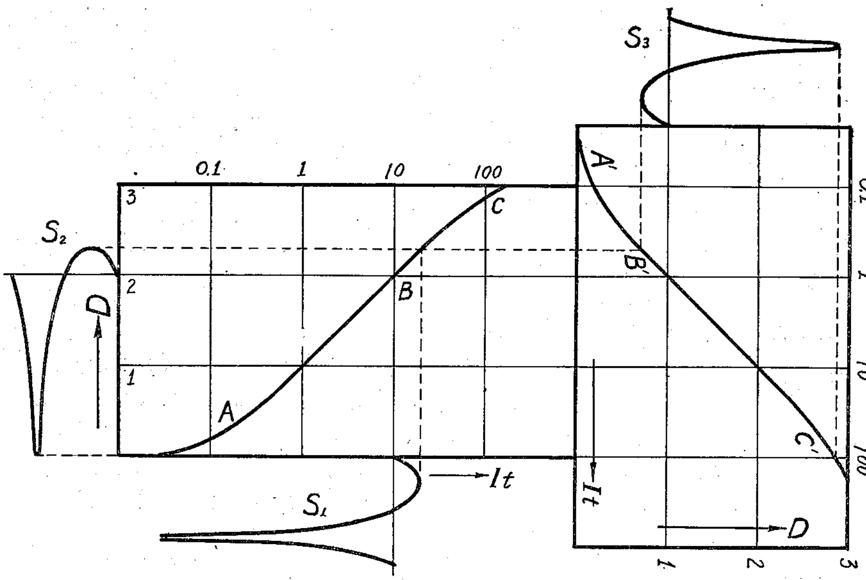

Fig. 4. Relation between Exposure and Blackness for Sinusoidal Light Intensity Variation.

negative, the abscissa It=light intensity×exposure time being measured in logarithmic scale as usual. A sinusoidal variation of light is then represented by a curve S_1 which causes a density variation shown by a similar curve S_2. Assuming Weber-Fechner's law to be valid so that the density is proportional to blackness as observed by the eye, in the negative one observes white lines on a black background. This circumstance seems to be unfavorable to a visual inspection, for small intensity variation are apt to be missed under dark illumination. This is due to the well-known fact that

Weber-Fechner's law fails for low intensity so that the minimum intensity discrimination $\Delta I/I$ of the eye increases rapidly as the light intensity I is diminished below a certain value. Thus the occasional missing is prevented simply by observing under sufficiently bright illumination. Moreover, the similar behavior of the eye and the ear found in their intensity discrimination may be utilised in an attempt to imitate the function of the ear by the present photographic method. The threshold value of sound intensity for the ear may be replaced by the corresponding minimum discrimination of intensity variation by the eye. The loudness as heard by the ear is proportional to the amplitude of blackness variation. If the amplitude of sound is desired, however, one must take recourse to a photometric opacity measurement with a negative of γ-value 1.

An alternative and perhaps a better way of presenting the results of an analysis is to make a positive print. The process is shown in the right half of Fig. 4. Black lines on a white background are observed in this case, so that the optimum condition for intensity discrimination is always satisfied for the eye. An imitation of the ear is also possible in this case, if one uses the "toe" portion in the characteristic curve of the printing paper, and an exposure is made in such a way that, corresponding to the threshold value of sound intensity the variation of light with amplitudes below some specified level comes under the toe of the characteristic curve. The Goldberg condition is also supposed to be satisfied in the linear portion of the curve, so that an exact parallelism between the loudness of sound and blackness amplitude in the analysed photograph may again be realised.

The analysing characteristics of the present method are completely determined by the analysing plate. The procedure for its preparation is now briefly described. In Fig. 5 is shown a simplified view of the arrangement used for this purpose.

An unexposed photographic plate A_1 is mounted on a large circular disk D, which is rotated about its centre by a driving rod R. The rod moves with constant velocity along a fixed line parallel to the plane of the disk, while a lever B with a straight edge and fixed to the disk is always kept in contact with the driving rod. This sliding edge of the lever must be parallel to the line which connects the centres of D and R, and also to one side of the plate A_1. A short focus cylindrical lense L_2 is placed in front of A_1, and a sharp straight line image passing through the centre of D and parallel to the line of motion of R is formed on the plate. The intensity of light is varied periodically by a small mirror M fixed to a pendulum, such that a photographic lense L_1 forms a vertically swinging image of a plate A_2 on the plane of the slit S which is placed in front of L_2. A_2 is a photographic plate having continuously varying density corresponding to a linear change of exposure time. The slit S is

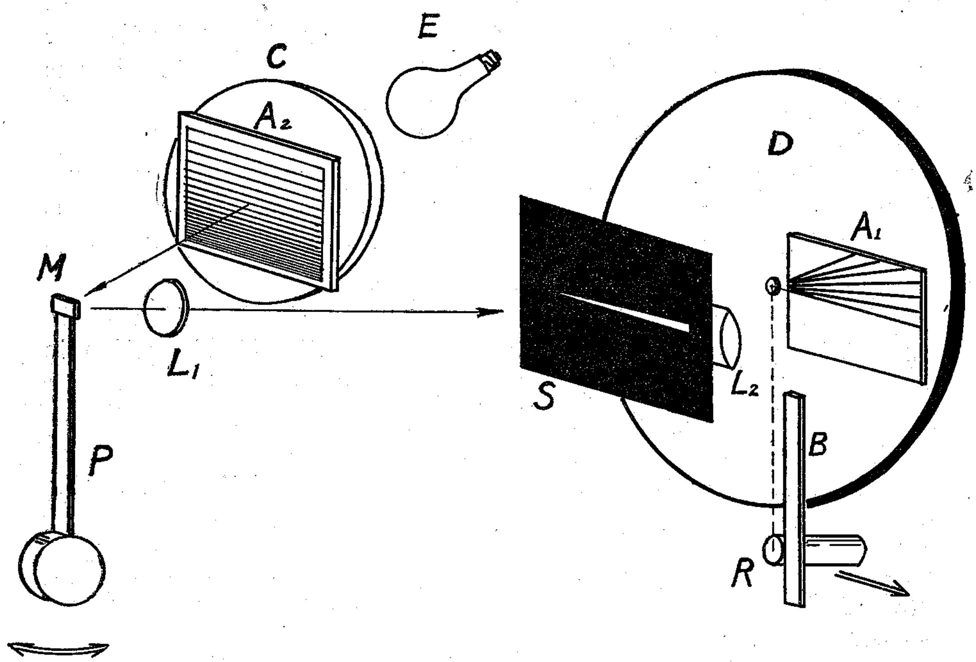

Fig. 5. Preparation of the Analysing Plate.

of wedge form, the point of contact corresponding to the centre of the disk, so that the light intensity along the straight line image formed by L_2 increases proportionally to the distance from the centre and an exposure is obtained which is independent of the distance. The angle of the slit must also be varied in accordance with the variation in angular velocity of the disk. But this effect is of no practical importance for small rotations not exceeding 45°. For wider angles of rotation an accessory device is needed for its correction.

By a proper adjustment of the intensity of light source E, the amplitude of the pendulum swing and the velocity of the driving rod, an analysing plate will be obtained having the desired characteristics. All the necessary technical procedures are the same as those employed in the recording system of sound film, so that any detailed description is unnecessary here.

In plate VI photographs are reproduced which show the time analysed spectra of two words "Ah!" and "Eh?" spoken in Japanese. The gradual ascending and descending in the word pitch are clearly shown by the corresponding curves which follow the frequency changes of each analysed component. To obtain more detailed information from these photographs, it is necessary to be well acquainted with the fundamental properties of the present method when applied to various typical examples. They are also of theoretical importance in a treatment of rapidly varying sounds, for erroneous arguments are sometimes found in dealing with such problems. In the remaining part of this paper, these problems of fundamental nature are treated with illustrations obtained by applying the present photographic analysis to corresponding model examples.

§ 4. Superposition- and Resonance- Principles

There are two principles of fundamental importance generally used in dealing with vibration and wave problems, the principle of superposition on the one side, and that of resonance on the other. Now the various existing devices for sound analysis may also be classified roughly into two, each employing one of the two principles. The present photographic method seems at first to belong to the class in which the superposition principle is employed, but the mathematical connection of the two principles enables one to obtain by photographic method results identical to those obtained by any other method. Thus the present method is the most general one, including others as special cases, and so is specially adapted to compare analysis obtained by different methods, and also to explain the intimate relation existing between the superposition- and the resonance-principles.

The principle of superposition on which a method of sound analysis is based, is involved in the expression (2, 3) :

$$A(\nu, t) = \int f(t')\Delta(\nu, t'-t)e^{-2\pi i\nu(t'-t)}dt' \tag{4,1}$$

$$= \int f(t')K(\nu, t'-t)dt'. \tag{4,1a}$$

Let $A_1(\nu, t)$ and $A_2(\nu, t)$ represent the spectra of two functions $f_1(t)$ and $f_2(t)$ such that

$$f(t) = a_1 f_1(t) + a_2 f_2(t) . \tag{4,2}$$

Then the spectrum of $f(t)$ is obtained by superposing $a_1 A_1(\nu, t)$ on $a_2 A_2(\nu, t)$, thus

$$A(\nu, t) = a_1 A_1(\nu, t) + a_2 A_2(\nu, t) . \tag{4,3}$$

This may be generalized to a sum of any number of functions and also to an integral with respect to a parameter. Thus the spectrum of a function defined by (4, 1) may be obtained by resolving $f(t')$ into any number of functions whose spectra are known and then by superposing the spectra of each of the components. Such a procedure is easily carried out in the photographic method by multiple printing on the same photographic paper. In fact the analysing process of the photographic method is in itself an automatic operation of these procedures. To use mathematical language, the function $f(t')$ is expressed by an integral such that

$$f(t') = \int f(t'')\delta(t''-t')dt'', \tag{4,4}$$

while the spectrum of Darac's δ-function is given by using (4, 1) or (4, 1a):

$$K(\nu, t'-t) = \int \delta(t''-t')K(\nu, t''-t')dt''. \tag{4,5}$$

Thus the equation (4, 1a) may be regarded as a mathematical expression for the superposition of "elementary spectra" $K(\nu, t'-t)$, which is of course another name for the analysing plate.

In Plates VII–X photographs are reproduced which show the successive processes by which the elementary spectra are superposed to give the final results. The operation of Dirac's function $\delta(t'-t_1)$ such as used in (4, 5) corresponds in the photographic method to a finite (unit) exposure while the film to be exposed is stopped at an instant $t' = t_1'$, so that these photographs, except those of Plate X, were obtained by repeated printing on a photographic paper. The analysing plate used is that shown in Plate V, Fig. 1. The same is used throughout this paper. The examples chosen are as follows:

Plate VII. Two impulses at various time intervals. Band spectra are obtained with frequency difference $1/T$ in the region where $T<\tau$ is satisfied. T is the time interval between two impulses. The maximum amplitude is found at $\nu=0,\ 1/T,\ 2/T,\ \ldots$.

Plate VIII. Series of impulses occurring at equal time interval T. The number of impulses is 1, 2, 4, 8, and ∞ for each photograph. Besides the principal bands at $\nu = 1/T,\ 2/T,\ \ldots\ldots$ there appear subsidary bands with smaller amplitudes. As the number of impulses increases the band width decreases, while the subsidary bands increase in number and decrease in amplitudes, their limits being determined by the characteristics of the analysing plate used. These circumstances are just analogous to the optical diffraction by a number of parallel slits.

Plate IX. Series of groups of impulses with equal time intervals. The principal bands of plate VIII split up into a number of components with frequency difference $1/T'$, the central components being at $\nu = 1/T,\ 2/T,\ \ldots$. T' is the time interval between successive groups and T that between successive impulses. Various stages of splitting may be observed by comparing those of different harmonics. When the number of groups is increased indefinitely, a case of amplitude modulation is obtained which will be treated separately in a later section.

Plate X. Dirichlet's discontinuous function,

$$\begin{aligned} f(t') &= 1 \qquad \text{for} \qquad t_1 \leq t' \leq t_2 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

Very similar to those of plate VII. But in this case the amplitude is zero for $\nu = 1/T,\ 2/T,\ \ldots$, and maximum for $\nu = 0,\ (1-\frac{1}{2})/T,\ \ldots$, were $T = t_2 - t_1$ is the time of exposure. The maximum amplitude decreases inversely proportional to ν. They are an exact analogue of light differaction by a slit of finite aperture. Similar cases to Plate VIII may also be considered.

The elementary spectrum $K(\nu, t'-t)$ defined by (4, 1a) is just analogous to the elementary wavelet used in Huygen's principle and more generally to Green's function in the theory of linear differential equations. The function $A(\nu, t)$ expressed by (4, 1a) may thus be regarded as a solution of some dynamical, electrical or optical problem. Now the resonance principle used in sound analysis may generally be formulated in a following mathematical form. A series of resonators having different responses is subject to an external force the form of which is to be analysed. The motion of each resonator is determined by linear differential equations of the form

$$L_\nu[x] = f(t), \tag{4,6}$$

where L_ν stands for a linear differential operator, the actual form of which is determined by the mechanical or other characteristics of the resonating system, and the suffix ν distinguishes different resonators. Let the solution of (4, 6) for

$$f(t) = e^{2\pi i\nu' t} \tag{4,7}$$

be given by the form

$$x_\nu(t) = D(\nu, \nu')e^{2\pi i\nu' t}, \tag{4,7a}$$

then the solution of (4, 6) for any $f(t)$ will be expressed as

$$x_\nu(t) = A(\nu, t) = \int A(\nu')D(\nu, \nu')e^{2\pi i\nu' t}d\nu', \tag{4,8}$$

where $A(\nu')$ is the Fourier component of $f(t)$, so that

$$A(\nu') = \int f(t')e^{-2\pi i\nu' t'}dt'. \tag{4,8a}$$

Equation (4, 8) represents how the applied force is filtered by each resonator, and when considered as a function of ν it shows the distribution of amplitude in different frequency regions. Moreover this equation can be brought to the same form as (4, 1). Since by denoting the solution of (4, 6) as

$$x_\nu(t) = K(\nu, t'-t), \tag{4,9}$$

for the case in which $f(t) = \delta(t'-t)$, one obtains the solution for an arbitrary function $f(t)$ in the form

$$A(\nu, t) = \int f(t')K(\nu, t'-t)dt, \tag{4,10}$$

which is identical to (4, 1a). A relation between $K(\nu, t'-t)$ and $D(\nu, \nu')$ defined by (4, 9) and (4, 7a) may easily be deduced which is expressed as

$$D(\nu, \nu') = \int \Delta(\nu, t'-t) e^{-2\pi i\nu'(t'-t)} dt', \tag{4,11}$$

where

$$K(\nu, t'-t) = \Delta(\nu, t'-t) e^{-2\pi i\nu'(t'-t)}, \tag{4,11a}$$

which corresponds to the relation already obtained in (2, 10).

Thus an exact parallelism between the method of superposition and that of resonance is established, so that it becomes possible to interpret one result of analysis in terms of another and vice versa. In particular, the present photographic method yields a means for representing and comparing different methods of analysis, as mentioned previously.

The photographs shown in Plates VII–X may be used to explain analysing process of the photographic method in terms of the resonance principle. Each sinusoidal variation of light in the analysing plate is compared to the possible motion in a set of resonators with various proper frequencies when they are acted upon by an instantaneous impulsive force. The number of periods in the plate corresponds to the reciprocal of the logarithmic decrement of the resonator. When a single impulse is applied, all the resonators are set in motion. (Plate VIII, 1) By another impulse following this, the motion of the resonator is modified so that a resultant motion occurs as long as the effect of the first impulse continues. (Plate VII). For a series of impulsive forces, the same process is repeated, and finite values of amplitude are found in the resonators where the motion occurs in the same phase. (Plates VIII, IX and X). Generally the photograph consists of a stationary part where the spectrum is constant for a time large compared with one period of the applied force, and another part where it is variable. The latter represents the transient state, an example of which is shown very clearly in Plate VIII (5). Also see Plate IX. It is to be noted that, according to the uncertainty relation described previously these transient states should not be considered as existing in reality, but they are only results of an analysis, different results being obtained by different methods.

§ 5. Amplitude Modulation and Interrupted Waves

The time variation of a sound wave may occur in the following four ways; 1) change of amplitude, 2) change of phase, 3) change of frequency and 4) change in overtone structure. The terms in these expressions are, however, not to be taken in their precise meaning. There is no change in frequencies, for example, where frequencies are determined exactly, so that infinite time is necessary for their measurement. In order that the above expressions may have a physical meaning, it must be admitted that some standard of judgment is used. The method of varying sound analysis formulated in this paper has shown this standard in a concrete form.

It is a well established fact that a modulated wave can be represented as several side-bands superposed on the main frequency. The wave is, in this point of view, analysed with infinite frequency discrimination. But considered as a change of amplitude, it must be treated in a different way. As a most simple case, consider a harmonic wave with its amplitude modulated by another such that

$$f(t) = (1+\cos 2\pi\nu_0 t)e^{2\pi i\nu' t} \tag{5,1}$$

The corresponding $A(\nu, t)$ is calculated as

$$A(\nu, t) = D(\nu, \nu-\nu')e^{2\pi i\nu' t} + \frac{1}{2}D(\nu, \nu-\nu'+\nu_0)e^{2\pi i(\nu'-\nu_0)t}$$
$$+\frac{1}{2}D(\nu, \nu-\nu'-\nu_0)e^{2\pi i(\nu'+\nu_0)t}, \tag{5,2}$$

where $$D(\nu, \nu'') = \int \Delta(\nu, t')e^{-2\pi i\nu'' t'}dt'.$$

When the range $\Delta\nu$ of ν'' in which $D(\nu, \nu'')$ is not sensibly equal to zero is small compared to ν_0, the three terms in (5, 2) appear separately, so that the spectrum is split up into three components, the central band of frequency ν' and two side-bands of frequency $\nu' \pm \nu_0$. In the other extreme case where $\Delta\nu$ is large compared to ν_0, so that $D(\nu, \nu'')$ is regarded as constant in the neighbourhood of $\nu''=0$, (5, 2) is reduced to

$$A(\nu, t) = (1+\cos 2\pi\nu_0 t)D(\nu, \nu-\nu')e^{2\pi i\nu' t}, \tag{5,2a}$$

An exact variation of amplitude is reproduced here. For the intermediate cases in which ν_0 is comparable to $\Delta\nu$, the spectrum depends very much on the functional form of $D(\nu, \nu'')$. Generally the splitting into components is obscured, varying periodically with time. The amplitude variation is also distorted in different degrees for different frequencies. A better idea will be obtained in this case when the process of analysis is considered by making use of the superposition principle as described in the preceeding section.

Examples are shown of these phenomena in Plate XI, in which impulse waves of constant frequency ν' are modulated by repeating a process such that the amplitude is constant for a number of periods and then the same interval of zero amplitude follows. The frequency of intermittence is varied for each example. Various stages of interference are amply illustrated. The photographs shown in Plates VIII–X may also be regarded as other examples of amplitude modulation, in which the modulation is effected for a limited time interval.

The examples described here, as well as those to be dealt with in the following sections, have an important bearing on the theory of hearing. As mentioned previously, the present photographic method enables one to simulate the ear in analysing variable sounds. The author is of the opinion that the tone quality of sound in its widest sense cannot be explained simply by an amplitude spectrum, as has been done successfully in the special case of stationary musical tones. Application of the amplitude spectrum to rapidly varying sound leads one to such an absurd conclusion as, for example, that the tone quality of a sound of vary short duration is the same as that found in carbon noise of a microphone. Backhaus[14] and others have pointed out the importance of non-stationary or transient elements especially at the beginning of sounds. The physical significance of these phenomena can only be understood by taking into account the analysing characteristics of the human ear. As different results are obtained by different methods of analysis, one is led to the conclusion that the tone quality of varying sounds is a result of a particular method of analysis, i.e. by the ear, so that it may be

14) H. Backhaus: Exgeb. d. Exakt. Naturwiss., 16, 237. 1937.

adequately expressed by a function $A(\nu, t)$ of (2,3) which it is now possible to obtain experimentally with close resemblance to the ear.[15]

Another special example of amplitude modulation which has frequently been dealt with in the theory of hearing is the effect of interrupting a continuous sound wave. In Plate XII is shown the effect of dropping 1, 2, 4, 6, 8, 12 and 16 consecutive periods in a train of impulses. Practically no effect is observed where the interval of pause is small compared to the time constant of the analysing plate, while a complete separation occurs when the former is made larger than the latter. In the intermediate stage the two waves before and after the interruption supperpose, resulting in a splitting up of the original band into a number of components with the frequency difference $\Delta\nu$ characteristic to the analysing plate ($\Delta\nu \approx 1/\Delta t$).[16]

If the number of intermittences is increased more than one, the same effect will be found as those shown in Plates IX and XI. The time constant of the analysing plate used in these examples is inversely proportional to frequency, so that as the time of pause is increased the harmonic components of higher order are separated before those of lower order, while on the other hand all the harmonic components will be separated simultaneously with the fundamental in case the time constant is independent of frequency.

§ 6. Phase Change, Beats and Other Related Phenomena.

The effect of phase reversal upon hearing has been investigated by Hartridge and others.[17] As one effect a corresponding interruption of the tone was heard. A simple treatment of this problem by the resonance theory of hearing is to consider a resonator tuned to the incoming sound and see what happens when the phase is reversed. Obviously the resonator comes to rest and then restarts in the new phase. The interval of time spent in this fall and rise in amplitude is of the order of the time constant of the resonator. But

15) The practical application of the present method to the theory of hearing will be treated in detail in a separate paper. It must be noticed that the photographs shown in this paper are no imitation of the ear.

16) Cf. Effect of phase change in § 6.

17) H. Hartridge: Proc. Phys. Soc. 48, 145. 1936. C.S. Hallpike, H. Hartridge and A. F. Rawdon-Smith: Proc. Phys. Soc. 49, 190. 1937. L. Hartshorn: Proc. Phys. Soc. 49, 194. 1937.

the situation is not so simple when one considers the effects produced on the adjacent resonators with proper frequencies above and below the one in tune. Békésy[18] has pointed out the importance of this phenomenon. A general consideration of this problem follows which is also applicable to other cases such as interrupted waves, transient phenomena at the beginning and end of a limited wave train, change of frequency and so on.

Consider a harmonic wave with frequency ν_0 which continues up to a time $t = t_1$ and then comes to rest. The analysis of such a wave will be represented by

$$\begin{aligned} A_1(\nu, t) &= \int_{-\infty}^{t_1} e^{2\pi i\nu_0 t'} \Delta(\nu, t'-t) e^{-2\pi i\nu(t'-t)} dt' \\ &= e^{2\pi i\nu_0 t} \int_{-\infty}^{t_1-t} \Delta(\nu, t') e^{-2\pi i(\nu-\nu_0)t'} dt'. \end{aligned} \tag{6,1}$$

By the property which has been ascribed to the Δ-function, a rough estimation may be made of this function.
Thus

$$\begin{aligned} \int_{-\infty}^{t_1-t} \Delta(\nu, t') e^{-2\pi i(\nu-\nu_0)t'} dt' &= D(\nu, \nu-\nu_0) \quad \text{for} \quad t_1-t \gg \Delta t \\ &= 0 \qquad\qquad\quad \text{for} \quad t-t_1 \gg \Delta t, \end{aligned} \tag{6,2}$$

where the function $D(\nu, \nu')$ is defined by (2,10) and Δt is the time constant of the Δ-function. For those cases in which $|t_1-t|$ is comparable to Δt, reasoning from a graph may be of use.[19] If the frequency difference $|\nu-\nu_0|$ is sufficiently large so that in the time interval Δt where the Δ-function is sensibly different from zero a number of periods of $e^{-2\pi i(\nu-\nu_0)t'}$ is involved, the integral (6,2) may be approximated by a harmonic function with an amplitude inversely proportional to $|\nu-\nu_0|$ and its phase changed by $\pi/2$, so that

$$\int_{-\infty}^{t_1-t} \Delta(\nu, t') e^{-2\pi i(\nu-\nu_0)t'} dt' = \frac{\Delta(\nu, t_1-t)}{2\pi i(\nu-\nu_0)} e^{-2\pi i(\nu-\nu_0)(t_1-t)} \tag{6,3}$$

$$\text{for} \quad |\nu-\nu_0| \gg \Delta\nu$$

$$\text{and} \quad |t_1-t| \ll \Delta t,$$

18) Békésy: Phys. ZS. 29, 793. 1928.
19) See Fig. 6.

where $\Delta\nu \approx 1/\Delta t$ is the characteristic frequency difference of the Δ-function. The lower limit of the frequency difference $|\nu-\nu_0|$ will be found in a case where about one complete period of $e^{-2\pi i(\nu-\nu_0)t'}$ coincides with Δt, so that $|\nu-\nu_0| = \Delta\nu$. For smaller values of

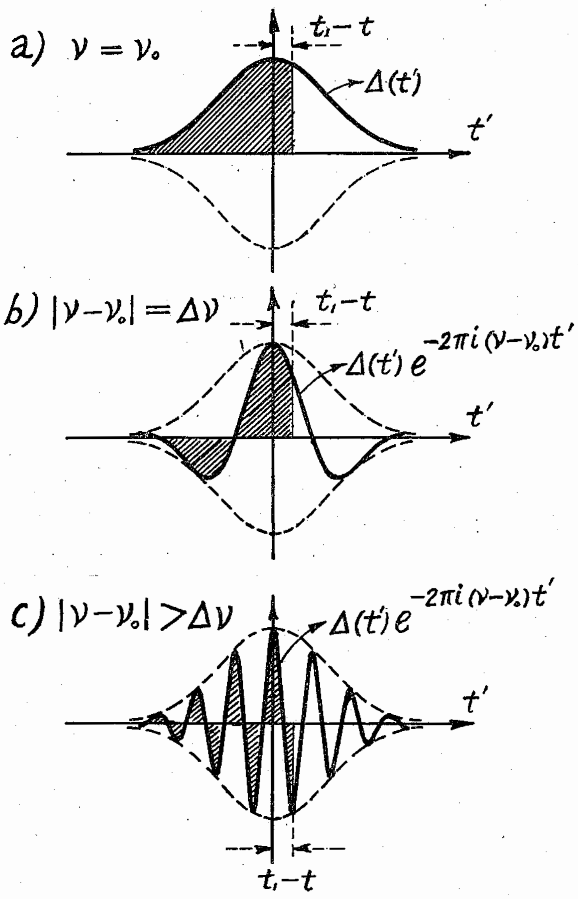

Fig. 6. Graphical Estimation of the Integal
$\int \Delta(t')e^{-2\pi i(\nu-\nu_0)t'}dt.$

$|\nu-\nu_0|$ than this the integral (6,2) becomes a monotonous function such that as $t'-t$ is increased the amplitude increases from zero up to a finite value of the order of

$$D(\nu, 0) = \int_{-\infty}^{\infty}\Delta(\nu;\, t')dt' .$$

Thus the spectrum given by (6,1) represents in the transient part a beat for large $|\nu-\nu_0|$ and a dampted oscillation for $\nu=\nu_0$. These phenomena are essentially the same as those well known in the case of osillators acted upon by an external periodic force. They are illustrated in Plate VIII.

Now a second harmonic wave with the same frequency but different phase is considered to start at an instant $t=t_2$. A similar treatment is applied. Its spectrum is now given by

$$A_2(\nu,\, t) = e^{2\pi i\nu_0 t+i\alpha}\int_{t_2-t}^{\infty}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt', \tag{6,4}$$

where
$$\int_{t_2-t}^{\infty}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt'$$
$$= D(\nu,\, \nu-\nu_0)-\int_{-\infty}^{t_2-t}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt'. \tag{6,5}$$

For $t_1=t_2$ two expressions (6,2) and (6,5) are complementary in the sense that when superposed they reduce to the function $D(\nu_1\, \nu-\nu_0)$.

Superposition of (6,1) and (6,4) for $\alpha=0$ explains the spectra of interrupted waves, while for $\alpha=\pi$ and $t_1=t_2$ the spectra of phase reversal are obtained. Thus for the latter case

$$\begin{aligned} A(\nu,\, t) &= A_1(\nu,\, t)+A_2(\nu,\, t) \\ &= e^{2\pi i\nu_0 t}\left[\int_{-\infty}^{t_1-t}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt'-\int_{t_1-t}^{\infty}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt'\right] \\ &= e^{2\pi i\nu_0 t}\left[2\int_{-\infty}^{t_1-t}\Delta(\nu,\, t')e^{-2\pi i(\nu-\nu_0)t'}dt'-D(\nu,\, \nu-\nu_0)\right], \end{aligned} \tag{6,6}$$

so that in the stationary part,

$$\begin{aligned} A(\nu,\, t) &= D(\nu,\, \nu-\nu_0)e^{2\pi i\nu_0 t} \quad \text{for} \quad t_1-t \gg \Delta t, \\ &= -D(\nu,\, \nu-\nu_0)e^{2\pi i\nu_0 t} \quad \text{for} \quad t-t_1 \gg \Delta t, \end{aligned} \tag{6,7}$$

corresponding to the phase reversal of the waves before and after the time $t=t_1$. For the transient part where $|t_1-t|\approx\Delta t$, the case of resonance is first considered. Putting $\nu=\nu_0$ in (6,6) one obtains

$$A(\nu,\, t) = e^{2\pi i\nu_0 t}\left[2\int_{-\infty}^{t_1-t}\Delta(\nu,\, t')dt'-D(\nu,\, 0)\right], \tag{6,8}$$

where the integral $\int_{-\infty}^{t_1-t} \Delta(\nu, t')\,dt'$ varies from $D(\nu, o)$ to zero as t goes through t_1, so that at a certain instant it will assume a value $\frac{1}{2}D(\nu, o)$. The amplitude of (6,8) will then become zero, and a corresponding pause must be observed.

If the frequency difference $|\nu-\nu_0|$ is other than zero but smaller than $\Delta\nu$, the integral in the bracket of (6,6) varies also from $D(\nu, \nu-\nu_0)$ to zero as t passes through t_1, but on account of the continuously varying phase of this integral, the two terms in the bracket of (6,6) give rise to a finite amplitude in the transient state, which by a closer graphical study is found to be greatest for $|\nu-\nu_1| = \frac{1}{2}\Delta\nu$. For $|\nu_1-\nu| > \Delta\nu$, both of the two terms in (6,6) are very small, so that this region of frequency is unimportant.

Thus it is shown that the effect of phase reversal in a train of waves is such that a separation of the spectrum into two components with frequency difference $\Delta\nu$ is found in an interval Δt where the reversal occurs, while the original band disappears in this interval. In Plate XIII, Fig. 3 illustrates these phenomena very clearly. The phase reversal was effected by an increase of time interval between two consecutive impulses by $\frac{1}{2}$ in a train of impulse waves. The other photographs show cases where the changes of phase are varied from zero to 2π. It will be noted that as the amount of phase change is varied, the position in the spectrum of maximum amplitude is continuously shifted, one appearing after another. The same is more clearly observed when one prepares two photographic plates such as shown by Fig. 5 of Plate VIII but with opposite time direction, and looks through them by sliding one plate over the other along the time direction of the photographs.

In Plates XIV and XV are shown further examples of similar nature. Plate XIV illustrates the combined effect of interruption and phase change, and in Plate XV is shown the case where a second impulse wave with different phase commences while the first is still in motion.

A similar general treatment of the effect of interruption shows that in this case also the spectrum splits up into components with frequency difference $\Delta\nu$, but there appears always the central band $\nu = \nu_0$, a marked decrease in the amplitude of which is not found until the interval between successive interruptions is increased up to the order of Δt, as previously explained by Plate XII.

A periodical repetition of phase reversal is found in the case of beat. The time analysed spectrum of beats may be classified into three groups. The first class contains those for which the beat frequency is so low that the amplitude of oscillation may be regarded as practically zero in the time interval Δt (time constant of the analyser) at the point of phase reversal. The spectrum shows in closest approximation the rise and fall in amplitude. The second is the case where finite amplitude is found in Δt which contains the time of phase reversal, but the beat frequency is still sufficiently low so that no two consecutive phase reversals fall in the same time interval Δt. Each phase reversal is independently analysed, and the corresponding periodical splitting with frequency difference $\Delta\nu$ is observed in the spectrum. Beats for which the beat frequency is higher than $1/\Delta t$ belong to the third class. The spectrum is then completely separated into two components corresponding to the component frequencies in the beat. Two harmonic waves with slightly different frequencies ν_1 and ν_2 are also completely separated in the spectum when $|\nu_1-\nu_2| > \Delta\nu$, as previously mentioned.

The photographs shown in Plate XVI were obtained by double exposures corresponding to two impulse waves with slightly different frequencies. The values of $(\nu_1-\nu_2)/{}^1\!/_2(\nu_1+\nu_2)$ for each photograph are

1) 1/40,5, 2) 1/20,5, 3) 1/10,5, 4) 1/5,5.

On the other hand the number of waves contained in the analysing plate is 18, so that from (2, 11a) the value of $\Delta\nu/\nu=1/9$. Thus Fig. 3 corresponds to the case where the separation of ν_1 and ν_2 is almost completed, while Fig. 1 corresponds to the first class, Fig. 2 to the second class and Fig. 4 to the third class in the above classification respectively. According to Schaefer and Guttmann[20] the frequency differences at which roughness due to beat is first perceived by the ear are approximately half of those where the first beginning of separation is recognised, in perfect agreement with the classification given above.

In Plate XVII is shown some examples in which two frequencies ν_1 and ν_2 are in simple whole number relations such as

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20) K. L. Schaefer and Guttmann: ZS. f. Psychol. u. Physiol. d. Sinnesorg., 32, 87, 1903. Handb. d. Norm. u. Pathol. Physiol. XI, 680. 1926. (A chapter on the theory of hearing by E. Waetzmann).

1) 1:2, octave; 2) 2: 3, fifth;
3) 3:4, fourth; 4) 4: 5, third;
5) 8:9, second; 6) 15:16, half tone.

To these are added two examples of triads,

7) major triad; 8) minor triad.

Beats produced by various overtones of adjacent frequencies may be clearly observed.

§ 7. Frequency Change and Triller.

One of the fundamental problems in the theory of hearing which has been frequently investigated by many authorities is the differential pitch sensitivity of the ear.[21] Discrimination by the ear of two pure tones with slightly different frequencies when sounded independently one after the other depends upon intensity, pitch and ear training. The effect of intensity will be treated separately in a later paper concerning the theory of hearing.[22] In this section a description is presented of the relation between frequency change and its time analysed spectrum.

In Plate XVIII photographs are given of the spectra obtained by analysing impulse waves the frequencies of which are suddenly changed by finite amounts. The ratio $\nu_1 : \nu_2$ of the frequencies and the corresponding intervals of tones before and after the changes are

1) 16:15, half tone; 2) 9:8, second; 3) 5:4, third;
4) 4: 3, fourth; 5) 3:2, fifth; 6) 5:3, sixth;
7) 15: 8, seventh; 8) 2:1, octave.

The value of $\Delta\nu/\nu$ of the analysing plate is the same as that used in the preceding photographs, viz. 1/9. For this value of $\Delta\nu/\nu$, the change of frequency is reproduced continuously in the spectrum up to a half tone. Of course the change may be detected for much

21) For example; V. O. Knudsen; Phys. Rev. 21, 84. 1923. H. Fletcher; Journ. Frankl. Inst. 295. 1923. E. G. Schower and R. Biddulph; J.A.S.A., 3, 275. 1931. W. E. Kock; J.A.S.A., 9, 129. 1937.

22) An analysing plate such as shown in Plate I, Figs. 3 and 4 shows different frequency discrimination for different amplitudes, in close analogy to the ear.

smaller relative frequency changes. When the relative frequency change exceeds 1/9, there appears in the transient part of the spectrum such a disturbance as is observed in Fig. 2. This disturbance is caused by an interference between transients of the two different waves, which are separated in the cases of Figs. 3–6. In Figs. 7 and 8 continuous frequency changes are observed between the octave of the lower note and the fundamental of the higher one. The same phenomena may also be observed in the higher harmonics in the other photographs.

Interesting phenomena are found in the case of triller, i.e. periodic repetitions of transitions between two fixed frequencies. In the model wave representing a triller, impulse waves were used in Plates XIX–XXII, such that a number of impulses with one fixed period (group A) is followed by the same number of impulses with another fixed period (group B). For slow trillers where the time interval between each successive group of the same signature is larger than Δt (time constant of the analyser), each change of frequency is independently analysed in the spectrum, so that the changes of frequency are exactly reproduced. It is to be noted here, and also in the other cases previously dealt with, that the precision in following the time variation of frequency is different for different harmonics, owing of course to the different time constant for different frequency of the analysing plate used. It would be an interesting problem in this connection to determine the exact behaviour of the time constant of the ear for various frequencies.

When the triller velocity is increased, and two successive groups of the same signature are found in the same time interval Δt, or in other words, the time interval between two consecutive changes of frequency becomes smaller than Δt, there occur interferences between adjacent transients. If the frequency difference of the tones on which the triller is made is sufficiently large so that the two frequencies are separated in the transient parts of the spectrum, they give rise to practically independent beats, because each group of the same signature is independently making periodically interrupted waves accompanied by phase changes. For small frequency differences comparable to $\Delta\nu \approx 1/\Delta t$, the two beats interfere with each other, and result in a single band with mean frequency of the two components in the triller, accompanied by periodically appearing and disappearing side bands. As the triller velocity is further increased, the interferences also proceed. In the limiting case where each group of impulses A and B is made up of but one or two periods, so that the number of groups entering into the same time interval Δt becomes sufficiently large, the original two bands which constituted the triller entirely disappear and a single band is observed with an intermediate frequency. The reason is simply explained as follows.

A group of impulses with small number of periods gives rise to a spectrum such as shown in Plate VIII. When a sufficiently large number of such spectra corresponding to those groups contained in Δt are superposed, the condition for maximum resultant is not satisfied in general by the original two frequencies unless both of the groups A and B always occur in the same phase. Maximum amplitude is instead found somewhere between them, the exact position depending upon the wave structure of the triller. It may be obtained from Fourier analysis of the given triller wave, the time interval between two successive groups of the same signature being regarded as the fundamental period. The amplitude becomes smaller as the frequency interval of the triller tones is increased. In case of sounds with overtone structure, one observes also in these extreme cases a band between each pair of adjacent harmonics of different sounds.

In Plates XIX–XXII are illustrated various stages of the phenomena described above. Examples are given for

Plate XIX, triller on semitone;
Plate XX, ,, ,, second;
Plate XXI, ,, ,, fifth;
Plate XXII, ,, ,, octave.

And the number of periods in a single group of impulses are for all plates as follows:

1) 1, 2) 2, 3) 4, 4) 6, 5) 8, 6) 12.

§ 8. Acknowledgements

In concluding this paper the author wishes to express his hearty thanks to Prof. Takeo Hori for his encouragement, and also to the Hattori Hokokai for financial support.

Analysing Plates

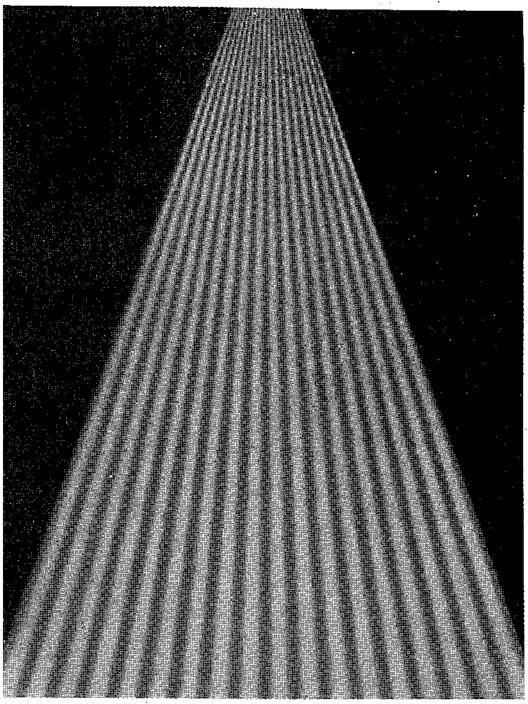


Fig. 1

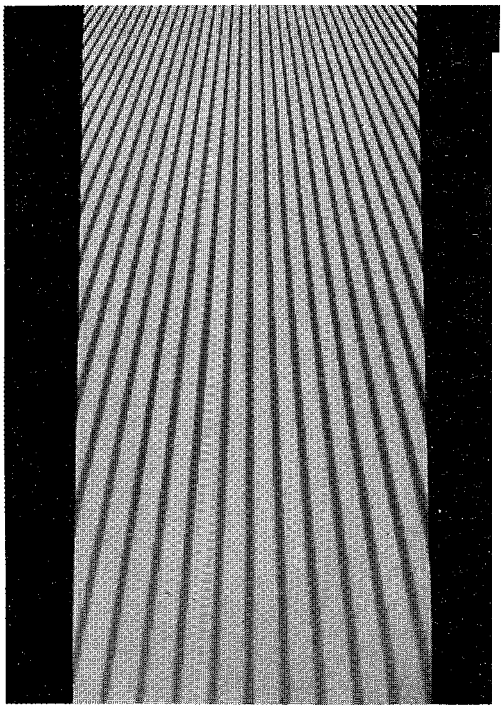
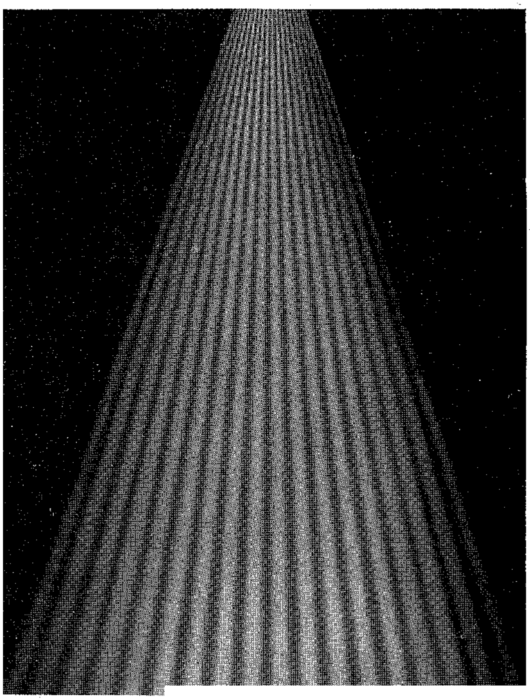
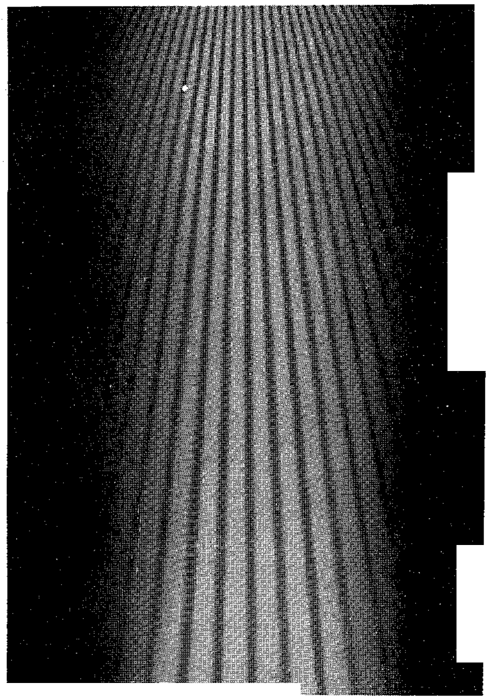
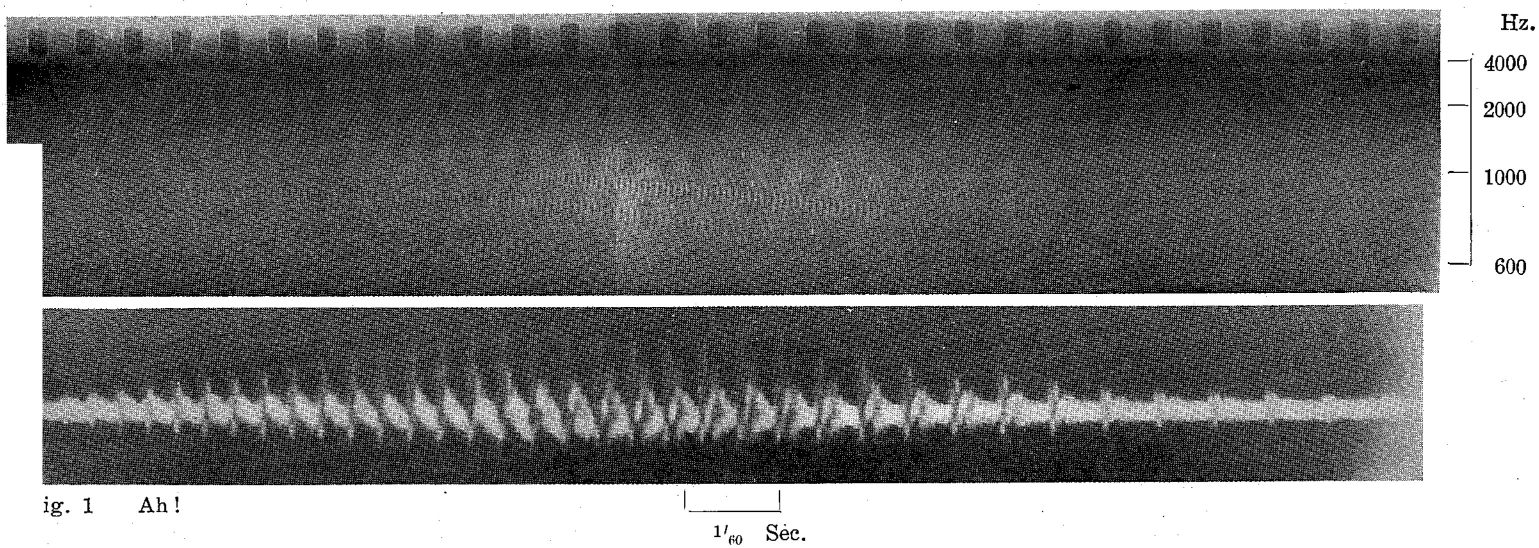


Fig. 2







ig. 1 Ah!

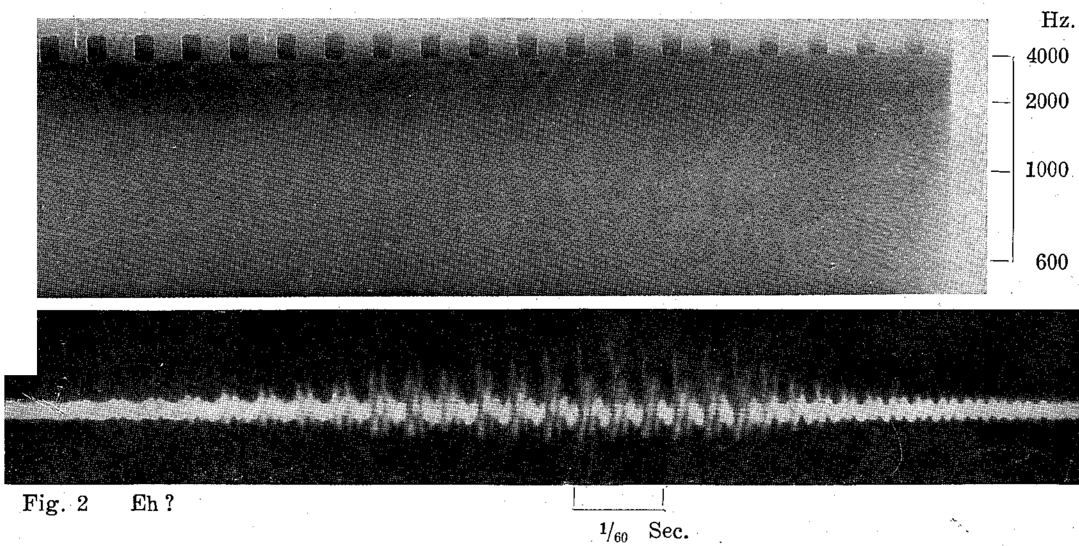


Fig. 2 Eh?

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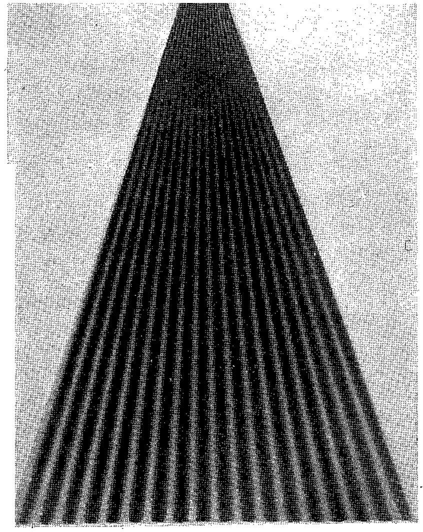


Fig. 1

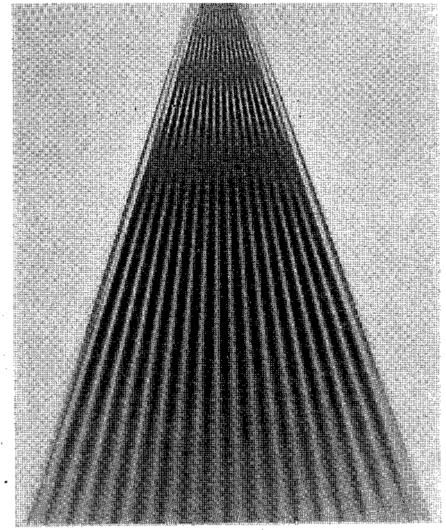


Fig. 2

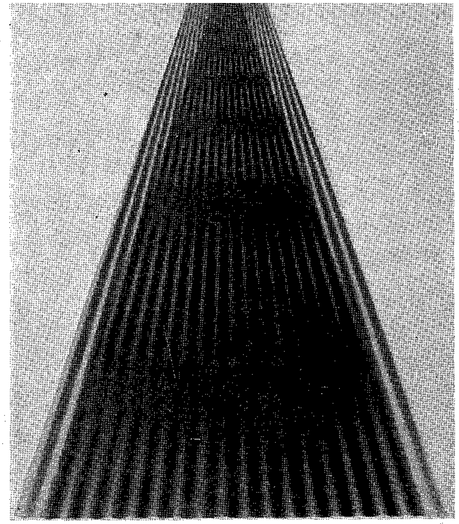
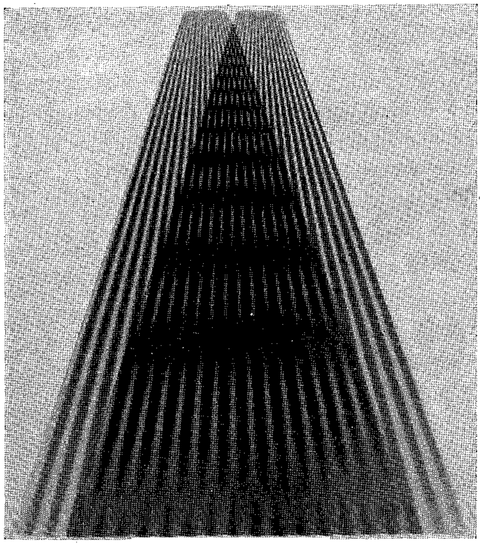


Fig. 3



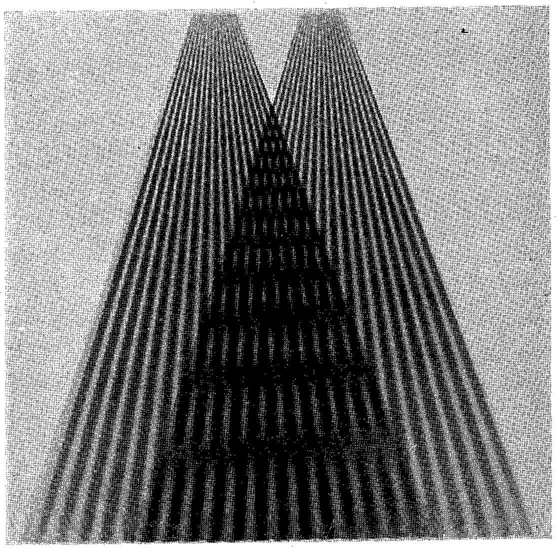


Fig. 5

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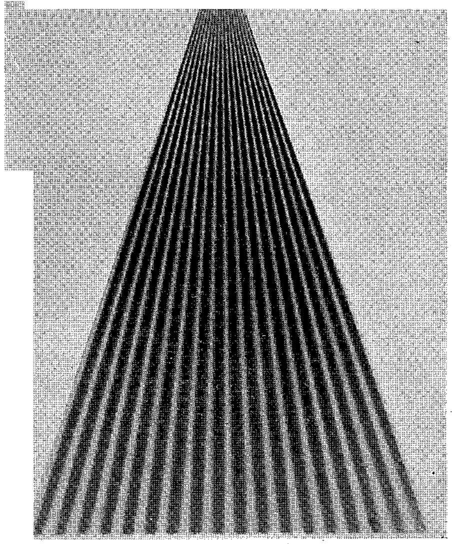
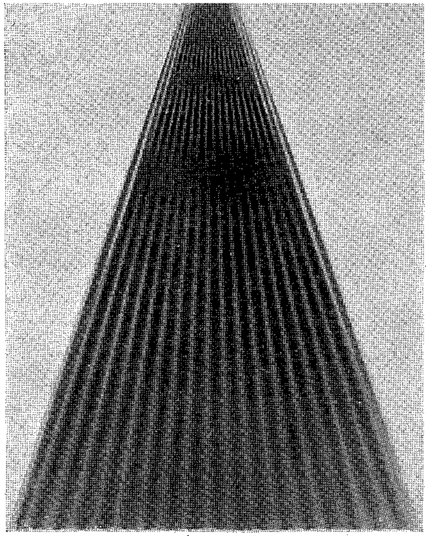


Fig. 1



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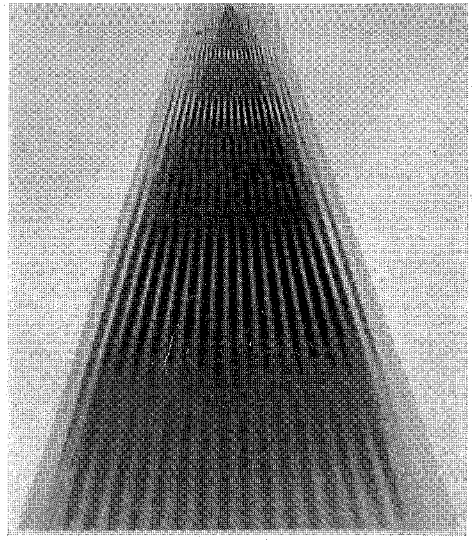

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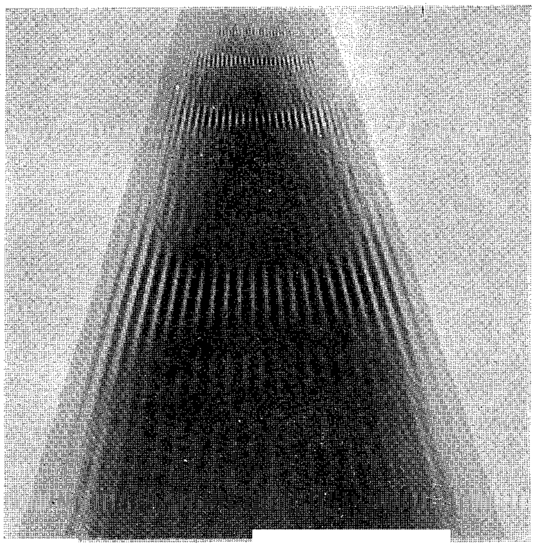

Fig.

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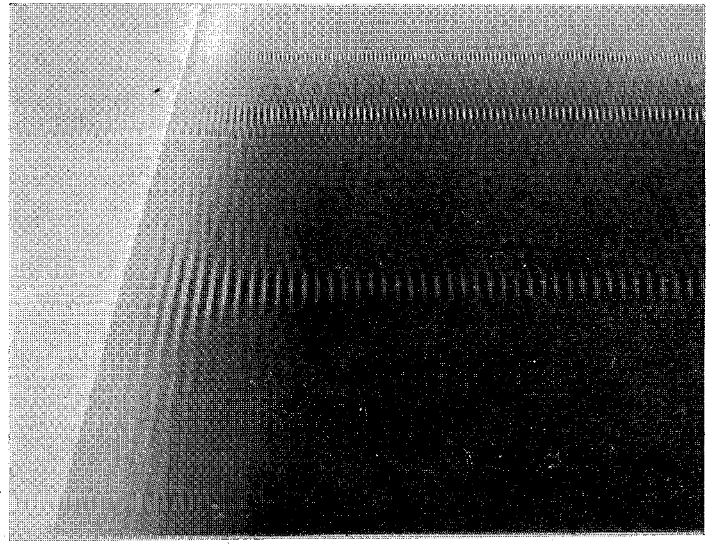

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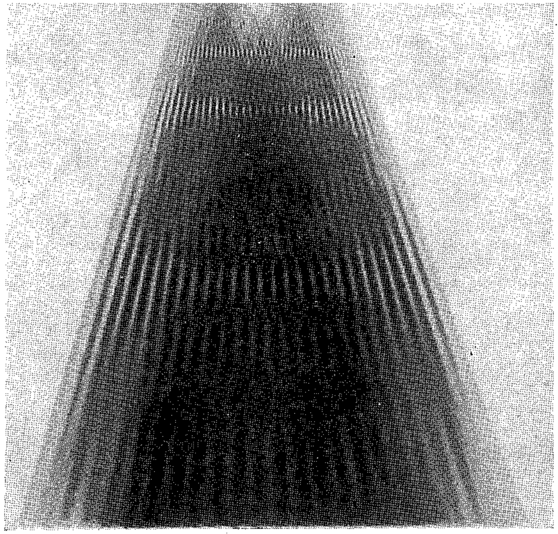
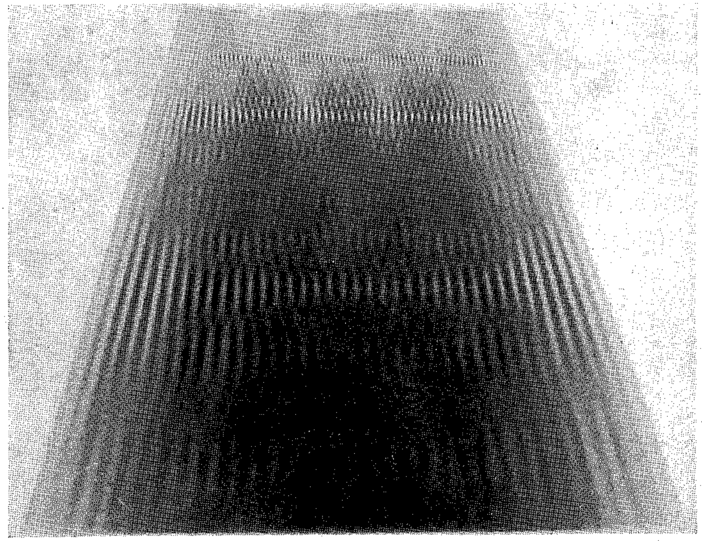


Fig. 1



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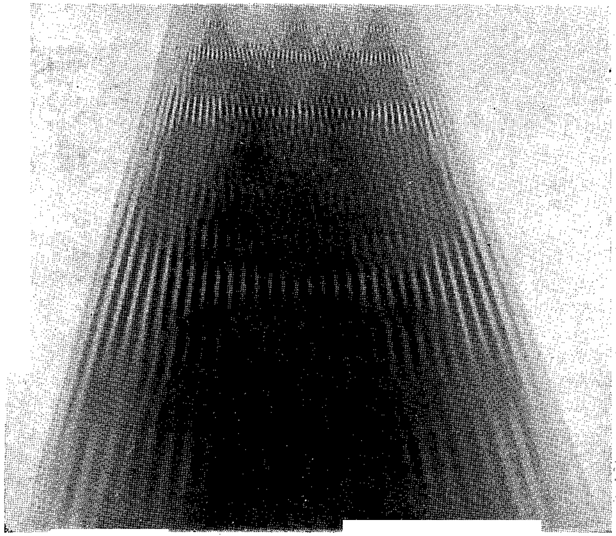

Fig. 3

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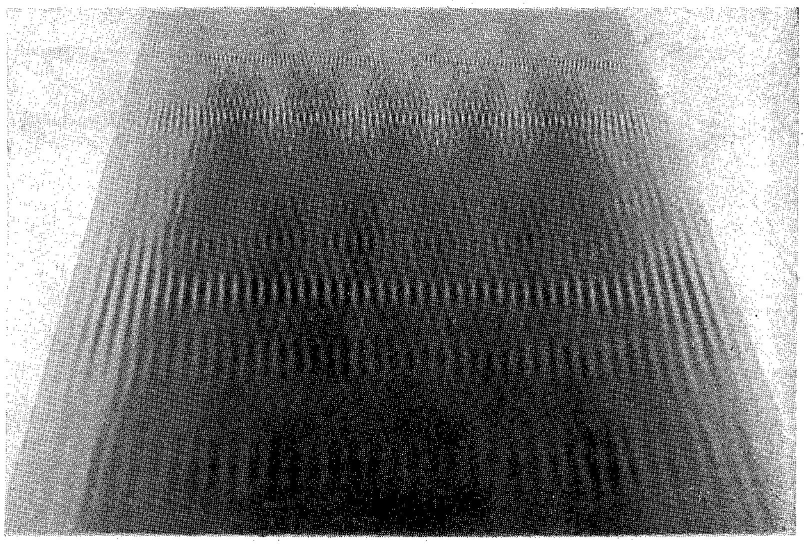

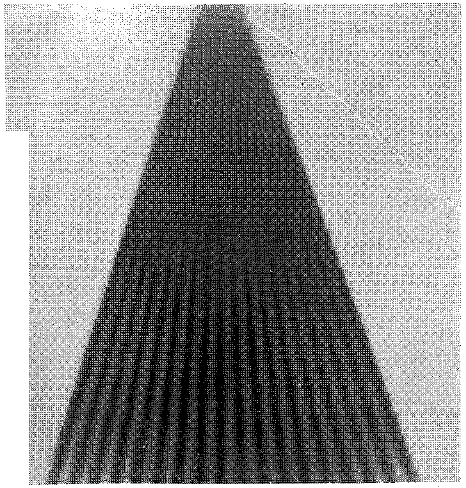

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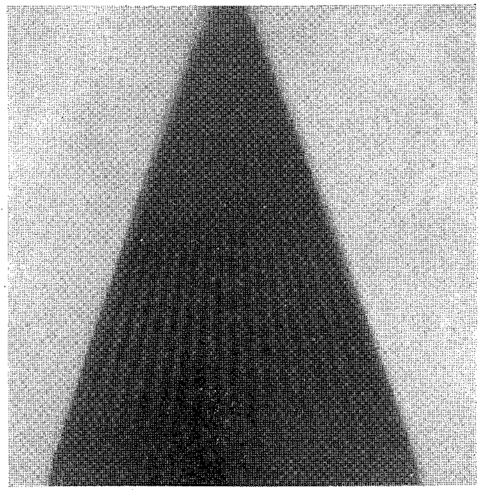

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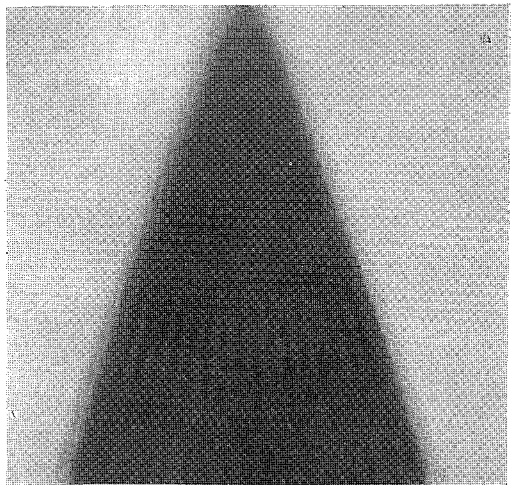

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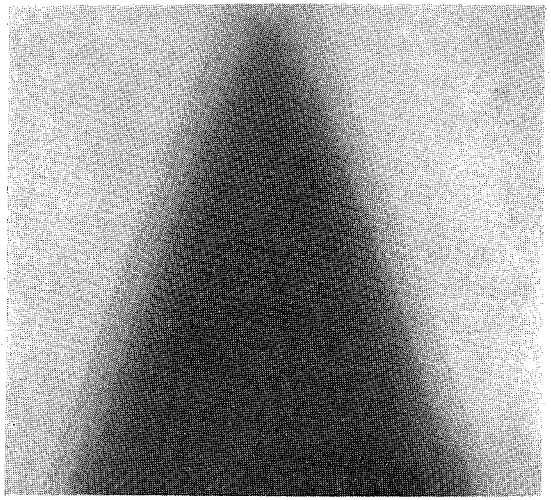

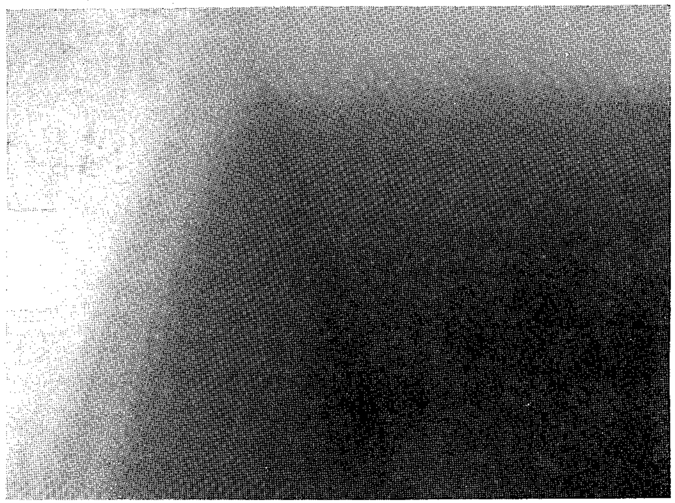

Fig. 5

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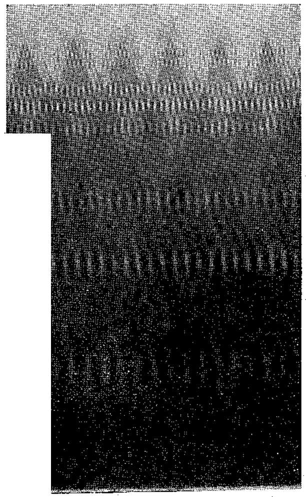
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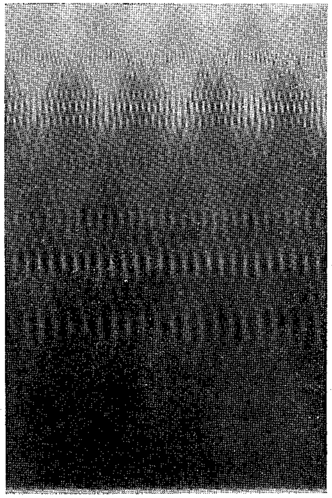
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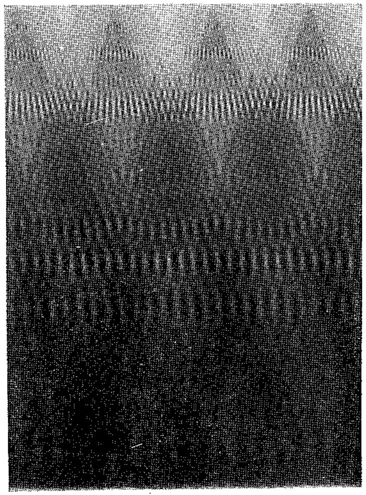
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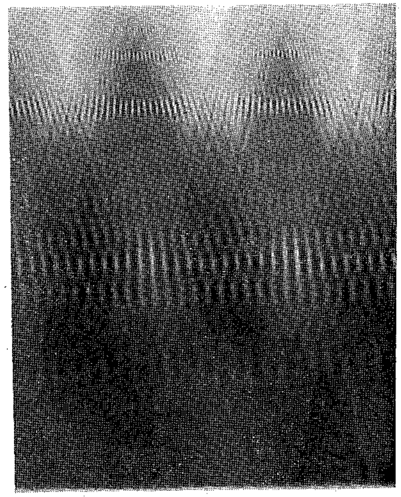
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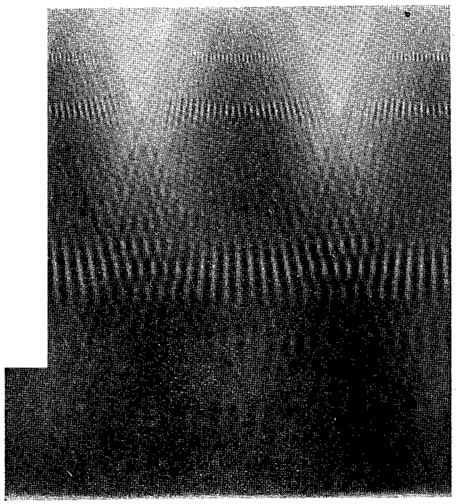
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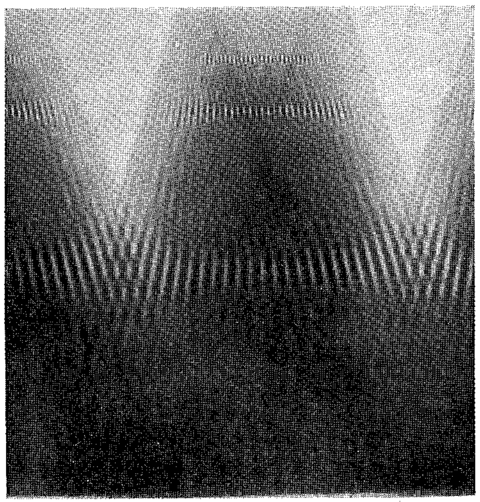
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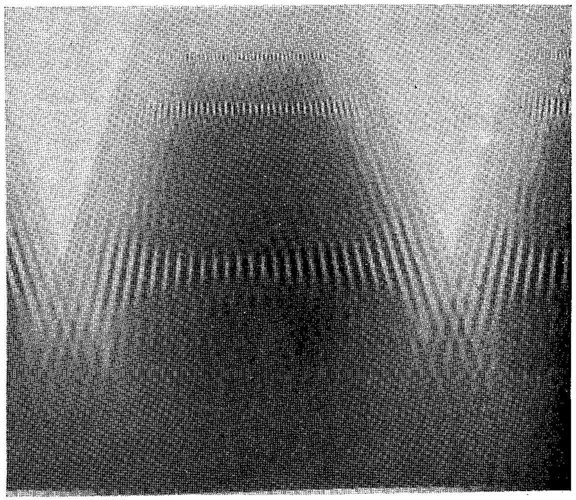
Fig. 7

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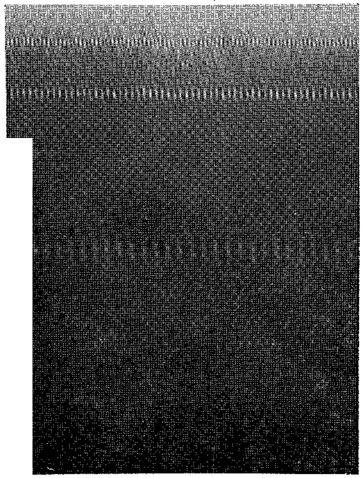
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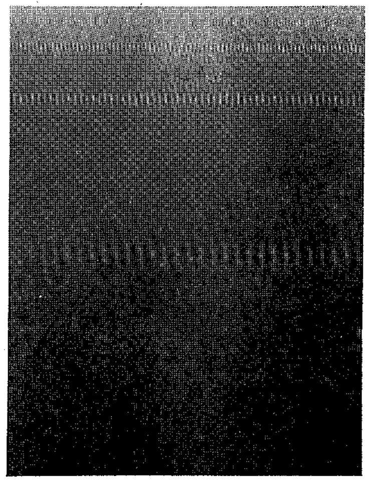
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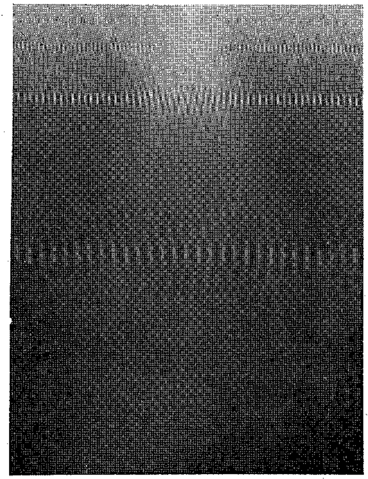
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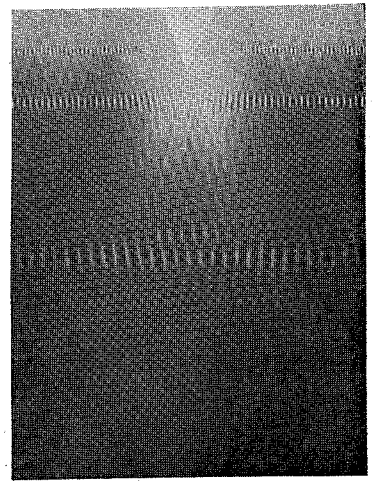
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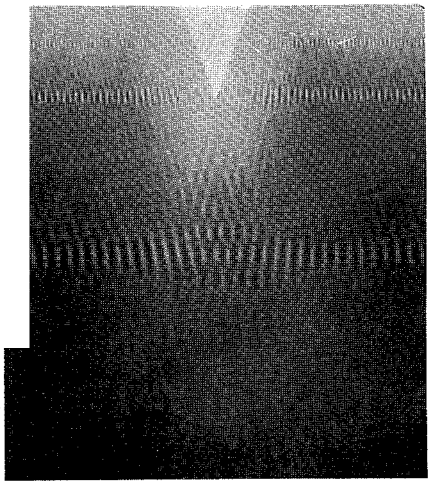
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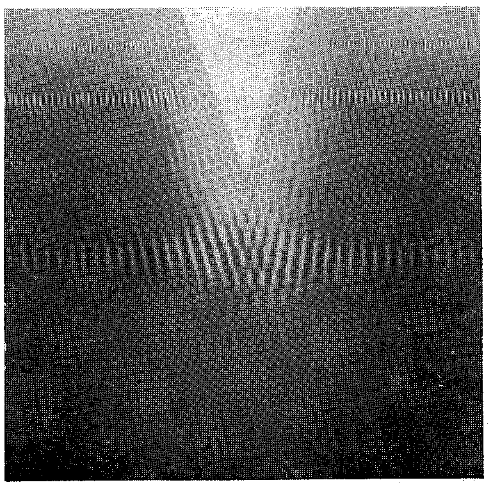
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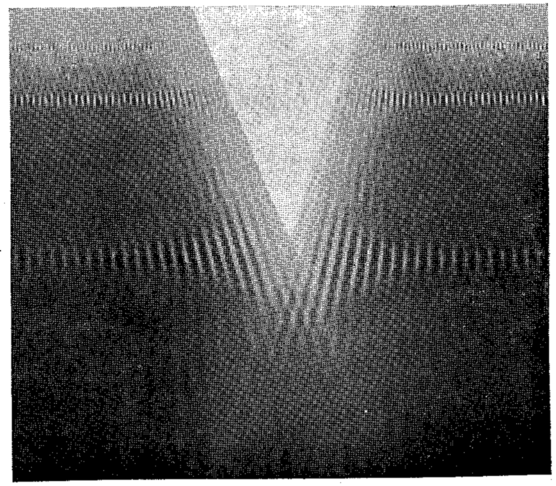
Fig. 7

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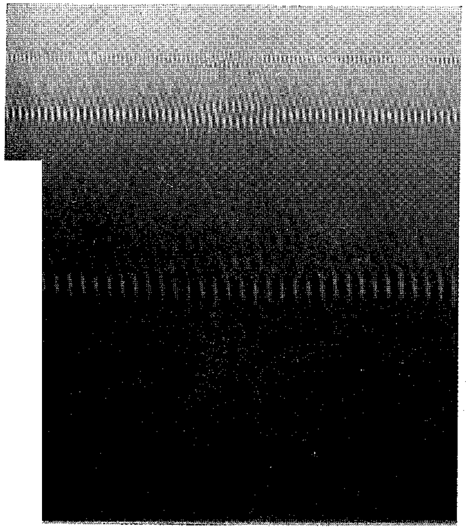

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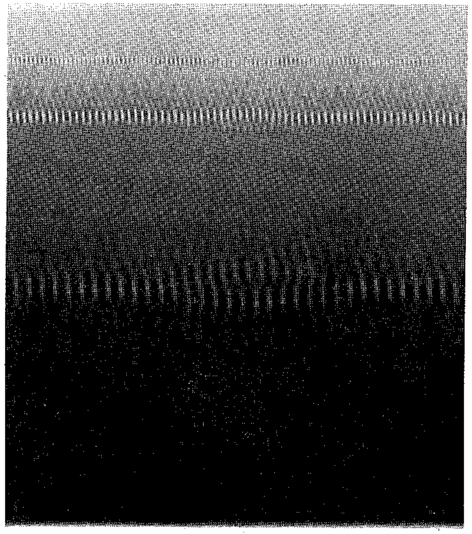

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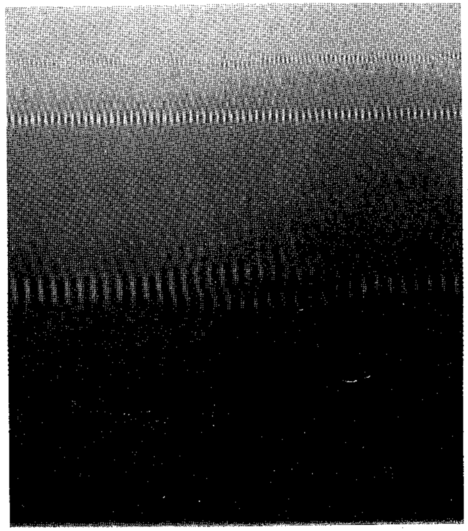

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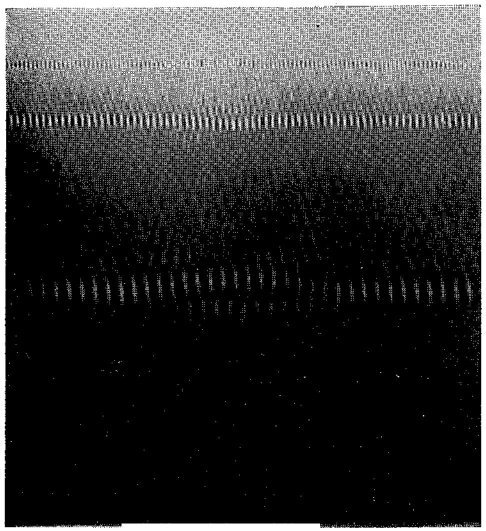

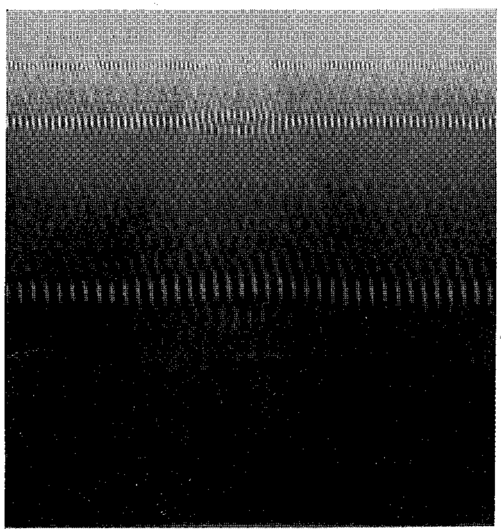

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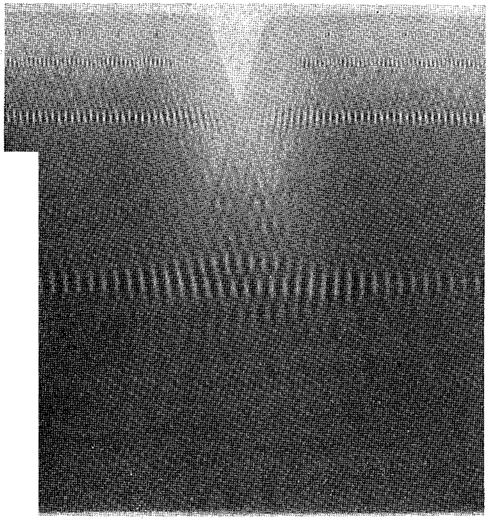

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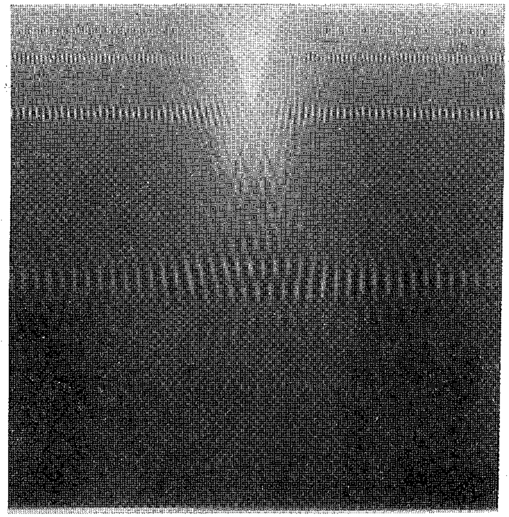
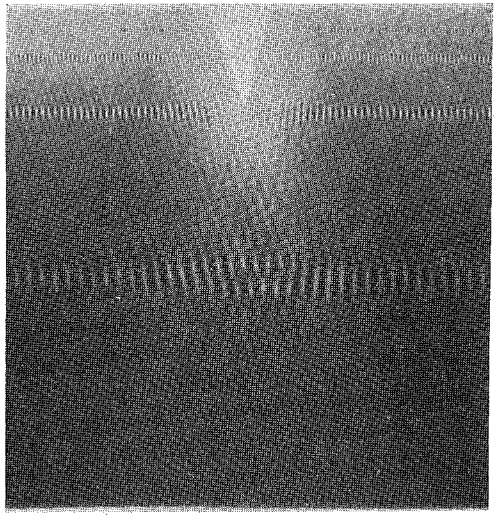


Fig. 2



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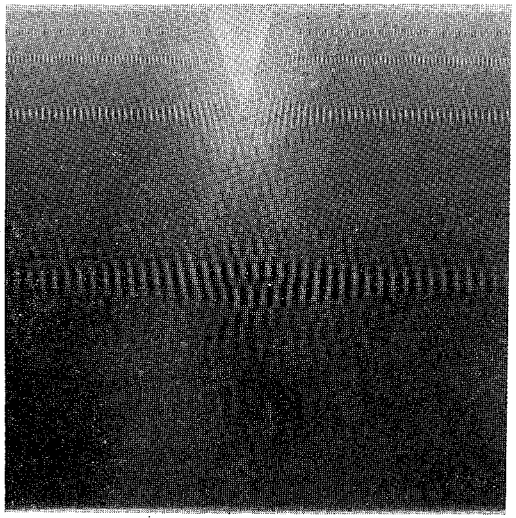

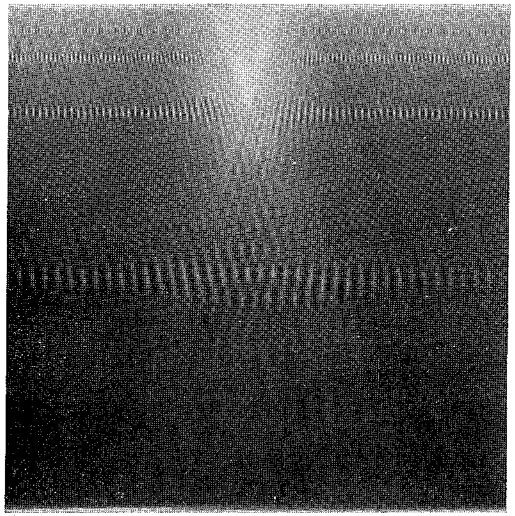

Fig. 5

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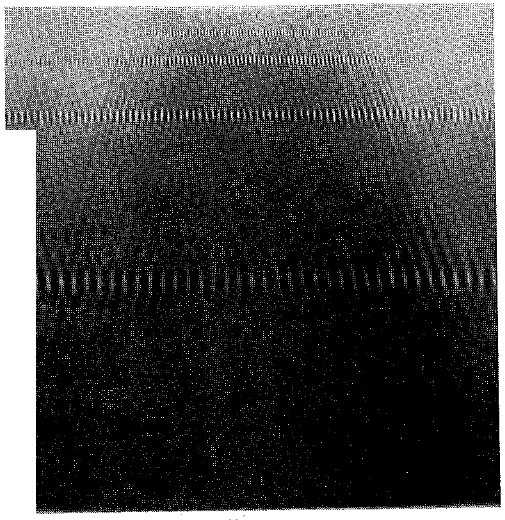
Ditto. The Two Waves Overlap Each Other.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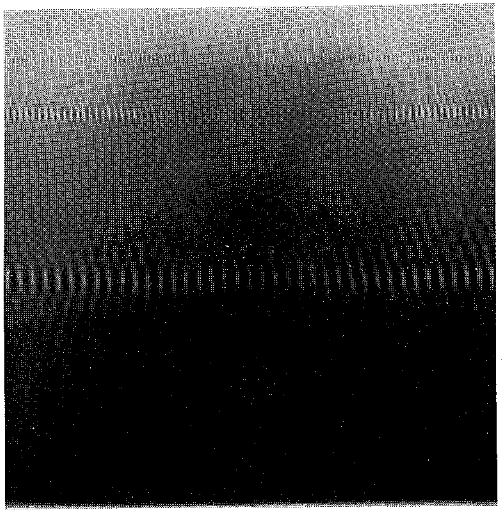
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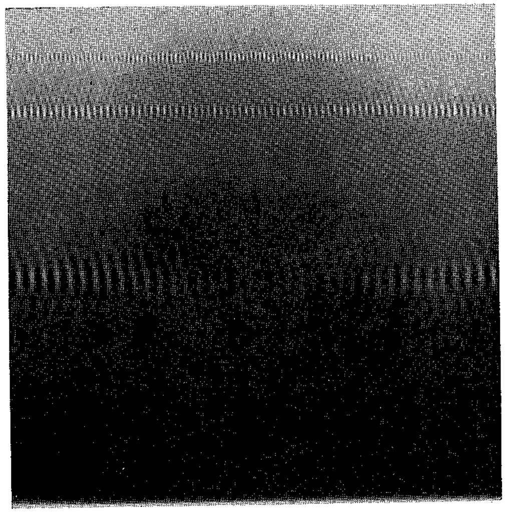
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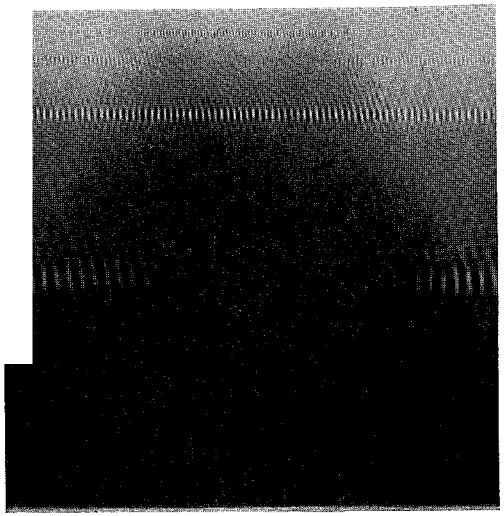
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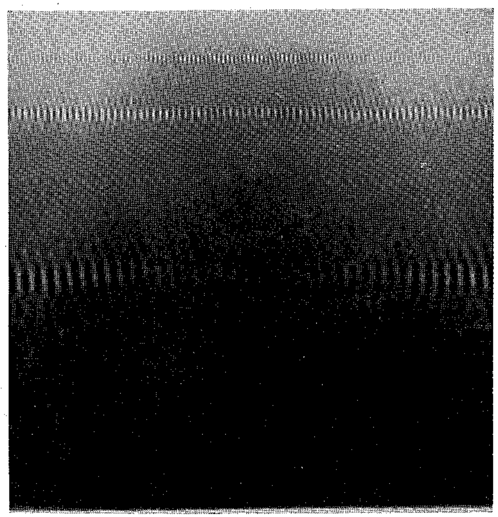
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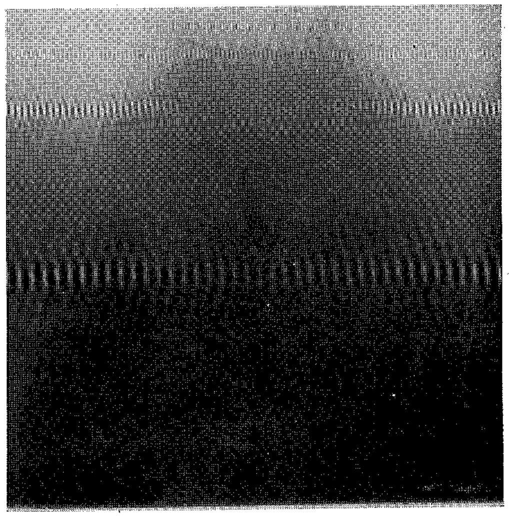
Fig. 6

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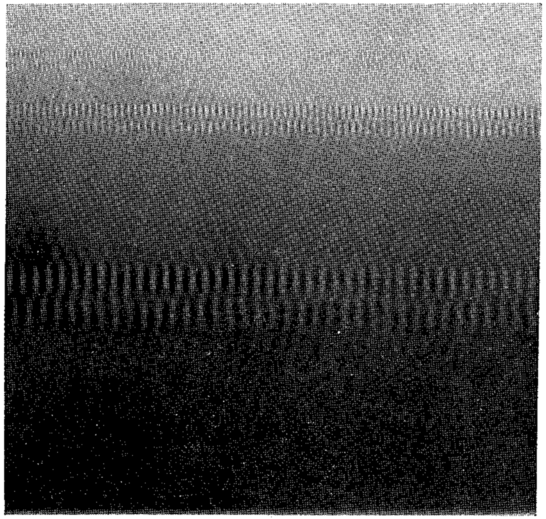

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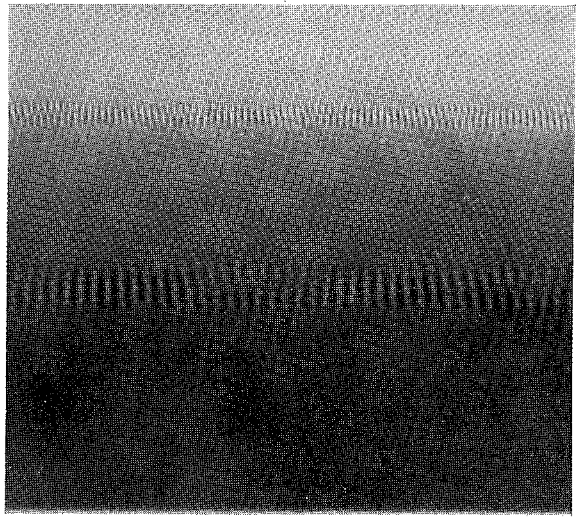
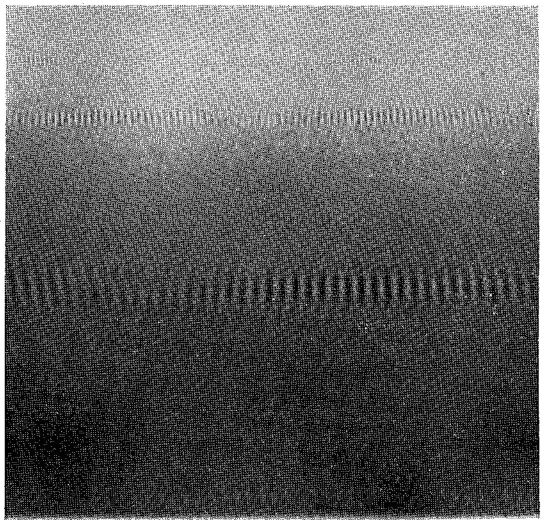


Fig. 2



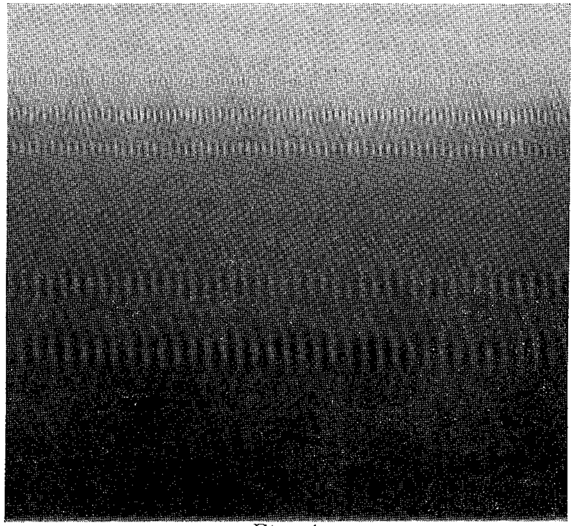


Fig. 4

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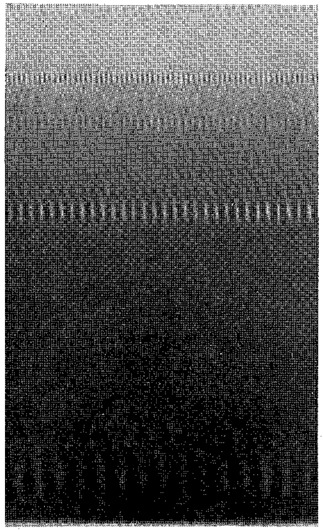

Fig. 1

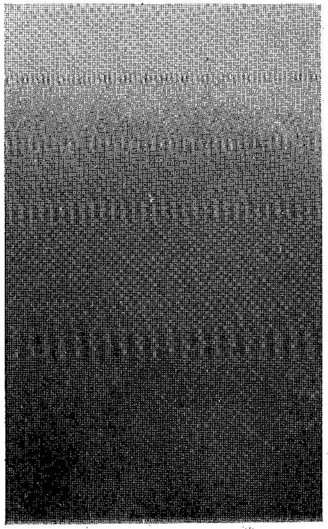

Fig. 2

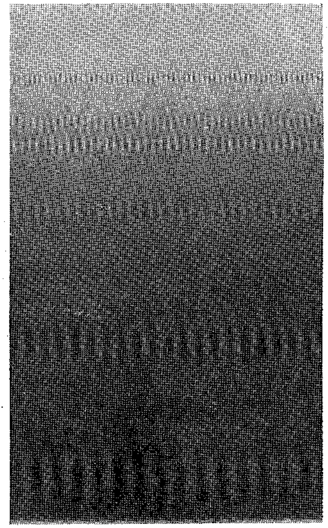

Fig. 3

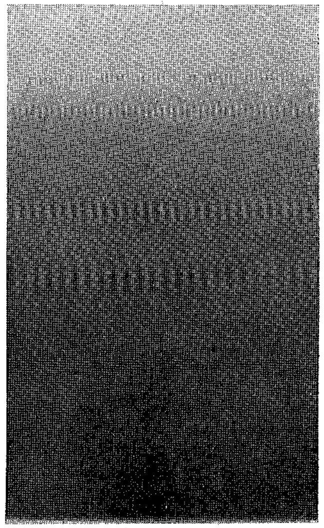

Fig. 4

ig. 5

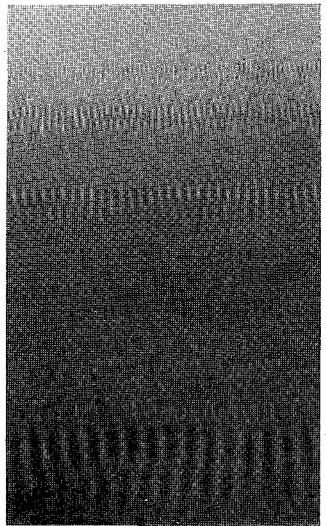

Fig. 6

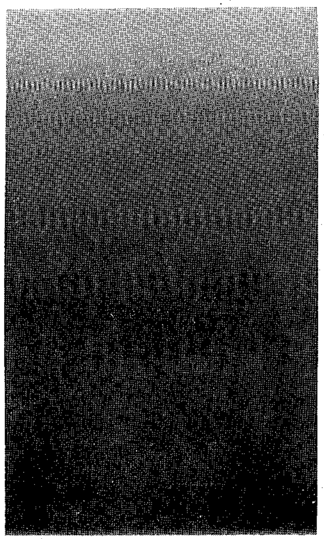

Fig. 7

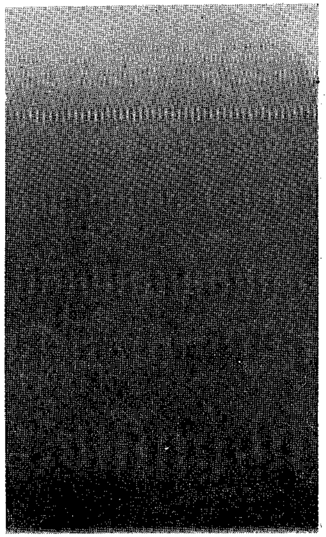

Fig. 8

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