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# A Study on Model Reference Adaptive Control In Economic Development (VI)

—Model Reference Adaptive Turnpike Theorem (II)—

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**Abstract**

The principal purpose of this paper is to improve upon the Model Reference Adaptive Turnpike Theorems presented in our previous paper 5) so that the theorems in 5) can be realized in the case of the relatively unstable system. As a follow up, the usefulness of the theorems obtained in this study is proved by some numerical simulations.

**Key Words:** Model reference adaptive turnpike theorem, Model reference adaptive system, Turnpike theorem, Dynamic input-output model, Relatively unstable system.

## 1. Introduction:

We have applied the theory of model reference adaptive system to the dynamic I-O system and developed some fundamental theories about model reference adaptive I-O system in the reference 3). However, as we pointed out in the reference 5), there were two major problems in our paper 3).

(1) How do we choose a reference model or a reference economic growth path?

(2) What adaptation laws can we apply when the reference model is unstable?

Our previous paper 5) have given some solutions to those problems, however, these solutions seem to be weak. Namely, as a reference path we adopted the economic growth path which converged to the turnpike. But the mathematical condition which realized that reference path gives some constrained conditions to the eigen values of the reference model, and it is not clear whether those conditions are always satisfied to actual dynamic I-O systems.

This article aims at extending the model reference adaptive turnpike theorems which have been obtained in 5) to be applied to a more general reference model, and also proving the usefulness of the theorems, which are obtained in this study, by some numerical simulations.

## 2. Reflection on the Previous Paper:

Let us simply review the main results which were obtained in our previous

paper 5). Now a dynamic I-O system under a fixed technology is represented as follows ;

$$Xm(t+1) = Bm^{-1}(I-Am+Bm)\, Xm(t) - Bm^{-1} H \qquad (1)$$

where, $Xm(t)$ : output vector $(n \times 1)$

$H$ : fixed final demand vector (exogenously given but excluding private investment) $(n \times 1)$

$Am$ : input-output coefficient matrix $(n \times n)$

$Bm$ : capital coefficient matrix $(n \times n)$

Let $\lambda$ be a Frobenius root of $(I-Am)^{-1} Bm$, then the turnpike of (1) is expressed as,

$$Xm(t) = \left(1+\frac{1}{\lambda}\right)^t \left(Xm(0) - (I-Am)^{-1} H\right) + (I-Am)^{-1} H \qquad (2)$$

Furthermore let us consider the economic growth path represented by (1) whose initial value is not on the turnpike, and indicate the distance between the path and the turnpike as $d(Xm(t))$. (see Figure 1.) We assume here that $\lim_{t\to\infty} d(Xm(t)) = 0$ for arbitrary economic growth path represented by (1). Then we can observe that,

$$d\left(Xm(t)\right) = ||Z(t) - Xm(t)|| = ||\Phi\, Xm(t) - \Phi(I-Am)^{-1} H|| \qquad (3)$$

where, $||\cdot||$ : Euclidian norm

$$\Phi = \eta\eta^T ||\eta||^{-2} - I$$

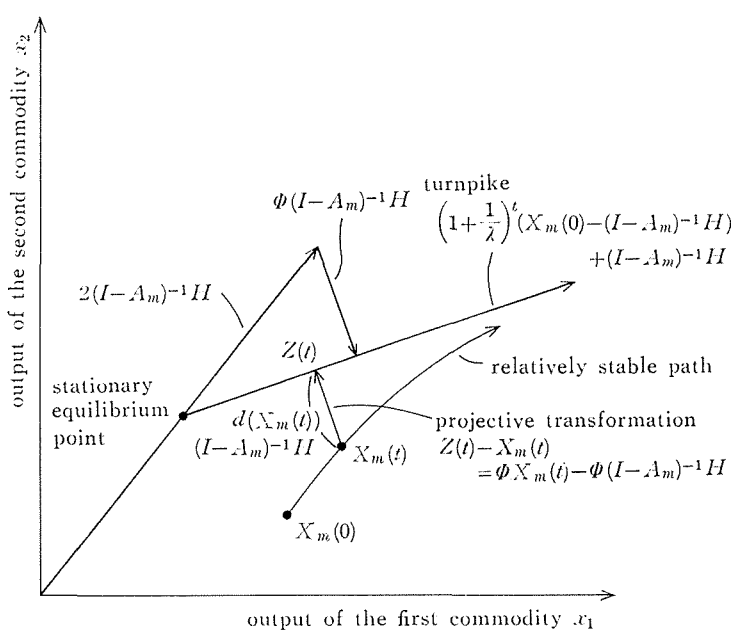


**Figure 1.** Concept of the Projective Transformation.

$\eta$: an eigen vector corresponding to the eigen value $1+\frac{1}{\lambda}$ of $Bm^{-1}(I-Am+Bm)$. Where $\lambda$ stands for a Frobenius root of $(I-Am)^{-1}Bm$.

In our previous paper 5) we called $n\times n$ matrix $\Phi$ the projective transformation. That is, $\Phi$ defines the linear transformation which draws a perpendicular line from $Xm(t)$ to the turnpike. Now let us define $\hat{X}m(t)$ as follows;

$$\hat{X}m(t)=Z(t)-\hat{X}m(t)+\Phi(I-Am)^{-1}H \tag{4}$$

It is easy to see that $\hat{X}m(t)=\Phi Xm(t)$. By the assumption mentioned above, $\lim_{t\to\infty}(Z(t)-Xm(t))=0$ holds, then $\lim_{t\to\infty}\hat{X}m(t)=\Phi(I-Am)^{-1}H$ is realized. Conversely when $\hat{X}m(t)$ is given $Xm(t)$ would be uniquely determined by the relationship $\Phi Xm(t)=\hat{X}m(t)$ if an inverse transformation of $\Phi$ could be defined. However, we can observe that the determinant of $\Phi$ is zero and also the rank of $\Phi$ is one. Therefore we can not define the ordinary inverse transformation of $\Phi$, that is to say, the inverse matrix of $\Phi$. Accordingly we had to introduce another transformation, i. e. the generalized inverse matrix $\Phi^+$ of $\Phi$, then solved $Xm(t)$ by $Xm(t)=\Phi^+\hat{X}m(t)$. Where the generalized inverse matrix $\Phi^+$ is a $n\times n$ matrix defined by $\Phi\Phi^+\Phi=\Phi$. And $\Phi^+$ is not uniquely determined in general, however, $\Phi^+$ can be defined uniquely with the initial condition $Xm(0)=\Phi^+\hat{X}m(0)$.

Now we multiply $\Phi$ to both sides of (1), then

$$\begin{aligned}&\Phi\,Xm(t+1)=\Phi\,Bm^{-1}(I-Am+Bm)\,Xm(t)-\Phi\,Bm^{-1}H\\&\therefore\quad \hat{X}m(t+1)=\Phi\,Bm^{-1}(I-Am+Bm)\,Xm(t)-\Phi\,Bm^{-1}H\end{aligned} \tag{5}$$

Substitute $\hat{X}m(t)=\Phi^+Xm(t)$ to (5), then

$$\hat{X}m(t+1)=\Phi\,Bm^{-1}(I-Am+Bm)\,\Phi^+\hat{X}m(t)-\Phi\,Bm^{-1}H \tag{6}$$

Because of $\lim_{t\to\infty}\hat{X}m(t)=\Phi(I-Am)^{-1}H$, $\Phi\,Bm^{-1}(I-Am+Bm)\,\Phi^+$ is a stable matrix. Accordingly when we set $Cm=\Phi\,Bm^{-1}(I-Am+Bm)\,\Phi^+$, the Lyapunov's equation mentioned below has a positive definite solution $P$. i. e. $\exists P=P^T>0$ s. t.

$$Cm^TPCm-Cm=-I \tag{7}$$

Then we can derive adaptation laws of $A(t)$ and $B(t)$ which make the gap between reference and adaptive models approach zero asymptotically in the below-stated model reference adaptive I-O system.

reference model

$$\begin{aligned}&\hat{X}m(t+1)=\Phi\,Bm^{-1}(I-Am+Bm)\,\Phi^+\hat{X}m(t)-\Phi\,Bm^{-1}H\\&Xm(t)=\Phi^+\hat{X}m(t)\end{aligned} \tag{8}$$

adaptive model

$$\begin{aligned}&\hat{X}(t+1)=\Phi B^{-1}(t+1)\,(I-A(t)+B(t))\,\Phi^+\hat{X}(t)-\Phi B^{-1}(t+1)\,H\\&X(t)=\Phi^+\hat{X}(t)\end{aligned} \tag{9}$$

However, a problem arising from this context is the assumption that $\lim_{t\to\infty} d(Xm(t))=0$. When that assumption is represented in terms of eigen value, the following condition should be realized.

$$\exists i \text{ such that } \mu_i > 1 > |\mu_j| \qquad (j \neq i) \tag{10}$$

where, $\mu_i$ : eigen value of $Bm^{-1}(I-Am+Bm)$ $(i=1, 2, \cdots, n)$

As written in the reference 14), the technological system in an actual dynamic I-O system which satisfies (10) has been hardly observed. And this fact gives us a strog constraint from an actually applicable point of view. Therefore it is very necessary to generalize the main results in 5) so that the results can be applied in the relatively unstable system.

## 3. Model Reference Adaptive Turnpike Theorem :

In this chapter we shall briefly explain model reference adaptive I-O system which is an application of MRAS to dynamic I-O system, and then show some theorems which were obtained related to the turnpike theorem.

Model reference adaptive I-O system is composed of two models, that is

reference model

$$Xm(t+1) = Bm^{-1}(I-Am+Bm)\, Xm(t) - Bm^{-1}\, H(t) \tag{11}$$

adaptive model

$$X(t+1) = B^{-1}(t+1)\left(I-A(t)+B(t)\right) X(t) - B^{-1}(t+1)\, H(t) \tag{12}$$

where, $Xm$ : $n$-dim. reference output vector
$Am$ : $n\times n$ reference input-output coefficient matrix
$Bm$ : $n\times n$ reference capital coefficient matrix
$H(t)$ : $n$-dim. final demand vector (exogenously given but excluding private investment)
$X(t)$ : $n$-dim. adaptive output vector
$A(t)$ : $n\times n$ adaptive input-output coefficient matrix
$B(t)$ : $n\times n$ adaptive capital coefficient matrix

The reference model represents an autonomous growth model under a fixed technology, and the adaptive model is a model which converges asymptotically and stably to the reference output through technological changes whose initial condition is given. (11) and (12) are transformed as

reference model

$$Xm(t+1) = CmXm(t) + Dm(t+1)\, H(t) \tag{13}$$

adaptive model

$$X(t+1) = C(t+1)\, X(t) + D(t+1)\, H(t) \tag{14}$$

where, $Cm=Bm^{-1}(I-Am+Bm)$

$Dm=-Bm^{-1}$

$C(t+1)=B^{-1}(t+1)\left(I-A(t)+B(t)\right)$

$D(t+1)=-B^{-1}(t+1)$

A problem arising from this context is how $C(t)$ and $D(t)$ are determined to make $\lim_{t\to\infty}||Xm(t)-X(t)||=0$. This problem is equivalent to proving the asymptotical stability of the below-mentioned error equation obtained by (13)-(14).

$$\varepsilon(t+1)=Cm\,\varepsilon(t)+\left(\mathrm{Cm}-C(t+1)\right)X(t)+\left(Dm-D(t+1)\right)H(t) \tag{15}$$

On the stability of (15) the next theorem holds.

Theorem 1.

reference model

$$Xm(t+1)=CmXm(t)+DmH(t) \tag{16}$$

adaptive model

$$X(t+1)=C(t+1)\,X(t)+D(t+1)\,H(t) \tag{17}$$

error equation

$$\varepsilon(t+1)=Cm\,\varepsilon(t)+\left(Cm-C(t+1)\right)X(t)+\left(Dm-D(t+1)\right)H(t) \tag{16}$$

When $k(t)$ is defined as $k(t)=1+||X(t)||^2+||H(t)||^2$ and assumed $\exists\tau>0$ such that $\lambda_{\max}<\sqrt{k(t)}$ for all $t>\tau$, the below-stated adaptation laws of $C(t)$ and $D(t)$ make the error equation asymptotically stable. i. e. $\lim_{t\to\infty}||Xm(t)-X(t)||=0$

$$C(t+1)=C(t)+\frac{1}{k(t)}\hat{\varepsilon}(t+1)\,X^T(t) \tag{19}$$

$$D(t+1)=D(t)+\frac{1}{k(t)}\hat{\varepsilon}(t+1)\,H^T(t) \tag{20}$$

Where $\lambda_{\max}$ stands for the eigen value of $Cm$ which has a maximum absolute value among the eigen values of $Cm$, and $\hat{\varepsilon}(t+1)$ is a priori error defined by

$$\hat{\varepsilon}(t+1)=Xm(t+1)-C(t)\,X(t)-D(t)\,H(t) \tag{21}$$

And $A(t)$ and $B(t)$ are solved as follows;

$$A(t)=I+D^{-1}(t+1)\,C(t+1)-D^{-1}(t) \tag{22}$$

$$B(t+1)=-D^{-1}(t+1) \tag{23}$$

Furthermore if there are sufficient linearly independent vectors in the sequence $\{H(t)\}_{t=0}^{\infty}$, then $\lim_{t\to\infty}||Am-A(t)||=0$, and $\lim_{t\to\infty}||Bm-B(t)||=0$ are realized.

(sketch of the proof)

Because the exact proof of this theorem is a complicated one, only a sketch

of the proof is presented here. Now the adaptation laws (19) and (20) are introduced from the next formulae.

$$C(t+1)=C(t)+\varepsilon(t+1)\ X^T(t) \tag{24}$$

$$D(t+1)=D(t)+\varepsilon(t+1)\ H^T(t) \tag{25}$$

These equations may be written as

$$C(t+1)=C(0)+\sum_{i=0}^{t}\varepsilon(i+1)\ X^T(i) \tag{26}$$

$$D(t+1)=D(0)+\sum_{i=0}^{t}\varepsilon(i+1)\ H^T(i) \tag{27}$$

Namely, the adaptation laws of $C(t)$ and $D(t)$ are determined by the past trend of $\varepsilon(t)$.

However, because (26) and (27) include $\varepsilon(t+1)$, that is, $X(t+1)$, these have no practical significance. Therefore let $\varepsilon(t+1)$ be transformed as follows;

$$\begin{aligned}
&\varepsilon(t+1)=Xm(t+1)-X(t+1)\\
&=Xm(t+1)-\Big(C(t+1)\ X(t)+D(t+1)\ H(t)\Big)\\
&=Xm(t+1)-\Big(C(t)+\varepsilon(t+1)\ X^T(t)\Big)\ X(t)-\Big(D(t)+\varepsilon(t+1)\ H^T(t)\Big)\ H(t)\\
&=Xm(t+1)-C(t)\ X(t)-D(t)\ H(t)-\Big(||X(t)||^2+||H(t)||^2\Big)\ \varepsilon(t+1)\\
&\therefore\quad \varepsilon(t+1)=\frac{1}{k(t)}\ \hat{\varepsilon}(t+1)
\end{aligned} \tag{28}$$

Substitute (28) into (24) and (25), then (19) and (20) are obtained. (19) and (20) can be practically implemented only using the information of before the $t$ th period.

Next we transform (19) and (20) as follows;

$$\begin{aligned}
&C(t+1)=C(t)+\frac{1}{k(t)}\Big(Xm(t+1)-C(t)\ X(t)-D(t)\ H(t)\Big)\ X^T(t)\\
&=C(t)+\frac{1}{k(t)}\Big(CmXm(t)+DmH(t)-C(t)\ X(t)-D(t)\ H(t)\Big)\ X^T(t)\\
&=C(t)+\frac{1}{k(t)}\Big(Cm\ \varepsilon(t)+\phi(t)\ X(t)+\psi(t)\ H(t)\Big)\ X^T(t)
\end{aligned} \tag{29}$$

where, $\phi(t)\equiv Cm-C(t)$, $\psi(t)\equiv Dm-D(t)$. Similarly

$$D(t+1)=D(t)+\frac{1}{k(t)}\Big(Cm\ \varepsilon(t)+\phi(t)\ X(t)+\psi(t)\ H(t)\Big)\ H^T(t) \tag{30}$$

$$\begin{aligned}
&\therefore\quad \varepsilon(t+1)=Xm(t+1)-X(t+1)\\
&=CmXm(t)+DmH(t)-C(t+1)\ X(t)-D(t+1)\ H(t)\\
&=\frac{1}{k(t)}\Big(Cm\ \varepsilon(t)+\phi(t)X(t)+\psi(t)\ H(t)\Big)
\end{aligned} \tag{31}$$

Therefore a proof of the asymptotical stability of (18) results in (31) being asymptoti-

cally stable. Setting a Lyapunov's function of $y(t+1)=\frac{1}{k(t)}Cmy(t)$ as $y^TPy$ ($P=P^T>0$ : positive definite $n\times n$ matrix), we define a Lyapunov's function candidate for (31) by using $P$ as

$$V(t)=\varepsilon^T(t)\,P\varepsilon(t)+tr\phi^T(t)\,P\phi(t)+tr\psi^T(t)\,P\psi(t) \tag{32}$$

Where '$tr$' stands for a trace operation of a matrix. $V(t)$ represents some kind of distance of outputs and parameters between the reference and the adaptive models. With a little complicated calculation, the increment of $V(t)$ is reduced as

$$V(t+1)-V(t)=\varepsilon^T(t)\left(\frac{Cm^T}{\sqrt{k(t)}}P\frac{Cm}{\sqrt{k(t)}}-P\right)\varepsilon(t)$$
$$-\frac{1}{k(t)}\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)^TP\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)\leqq 0 \tag{33}$$

Consequently $V(t)$ is proved as a Lyapunov's function and $\lim_{t\to\infty}V(t)$ is bounded.

$$\therefore\quad \left|\sum_{t=0}^{\infty}\Big(V(t+1)-V(t)\Big)\right|=\Big|V(\infty)-V(0)\Big|<\infty$$

$$\therefore\quad \left|\sum_{t=0}^{\infty}\Big(V(t+1)-V(t)\Big)\right|=\left|\sum_{t=0}^{\infty}\left\{\varepsilon^T(t)\left(\frac{Cm^T}{\sqrt{k(t)}}P\frac{Cm}{\sqrt{k(t)}}-P\right)\varepsilon(t)\right.\right.$$
$$\left.\left.-\frac{1}{k(t)}\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)^TP\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)\right\}\right|<\infty$$

$$\therefore\quad \lim_{t\to\infty}\left\{\varepsilon^T(t)\left(\frac{Cm^T}{\sqrt{k(t)}}P\frac{Cm}{\sqrt{k(t)}}-P\right)\varepsilon(t)\right.$$
$$\left.-\frac{1}{k(t)}\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)^TP\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)\right\}=0$$

$$\therefore\quad \lim_{t\to\infty}\varepsilon(t)=0 \text{ and } \lim_{t\to\infty}\Big(\phi(t)\,X(t)+\psi(t)\,H(t)\Big)=0$$

$$\therefore\quad \lim_{t\to\infty}||Xm(t)-X(t)||=0$$

If the sequence $\{H(t)\}_{t=0}^{\infty}$ contains sufficient linearly independent vectors, then $\lim_{t\to\infty}(\phi(t)\,X(t)+\psi(t)\,H(t))=0\Leftrightarrow$

$$\lim_{t\to\infty}\phi(t)=0 \text{ and } \psi(t)=0$$

$$\therefore\quad \lim_{t\to\infty}||Am-A(t)||=0 \text{ and } \lim_{t\to\infty}||Bm-B(t)||=0$$

(The exact proof of this theorem is shown in 7).) Q. E. D.

By a similar proof mentioned above, we obtain

Theorem 2

reference model

$$Xm(t+1)=CmXm(t)+DmH(t) \tag{34}$$

adaptive model

$$X(t+1)=C(t+1)\,X(t)+D(t+1)\,H(t) \tag{35}$$

error equation

$$\varepsilon(t+1)=Cm\varepsilon(t)+\Big(Cm-C(t+1)\Big)\,X(t)+\Big(Dm-D(t+1)\Big)\,H(t) \tag{36}$$

If $k(t)$ is getting sufficiently large as $t\to\infty$, then the below-stated adaptation laws of $C(t)$, $D(t)$ make the error equation asymptotically stable.

$$C(t+1)=C(t)+\Big(I+\Gamma(t)\Big)^{-1}Kc\otimes\hat{\varepsilon}(t+1)\,X^{T}(t) \tag{37}$$

$$D(t+1)=D(t)+\Big(I+\Gamma(t)\Big)^{-1}Kd\otimes\hat{\varepsilon}(t+1)H^{T}(t) \tag{38}$$

Where $Kc$ and $Kd$ are the matrices which have positive elements i. e. $Kc=(kc_{ij})$, $Kd=(kd_{ij})$, $kc_{ij}$ and $kd_{ij}>0$. $\otimes$ stands for a matrix operation as follows;

$$\begin{pmatrix}a_{11}\cdots a_{1n}\\ \vdots\quad\vdots\\ a_{n1}\cdots a_{nn}\end{pmatrix}\otimes\begin{pmatrix}b_{11}\cdots b_{1n}\\ \vdots\quad\vdots\\ b_{n1}\cdots b_{nn}\end{pmatrix}=\begin{pmatrix}a_{11}\ b_{11}\cdots a_{1n}\ b_{1n}\\ \vdots\qquad\qquad\vdots\\ a_{n1}\ b_{n1}\cdots a_{nn}\ b_{nn}\end{pmatrix} \tag{39}$$

And $\Gamma(t)$ represents the next $n\times n$ diagonal matrix.

$$\Gamma(t)=\begin{pmatrix}\sum_{j=1}^{n}\Big\{kc_{1j}\,x_j^2(t)+kd_{1j}\,h_j^2(t)\Big\} & & 0\\ & \ddots & \\ 0 & & \sum_{j=1}^{n}\Big\{kc_{nj}\,x_j^2(t)+kd_{nj}h_j^2(t)\Big\}\end{pmatrix} \tag{40}$$

Furthermore if the sequence $\{H(t)\}_{t=0}^{\infty}$ include sufficient linearly independent vectors, then

$$\lim_{t\to\infty}||Am-A(t)||=0 \quad\text{and}\quad \lim_{t\to\infty}||Bm-B(t)||=0$$

When a tunpike is chosen as a reference model in Theorem 2, the following theorem is obtained.

Theorem 3.

Let a dynamic I-O system with constant technology be set as

$$Xm(t+1)=Bm^{-1}(I-Am+Bm)\,Xm(t)-Bm^{-1}\,H \tag{41}$$

The turnpike from the stationary equilibrium point is represented as

$$Xm(t)=\Big(1+\frac{1}{\lambda}\Big)^{t}\Big(Xm(0)-(I-Am)^{-1}\,H\Big)+(I-Am)^{-1}\,H \tag{42}$$

Where $\lambda$ stands for the Frobenius root of $(I-Am)^{-1}\,Bm$, and $Xm(0)$ is the initial value of $Xm(t)$ which is also an eigen vector corresponding to $\lambda$. Next let a dynamic I-O system with variable technology be represented as,

$$X(t+1)=B^{-1}(t+1)\Big(I-A(t)+B(t)\Big)\,X(t)-B^{-1}(t+1)\,H \tag{43}$$

When the adaptation laws of $A(t)$ and $B(t)$ are adopted as

$$C(t+1)=C(t)+\Big(I+\Gamma(t)\Big)^{-1}\,Kc\otimes\Big(\hat{\varepsilon}(t+1)\,X^{T}(t)\Big) \tag{44}$$

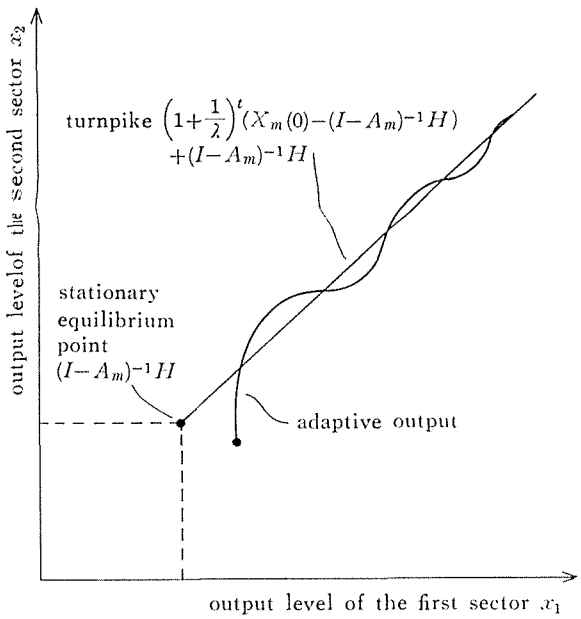


**Figure 2.** Adaptation process of the adaptive output to the turnpike.

$$D(t+1)=D(t)+\left(I+\Gamma(t)\right)^{-1}Kd\otimes\left(\hat{\varepsilon}(t+1)\,H^T\right) \tag{45}$$

then $X(t)$ will converge to the turnpike. Where the notation is the same as that one of Theorem 2.

We shall explain Theorems 1, 2, and 3 in the following context briefly. Fist of all, the three theorems are realized even in the case that the reference model is relatively unstable. This fact shows that Theorems 1, 2, and 3 are extensions of the theorem which was proved in our previous paper 5). And also this result denotes that Theorems 1, 2, and 3 can be practically implemented to any actual dynamic I-O systems.

Secondly, the adaptation laws of $C(t)$ and $D(t)$ only use the reference outputs and exogeneously derived final demand. Namely, these do not use the information relating to the reference technology, despite the fact that the reference technology is used in the proof. From the fact that the adaptation laws can identify unknown technologies of a dynamic I-O system, the adaptation process in the theorems are called "adaptive observer" or "adaptive identifier" in the system theory. Furthermore, the reference technology is not limited to be fixed. If the absolute values of the eigen values of $Cm$ are bounded when $Cm$ is variable, the adaptation laws will be effective.

Thirdly, a convergence speed of the adaptation process depends on the absolute values of the eigen values of $\frac{1}{k(t)}\,Cm$ or $(I+\Gamma(t))^{-1}Cm$. When absolute values of the eigen values are smaller, the convergence speed is faster. In Theorem 2 and 3, the convergence speed can be controled by $Kc$ and/or $Kd$, that is, the larger

$Kc$ and/or $Kd$, the slower the speed. However, if $Kc$ and/or $Kd$ are set to be too big, the adaptive technology seems to change sharply in its initial stages. Therefore, it is difficult but important to select $Kc$ and $Kd$ so that the adaptation process does not lose economic sense. In the dynamic I-O system there are some constraints that are well known, e. g., non-negativity of $A(t)$ and $B(t)$, Solow's condition for the summuation of columns of $A(t)$ and so on.

In our experience, some numerical simulations have shown that some $Kc$ and/or $Kd$ have made the above-mentioned constrained conditions of $A(t)$ or $B(t)$ not to be satisfied. At present, $Kc$ and $Kd$ may be chosen by trial and error.

Fourthly, it is possible that $Kc$ and $Kd$ are variables. Now let variable $Kc$ and $Kd$ be noted as $Kc(t)$ and $Kd(t)$. These can be substituted into (37) and (38), then

$$C(t+1)=\sum_{k=0}^{t}\left(I+\Gamma(k)\right)^{-1}\left(Kc(k+1)\otimes\hat{\varepsilon}(k+1)\ X^{T}(k)\right) \tag{46}$$

$$D(t+1)=\sum_{k=0}^{t}\left(I+\Gamma(k)\right)^{-1}\left(Kd(k+1)\otimes\hat{\varepsilon}(k+1)\ H^{T}(k)\right) \tag{47}$$

If $Kc(k)$ and $Kd(k)$ are denoted as small values in the initial stages and varied to larger ones for the present period, the adaptation algorithms may mean an adjustment process which neglects the past and stresses the present. Finally, it may be impossible to convert Theorems 1, 2, and 3 into a continuous format.

## 4. Numerical Simulations:

In this chapter we present some numerical simulations of the model reference adaptive I-O system.

### 4. 1. *Model reference adaptive I-O system with respect to the ordinary turnpike*

Let the first numerical simulation defined as follows;

$$Am=\begin{pmatrix}0.3 & 0.3\\ 0.2 & 0.2\end{pmatrix},\quad Bm=\begin{pmatrix}1.9 & 2.1\\ 2.6 & 3.4\end{pmatrix},\quad H(t)=\begin{pmatrix}10\\ 10\end{pmatrix}$$

Then $(I-Am)^{-1}Bm=\begin{pmatrix}4.6 & 5.4\\ 4.4 & 5.6\end{pmatrix}$ and its eigen values are $\lambda_1=10$ and $\lambda_2=0.2$, and $\lambda_1=10$ is the Frobenius root. Associated eigen vectors of $\lambda_1$ and $\lambda_2$ are $(1,\ 1)^T$ and $(27,\ -22)^T$. Then the eigen values of $Bm^{-1}(I-Am+Bm)$ are $1+\frac{1}{\lambda_1}=1.1$ and $1+\frac{1}{\lambda_2}=6.0$ with the eigen vectors $(1,\ 1)^T$ and $(27,\ -22)^T$. We observe that the reference system is relatively unstable because the eigen value $1+\frac{1}{\lambda_1}$ corresponding to the Frobenius root does not have maximum radius between two eigen values. Setting an initial condition $Xm(0)=(10,\ 10)^T+(22,\ 18)^T$, the turnpike from the stationary equilibrium point $(I-Am)^{-1}H=(22,\ 18)^T$ is represented as follows;

$$\begin{pmatrix} xm_1(t) \\ xm_2(t) \end{pmatrix} = 1.1^t \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \qquad (t = 0, 1, \cdots)$$

Let us simulate an adaptation process whose reference model is the above-stated turnpike.

reference model

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = 1.1^{t+1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix}, \quad \begin{pmatrix} xm_1(0) \\ xm_2(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \tag{48}$$

adaptive model

$$X(t+1) = B^{-1}(t+1)\left(I - A(t) + B(t)\right) X(t) - B^{-1}(t+1)\, H(t) \tag{49}$$

where, $A(-1) = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$ $\quad B(-1) = B(0) = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$X(0) = (31,\ 19)^T$, $H(t) = (10,\ 10)^T$

By Theorem 2 or 3 we adopt the next adaptation laws.

$$C(t+1) = C(t) + \left(I + \Gamma(t)\right)^{-1} \left(Kc \otimes \hat{\varepsilon}(t+1)\, X^T(t)\right) \tag{50}$$

$$D(t+1) = D(t) + \left(I + \Gamma(t)\right)^{-1} \left(Kd \otimes \hat{\varepsilon}(t+1)\, H^T(t)\right) \tag{51}$$

where, $C(t+1) = B^{-1}(t+1)\,(I - A(t) + B(t))$

$$D(t+1) = -B^{-1}(t+1)$$

$$Kc = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad Kd = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$

$$\Gamma(t) = \begin{pmatrix} \sum_{j=1}^{2} \left\{kc_{1j}\, x_j^2(t) + kd_{1j}\, h_j^2(t)\right\} & 0 \\ 0 & \sum_{j=1}^{2} \left\{kc_{2j}\, x_j^2(t) + kd_{2j}\, h_j^2(t)\right\} \end{pmatrix} \tag{52}$$

The results of the simulation is presented in tables 1~4. These simulations show that the adaptive outputs almost converge to the turnpike prior to the tenth period, and that the variation of $A(t)$ is larger in its initial stages. We also observe that $A(t)$ and $B(t)$ do not approach $Am$ and $Bm$, respectively, because $H(t)$ is a constant vector, and that the adaptive model chooses another turnpike technology which has the same turnpike as that of the reference model.

### 4. 2. *Model reference adaptive I-O system with respect to the turnpike which has a variable growth rate*

In this section we consider the turnpike which has a variable growth rate, however, a growth rate of a conventional turnpike is constant. Let the reference

**Table 1.** Simulation results of output levels of the reference and the adaptive models (section 4. 1.)

| period | $xm_1(t)$ | $x_1(t)$ | $xm_2(t)$ | $x_2(t)$ |
|---|---|---|---|---|
| 0. | 32.000 | 31.000 | 28.000 | 19.000 |
| 1. | 33.000 | 38.289 | 29.000 | 21.905 |
| 2. | 34.100 | 35.012 | 30.100 | 29.660 |
| 3. | 35.310 | 33.869 | 31.310 | 32.645 |
| 4. | 36.641 | 36.046 | 32.641 | 33.126 |
| 5. | 38.105 | 38.288 | 34.105 | 33.953 |
| 6. | 39.716 | 39.878 | 35.716 | 35.586 |
| 7. | 41.487 | 41.483 | 37.487 | 37.493 |
| 8. | 43.436 | 43.403 | 39.436 | 39.465 |
| 9. | 45.580 | 45.571 | 41.580 | 41.589 |
| 10. | 47.937 | 47.938 | 43.937 | 43.940 |
| 11. | 50.531 | 50.529 | 46.531 | 46.535 |
| 12. | 53.384 | 53.381 | 49.384 | 49.389 |
| 13. | 56.523 | 56.520 | 52.523 | 52.527 |
| 14. | 59.975 | 59.972 | 55.975 | 55.979 |
| 15. | 63.773 | 63.770 | 59.773 | 59.776 |
| 16. | 67.950 | 67.948 | 63.950 | 63.953 |
| 17. | 72.545 | 72.543 | 68.545 | 68.548 |
| 18. | 77.599 | 77.597 | 73.599 | 73.602 |
| 19. | 83.159 | 83.157 | 79.159 | 79.162 |
| 20. | 89.275 | 89.274 | 85.275 | 85.277 |

**Table 2.** Simulation results of the adaptive input-output coefficient matrix $A(t)$ (section 4. 1.)

| period | $a_{11}(t)$ | $a_{12}(t)$ | $a_{21}(t)$ | $a_{22}(t)$ |
|---|---|---|---|---|
| 0. | 0.30000 | 0.30000 | 0.20000 | 0.20000 |
| 1. | 0.62715 | 0.56879 | 0.23612 | 0.15931 |
| 2. | 0.39777 | 0.28106 | 0.52595 | 0.43130 |
| 3. | 0.32726 | 0.20064 | 0.38482 | 0.24505 |
| 4. | 0.37502 | 0.25782 | 0.31059 | 0.15719 |
| 5. | 0.39009 | 0.27290 | 0.30446 | 0.15301 |
| 6. | 0.38297 | 0.26358 | 0.31867 | 0.16989 |
| 7. | 0.38042 | 0.26063 | 0.32585 | 0.17732 |
| 8. | 0.38261 | 0.26311 | 0.32749 | 0.17846 |
| 9. | 0.38472 | 0.26527 | 0.32927 | 0.18002 |
| 10. | 0.38604 | 0.26651 | 0.33185 | 0.18264 |
| 11. | 0.38724 | 0.26765 | 0.33469 | 0.18532 |
| 12. | 0.38852 | 0.26889 | 0.33729 | 0.18782 |
| 13. | 0.38978 | 0.27011 | 0.33979 | 0.19024 |
| 14. | 0.39101 | 0.27130 | 0.34224 | 0.19261 |
| 15. | 0.39219 | 0.27244 | 0.34462 | 0.19491 |
| 16. | 0.39334 | 0.27355 | 0.34692 | 0.19714 |
| 17. | 0.39444 | 0.27463 | 0.34913 | 0.19929 |
| 18. | 0.39549 | 0.27565 | 0.35125 | 0.20135 |
| 19. | 0.39651 | 0.27664 | 0.35328 | 0.20333 |
| 20. | 0.39747 | 0.27757 | 0.35521 | 0.20520 |

**Table 3.** Simulation results of the adaptive capital coefficient matrix $B(t)$ (section 4.1.)

| period | $b_{11}(t)$ | $b_{12}(t)$ | $b_{21}(t)$ | $b_{22}(t)$ |
|---|---|---|---|---|
| 0. | 1.90000 | 2.10000 | 2.60000 | 3.40000 |
| 1. | 1.67496 | 1.85654 | 2.11880 | 2.87940 |
| 2. | 1.43769 | 1.56034 | 1.89652 | 2.60191 |
| 3. | 1.43415 | 1.55592 | 1.66675 | 2.31509 |
| 4. | 1.46470 | 1.59405 | 1.62632 | 2.26462 |
| 5. | 1.45709 | 1.58456 | 1.64126 | 2.28326 |
| 6. | 1.44738 | 1.57243 | 1.65029 | 2.29453 |
| 7. | 1.44629 | 1.57108 | 1.64811 | 2.29181 |
| 8. | 1.44689 | 1.57183 | 1.64427 | 2.28702 |
| 9. | 1.44626 | 1.57103 | 1.64209 | 2.28429 |
| 10. | 1.44519 | 1.56970 | 1.64054 | 2.28236 |
| 11. | 1.44429 | 1.56858 | 1.63895 | 2.28037 |
| 12. | 1.44352 | 1.56761 | 1.63736 | 2.27839 |
| 13. | 1.44279 | 1.56671 | 1.63587 | 2.27654 |
| 14. | 1.44211 | 1.56586 | 1.63451 | 2.27484 |
| 15. | 1.44149 | 1.56508 | 1.63326 | 2.27327 |
| 16. | 1.44092 | 1.56436 | 1.63211 | 2.27184 |
| 17. | 1.44039 | 1.56371 | 1.63106 | 2.27053 |
| 18. | 1.43992 | 1.56312 | 1.63012 | 2.26935 |
| 19. | 1.43950 | 1.56260 | 1.62927 | 2.26829 |
| 20. | 1.43912 | 1.56212 | 1.62851 | 2.26735 |

**Table 4.** Simulation results of the eigen values of $B^{-1}(t+1)(I-A(t)+B(t))$ (section 4.1.)

| period | $r_1(t)$ | $r_2(t)$ |
|---|---|---|
| 0. | 6.00000 | 1.10000 |
| 1. | 5.73652 | 1.29381 |
| 2. | 5.58812 | 1.18921 |
| 3. | 5.75979 | 1.12612 |
| 4. | 5.80414 | 1.12497 |
| 5. | 5.78705 | 1.12671 |
| 6. | 5.76966 | 1.12592 |
| 7. | 5.77029 | 1.12479 |
| 8. | 5.77411 | 1.12373 |
| 9. | 5.77527 | 1.12268 |
| 10. | 5.77542 | 1.12162 |
| 11. | 5.77581 | 1.12056 |
| 12. | 5.77637 | 1.11951 |
| 13. | 5.77692 | 1.11846 |
| 14. | 5.77742 | 1.11743 |
| 15. | 5.77790 | 1.11643 |
| 16. | 5.77837 | 1.11544 |
| 17. | 5.77882 | 1.11449 |
| 18. | 5.77924 | 1.11356 |
| 19. | 5.77965 | 1.11267 |
| 20. | 5.78003 | 1.11181 |

model be set as follows;

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = \prod_{s=0}^{t} (1.1+0.001 \cdot s) \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix}, \quad \begin{pmatrix} xm_1(0) \\ xm_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix} \tag{53}$$

The growth rate of the reference model from the $t$ th period to the $t+1$ th period is $(10+0.1 \cdot t)\%$. Of course the technology of the reference model is not fixed. Accordingly the technology is assumed to be unidentified because we can apply Theorem 3 even in the case of unidentified technology.

Next we set the adaptive model as follows;

adaptive model

$$X(t+1) = B^{-1}(t+1)\left(I - A(t) + B(t)\right) X(t) - B^{-1}(t+1)\, H(t) \tag{54}$$

where, $A(-1) = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$, $\quad B(-1) = B(0) = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$$H(t) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} 220 \\ 180 \end{pmatrix}$$

If $A(t)$ and $B(t)$ is fixed as the initial values, the adaptive model has the turnpike with the same direction as the reference, however, the annual growth rate is constant, that is $10\%$.

When the adaptation laws are chosen as,

$$C(t+1) = C(t) + \left(I + \Gamma(t)\right)^{-1} \left(Kc \otimes \hat{\varepsilon}(t+1)\, X^T(t)\right) \tag{55}$$

$$D(t+1) = D(t) + \left(I + \Gamma(t)\right)^{-1} \left(Kd \otimes \hat{\varepsilon}(t+1)\, H^T(t)\right) \tag{56}$$

where, $Kc = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$, $\quad Kd = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$

The adaptation process is depicted in table 5~8. In table 5 it is shown that the adaptive outputs almost coincide with the reference at every period. In table 6 and 7 $A(t)$ and $B(t)$ are uniformly decreasing. These trends depict a technological progress aimed at catching up with the variable growth rate turnpike.

### 4. 3. *Model reference adaptive I-O system associated stock adjustment activities and temporary trade activities with respect to the ordinary turnpike*

In this section we introduce two types of additional activities for the adaptation to the turnpike. The first is the stock adjustment activities which transfer the unused stock of goods in some period to the next period without any cost. The second type is the temporary trade activities which involve the export of unused stock of goods and the import action of urgently needed commodities. These

**Table 5.** Simulation results of output levels of the reference and the adaptive models (section 4. 2.)

| period | $xm_1(t)$ | $x_1(t)$ | $xm_2(t)$ | $x_2(t)$ |
|---|---|---|---|---|
| 0. | 320.000 | 320.000 | 280.000 | 280.000 |
| 1. | 330.000 | 330.000 | 290.000 | 290.000 |
| 2. | 341.110 | 341.110 | 301.110 | 301.110 |
| 3. | 353.463 | 353.463 | 313.463 | 313.463 |
| 4. | 367.210 | 367.210 | 327.210 | 327.210 |
| 5. | 382.520 | 382.520 | 342.520 | 342.520 |
| 6. | 399.584 | 399.585 | 359.584 | 359.584 |
| 7. | 418.620 | 418.620 | 378.620 | 378.620 |
| 8. | 439.873 | 439.873 | 399.873 | 399.873 |
| 9. | 463.619 | 463.619 | 423.619 | 423.619 |
| 10. | 490.174 | 490.174 | 450.174 | 450.174 |
| 11. | 519.893 | 519.893 | 479.893 | 479.893 |
| 12. | 553.181 | 553.181 | 513.181 | 513.181 |
| 13. | 590.497 | 590.497 | 550.497 | 550.497 |
| 14. | 632.363 | 632.363 | 592.363 | 592.363 |
| 15. | 679.373 | 679.373 | 639.373 | 639.373 |
| 16. | 732.201 | 732.200 | 692.201 | 692.201 |
| 17. | 791.616 | 791.616 | 751.616 | 751.616 |
| 18. | 858.495 | 858.495 | 818.495 | 818.495 |
| 19. | 933.837 | 933.838 | 893.837 | 893.837 |
| 20. | 1,018.780 | 1,018.780 | 978.784 | 978.784 |

**Table 6.** Simulation results of the adaptive input-output coefficient matrix $A(t)$ (section 4. 2.)

| period | $a_{11}(t)$ | $a_{12}(t)$ | $a_{21}(t)$ | $a_{22}(t)$ |
|---|---|---|---|---|
| 0. | 0.30000 | 0.30000 | 0.20000 | 0.20000 |
| 1. | 0.29829 | 0.29816 | 0.19743 | 0.19724 |
| 2. | 0.29730 | 0.29725 | 0.19596 | 0.19588 |
| 3. | 0.29622 | 0.29625 | 0.19433 | 0.19437 |
| 4. | 0.29502 | 0.29514 | 0.19254 | 0.19270 |
| 5. | 0.29372 | 0.29393 | 0.19058 | 0.19089 |
| 6. | 0.29230 | 0.29262 | 0.18846 | 0.18892 |
| 7. | 0.29077 | 0.29120 | 0.18616 | 0.18680 |
| 8. | 0.28914 | 0.28968 | 0.18371 | 0.18452 |
| 9. | 0.28739 | 0.28806 | 0.18109 | 0.18209 |
| 10. | 0.28555 | 0.28635 | 0.17833 | 0.17952 |
| 11. | 0.28361 | 0.28453 | 0.17541 | 0.17680 |
| 12. | 0.28157 | 0.28264 | 0.17236 | 0.17396 |
| 13. | 0.27946 | 0.28066 | 0.16918 | 0.17099 |
| 14. | 0.27726 | 0.27861 | 0.16590 | 0.16791 |
| 15. | 0.27500 | 0.27648 | 0.16250 | 0.16472 |
| 16. | 0.27268 | 0.27429 | 0.15902 | 0.16144 |
| 17. | 0.27031 | 0.27205 | 0.15546 | 0.15808 |
| 18. | 0.26789 | 0.26976 | 0.15184 | 0.15464 |
| 19. | 0.26544 | 0.26743 | 0.14816 | 0.15115 |
| 20. | 0.26296 | 0.26507 | 0.14445 | 0.14760 |

**Table 7.** Simulation results of the adaptive capital coefficient matrix $B(t)$ (section 4. 2.)

| period | $b_{11}(t)$ | $b_{12}(t)$ | $b_{21}(t)$ | $b_{22}(t)$ |
|---|---|---|---|---|
| 0. | 1.90000 | 2.10000 | 2.60000 | 3.40000 |
| 1. | 1.90000 | 2.10000 | 2.60000 | 3.40000 |
| 2. | 1.90093 | 2.10113 | 2.60139 | 3.40170 |
| 3. | 1.90195 | 2.10239 | 2.60293 | 3.40358 |
| 4. | 1.90306 | 2.10374 | 2.60459 | 3.40561 |
| 5. | 1.90426 | 2.10521 | 2.60639 | 3.40781 |
| 6. | 1.90554 | 2.10677 | 2.60830 | 3.41015 |
| 7. | 1.90688 | 2.10841 | 2.61032 | 3.41262 |
| 8. | 1.90829 | 2.11013 | 2.61244 | 3.41520 |
| 9. | 1.90975 | 2.11192 | 2.61462 | 3.41787 |
| 10. | 1.91125 | 2.11374 | 2.61687 | 3.42062 |
| 11. | 1.91276 | 2.11560 | 2.61915 | 3.42340 |
| 12. | 1.91429 | 2.11747 | 2.62144 | 3.42621 |
| 13. | 1.91582 | 2.11933 | 2.62373 | 3.42900 |
| 14. | 1.91732 | 2.12117 | 2.62598 | 3.43176 |
| 15. | 1.91879 | 2.12297 | 2.62819 | 3.43446 |
| 16. | 1.92022 | 2.12472 | 2.63033 | 3.43708 |
| 17. | 1.92160 | 2.12639 | 2.63239 | 3.43959 |
| 18. | 1.92290 | 2.12799 | 2.63435 | 3.44199 |
| 19. | 1.92414 | 2.12951 | 2.63621 | 3.44426 |
| 20. | 1.92530 | 2.13093 | 2.63795 | 3.44639 |

**Table 8.** Simulation results of the eigen values of $B^{-1}(t+1)(I-A(t)+B(t))$ (section 4. 2.)

| period | $r_1(t)$ | $r_2(t)$ |
|---|---|---|
| 0. | 6.00000 | 1.10000 |
| 1. | 6.00000 | 1.10000 |
| 2. | 6.00000 | 1.10031 |
| 3. | 6.00000 | 1.10068 |
| 4. | 6.00000 | 1.10109 |
| 5. | 6.00000 | 1.10155 |
| 6. | 6.00000 | 1.10207 |
| 7. | 6.00000 | 1.10263 |
| 8. | 6.00000 | 1.10325 |
| 9. | 6.00000 | 1.10392 |
| 10. | 6.00000 | 1.10465 |
| 11. | 6.00000 | 1.10543 |
| 12. | 6.00000 | 1.10627 |
| 13. | 6.00000 | 1.10716 |
| 14. | 6.00000 | 1.10810 |
| 15. | 6.00000 | 1.10908 |
| 16. | 6.00000 | 1.11010 |
| 17. | 6.00000 | 1.11116 |
| 18. | 6.00000 | 1.11226 |
| 19. | 6.00000 | 1.11338 |
| 20. | 6.00000 | 1.11453 |

activities are formulated as follows;

$$K(t) = S(t)\, X(t) \tag{57}$$

$$EM(t) = W(t)\, X(t) \tag{58}$$

where, $K(t)$: stock adjustment activity vector $(n \times 1)$
$S(t)$: stock adjustment coefficient matrix $(n \times n)$
$EM(t)$: temporary trade activity vector $(n \times 1)$
$W(t)$: temporary trade coefficient matrix $(n \times n)$

Then the balance equation of the adaptive model is transformed as,

$$\begin{aligned} X(t) &= A(t)\, X(t) + B(t+1)\, X(t+1) - B(t)\, X(t) \\ &\quad + S(t+1)\, X(t+1) - S(t)\, X(t) + W(t)\, X(t) + H(t) \\ \therefore \quad X(t+1) &= \big(B(t+1)+S(t+1)\big)^{-1}\big(I - A(t) + B(t) - W(t) + S(t)\big)\, X(t) \\ &\quad - \big(B(t+1)+S(t+1)\big)^{-1} H(t) \end{aligned} \tag{59}$$

When we adopt the reference model which is the same as in section 4. 1. and assume that $A(t)=Am$, $B(t)=Bm$ then the system can be represented as follows;

reference model

$$\begin{pmatrix} xm_1(t+1) \\ xm_2(t+1) \end{pmatrix} = 1.1^{t+1}\begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} 22 \\ 18 \end{pmatrix} \tag{60}$$

adaptive model

$$\begin{aligned} X(t+1) &= \big(Bm+S(t+1)\big)^{-1}\big(I - Am + Bm + S(t) - W(t)\big)\, X(t) \\ &\quad - \big(Bm+S(t+1)\big)^{-1} H(t) \end{aligned} \tag{61}$$

where, $Am = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix}$, $Bm = \begin{pmatrix} 1.9 & 2.1 \\ 2.6 & 3.4 \end{pmatrix}$

$X(0) = (31,\ 19)^T$, $H(t) = (10,\ 10)^T$

Setting $C(t+1)$ and $D(t+1)$ as,

$$C(t+1) = \big(Bm+S(t+1)\big)^{-1}\big(I - Am + Bm + S(t) - W(t)\big) \tag{62}$$

$$D(t+1) = -\big(Bm+S(t+1)\big)^{-1} \tag{63}$$

the same adaptation algorithm in Theorem 2 or 3 can be utilized for this problem and $S(t)$ and $W(t)$ are obtained. However, it should be noted here that $K(t)$ must be positive, therefore some amount of caution is needed for setting up $Kc$ and $Kd$. Here $Kc$ and $Kd$ are applied as,

$$Kc = \begin{pmatrix} 0.0001 & 0.0001 \\ 0.0001 & 0.0001 \end{pmatrix}, \quad Kd = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $S(t) \geqq 0$ is taken into account in the constrained condition of the adaptation.

The results of the simulations are presented in tables 9～13. In table 9 it is shown that the adaptive outputs almost converge to the turnpike before the 5 th period, however, the behavior of the adaptive outputs fluctuate a little larger in the initial stages. The reason may be that the stock adjustment activities do not work because of the constrained condition $S(t) \geqq 0$. From the results depicted in table 10～13, we see that $K(t)$ and $EM(t)$ work effectively in only three periods after the initial time and take on very little values after the third period. This result shows that the adjustment of the adaptive model to the turnpike only works in the pre-third period and the adaptive model takes on the turnpike technology after the third period.

## 5. Conclusion

This study investigated Model Reference Adaptive Turnpike Theorem following the results in our previous paper 5). In this paper we have succeeded in removing the assumption of the relative stability which was a major problem in 5), and developed the theorems which could be applied to a more general dynamic I-O system. Furthermore the usefulness of the theorems obtained in this article was proved by some numerical simulations. And it is hoped that those theorems would play an

**Table 9.** Simulation results of output levels of the reference and the adaptive models (section 4. 3.)

| period | $xm_1(t)$ | $x_1(t)$ | $xm_2(t)$ | $x_2(t)$ |
|---|---|---|---|---|
| 0. | 32.000 | 31.000 | 28.000 | 19.000 |
| 1. | 33.000 | 53.500 | 29.000 | 1.500 |
| 2. | 34.100 | 34.856 | 30.100 | 29.425 |
| 3. | 35.310 | 35.360 | 31.310 | 31.270 |
| 4. | 36.641 | 36.632 | 32.641 | 32.649 |
| 5. | 38.105 | 38.104 | 34.105 | 34.106 |
| 6. | 39.716 | 39.715 | 35.716 | 35.716 |
| 7. | 41.487 | 41.486 | 37.487 | 37.488 |
| 8. | 43.436 | 43.435 | 39.436 | 39.437 |
| 9. | 45.580 | 45.579 | 41.580 | 41.580 |
| 10. | 47.937 | 47.937 | 43.937 | 43.938 |
| 11. | 50.531 | 50.530 | 46.531 | 46.532 |
| 12. | 53.384 | 53.383 | 49.384 | 49.385 |
| 13. | 56.523 | 56.522 | 52.523 | 52.524 |
| 14. | 59.975 | 59.974 | 55.975 | 55.976 |
| 15. | 63.773 | 63.771 | 59.773 | 59.774 |
| 16. | 67.950 | 67.949 | 63.950 | 63.951 |
| 17. | 72.545 | 72.543 | 68.545 | 68.546 |
| 18. | 77.599 | 77.598 | 73.599 | 73.600 |
| 19. | 83.159 | 83.158 | 79.159 | 79.160 |
| 20. | 89.275 | 89.274 | 85.275 | 85.276 |

**Table 10.** Simulation results of the stock adjustment coefficient matrix $S(t)$ (section 4. 3.)

| period | $s_{11}(t)$ | $s_{12}(t)$ | $s_{21}(t)$ | $s_{22}(t)$ |
|---|---|---|---|---|
| 0. | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 1. | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 2. | 0.40360 | 0.49329 | 0.00000 | 0.00000 |
| 3. | 0.00000 | 0.00000 | 0.21650 | 0.26461 |
| 4. | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 5. | 0.00329 | 0.00403 | 0.00000 | 0.00000 |
| 6. | 0.00174 | 0.00212 | 0.00000 | 0.00000 |
| 7. | 0.00168 | 0.00205 | 0.00000 | 0.00000 |
| 8. | 0.00177 | 0.00217 | 0.00000 | 0.00000 |
| 9. | 0.00186 | 0.00227 | 0.00000 | 0.00000 |
| 10. | 0.00196 | 0.00240 | 0.00000 | 0.00000 |
| 11. | 0.00206 | 0.00252 | 0.00000 | 0.00000 |
| 12. | 0.00218 | 0.00266 | 0.00000 | 0.00000 |
| 13. | 0.00229 | 0.00280 | 0.00000 | 0.00000 |
| 14. | 0.00243 | 0.00297 | 0.00000 | 0.00000 |
| 15. | 0.00257 | 0.00314 | 0.00000 | 0.00000 |
| 16. | 0.00273 | 0.00334 | 0.00000 | 0.00000 |
| 17. | 0.00292 | 0.00357 | 0.00000 | 0.00000 |
| 18. | 0.00311 | 0.00380 | 0.00000 | 0.00000 |
| 19. | 0.00330 | 0.00404 | 0.00000 | 0.00000 |
| 20. | 0.00354 | 0.00433 | 0.00000 | 0.00000 |

**Table 11.** Simulation results of levels of the stock adjustment activities $K(t)$ (section 4. 3.)

| period | $K_1(t)$ | $K_2(t)$ |
|---|---|---|
| 0. | 0.000 | 0.000 |
| 1. | 0.000 | 0.000 |
| 2. | 28.583 | 0.000 |
| 3. | 0.000 | 15.930 |
| 4. | 0.000 | 0.000 |
| 5. | 0.263 | 0.000 |
| 6. | 0.145 | 0.000 |
| 7. | 0.147 | 0.000 |
| 8. | 0.162 | 0.000 |
| 9. | 0.179 | 0.000 |
| 10. | 0.199 | 0.000 |
| 11. | 0.221 | 0.000 |
| 12. | 0.247 | 0.000 |
| 13. | 0.277 | 0.000 |
| 14. | 0.313 | 0.000 |
| 15. | 0.352 | 0.000 |
| 16. | 0.399 | 0.000 |
| 17. | 0.457 | 0.000 |
| 18. | 0.521 | 0.000 |
| 19. | 0.594 | 0.000 |
| 20. | 0.686 | 0.000 |

**Table 12.** Simulation results of the temporary trade coefficient matrix $W(t)$ (section 4. 3.)

| period | $w_{11}(t)$ | $w_{12}(t)$ | $w_{21}(t)$ | $w_{22}(t)$ |
|---|---|---|---|---|
| 0. | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 1. | −0.44850 | −0.53814 | −0.00177 | −0.00005 |
| 2. | 0.40374 | 0.49333 | −0.24241 | −0.28873 |
| 3. | 0.00014 | 0.00004 | 0.21473 | 0.26456 |
| 4. | −0.00352 | −0.00436 | −.000177 | −0.00005 |
| 5. | 0.00150 | 0.00175 | −0.00176 | −0.00005 |
| 6. | 0.00001 | −0.00008 | −0.00176 | −0.00005 |
| 7. | −0.00015 | −0.00027 | −0.00176 | −0.00005 |
| 8. | −0.00015 | −0.00028 | −0.00176 | −0.00004 |
| 9. | −0.00018 | −0.00031 | −0.00176 | −0.00004 |
| 10. | −0.00019 | −0.00031 | −0.00175 | −0.00004 |
| 11. | −0.00022 | −0.00035 | −0.00175 | −0.00004 |
| 12. | −0.00023 | −0.00036 | −0.00175 | −0.00004 |
| 13. | −0.00027 | −0.00041 | −0.00175 | −0.00003 |
| 14. | −0.00028 | −0.00042 | −0.00174 | −0.00003 |
| 15. | −0.00032 | −0.00046 | −0.00174 | −0.00003 |
| 16. | −0.00038 | −0.00052 | −0.00174 | −0.00002 |
| 17. | −0.00039 | −0.00054 | −0.00173 | −0.00002 |
| 18. | −0.00042 | −0.00057 | −0.00173 | −0.00002 |
| 19. | −0.00049 | −0.00065 | −0.00172 | −0.00001 |
| 20. | −0.00054 | −0.00070 | −0.00171 | −0.00000 |

**Table 13.** Simulation results of levels of the temporary trade activities $EM(t)$ (section 4. 3.)

| period | $EM_1(t)$ | $EM_2(t)$ |
|---|---|---|
| 0. | 0.000 | 0.000 |
| 1. | −24.802 | −0.095 |
| 2. | 28.589 | −16.945 |
| 3. | 0.006 | 15.866 |
| 4. | −0.271 | −0.066 |
| 5. | 0.117 | −0.069 |
| 6. | −0.003 | −0.072 |
| 7. | −0.016 | −0.075 |
| 8. | −0.018 | −0.078 |
| 9. | −0.021 | −0.082 |
| 10. | −0.022 | −0.086 |
| 11. | −0.027 | −0.090 |
| 12. | −0.030 | −0.095 |
| 13. | −0.037 | −0.100 |
| 14. | −0.040 | −0.106 |
| 15. | −0.048 | −0.113 |
| 16. | −0.059 | −0.120 |
| 17. | −0.066 | −0.127 |
| 18. | −0.074 | −0.135 |
| 19. | −0.093 | −0.144 |
| 20. | −0.107 | −0.153 |

important role in a stage of a corroborative study.

Areas worth examining for futrher research and analysis include introduction of the adaptation process by a control of the private capital investment and/or R & D investment to make the adaptation law endogeneously possible.

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