Further Analysis of Rectangular Building Frames by the Mechanical Tabulation Method.

By

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In a previous paper the author described an exact solution of rectangular building frames by the Mechanical Tabulation Method. This may be very instructive to show how easily the simultaneous equations can be mechanically obtained, from which the redundancies are to be computed.

The merits of this method consist in the large reduction in the number of redundancies and no mistakes can be made in writing the fundamental table, on account of the systematic arrangement of the suffix-number and the symmetrical property of the places where the coefficients of unknown quantities are to be written.

With all its accuracy of the calculation, the method is very simple and a knowledge of elementary algebra is sufficient for the calculation of the stresses of higher structures.

While the solution of these simultaneous equations is generally too long to be used in the actual design of a high building we have described in the summary and conclusion of the former paper how it has great value as a standard calculation for checking the accuracy of another approximate rapid method.

But we can now say, for this solution after repeated trials, that it has indeed great value also as a practical calculation of the stresses of high buildings with speed and precision.

The solution of the set of 18 simultaneous equations for unknown slope φ, solved in the previous paper, i.e. the problem of a department store building symmetrical about the vertical center line of the frame, six-stories high and six-spans long, carrying loads concentrated symmetrically and distributed uniformly, was effected by repeated trial and checked comparing with the true value.

This repeated trial is accurate enough to obtain the approximate value of redundant stresses.

1. Recapitulation of the Result Obtained in the Previous Example.

Fig. 1 represents a six storied symmetrical bent with six equal spans. All the systems of load are symmetrical and all legs of the bent are fixed at the bases. It is required to find the bending moments at all of the joints.

Representing by $\xi = I : l$ and by ρ_r two times the sum of ξ of all the members which intersect at the joint r, e.g. in Fig. 1

$$\rho_1 = 2\left\{\xi_I + \xi_1 + \xi'_1\right\}$$

$$= 2\left\{98 + 55 + 88.5\right\}$$

$$= 483 \ in^3,$$

$$\rho_2 = 2\left\{\xi_{II} + \xi_1 + \xi_2 + \xi'_2\right\}$$

$$= 2\left\{157.8 + 55 + 55 + 156\right\}$$

$$= 847.6 \ in^3$$

etc.,

we obtain the following table of general equations to be used in determining the unknown slopes for the bent in question (Table 1).

TABLE 1.

No. of Equation	Left-Hand Member of Equation																		Right-Hand Member of Equation
	Coefficients of Unknown Slopes φ																		Coefficient of $\frac{1}{2E}$
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	φ_{17}	φ_{18}	
1	ρ_1	ξ_1				ξ'_1													C_{12}
2	ξ_1	ρ_2	ξ_2		ξ'_2														0
3		ξ_2	ρ_3	ξ_3															0
4			ξ_3	ρ_4	ξ_4				ξ'_4										0
5		ξ'_2		ξ_4	ρ_5	ξ_5		ξ'_5											0
6	ξ'_1				ξ_5	ρ_6	ξ_6												C_{65}
7						ξ_6	ρ_7	ξ_7				ξ'_7							C_{78}
8					ξ'_5		ξ_7	ρ_8	ξ_8		ξ'_8								0
9				ξ'_4				ξ_8	ρ_9	ξ_9									0
10									ξ_9	ρ_{10}	ξ_{10}				ξ'_{10}				0
11								ξ'_8		ξ_{10}	ρ_{11}	ξ_{11}		ξ'_{11}					0
12							ξ'_7				ξ_{11}	ρ_{12}	ξ_{12}						$C_{12,11}$
13												ξ_{12}	ρ_{13}	ξ_{13}				ξ'_{13}	$C_{13,14}$
14											ξ'_{11}		ξ_{13}	ρ_{14}	ξ_{14}		ξ'_{14}		0
15										ξ'_{10}				ξ_{14}	ρ_{15}	ξ_{15}			0
16															ξ_{15}	ρ_{16}	ξ_{16}		0
17														ξ'_{14}		ξ_{16}	ρ_{17}	ξ_{17}	0
18													ξ'_{13}				ξ_{17}	ρ_{18}	$C_{18,17}$

Substituting each value of ρ and ξ in Table 1 gives Table 2.

The figures given in the right-hand column are coefficients of *.00001** as indicated at the head of the column.

These simultaneous equations have been solved by process of elimination and the results are as follows:

* Erratum in Memoirs Vol. I. No. 3, Appendix, for *0.0001* read *.00001*.

TABLE 2.

No. of Equation	Left-Hand Member of Equation: Coefficients of Unknown Slopes φ																		Right-Hand Member of Equation Coefficient of .00001
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	φ_{17}	φ_{18}	
1	483	55				28.5													1065
2	55	847.6	55		156														0
3		55	847.6	156															0
4			156	736	55				102										0
5		156		55	736	55		102											0
6	88.5				55	432	72.5												1065
7						72.5	400	55				72.5							1065
8					102		55	530	55		53								0
9				102				55	530	53									0
10									53	402	55				38				0
11								53		55	402	55		38					0
12							72.5				55	400	72.5						1065
13												72.5	362	55				53.5	1065
14											38		55	366	55		35		0
15										38				55	366	35			0
16															35	380	77.5		0
17														35		77.5	380	77.5	0
18													53.5				77.5	262	1195

$\varphi_1 = 1.902\ 23$ $\qquad\varphi_{10} = .029\ 95$

$\varphi_2 = -.110\ 09$ $\qquad\varphi_{11} = -.236\ 24$

$\varphi_3 = .006\ 63$ $\qquad\varphi_{12} = 1.982\ 78$

$\varphi_4 = .002\ 72$ $\qquad\varphi_{13} = 1.914\ 58$

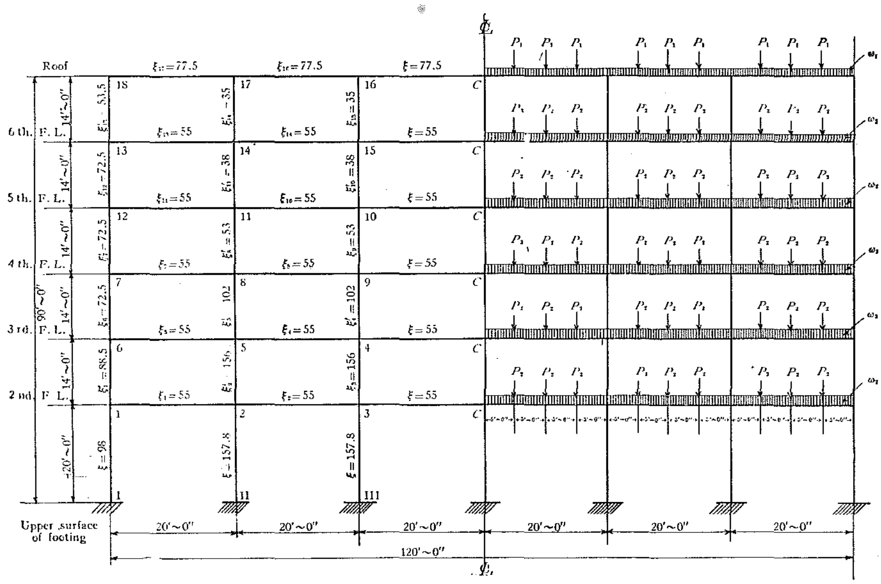
$\varphi_5 = -.081\ 71$ $\qquad\varphi_{14} = -.175\ 12$

$\varphi_6 = 1.720\ 53$ $\qquad\varphi_{15} = .005\ 13$

$\varphi_7 = 2.014\ 81$ $\qquad\varphi_{16} = .188\ 98$

$\varphi_8 = -.171\ 21$ $\qquad\varphi_{17} = -.928\ 93$

$\varphi_9 = .014\ 24$ $\qquad\varphi_{18} = 4.444\ 89$



ELEVATION OF FRAME.

$P_1 = 8200^{\#}$, $P_2 = 7450^{\#}$,
$\omega_1 = 255^{\#}/l.ft$, $\omega_2 = 200^{\#}/l.ft.$

Fig. 1.

The moments at the ends of the columns and girders given by the fundamental equations

$$\left.\begin{aligned} M_{rs} &= 2E\xi_r\Big\{2\varphi_r + \varphi_s - 3\mu\Big\} - C_{rs}, \\ M_{sr} &= 2E\xi_r\Big\{2\varphi_s + \varphi_r - 3\mu\Big\} + C_{sr} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \; (1)$$

are indicated in Memoirs Vol. I. No. 3 p. 187, Table 3, where the "*Total Moment*" means the total sum of the moments due to concentrated loads and uniformly distributed loads.

The accuracy of the computation has been checked by the statical condition of equilibrium of the joint-moments and the percentage of error for the least moment is given in the right-hand column of the same table.

2. Computation Procedure by Repeated Trial.

The solution of the set of 18 simultaneous equations is too long to be done, by process of elimination, in the actual design.

For the process of elimination we are forced to use a calculating machine of high capacity and the neglect of a small decimal part in calculation effects the result with a considerable percentage of error.

The successive approximations by repeated trial give good results in the case of our problem. Each equation has generally one term whose coefficient is relatively large compared with the others, i.e. the value of ρ is in general much larger than that of ξ.

If we therefore obtain an approximate value of φ for each joint, the substitution of these values in each equation in the terms with ξ, and the solution of the equation for the term with ρ, considering the φ with ρ to be unknown, leads to a much closer approximation.

Thus for joint 18 the last equation of Table 2 gives:

$$53.5\,\varphi_{13}+77.5\,\varphi_{17}+262\,\varphi_{18}=1195,$$

in which we have roughly approximate values, φ_{13}, φ_{17} and φ_{18} with coefficient *.00001* as

$$\varphi_{13}=\ 2.266$$
$$\varphi_{17}=-0.970.$$
$$\varphi_{18}=\ 4.561^{*}$$

Substituting φ_{13} and φ_{17} in the last equation and solving for φ_{18} we get *4.385*, a value much nearer the true value *4.44489* with coefficient *.00001*.

This same procedure is carried out for each joint and thus a full set of new values for φ is obtained and the process repeated until there is no practical percentage of error when the accuracy of the computation is checked by the statical condition of equilibrium of the joint-moment, i.e.:

$$\sum M=0.$$

* The computation of these approximate values is treated in section 3.

Table 3 shows the values of φ obtained by the repeated trial and in Table 4 we tabulate the joint-moment computed by (1) with the nearest values of φ in Table 3.

TABLE 3.

Table of φ Obtained by Repeated Trial

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
Assumed Value	2.250	-0.289	0.037	0.037	-0.289	2.250	2.250	-0.289	0.037
1st Approximation	1.905	-0.138	0.020	-0.007	-0.047	1.718	1.922	-0.158	0.020
2nd Approximation	1.898	-0.110	0.009	-0.0004	-0.077	1.743	2.012	-0.168	0.014
3rd Approximation	1.896	-0.108	0.0067	0.0023	-0.083	1.748	2.010	-0.171	0.014
4th Approximation	1.8965	-0.1079	0.0064	0.0029	-0.0844	1.7502	2.0095	-0.1704	0.0143

	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	φ_{17}	φ_{18}
Assumed Value	0.037	-0.289	2.250	2.266	-0.250	0.018	0.197	-0.970	4.561
1st Approximation	0.022	-0.213	1.954	1.877	-0.225	0.016	0.184	0.911	4.385
2nd Approximation	0.028	-0.238	1.995	1.919	-0.173	0.011	0.186	-0.923	4.447
3rd Approximation	0.029	-0.236	1.984	1.912	-0.175	0.005	0.188	-0.927	4.442
4th Approximation	0.0299	-0.2362	1.9836	1.9142	-0.1747	0.0052	0.1889	-.9287	4.4448

TABLE 4.

TABLE OF JOINT-MOMENT

(Moments are Expressed in Ft.-lbs.)

	Total moment	Percentage of error for the least moment*
M_{18-17}	−28901.5	
M_{18-13}	28900.1	
M_{17-18}	69776.1	
M_{17-16}	−66215.8	

* Percentage of error less than 0.5% is not written in the table.

TABLE 4 (Continued)

	Total moment	Percentage of error for the least moment
M_{17-14}	− 3556.1	
M_{16-17}	57615.2	
M_{16-c}	−58286.0	
M_{16-15}	670.2	
M_{15-14}	52777.3	
M_{15-c}	− 53200.6	
M_{15-16}	348.7	
M_{15-10}	76.5	2.4 %
M_{14-13}	57532.4	
M_{14-15}	− 54175.7	
M_{14-17}	− 2236.6	
M_{14-11}	− 1112.6	0.6 %
M_{13-14}	− 43181.5	
M_{13-18}	22130.8	
M_{13-12}	21068.5	
M_{12-11}	− 42968.9	
M_{12-13}	21320.0	
M_{12-7}	21665.5	

TABLE 4 (Continued)

	Total moment	Percentage of error for the least moment
M_{11-12}	57385.0	
M_{11-10}	−54446.0	
M_{11-14}	− 1229.4	0.5 %
M_{11-8}	− 1703.4	
M_{10-11}	52744.1	
M_{10-c}	−53064.7	
M_{10-15}	123.5	0.6 %
M_{10-9}	196.3	
M_{9-8}	52839.2	
M_{9-c}	−53150.5	
M_{9-10}	155.0	2.7 %
M_{9-4}	160.6	
$M_{8\ 9}$	−54127.0	
M_{8-7}	57818.1	
M_{8-11}	− 1529.0	
M_{8-5}	− 2168.5	
M_{7-8}	−42645.5	
M_{7-12}	21759.4	

TABLE 4 (Continued)

	Total moment	Percentage of error for the least moment
M_{7-6}	20913.3	
M_{6-5}	−43835.2	
M_{6-7}	19973.3	
M_{6-1}	23881.2	
M_{5-6}	57578.0	
M_{5-4}	−53685.4	
M_{5-8}	− 1729.9	
M_{5-2}	− 2158.2	
M_{4-5}	53013.0	
M_{4-c}	−53213.2	
M_{4-9}	102.5	
M_{4-3}	95.1	2.7 %
M_{3-2}	52967.6	
M_{3-c}	−53194.0	
M_{3-4}	122.4	
M_{3-III}	100.9	3.0 %
M_{2-3}	− 53805.0	
M_{2-1}	57851.1	

TABLE 4 (Continued)

	Total moment	Percentage of error for the least moment
M_{2-II}	− 1702.6	
M_{2-5}	− 2341.5	
M_{1-I}	18585.7	
M_{1-6}	24528.6	
M_{1-2}	−43095.1	
M_{I-1}	9292.8	
M_{II-2}	− 851.3	
M_{III-3}	50.4	
M_{c-3}	53246.8	
M_{c-4}	53237.1	
M_{c-9}	53268.5	
M_{c-10}	53311.4	
M_{c-15}	53243.5	
M_{c-16}	60481.9	

3. Method of Finding the Approximate Value of φ.

For the approximate values of φ, which are to be used in the first approximation, a rough computation is sufficient or in other words they may be assumed.

As it is convenient to get the value as nearly as possible to the true value we have calculated the values of φ as in the following. Since this procedure is not absolute, other proper methods of finding the approximate value of φ can be applicable here, as the object is nothing but to find the approximate value of φ. When the more approximate values are applied at first, the more accurate results are obtained with less effort of repeated trial.

i) Approximate Value of φ_{18}.

To obtain the approximate value of φ_{18}, we assume the bent as shown in Fig. 2 and the equation to be used in determining the unknown slope φ_{18} is:

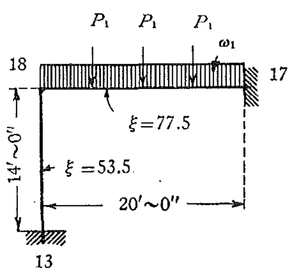


Fig. 2.

$$\left.\begin{aligned} \varphi_{18} &= \frac{C}{2E\rho_{18}} = \frac{p}{\rho_{18}} \\ \text{where} \quad p &= \frac{C}{2E} = .01195\ in^3 \\ \rho_{18} &= 2\left\{\xi_{17} + \xi'_{13}\right\} \\ &= 262\ in^3 \end{aligned}\right\} \qquad \ldots\ldots\ldots(2)$$

From (2) we get therefore:

$$\varphi_{18} = 4.561$$

with coefficient of *.00001.**

ii) Approximate Values of φ_{17} *and* φ_{16}.

To obtain the approximate value of φ_{17} and φ_{16}, we assume the bent as shown in Fig. 3 and equations to be used in determining the unknown slopes φ_{17} and φ_{16} are obtained from Table 5.

* We omit the coefficient in the calculation note below.

TABLE 5.

Left-Hand Member of Equation		Right-Hand Member of Equation
φ_{16}	φ_{17}	(Known Term)
ρ_{16}	ξ	0
ξ	ρ_{17}	$-\xi\varphi_{18}$

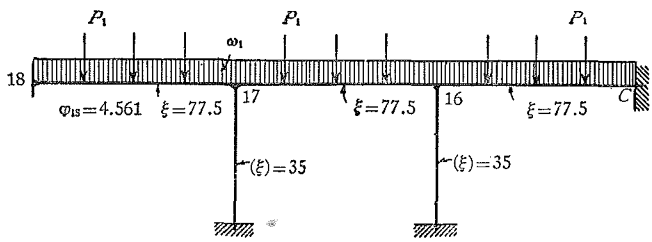


Fig. 3.

We get the following two equations from Table 5:

$$\left.\begin{aligned} &\varphi_{16}\rho_{16}+\varphi_{17}\xi=0, \\ &\varphi_{16}\xi+\varphi_{17}\rho_{17}=-\varphi_{18}\xi, \\ \text{where}\quad & \\ &\rho_{16}=\rho_{17}=380\ in^3, \\ &\xi=77.5\ in^3, \\ &\varphi_{18}=4.561. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From these two equations we get:

$$\varphi_{16}=\ 0.197,$$
$$\varphi_{17}=-0.970.$$

iii) Approximate Values of φ_{13}, φ_{12}, φ_7, φ_6 *and* φ_1.

To obtain the approximate values of φ_{13}, φ_{12}, φ_7, φ_6 and φ_1, we assume the bent as shown in Fig. 4. The equation to be used in determining φ is:

$$\varphi_A=\frac{1}{\rho_A}\left\{p-\varphi_C\xi'\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where φ_C is the known value computed directly previously.

For φ_{13} we have:

$$\varphi_{13}=\frac{1}{\rho_{13}}\left\{p-\varphi_{18}\xi'_{13}\right\}$$

$$=2.266$$

In a similar way

$$\varphi_{12}=2.250$$

and for φ_7, φ_6, φ_1 we assumed the same value as φ_{12}.

i.e. $\varphi_7=\varphi_6=\varphi_1=2.250$

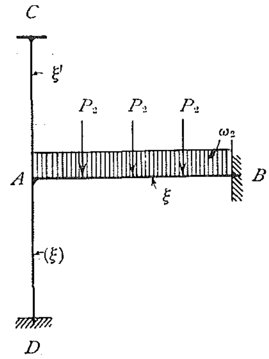


Fig. 4.

iv). Approximate Values of φ_{15}, φ_{14}, φ_{11}, φ_{10}, φ_9, φ_8, φ_5, φ_4, φ_3 and φ_2.

To obtain the approximate values of remaining φ, we assume the bent as shown in Fig. 5.

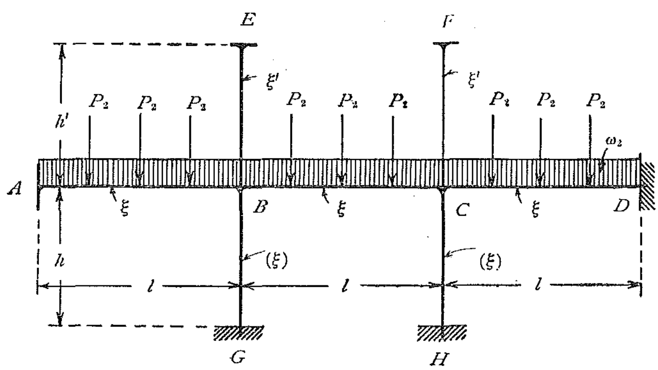


Fig. 5.

The equations to be used in determining φ are:

$$\left.\begin{aligned}\varphi_B&=-\frac{\varphi_A\xi\rho_C+\varphi_E\xi'\rho_C-\varphi_F\xi'\xi}{\rho_C^2-\xi^2},\\ \varphi_C&=\frac{\varphi_A\xi^2+\varphi_E\xi'\xi-\varphi_F\xi'\rho_C}{\rho_C^2-\xi^2}\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(5)$$

where φ_A, φ_E, φ_F are the known values computed directly previously.

Applying (5) gives :

$$\varphi_{15}=0.018$$
$$\varphi_{14}=-0.250$$

With these values of φ_{15}, φ_{14} and also φ_{12} previously computed, we get :

$$\varphi_{11}=-0.289$$
$$\varphi_{10}=0.037$$

For φ_8, φ_5, φ_2 we assumed the same value as φ_{11} and for φ_9, φ_4, φ_3 the same value as φ_{10}, as the former find themselves in almost the same boundary condition as the latter.

i.e. $$\varphi_8=\varphi_5=\varphi_2=-0.289$$
$$\varphi_9=\varphi_4=\varphi_3=0.037.$$

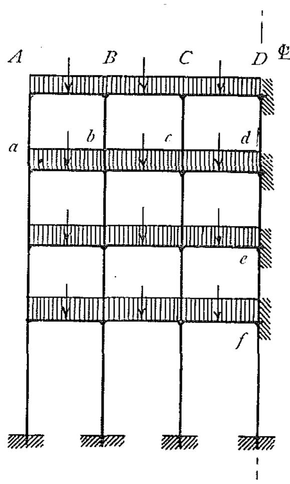
Summary and Conclusions.

The principal conclusions to be drawn from the investigation of the rectangular building frames by the Mechanical Tabulation Method have already been given (Page 183).

The solution of simultaneous equations tabulated mechanically is very much simplified by the method of repeated trial. We have explained the solution with respect to the numerical example treated in these Memoirs Vol. I. No. 3.

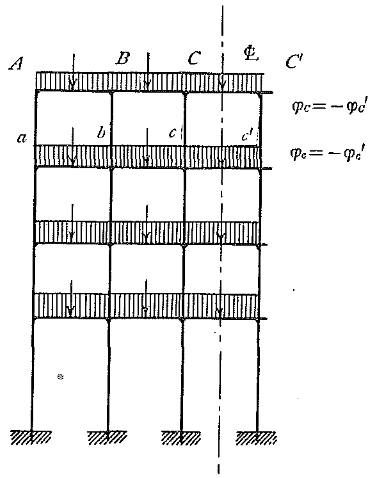
Together with some additional facts, the method of finding the first assumed values of φ will be here restated briefly but more generally than described in the previous section.

For frames in symmetrical condition with multiple span of even number, the slopes at the joints in the center line of the building are zero, thus in Fig. 6 $\varphi_D=\varphi_d=\varphi_e=\varphi_f=0$; while in frames of the same kind with multiple span of odd number, the slopes of the joints in one vertical plane nearest left to the center line of the building, must be equal and opposite to those of the joints in another vertical plane nearest right to the center line of the building. Thus in Fig. 7



Number of Spans $=2n$

Fig. 6.



Number of Spans $=2n+1$

Fig. 7.

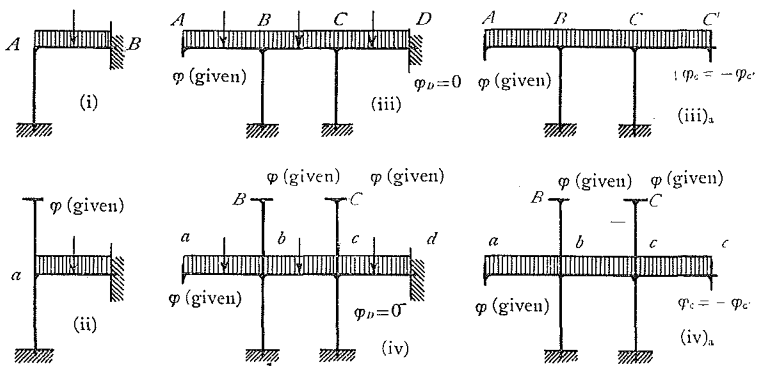


Fig. 8.

$$\varphi_C=-\varphi'_C,$$
$$\varphi_c=-\varphi'_c \quad etc.$$

and in general

$$\varphi=-\varphi'.$$

For frames in symmetrical condition with multiple span of even or odd number, we calculate for both, firstly the value of φ_A with respect to the frame in Fig. 8 (*i*). Secondly we calculate, with this value of φ_A, the value of φ_a with respect to the frame in Fig. 8 (*ii*).

We find likewise with the above computed value of φ_A the values of φ_B and φ_C with respect to the frames in Fig. 8 *(iii)* and $(iii)_a$ for frames with span of even and odd number respectively.

In a similar way, with the known values of φ_a, φ_B, φ_C computed previously, we proceed to carry out the calculation for the frames in Fig. 8 *(iv)* and $(iv)_a$ respectively corresponding to the given frames with span of even or odd number.

The computation in a similar way is to be continued until one obtains the approximate values of all φ's.

In Table 3 for the values of φ which are firstly computed as explained above, we use the term *assumed values* of φ and for the values of φ computed by the method of approximation of repeated trial we use the term *approximate values* of φ.

The principal object is to obtain, even roughly but easily, the assumed values of φ, which can be hereafter gradually corrected, or as it were, refined by the test of repeated trial which leads to a much closer approximation. This process must be repeated until there is practical agreement of the condition of equilibrium or until the difference is small enough to satisfy the desired accuracy of the given problem.

In the problem treated above, the percentage of error for the least moment has been computed and is shown in Table 4, where we see no large percentage of error greater than 3.0%. This error is not considerable to the purposes of practical design and if we distribute the error of discrepancy over all the joint-moments, which must satisfy the condition of equilibrium, we can obtain thereby an even less percentage of error.

It is noticeable that the process of elimination for the solution of simultaneous equations may perhaps be considered as a good method of finding unknowns, notwithstanding that we often find there some inevitable error of accumulation caused by the neglect of some small decimal part in calculation. The process of repeated trial is generally in the strict sense the method of correcting the computed results.

The percentage of error for the least moment due to the joint-moments shown in *Memoirs*, Vol. I. No. 3, Table 3 may become zero by the process of repeated trial.

The method here proposed however, starting upon the approximate repeated trial, lends itself well to an exact solution. A complete solution of the 18 simultaneous equations by process of elimination requires, in our experiments, at least one month-involved labour of five continuous hours a day in virtue of the necessity of using a calculating machine of high capacity; the method of repeated trial however solved the same problem with only two day's work with a machine of low capacity; the rapid computation by sliderules may also give good results on the same problem.

It is worthy of note that for all the calculation of secondary stresses the Mechanical Tabulation Method is the most exact and simple method to get the simultaneous equations to be used in determining the redundancies, the saving in time and labour being very considerable, and it gives results, by the process of repeated trial with speed and precision, for all values substantially identical with those of the exact method of elimination.

This procedure may be well applicable to the investigation of building frames in asymmetrical condition with more or less consideration to the different boundary conditions.

Sapporo, January, 1928.